MSc. High Performance Computing

TCD

Quantitative Finance

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Time

- Throughout this course we treat time as a continuous, indivisible variable and we look at prices of instruments that are functions of this continuous time.
- Is this valid?
- Modern Quantitative Finance is evolving with current active research and development in jumpy, discontinuous and illiquid markets. This is beyond the scope of an introductory course but should be borne in mind.

Intraday, Day End, Fixing

- Trading takes place in discrete locations (Tokyo, Frankfurt, London and New York being the biggest and best known.)
- Prices continually fluctuate intraday (intraday time)

$$t.$$

- Every financial institution must mark its trading positions to market once a day. For the purposes of closing prices, time is discrete, daily and we can happily talk about t_j meaning the close of business time on the jth day.

$$t_j.$$

- There are other important intraday times, such as 11am London time when Libor (an index we will come accross many times) is fixed for the purposes of settling Swaps and other derivatives that depend on it. So Libor rates are traded continuously throughout the day, but the 11am snapshot data, cleaned according to the Libor rules, is used for settling trades that reference Libor as an index.

Calendars and Business Days

- Trading is National
- A location must be open (business day) to trade
- Holidays need to be known for years in advance
- Some holidays are Global, some regional, some local

Interest

We spend much time dealing with interest and the associated concept of the time value of money. The calculation of interest is straightforward once the rules for computing it are understood. However it is a cause of confusion and so we spend a little time on the rules for computing interest.

Interest accrues between two dates. Interest rates are generally quoted as annual simple rates and in order to calculate them we need to know what one year, or a fraction of a year is. To do this we use a day count fraction (also known as year fraction). The day count fraction is simply (number of days)/ (number of days in a year). Put in symbols:

$$\delta = \frac{days(t_1, t_2)}{basis}$$

The confusion arises because historically, different locations developed different ways of measuring the number of days in a year (month, week). All of the methods seem rational but all give rise to different fractions.

Common Day Counts

Some common ones are

- Actual 360: $basis = 360$, $days(t_1, t_2)$ is the number of calender (not business) days in the interest period spanning t_1 and t_2.
- Thirty 360, also known as 30/360 or Bond Basis: $basis = 360$, every full month is counted as 30 days (US, European and Italian implementations differ slightly on the treatment of month ends)
- Actual 365: $basis = 365$, $days(t_1, t_2)$ is the number of calender (not business) days in the interest period spanning t_1 and t_2
- ActualActual(ISDA): Period t_1 to t_2 is decomposed into three sections spanning t_1 to year end, a period of n full years, and final year start to t_2. $basis_1$ ($basis_2$) for the first (last) section is 366 if t_1 (t_2) is in a leap year, otherwise 365. The overall year fraction is then computed as

$$days/basis = days_1/basis_1 + n + days_2/basis_2.$$

There are slightly deviating implementations of this method (according to ISMA, AFB),
see http://www.isda.org/publications/pdf/Day-Count-Fraction1999.pdf

Simple Interest

The interest rate quoted needs to be interpreted together with its *day count convention*. The interest amount A is computed with *simple compounding* as follows,

$$A = N \cdot r \cdot \delta, \qquad \delta = \frac{days(t_1, t_2)}{basis} \tag{1}$$

where N is the notional amount, r is the interest rate, and δ stands for the *year fraction* or *day count fraction*. The notional plus interest is

$$N(1 + r\delta(t_1, t_2)) \tag{2}$$

Compound Interest to Continuous Interest

In compound interest the period from t_1 to t_n is split up into n interest periods and a simple rate applied in each period to the principal plus interest from the previous. So the notional plus interest after n periods is

$$N(1 + r\delta(t_1, t_2)) \cdot (1 + r\delta(t_2, t_3)) \dots (1 + r\delta(t_{n-1}, t_n)).$$

One can then ask what is the simple interest rate applied over t_1, t_n that would give the same notional plus interest. This simple simple rate that is equivalent the compounded amount is the basis for common equivalent rates like AER and APR that have to be notified to consumers to avoid confusion over basis and compounding.

