Trinity Centre for High Performance Computing

MSc in HPC course 5635b

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January 26, 2017

An Introduction to Mathematical Finance (5635b)

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Course Outline

1. Brownian Motion
2. Integration
3. Itô Integral
4. Itô Formula

Symmetric Random Walk

Symmetric Random Walk

- Start with a “fair” coin
- Result of a coin toss can be a head (H) or a tail (T)
- Since the coin is fair,

$$\begin{aligned} P(H) &= p = \frac{1}{2}, \\ P(T) &= q = 1 - p = \frac{1}{2}. \end{aligned}$$

- Take a sequence of coin tosses $\omega = \omega_1\omega_2\omega_3\omega_4$, where each ω_i is a coin toss
- Each coin toss is independent of the others

▶ Steven E. Shreve.
Stochastic Calculus for Finance II: Continuous-Time Models (Springer Finance).
Springer, 1st ed. 2004. corr. 2nd printing edition, June 2004.

Symmetric Random Walk

Define a random variable X_i

$$X_i = \begin{cases} +1, & \omega_i = H \\ -1, & \omega_i = T \end{cases}$$

$$\begin{aligned} \mathbb{E}[X_i] &= p \cdot 1 + q \cdot (-1) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1) = 0\,, \\ \mathrm{Var}[X_i] &= \mathbb{E}[X_i^2] = p \cdot 1 + q \cdot (1) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (1) = 1\,, \end{aligned}$$

where $\mathrm{Var}[X_i] = \mathbb{E}[X_i^2]$ because $\mathbb{E}[X_i] = 0$

Symmetric Random Walk

Define a process M_k, where $M_0 = 0$ and

$$M_k = \sum_{i=0}^{k} X_i\,.$$

The process $M_0, M_1, M_2, \cdots$ is called a (symmetric) random walk

Properties of the Random Walk

Increments

Let $0 = k_0 < k_1 < k_2 < \cdots < k_m$ be a set of integers

Example

$0 = k_0 < 5(k_1) < 9(k_2) < 15(k_3) < \cdots < 38(k_m)$.

Then the *increment*

$$\begin{aligned} M_{k_{i+1}} - M_{k_i} &= \sum_{j=1}^{k_{i+1}} X_j - \sum_{j=1}^{k_i} X_j\,, \\ &= \left(X_1 + X_2 + \cdots + X_{k_{i+1}}\right) - \left(X_1 + X_2 + \cdots X_{k_i}\right)\,, \\ &= X_{k_i+1} + X_{k_i+2} + \cdots + X_{k_{i+1}}\,, \\ &= \sum_{j=k_i+1}^{k_{i+1}} X_j\,. \end{aligned}$$

Properties of the Random Walk

Increments

Example

Letting $0 = k_0 < 5(k_1) < 9(k_2) < 15(k_3) < \cdots < 38(k_m)$,

$$\begin{aligned} M_{k_2} - M_{k_1} &= M_9 - M_5 = \sum_{j=1}^{9} X_j - \sum_{j=1}^{5} X_j\,, \\ &= (X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7 + X_8 + X_9) \\ &\quad - (X_1 + X_2 + X_3 + X_4 + X_5)\,, \\ &= (X_6 + X_7 + X_8 + X_9)\,, \\ &= \sum_{j=6}^{9} X_j = \sum_{j=5+1}^{9} X_j = \sum_{j=k_1+1}^{k_2} X_j\,. \end{aligned}$$

Independence of Increments

For $0 = k_0 < k_1 < k_2 < \cdots < k_m$, the increments

$$M_{k_1} - M_{k_0}, M_{k_2} - M_{k_1}, M_{k_3} - M_{k_2}, \cdots,$$

are independent of each other.

$$\sum_{j=k_0+1}^{k_1} X_j, \sum_{j=k_1+1}^{k_2} X_j, \sum_{j=k_2+1}^{k_3} X_j, \cdots$$

This is because each increment is based on different groups of coin tosses and all the coin tosses are independent of each other

Example

Letting $0 = k_0 < 5(k_1) < 9(k_2) < 15(k_3) < \cdots < 38(k_m)$,
Then

$$M_{k_2} - M_{k_1} = M_9 - M_5 = X_6 + X_7 + X_8 + X_9,$$

and

$$M_{k_3} - M_{k_2} = M_{15} - M_9 = X_{10} + X_{11} + X_{12} + X_{13} + X_{14} + X_{15},$$

Since all the coin tosses are independent of each other, the increments are independent of each other

Expectation and Variance

$$\begin{aligned}
\mathbb{E}\left(M_{k_{i+1}} - M_{k_i}\right) &= \sum_{j=1}^{k_{i+1}} \mathbb{E}X_j - \sum_{j=1}^{k_i} \mathbb{E}X_j, \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}\left(M_{k_{i+1}} - M_{k_i}\right) &= \mathrm{Var}\left(\sum_{j=1}^{k_{i+1}} X_j - \sum_{j=1}^{k_i} X_j\right), \\
&= \mathrm{Var}\left(\sum_{j=k_i+1}^{k_{i+1}} X_j\right), \\
&= \sum_{j=k_i+1}^{k_{i+1}} \mathrm{Var}\left(X_j\right), \\
&= \sum_{j=k_i+1}^{k_{i+1}} 1 = k_{i+1} - k_i \qquad (\text{because} \sum_{i=1}^{n} 1 = n)
\end{aligned}$$

Expectation and Variance

Example

Letting $0 = k_0 < 5(k_1) < 9(k_2) < 15(k_3) < \cdots < 38(k_m)$, Then

$$\begin{aligned}
\mathrm{Var}\left(M_{k_2} - M_{k_1}\right) &= \mathrm{Var}\left(M_9 - M_5\right), \\
&= \mathrm{Var}\left(X_6 + X_7 + X_8 + X_9\right), \\
&= (1 + 1 + 1 + 1), \\
&= 4 = 9 - 5 = k_2 - k_1.
\end{aligned}$$

Martingale Property for symmetric random walk

Let $0 \leq k < l$ be integers (times). Then

$$\begin{aligned} \mathbb{E}_k[M_l] &= \mathbb{E}_k[M_l - M_k + M_k], \\ &= \mathbb{E}_k[M_l - M_k] + \mathbb{E}_k[M_k], \end{aligned}$$

At step k M_k is known, so $\mathbb{E}_k[M_k] = M_k$.
Also, the quantity $M_l - M_k$ is based only on coin tosses greater than k, so is independent of all coin tosses up to and including step k. So $\mathbb{E}_k[M_l - M_k] = \mathbb{E}[M_l - M_k]$.
Therefore,

$$\begin{aligned} \mathbb{E}_k[M_l] &= \mathbb{E}_k[M_l - M_k] + \mathbb{E}_k[M_k], \\ &= \mathbb{E}[M_l - M_k] + M_k, \\ &= 0 + M_k, \\ &= M_k. \end{aligned}$$

So the symmetric random walk is a Martingale.

Martingale Property for symmetric random walk

Same calculation, Different notation

Let $0 \leq k < l$ be integers (times). Then

$$\begin{aligned} \mathbb{E}[M_l|\mathcal{F}_k] &= \mathbb{E}[M_l - M_k + M_k|\mathcal{F}_k], \\ &= \mathbb{E}[M_l - M_k|\mathcal{F}_k] + \mathbb{E}[M_k|\mathcal{F}_k], \\ &= \mathbb{E}[M_l - M_k|\mathcal{F}_k] + M_k, \\ &= \mathbb{E}[M_l - M_k] + M_k, \\ &= 0 + M_k, \\ &= M_k. \end{aligned}$$

So the symmetric random walk is a Martingale.

