Trinity Centre for High Performance Computing

MSc in HPC course 5635b

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An Introduction to Mathematical Finance (5635b)

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Course Outline

Introduction

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Course Details

- I will be covering (primarily) Stochastic Calculus over 3-4 weeks.
- Donal Gallagher and then Roland Lichters will finish off the course.
- For my part of the course I will hand out one or more exercise sheets which will form part of the assessment for the course
- Rough solutions will be posted after the due date for the exercises

References / Acknowledgements

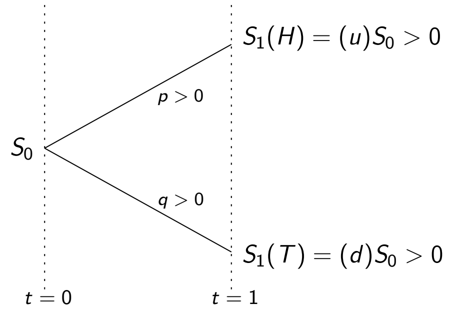
Much (not all) of the material in these slides was taken or adapted from

- Steven E. Shreve.
  Stochastic Calculus for Finance I: The Binomial Asset Pricing Model (Springer Finance).
  Springer, 1 edition, June 2005.
- Steven E. Shreve.
  Stochastic Calculus for Finance II: Continuous-Time Models (Springer Finance).
  Springer, 1st ed. 2004. corr. 2nd printing edition, June 2004.

These authors are not responsible for any errors in this course

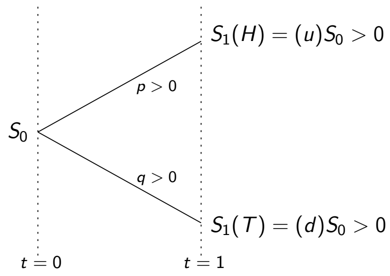
One Step Binomial Model (OSBM)

- Consider the value of one share of a stock at just two times, $t = 0$ and $t = 1$
- Has value S_0 at $t = 0$
- At $t = 1$ takes one of *only two*, positive, possible values:
  - $S_1(H)$ with probability $p > 0$
  - $S_1(T)$ with probability $q > 0$
- may be viewed as a weighted coin toss resulting in heads (H), or tails (T)
- We take $S_1(H) = uS_0$ and $S_1(T) = dS_0$, where $0 < d < u$
- if $S_1(H) = S_1(T)$ there is no uncertainty at $t = 1$



Expected Value of S_1

$$\mathbb{E}[S_1] = pS_1(H) + qS_1(T)\,.$$



Assumptions

- No taxes, transaction costs, margin costs
- stocks can be shorted at no additional cost
- shares or fractions of shares may be purchased without affecting price of share

Interest Rate r
Money Market

- In addition to the existence of the stock, there is also a *money market*
- Money may be invested or borrowed from the money market
- An amount X invested at time $t = 0$ yields $(1 + r)X$ at time $t = 1$
- An amount X borrowed at $t = 0$ results in a debt of $(1 + r)X$ at $t = 1$
- The rate r is usually assumed to be greater than 0, but is required only to be greater than -1

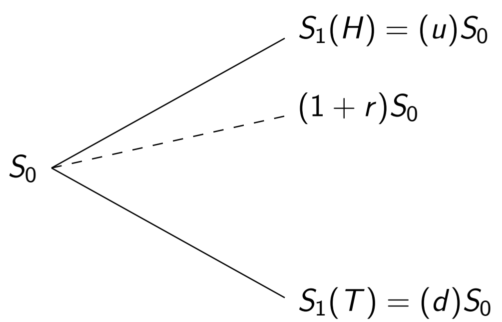
Arbitrage

Definition (Arbitrage)

Arbitrage may be defined as a trading strategy which begins with no money which has a zero probability of losing money and a positive probability of making money at some later time

- Steven E. Shreve.
  Stochastic Calculus for Finance I: The Binomial Asset Pricing Model (Springer Finance).
  Springer, 1 edition, June 2005.

No Arbitrage Condition for Binomial Model
$d < 1 + r < u$



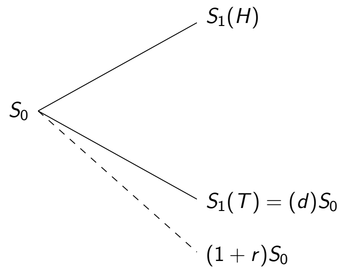
Suppose $d \geq 1 + r$

$t = 0$

- Borrow S_0 from money market
- Buy stock for S_0

$t = 1$

- Owe $(1 + r)S_0$
- Price of stock either $S_1(H)$ or $S_1(T)$
- But $S_1(H), S_1(T) \geq (1 + r)S_0$
- Since $u > d \geq 1 + r$, there is a positive probability of profit
- So there is an arbitrage



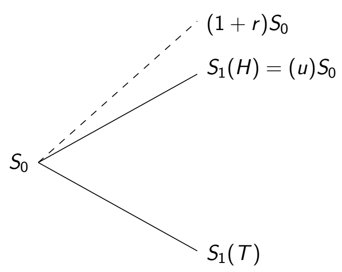
Suppose $u \leq 1 + r$

$t = 0$

- Short the stock for S_0
- Invest cash S_0 in money market

$t = 1$

- Receive $(1 + r)S_0$
- Price of stock either $S_1(H)$ or $S_1(T)$
- But $S_1(H), S_1(T) \leq (1 + r)S_0$
- Since $1 + r \geq u > d$, there is a positive probability of profit
- So there is an arbitrage



Derivatives in OSBM

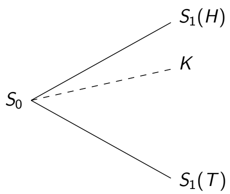
- Consider contracts which have payoffs at $t = 1$ which are contingent on the value of the stock
- So their value *derives* from the value of the *underlying* stock
- Examples
  - European call option
  - European put option
  - Forward contract

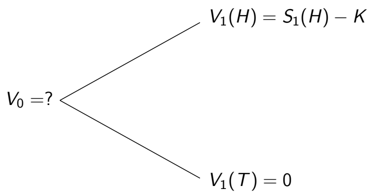
European Call Option in OSBM

Definition (European Call Option)

A contract entered into at time $t = 0$ which gives the holder the right but not the obligation to purchase the stock at time $t = 1$ for *strike price* K.

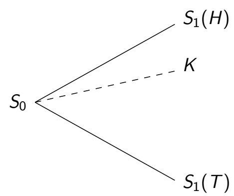
- Value at $t = 1$ is $S_1 = S_1(H)$ or $S_1(T)$
- Assume that $S_1(T) < K < S_1(H)$
- Value at time $t = 1$ known: $(S_1 - K)^+$
- What, if anything, is a fair value at $t = 0$?

