Martingale:
$\mathbb{E}[M_{n+1}] = M_n$
$\mathbb{E}[M(t) | \mathcal{F}(s)] = M(s)$

Problem 1

a Let's change notation (but with the same conceptual meaning):

$$E[M_\ell | \mathcal{F}_k] = E_k[M_\ell]$$

At step k M_k is known: $M_k = E_k[M_k]$.

$(M_\ell - M_k)$ is independent of M_σ $(\sigma \leq k)$, so:

$$E_k[M_\ell - M_k] = E_0[M_\ell - M_k] = E[M_\ell - M_k]$$

It's known that $M_0 = E[M_k]$ $(n=0,1,\dots,N)$ $\Rightarrow E[M_k] = 0$

Combining all the previous with $M_\ell = M_\ell + (M_k - M_k)$, then:

$$E_k[M_\ell] = E_k[M_\ell - M_k + M_k] = E_k[M_\ell - M_k] + E_k[M_k]$$
$$= 0 + M_k = M_k$$

$\Rightarrow$ the symmetric random walk is a Martingale.

b $[M,M]_k = \sum_{i=1}^{k} (M_i - M_{i-1})^2$, but from the explicit form of the symmetric random walk:

$$M_i - M_{i-1} = \sum_{j=1}^{i} X_j - \sum_{j=1}^{i-1} X_j = X_i$$

$$\Rightarrow \sum_{i=1}^{k} (M_i - M_{i-1})^2 = \sum_{i=1}^{k} X_i^2, \text{ but: } X_i = \begin{cases} +1 & H \\ -1 & T \end{cases}$$

$$\Rightarrow [M,M]_k = \sum_{i=1}^{k} (\pm 1)^2 = k$$

Problem 2

S.R.W. $\quad s<t$

a

- $W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt} \Rightarrow W^{(n)}(t) - W^{(n)}(s) = \frac{1}{\sqrt{n}}(M_{nt} - M_{ns})$

As M_{nt} is a symmetric random walk, ~~and~~ then $M_{nt} - M_{ns}$ is ~~also a symmetric random walk~~, (from the previous problem) independent of everything before "time" s (included).

Because $(M_{nt} - M_{ns})$ is independent of information before s $\Rightarrow E[M_{nt} - M_{ns}] = 0$

$\Rightarrow E[W^{(n)}(t) - W^{(n)}(s)] = 0$ as n is fixed

- $Var[W^{(n)}(t) - W^{(n)}(s)] = Var\Big[\underbrace{\frac{1}{\sqrt{n}}(M_{nt} - M_{ns})}_{\equiv U}\Big]$

$= E[(U - \cancelto{0}{E[U]})^2]$

$= E[U^2] = \frac{1}{n} E[M_{nt} - M_{ns}]$

and from part b. of previous problem:

$E[M_{nt} - M_{ns}] = nt - ns$

$\Rightarrow Var[W^{(n)}(t) - W^{(n)}(s)] = \frac{1}{n}(nt - ns) = t - s$

b Similarly to part a. of previous problem:

$\cancel{E[W^{(n)}(t) - W^{(n)}(s)]}$

$E[W^{(n)}(t) \mid \mathcal{F}(s)] = E[W^{(n)}(t) - W^{(n)}(s) \mid \mathcal{F}(s)] + E[W^{(n)}(s) \mid \mathcal{F}(s)]$

Because of $W^{(n)}(t) - W^{(n)}(s)$ being independent of everything before s (included):

$E[W^{(n)}(t) - W^{(n)}(s) \mid \mathcal{F}(s)] = E[W^{(n)}(t) - W^{(n)}(s)]$

$= 0$

$\Rightarrow E[W^{(n)}(t) \mid \mathcal{F}(s)] = E[W^{(n)}(s) \mid \mathcal{F}(s)] = W(s)$

$\Rightarrow$ Brownian motion is a Martingale.