Scaled Random Walk

Limiting Behaviour

- With the random walk defined in the previous slides there is no useful idea of limiting
- There is only one variable to limit: k, in M_k
- Will now define a *scaled* random walk

Scaled (Symmetric) Random Walk

Symmetric if $p = 1 - q = \frac{1}{2}$

Define

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt} \,.$$

- $W^{(n)}(t)$ is defined for n, t where nt is an integer
- For $n = 100$ and $t = 0.25$, $nt = 25$; an integer
- For $n = 100$ and $t = 0.00000001$, $nt = 0.000001$, not an integer
- Each unit interval in $[0, t]$ split into n parts of length $\frac{1}{n}$

Scaled (Symmetric) Random Walk

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt} = \sum_{j=1}^{nt} \frac{1}{\sqrt{n}} X_j$$

For each X_j term,

$$\frac{1}{\sqrt{n}} X_i = \begin{cases} +\frac{1}{\sqrt{n}}, & \omega_i = H \\ -\frac{1}{\sqrt{n}}, & \omega_i = T \end{cases}$$

So the step size is smaller as n gets larger

Independence of Increments of $W^{(n)}(t)$

$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt}$ is defined as a random walk, so its increments are independent from previous slides

Expectation and Variance of $\frac{1}{\sqrt{n}} X_j$

$$\mathbb{E}\left(\frac{1}{\sqrt{n}} X_j\right) = 0$$

$$\text{Var}\left(\frac{1}{\sqrt{n}} X_j\right) = \mathbb{E}\left[\left(\frac{1}{\sqrt{n}} X_j\right)^2\right],$$

since $\mathbb{E}\left(\frac{1}{\sqrt{n}} X_j\right) = 0$.

$$\begin{aligned} \mathbb{E}\left[\left(\frac{1}{\sqrt{n}} X_j\right)^2\right] &= \frac{1}{n}\mathbb{E}(X_j^2)\,, \\ &= \frac{1}{n} \cdot 1 = \frac{1}{n}\,. \end{aligned}$$

Expectation and Variance $W^{(n)}(t) - W^{(n)}(s)$

$s < t$

By definition of $W^{(n)}(t)$ as a symmetric random walk,

$$\mathbb{E}\left[W^{(n)}(t) - W^{(n)}(s)\right] = 0$$

and

$$\begin{aligned}
\mathrm{Var}\left[W^{(n)}(t) - W^{(n)}(s)\right] &= \mathrm{Var}\left[\frac{1}{\sqrt{n}}\left(M_{nt} - M_{ns}\right)\right], \\
&= \frac{1}{n}\mathrm{Var}\left[M_{nt} - M_{ns}\right], \\
&= \frac{1}{n}(nt - ns) \quad \text{(from earlier slides)}, \\
&= t - s\,.
\end{aligned}$$

Limit of Scaled Symmetric Random Walk

In the limit as $n \to \infty$, $W^{(n)}(t)$ limits to *Brownian Motion*, $W(t)$.

Limit of Scaled Symmetric Random Walk to Normal distribution

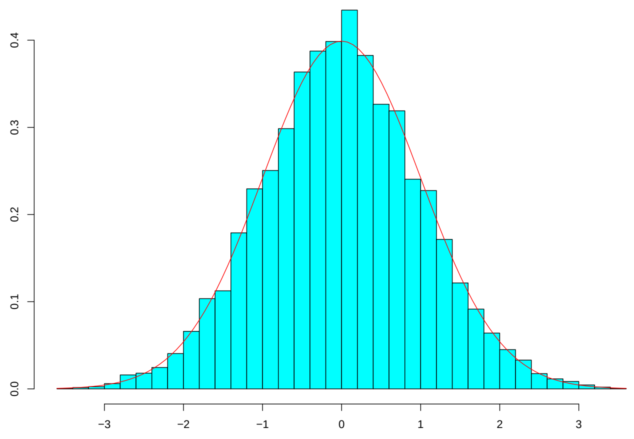


Figure: Histogram of values at $t = 1$ of 10000 scaled random walks, each of length 5000. Red curve; density function of normal distribution $N(0, 1)$. (Sample mean: $\mu = -0.00182$; sample variance: $\sigma = 1.00222$)

Brownian Motion

Definition of Brownian Motion

Wiener Process

Definition (Brownian Motion)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Then for each $\omega \in \Omega$ a Brownian motion is a *continuous* function $W(t)$, $t > 0$ which depends on ω, which has the properties that

1. $W(0) = 0$,
2. $W(t)$ is continuous (almost surely)
3. For $0 = t_0 < t_1 < t_2 < \cdots t_k < \cdots$, the variables

$$W(t_1) = (W(t_1) - W(t_0)), (W(t_2) - W(t_1)), \cdots, (W(t_{k+1}) - W(t_k)), \cdots,$$

are independent of each other. Thus $W(t)$ has *independent increments*. Moreover each increment $(W(t_{j+1}) - W(t_j))$ is normally distributed with

$$\mathbb{E}(W(t_{j+1}) - W(t_j)) = 0,$$

and

$$\mathrm{Var}(W(t_{j+1}) - W(t_j)) = t_{j+1} - t_j.$$

Martingale

Using the filtration notation, we give a definition of a martingale by analogy with the one we have already seen

Definition (Martingale)

A process $X(t)$ (e.g., Brownian Motion) is a martingale if, for $0 \leq s < t$,

$$\mathbb{E}[X(t) \mid \mathcal{F}(s)] = X(s).$$

Such a process is drift free.

Some Properties of Brownian Motion

Moments of $W(t)$

Let $0 \leq s < t$, then

- Moments, by definition:

$$W(t) - W(s) \sim N(0, t-s).$$

  and, clearly $W(t) = W(t) - W(0) \sim N(0, t)$

$$\mathbb{E}[W(t)] = \mathbb{E}[W(s)] = 0, \qquad \mathbb{E}[(W(t) - W(s))^2] = t - s$$

- Covariance: $W(s)$ and $W(t) - W(s)$ are independent.
  So:

$$\begin{aligned} Cov[W(t), W(s)] &= \mathbb{E}[W(t)\, W(s)] - \mathbb{E}[W(t)]\, \mathbb{E}[W(s)] \\ &= \mathbb{E}[W(t)\, W(s)] \\ &= \mathbb{E}[(W(t) - W(s) + W(s))\, W(s)] \\ &= \mathbb{E}[(W(t) - W(s))\, W(s)] + \mathbb{E}[W^2(s)] \\ &= \mathbb{E}[W(t) - W(s)]\, \mathbb{E}[W(s)] + \mathbb{E}[W^2(s)] \\ &= s \end{aligned}$$

Brownian Motion is a Martingale

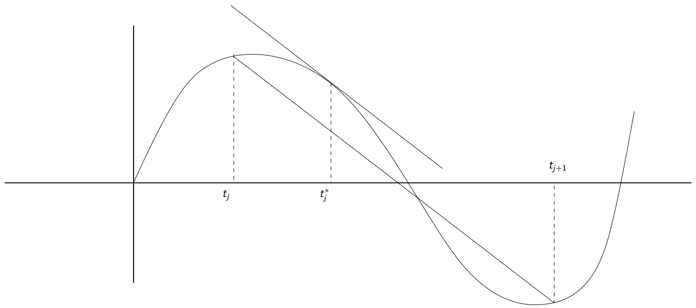
- $\mathbb{E}[W(t)] = W(0) = 0$.
- Likewise, conditional upon information up to time s ($0 < s < t$):

$$\begin{aligned}
\mathbb{E}[W(t)|\mathcal{F}(s)] &= \mathbb{E}[W(t) - W(s) + W(s)|\mathcal{F}(s)] \\
&= \mathbb{E}[W(t) - W(s)|\mathcal{F}(s)] + \mathbb{E}[W(s)|\mathcal{F}(s)] \\
&= 0 + W(s) \\
&= W(s)
\end{aligned}$$

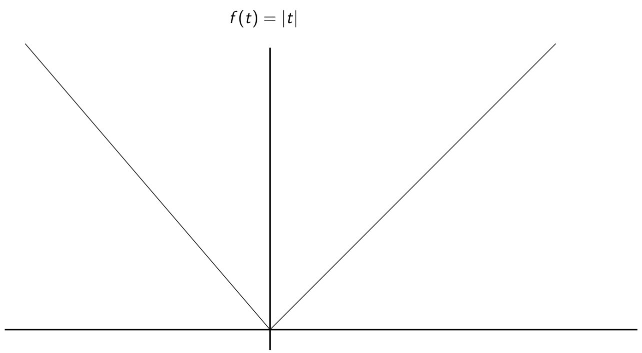
The expected future value equals the current value, the process is *drift-free*.