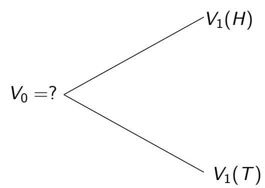




General Derivative in OSBM

- Value at $t = 0$ is V_0
- Value at $t = 1$ is $V_1 = V_1(H)$ or $V_1(T)$
- Payoff at time $t = 1$ is known in terms of S_1 – uncertainty occurs due to uncertainty as to which value S_1 will take
- What is V_0?





General Derivative in OSBM

Replicate the Option

- Create a "portfolio" of the stock and money market investment at $t = 0$
- Tune the relative amounts of stock and money market investment such that at $t = 1$ the portfolio takes the value of the derivative *no matter which value the stock takes* $t = 1$

Replicate the Option (ctd)

$t = 0$

- Start with cash X_0 at $t = 0$
- Purchase Δ_0 shares of the stock
- The cash position[a] is $(X_0 - \Delta_0 S_0)$

[a] If this is positive then $(1+r)(X_0 - \Delta_0 S_0)$ will be obtained at $t = 1$. If it is negative, then the same amount will be owed at $t = 1$

$t = 1$

- The cash position is $\Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0)$
- As usual $S_1 = S_1(H)$ or $S_1(T)$, so

$$\Delta_0 S_1(H) + (1+r)(X_0 - \Delta_0 S_0)\,,$$
$$\Delta_0 S_1(T) + (1+r)(X_0 - \Delta_0 S_0)$$

...

Replicate the Option (ctd)

$t = 1$

So, in order that the portfolio replicates the value of the derivative at $t = 1$, set

$$\Delta_0 S_1(H) + (1+r)(X_0 - \Delta_0 S_0) = V_1(H)\,,$$
$$\Delta_0 S_1(T) + (1+r)(X_0 - \Delta_0 S_0) = V_1(T)\,;$$

two equations in two unknowns $(X_0\,,\Delta_0)$. Solving gives

$$X_0 = \frac{1}{1+r}\left[\tilde{p}V_1(H) + \tilde{q}V_1(T)\right]\,,$$

and

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}\,.$$

Replicate the Option (ctd)

$\tilde{q}$ and $\tilde{p}$

where

$$\tilde{p} = \frac{1+r-d}{u-d}\,,$$
$$\tilde{q} = \frac{u-(1+r)}{u-d}\,,$$

$$\tilde{p} + \tilde{q} = 1\,.$$

and

$$\tilde{p}, \tilde{q} > 0 \quad \text{by assumption of no arbitrage – check this}$$

Example: European Call Option

Example

Suppose $S_0 = 4$, and $u = \frac{1}{d} = 2$. Also suppose $r = \frac{1}{4}$. Then $S_1(H) = 8$ and $S_1(T) = 2$.
Then

$$\frac{1}{1+r} = \frac{1}{1+\frac{1}{4}} = \frac{4}{5},$$

so

$$\begin{aligned} \tilde{p} &= \frac{1+r-d}{u-d} = \frac{1}{2}, \\ \tilde{q} &= \frac{u-(1+r)}{u-d} = \frac{1}{2}. \end{aligned}$$

Consider a European call option expiring at $t = 1$ with strike price $K = 5$. At $t = 1$, possible payoffs are:

$$\begin{aligned} V_1(H) &= (S_1(H) - K)^+ = (8-5)^+ = 3\,, \\ V_1(T) &= (S_1(T) - K)^+ = (2-5)^+ = 0\,. \end{aligned}$$

Example (ctd)

Example

So

$$\begin{aligned} X_0 &= \frac{1}{1+r}\left[\tilde{p}V_1(H) + \tilde{q}V_1(T)\right], \\ &= \frac{4}{5}\left[\frac{1}{2}\cdot 3 + \frac{1}{2}\cdot 0\right], \\ &= \frac{6}{5} = 1.20\,. \end{aligned}$$

Also,

$$\begin{aligned} \Delta_o &= \frac{V_1(H) - V_1(T)}{S_0(u-d)}, \\ &= \frac{3-0}{4\cdot\frac{3}{2}}, \\ &= \frac{3}{6} = \frac{1}{2}\,. \end{aligned}$$

Example (ctd)

Replication Process for writer of call option

$t = 0$

- Start with $X_0 = 1.2$
- purchase $\Delta_0 = \frac{1}{2}$ units of underlying asset for $\frac{1}{2}\cdot 4 = 2$ euro
- In order to do this, must borrow $2 - 1.2 = 0.8$ euro
- value of portfolio at $t = 0$ is

$$X_0 - \Delta_0 S_0 = 1.2 - 2 = -0.8$$

$t = 1$

- Owe $(1+r)\cdot 0.8 = \frac{5}{4}\cdot 0.8 = 1$ euro
- Value of portfolio

  H: $\frac{1}{2}\cdot 8 + \frac{5}{4}(1.2 - \frac{1}{2}\cdot 4) = 3\,,$
  T: $\frac{1}{2}\cdot 2 + \frac{5}{4}(1.2 - \frac{1}{2}\cdot 4) = 0\,,$

- Payoff of derivative contract replicated by portfolio at $t = 1$ whether a head or a tail is tossed

No-Arbitrage Price?

- The "price" $X_0 = 1.2$ is the starting capital required by the seller to create a portfolio to hedge the payoff of the call option regardless of whether a head or a tail is tossed at $t = 1$
- It is also a no-arbitrage price:
  - Suppose the option seller could sell for a greater price, e.g., $C = 1.21$.
  - Then the seller could take the 0.01 cents and invest in a separate money market account at $t = 0$
  - Then the seller could use to remaining 1.20 to hedge the option as before
  - At $t = 1$ the portfolio created starting from $X_0 = 1.2$ would cover the option payoff no matter what
  - And there would be an additional $(1+r)(0.01)$ for the seller
  - So, positive probability of gain, and no possibility of loss – arbitrage

No-Arbitrage Price? (ctd)

Suppose the seller was selling for 1.19 euro

$t = 0$

- Then the *buyer* could buy for 1.19
- Reverse the portfolio strategy of the seller:
  - Sell short $\Delta_0 = \frac{1}{2}$ of the stock ($\frac{1}{2} \cdot 4 = 2$)
  - Use 1.19 of the 2 euro to buy the option
  - Invest 0.8 euro in one money market account
  - Invest remaining 0.01 euro in a separate money market account

No-Arbitrage Price? (ctd)