One can take this process to its logical conclusion by adding more and more time periods. Say we use and Actual 365 basis and for n periods in a year at a rate of z, then we get notional plus interest of

$$N(1 + \frac{z}{n}).(1 + \frac{z}{n}) \dots (1 + \frac{z}{n}) = N(1 + \frac{z}{n})^n.$$

Now we take this process to its natural conclusion by taking the limit as n tends to infinity (compounding over time intervals that add up to one year) we get

$$N \lim_{n \to \infty} (1 + \frac{z}{n})^n = Ne^z.$$

More generally, if we take t years, we get

$$Ne^{zt} \tag{3}$$

This rate, z, in (3) is known as the continuously compounded or zero interest rate. The reason it's known as the zero rate will hopefully become clearer soon.

It is common in Quantitative Finance to use continuously compounded rates as it avoids the need to define interest conventions[1] and is more computationally

Settlement

Settlement is the business of paying money or equivalents that fall due under the terms of a financial contract. Settlement can only take place if both counterparties are at work (business days in both locations). Leaving aside the details of how settlement occurs, there are two basic types

- Cash Settlement
- Physical Settlement

OTC vs Exchange Traded

- OTC, Exchange Traded, Standard Indices
- Exchange Traded Highly standardised and generally highly liquid
- OTC Over the Counter: Private, often non-standard

Replication and Arbitrage

"In an efficient market there should be no free money. Put another way, you should not be able to construct a set of trades today that lock in a profit today without taking risk."

- The key phrase is actually the "today" in the last sentence.
- We do not, as financial modellers, aim to predict the future.
- What we do is aim to eliminate the possibility of locking in a risk free profit today based on the knowable state of financial markets today
- Anyone who wishes or thinks they can predict the future should seek their fortunes as a trader.

  This principle gives us the so-called replication argument: If two portfolios have the same cashflows under all circumstances then they have the same price.

  This allows us to a decompose complex trade into a sum of simpler trades that we know how to price. The complex trade has the same price as the sum of simpler trades that replicate the payoff.

Cash-flow Discounting

The simplest form of the replication argument is used to price instruments with known cashflows in the future. The instrument is decomposed into a sum of the individual known cashflows. We know the value of a fixed cashflow in the future today, it is

$$P(t, t_i) \cdot cashflow(t_i)$$

and so the instrument is priced. $P(t, t_i)$ denotes the value at time t of an elementary financial instrument that pays the amount 1 at its maturity t_i. For the moment we assume it is given. Later in this section, we will come back to the question how $P(t, t_i)$ is expressed in terms of financial instrument prices observed in the market. For example, one choice is to associate $P(t, t_i)$ with the value of a zero bond with maturity t_i.

Notation: We use $P(t, t_j)$ to denote non-stochastic discount factors. We reserve $D(t, t_j)$ for stochastic discount factors in later applications where the distinction is important.

Deposit

A Deposit is the most basic type of *over the counter (OTC)* trade: The counterparties enter into an agreement on date t_1 (*trade date*) to exchange a cash amount N (*notional*) on date $t_2 \geq t_1$ (*effective date*) and to return the amount N plus interest on date t_3 (*maturity date*).

Deposits are done as inter-bank trades for short term liquidity management, but of course also taken from customers.

MNKR45

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Cash Deposit Composite (Bid Rates)

	USD	EUR	CHF	JPY	AUD	CAD
ON	0.68	-0.50				
TN	0.68	-0.50		-0.150		
SN	0.68	-0.50		-0.250		-----
1W	0.72	-0.50		-0.250		
2W	0.72	-0.50		-----		-----
3W	0.72	-0.50		-----		-----
1M	0.70	-0.51		-0.250		
2M	0.85	-0.47		-0.300		
3M	1.05	-0.45		-0.250		
4M	1.10	-0.42		-0.250		-----
5M	1.15	-0.39		-0.200		-----
6M	1.20	-0.36		-0.200		
7M	-----	-0.32		-0.380		-----
8M	-----	-0.29		-0.350		-----
9M	1.45	-0.30		-0.200		
10M	-----	-0.33		-0.310		-----
11M	-----	-0.30		-0.270		-----
12M	1.60	-0.27		-0.200		