Variations of Brownian Motion

Differentiable everywhere



Not differentiable everywhere



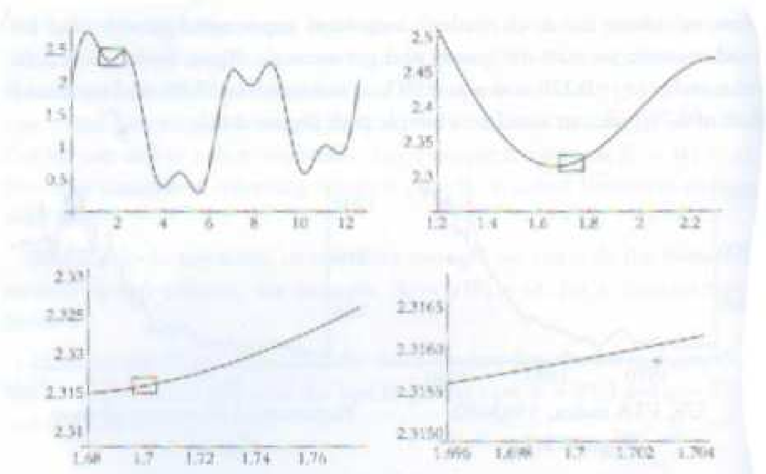


Figure 3.7 Progressive magnification around the point 1.7

- Martin Baxter and Andrew Rennie. *Financial Calculus: an introduction to derivative pricing.* CUP, 1996.

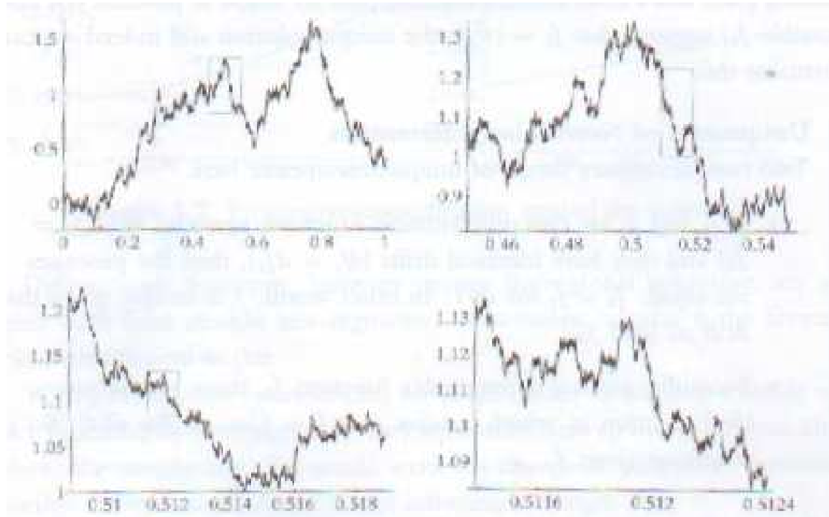


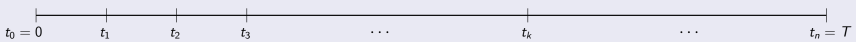
Figure 3.8 'Zooming in' on Brownian motion

- Martin Baxter and Andrew Rennie. *Financial Calculus: an introduction to derivative pricing.* CUP, 1996.

A Partition of the interval $[0, T]$

Definition (Partition)

A *partition* $\Pi = \{t_0\, , t_1\, , \cdots\, , t_n\}$ is a set of points in the interval $[0\, , T]$ such that $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$



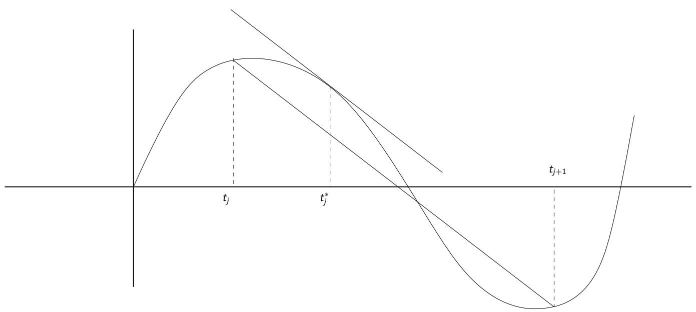
The *mesh* of the partition is defined as

$$||\Pi|| = \max_{k=0,\ldots,n-1} (t_{k+1} - t_k)$$

Mean Value Theorem

If $f(t)$ continuous on $[t_j, t_{j+1}]$ and differentiable on the interval (t_j, t_{j+1}), then there is some t_j^* in (t_j, t_{j+1}) such that

$$f'(t_j^*) = \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j}$$



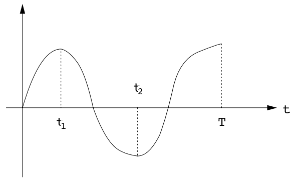
Note this does not in hold in the absence of differentiability e.g., $f(t) = |t|$.

First Variation of a differentiable function f

Definition

$$
\begin{aligned}
FV_f(T) &= \lim_{||\Pi||\to 0} \sum_{k=0}^{n-1} |f(t_{k+1}) - f(t_k)| \\
&= \lim_{||\Pi||\to 0} \sum_{k=0}^{n-1} |f'(t_k^*)|\,(t_{k+1} - t_k) \qquad \text{(mean value theorem)} \\
&= \int_0^T |f'(t)|dt.
\end{aligned}
$$

$FV_f(T)$ is a measure of up and down movement on the y axis (note the absolute value: $|f(t)|$). See also: http://en.wikipedia.org/wiki/Total_variation



Second (Quadratic) Variation of a differentiable function f

Definition

$$
\begin{aligned}
QV_f(T) &= \lim_{||\Pi||\to 0} \sum_{k=0}^{n-1} [f(t_{j+1}) - f(t_j)]^2 \\
&= \lim_{||\Pi||\to 0} \sum_{k=0}^{n-1} |f'(t_k^*)|^2\,(t_{k+1} - t_k)^2 \\
&\leq \lim_{||\Pi||\to 0} \left(\max_{0\leq k<n} (t_{k+1} - t_k) \right) \sum_{k=0}^{n-1} |f'(t_k^*)|^2\,(t_{k+1} - t_k) \\
&= \lim_{||\Pi||\to 0} ||\Pi|| \sum_{k=0}^{n-1} |f'(t_k^*)|^2\,(t_{k+1} - t_k) \\
&= \lim_{||\Pi||\to 0} ||\Pi|| \int_0^T |f'(t)|^2 dt \\
&= 0\,, \ \text{assuming} \int_0^T |f'(t)|^2 dt < \infty
\end{aligned}
$$

Quadratic Variation of Brownian Motion

Consider a partition $\Pi = \{t_0\,, t_1\,, \cdots\,, t_n\}$ of the interval $[0, t]$ such that $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$. The quadratic variation is defined to be

$$QV_W(t) = \lim_{||\Pi||\to 0} \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2\ ,$$

where $||\Pi|| = \max_{0\leq k<n}(t_{k+1} - t_k)$ is referred to as the mesh of the partition.

Quadratic Variation of Brownian Motion

We want to prove that, for $\Pi = \{t_0\,, t_1\,, \cdots\,, t_n\}$,

$$QV_W(t) = \lim_{||\Pi||\to 0} \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2 = t$$

Procedure:

1. Show that $\mathbb{E}\left[\sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2\right] = t$
2. Because $QV_W(t)$ itself is stochastic, it has a variance. We need to show this variance is zero (in the limit):

$$\text{Var}\left[\sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2\right] = 0 \text{ (as } ||\Pi|| \to 0)$$

Show that $\mathbb{E}\left[\sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))^2\right]=t$

$$\mathbb{E}\left[\sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))^2\right]=\sum_{j=0}^{n-1}\mathbb{E}\left[(W(t_{j+1})-W(t_j))^2\right].$$

Consider individual terms:

$$\mathbb{E}\left[(W(t_{j+1})-W(t_j))^2\right]=\mathrm{Var}\left[(W(t_{j+1})-W(t_j))\right]=t_{j+1}-t_j\,.$$

Therefore

$$\mathbb{E}\left[\sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))^2\right]=\sum_{j=0}^{n-1}(t_{j+1}-t_j)=t\,.$$

