Trinity Centre for High Performance Computing

MSc in HPC course 5635b

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An Introduction to Mathematical Finance (5635b)

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Course Outline

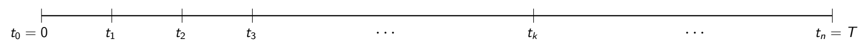
Itô Integral

Integration

For an ordinary function $f(x)$, we can define an integral as the limit of a sum:

$$\int_0^T f(t)dt = \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} f(t_j^*)(t_{j+1} - t_j)\,,$$

where t_j^* is in $[t_i\,, t_{j+1}]$.



Remember:

$$||\Pi|| = \max_{k=0,\dots,n-1} (t_{k+1} - t_k)$$

http://en.wikipedia.org/wiki/Riemann_integral

Stochastic Integral

We want to define an integral where the integrator is a Wiener process,

$$I(t) = \int_0^t \Delta(s)\, dW(s)$$

where $\Delta(s)$ is square-integrable. $\Delta(t)$ is determined based on information collected up to time t and may be stochastic.

In ordinary calculus, with differentiable function $f(t)$ instead of $W(t)$, we could define

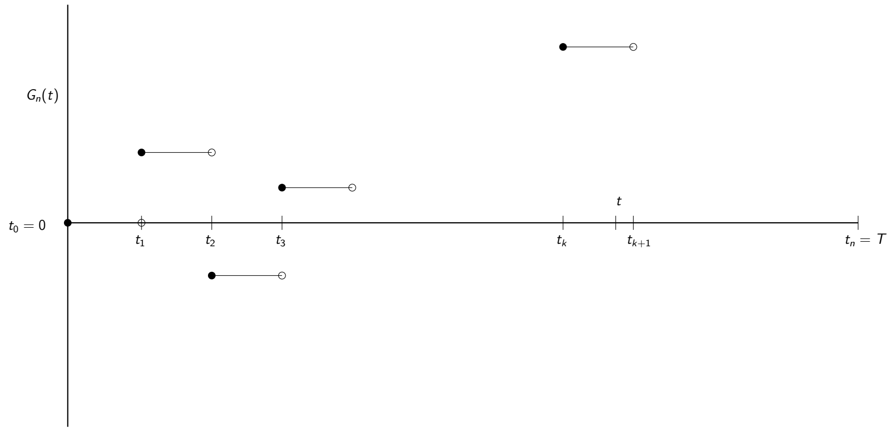
$$\int_0^t \Delta(s)\, df(s) = \int_0^t \Delta(s)\, f'(s)\, ds.$$

This does not work here, because W is not differentiable.

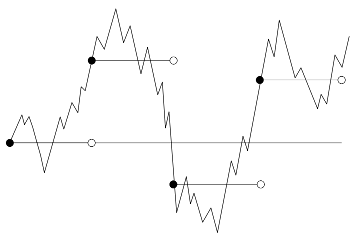
Instead we discretize, choose a partition first, define what we mean, and then shrink the mesh.

Step Function $\Delta(t)$

For a partition $\Pi = \{t_0\,, t_1\,, \cdots, t_n\}$ of the interval $[0, T]$, where $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$, define a "step function" $\Delta_n(t)$, on Π to be a function which holds a constant value in each interval $[t_j\,, t_{j+1})$.



Step function approximating general function



Stochastic Integral, Definition

We choose a partition $\Pi = t_0, t_1, \ldots, t_n$ of the time interval $[0, T]$,

$$0 = t_0 \le t_1 \le \cdots \le t_n = T, \qquad ||\Pi|| = \max_{k=0,\ldots,n-1} (t_{k+1} - t_k).$$

We then define the stochastic integral of a step function $\Delta_\Pi(t)$ as

$$I_\Pi(t) = \sum_{j=0}^{n-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) = \int_0^T \Delta_\Pi(t)\, dW(t)\,,$$

and an integral for a general function $\Delta(t)$,

$$I(T) = \int_0^T \Delta(t)\, dW(t) = \lim_{||\Pi|| \to 0} I_\Pi(T),$$

where

$$\lim_{||\Pi|| \to 0, n \to \infty} \Delta_\Pi(T) = \Delta(T)\,.$$

$$\left[\text{Actually:} \qquad \lim_{n\to\infty} \mathbb{E} \int_0^T |\Delta_\Pi(t) - \Delta(t)|^2 dt = 0\,.\right]$$

Itô and Stratonovich

The position in time inteval $[t_k, t_{k+1}]$ where we evaluate $\Delta(t)$ is crucial, we obtain different values of $I(t)$ in the limit depending on this choice:

- Left point: popular in Finance (think of Δ as asset holdings chosen due information up to time t_k and then exposed to random movements of the price W per unit holding over the next time period). The resulting integral is called *Itô integral*, to be used in the following.
- Mid point: popular in Physics, the resulting integral is called Stratonovich integral

$\int_0^T W(t)\,dW(t)$

In ordinary calculus we have for $f(0) = 0$

$$\int_0^T f(t)\,df(t) = \int_0^T f(t) f'(t)\,dt = \frac{1}{2}\int_0^T \frac{d}{dt}(f^2(t))dt = \frac{1}{2}f^2(T)$$

For the Itô integral we will show that

$$I(T) = \int_0^T \Delta(t)\,dW(t) = \int_0^T W(t)\,dW(t) = \frac{1}{2}\left(W^2(T) - T\right)$$