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Figure: Deposit quotes as of 24/02/17

Short Term Deposits

In the Euro zone the conventional gap between trade date t_1 and effective date t_2 is 2 days, called *spot* days. Trade periods (terms) typically range from one week (1W) up to 12 months (12M). Special short deposits are:

- Spot Next (SN): Effective in 2 days, maturity in 3 days
- Tom Next (TN): Effective tomorrow, maturity in 2 days
- Overnight (ON): Effective today, maturity tomorrow

For example, spot days are zero in the UK (GBP zone).

Calendar and Business Day Convention

Periods are often quoted generically in days, months or years from today, such as 1D, 2D, 1W, 3M, 3Y. To figure out which exact date is meant, additional information is required-

- business day calendar
- business day convention

Examples:

1. Spot start, period 7D: Use the calendar to find the effective date two business days from today, then count on to the maturity date 7 business days from the effective date
2. Spot start, period 3W: Find the effective date as above. Add 3 weeks (21 calendar days) to find the maturity date. If this date is a holiday or falls on a weekend, roll the maturity date to the previous or next business day (this is decided by the business day convention).
3. Spot start, period 3M: Find effective date as above. Add 3 months to find the maturity date, keeping the day of months (e.g. 7 May $\rightarrow$ 7 August). Roll the maturity date as above.
4. Spot start, period 2Y: Find the effective date as above. Add 2 years (24 months) to find the maturity date, keeping the day of months as above, roll the maturity date as required.

Discounting

The present value of the deposit is the *discounted* cash flow (2)

$$\Pi(t) = N \cdot (\, 1 + r(t_1, t_2) \cdot \delta(t_1, t_2)\,) \cdot P(t, t_2) \tag{4}$$

Let us assume that $t_1 = t$ is today, and the deposit above represents investment of amount N today and repayment of amount $N \cdot (1 + r\delta)$ at date t_2. If the deposit rate r is the fair rate for period $[t_1, t_2]$, the present value of deposit flow should be equal to the investment N. Hence we solve (4) for the *discount factor*

$$P(t_1, t_2) = \frac{1}{1 + r(t_1, t_2)\, \delta(t_1, t_2)}. \tag{5}$$

We come back to the cases where t_1 is later than today in section 11

As the broker screen 1 shows, there are actually two quoted Deposit rates for each maturity, e.g. 3.10 and 2.90 for maturity 6M:

A broker or market maker

- *offers* or *asks* 3.10 for a 6M Deposit where she *lends* or pays the funds initially (receives the interest), and
- *bids* 2.90 where she *borrows* or receives the funds initially (pays interest).

The bid/offer spread is a measure of the liquidity in the market. Little liquidity (few parties offering funds) will allow the broker to quote at a large bid/offer spread. High liquidity will make the spread shrink.

Zero Bond

A Zero Bond is an exchange traded security with a payoff similar to a Deposit- it pays only once at maturity date, T, the notional amount N. Maturities can be significantly longer, on the order of years and decades, than maturities of the relatively short term Deposits. Counterparty for a Zero Bond purchase is an exchange. The final payment N is ultimately received from the bond *issuer*.

Since Zero Bonds do not pay any interim interest they trade at a discount, i.e. if purchased or sold, their value $\Pi(t)$ (price $\Pi(t)/N$) will be significantly less than the notional or face amount N (price $\Pi(t)/N < 1$). The fair price as of today (t) of a Zero Bond maturing at T represents the most direct observation of a discount factor $P(t,T)$[2].

The Zero Bond price $P(t,T)$ is often expressed as a zero rate $z(T)$ as follows,

$$P(t,T) = \exp\{-z(T) \cdot \delta(t,T)\} \tag{6}$$

where $\delta(t,T)$ is the year fraction between today and maturity date T. The day count convention used here is usually Actual 365. The reason z is called a zero rate will become clear in the next section, it is a regular fixed rate bond paying an interest rate of zero percent.