$t = 1$

- Must purchase $\Delta_0 = \frac{1}{2}$ stock on open market after selling short at $t = 0$
- If value of stock is $S_1(H) = 8$, then $\frac{1}{2}S_1 = 4$
  - receive $(1 + r) \cdot 0.8 = \frac{5}{4} \cdot 0.8 = 1$ euro from investment
  - Use call option to purchase asset for 5 euro
  - Sell half asset for 4 euro (so, cost is 1 euro).
  - But this matches the gain from the investment of 0.8 euro at $t = 0$
  - so break even
- If value of stock is $S_1(T) = 2$, then $\frac{1}{2}S_1 = 1$
  - Receive one euro from investment of 0.8 at $t = 0$
  - Call option worthless
  - But $\frac{1}{2} \cdot S_1(T) = 1$, which is the amount available
  - so break even

Separate from all of the above at $t = 1$ the investment of 0.01 returns $(1 + r)(0.01)$; implies arbitrage
So in this case $X_0 = 1.2$ is a no arbitrage price

Two Step Binomial

We assume now that the stock evolves in two steps from today to option expiry, taking three possible values at expiry, $S_2(HH) = S_0u^2$, $S_2(HT) = S_2(TH) = S_0ud$ and $S_2(TT) = S_0d^2$, and two possible values after step one, $S_1(H) = S_0u$ and $S_1(T) = S_0d$. Following the arguments of the single step case, we have therefore the two possible option values after step one,

$$\begin{aligned} V_1(H) &= \frac{1}{1+r}\{\tilde{p}\, V_2(HH) + (1-\tilde{p})\, V_2(HT)\} \\ V_1(T) &= \frac{1}{1+r}\{\tilde{p}\, V_2(TH) + (1-\tilde{p})\, V_2(TT)\} \end{aligned}$$

where $V_2(HH)$, $V_2(TH) = V_2(HT)$ and $V_2(TT)$ at expiry are known and determined by the underlying stock values at expiry.
Now, given $V_1(H)$ and $V_1(T)$ we can determine the current option value using

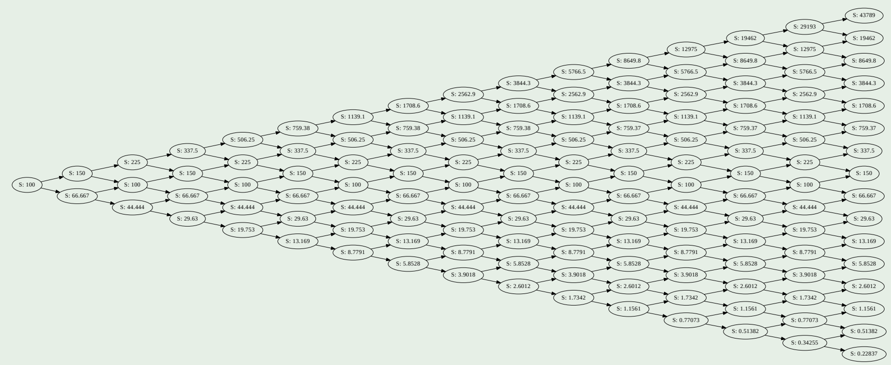
$$X_0 = \frac{1}{(1+r)^2}\left\{\tilde{p}^2\, V_2(HH) + 2\tilde{p}(1-\tilde{p})\, V_2(HT) + (1-\tilde{p})^2 V_2(TT)\right\},$$

where, for a European call option, $V_2(HH) = \max(S_0u^2 - K, 0)$ etc.

15 step tree

You can keep going...

Example (15 steps; $S_0 = 100$, $u = \frac{1}{d} = 1.5$)



Many Step Binomial for European Call Option

We can now generalize the previous section's result for a European Call option to n binomial steps, where the stock evolves over n steps from today to option expiry, forming a recombining binomial tree so that the stock assumes values between $S_0 u^n$ and $S_0 d^n$. The call (put) option value C (P) is then given by:

$$\begin{aligned} X_0 = C &= \frac{1}{(1+r)^n} \sum_{i=0}^{n} \binom{n}{i} \tilde{p}^i (1-\tilde{p})^{n-i} \max\left\{S_0 u^i d^{n-i} - K, 0\right\} , \\ P &= \frac{1}{(1+r)^n} \sum_{i=0}^{n} \binom{n}{i} \tilde{p}^i (1-\tilde{p})^{n-i} \max\left\{K - S_0 u^i d^{n-i}, 0\right\} . \end{aligned}$$

The expected value of S_0 after n steps is

$$\begin{aligned} \tilde{\mu}_n &= S_0 \sum_{i=0}^{n} \binom{n}{i} \tilde{p}^i (1-\tilde{p})^{n-i} u^i d^{n-i} \\ &= S_0 \left(\tilde{p}\, u + (1-\tilde{p})\, d\right)^n \\ &= S_0 \left(1+r\right)^n \end{aligned}$$

Replication in Multi step Binomomial model

Consider an N-step binomial model with

$$\tilde{p} = \frac{1+r-d}{u-d}, \qquad \tilde{q} = \frac{u-(1+r)}{u-d} .$$

Let V_N represent the payoff of a derivative contract at step N. V_N is dependent on the first N coin tosses, $\omega_1\omega_2\cdots\omega_N$. Define random variables $V_{N-1},\ V_{N-2},\ \cdots,\ V_0$ recursivly:

$$V_n(\omega_1\omega_2\cdots\omega_n) = \frac{1}{1+r}\left[\tilde{p}\, V_{n+1}(\omega_1\omega_2\cdots\omega_n H) + \tilde{q}\, V_{n+1}(\omega_1\omega_2\cdots\omega_n T)\right],$$

where $n = N-1, N-2, \cdots, 0$. Each V_n depends on the first n coin tosses.

- Steven E. Shreve.
  Stochastic Calculus for Finance I: The Binomial Asset Pricing Model (Springer Finance).
  Springer, 1 edition, June 2005.

Replication in Multi step Binomomial model

- Henceforth we'll write $V_{n+1}(\omega_1\omega_2\cdots\omega_n H) = V_{n+1}(H)$ and similarly for $V_{n+1}(T)$.
- The same style of notation will be used for the asset prices: $S_{n+1}(H), S_{n+1}(T)$.
- Define
  $$\Delta_n = \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)},$$
  where $n = 0, 1, \cdots, N-1$.
- Finally define a portfolio value process $X_0, X_1, \cdots, X_N$ by
  $$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n) .$$
  If $X_0 = V_0$, then
  $$X_n(\omega_1\omega_2\cdots\omega_n) = V_n(\omega_1\omega_2\cdots\omega_n),$$
  for $0 \le n \le N$.

