Trinity College Dublin
Coláiste na Tríonóide, Baile Átha Cliath
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MSc. in HPC

Quantitative Finance (5635b)

Binomial Trees

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January 26, 2017

1 A note on exercises involving programming

- Use any of C, C++, python, ruby, fortran, R, matlab, mathematica to do these exercises. If you want to use another package or language ask me first
- If the language you are using has any implementations of the models referred to in the exercises below (e.g., matlab) then you may **not** use those implementations to carry out the exercises. Obviously you can use them to check your answers
- You **may** use any random number generators that are provided by the package or language
- In all simulations below, it is acceptable to use equal time length intervals when partitioning any interval on the real line

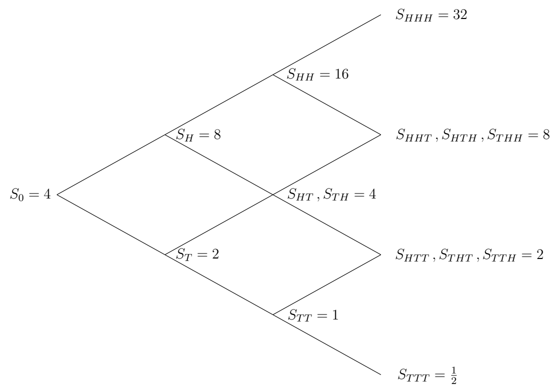


Figure 1: Tree used for Conditional Probability Questions

1. Pricing a call option on a binomial tree

   Let S be an asset which evolves over time on a four step binomial tree. The up and down parameters of the evolution are $u = \frac{1}{d} = 1.6$. The value of S at $t = 0$ is $S_0 = 10$. The prevailing risk free interest rate for borrowing and lending money is $r = 20\%$.

   a) What is the value at $t = 0$ of a European call option on this asset with strike price $K = 11$ which expires at the end of four time steps?
   b) Calculate values of Δ for $t = 0$ and $t = 1$ on this tree

2. Using the parameters from the previous question, what is the value at $t = 0$ of a European put option on this asset with strike price $K = 11$ which expires at the end of four time steps?

3. Conditional probabilities on binomial trees

   Given probabilities $\tilde{p} = \frac{2}{3}$ and $\tilde{q} = \frac{1}{3}$, on the binomial tree above, where $u = \frac{1}{d} = 2$, calculate

   a) $\tilde{\mathbb{E}}_1[S_2](H)$
   b) $\tilde{\mathbb{E}}_1[S_2](T)$
   c) $\tilde{\mathbb{E}}_1[S_3](T)$

4. Properties of conditional probabilities

   In the three step binomial model above in $u = \frac{1}{d} = 2$. Let $p = \frac{2}{3}$, $q = \frac{1}{3}$

   a) Show that

$$\mathbb{E}_1[S_2 + S_3](H) = \mathbb{E}_1[S_2](H) + \mathbb{E}_1[S_3](H)\,,$$
$$\mathbb{E}_1[S_2 + S_3](T) = \mathbb{E}_1[S_2](T) + \mathbb{E}_1[S_3](T)\,.$$

b) Show that

$$\mathbb{E}_1[S_1 S_2](H) = S_1(H)\mathbb{E}_1[S_2](H)\,,$$
$$\mathbb{E}_1[S_1 S_2](T) = S_1(T)\mathbb{E}_1[S_2](T)\,,$$

c) Calculate

$$\mathbb{E}_2[S_3](HH)\,,$$
$$\mathbb{E}_2[S_3](HT)\,,$$
$$\mathbb{E}_2[S_3](TH)\,,$$
$$\mathbb{E}_2[S_3](TT)\,,$$

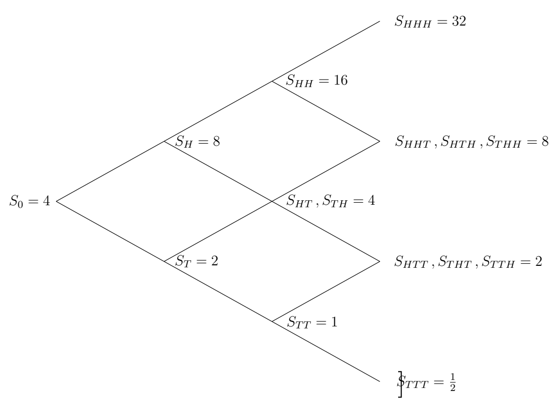
and then calculate

$$\mathbb{E}_1[\mathbb{E}_2[S_3]](H)\,,$$
$$\mathbb{E}_1[\mathbb{E}_2[S_3]](T)\,,$$

Finally, show that

$$\mathbb{E}_1[S_3](H) = \mathbb{E}_1[\mathbb{E}_2[S_3]](H)\,,$$
$$\mathbb{E}_1[S_3](T) = \mathbb{E}_1[\mathbb{E}_2[S_3]](T)\,,$$

5. Consider the three step binomial model with $u = \frac{1}{d} = 2$ and $r = \frac{1}{4}$.



a) Letting $M_n = S_n$ and considering $\mathbb{E}_n[M_{n+1}]$ show that S_n is a martingale when $p = \frac{1}{3}\,, q = \frac{2}{3}$

b) Now let

$$M_n = \frac{S_n}{(1+r)^n}\,.$$

Show that M_n is a martingale under the measure $p = \frac{1}{2}\,, q = \frac{1}{2}$

c) Is $M_n = S_n$ a martingale under $p = \frac{1}{2}\,, q = \frac{1}{2}$?

d) Are either S_n or $\frac{S_n}{(1+r)^n}$ martingales under $p = \frac{2}{3}\,, q = \frac{1}{3}$

e) Which of these pairs $(p\,, q)$ is a risk-neutral measure?

6. For a general binomial model with $0 < d < 1 + r < u$, with risk neutral probabilities given by

$$\begin{aligned}\tilde{p} &= \frac{1+r-d}{u-d}\,,\\ \tilde{q} &= \frac{u-(1+r)}{u-d}\,,\end{aligned}$$

the discounted stock price process is

$$\frac{S_n}{(1+r)^n}\,.$$

By considering

$$\tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right]\,,$$

show that the discounted stock process is a martingale under the risk neutral probabilities

7. a) Implement a scaled random walk $W^{(n)}(t)$

b) Plot the scaled random walk for $n = 10, 200, 1000$ out to times $t = 0.5, 1, 2$ on separate plots

c) For $t = 2$, plot 400 paths of the walk on one plot for $n = 500$

d) Take the multiple (e.g., 10000) final values at $t = 2$ of multiple scaled random walks out to $t = 2$ for $n = 1000$.

- Make a normed histogram of the distribution of final values
- Indicate which normal distribution the histogram should be approximating
- Indicate that your distribution of values does in fact approximate this normal distribution
- Do the same for $t = 0.5$