$\boxed{c}$ $$\sum_{j=1}^{nt} \Big(\underbrace{W^{(100)}\left(\tfrac{j}{100}\right) - W^{(100)}\left(\tfrac{j-1}{100}\right)}_{\frac{1}{\sqrt{100}}\left(\sum_{i=1}^{j} x_i - \sum_{i=1}^{j-1} x_i\right) = \frac{1}{\sqrt{100}}(x_j)} \Big)^2 \equiv S$$

$$\Rightarrow S = \frac{1}{100}\sum_{j=1}^{nt} (x_j)^2 = \frac{nt}{100} = \frac{100 \cdot 1{,}48}{100} = 1{,}48$$

$\boxed{d}$ Similar to the previous part:

$$S = \sum_{j=1}^{nt} \Big(\underbrace{W^{(n)}\left(\tfrac{j}{n}\right) - W^{(n)}\left(\tfrac{j-1}{n}\right)}_{\frac{1}{\sqrt{n}} x_j} \Big)^2$$

$$\Rightarrow S = \sum_{j=1}^{nt} \left(\frac{1}{\sqrt{n}} x_j \right)^2 = t$$

Problem 3

a The form of Z_i is:

$$Z_i = f''(W(t_i))\left(\left(W(t_{i+1}) - W(t_i)\right)^2 - (t_{i+1} - t_i)\right)$$

which is clearly $\mathcal{F}(t_{i+1})$ - measurable, as all stochastic variables are known at time t_{i+1}, i.e. $W(t_{i+1})$ and $W(t_i)$ are known at time t_{i+1}.

From the form of Z_i:

known at t_i

$$E[Z_i \mid \mathcal{F}(t_i)] = f''(W(t_i))\left(E\left[\left(W(t_{i+1}) - W(t_i)\right)^2 \mid \mathcal{F}(t_i)\right] - (t_{i+1} - t_i)\right)$$

non-stochastic part

$E[(W(t_{i+1}) - W(t_i))^2 \mid \mathcal{F}(t_i)] = t_{i+1} - t_i$, as the starting point for measuring is t_i, so $t_{i+1} - t_i$ is the variance.

$$\Rightarrow E[Z_i \mid \mathcal{F}(t_i)] = 0$$

$$Z_i^2 = \underbrace{\left(f''(W(t_i))\right)^2}_{\text{known at } t_i} \underbrace{\left[\left(W(t_{i+1}) - W(t_i)\right)^2 - (t_{i+1} - t_i)\right]^2}_{\left(W(t_{i+1}) - W(t_i)\right)^4 - 2\left(W(t_{i+1}) - W(t_i)\right)^2(t_{i+1} - t_i) + (t_{i+1} - t_i)^2}$$

Since $W(t_{i+1}) - W(t_i)$ is normally distributed:

$$E[(W(t_{i+1}) - W(t_i))^4 \mid \mathcal{F}(t_i)] = 3(t_{i+1} - t_i)^2$$

non-stochastic

Also: $E[\underbrace{(W(t_{i+1}) - W(t_i))^2}_{\text{mean} = t_{i+1} - t_i}(t_{i+1} - t_i) \mid \mathcal{F}(t_i)] = (t_{i+1} - t_i)^2$

$$\Rightarrow E[Z_i^2 \mid \mathcal{F}(t_i)] = \left(f''(W(t_i))\right)^2 \left(\underbrace{3(t_{i+1} - t_i)^2 - 2(t_{i+1} - t_i) + (t_{i+1} - t_i)}_{2\cdot(t_{i+1} - t_i)^2}\right)$$

$\boxed{b}$

The result from this part implies:

$$\lim_{\|\Pi\|\to 0} \frac{1}{2}\sum_{i=0}^{n-1} f''(W(t_i))\left(W(t_{i+1}) - W(t_i)\right)$$

$$= \int_0^T f''(W(t_i))\,dt$$

As $E[Z_i \mid \mathcal{F}(t_i)] = 0 \Rightarrow E\sum_{i=0}^{n-1} Z_i = \sum_{i=0}^{n-1} E[Z_i] = 0$

On the other hand:

$$\lim_{\|\Pi\|\to 0} E\left[\left(\sum_{i=0}^{n-1} Z_i - \underbrace{E\left[\sum_{i=0}^{n-1} Z_i\right]}_{0}\right)^2\right] = \lim_{\|\Pi\|\to 0} E\left[\sum_{i=0}^{n-1} Z_i^2 + \sum_{0\le i\le j\le k} Z_i Z_j\right]$$

$$= \lim_{\|\Pi\|\to 0}\left(\sum E[Z_i^2] + \sum_{0\le i\le j\le k} E[Z_i Z_j]\right)$$