Show that $\mathrm{Var}\left[\sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))^2\right]=0$ (as $||\Pi||\to 0$)

We have

$$\mathrm{Var}\left[\sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))^2\right].$$

Since individual terms of sum are independent of each other (independence of increments),

$$\mathrm{Var}\left[\sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))^2\right]=\sum_{j=0}^{n-1}\mathrm{Var}\left[(W(t_{j+1})-W(t_j))^2\right].$$

Individual terms of $\mathrm{Var}\left[\sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))^2\right]$

Take individual terms and let $\Delta W_j=(W(t_{j+1})-W(t_j))$
Which means that $(W(t_{j+1})-W(t_j))^2$ is written as ΔW_j^2, so

$$\mathrm{Var}\left[(W(t_{j+1})-W(t_j))^2\right]=\mathrm{Var}\left[\Delta W_j^2\right]$$

$$\begin{aligned}
\mathrm{Var}\left[\Delta W_j^2\right] &= \mathbb{E}\left[\left(\Delta W_j^2-\mathbb{E}\left[\Delta W_j^2\right]\right)^2\right],\\
&= \mathbb{E}\left[\left(\Delta W_j^2-(t_{j+1}-t_j)\right)^2\right],\\
&= \mathbb{E}\left[(\Delta W_j)^4-2\Delta W_j^2(t_{j+1}-t_j)+(t_{j+1}-t_j)^2\right],\\
&= \mathbb{E}[(\Delta W_j)^4]-2\mathbb{E}[\Delta W_j^2](t_{j+1}-t_j)+(t_{j+1}-t_j)^2,\\
&= \mathbb{E}[(\Delta W_j)^4]-2\underbrace{(t_{j+1}-t_j)}_{\mathbb{E}[\Delta W_j^2]}(t_{j+1}-t_j)+(t_{j+1}-t_j)^2,\\
&= \mathbb{E}[(\Delta W_j)^4]-(t_{j+1}-t_j)^2\,.
\end{aligned}$$

Individual terms of $\mathrm{Var}\left[\sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))^2\right]$ (ctd)

We have

$$\mathbb{E}\left[(\Delta W_j)^4\right]=\mathbb{E}\left[(W(t_{j+1})-W(t_j))^4\right],$$

where $X=(W(t_{j+1})-W(t_j))$ is normally distributed with mean 0 and variance $(t_{j+1}-t_j)$, i.e.,

$$X\sim N(0\,,\sigma^2=(t_{j+1}-t_j))\,.$$

Based on the properties of the normal distribution,

$$\begin{aligned}
\mathbb{E}[X^4] &= 3\sigma^4\,,\ \text{(since the mean is zero)}\\
&= 3(t_{j+1}-t_j)^2\,.
\end{aligned}$$

So,

$$\mathbb{E}\left[(\Delta W_j)^4\right]=3(t_{j+1}-t_j)^2,$$

which means that

$$\begin{aligned}
\mathrm{Var}\left[(W(t_{j+1})-W(t_j))^2\right] &= \mathbb{E}[(\Delta W_j)^4]-(t_{j+1}-t_j)^2,\\
&= 3(t_{j+1}-t_j)^2-(t_{j+1}-t_j)^2,\\
&= 2(t_{j+1}-t_j)^2\,.
\end{aligned}$$

Sum over all individual terms and take limit

So,

$$\begin{aligned}
\operatorname{Var}\left[\sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))^2\right] &= 2\sum_{j=0}^{n-1}(t_{j+1}-t_j)^2\,,\\
&\le 2\max_{0\le k<n}(t_{k+1}-t_k)\sum_{j=0}^{n-1}(t_{j+1}-t_j)\,,\\
&= 2||\Pi||\sum_{j=0}^{n-1}(t_{j+1}-t_j) \;=\; 2||\Pi||\cdot t\,.
\end{aligned}$$

In the limit as $||\Pi|| \to 0$,

$$\lim_{||\Pi||\to 0}\operatorname{Var}\left[\sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))^2\right] = 0\cdot t = 0\,.$$

Note that $\operatorname{Var}\left[\sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))^2\right]$ <u>is only zero in the limit</u>

Therefore

$$\boxed{QV_W(t) = t\,.}$$

Recap

We wanted to prove that

$$QV_W(t) = \lim_{||\Pi||\to 0}\sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))^2 = t$$

We showed that

1. Expected value is t;

$$\mathbb{E}\left[\sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))^2\right] = t$$

2. Variance is zero;

$$\operatorname{Var}\left[\sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))^2\right] = 0 \text{ (as } ||\Pi||\to 0)$$

Brownian motion accumulates 1 unit of quadratic variation per unit time

Differential Notation

The statement about the quadratic variation of Brownian motion

$$QV_W(T) = \lim_{||\Pi||\to 0}\sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))^2 = T\,,$$

is informally referred to as

$$dW(t)dW(t) = dt\,.$$

This notation proves convenient later on as a shorthand
Other limits are referred to using a similar shorthand, and one which is also similar to the notation used in ordinary calculus;

The notation $dW(t)dt = 0$ is used to refer to the fact that the following limit vanishes

$$\lim_{||\Pi||\to 0}\sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))\,(t_{j+1}-t_j) = 0\,, \tag{1}$$

and

$$\lim_{||\Pi||\to 0}\sum_{j=0}^{n-1}(t_{j+1}-t_j)^2 = 0\,, \tag{2}$$

has the notation $dtdt = 0$ assigned to it

Differential Notation (ctd)

$dW(t)dt = 0$

$$\begin{aligned}
\left|\sum_{j=0}^{n-1}\Big(W(t_{j+1})-W(t_j)\Big)\,(t_{j+1}-t_j)\right| &\le \max_{0\le k<n}\left|\Big(W(t_{k+1})-W(t_k)\Big)\right|\sum_{j=0}^{n-1}(t_{j+1}-t_j)\\
&= \max_{0\le k<n}\left|\Big(W(t_{k+1})-W(t_k)\Big)\right|\cdot T\\
&\to 0\cdot T \quad (\text{as } ||\Pi||\to 0)\,,
\end{aligned}$$

by continuity of $W(t)$ (which is continuous by definition).

$dtdt = 0$

$$\begin{aligned}
\left|\sum_{j=0}^{n-1}(t_{j+1}-t_j)^2\right| &\le ||\Pi||\sum_{j=0}^{n-1}(t_{j+1}-t_j)\\
&= ||\Pi||\cdot T\\
&\to 0\cdot T \quad (\text{as } ||\Pi||\to 0)\,.
\end{aligned}$$

First Variation, Brownian Motion

Since the quadratic or second variation of a brownian motion process is finite- what does this imply for the first variation?

$$
\begin{aligned}
FV_W(t) &= \lim_{||\Pi||\to 0} \sum_{j=0}^{n-1} \left| (W(t_{j+1}) - W(t_j)) \right| , \\
&\geq \lim_{||\Pi||\to 0} \frac{\sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2}{\max_{0\leq k<n} |(W(t_{k+1}) - W(t_k))|} , \\
&= \lim_{||\Pi||\to 0} \frac{QV_W(t)}{\max_{0\leq k<n} |(W(t_{k+1}) - W(t_k))|} , \\
&\to \infty \; (\text{as } ||\Pi|| \to 0) ,
\end{aligned}
$$

The denominator goes to zero, because the Brownian motion is continuous almost surely.
This result indicates how strange a "function" Brownian motion is

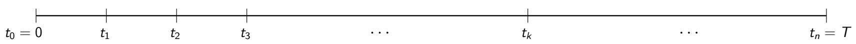
Itô Integral

Integration

For an ordinary function $f(x)$, we can define an integral as the limit of a sum:

$$\int_0^T f(t)dt = \lim_{||\Pi||\to 0} \sum_{j=0}^{n-1} f(t_j^*)(t_{j+1} - t_j) ,$$

where t_j^* is in $[t_i , t_{j+1}]$.



Remember:

$$||\Pi|| = \max_{k=0,\ldots,n-1} (t_{k+1} - t_k)$$

`http://en.wikipedia.org/wiki/Riemann_integral`

Stochastic Integral

We want to define an integral where the integrator is a Wiener process,

$$I(t) = \int_0^t \Delta(s)\, dW(s)$$

where $\Delta(s)$ is square-integrable. $\Delta(t)$ is determined based on information collected up to time t and may be stochastic.