Replication in Multi step Binomomial model

Proof by induction. Let $X_0 = V_0$. Assume $X_n = V_n$. What is situation for $n+1$?
H : If a head is tossed for step $n+1$, then

$$X_{n+1}(H) = \Delta_n u S_n + (1+r)(X_n - \Delta_n S_n) .$$

Substiting for Δ_n gives

$$\begin{aligned} X_{n+1}(H) =& \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)} u S_n \\ &+ (1+r)\left(X_n - \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)} S_n\right) . \end{aligned}$$

$S_{n+1}(H) - S_{n+1}(T) = uS_n - dS_n = (u-d)S_n$, so

$$\begin{aligned} X_{n+1}(H) =& \frac{V_{n+1}(H) - V_{n+1}(T)}{u-d} u \\ &+ (1+r)\left(X_n - \frac{V_{n+1}(H) - V_{n+1}(T)}{u-d}\right) . \end{aligned}$$

By induction, $X_n = V_n$, and by the formula some slides above

$$V_n = \frac{1}{1+r}[\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)].$$

So

$$\begin{aligned}
X_{n+1}(H) =& \frac{V_{n+1}(H) - V_{n+1}(T)}{u-d}u \\
& + (1+r)V_n - (1+r)\frac{V_{n+1}(H) - V_{n+1}(T)}{u-d}, \\
=& \frac{V_{n+1}(H) - V_{n+1}(T)}{u-d}u + \underbrace{\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)}_{\text{substituted for } (1+r)V_n} \\
& - (1+r)\frac{V_{n+1}(H) - V_{n+1}(T)}{u-d}, \\
=& \frac{1}{u-d}V_{n+1}(H)\,[u + 1 + r - d - 1 - r] \\
& + \frac{1}{u-d}V_{n+1}(T)\,[-u + u - 1 - r + 1 + r], \\
=& V_{n+1}(H).
\end{aligned}$$

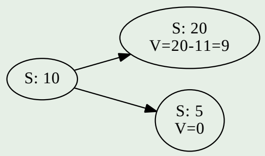
Similarly, in the case that a tail is tossed at step $n+1$, $X_{n+1}(T) = V_{n+1}(T)$.

Hedging a European call option

Mechanics of Hedging an Option

Example (Hedging a European call option)

- Suppose a European call option is sold at $t = 0$
- The value of the underlying asset at $t = 0$ is $S_0 = 10$
- The parameters of the tree are: $u = \frac{1}{d} = 2$
- The risk free interest rate between $t = 0$ and $t = 1$ is $r = 10\%$
- The strike price of the option is $K = 11$



The question is: If you are the seller of this European call option, how do you hedge your position so that you will suffer no loss regardless of the outcome at time $t = 1$

Hedging a European call option

Example (Hedging a European call option)

From the previous slide

$$\tilde{p} = \frac{1+r-d}{u-d} = \frac{1+0.1-\frac{1}{2}}{2-\frac{1}{2}} = 0.4$$

$$\tilde{q} = \frac{u-1-r}{u-d} = \frac{2-1-0.1}{2-\frac{1}{2}} = 0.6$$

Hedging a European call option

Example (Hedging a European call option)

Use the formula already derived to value V at $t = 0$:

$$\begin{aligned}
V = X_0 &= \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)] \\
&= \frac{1}{1.1}\big[(0.4)(9) + (0.6)(0)\big] \\
&= 3.2727
\end{aligned}$$

The call option is sold at $t = 0$ for this amount

Hedging a European call option

Example (Hedging a European call option)

- Having sold the call option, the seller is now exposed to variations in the price of the underlying asset in the next time step
- The derivation of the formula for the value of the call option indicates how the seller can hedge the option

Hedging a European call option

Example (Hedging a European call option)

At $t = 0$

- Sell option for $V = 3.2727$
- Calculate Δ based on the formula

$$\Delta = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{9 - 0}{20 - 5} = 0.6$$

- purchase 0.6 shares of underlying asset at 10 euros per share
- This requires $10 * 0.6 = 6$ euros. However have only charged 3.2727 euros for call option, so must borrow $6 - 3.2727 = 2.7273$ euros at $r = 0.1$
- Thus at $t = 0$
  - have portfolio of 0.6 units of underlying
  - have sold 2.7273 of bonds maturing at $t = 1$

Hedging a European call option

$t = 1$; $S_1 = 20$

Example (Hedging a European call option)

At $t = 1$:

If $S_1 = 20$

- The strike price of the option is 11 which is less than the current value, so the option will be exercised
- Therefore the option seller is obliged to sell 1 unit of underlying for price $K = 11$
- The value of the existing holding in underlying asset is now $(0.6)(20) = 12$
- Owe $2.7273 * 1.1 = 3.0$ euro
- Take payment from option holder of 11 euro
- Must purchase 0.4 units of underlying in order to have 1 unit of asset to deliver to option holder. This has cost $(0.4)(20) = 8$ euro
- Purchase 0.4 units; deliver 1 unit to holder. This leaves $11 - 8 = 3$ euro remaining, which serves to pay off the loan amount

Hedging a European call option

$t = 1$; $S_1 = 5$

Example (Hedging a European call option)

At $t = 1$:

If $S_1 = 5$

- The strike price of the option is 11 which is more than the current value of the underlying, so the option will not be exercised
- The value of the existing holding in underlying asset is now $(0.6)(5) = 3$ euro
- Owe $2.7273 * 1.1 = 3.0$ euro
- Sell the holding of the underlying and pay off the loan

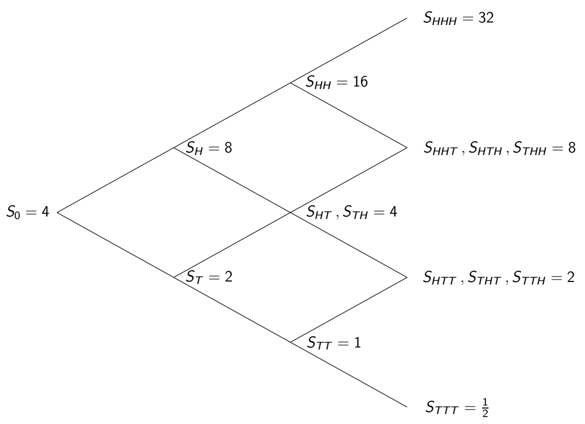
Conditional Expected Values on a k-step Tree

A k-Step Binomial Model
$k > n$

- In a k-step model we could have n coin tosses $\omega_1, \cdots \omega_n$ where $k > n$ and, $w_i = H$ or T
- At step n we will have $S_n(\omega_1 \cdots \omega_n)$, where the value of S_n will depend on the first n coin tosses
- And at step $n+1$ we will have $S_{n+1}(\omega_1 \cdots \omega_n H)$ (with probability p) **or** $S_{n+1}(\omega_1 \cdots \omega_n T)$ (with probability q)
- where

$$S_{n+1}(\omega_1 \cdots \omega_n H) = uS_n\,,$$
$$S_{n+1}(\omega_1 \cdots \omega_n T) = dS_n\,,$$

3 step binomial model



Conditional Expectation of S_{n+1}
Between step n and $n+1$

We call

$$\mathbb{E}_n[S_{n+1}] = [pS_{n+1}(H) + qS_{n+1}(T)]\,,$$

the conditional expectation of S_{n+1} based on the information available at step (time) n.