Approximate $W(t)$ with a step function

$$\Delta_\Pi(t) = W_\Pi(t) = \begin{cases} W(0) = 0 & \text{if } 0 \le t < \frac{T}{n}, \\ W(\frac{T}{n}) & \text{if } \frac{T}{n} \le t < \frac{2T}{n}, \\ W(\frac{2T}{n}) & \text{if } \frac{2T}{n} \le t < \frac{3T}{n}, \\ \vdots & \\ W(\frac{(n-1)T}{n}) & \text{if } \frac{(n-1)T}{n} \le t < T, \end{cases}$$

So,

$$\begin{aligned} \int_0^T W(t)dW(t) &= \lim_{n\to\infty} \int_0^T \Delta_\Pi(t)dW(t) \\ &= \lim_{n\to\infty} \sum_{j=0}^{n-1} W\left(\frac{jT}{n}\right)\left[W\left(\frac{(j+1)T}{n}\right) - W\left(\frac{jT}{n}\right)\right]. \end{aligned}$$

$\int_0^T W(t)\,dW(t)$

Letting $W_j = W\left(\frac{jT}{n}\right)$, consider the sum:

$$\begin{aligned} \frac{1}{2}\sum_{j=0}^{n-1}(W_{j+1} - W_j)^2 &= \frac{1}{2}\sum_{j=0}^{n-1} W_{j+1}^2 - \sum_{j=0}^{n-1} W_{j+1}W_j + \frac{1}{2}\sum_{j=0}^{n-1} W_j^2 \\ &= \frac{1}{2}W_n^2 + \frac{1}{2}\sum_{j=0}^{n-1} W_j^2 - \sum_{j=0}^{n-1} W_{j+1}W_j + \frac{1}{2}\sum_{j=0}^{n-1} W_j^2 \\ &= \frac{1}{2}W_n^2 + \sum_{j=0}^{n-1} W_j^2 - \sum_{j=0}^{n-1} W_{j+1}W_j \\ &= \frac{1}{2}W_n^2 - \sum_{j=0}^{n-1} W_j(W_{j+1} - W_j) \end{aligned}$$

$\int_0^T W(t)\,dW(t)$

Take the limit as $||\Pi|| \to 0$ gives:

$$\frac{1}{2}T = \frac{1}{2}W^2(T) - \int_0^T W(t)dW(t)\,,$$

so,

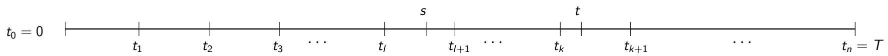
$$\int_0^T W(t)dW(t) = \frac{1}{2}W^2(T) - \frac{1}{2}T\,.$$

$I_\Pi(t)$ is a Martingale

In order to show $I_\Pi(t)$ is a Martingale, we need to show that for $0 \le s \le t \le T$,

$$\mathbb{E}\left[\, I_\Pi(t) \mid \mathcal{F}(s)\right] = I_\Pi(s)\,.$$

Set up a partition as follows



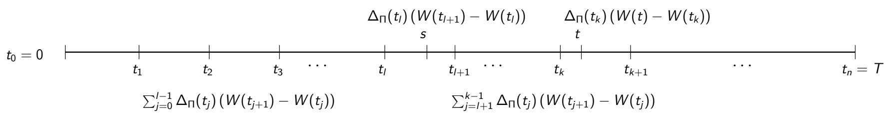
where we have $0 \le s < t \le T$, such that for $l < k$ (i.e., $t_l < t_k$), $s \in [t_l, t_{l+1})$ and $t \in [t_k, t_{k+1})$. As before, we have

$$I_\Pi(t) = \sum_{j=0}^{k-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) + \Delta_\Pi(t_k)\,(W(t) - W(t_k))$$

$I_\Pi(t)$ is a Martingale (ctd)

We can split $I_\Pi(t)$ up into four parts:

$$\begin{aligned} I_\Pi(t) &= \sum_{j=0}^{k-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) + \Delta_\Pi(t_k)\,(W(t) - W(t_k))\;, \\ &= \sum_{j=0}^{l-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) + \Delta_\Pi(t_l)\,(W(t_{l+1}) - W(t_l)) \\ &\quad + \sum_{j=l+1}^{k-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) + \Delta_\Pi(t_k)\,(W(t) - W(t_k)) \end{aligned}$$



$I_\Pi(t)$ is a Martingale (ctd)

So $\mathbb{E}\left[\, I_\Pi(t) \mid \mathcal{F}(s)\right]$ becomes

$$\begin{aligned} &\mathbb{E}\left[\sum_{j=0}^{l-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j))\,\middle|\,\mathcal{F}(s)\right] \\ &+ \mathbb{E}\left[\Delta_\Pi(t_l)\,(W(t_{l+1}) - W(t_l)) \mid \mathcal{F}(s)\right] \\ &+ \mathbb{E}\left[\sum_{j=l+1}^{k-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j))\,\middle|\,\mathcal{F}(s)\right] \\ &+ \mathbb{E}\left[\Delta_\Pi(t_k)\,(W(t) - W(t_k)) \mid \mathcal{F}(s)\right] \end{aligned}$$

By taking out what is known, this becomes:

$$\begin{aligned} &\sum_{j=0}^{l-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) \\ &+ \Delta_\Pi(t_l)\,(\mathbb{E}\left[W(t_{l+1}) \mid \mathcal{F}(s)\right] - W(t_l)) \\ &+ \cdots \end{aligned}$$

Using the fact that $W(t)$ is a martingale ($\mathbb{E}\left[W(t) \mid \mathcal{F}(s)\right] = W(s)$) gives

$$\sum_{j=0}^{l-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) + \Delta_\Pi(t_l)\,(W(s) - W(t_l)) + \cdots$$