In ordinary calculus, with differentiable function $f(t)$ instead of $W(t)$, we could define

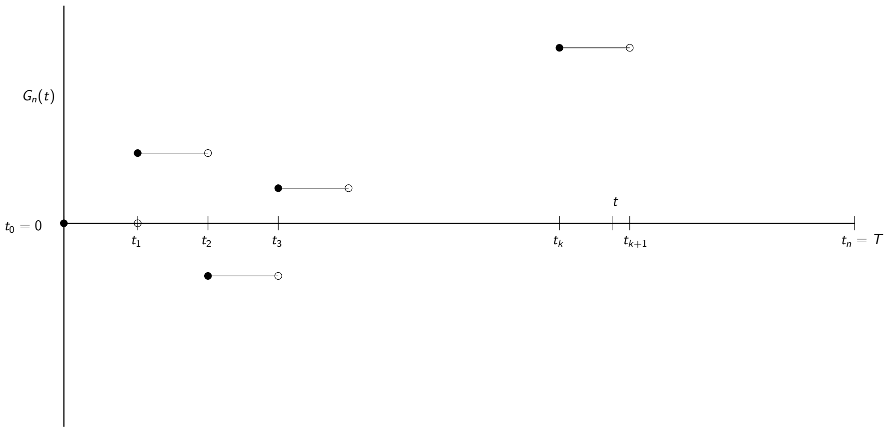
$$\int_0^t \Delta(s)\, df(s) = \int_0^t \Delta(s)\, f'(s)\, ds.$$

This does not work here, because W is not differentiable.

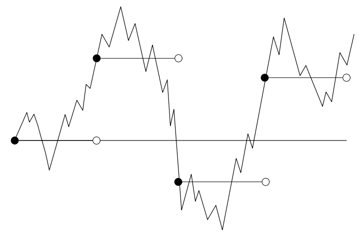
Instead we discretize, choose a partition first, define what we mean, and then shrink the mesh.

Step Function $\Delta(t)$

For a partition $\Pi = \{t_0, t_1, \cdots, t_n\}$ of the interval $[0, T]$, where $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$, define a "step function" $\Delta_n(t)$, on Π to be a function which holds a constant value in each interval $[t_j, t_{j+1})$.



Step function approximating general function



Stochastic Integral, Definition

We choose a partition $\Pi = t_0, t_1, ..., t_n$ of the time interval $[0, T]$,

$$0 = t_0 \le t_1 \le \cdots \le t_n = T, \qquad ||\Pi|| = \max_{k=0,\ldots,n-1} (t_{k+1} - t_k).$$

We then define the stochastic integral of a step function $\Delta_\Pi(t)$ as

$$I_\Pi(t) = \sum_{j=0}^{n-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) = \int_0^T \Delta_\Pi(t)\, dW(t),$$

and an integral for a general function $\Delta(t)$,

$$I(T) = \int_0^T \Delta(t)\, dW(t) = \lim_{||\Pi|| \to 0} I_\Pi(T),$$

where

$$\lim_{||\Pi|| \to 0, n \to \infty} \Delta_\Pi(T) = \Delta(T).$$

$$\left[\text{Actually:} \qquad \lim_{n \to \infty} \mathbb{E} \int_0^T |\Delta_\Pi(t) - \Delta(t)|^2 dt = 0 .\right]$$

Itô and Stratonovich

The position in time inteval $[t_k, t_{k+1}]$ where we evaluate $\Delta(t)$ is crucial, we obtain different values of $I(t)$ in the limit depending on this choice:

- Left point: popular in Finance (think of Δ as asset holdings chosen due information up to time t_k and then exposed to random movements of the price W per unit holding over the next time period). The resulting integral is called *Itô integral*, to be used in the following.
- Mid point: popular in Physics, the resulting integral is called Stratonovich integral

$\int_0^T W(t)\,dW(t)$

In ordinary calculus we have for $f(0)=0$

$$\int_0^T f(t)\,df(t) = \int_0^T f(t)\,f'(t)\,dt = \frac{1}{2}\int_0^T \frac{d}{dt}(f^2(t))dt = \frac{1}{2}f^2(T)$$

For the Itô integral we will show that

$$I(T) = \int_0^T \Delta(t)\,dW(t) = \int_0^T W(t)\,dW(t) = \frac{1}{2}\,(W^2(T)-T)$$

Approximate $W(t)$ with a step function

$$\Delta_\Pi(t) = W_\Pi(t) = \begin{cases} W(0)=0 & \text{if } 0 \le t < \frac{T}{n}, \\ W(\frac{T}{n}) & \text{if } \frac{T}{n} \le t < \frac{2T}{n}, \\ W(\frac{2T}{n}) & \text{if } \frac{2T}{n} \le t < \frac{3T}{n}, \\ \vdots & \\ W(\frac{(n-1)T}{n}) & \text{if } \frac{(n-1)T}{n} \le t < T, \end{cases}$$

So,

$$\begin{aligned} \int_0^T W(t)dW(t) &= \lim_{n\to\infty}\int_0^T \Delta_\Pi(t)dW(t) \\ &= \lim_{n\to\infty}\sum_{j=0}^{n-1} W\left(\frac{jT}{n}\right)\left[W\left(\frac{(j+1)T}{n}\right) - W\left(\frac{jT}{n}\right)\right]. \end{aligned}$$

$\int_0^T W(t)\,dW(t)$

Letting $W_j = W\left(\frac{jT}{n}\right)$, consider the sum:

$$\begin{aligned} \frac{1}{2}\sum_{j=0}^{n-1}(W_{j+1}-W_j)^2 &= \frac{1}{2}\sum_{j=0}^{n-1} W_{j+1}^2 - \sum_{j=0}^{n-1} W_{j+1}\,W_j + \frac{1}{2}\sum_{j=0}^{n-1} W_j^2 \\ &= \frac{1}{2}\,W_n^2 + \frac{1}{2}\sum_{j=0}^{n-1} W_j^2 - \sum_{j=0}^{n-1} W_{j+1}\,W_j + \frac{1}{2}\sum_{j=0}^{n-1} W_j^2 \\ &= \frac{1}{2}\,W_n^2 + \sum_{j=0}^{n-1} W_j^2 - \sum_{j=0}^{n-1} W_{j+1}\,W_j \\ &= \frac{1}{2}\,W_n^2 - \sum_{j=0}^{n-1} W_j(W_{j+1}-W_j) \end{aligned}$$

$\int_0^T W(t)\,dW(t)$

Take the limit as $||\Pi|| \to 0$ gives:

$$\frac{1}{2}T = \frac{1}{2}\,W^2(T) - \int_0^T W(t)dW(t),$$

so,

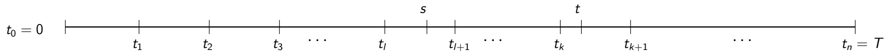
$$\int_0^T W(t)dW(t) = \frac{1}{2}\,W^2(T) - \frac{1}{2}T.$$

$I_\Pi(t)$ is a Martingale

In order to show $I_\Pi(t)$ is a Martingale, we need to show that for $0 \le s \le t \le T$,

$$\mathbb{E}\left[\, I_\Pi(t) \mid \mathcal{F}(s)\right] = I_\Pi(s)\,.$$

Set up a partition as follows



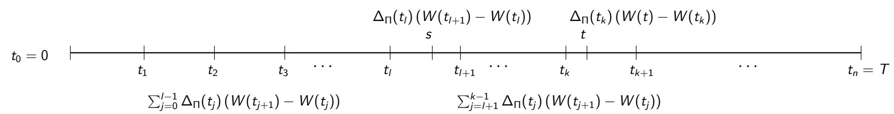
where we have $0 \le s < t \le T$, such that for $l < k$ (i.e., $t_l < t_k$), $s \in [t_l, t_{l+1})$ and $t \in [t_k, t_{k+1})$. As before, we have

$$I_\Pi(t) = \sum_{j=0}^{k-1} \Delta_\Pi(t_j)\left(W(t_{j+1}) - W(t_j)\right) + \Delta_\Pi(t_k)\left(W(t) - W(t_k)\right)$$