Since r is constant here, we also have

$$\mathbb{E}_n\left[\frac{S_{n+1}}{1+r}\right] = \frac{1}{1+r}[pS_{n+1}(H) + qS_{n+1}(T)]\,.$$

Examples
For $p = q = \frac{1}{2}$

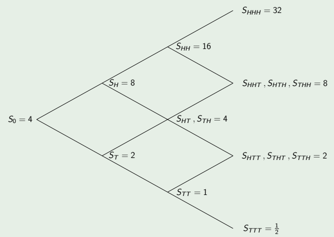
Example

Expected value of S_2 given that S_1 is a head

$$\mathbb{E}_1[S_2](H) = \frac{1}{2} \cdot 16 + \frac{1}{2} \cdot 4 = 10\,,$$

Expected value of S_2 given that S_1 is a tail

$$\mathbb{E}_1[S_2](T) = \frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 1 = \frac{5}{2}\,,$$



$\mathbb{E}_n[X](\omega_1 \cdots \omega_n)$

So far we've defined a conditional expectation between two consecutive steps. It can also be defined for steps separated by a larger gap. Let $0 \le n \le N$. Let X be a random variable depending on the first N coin tosses.

Definition ($\mathbb{E}_n(X)(\omega_1 \cdots \omega_n)$)

Let $0 \le n \le N$ and $\omega_1 \cdots \omega_n$ be a given sequence of coin tosses (assume we are at step n). There are 2^{N-n} possible sequences of coin tosses $\omega_{n+1} \cdots \omega_N$ between step n and step N. Let $\#H(\omega_{n+1} \cdots \omega_N)$ denote the number of heads in the sequences $\omega_{n+1} \cdots \omega_N$ and let $\#T(\omega_{n+1} \cdots \omega_N)$ denote the number of tails. Then we have the expected value of X (at step N) based on the information available at step n.

$$\mathbb{E}_n[X](\omega_1 \cdots \omega_n) = \sum_{\omega_{n+1} \cdots \omega_N} p^{\#H(\omega_{n+1} \cdots \omega_N)} q^{\#T(\omega_{n+1} \cdots \omega_N)} X(\omega_1 \cdots \omega_n \cdots \omega_N)$$

- Steven E. Shreve.
  Stochastic Calculus for Finance I: The Binomial Asset Pricing Model (Springer Finance).
  Springer, 1 edition, June 2005.

$\mathbb{E}_n[X](\omega_1 \cdots \omega_n)$
Special Cases

For a random variable $X(\omega_1 \cdots \omega_N)$ dependent on $N > n$ coin tosses

$$\mathbb{E}_0[X] = \mathbb{E}[X]\,,$$

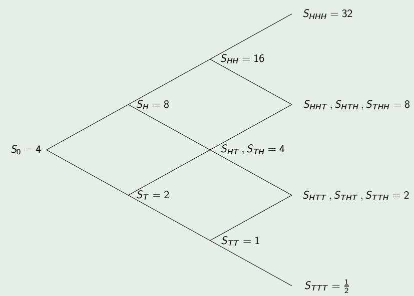
$$\mathbb{E}_N[X] = X\,.$$

Examples
For $p = q = \frac{1}{2}$

Example

Expected value of S_3 based on information available at step 1; i.e., given that S_1 is H

$$\mathbb{E}_1[S_3](H) = \left(\frac{1}{2}\right)^2 \cdot 32 + 2\left(\frac{1}{2}\right)^2 \cdot 8 + \left(\frac{1}{2}\right)^2 \cdot 2 = 12.5\,,$$



What is the value of $\mathbb{E}_1[S_3](T)$?

$\mathbb{E}_1[S_3]$ as a random variable

Note that $\mathbb{E}_1[S_3]$ can take two different values ($\mathbb{E}_1[S_3](H)$ or $\mathbb{E}_1[S_3](T)$) depending on whether a S_1 took the value H or T at step one. So $\mathbb{E}_1[S_3]$ is itself a random variable

Properties of Conditional Expectations

Discrete case

As before let $0 \le n \le N$ and let X, Y be random variables which are dependent on the first N coin tosses. Suppose that we know the first n coin tosses $\omega_1 \cdots \omega_n$, but not the remaining tosses $\omega_{n+1} \cdots \omega_N$. Then

Linearity of conditional expectations For constants c_1, c_2

$$\boxed{\mathbb{E}_n(c_1 X + c_2 Y) = c_1 \mathbb{E}_n(X) + c_2 \mathbb{E}_n(Y)\,.}$$

Taking out what is known If X only depends on the first n tosses (known),

$$\boxed{\mathbb{E}_n(XY) = X \cdot \mathbb{E}_n(Y)\,.}$$

Iterated conditioning if $0 \le n \le m \le N$,

$$\boxed{\mathbb{E}_n(\mathbb{E}_m(X)) = \mathbb{E}_n(X)\,.}$$

This means in particular that $\mathbb{E}(\mathbb{E}_m(X)) = \mathbb{E}_0(\mathbb{E}_m(X)) = \mathbb{E}_0(X) = \mathbb{E}(X)$.

Independence If X depends *only* on coin tosses $n+1, \cdots, N$, then

$$\boxed{\mathbb{E}_n(X) = \mathbb{E}(X)\,.}$$

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Stochastic Process

Definition (Stochastic Process)

A *stochastic process* is a sequence of random variables indexed by time

In the context of a binomial model, the random variables of a stochastic process take values at each time step $(0, 1, \cdots)$.

Example

The stock process on a binomial tree is a stochastic process

Adapted Stochastic Process

Definition

Consider a binomial tree model. Let $M_0, M_1, \cdots, M_N$ be a sequence of random variables indexed by time step on the tree. The sequence M_i forms a stochastic process.

Suppose in addition that at each step n, M_n depends only on the first n coin tosses. Then this sequence is called an *adapted* stochastic process

Example

The stock process on a binomial tree is an adapted stochastic process because at each step n, the value of S_n is based on the first n coin tosses

Martingales

Definition (Martingale)

Let $M_0\,, M_1\,, \cdots\,, M_n\,, \cdots\,, M_N$ be an adapted stochastic process (each M_n is a random variable which is dependent on the first n coin tosses)

1. If, for each n
$$M_n = \mathbb{E}_n[M_{n+1}]\,, \quad 0, 1, \cdots, N-1$$
then this stochastic process is a *martingale*
2. If, for each n
$$M_n \leq \mathbb{E}_n[M_{n+1}]\,, \quad 0, 1, \cdots, N-1$$
then this stochastic process is a *submartingale* (tendancy to increase)
3. If, for each n
$$M_n \geq \mathbb{E}_n[M_{n+1}]\,, \quad 0, 1, \cdots, N-1$$
then this stochastic process is a *supermartingale* (tendancy to decrease)

We will be interested here in martingales only

Results following from Martingale definition

Let a stochastic process $\{M_i\}$ be a martingale. So $M_n = \mathbb{E}_n[M_{n+1}]\,, n = 0, 1, \cdots, N-1$. This means that

$$M_n = \mathbb{E}_n[M_{n+1}] = \mathbb{E}_n[\mathbb{E}_{n+1}[M_{n+2}]] = \underbrace{\mathbb{E}_n[M_{n+2}]}_{\text{by iterated conditioning}}$$

Extending this, we have for $m > n$, $M_n = \mathbb{E}_n[M_m]$.