Z_i and Z_j are independent, as they depend on different values of $W(t_{i+1}) - W(t_i)$, and as $E[Z_i] = 0$, then:

$$\lim_{\|\Pi\|\to 0} \operatorname{Var}\left[\sum_{i=0}^{n-1} Z_i\right] = \lim_{\|\Pi\|\to 0}\left(\sum_{i=0}^{n-1} E[Z_i^2]\right)$$

$$\le \lim_{\|\Pi\|\to 0} \max(t_{i+1} - t_i) \sum f''(W(t_i))(t_{i+1} - t_i)$$

$$\underbrace{\|\Pi\|}_{=0} \int f''(W(t_i))\,dt = 0$$

Therefore, as μ and σ^2 are zero for $\sum_{i=0}^{n-1} Z_i$, it's zero!

$\boxed{\text{Problem 4.}}$

From scaled symmetric random walk; $M_0 = E[M_{nt}] \Rightarrow 0 = E[W^{(n)}(t)]$

and then: $E[W(t)] = 0$ for Brownian motion. Then:

$0<s<t$

$$E[W(t) \mid \mathcal{F}(s)] = E[W(t) - W(s) + W(s) \mid \mathcal{F}(s)]$$

$$= E[\underbrace{W(t) - W(s)}_{\text{independent of } \mathcal{F}(s)} \mid \mathcal{F}(s)] + \underbrace{E[W(s) \mid \mathcal{F}(s)]}_{W(s)}$$

independent of $\mathcal{F}(s)$, which implies:

$$E[W(t) \mid \mathcal{F}(s)] = \underbrace{E[W(t) - W(s)]}_{0} + W(s)$$

$$= W(s) \Rightarrow \text{Brownian motion is a Martingale.}$$

Problem 5

$$dX(t) = P(t)\,dt + 4\,Q(t)\,dW(t)$$

$$\Rightarrow dX(t)\,dX(t) = (P(t))^2\,\cancel{dt\,dt}^{\,0} + 8\,P(t)\,Q(t)\,\cancel{dW\,dt}^{\,0}$$

$$+ 16\,(Q(t))^2\,\cancel{dW\,dW}^{\,dt}$$

$$= (P(t))^2 \cdot 0 + 8 \cdot (P(t)\,Q(t)) \cdot 0$$

$$+ 16\,(Q(t))^2\,dt$$

$$\Rightarrow df = 16\,(Q(t))^2\,dt$$

Problem 6

$$df = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}\right)dt + \frac{\partial f}{\partial W}dW$$

a

$$\frac{\partial f}{\partial t} = 0 \qquad \frac{\partial f}{\partial W} = 2W \qquad \frac{\partial^2 f}{\partial W^2} = 2$$

$$\Rightarrow df = dt + 2W\,dW$$

b

$$\frac{\partial f}{\partial t} = -1 \qquad \frac{\partial f}{\partial W} = 2W \qquad \frac{\partial^2 f}{\partial W^2} = 2$$

$$\Rightarrow df = 2W\,dW$$

c

$$\frac{\partial f}{\partial t} = -W \qquad \frac{\partial f}{\partial W} = W^2 - t \qquad \frac{\partial^2 f}{\partial W^2} = 2W$$

$$\Rightarrow df = \left(-W + \frac{1}{2}\cdot 2W\right)dt + (W^2 - t)\,dW$$

$$= (W^2 - t)\,dW$$

d

$$\frac{\partial f}{\partial t} = -re^{-rt}W \qquad \frac{\partial f}{\partial W} = e^{-rt} \qquad \frac{\partial^2 f}{\partial W^2} = 0$$

$$\Rightarrow df = (-re^{-rt}W)\,dt + (e^{-rt})\,dW$$

$$= e^{-rt}(-rW\,dt + dW)$$

e

$$\frac{\partial f}{\partial t} = 0 \qquad \frac{\partial f}{\partial W} = 2\cdot e^{2W} \qquad \frac{\partial^2 f}{\partial W^2} = 4e^{2W}$$