$I_\Pi(t)$ is a Martingale (ctd)

We can split $I_\Pi(t)$ up into four parts:

$$\begin{aligned} I_\Pi(t) &= \sum_{j=0}^{k-1} \Delta_\Pi(t_j)\left(W(t_{j+1}) - W(t_j)\right) + \Delta_\Pi(t_k)\left(W(t) - W(t_k)\right)\,, \\ &= \sum_{j=0}^{l-1} \Delta_\Pi(t_j)\left(W(t_{j+1}) - W(t_j)\right) + \Delta_\Pi(t_l)\left(W(t_{l+1}) - W(t_l)\right) \\ &\quad + \sum_{j=l+1}^{k-1} \Delta_\Pi(t_j)\left(W(t_{j+1}) - W(t_j)\right) + \Delta_\Pi(t_k)\left(W(t) - W(t_k)\right) \end{aligned}$$



$I_\Pi(t)$ is a Martingale (ctd)

So $\mathbb{E}\left[\, I_\Pi(t) \mid \mathcal{F}(s)\right]$ becomes

$$\begin{aligned} &\mathbb{E}\left[\sum_{j=0}^{l-1} \Delta_\Pi(t_j)\left(W(t_{j+1}) - W(t_j)\right) \,\middle|\, \mathcal{F}(s)\right] \\ &+ \mathbb{E}\left[\Delta_\Pi(t_l)\left(W(t_{l+1}) - W(t_l)\right) \mid \mathcal{F}(s)\right] \\ &+ \mathbb{E}\left[\sum_{j=l+1}^{k-1} \Delta_\Pi(t_j)\left(W(t_{j+1}) - W(t_j)\right) \,\middle|\, \mathcal{F}(s)\right] \\ &+ \mathbb{E}\left[\Delta_\Pi(t_k)\left(W(t) - W(t_k)\right) \mid \mathcal{F}(s)\right] \end{aligned}$$

By taking out what is known, this becomes:

$$\begin{aligned} &\sum_{j=0}^{l-1} \Delta_\Pi(t_j)\left(W(t_{j+1}) - W(t_j)\right) \\ &+ \Delta_\Pi(t_l)\left(\mathbb{E}\left[W(t_{l+1}) \mid \mathcal{F}(s)\right] - W(t_l)\right) \\ &+ \cdots \end{aligned}$$

Using the fact that $W(t)$ is a martingale ($\mathbb{E}\left[W(t) \mid \mathcal{F}(s)\right] = W(s)$) gives

$$\sum_{j=0}^{l-1} \Delta_\Pi(t_j)\left(W(t_{j+1}) - W(t_j)\right) + \Delta_\Pi(t_l)\left(W(s) - W(t_l)\right) + \cdots$$

$I_\Pi(t)$ is a Martingale (ctd)

So far we have

$$\begin{aligned} \mathbb{E}\left[\, I_\Pi(t) \mid \mathcal{F}(s)\right] = I_\Pi(s) & \\ &+ \mathbb{E}\left[\sum_{j=l+1}^{k-1} \Delta_\Pi(t_j)\left(W(t_{j+1}) - W(t_j)\right) \,\middle|\, \mathcal{F}(s)\right] \\ &+ \mathbb{E}\left[\Delta_\Pi(t_k)\left(W(t) - W(t_k)\right) \mid \mathcal{F}(s)\right] \end{aligned}$$

$I_\Pi(t)$ is a Martingale (ctd)

What is

$$\mathbb{E}\left[\sum_{j=l+1}^{k-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j))\,\middle|\, \mathcal{F}(s)\right]\ ?$$

Looking at terms individually

$$\begin{aligned}
\mathbb{E}\,[\Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j))\,|\,\mathcal{F}(s)] &= \mathbb{E}\,[\mathbb{E}\,[\Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j))\,|\,\mathcal{F}(t_j)]\,|\,\mathcal{F}(s)] \\
&= \mathbb{E}\,[\Delta_\Pi(t_j)\,(\mathbb{E}\,[W(t_{j+1})\,|\,\mathcal{F}(t_j)] - W(t_j))\,|\,\mathcal{F}(s)] \\
&= \mathbb{E}\,[\Delta_\Pi(t_j)\,(W(t_j) - W(t_j))\,|\,\mathcal{F}(s)] \\
&= 0\,,
\end{aligned}$$

where we used the iterated conditioning rule along with the fact that $s < t_j$.
So,

$$\mathbb{E}\left[\sum_{j=l+1}^{k-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j))\,\middle|\, \mathcal{F}(s)\right] = 0\,.$$

$I_\Pi(t)$ is a Martingale (ctd)

Now we have

$$\begin{aligned}
\mathbb{E}\,[I_\Pi(t)\,|\,\mathcal{F}(s)] = I_\Pi(s)& \\
+\,\mathbb{E}\,[\Delta_\Pi(t_k)\,(W(t) - W(t_k))\,|\,\mathcal{F}(s)]&
\end{aligned}$$

Using a similar iterated conditioning argument to the one used on the previous slide,

$$\mathbb{E}\,[\Delta_\Pi(t_k)\,(W(t) - W(t_k))\,|\,\mathcal{F}(s)] = 0$$

Therefore

$$\mathbb{E}\,[I_\Pi(t)\,|\,\mathcal{F}(s)] = I_\Pi(s)\,,$$

So $I_\Pi(t)$ is a Martingale.

Itô Isometry

Since $I_\Pi(t)$ is a Martingale,

$$\mathbb{E}(I_\Pi(t)) = I(0) = 0\,,$$

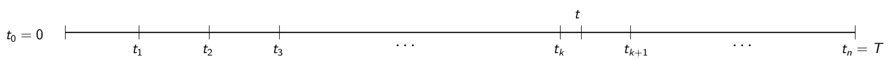
So

$$\mathrm{Var}(I_\Pi(t)) = \mathbb{E}(I_\Pi(t)^2)\,.$$

We will show that

$$\mathbb{E}((I_\Pi(t))^2) = \mathbb{E}\int_0^t (\Delta_\Pi(u))^2 du$$

Use a similar partition to the one used before,



So,

$$I_\Pi(t) = \sum_{j=0}^{k-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) + \Delta_\Pi(t_k)(W(t) - W(t_k))$$

Itô Isometry (ctd)

$$I_\Pi(t) = \sum_{j=0}^{k-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) + \Delta_\Pi(t_k)(W(t) - W(t_k))\,.$$

Let $\Delta W_j = (W(t_{j+1}) - W(t_j))\,, 0 \le j < k$, and let $\Delta W_k = (W(t) - W(t_k))$.
Then rewrite the Itô integral as,

$$I_\Pi(t) = \sum_{j=0}^{k} \Delta_\Pi(t_j)\Delta W_j\,.$$

So,

$$\begin{aligned}
\mathbb{E}((I_\Pi(t))^2) &= \mathbb{E}\left[\left(\sum_{j=0}^{k} \Delta_\Pi(t_j)\Delta W_j\right)\left(\sum_{i=0}^{k} \Delta_\Pi(t_i)\Delta W_i\right)\right]\,, \\
&= \mathbb{E}\left[\sum_{j=0}^{k} (\Delta_\Pi(t))^2 \Delta W_j^2 + 2 \sum_{0 \le i < j \le k} \Delta_\Pi(t_i)\Delta_\Pi(t_j)\Delta W_i \Delta W_j\right]\,,
\end{aligned}$$

Itô Isometry (ctd)

Taking the second term first

$$
\begin{aligned}
&\mathbb{E}\left[2\sum_{0\le i<j\le k}\Delta_\Pi(t_i)\Delta_\Pi(t_j)\Delta W_i\Delta W_j\right]\\
&=2\sum_{0\le i<j\le k}\mathbb{E}\left[\Delta_\Pi(t_i)\Delta_\Pi(t_j)\Delta W_i\Delta W_j\right]\\
&=2\sum_{0\le i<j\le k}\mathbb{E}[\Delta_\Pi(t_i)\Delta_\Pi(t_j)\Delta W_i]\underbrace{\mathbb{E}[\Delta W_j]}_{=0}\\
&=0\,.
\end{aligned}
$$