Results following from Martingale definition

From the definition of a martingale

$$M_n = \mathbb{E}_n[M_{n+1}]\,.$$

Take $\mathbb{E}$ of both sides:

$$\mathbb{E}M_n = \mathbb{E}_0 M_n = \mathbb{E}_0[\mathbb{E}_n[M_{n+1}]] = \underbrace{\mathbb{E}_0 M_{n+1}}_{\text{iterated conditioning}} = \mathbb{E}[M_{n+1}]\,,$$

which becomes

$$\mathbb{E}M_0 = \mathbb{E}M_1 = \mathbb{E}M_2 = \cdots\,,$$

but $\mathbb{E}M_0 = M_0$, so

$$M_0 = \mathbb{E}M_1 = \mathbb{E}M_2 = \cdots = \mathbb{E}M_N\,.$$

$$M_0 = \mathbb{E}M_n\,, n = 0\,, 1\,, \cdots\,, N$$

Conditional Expectations in continuous time

$$\mathbb{E}_s[M_t] \approx \mathbb{E}[M(t)|\mathcal{F}(s)] \;\; (0 \leq s \leq t \leq T)$$

Properties of Conditional Expectations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G}$ be a sub σ-algebra of $\mathcal{F}$. Let X and Y be random variables.

Linearity Let c_1 and c_2 be constants. Then

$$\mathbb{E}[c_1 X + c_2 Y|\mathcal{G}] = c_1\mathbb{E}[X|\mathcal{G}] + c_2\mathbb{E}[Y|\mathcal{G}]\,.$$

Taking out what is known Let X be $\mathcal{G}$-measurable, so X may be ascertained based on the information contained in $\mathcal{G}$. Then

$$\mathbb{E}[XY|\mathcal{G}] = X\mathbb{E}[Y|\mathcal{G}]\,.$$

Iterated conditioning If $\mathcal{H}$ is a sub-σ-algebra of $\mathcal{G}$ (i.e., H contains less information than $\mathcal{G}$). Then,

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]\,.$$

Independence Let X be independent of information available in $\mathcal{G}$. Then

$$\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]\,.$$

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Binomial

Linearity For constants c_1, c_2

$$\mathbb{E}_n(c_1X + c_2Y) = c_1\mathbb{E}_n(X) + c_2\mathbb{E}_n(Y)\,.$$

Taking out what is known If X only depends on the first n tosses (known),

$$\mathbb{E}_n(XY) = X \cdot \mathbb{E}_n(Y)\,.$$

Iterated conditioning if $0 \le n \le m \le N$,

$$\mathbb{E}_n(\mathbb{E}_m(X)) = \mathbb{E}_n(X)\,.$$

Independence If X depends *only* on coin tosses $n+1, \cdots, N$, then

$$\mathbb{E}_n(X) = \mathbb{E}(X)\,.$$

Measure Theoretic

Linearity Let c_1 and c_2 be constants. Then

$$\mathbb{E}[c_1X + c_2Y|\mathcal{G}] = c_1\mathbb{E}[X|\mathcal{G}] + c_2\mathbb{E}[Y|\mathcal{G}]$$

Taking out what is known Let X be $\mathcal{G}$-measurable, so X may be ascertained based on the information contained in $\mathcal{G}$. Then

$$\mathbb{E}[XY|\mathcal{G}] = X\mathbb{E}[Y|\mathcal{G}]\,.$$

Iterated conditioning If $\mathcal{H}$ is a sub-σ-algebra of $\mathcal{G}$ (i.e., H contains less information than $\mathcal{G}$). Then,

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]\,.$$

Independence Let X be independent of information available in $\mathcal{G}$. Then

$$\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]\,.$$

Martingales Again

We've already had one definition of a martingale for a stochastic process M_n defined on a binomial tree using conditional expectations.

$$\mathbb{E}_n[M_{n+1}] = M_n\,.$$

An analgous definition using the measure theoretic conditional expectations that we have just encountered is

Definition

Let $M(t)$ be a stochastic process which is adapted to a filtration $\mathcal{F}(t)$. Then for $0 \le s \le t \le T$, $M(t)$ is a martingale if

$$\mathbb{E}[M(t)|\mathcal{F}(s)] = M(s)\,.$$

Discounted Asset Price

Risk-Neutral Measure

Let $\omega_1, \omega_2, \cdots, \omega_n$ be the first n coin tosses. So

$$S_n = S_n(\omega_1, \omega_2, \cdots, \omega_n)$$

$$\tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] =$$

$$\begin{aligned}\tilde{p} &= \frac{1+r-d}{u-d}\,,\\ \tilde{q} &= \frac{u-(1+r)}{u-d}\,,\end{aligned}$$

$$\begin{aligned}\tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] &= \frac{1}{(1+r)^{n+1}}\left[\tilde{p}S_{n+1}(\omega_1, \omega_2, \cdots, \omega_n H) + \tilde{q}S_{n+1}(\omega_1, \omega_2, \cdots, \omega_n T)\right]\\ &= \frac{1}{(1+r)^{n+1}}\left[\tilde{p}uS_n(\omega_1, \omega_2, \cdots, \omega_n) + \tilde{q}dS_n(\omega_1, \omega_2, \cdots, \omega_n)\right]\\ &= \frac{S_n(\omega_1, \omega_2, \cdots, \omega_n)}{(1+r)^{n+1}}\left[\tilde{p}u + \tilde{q}d\right]\\ &= \frac{S_n(\omega_1, \omega_2, \cdots, \omega_n)}{(1+r)^{n}}\left[\frac{\tilde{p}u + \tilde{q}d}{1+r}\right]\\ &= \frac{S_n(\omega_1, \omega_2, \cdots, \omega_n)}{(1+r)^{n}} \implies \text{martingale}\,.\end{aligned}$$

Therefore the discounted asset price process is a martingale under the risk-neutral measure.