$I_\Pi(t)$ is a Martingale (ctd)

So far we have

$$\mathbb{E}\,[\,I_\Pi(t)\,|\,\mathcal{F}(s)] = I_\Pi(s) + \mathbb{E}\left[\sum_{j=l+1}^{k-1}\Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j))\,\middle|\,\mathcal{F}(s)\right] + \mathbb{E}\,[\Delta_\Pi(t_k)\,(W(t) - W(t_k))\,|\,\mathcal{F}(s)]$$

$I_\Pi(t)$ is a Martingale (ctd)

What is

$$\mathbb{E}\left[\sum_{j=l+1}^{k-1}\Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j))\,\middle|\,\mathcal{F}(s)\right] ?$$

Looking at terms individually

$$\begin{aligned}\mathbb{E}\,[\Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j))\,|\,\mathcal{F}(s)] &= \mathbb{E}\,[\mathbb{E}\,[\Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j))\,|\,\mathcal{F}(t_j)]\,|\,\mathcal{F}(s)]\\ &= \mathbb{E}\,[\Delta_\Pi(t_j)\,(\mathbb{E}\,[\,W(t_{j+1})\,|\,\mathcal{F}(t_j)] - W(t_j))\,|\,\mathcal{F}(s)]\\ &= \mathbb{E}\,[\Delta_\Pi(t_j)\,(W(t_j) - W(t_j))\,|\,\mathcal{F}(s)]\\ &= 0\,,\end{aligned}$$

where we used the iterated conditioning rule along with the fact that $s < t_j$.
So,

$$\mathbb{E}\left[\sum_{j=l+1}^{k-1}\Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j))\,\middle|\,\mathcal{F}(s)\right] = 0\,.$$

$I_\Pi(t)$ is a Martingale (ctd)

Now we have

$$\mathbb{E}\,[\,I_\Pi(t)\,|\,\mathcal{F}(s)] = I_\Pi(s) + \mathbb{E}\,[\Delta_\Pi(t_k)\,(W(t) - W(t_k))\,|\,\mathcal{F}(s)]$$

Using a similar iterated conditioning argument to the one used on the previous slide,

$$\mathbb{E}\,[\Delta_\Pi(t_k)\,(W(t) - W(t_k))\,|\,\mathcal{F}(s)] = 0$$

Therefore

$$\mathbb{E}\,[\,I_\Pi(t)\,|\,\mathcal{F}(s)] = I_\Pi(s)\,,$$

So $I_\Pi(t)$ is a Martingale.

Itô Isometry

Since $I_\Pi(t)$ is a Martingale,

$$\mathbb{E}(I_\Pi(t)) = I(0) = 0\,,$$

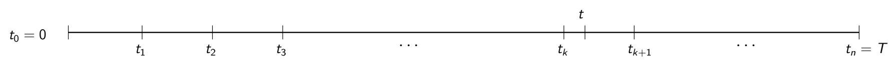
So

$$\mathrm{Var}(I_\Pi(t)) = \mathbb{E}(I_\Pi(t)^2)\,.$$

We will show that

$$\mathbb{E}((I_\Pi(t))^2) = \mathbb{E}\int_0^t (\Delta_\Pi(u))^2 du$$

Use a similar partition to the one used before,



So,

$$I_\Pi(t) = \sum_{j=0}^{k-1}\Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) + \Delta_\Pi(t_k)(W(t) - W(t_k))$$

Itô Isometry (ctd)

$$I_\Pi(t) = \sum_{j=0}^{k-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) + \Delta_\Pi(t_k)(W(t) - W(t_k))\,.$$

Let $\Delta W_j = (W(t_{j+1}) - W(t_j))\,, 0 \le j < k$, and let $\Delta W_k = (W(t) - W(t_k))$.
Then rewrite the Itô integral as,

$$I_\Pi(t) = \sum_{j=0}^{k} \Delta_\Pi(t_j)\Delta W_j\,.$$

So,

$$\begin{aligned} \mathbb{E}((I_\Pi(t))^2) &= \mathbb{E}\left[\left(\sum_{j=0}^{k} \Delta_\Pi(t_j)\Delta W_j\right)\left(\sum_{i=0}^{k} \Delta_\Pi(t_i)\Delta W_i\right)\right], \\ &= \mathbb{E}\left[\sum_{j=0}^{k}(\Delta_\Pi(t))^2 \Delta W_j^2 + 2\sum_{0\le i<j\le k} \Delta_\Pi(t_i)\Delta_\Pi(t_j)\Delta W_i \Delta W_j\right], \end{aligned}$$

Itô Isometry (ctd)

Taking the second term first

$$\begin{aligned} &\mathbb{E}\left[2\sum_{0\le i<j\le k} \Delta_\Pi(t_i)\Delta_\Pi(t_j)\Delta W_i\Delta W_j\right] \\ &= 2\sum_{0\le i<j\le k} \mathbb{E}\left[\Delta_\Pi(t_i)\Delta_\Pi(t_j)\Delta W_i\Delta W_j\right] \\ &= 2\sum_{0\le i<j\le k} \mathbb{E}[\Delta_\Pi(t_i)\Delta_\Pi(t_j)\Delta W_i]\,\underbrace{\mathbb{E}[\Delta W_j]}_{=0} \\ &= 0\,. \end{aligned}$$

Because

- $\Delta_\Pi(t_i)\Delta_\Pi(t_j)\Delta W_i$ $\mathcal{F}(t_j)$-measurable
- ΔW_j independent of $\mathcal{F}(t_j)$