$$\Rightarrow df = 2\cdot e^{2W}(dt + dW)$$

$\boxed{f}$ $f = e^{\sigma W} \cdot e^{-\frac{1}{2}\sigma^2 t}$

$$\frac{\partial f}{\partial t} = -\frac{1}{2}\sigma^2 \cdot e^{\sigma W} e^{-\frac{1}{2}\sigma^2 t}$$

$$\frac{\partial f}{\partial W} = \sigma \cdot e^{\sigma W} \cdot e^{-\frac{1}{2}\sigma^2 t} \qquad \frac{\partial^2 f}{\partial W^2} = \sigma^2 \cdot e^{\sigma W} \cdot e^{-\frac{1}{2}\sigma^2 t}$$

$$\Rightarrow df = \left(-\frac{1}{2}\sigma^2 \cdot e^{\sigma W} e^{-\frac{1}{2}\sigma^2 t} + \frac{1}{2}\sigma^2 e^{\sigma W} \cdot e^{-\frac{1}{2}\sigma^2 t}\right) dt + \sigma e^{\sigma W} \cdot e^{-\frac{1}{2}\sigma^2 t} dW$$

$$\Rightarrow df = \sigma e^{\sigma W} e^{-\frac{1}{2}\sigma^2 t} dW$$

Any process involving a time integral is not a Martingale, as it has extra information after $\mathcal{F}(s)$.

Therefore, processes b., c., f.

Equivalently, they are Martingales as they are Itô integrals.

$\boxed{\text{Problem 7}}$ $f = \frac{1}{2}W^2 \Rightarrow f(T, W(T)) = \frac{1}{2}W(T)^2$

$f(0) = 0$

And also: $\frac{\partial f}{\partial t} = 0 \quad \frac{\partial f}{W} = W \quad \frac{\partial^2 f}{\partial W^2} = 1$

$\Rightarrow df = \left(\frac{1}{2}\cdot 1\right)dt + W\,dW$

$\Rightarrow f(T) - f(0) = \frac{1}{2}T + \int_0^T W\,dW$

$\Rightarrow \int_0^T W\,dW = \frac{1}{2}\left((W(T))^2 - T\right)$

$\boxed{\text{Problem 8}}$

Given $f = xy$ (or more ~~specifically~~ explicitly $f(t, x(t), y(t)) = x(t) \cdot y(t)$)

then:

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2} dx\,dx + \frac{\partial^2 f}{\partial x \partial y} dx\,dy + \frac{\partial^2 f}{\partial y^2} dy\,dy \right)$$

$$= 0 \cdot dt + y \cdot dx + x \cdot dy + \frac{1}{2}\left(0 \cdot dx\,dx + 1 \cdot dx\,dy + 0 \cdot dy\,dy \right)$$

$$\Rightarrow df = d(xy) = y\,dx + y\,dx + dx\,dy$$

$\boxed{\text{Problem 9}}$

For an Itô process: $dS = \overset{\nearrow A}{\mu S}\,dt + \overset{\nearrow B}{\sigma S}\,dW$

$$\Rightarrow df = f_t\,dt + f_S\,dS + \frac{1}{2} f_{SS} (\sigma S)^2\,dt$$

With: $f = e^{-rt} S$

then:

$$df = \left(-re^{-rt} S + \frac{1}{2} \cdot \overset{f_{SS}}{0} \cdot (\sigma S)^2 \cdot \right) dt + e^{-rt} \cdot dS$$

$$\Rightarrow df = \left(-r \cdot S\,dt + dS\right) \cdot e^{-rt}$$

$$= \left(-r \cdot S\,dt + (\mu S\,dt + \sigma S\,dW)\right) e^{-rt}$$

$$= e^{-rt} \cdot \left((\mu - r) S\,dt + (\sigma S)\,dW \right)$$

$\boxed{\text{Problem 10}}$

$$dS = \mu S\, dt + \sigma S\, dW$$

$\boxed{a}$

$$dc = \frac{\partial c}{\partial t} \cdot dt + \frac{\partial c}{\partial S} \cdot dS + \frac{1}{2} \frac{\partial^2 c}{\partial S^2} (\sigma S)^2 dt$$

$$= \underbrace{\left(\frac{\partial c}{\partial t} + \mu S \frac{\partial c}{\partial S} + \frac{1}{2} \frac{\partial^2 c}{\partial S^2} (\sigma S)^2 \right)}_{A_c} dt$$

$$+ \underbrace{\sigma S \frac{\partial c}{\partial S}}_{B_c} dW$$

Then:

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial c} dc + \frac{1}{2} \frac{\partial^2 f}{\partial c^2} (B_c)^2 dt$$

$$\frac{\partial f}{\partial t} = -r \cdot e^{-rt} c \qquad \frac{\partial f}{\partial c} = e^{-rt} \qquad \frac{\partial^2 f}{\partial c^2} = 0$$

$$\Rightarrow df = \left(-r e^{-rt} c\right) dt + \left(e^{-rt}\right) dc$$

$$= e^{-rt} \left(-r \cdot c + \frac{\partial c}{\partial t} + \mu S \frac{\partial c}{\partial S} + \frac{1}{2} \frac{\partial^2 c}{\partial S^2} (\sigma S)^2 \right) dt$$

$$+ e^{-rt} \cdot \sigma S \cdot \frac{\partial c}{\partial S} dW$$

Problem 11

a $\quad dS(t) = \alpha\, dt + \sigma\, dW(t)$

From direct integration:

$$S(T) - S(0) = \alpha T + \sigma W(T)$$

$$\Rightarrow S(T) = S_0 + \alpha T + \sigma W(T)$$

b $\quad dS(t) = \cancel{\alpha dt / \sigma}\ \alpha S(t)\, dt + \sigma S(t)\, dW(t)$

Take: $f = \ln S \Rightarrow \frac{\partial f}{\partial t} = 0 \, , \quad \frac{\partial f}{\partial S} = \frac{1}{S} \, , \quad \frac{\partial^2 f}{\partial S^2} = \frac{-1}{S^2}$

$$\Rightarrow df(S) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma\, dW$$

$$\Rightarrow f(T) = f(t=0) + \left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma W(T)$$

$$f(t=0) = \ln(S(t=0)) = \ln S_0$$

$$\Rightarrow S(T) = e^{f(T)} = S_0 \cdot \exp\left\{ \left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma W(T) \right\}$$

c Similar to the previous case, but just skip one integration step:

$$S(T) = S_0 \cdot \exp\left\{ f(T) - f(0) \right\}$$

$$f(T) - f(0) = \int_0^T \left(\mu(t) - \frac{1}{2}\sigma^2(t)\right) dt + \int_0^T \sigma(t)\, dW(t)$$

$$\Rightarrow S(T) = S_0 \cdot \exp\left\{ \int_0^T \left(\mu(t) - \frac{1}{2}\sigma^2(t)\right) dt + \int_0^T \sigma(t)\, dW(t) \right\}$$

$\boxed{\text{Problem 12}}$

$\boxed{a}$ $S(t) = S_0 + \alpha t + \sigma W(t)$

$$\frac{\partial S}{\partial t} = \alpha \quad \frac{\partial S}{\partial W} = \sigma \quad \frac{\partial^2 S}{\partial W^2} = 0$$

$$\Rightarrow dS = \alpha\, dt + \sigma\, dW$$

$\boxed{b}$

$$S(t) = S_0 \cdot \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right\}$$

$$\frac{\partial S}{\partial t} = \left(\mu - \frac{1}{2}\sigma^2\right) S(t) \quad \frac{\partial S}{\partial W} = \sigma S(t) \quad \frac{\partial^2 S}{\partial W^2} = \sigma^2 S(t)$$

$$\Rightarrow dS = \left(\mu \cancel{- \frac{1}{2}\sigma^2}\right) S(t)\, dt + \sigma S(t)\, dW + \cancel{\frac{1}{2}\sigma^2 S(t) dt}$$

$$= \mu S(t)\, dt + \sigma S(t)\, dW$$

$\boxed{c}$ $S(t) = S_0 \cdot \exp\left\{ \int_0^t \left(\mu(u) - \frac{1}{2}\sigma^2(u)\right) dt + \int_0^t \sigma(u)\, dW \right\}$

and since: $\frac{\partial}{\partial t}\int_0^t f(u)\,du = f(t)$, then:

$$\cancel{\frac{\partial S}{\partial t} = \mu(t) - \frac{1}{2}\sigma(t)^2 \quad \frac{\partial S}{\partial W} =}$$