Because

- $\Delta_\Pi(t_i)\Delta_\Pi(t_j)\Delta W_i$ $\mathcal{F}(t_j)$-measurable
- ΔW_j independent of $\mathcal{F}(t_j)$

Itô Isometry (ctd)

So,

$$
\mathbb{E}((I_\Pi(t))^2)=\mathbb{E}\left[\sum_{j=0}^{k}(\Delta_\Pi(t))^2\Delta W_j^2\right]+0
$$

and

$$
\begin{aligned}
\mathbb{E}\left[\sum_{j=0}^{k}(\Delta_\Pi(t_j))^2\Delta W_j^2\right]&=\sum_{j=0}^{k}\mathbb{E}\left[(\Delta_\Pi(t_j))^2\Delta W_j^2\right]=\sum_{j=0}^{k}\mathbb{E}\left[(\Delta_\Pi(t_j))^2\right]\mathbb{E}\left[\Delta W_j^2\right]\\
&=\sum_{j=0}^{k-1}\mathbb{E}\left[(\Delta_\Pi(t_j))^2\right](t_{j+1}-t_j)+\mathbb{E}\left[(\Delta_\Pi(t_k))^2\right](t-t_j)\\
&=\mathbb{E}\left[\sum_{j=0}^{k-1}(\Delta_\Pi(t_j))^2(t_{j+1}-t_j)\right]+\mathbb{E}[(\Delta_\Pi(t_k))^2](t-t_j)\\
&=\mathbb{E}\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}(\Delta_\Pi(u))^2du+\mathbb{E}\int_{t_k}^{t}(\Delta_\Pi(u))^2du\\
&=\boxed{\mathbb{E}\int_0^t(\Delta_\Pi(u))^2du}
\end{aligned}
$$

Quadratic variation of Itô integral

Since the Itô integral is written as

$$
I(t)=\int_0^t G(u)dW(u)\,,
$$

In informal notation, this can be written as

$$
dI(t)=G(u)dW(t)
$$

Again informally, the quadratic variation is written,

$$
dIdI=G(t)dW(t)G(t)dW(t)=(G(t))^2dt
$$

So, the quadratic variation of the Itô integral is

$$
QV_I(t)=\int_0^t(G(u))^2du
$$

Summary: Properties of Itô Integral

For an Itô integral

$$
I(T)=\int_0^T G(t)dW(t)\,,
$$

- Expected Value:
$$
\mathbb{E}[I(T)]=0
$$
,
- Variance: (Itô Isometry):
$$
\mathrm{Var}[I(T)]=\int_0^T\mathbb{E}[G^2(t)]dt\,,
$$
- Quadratic variation:
$$
\mathrm{QV}_I(T)=\int_0^T[G^2(t)]dt\,,
$$
- Martingale: for $0\le s<t$,
$$
\mathbb{E}[I(t)|\mathcal{F}(s)]=I(s)
$$

Chain Rule

$f(W(t))$

For an expression of the form $f(W(t))$, if asked for

$$\frac{d}{dt} f(W(t)),$$

would normally write

$$\frac{d}{dt} f(W(t)) = \frac{df(W(t))}{dW} \frac{dW(t)}{dt}$$

or

$$df(W(t)) = \frac{df(W(t))}{dW} \frac{dW(t)}{dt} dt$$

or

$$df(W(t)) = \frac{df(W(t))}{dW} dW(t)$$

But $\frac{dW(t)}{dt}$ does not exist

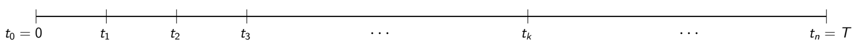
df(t,W(t))

For a function $f(t, W(t))$ of time and $W(t)$, we would write

$$df(t, W(t)) = \frac{\partial f(t, W(t))}{\partial t} dt + \frac{\partial f(t, W(t))}{\partial W(t)} dW(t),$$

But $\frac{dW(t)}{dt}$ does not exist

Partition of interval $[0, T]$



Taylor Series

Given a differentiable function $f(x)$ and two points x_j and x_{j+1}, then

$$f(x_{j+1}) = f(x_j) + f'(x_j)(x_{j+1} - x_j) + \frac{1}{2} f''(x_j)(x_{j+1} - x_j)^2 + \cdots,$$

where $x_{j+1} = x_j + (x_{j+1} - x_j)$.

For a function $f(t, x(t))$ and points $(t_j, x(t_j))$ and $(t_{j+1}, x(t_{j+1}))$

$$\begin{aligned} f(t_{j+1}, x(t_{j+1})) = f(t_j, x(t_j)) \\ &+ f_t(t_j, x(t_j))(t_{j+1} - t_j) + f_x(t_j, x(t_j))(x(t_{j+1}) - x(t_j)) \\ &+ \frac{1}{2} f_{tt}(t_j, x(t_j))(t_{j+1} - t_j)^2 \\ &+ f_{tx}(t_j, x(t_j))(t_{j+1} - t_j)(x(t_{j+1}) - x(t_j)) \\ &+ \frac{1}{2} f_{xx}(t_j, x(t_j))(x(t_{j+1}) - x(t_j))^2 \\ &+ \text{higher order terms} \cdots, \end{aligned}$$

where

$$f_t = \frac{\partial f(t,x)}{\partial t}, \qquad f_x = \frac{\partial f(t,x)}{\partial x},$$

$$f_{tt} = \frac{\partial^2 f(t,x)}{\partial t^2}, \qquad f_{tx} = \frac{\partial^2 f(t,x)}{\partial t \partial x},$$

$$f_{xx} = \frac{\partial^2 f(t,x)}{\partial x^2}.$$

Itô's Formula for $f(t, W(t))$

The function $f(x)$ is differentiable, so we can expand it as before

$$
\begin{aligned}
f(t_{j+1}, W(t_{j+1})) &= f(t_j, W(t_j)) \\
&+ f_t(t_j, W(t_j))(t_{j+1} - t_j) + f_x(t_j, W(t_j))(W(t_{j+1}) - W(t_j)) \\
&+ \frac{1}{2} f_{tt}(t_j, W(t_j))(t_{j+1} - t_j)^2 \\
&+ f_{tx}(t_j, W(t_j))(t_{j+1} - t_j)(W(t_{j+1}) - W(t_j)) \\
&+ \frac{1}{2} f_{xx}(t_j, W(t_j))(W(t_{j+1}) - W(t_j))^2 \\
&+ \text{higher order terms} \cdots ,
\end{aligned}
$$

where

$$
\begin{aligned}
f_t &= \frac{\partial f(t,x)}{\partial t}, & f_x &= \frac{\partial f(t,x)}{\partial W(t)}, \\
f_{tt} &= \frac{\partial^2 f(t,x)}{\partial t^2}, & f_{tx} &= \frac{\partial^2 f(t,x)}{\partial t \partial W(t)}, \\
f_{xx} &= \frac{\partial^2 f(t,x)}{\partial W^2(t)}.
\end{aligned}
$$

Itô's Formula for $f(t, W(t))$ (ctd)

Summing, we have

$$
\begin{aligned}
f(T, W(T)) - f(0, W(0)) &= \sum_{j=0}^{n-1} \left[f(t_{j+1}, W(t_{j+1})) - f(t_j, W(t_j)) \right] \\
&= \sum_{j=0}^{n-1} f_t(t_j, W(t_j))(t_{j+1} - t_j) \\
&+ \sum_{j=0}^{n-1} f_x(t_j, W(t_j))(W(t_{j+1}) - W(t_j)) \\
&+ \sum_{j=0}^{n-1} \frac{1}{2} f_{xx}(t_j, W(t_j))(W(t_{j+1}) - W(t_j))^2 \\
&+ \sum_{j=0}^{n-1} \frac{1}{2} f_{tt}(t_j, W(t_j))(t_{j+1} - t_j)^2 \\
&+ \sum_{j=0}^{n-1} f_{tx}(t_j, W(t_j))(t_{j+1} - t_j)(W(t_{j+1}) - W(t_j)) \\
&+ \text{higher order terms} \cdots .
\end{aligned}
$$

Itô's Formula for $f(t, W(t))$ (ctd)

$\lim_{||\Pi|| \to 0}$

In the limit as $||\Pi|| \to 0$, this becomes,

$$
\begin{aligned}
& f(T, W(T)) - f(0, W(0)) = \\
&= \int_0^T f_t(t, W(t))dt \leftarrow \left(\lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} f_t(t_j, W(t_j))(t_{j+1} - t_j) \right) \\
&+ \int_0^T f_x(t, W(t))dW(t) \leftarrow \left(\lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} f_x(t_j, W(t_j))(W(t_{j+1}) - W(t_j)) \right) \\
&+ \frac{1}{2} \int_0^T f_{xx}(t, W(t))dt \leftarrow \left(\lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} \frac{1}{2} f_{xx}(t_j, W(t_j))(W(t_{j+1}) - W(t_j))^2 \right) \\
&+ 0 \leftarrow \left(\lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} \frac{1}{2} f_{tt}(t_j, W(t_j))(t_{j+1} - t_j)^2 \right) \\
&+ 0 \leftarrow \left(\lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} f_{tx}(t_j, W(t_j))(W(t_{j+1}) - W(t_j))(t_{j+1} - t_j) \right),
\end{aligned}
$$

using arguments very like ones we have seen before.