Itô Isometry (ctd)

So,

$$\mathbb{E}((I_\Pi(t))^2) = \mathbb{E}\left[\sum_{j=0}^{k}(\Delta_\Pi(t))^2\Delta W_j^2\right] + 0$$

and

$$\begin{aligned} \mathbb{E}\left[\sum_{j=0}^{k}(\Delta_\Pi(t_j))^2\Delta W_j^2\right] &= \sum_{j=0}^{k}\mathbb{E}\left[(\Delta_\Pi(t_j))^2\Delta W_j^2\right] = \sum_{j=0}^{k}\mathbb{E}\left[(\Delta_\Pi(t_j))^2\right]\mathbb{E}\left[\Delta W_j^2\right] \\ &= \sum_{j=0}^{k-1}\mathbb{E}\left[(\Delta_\Pi(t_j))^2\right](t_{j+1} - t_j) + \mathbb{E}\left[(\Delta_\Pi(t_k))^2\right](t - t_j) \\ &= \mathbb{E}\left[\sum_{j=0}^{k-1}(\Delta_\Pi(t_j))^2(t_{j+1} - t_j)\right] + \mathbb{E}[(\Delta_\Pi(t_k))^2](t - t_j) \\ &= \mathbb{E}\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}(\Delta_\Pi(u))^2 du + \mathbb{E}\int_{t_k}^{t}(\Delta_\Pi(u))^2 du \\ &= \boxed{\mathbb{E}\int_0^t (\Delta_\Pi(u))^2 du} \end{aligned}$$

Quadratic variation of Itô integral

Since the Itô integral is written as

$$I(t) = \int_0^t G(u)dW(u)\,,$$

In informal notation, this can be written as

$$dI(t) = G(u)dW(t)$$

Again informally, the quadratic variation is written,

$$dIdI = G(t)dW(t)G(t)dW(t) = (G(t))^2 dt$$

So, the quadratic variation of the Itô integral is

$$QV_I(t) = \int_0^t (G(u))^2 du$$

Summary: Properties of Itô Integral

For an Itô integral

$$I(T) = \int_0^T G(t)dW(t)\,,$$

- Expected Value:

$$\mathbb{E}[I(T)] = 0$$

,

- Variance: (Itô Isometry):

$$\mathrm{Var}[I(T)] = \int_0^T \mathbb{E}[G^2(t)]dt\,,$$

- Quadratic variation:

$$\mathrm{QV}_I(T) = \int_0^T [G^2(t)]dt\,,$$

- Martingale: for $0 \le s < t$,

$$\mathbb{E}[I(t)|\mathcal{F}(s)] = I(s)$$

Chain Rule

$f(W(t))$

For an expression of the form $f(W(t))$, if asked for

$$\frac{d}{dt}f(W(t))\,,$$

would normally write

$$\frac{d}{dt}f(W(t)) = \frac{df(W(t))}{dW}\frac{dW(t)}{dt}$$

or

$$df(W(t)) = \frac{df(W(t))}{dW}\frac{dW(t)}{dt}dt$$

or

$$df(W(t)) = \frac{df(W(t))}{dW}dW(t)$$

But $\frac{dW(t)}{dt}$ does not exist

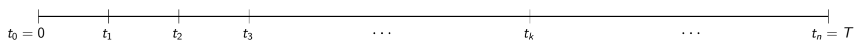
df(t,W(t))

For a function $f(t, W(t))$ of time and $W(t)$, we would write

$$df(t, W(t)) = \frac{\partial f(t, W(t))}{\partial t}dt + \frac{\partial f(t, W(t))}{\partial W(t)}dW(t)\,,$$

But $\frac{dW(t)}{dt}$ does not exist

Partition of interval $[0, T]$



Taylor Series

Given a differentiable function $f(x)$ and two points x_j and x_{j+1}, then

$$f(x_{j+1}) = f(x_j) + f'(x_j)(x_{j+1} - x_j) + \frac{1}{2} f''(x_j)(x_{j+1} - x_j)^2 + \cdots ,$$

where $x_{j+1} = x_j + (x_{j+1} - x_j)$.

For a function $f(t, x(t))$ and points $(t_j\,, x(t_j))$ and $(t_{j+1}\,, x(t_{j+1}))$

$$\begin{aligned} f(t_{j+1}, x(t_{j+1})) = f(t_j, x(t_j)) & \\ &+ f_t(t_j, x(t_j))(t_{j+1} - t_j) + f_x(t_j, x(t_j))(x(t_{j+1}) - x(t_j)) \\ &+ \frac{1}{2} f_{tt}(t_j, x(t_j))(t_{j+1} - t_j)^2 \\ &+ f_{tx}(t_j, x(t_j))(t_{j+1} - t_j)(x(t_{j+1}) - x(t_j)) \\ &+ \frac{1}{2} f_{xx}(t_j, x(t_j))(x(t_{j+1}) - x(t_j))^2 \\ &+ \text{higher order terms} \cdots , \end{aligned}$$

where

$$f_t = \frac{\partial f(t,x)}{\partial t}, \qquad f_x = \frac{\partial f(t,x)}{\partial x},$$
$$f_{tt} = \frac{\partial^2 f(t,x)}{\partial t^2}, \qquad f_{tx} = \frac{\partial^2 f(t,x)}{\partial t \partial x},$$
$$f_{xx} = \frac{\partial^2 f(t,x)}{\partial x^2}.$$