[2] Later on, we will treat interest rates as stochastic, so that also the discount factor will become stochastic. In this generalization we will claim that the *expected* discount factor $\mathbb{E}[D(t,T)]$ will match the observed prices of Zero Bonds with maturity T.

Fixed Coupon Bond

A Coupon Bond is an exchange traded security paying a fixed interest rate. Most government and public sector debt is in Coupon Bond format.

The simplest fixed rate bond pays a constant coupon (bond terminology for interest rate), k, on a constant notional, N. This simple constant notional is known as a "Bullet". Discounting the cash-flows we get

$$\Pi(t) = \sum_{i=1}^{n} N\, k\, \delta(t_{i-1}, t_i)\, P(t, t_i) + N\, P(t, t_n) \tag{7}$$

A fixed coupon bond can of course have an interest rate that is fixed but is different in each period, and also allows fixed repayments (amortisations.) The pricing formula for a fixed rate amortising bond is

$$\Pi(t) = \sum_{i=1}^{n} \{N_i\, k_i\, \delta(t_{i-1}, t_i) + N_{i-1} - N_i\}\; P(t, t_i) + N_n\, P(t, t_n) \tag{8}$$

where k_i and N_i denote fixed coupon rates and notional amounts for period i, N_0 is the initial notional. The terms $N_{i-1} - N_i$ represent amortization payments. The sum starts with the current interest period, i.e. we assume $t_0 \leq today < t_1$. When notional amounts do not change, $N_i = N$, and coupon rates $k_i = 0$, the Coupon Bond reduces to a Zero Bond.

Cashflow Schematic

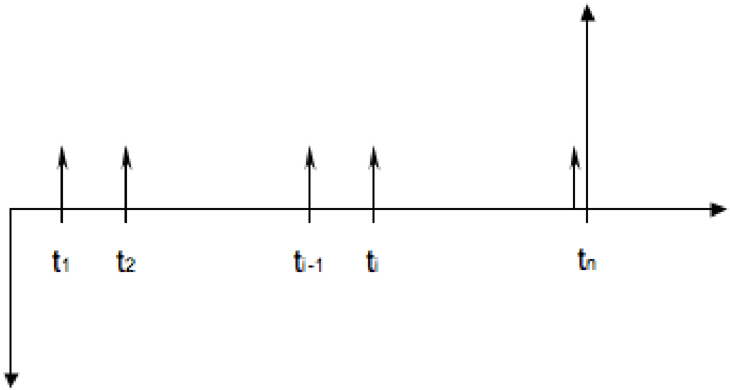


Figure: Fixed Coupon Bond cashflow schematic

Annuity and Par Rates

Let us consider a Fixed Coupon Bond with constant rate and notional, so that

$$\Pi = N\,k \sum_{i=1}^{n} \delta(t_{i-1}, t_i)\, P(t, t_i) + N\, P(t, t_n)$$

The sum $\sum_{i=1}^{n} \delta(t_{i-1}, t_i)\, P(t, t_i)$ is also called *annuity*. The rate k^* that turns the Coupon Bond value into $\Pi = N$ (or price $\Pi/N = 1$) is called *par* rate,

$$k^* \sum_{i=1}^{n} \delta(t_{i-1}, t_i)\, P(t, t_i) + P(t, t_n) = 1$$

Note if $P(t, t_i)$ are known up to index $n-1$, then $P(t, t_n)$ is determined by k^*,

$$P(t, t_n) = \frac{1 - k^* \sum_{i=1}^{n-1} \delta(t_{i-1}, t_i)\, P(t, t_i)}{1 + k^* \delta(t_{n-1}, t_n)} \tag{9}$$

In financial language, par and fair are synonymous, so par rate means the rate that would make the bond fair.

Libor

LIBOR stands for *London Interbank Offered Rate* and is a benchmark for the interest rates at which prime banks lend unsecured funds in the London interbank market. LIBOR rates are determined daily by the British Bankers' Association (BBA) since 1 January 1986. They define LIBOR as follows:

BBA LIBOR is the BBA fixing of the London Inter-Bank Offered Rate. It is based on offered interbank deposit rates contributed in accordance with the Instructions to BBA LIBOR Contributor Banks.