Discounted Portfolio Value
Risk-Neutral Measure

From earlier on, the portfolio value process is defined by

$$X_{n+1}(H) = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)\,,$$

- where Δ_n is the amount of asset to be held on step n,
- the value S_{n+1} is one of two values $S_{n+1}(\omega_1\omega_2\cdots\omega_n H)$ or $S_{n+1}(\omega_1\omega_2\cdots\omega_n T)$
- These two values are written as $S_{n+1}(H)$, $S_{n+1}(T)$.

Then

$$\begin{aligned}
\tilde{\mathbb{E}}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right] &= \frac{1}{(1+r)^{n+1}}\tilde{\mathbb{E}}_n\left[\Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)\right]\,,\\
&= \underbrace{\frac{1}{(1+r)^{n+1}}\tilde{\mathbb{E}}_n\left[\Delta_n S_{n+1}\right] + \frac{1}{(1+r)^{n}}\tilde{\mathbb{E}}_n\left[(X_n - \Delta_n S_n)\right]}_{\text{linearity of conditional expectations}}\,,\\
&= \underbrace{\frac{1}{(1+r)^{n+1}}\Delta_n\tilde{\mathbb{E}}_n\left[S_{n+1}\right] + \frac{1}{(1+r)^{n}}(X_n - \Delta_n S_n)}_{\text{taking out what is known at step } n}\,,\\
&= \Delta_n\tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] + \frac{1}{(1+r)^{n}}(X_n - \Delta_n S_n)\,,\\
&= \Delta_n \underbrace{\frac{S_n}{(1+r)^n}}_{\frac{S_n}{(1+r)^n}\text{ is a martingale}} + \frac{1}{(1+r)^{n}}(X_n - \Delta_n S_n)\,,\\
&= \frac{X_n}{(1+r)^n} \implies \text{martingale}\,.
\end{aligned}$$

Symmetric Random Walk

Symmetric Random Walk

- Start with a “fair” coin
- Result of a coin toss can be a head (H) or a tail (T)
- Since the coin is fair,

$$\begin{aligned}
P(H) &= p = \frac{1}{2}\,,\\
P(T) &= q = 1 - p = \frac{1}{2}\,.
\end{aligned}$$

- Take a sequence of coin tosses $\omega = \omega_1\omega_2\omega_3\omega_4$, where each ω_i is a coin toss
- Each coin toss is independent of the others

▶ Steven E. Shreve.
Stochastic Calculus for Finance II: Continuous-Time Models (Springer Finance).
Springer, 1st ed. 2004. corr. 2nd printing edition, June 2004.

Symmetric Random Walk

Define a random variable X_i

$$X_i = \begin{cases} +1, & \omega_i = H \\ -1, & \omega_i = T \end{cases}$$

$$\begin{aligned}
\mathbb{E}[X_i] &= p\cdot 1 + q\cdot(-1) = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot(-1) = 0\,,\\
\mathrm{Var}[X_i] &= \mathbb{E}[X_i^2] = p\cdot 1 + q\cdot(1) = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot(1) = 1\,,
\end{aligned}$$

where $\mathrm{Var}[X_i] = \mathbb{E}[X_i^2]$ because $\mathbb{E}[X_i] = 0$

Symmetric Random Walk

Define a process M_k, where $M_0 = 0$ and

$$M_k = \sum_{i=0}^{k} X_i \,.$$

The process $M_0, M_1, M_2, \cdots$ is called a (symmetric) random walk

Properties of the Random Walk

Increments

Let $0 = k_0 < k_1 < k_2 < \cdots < k_m$ be a set of integers

Example

$0 = k_0 < 5(k_1) < 9(k_2) < 15(k_3) < \cdots < 38(k_m)$.

Then the *increment*

$$\begin{aligned} M_{k_{i+1}} - M_{k_i} &= \sum_{j=1}^{k_{i+1}} X_j - \sum_{j=1}^{k_i} X_j \,, \\ &= \left(X_1 + X_2 + \cdots + X_{k_{i+1}}\right) - \left(X_1 + X_2 + \cdots X_{k_i}\right) \,, \\ &= X_{k_i+1} + X_{k_i+2} + \cdots + X_{k_{i+1}} \,, \\ &= \sum_{j=k_i+1}^{k_{i+1}} X_j \,. \end{aligned}$$

Properties of the Random Walk

Increments

Example

Letting $0 = k_0 < 5(k_1) < 9(k_2) < 15(k_3) < \cdots < 38(k_m)$,

$$\begin{aligned} M_{k_2} - M_{k_1} &= M_9 - M_5 = \sum_{j=1}^{9} X_j - \sum_{j=1}^{5} X_j \,, \\ &= (X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7 + X_8 + X_9) \\ &\quad - (X_1 + X_2 + X_3 + X_4 + X_5) \,, \\ &= (X_6 + X_7 + X_8 + X_9) \,, \\ &= \sum_{j=6}^{9} X_j = \sum_{j=5+1}^{9} X_j = \sum_{j=k_1+1}^{k_2} X_j \,. \end{aligned}$$

Independence of Increments

For $0 = k_0 < k_1 < k_2 < \cdots < k_m$, the increments

$$M_{k_1} - M_{k_0}, M_{k_2} - M_{k_1}, M_{k_3} - M_{k_2}, \cdots \,,$$

are independent of each other.

$$\sum_{j=k_0+1}^{k_1} X_j, \sum_{j=k_1+1}^{k_2} X_j, \sum_{j=k_2+1}^{k_3} X_j, \cdots$$

This is because each increment is based on different groups of coin tosses and all the coin tosses are independent of each other

Example

Letting $0 = k_0 < 5(k_1) < 9(k_2) < 15(k_3) < \cdots < 38(k_m)$,
Then
$$M_{k_2} - M_{k_1} = M_9 - M_5 = X_6 + X_7 + X_8 + X_9 \,,$$
and
$$M_{k_3} - M_{k_2} = M_{15} - M_9 = X_{10} + X_{11} + X_{12} + X_{13} + X_{14} + X_{15} \,,$$
Since all the coin tosses are independent of each other, the increments are independent of each other

Expectation and Variance

$$\begin{aligned}
\mathbb{E}\big(M_{k_{i+1}} - M_{k_i}\big) &= \sum_{j=1}^{k_{i+1}} \mathbb{E}X_j - \sum_{j=1}^{k_i} \mathbb{E}X_j \,, \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}\big(M_{k_{i+1}} - M_{k_i}\big) &= \mathrm{Var}\left(\sum_{j=1}^{k_{i+1}} X_j - \sum_{j=1}^{k_i} X_j\right) \,, \\
&= \mathrm{Var}\left(\sum_{j=k_i+1}^{k_{i+1}} X_j\right) \,, \\
&= \sum_{j=k_i+1}^{k_{i+1}} \mathrm{Var}\,(X_j) \,, \\
&= \sum_{j=k_i+1}^{k_{i+1}} 1 = k_{i+1} - k_i \qquad \left(\text{because} \sum_{i=1}^{n} 1 = n\right)
\end{aligned}$$