Itô's Formula for $f(t, W(t))$

The function $f(x)$ is differentiable, so we can expand it as before

$$\begin{aligned} f(t_{j+1}, W(t_{j+1})) = f(t_j, W(t_j)) & \\ &+ f_t(t_j, W(t_j))(t_{j+1} - t_j) + f_x(t_j, W(t_j))(W(t_{j+1}) - W(t_j)) \\ &+ \frac{1}{2} f_{tt}(t_j, W(t_j))(t_{j+1} - t_j)^2 \\ &+ f_{tx}(t_j, W(t_j))(t_{j+1} - t_j)(W(t_{j+1}) - W(t_j)) \\ &+ \frac{1}{2} f_{xx}(t_j, W(t_j))(W(t_{j+1}) - W(t_j))^2 \\ &+ \text{higher order terms} \cdots , \end{aligned}$$

where

$$f_t = \frac{\partial f(t,x)}{\partial t}, \qquad f_x = \frac{\partial f(t,x)}{\partial W(t)},$$
$$f_{tt} = \frac{\partial^2 f(t,x)}{\partial t^2}, \qquad f_{tx} = \frac{\partial^2 f(t,x)}{\partial t \partial W(t)},$$
$$f_{xx} = \frac{\partial^2 f(t,x)}{\partial W^2(t)}.$$

Itô's Formula for $f(t, W(t))$ (ctd)

Summing, we have

$$\begin{aligned} f(T\,, W(T)) - f(0\,, W(0)) &= \sum_{j=0}^{n-1} \Big[f(t_{j+1}, W(t_{j+1})) - f(t_j, W(t_j)) \Big] \\ &= \sum_{j=0}^{n-1} f_t(t_j, W(t_j))(t_{j+1} - t_j) \\ &\quad + \sum_{j=0}^{n-1} f_x(t_j, W(t_j))(W(t_{j+1}) - W(t_j)) \\ &\quad + \sum_{j=0}^{n-1} \frac{1}{2} f_{xx}(t_j, W(t_j))(W(t_{j+1}) - W(t_j))^2 \\ &\quad + \sum_{j=0}^{n-1} \frac{1}{2} f_{tt}(t_j, W(t_j))(t_{j+1} - t_j)^2 \\ &\quad + \sum_{j=0}^{n-1} f_{tx}(t_j, W(t_j))(t_{j+1} - t_j)(W(t_{j+1}) - W(t_j)) \\ &\quad + \text{higher order terms} \cdots . \end{aligned}$$

Itô's Formula for $f(t, W(t))$ (ctd)

$\lim_{||\Pi|| \to 0}$

In the limit as $||\Pi|| \to 0$, this becomes,

$$\begin{aligned} & f(T\,, W(T)) - f(0\,, W(0)) = \\ & = \int_0^T f_t(t, W(t))dt \leftarrow \left(\lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} f_t(t_j, W(t_j))(t_{j+1} - t_j) \right) \\ & + \int_0^T f_x(t, W(t))dW(t) \leftarrow \left(\lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} f_x(t_j, W(t_j))(W(t_{j+1}) - W(t_j)) \right) \\ & + \frac{1}{2} \int_0^T f_{xx}(t, W(t))dt \leftarrow \left(\lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} \frac{1}{2} f_{xx}(t_j, W(t_j))(W(t_{j+1}) - W(t_j))^2 \right) \\ & + 0 \leftarrow \left(\lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} \frac{1}{2} f_{tt}(t_j, W(t_j))(t_{j+1} - t_j)^2 \right) \\ & + 0 \leftarrow \left(\lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} f_{tx}(t_j, W(t_j))(W(t_{j+1}) - W(t_j))(t_{j+1} - t_j) \right) , \end{aligned}$$

using arguments very like ones we have seen before.

Itô's Formula for $f(t, W(t))$

So we have,

$$f(T, W(T)) - f(0, W(0)) = \int_0^T df(t, W(t)) = \int_0^T f_t(t, W(t))dt + \int_0^T f_W(t, W(t))dW(t) + \frac{1}{2}\int_0^T f_{WW}(t, W(t))dt$$

In informal differential notation,

$$\boxed{df(t, W(t)) = f_t dt + f_W dW(t) + \frac{1}{2} f_{WW} dt\,,}$$

or

$$\boxed{df(t, W(t)) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial W} dW(t) + \frac{1}{2}\frac{\partial^2 f}{\partial W^2} dt\,.}$$

or, if you like,

$$\boxed{df(t, W(t)) = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}\right) dt + \frac{\partial f}{\partial W} dW(t)\,.}$$

$\int_0^T WdW$ again

The integral can be quickly evaluated using Itô's formula
Let $f(x) = \frac{1}{2}x^2$. Then

$$\frac{\partial f(x)}{\partial x} = f_x(x) = x\,,$$
$$\frac{\partial^2 f(x)}{\partial x^2} = f_{xx}(x) = 1\,.$$

If we replace x by W, the Itô formula gives

$$df(W) = \underbrace{f_t}_{f_t=0} dt + f_W dW + \frac{1}{2} f_{WW} dt\,,$$
$$= WdW + \frac{1}{2}\cdot 1 \cdot dt\,,$$

So

$$\int_0^T df(W) = f(W(T)) - f(W(0)) = \frac{1}{2}(W(T))^2 + \underbrace{0}_{W(0)=0}$$
$$= \int_0^T WdW + \int_0^T \frac{1}{2}\cdot dt = \int_0^T WdW + \frac{1}{2}T\,,$$