$$\frac{\partial S}{\partial t} = \left(\mu(t) - \frac{1}{2}\sigma^2(t)\right) S(t), \quad \frac{\partial S}{\partial W} = \sigma(t) S(t), \quad \frac{\partial^2 S}{\partial W^2} = \sigma^2(t) S(t)$$

$$\Rightarrow dS = \mu(t) S(t)\, dt + \sigma(t) S(t)\, dW$$

Problem 13

a) It is known that: $d(xy) = x\,dy + y\,dx + dx\,dy$

Then:

$$d(e^{\beta t}R(t)) = d(e^{\beta t})R(t) + e^{\beta t}d(R(t)) + d(e^{\beta t})\,d(R(t))$$

$$d(e^{\beta t}) = \beta e^{\beta t}dt$$

$$d(R(t)) = (\alpha - \beta R(t))dt + \sigma\sqrt{R(t)}\,dW$$

$$\Rightarrow d(e^{\beta t}R) = \cancel{\beta R(t) e^{\beta t}dt} + e^{\beta t}\left((\alpha - \cancel{\beta R(t)})dt + \sigma\sqrt{R(t)}\,dW\right) + \underbrace{\beta e^{\beta t}dt\left((\ldots)dt + (\ldots)dW\right)}_{=0}$$

$$\Rightarrow d(e^{\beta t}R) = \sigma e^{\beta t}\sqrt{R(t)}\,dW + \alpha e^{\beta t}dt$$

$$\int_0^t e^{\beta u}du = \frac{1}{\beta}(e^{\beta t}-1)$$

b) Integration last expression from previous part:

$$e^{\beta t}R(t) - R(0) = \sigma\int_0^t e^{\beta u}\sqrt{R(u)}\,dW(u) + \alpha\int_0^t e^{\beta u}du$$

$$\Rightarrow e^{\beta t}R(t) = R(0) + \sigma\underbrace{\int_0^t e^{\beta u}\sqrt{R(u)}\,dW(u)}_{=I} + \frac{\alpha}{\beta}(e^{\beta t}-1)$$

an Itô integral

c) For an Itô integral:

$$E(I) = 0$$

$$\Rightarrow E[e^{\beta t}R(t)] = e^{\beta t}E[R(t)] = E[R(0)] + \sigma E[I] + \underbrace{E\left[\frac{\alpha}{\beta}(e^{\beta t}-1)\right]}_{\text{deterministic}}$$

deterministic

Itô

$$= R(0) + \frac{\alpha}{\beta}(e^{\beta t}-1)$$

$\boxed{\text{Problem 14}}$

$\boxed{a}$ As stated before:

$$d(xy) = x\,dy + y\,dx + dx\,dy.$$

$$\Rightarrow d\Bigg(\underbrace{\exp\left\{\int_0^u b(s)\,ds\right\}}_{\equiv A} R(u)\Bigg) = A\,dR + R\,dA + dR\,dA$$

$$dA = \frac{\partial A}{\partial u}\,du = b(u)\,A\,du$$

$$dR = (\alpha - bR)\,du + \sigma\,d\tilde{W}$$

$$\Rightarrow d(AR) = e^{\int_0^u b(s)ds}\Big((\alpha - \cancel{bR})\,du + \sigma\,d\tilde{W}\Big) + \cancel{Rb}\,e^{\int_0^u b(s)ds}\,du + \underbrace{(bA\,du)\big((\sim)\,du + (\sim)\,d\tilde{W}\big)}_{=0}$$

$$d\left(e^{\int_0^u b(s)ds} R\right) = e^{\int_0^u b(s)ds}\left(\alpha\,du + \sigma\,d\tilde{W}\right)$$

$\boxed{b}$ Integrating from t to T: (last expression before)

$$e^{\int_0^T b(s)ds} R(T) = e^{-\int_0^t b(s)ds}\,\cancelto{r}{R(t)} + \int_t^T e^{\int_0^u b(s)ds}\left(\alpha\,du + \sigma\,d\tilde{W}\right)$$