LIBOR rates have tenors in the range from overnight up to 12 months in ten currencies, GBP, USD, JPY, CHF, CAD etc. The BBA provides daily LIBOR fixings by polling quotes from a panel of 8, 12 or 16 banks and computing interquartile means (i.e. dropping the highest and lowest 25%).

There are similar alternative national/regional money market references such as Euribor published by the European Banking Federation as benchmark for the Euro, JPY TIBOR (Tokyo), DKK Cibor (Copenhagen), NOK Nibor (Norway), SEK Stibor (Stockholm) etc.

Fixings

11:45 02JAN09 THOMSON REUTERS BBA LIBOR RATES UK67516 LIBOR01
BRITISH BANKERS ASSOCIATION INTEREST SETTLEMENT RATES Alternative to <3750>
[02/01/09] RATES AT 11:00 LONDON TIME 02/01/2009 Disclaimer <LIBORDISC>
BBA Guide <BBAMENU>

	USD	GBP	CAD	EUR	JPY	EUR 365
O/N	0.12375	2.00000	1.71667	2.14250 SN	0.31125	2.17226
1WK	0.31375	2.00875	1.84167	2.34875	0.44375	2.38137
2WK	0.36500	2.07500	1.90333	2.41625	0.50250	2.44981
1MO	0.43000	2.11375	1.99167	2.55500	0.67125	2.59049
2MO	1.09875	2.50500	2.04500	2.73813	0.75625	2.77616
3MO	1.41250	2.70500	2.11500	2.84875	0.82375	2.88832
4MO	1.54625	2.77875	2.16000	2.88563	0.87250	2.92571
5MO	1.65000	2.83750	2.18667	2.90500	0.90750	2.94535
6MO	1.75250	2.90250	2.21667	2.94563	0.94625	2.98654
7MO	1.80250	2.93250	2.25333	2.96538	0.98250	3.00657
8MO	1.85625	2.95250	2.28167	2.97688	1.00500	3.01823
9MO	1.90625	2.97375	2.31667	2.99375	1.02875	3.03533
10MO	1.94750	2.98875	2.35000	3.00438	1.05250	3.04611
11MO	1.98750	3.00063	2.37833	3.01563	1.06875	3.05751
12MO	2.02375	3.01125	2.41667	3.02688	1.08250	3.06892

<0#LIBORSUPERRICS> RICs for above <0#LIBORRICS> Contributor RICs

Figure: Libor fixings as of 02/01/09

We will refer to the BBA LIBOR, Euribor and alternative local benchmark Deposits in the following as *Libor*. These Libor rates are widely used as reference rates for financial instruments with floating payments such as Swaps (see below).

Euribor Fixings

```
10:03 02JAN09    FBE EURIBOR RATES                        UK67516          EURIBOR01

    EURIBOR RATES ACT/360 AT 11H00 BRUSSELS TIME 02/01/2009      02/01 10:03 GMT
                  ACT/ 360                          VALUE DATE 06/01/09
         <EURIBORSWD=>   1WK      2.357          FIXING ALERTS <FIXALERT>
         <EURIBOR2WD=>   2WK      2.429  ======================================
         <EURIBOR3WD=>   3WK      2.481  EURIBOR  [Euro Interbank Offered Rate]
         <EURIBOR1MD=>   1MO      2.570  is the rate at which Euro interbank
         <EURIBOR2MD=>   2MO      2.754  term deposits within the Euro zone are
         <EURIBOR3MD=>   3MO      2.859  offered by one Prime Bank to another
                                         Prime Bank.It is computed as an average
         <EURIBOR4MD=>   4MO      2.893  of daily quotes provided for fifteen
         <EURIBOR5MD=>   5MO      2.908  maturities by a panel of 43 of the most
         <EURIBOR6MD=>   6MO      2.945  active Banks in the Euro zone.
                                         It is quoted on an act/360 day count
         <EURIBOR7MD=>   7MO      2.962  convention,and is fixed at 11:00am[CET]
         <EURIBOR8MD=>   8MO      2.976  displayed to three decimal places.
         <EURIBOR9MD=>   9MO      2.991  ======================================
                                         See <EURIBOR> for details of Panel Bank
         <EURIBOR10MD=> 10MO      3.002  contributions and historical recap
         <EURIBOR11MD=> 11MO      3.013  displays
         <EURIBOR1YD=>  12MO      3.025  ======================================
EUREPO <EUREPO> EONIA <EONIA>, LIBOR master index see <BBALIBORS> Composite
displays: [a/360] see <EURIBOR=>, [a/365] <EURIBOR365>,Disclaimer <EURIBORDISC>
EONIA SWAP INDEX <EONIAINDEX>
```