Therefore

$$\int_0^T WdW = \frac{1}{2}W^2(T) - \frac{1}{2}T\,.$$

Product Rule

Let $X(t, W(t))$ and $Y(t, W(t))$, so that $X \cdot Y$ is a function of t and $W(t)$, too.

$$\begin{aligned} d[XY] &= \left(\frac{\partial XY}{\partial t} + \frac{1}{2}\frac{\partial^2 XY}{\partial W^2}\right) dt + \frac{\partial XY}{\partial W} dW \\ &= \ldots \\ &= X\,dY + Y\,dX + \frac{\partial X}{\partial W}\frac{\partial Y}{\partial W} dt \\ &= X\,dY + Y\,dX + dX\,dY \end{aligned}$$

Itô Process

An Itô process $X(t)$ is defined

$$X(t) = X(0) + \int_0^t A(t)\,dt + \int_0^t B(t)\,dW\,,$$

or, informally,

$$dX(t) = A(t)\,dt + B(t)\,dW\,.$$

Conditions are imposed on the functions $A(t)$ and $B(t)$

$$\begin{aligned} \mathbb{E}\int_0^t B^2(u)du &< \infty\,, \\ \int_0^t |A(u)|du &< \infty\,. \end{aligned}$$

Quadratic Variation for $X(t)$
informally

Using the rules we have already described, the quadratic variation can be obtained informally as follows

$$\begin{aligned} QV_X(t) &= dX(t)dX(t) \\ &= (A(t)\,dt + B(t)\,dW)^2 \\ &= A^2(t)dtdt + 2A(t)B(t)dWdt + B^2(t)dWdW \\ &= 0 + 0 + B^2(t)dWdW \\ &= B^2(t)dt\,. \end{aligned}$$

Integral with respect to Itô Process

We've seen Itô integrals with respect to Brownian Motion:

$$\int_0^t G(u)dW(u)\,.$$

We can also define an integral with respect to an Itô process by splitting up the $A(t)$ and $B(t)$ terms

$$\int_0^t G(u)dX(u) = \int_0^t G(u)A(u)du + \int_0^t G(u)B(u)dW(u)\,.$$

Itô's Formula for $f(t, X(t))$
instead of $f(t, W(t))$

Proceeding as before, we have (replace $W(t)$ by $X(t)$):

$$\begin{aligned} f(t_{j+1}, X(t_{j+1})) = f(t_j, X(t_{j+1})) & \\ &+ f_t(t_j, X(t_{j+1}))(t_{j+1} - t_j) + f_x(t_j, X(t_{j+1}))(X(t_{j+1}) - X(t_{j+1})) \\ &+ \frac{1}{2} f_{tt}(t_j, X(t_{j+1}))(t_{j+1} - t_j)^2 \\ &+ f_{tx}(t_j, X(t_{j+1}))(t_{j+1} - t_j)(X(t_{j+1}) - X(t_{j+1})) \\ &+ \frac{1}{2} f_{xx}(t_j, X(t_{j+1}))(X(t_{j+1}) - X(t_{j+1}))^2 \\ &+ \text{higher order terms} \cdots, \end{aligned}$$

where

$$f_t = \frac{\partial f(t,x)}{\partial t}, \qquad f_x = \frac{\partial f(t,x)}{\partial X(t)},$$

$$f_{tt} = \frac{\partial^2 f(t,x)}{\partial t^2}, \qquad f_{tx} = \frac{\partial^2 f(t,x)}{\partial t \partial X(t)},$$

$$f_{xx} = \frac{\partial^2 f(t,x)}{\partial X(t)^2}.$$

Itô's Formula for $f(t, X(t))$

So, for $f(t, X(t))$,

$$\boxed{df(t, X(t)) = f_t(t, X(t))dt + f_x(t, X(t))dX + \frac{1}{2} f_{xx}(t, X(t))dXdX\,,}$$

where

$$dX(t)dX(t) = B^2(t)dt\,.$$

Summary
Itô's Formula for $f(t, X(t))$

For $dX(t) = A(t)\,dt + B(t)\,dW$ and a function $f(t, X(t))$

$$\boxed{df(t,X) = f_t(t,X)dt + f_x(t,X)dX + \frac{1}{2}f_{xx}(t,X)dXdX\,,}$$

$$\boxed{df(t,X) = f_t dt + f_x dX + \frac{1}{2}f_{xx}B^2(t)dt\,,}$$

$$\boxed{df(t,X) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}dX + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}B^2(t)dt\,.}$$

$$\boxed{df(t,X) = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}B^2(t)\right)dt + \frac{\partial f}{\partial X}dX\,.}$$

Itô's Formula for $f(t, X(t))$
In terms of $W(t)$...

$$\begin{aligned}
df(t,X) =& f_t(t,X(t))dt + f_x(t,X(t))dX + \frac{1}{2}f_{xx}(t,X(t))B^2(t)dt\,,\\
=& f_t(t,X(t))dt + f_x(t,X(t))(A(t)\,dt + B(t)\,dW) + \frac{1}{2}f_{xx}(t,X(t))B^2(t)dt\,,\\
=& f_t(t,X(t))dt + f_x(t,X(t))A(t)\,dt + f_x(t,X(t))B(t)\,dW\\
&+ \frac{1}{2}f_{xx}(t,X(t))B^2(t)dt\,.
\end{aligned}$$

Stochastic Differential Equations

Basic Differential Equations
Reminder

Example ($df(t) = \mu f(t)dt$)

Consider the differential equation

$$df(t) = \mu f(t)dt\,,$$

where μ is a constant.
We can rewrite this as

$$\frac{1}{f(t)}df(t) = \mu dt\,,$$

Let $f(t=0) = f_0$. We want a formula for the value of $f(T)$, for $T > 0$, i.e., on the interval $[0, T]$. So again we integrate the equation:

$$\int_0^T \frac{df(t)}{f(t)} = \mu\int_0^T dt = \mu(T-0)\,,$$

$$\begin{aligned}
\int_0^T \frac{df(t)}{f(t)} &= \log f(t)\Big|_0^T\\
&= \log f(T) - \log f(0)\\
&= \log\frac{f(T)}{f(0)} = \log\frac{f(T)}{f_0} \text{ (using the properties of } \log x)
\end{aligned}$$

Basic Differential Equations

Example ($df(t) = \mu f(t) dt$ (ctd.))