Because: $\left.\dfrac{e^{\int_0^a f(x)dx}}{e^{\int_0^b f(x)dx}}\right|_{b<a} = e^{\int_b^a f(x)dx}$, then: $\left(\begin{array}{c}\text{dividing}\\ \text{both sides}\\ \text{by: } e^{\int_0^T b(s)ds}\end{array}\right)$

$$R(T) = r\cdot\frac{e^{-\int_0^t b(s)ds}}{e^{\int_0^T b(s)ds}} + \frac{1}{e^{\int_0^T b(s)ds}}\int_t^T e^{\int_0^u b(s)ds}\left(\alpha\,du + \sigma\,d\tilde{W}\right)$$

$$\Rightarrow R(T) = r e^{-\int_t^T b(s)\,ds} + \int_t^T e^{-\int_u^T b(s)\,ds} \alpha(u)\,du + \int_t^T e^{-\int_u^T b(s)\,ds} \sigma(u)\,d\tilde{W}(u)$$

$\boxed{c}$ Because for an Itô integral $E(I) = 0$ and:

$$Var[I] = \int_{a_0}^{b_0} G^2(s)\,ds, \quad \text{where } I = \int_{a_0}^{b_0} G(s)\,dW(s)$$

then:

$$E[R] = r e^{-\int_t^T b(s)\,ds} + \int_t^T e^{-\int_u^T b(s)\,ds} \alpha(u)\,du$$

$$Var[R] = \int_t^T \left(e^{-\int_u^T b(s)\,ds} \sigma(u) \right)^2 du$$

Normal random variable: (conditions)

$$\left. \begin{aligned} M &= E[X] \\ V &= Var[X] = \sigma^2 \end{aligned} \right\} \Rightarrow E[e^{uX}] = e^{\frac{1}{2}u^2\sigma^2}$$

Then, it is clear that R is normally distributed around $E[R]$, because of the normal nature of the Itô integral in R

$$I(T) = \int_t^T e^{-\int_u^T b(s)\,ds} \sigma(u)\,d\tilde{W}(u)$$

as it's known that Itô integrals are normally distributed.

$\boxed{\text{Problem 15}}$

$\boxed{a}$

$$\frac{1}{Z} = \underbrace{e^{+\int_0^t \Theta(u)\,dW(u)}}_{A} \cdot \underbrace{e^{+\frac{1}{2}\int_0^t \Theta^2(u)\,du}}_{B}$$

$$d(AB) = A\,dB + B\,dA + dA\,dB$$

As $dA \propto (dW \| dt)$ and $dB \propto dt \Rightarrow dA\,dB = 0$

$$\Rightarrow d(AB) = d\left(\frac{1}{Z}\right) = A\cdot\left(\frac{1}{2}\Theta^2(t)\,B\right)dt \; +$$

$$B\cdot\left(\frac{1}{2}\Theta^2 A\,dt + \Theta A\,dW\right)$$

$$= \frac{1}{2}\Theta^2(t)\cdot\frac{1}{Z}\cdot 2\cdot dt + \Theta\cdot\frac{1}{Z}\,dW$$

$\boxed{b}$

$$d\left(\frac{M}{Z}\right) = M\,d\left(\frac{1}{Z}\right) + \frac{1}{Z}\,dM + dM\cdot d\left(\frac{1}{Z}\right)$$

$$= \frac{M}{Z}\left(\Theta^2 dt + \Theta\,dW\right) + \frac{1}{Z}\cdot\Gamma\,dW \; +$$

$$\frac{\Gamma}{Z}\,dW\left(\Theta^2 dt + \Theta\,dW\right)$$

$$\Rightarrow d\left(\frac{M}{Z}\right) = \left(\frac{M}{Z}\Theta^2 + \frac{\Gamma}{Z}\Theta\right)dt \; +$$

$$\left(\frac{M}{Z}\Theta + \frac{\Gamma}{Z}\right)dW$$

$$= \frac{1}{Z}\left(M\Theta + \Gamma\right)\left(\Theta\,dt + dW\right)$$

$\boxed{\text{Problem 16}}$ $\quad Z = \frac{S}{S_h}$

using Itô product

$$dS = Z\,dS_h + S_h\,dZ + dZ\,dS_h$$

$$= S_h\,dZ + \frac{S}{S_h}\left(\mu S_h\,dt + \sigma S_h\,dW\right) + dZ\,dS_h$$