Itô's Formula for $f(t, W(t))$

So we have,

$$
f(T, W(T)) - f(0, W(0)) = \int_0^T df(t, W(t)) = \int_0^T f_t(t, W(t))dt + \int_0^T f_W(t, W(t))dW(t) + \frac{1}{2} \int_0^T f_{WW}(t, W(t))dt
$$

In informal differential notation,

$$
\boxed{df(t, W(t)) = f_t dt + f_W dW(t) + \frac{1}{2} f_{WW} dt,}
$$

or

$$
\boxed{df(t, W(t)) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial W} dW(t) + \frac{1}{2} \frac{\partial^2 f}{\partial W^2} dt.}
$$

or, if you like,

$$
\boxed{df(t, W(t)) = \left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial W^2} \right) dt + \frac{\partial f}{\partial W} dW(t).}
$$

$\int_0^T WdW$ again

The integral can be quickly evaluated using Itô's formula
Let $f(x) = \frac{1}{2}x^2$. Then

$$\frac{\partial f(x)}{\partial x} = f_x(x) = x\,,$$
$$\frac{\partial^2 f(x)}{\partial x^2} = f_{xx}(x) = 1\,.$$

If we replace x by W, the Itô formula gives

$$df(W) = \underbrace{f_t}_{f_t=0} dt + f_W dW + \frac{1}{2} f_{WW} dt\,,$$
$$= WdW + \frac{1}{2}\cdot 1 \cdot dt\,,$$

So

$$\int_0^T df(W) = f(W(T)) - f(W(0)) = \frac{1}{2}(W(T))^2 + \underbrace{0}_{W(0)=0}$$
$$= \int_0^T WdW + \int_0^T \frac{1}{2}\cdot dt = \int_0^T WdW + \frac{1}{2}T\,,$$

Therefore

$$\int_0^T WdW = \frac{1}{2}W^2(T) - \frac{1}{2}T\,.$$

Product Rule

Let $X(t, W(t))$ and $Y(t, W(t))$, so that $X \cdot Y$ is a function of t and $W(t)$, too.

$$\begin{aligned} d[XY] &= \left(\frac{\partial XY}{\partial t} + \frac{1}{2}\frac{\partial^2 XY}{\partial W^2}\right) dt + \frac{\partial XY}{\partial W} dW \\ &= \ldots \\ &= X\,dY + Y\,dX + \frac{\partial X}{\partial W}\frac{\partial Y}{\partial W} dt \\ &= X\,dY + Y\,dX + dX\,dY \end{aligned}$$

Itô Process

An Itô process $X(t)$ is defined

$$X(t) = X(0) + \int_0^t A(t)\,dt + \int_0^t B(t)\,dW\,,$$

or, informally,

$$dX(t) = A(t)\,dt + B(t)\,dW\,.$$

Conditions are imposed on the functions $A(t)$ and $B(t)$

$$\begin{aligned} \mathbb{E}\int_0^t B^2(u)du &< \infty\,, \\ \int_0^t |A(u)|du &< \infty\,. \end{aligned}$$

Quadratic Variation for $X(t)$

informally

Using the rules we have already described, the quadratic variation can be obtained informally as follows

$$\begin{aligned} QV_X(t) &= dX(t)dX(t) \\ &= (A(t)\,dt + B(t)\,dW)^2 \\ &= A^2(t)dtdt + 2A(t)B(t)dWdt + B^2(t)dWdW \\ &= 0 + 0 + B^2(t)dWdW \\ &= B^2(t)dt\,. \end{aligned}$$

Integral with respect to Itô Process

We've seen Itô integrals with respect to Brownian Motion:

$$\int_0^t G(u)dW(u)\,.$$

We can also define an integral with respect to an Itô process by splitting up the $A(t)$ and $B(t)$ terms

$$\int_0^t G(u)dX(u) = \int_0^t G(u)A(u)du + \int_0^t G(u)B(u)dW(u)\,.$$

Itô's Formula for $f(t, X(t))$

instead of $f(t, W(t))$

Proceeding as before, we have (replace $W(t)$ by $X(t)$):

$$\begin{aligned} f(t_{j+1}, X(t_{j+1})) = f(t_j, X(t_{j+1})) & \\ &+ f_t(t_j, X(t_{j+1}))(t_{j+1} - t_j) + f_x(t_j, X(t_{j+1}))(X(t_{j+1}) - X(t_{j+1})) \\ &+ \frac{1}{2} f_{tt}(t_j, X(t_{j+1}))(t_{j+1} - t_j)^2 \\ &+ f_{tx}(t_j, X(t_{j+1}))(t_{j+1} - t_j)(X(t_{j+1}) - X(t_{j+1})) \\ &+ \frac{1}{2} f_{xx}(t_j, X(t_{j+1}))(X(t_{j+1}) - X(t_{j+1}))^2 \\ &+ \text{higher order terms} \cdots, \end{aligned}$$

where

$$f_t = \frac{\partial f(t,x)}{\partial t}, \qquad f_x = \frac{\partial f(t,x)}{\partial X(t)},$$

$$f_{tt} = \frac{\partial^2 f(t,x)}{\partial t^2}, \qquad f_{tx} = \frac{\partial^2 f(t,x)}{\partial t \partial X(t)},$$

$$f_{xx} = \frac{\partial^2 f(t,x)}{\partial X(t)^2}.$$

Itô's Formula for $f(t, X(t))$

So, for $f(t, X(t))$,

$$\boxed{df(t, X(t)) = f_t(t, X(t))dt + f_x(t, X(t))dX + \frac{1}{2} f_{xx}(t, X(t))dXdX\,,}$$

where

$$dX(t)dX(t) = B^2(t)dt\,.$$

Summary

Itô's Formula for $f(t, X(t))$

For $dX(t) = A(t)\,dt + B(t)\,dW$ and a function $f(t, X(t))$

$$\boxed{df(t, X) = f_t(t, X)dt + f_x(t, X)dX + \frac{1}{2} f_{xx}(t, X)dXdX\,,}$$

$$\boxed{df(t, X) = f_t dt + f_x dX + \frac{1}{2} f_{xx} B^2(t)dt\,,}$$

$$\boxed{df(t, X) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}dX + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}B^2(t)dt\,.}$$

$$\boxed{df(t, X) = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}B^2(t)\right)dt + \frac{\partial f}{\partial X}dX\,.}$$

Itô's Formula for $f(t, X(t))$

In terms of $W(t)$...

$$
\begin{aligned}
df(t,X) =& f_t(t,X(t))dt + f_x(t,X(t))dX + \frac{1}{2} f_{xx}(t,X(t))B^2(t)dt\,, \\
=& f_t(t,X(t))dt + f_x(t,X(t))(A(t)\,dt + B(t)\,dW) + \frac{1}{2} f_{xx}(t,X(t))B^2(t)dt\,, \\
=& f_t(t,X(t))dt + f_x(t,X(t))A(t)\,dt + f_x(t,X(t))B(t)\,dW \\
& + \frac{1}{2} f_{xx}(t,X(t))B^2(t)dt\,.
\end{aligned}
$$

Review of Course Topics

1. Brownian Motion
2. Integration
3. Itô Integral
4. Itô Formula