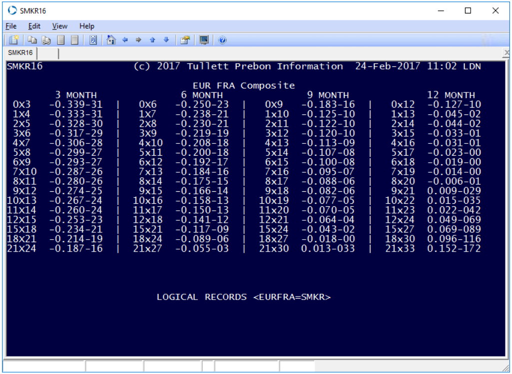
Figure: Euribor fixings as of 02/01/09

Taking Libor as a reference for spot-starting Deposit rate, we can invert (5) to express the Libor rate at time t_1 for maturity t_2 in terms of the discount factor $P(t_1, t_2)$ as

$$P(t_1, t_2) = \frac{1}{1 + L(t_1, t_2)\,\delta(t_1, t_2)} \quad \Leftrightarrow \quad L(t_1, t_2) = \frac{1 - P(t_1, t_2)}{\delta(t_1, t_2)\,P(t_1, t_2)}. \tag{10}$$

Forward Rates

In this section we will define forward rates. Forward rates are fair interest rates for investments over future rather than spot starting interest periods. We define forward rates through a traded instrument, the Forward Rate Agreement (FRA). The FRA involves three times, trade date t_1, effective date t_2 and maturity date t_3. FRAs are short term instruments spanning interest periods $[t_2, t_3]$ of typically three and six months and starting at t_2 between one and eighteen months in the future, as shown e.g. in figure 5.



SMKR16
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SMKR16 (c) 2017 Tullett Prebon Information 24-Feb-2017 11:02 LDN

EUR FRA Composite

	3 MONTH		6 MONTH		9 MONTH		12 MONTH
0X3	-0.339-31	0X6	-0.250-23	0X9	-0.183-16	0x12	-0.127-10
1X4	-0.333-31	1X7	-0.238-21	1x10	-0.125-10	1x13	-0.045-02
2X5	-0.328-30	2X8	-0.230-21	2x11	-0.122-10	2x14	-0.044-02
3X6	-0.317-29	3X9	-0.219-19	3x12	-0.120-10	3x15	-0.033-01
4X7	-0.306-28	4X10	-0.208-18	4x13	-0.113-09	4x16	-0.031-01
5X8	-0.299-27	5X11	-0.200-18	5x14	-0.107-08	5x17	-0.023-00
6X9	-0.293-27	6X12	-0.192-17	6x15	-0.100-08	6x18	-0.019-00
7X10	-0.287-26	7X13	-0.184-16	7x16	-0.095-07	7x19	-0.014-00
8X11	-0.280-26	8X14	-0.175-15	8x17	-0.088-06	8x20	-0.006-01
9X12	-0.274-25	9X15	-0.166-14	9x18	-0.082-06	9x21	0.009-029
10X13	-0.267-24	10X16	-0.158-13	10x19	-0.077-05	10x22	0.015-035
11X14	-0.260-24	11X17	-0.150-13	11x20	-0.070-05	11x23	0.022-042
12X15	-0.253-23	12X18	-0.141-12	12x21	-0.064-04	12x24	0.049-069
15X18	-0.234-21	15X21	-0.117-09	15x24	-0.043-02	15x27	0.069-089
18X21	-0.214-19	18X24	-0.089-06	18x27	-0.018-00	18x30	0.096-116
21X24	-0.187-16	21X27	-0.055-03	21x30	0.013-033	21x33	0.152-172