Using Itô formula for $Z = \frac{S}{S_h} = Z(t, S(t), S_h(t))$:

$$dZ = \cancel{\frac{\partial Z}{\partial t}}^{0} dt + \frac{\partial Z}{\partial S}\,dS + \frac{\partial Z}{\partial S_h}\,dS_h + \frac{1}{2}\cancel{\frac{\partial^2 Z}{\partial S^2}}^{0} dt + \frac{1}{2}\frac{\partial^2 Z}{\partial S_h^2}\,dt$$

$$= \frac{1}{S_h}\,dS + \left(\frac{-S}{S_h^2}\right) dS_h + \frac{1}{2}\left(\frac{S}{S_h^3}\right) B^2\,dt$$

$B = \sigma S_h$

$$= \frac{1}{S_h}\left(f\,dt + g\,dW\right) + \frac{1}{2}\cdot\frac{S}{S_h}\sigma^2\,dt)$$

$$\Rightarrow dZ\,dS_h = g\sigma\,dt$$

$$\Rightarrow dS = S_h\,dZ + (S\mu\,dt + \sigma S\,dW) + g\sigma\,dt$$

$$\Rightarrow dZ = \frac{1}{S_h}\left(dS - (\mu S\,dt + \sigma S\,dW) - g\sigma\,dt\right)$$

$$= \frac{1}{S_h}\left((f - g\sigma)\,dt + g\,dW\right)$$

Integrating dZ:

$$Z(t) - Z(0) = \int_0^t \frac{1}{S_h(u)} \cdot \Big(\big[f(u) - g(u)\,\sigma(u) \big] \Big) du + \int_0^t \frac{g(u)\,dW(u)}{S_h(u)} \Big)$$

Where:

$$S_h(u) = \cancel{S_h(0)}^{\,1} \cdot \exp\left\{ \int_0^u \left(\mu(x) - \frac{1}{2}\sigma^2(x) \right) dx + \int_0^u \sigma(x)\, dW(x) \right\}$$

$$\frac{S}{S_h} = Z \Rightarrow \frac{S(t)}{S_h(t)} - \frac{S(0)}{\cancel{S_h(0)}^{\,1}} = Z(t) - Z(0)$$

$$\Rightarrow S(t) = S_h(t) \cdot \left(S(0) + \int_0^t \frac{f(u) - g(u)\,\sigma(u)}{S_h(u)}\, du + \int_0^t \frac{g(u)}{S_h(u)}\, dW(u) \right)$$

Problem 17.

a

It is known that, for:

$$dS = rS\,dt + \sigma S\,d\tilde{W}$$

the solution $S(T)$ is of the form:

$$S(T) = S(0)\cdot \exp\left\{ \underbrace{\int_0^T \left(r(t) - \frac{1}{2}\sigma^2(t)\right) dt}_{A:\ \text{deterministic}} + \underbrace{\int_0^T \sigma(t)\,d\tilde{W}(t)}_{B:\ \text{non-deterministic}} \right\}$$

The non-deterministic part is an Itô integral, i.e. $B = I$.
As B is an Itô integral, it is normally distributed, therefore:

$$X \equiv A + B$$

is normally distributed. Also:

$$E[X] = \int_0^T \left(r(t) - \frac{1}{2}\sigma^2(t)\right) dt$$

$$Var[X] = \int_0^T \sigma^2(t)\,dt$$

b From the given form for the standard BSM:

$$BSM(T, S(0), K, R, \Sigma) =$$

$$S(0)\cdot N\left(\frac{\log\left(\frac{S(0)}{K}\right) + RT + \frac{1}{2}T\left(\frac{1}{T}\int_0^T \sigma^2(u)\,du\right)}{\sqrt{T}\sqrt{\frac{1}{T}\int_0^T \sigma^2(u)\,du}} \right)$$

$$- e^{-RT}\cdot K\cdot N\left(\frac{\log\left(\frac{S(0)}{K}\right) + RT - \frac{1}{2}\cdot T\left(\frac{1}{T}\int_0^T \sigma^2(u)\,du\right)}{\sqrt{T}\sqrt{\frac{1}{T}\int_0^T \sigma^2(u)\,du}} \right)$$

On the other hand:

$$c(0, S(0)) = e^{-\int_0^T r(u)\,du} \cdot E\left(S(T) - K\right)^+$$