From the previous slide

$$\log \frac{f(T)}{f_0} = \mu T \,,$$

Raising both sides to the power of e (remember e^x and $\exp(x)$ are the same thing)

$$\exp\left(\log \frac{f(T)}{f_0}\right) = \exp(\mu T) \,,$$

Solution:

$$\boxed{f(T) = f_0 e^{\mu T} \,,}$$

because $\exp\left(\log \frac{f(T)}{f_0}\right) = \frac{f(T)}{f_0}$.

Brownian Motion

First regular Brownian Motion, or SDE's of the form

$$dS = \mu(t)dt + \sigma(t)dW \,.$$

In fact we know the solution already, we just integrate to get

$$S(t) - S(0) = \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s)$$

When $\mu(t), \sigma(t)$ are constant, we have

$$S(t) - S(0) = \mu t + \sigma W(t)$$

Geometric Brownian Motion

A special case of general SDE

Consider

$$dS = \mu(t)S(t)dt + \sigma(t)S(t)dW \,.$$

which is special case of

$$dS = (f(t) + \mu(t)S(t))dt + (g(t) + \sigma(t)S(t))dW \,.$$

with $f(t) = g(t) = 0$

This is known as a homogeneous equation (recall from ODE's) because all the terms depend on S.

Geometric Brownian Motion

From the previous slide

$$dS = \mu(t)S(t)dt + \sigma(t)S(t)dW(t) \,.$$

Need to solve to obtain $S(t)$. Try solution of form $S = e^f$. Then $f = \ln S$. $df(S(t)) = ?$.
Apply Itô:

$$\begin{aligned} f_t &= 0 \,, \\ f_S &= \frac{1}{S} \,, \\ f_{SS} &= \frac{-1}{S^2} \,. \end{aligned}$$

Therefore

$$\begin{aligned} df(S) &= f_S dS + \frac{1}{2} f_{SS} dS\, dS \,, \\ &= \frac{1}{S} dS - \frac{1}{2}\frac{1}{S^2} dS\, dS \,, \\ &= \frac{1}{S}\left(\mu(t)S(t)dt + \sigma(t)S(t)dW(t)\right) - \frac{1}{2}\frac{1}{S^2}\sigma^2(t)S^2 dt \,, \\ &= \left(\mu(t) - \frac{1}{2}\sigma(t)^2\right) dt + \sigma(t)dW(t) \,. \end{aligned}$$

Geometric Brownian Motion (ctd)

From the previous slide

$$\int_0^T df(S(t)) = f(T) - f(0) = \log(S(T)) - \log(S(0))$$

$$= \int_0^T \left(\mu(t) - \frac{1}{2}\sigma(t)^2\right) dt + \int_0^T \sigma(t) dW(t) \,.$$

Therefore

$$S(T) = S(0) \exp\left\{ \int_0^T \left(\mu(t) - \frac{1}{2}\sigma(t)^2\right) dt + \int_0^T \sigma(t) dW(t) \right\}$$

When μ, σ are constant, this becomes

$$S(T) = S(0) \exp\left\{ \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t) \right\}$$

An Interesting Martingale

Consider the Generalised Geometric Brownian Motion with $\mu(t) = 0$

$$S(t) = S(0) \exp\left(-\frac{1}{2} \int_0^t \sigma^2(t') dt' + \int_0^t \sigma(t')\, dW(t') \right) .$$

Use Ito's formula to derive the SDE for $S(t)$:

$$dS(t) = \sigma(t)\, S(t)\, dW(t)$$

so that

$$S(t) - S(t_0) = \int_{t_0}^t \sigma(t')\, S(t')\, dW(t')$$

and the expectation conditional on information up to t_0 is

$$\mathbb{E}[S(t)|\mathcal{F}(t_0)] = \mathbb{E}[S(t_0)|\mathcal{F}(t_0)] + \mathbb{E}\left[\int_{t_0}^t \sigma(t')\, S(t')\, dW(t') | \mathcal{F}(t_0) \right] = S(t_0).$$

$\Rightarrow$ $S(t)$ is a Martingale – i.e., drift-free (generalised) Geometric Brownian Motion is a Martingale

Distribution of Itô integral with deterministic integrand

An Itô integral

$$I(t) = \int_0^t G(s) dW(s) \,,$$

where $G(t)$ is deterministic is normally distributed with mean 0 and variance $\int_0^t G^2(s) ds$.