LOGICAL RECORDS <EURFRA=SMKR>

Figure: FRA quotes as of 24/02/17

If we are expecting a future Libor payment $L(t_2,t_3)\delta(t_2,t_3)$ at t_3 based on Libor resetting at t_2 and unknown at times $t < t_2$, a FRA contract allows locking in a future fixed interest rate K: The FRA contract has the following value at maturity date t_3,

$$\Pi(t_3) = N\,\delta(t_2,t_3)\,\left(K - L(t_2,t_3)\right). \tag{11}$$

Hence FRA contract payoff plus Libor payment will net to a deterministic fixed interest payment $N\,\delta(t_2,t_3)\,K$ at t_3. When the Libor rate resets at t_2, the FRA contract payoff is fully determined. Therefore, FRA contracts are usually settled already at effective date t_2. The contract value at t_2 is given by the discounted payoff, known at t_2:

$$\Pi(t_2) = N\,\delta(t_2,t_3)\,\left(K - L(t_2,t_3)\right)\,P(t_2,t_3) = N\,\delta(t_2,t_3)\,\frac{K - L(t_2,t_3)}{1 + L(t_2,t_3)\,\delta(t_2,t_3)}.$$

To figure out the FRA contract's value on trade date t_1, we go back to the FRA payoff at maturity (11) and start with replacing $L(t_2,t_3)$ by its definition in terms of the discount factor (10),

$$\Pi(t_3) = N\,\delta(t_2,t_3)\,\left(K - \frac{1 - P(t_2,t_3)}{\delta(t_2,t_3)\,P(t_2,t_3)}\right) = N\left(\delta(t_2,t_3)\,K + 1 - \frac{1}{P(t_2,t_3)}\right).$$

We now consider the amount represented by last term in brackets, $-1/P(t_2,t_3)$. Its value at time t_2 is obtained by discounting over period $[t_2,t_3]$, i.e. by multiplying with $P(t_2,t_3)$, so that its value at time t_2 is -1. An amount with value -1 at time t_2 is worth $-P(t_1,t_2)$ at trade time t_1.

Next, we consider the amount represented by first two terms in brackets, $\delta(t_2, t_3)\, K + 1$. It is deterministic and fixed on trade date already, so that its value at t_1 is by discounting $(\delta(t_2, t_3)\, K + 1)\, P(t_1, t_3)$.

In summary, the FRA contract value at trade time t_1 is

$$\Pi(t_1) = N\,\{\delta(t_2, t_3)\, K\, P(t_1, t_3) + P(t_1, t_3) - P(t_1, t_2)\} \tag{12}$$

The fixed rate K that makes the FRA contract fair at trade time, i.e. $\Pi(t_1) = 0$, can now be resolved to

$$K = \frac{1}{\delta(t_2, t_3)} \left(\frac{P(t_1, t_2)}{P(t_1, t_3)} - 1 \right) =: F(t_1; t_2, t_3) \tag{13}$$

and defines the *forward Libor* rate.

Finally, we rewrite (12) in terms of the just defined forward rate,

$$\Pi(t_1) = N\, P(t_1, t_3)\, \delta(t_2, t_3)\, \{\, K - F(t_1; t_2, t_3)\, \} \tag{14}$$

Comparing to the FRA payoff/value at t_3 in (11), we have done the following to obtain the FRA value at time t_1:

- multiply by discount factor $P(t_1, t_3)$
- replace $L(t_2, t_3)$ by $F(t_1; t_2, t_3)$

The latter recipe will be applied in the subsequent sections again. It is justified as long as natural Libor payments occur, i.e. the Libor rate is reset at the beginning of an interest period, paid at the end of the period, and the period length matches the Libor tenor.