Expectation and Variance

Example

Letting $0 = k_0 < 5(k_1) < 9(k_2) < 15(k_3) < \cdots < 38(k_m)$, Then
$$\begin{aligned}
\mathrm{Var}\,(M_{k_2} - M_{k_1}) &= \mathrm{Var}\,(M_9 - M_5) \,, \\
&= \mathrm{Var}\,(X_6 + X_7 + X_8 + X_9) \,, \\
&= (1 + 1 + 1 + 1) \,, \\
&= 4 = 9 - 5 = k_2 - k_1 \,.
\end{aligned}$$

Martingale Property for symmetric random walk

Let $0 \le k < l$ be integers (times). Then

$$\begin{aligned}
\mathbb{E}_k[M_l] &= \mathbb{E}_k[M_l - M_k + M_k] \,, \\
&= \mathbb{E}_k[M_l - M_k] + \mathbb{E}_k[M_k] \,,
\end{aligned}$$

At step k M_k is known, so $\mathbb{E}_k[M_k] = M_k$.
Also, the quantity $M_l - M_k$ is based only on coin tosses greater than k, so is independent of all coin tosses up to and including step k. So $\mathbb{E}_k[M_l - M_k] = \mathbb{E}[M_l - M_k]$.
Therefore,

$$\begin{aligned}
\mathbb{E}_k[M_l] &= \mathbb{E}_k[M_l - M_k] + \mathbb{E}_k[M_k] \,, \\
&= \mathbb{E}[M_l - M_k] + M_k \,, \\
&= 0 + M_k \,, \\
&= M_k \,.
\end{aligned}$$

So the symmetric random walk is a Martingale.

Martingale Property for symmetric random walk

Same calculation, Different notation

Let $0 \leq k < l$ be integers (times). Then

$$\begin{aligned}
\mathbb{E}[M_l|\mathcal{F}_k] &= \mathbb{E}[M_l - M_k + M_k|\mathcal{F}_k]\,, \\
&= \mathbb{E}[M_l - M_k|\mathcal{F}_k] + \mathbb{E}[M_k|\mathcal{F}_k]\,, \\
&= \mathbb{E}[M_l - M_k|\mathcal{F}_k] + M_k\,, \\
&= \mathbb{E}[M_l - M_k] + M_k\,, \\
&= 0 + M_k\,, \\
&= M_k\,.
\end{aligned}$$

So the symmetric random walk is a Martingale.

Scaled Random Walk

Limiting Behaviour

- With the random walk defined in the previous slides there is no useful idea of limiting
- There is only one variable to limit: k, in M_k
- Will now define a *scaled* random walk

Scaled (Symmetric) Random Walk

Symmetric if $p = 1 - q = \frac{1}{2}$

Define

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt}\,.$$

- $W^{(n)}(t)$ is defined for n, t where nt is an integer
- For $n = 100$ and $t = 0.25$, $nt = 25$; an integer
- For $n = 100$ and $t = 0.00000001$, $nt = 0.000001$, not an integer
- Each unit interval in $[0, t]$ split into n parts of length $\frac{1}{n}$

Scaled (Symmetric) Random Walk

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt} = \sum_{j=1}^{nt} \frac{1}{\sqrt{n}} X_j$$

For each X_j term,

$$\frac{1}{\sqrt{n}} X_i = \begin{cases} +\frac{1}{\sqrt{n}}, & \omega_i = H \\ -\frac{1}{\sqrt{n}}, & \omega_i = T \end{cases}$$

So the step size is smaller as n gets larger

Independence of Increments of $W^{(n)}(t)$

$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt}$ is defined as a random walk, so its increments are independent from previous slides

Expectation and Variance of $\frac{1}{\sqrt{n}} X_j$

$$\mathbb{E}\left(\frac{1}{\sqrt{n}} X_j\right) = 0$$

$$\text{Var}\left(\frac{1}{\sqrt{n}} X_j\right) = \mathbb{E}\left[\left(\frac{1}{\sqrt{n}} X_j\right)^2\right],$$

since $\mathbb{E}\left(\frac{1}{\sqrt{n}} X_j\right) = 0$.

$$\begin{aligned} \mathbb{E}\left[\left(\frac{1}{\sqrt{n}} X_j\right)^2\right] &= \frac{1}{n}\mathbb{E}(X_j^2), \\ &= \frac{1}{n} \cdot 1 = \frac{1}{n}. \end{aligned}$$

Expectation and Variance $W^{(n)}(t) - W^{(n)}(s)$
$s < t$

By definition of $W^{(n)}(t)$ as a symmetric random walk,

$$\mathbb{E}\left[W^{(n)}(t) - W^{(n)}(s)\right] = 0$$

and

$$\begin{aligned} \text{Var}\left[W^{(n)}(t) - W^{(n)}(s)\right] &= \text{Var}\left[\frac{1}{\sqrt{n}}(M_{nt} - M_{ns})\right], \\ &= \frac{1}{n}\text{Var}\left[M_{nt} - M_{ns}\right], \\ &= \frac{1}{n}(nt - ns) \quad \text{(from earlier slides)}, \\ &= t - s. \end{aligned}$$

Limit of Scaled Symmetric Random Walk

In the limit as $n \to \infty$, $W^{(n)}(t)$ limits to *Brownian Motion*, $W(t)$.

Limit of Scaled Symmetric Random Walk to Normal distribution

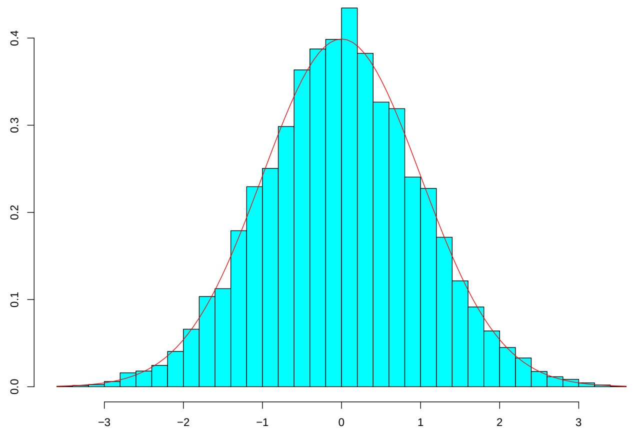

Figure: Histogram of values at $t = 1$ of 10000 scaled random walks, each of length 5000. Red curve; density function of normal distribution $N(0, 1)$. (Sample mean: $\mu = -0.00182$; sample variance: $\sigma = 1.00222$)

Review of Course Topics

1. Introduction
2. Assumptions
3. Binomial Model
4. European Stock Option: Binomial Model
5. Conditional Expected Values on a Tree