Mean $I(t)$ is a martingale, so $EI(t) = I(0) = 0$

Variance $\mathbb{E}I(t) = 0$, so

$$\text{Var}\,[I(t)] = \mathbb{E}I^2(t) = \mathbb{E} \int_0^t G^2(s) ds$$

Distribution *If $I(t)$ is normally distributed, then its moment generating function (MGF) will be of the same form as that of a normally distributed random variable*

The MGF of a normally distributed random variable, X, with mean 0 and variance σ^2 is

$$\text{MGF}(X) = \mathbb{E}\left[e^{uX}\right] = e^{\frac{1}{2}u^2\sigma^2}$$

If $I(t)$ is normally distributed with mean 0 and variance $\int_0^t I^2(s) ds$, then it should have

$$\text{MGF}(I(t)) = \mathbb{E}\left[e^{uI(t)}\right] = e^{\frac{1}{2}u^2 \int_0^t G^2(s) ds}$$

Distribution of Itô integral with deterministic integrand

If

$$\mathbb{E}\left[e^{uI(t)}\right] = e^{\frac{1}{2}u^2 \int_0^t G^2(s) ds} ,$$

then

$$\mathbb{E}\left[e^{uI(t)}\right] e^{-\frac{1}{2}u^2 \int_0^t G^2(s) ds} = 1 .$$

Since $G(t)$ is deterministic, this can be written as

$$\mathbb{E}\left[e^{uI(t)} e^{-\frac{1}{2}u^2 \int_0^t G^2(s) ds} \right] = 1 ,$$

or

$$\mathbb{E}\left[e^{uI(t) - \frac{1}{2}u^2 \int_0^t G^2(s) ds} \right] = 1 .$$

Writing $\sigma(t) = uG(t)$, and $I(t) = \int_0^t G(s) dW(s)$,

$$\begin{aligned} \mathbb{E}\left[e^{u \int_0^t G(s) dW(s) - \frac{1}{2}u^2 \int_0^t G^2(s) ds} \right] &= 1 , \\ = \quad \mathbb{E}\left[e^{\int_0^t uG(s) dW(s) - \frac{1}{2} \int_0^t u^2 G^2(s) ds} \right] &, \\ = \quad \mathbb{E}\left[e^{\int_0^t \sigma(s) dW(s) - \frac{1}{2} \int_0^t \sigma^2 ds} \right] &= 1 , \end{aligned}$$

Distribution of Itô integral with deterministic integrand

But

$$S(t) = e^{\int_0^t \sigma(s)dW(s) - \frac{1}{2}\int_0^t \sigma^2 ds},$$

is a martingale. Also $S(0) = 1$. Therefore

$$\mathbb{E}(S(t)) = 1,$$

as required. So $I(t)$ is normally distributed with mean 0 and variance $\int_0^t G^2(s)ds$.

Euler Method for Ordinary Differential Equations

Consider a (deterministic) initial value problem:

$$\frac{dx(t)}{dt} = a(t, x), \qquad x(t_0) = x_0.$$

- It is not possible in general to find explicit solutions to such an equation
- Numerical approximations to the solution are often required
- one type of numerical approximation is a discrete time approximation
- the continuous time differential equation is replaced by a discrete time difference equation which generates values $y_1, y_2, \cdots, y_n, \cdots$ which are approximations to $x(t_1; t_0, x_0), x(t_2; t_0, x_0), \cdots, x(t_n; t_0, x_0), \cdots$, at the time points $t_0 < t_1 < t_2 < \cdots < t_n < \cdots$
- if the differences between the time points, $\Delta_n = t_{n+1} - t_n$ are small enough then the method should be reasonably accurate

Euler Method:

$$y_{n+1} = y_n + a(t_n, y_n)\Delta_n,$$

where $y_0 = x_0$.

Euler-Maruyama Method for Stochastic Differential Equations

Consider a SDE

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW, \qquad X(t_0) = X_0.$$

For $t_0 < t_1 < t_2 < \cdots < t_n < \cdots$, the Euler-Maruyama (discrete-time) approximation to the solution to this SDE is:

$$Y_{n+1} = Y_n + a(t_n, Y_n)(t_{n+1} - t_n) + b(t_n, Y_n)(W(t_{n+1}) - W(t_n)),$$

with $Y_0 = X_0$.
Writing $(t_{n+1} - t_n)$ as Δ_n and $(W(t_{n+1}) - W(t_n))$ as ΔW_n, this becomes

$$Y_{n+1} = Y_n + a(t_n, Y_n)\Delta_n + b(t_n, Y_n)\Delta W_n,$$

Strong Convergence

For the SDE

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW, \qquad X(t_0) = X_0.$$

Let $X(T)$ be the solution of the SDE at some time $T > 0$. Let $Y(T)$ be the solution at time T obtained using the Euler-Maruyama method. Additionally assume that $X(T)$ and $Y(T)$ are evaluated using the same underlying Brownian Motion path. We can obtain an estimate of pathwise closeness between the actual solution and the Euler-Maruyama approximation by calculating

$$\epsilon = \mathbb{E}(|X(T) - Y(T)|).$$

The expected value is required by the presence of the Brownian motion term

Weak Convergence

For the SDE

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW\,, \qquad X(t_0) = X_0\,.$$

Let $X(T)$ be the solution of the SDE at some time $T > 0$. Let $Y(T)$ be the solution at time T obtained using the Euler-Maruyama method.
We can obtain an estimate of closeness of moments of the actual solution to those of the Euler-Maruyama approximation may be obtained by calculating

$$|\mathbb{E}[X(T)] - \mathbb{E}[Y(T)]|\,.$$

For the Euler-Maruyama method, the order of strong convergence is generally different from the order of weak convergence

Order of Convergence

Suppose an error ϵ has the following form in terms of a time interval size Δ (assuming equal interval sizes)

$$\epsilon(\Delta) = C\Delta^{\gamma}\,.$$

Then

$$\log \epsilon = \gamma \log \Delta + \log C\,.$$

Review of Course Topics