TRINITY COLLEGE DUBLIN, SCHOOL OF MATHEMATICS

Assignment #1, Module: MA5635

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1 PROB1

For a four step binomial tree, the value of a European call option at $t = 0$ is:

$$X_0 = \frac{1}{(1+r)^4} \sum_{i=0}^{4} \binom{4}{i} \widetilde{p}^i (1-\widetilde{p})^{4-i} \max\{S_0 u^i d^{4-i} - K, 0\}$$

where:

$$\widetilde{p} = \frac{1+r-d}{u-d} = 0.5897$$

and:

$$1 - \widetilde{p} = 0.4103$$

Also, for the amount of shares to buy, at $t = 0$ and $t = 1$, respectively:

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}$$

$$\Delta_1(\omega) = \frac{V_2(\omega H) - V_2(\omega T)}{S_2(\omega H) - S_2(\omega T)} \Rightarrow$$

$$\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)}$$

$$\Delta_1(T) = \frac{V_2(TH) - V_2(TT)}{S_2(TH) - S_2(TT)}$$

with:

$$V_2(HH) = \max\{S_0 u^2 - K, 0\}$$

$$V_2(TH) = \max\{S_0 ud - K, 0\}$$

$$V_2(HT) = \max\{S_0 du - K, 0\}$$

$$V_2(TT) = \max\{S_0 d^2 - K, 0\}$$

and substituting values:

$$S_0 u^2 = 10 \cdot 1.6^2 = 25.6 \Rightarrow V_2(HH) = 14.6$$

$$S_0 ud = 10 \cdot 1.6 \cdot 1.6^{-1} = 10 \Rightarrow V_2(HT) = V_2(TH) = 0$$

$$S_0 d^2 = 10 \cdot 1.6^{-2} \Rightarrow V_2(TT) = 0$$

which leads to:

$$\Delta_1(H) = \frac{14.6}{S_0(u^2 - ud)} = 0.9359$$

$$\Delta_1(T) = 0$$

On the other hand:

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_0(u - d)}$$

for which:

$$V_1(H) = \max\{S_0 u - K, 0\} = 5$$

$$V_1(T) = \max\{S_0 d - K, 0\} = 0$$

and then:

$$\Delta_0 = 0.5128$$

Finally, for the call option:

$$X_0 = \frac{1}{(1+0.2)^4}\left(0 + 0 + 0 + \binom{4}{3}\widetilde{p}^3(1-\widetilde{p})^1(S_0 u^2 - K) + \binom{4}{4}\widetilde{p}^4(S_0 u^4 - K)\right) = 5.55$$

2 PROB2

The put option is similar to call option, but when selling short, K becomes an upper limit:

$$X_0 = \frac{1}{(1+r)^4} \sum_{i=0}^{4} \binom{4}{i} \widetilde{p}^i (1-\widetilde{p})^{4-i} \max\{K - S_0 u^i d^{4-i}, 0\}$$

and then:

$$X_0 = \frac{1}{(1+0.2)^4} \left(\binom{4}{0} (1-\widetilde{p})^4 (K - S_0 d^4) + \binom{4}{1} \widetilde{p}(1-\widetilde{p})^3 (K - S_0 d^2) + \binom{4}{2} \widetilde{p}^2 (1-\widetilde{p})^2 (K - S_0) + 0 + 0 \right)$$

$$= 0.856$$

3 Prob3

3.1 $\widetilde{E}_1[S_2](H)$

As the obtained toss at $t = 1$ was H, then:

$$S_1(H) = uS_0 = 2 \cdot 4 = 8$$

$$\Rightarrow \widetilde{E}_1[S_2](H) = \widetilde{p}S_2(H) + \widetilde{q}S_2(T) = S_1(H)(\widetilde{p}u + \widetilde{q}d) = 8\left(\frac{2}{3}2 + \frac{1}{3}\frac{1}{2}\right) = 12$$

3.2 $\widetilde{E}_1[S_2](T)$

$$S_1(T) = dS_0 = 2^{-1} \cdot 4 = 2$$

$$\Rightarrow \widetilde{E}_1[S_2](T) = \widetilde{p}S_2(H) + \widetilde{q}S_2(T) = S_1(T)(\widetilde{p}u + \widetilde{q}d) = 2\left(\frac{2}{3}2 + \frac{1}{3}\frac{1}{2}\right) = 3$$

3.3 $\widetilde{E}_1[S_3](T)$

$$S_1(T) = dS_0 = 2^{-1} \cdot 4 = 2$$

$$\Rightarrow \widetilde{E}_1[S_3](T) = \left(8 \cdot \widetilde{p}^2 + 2(\cdot 2 \cdot \widetilde{p}\widetilde{q}) + \frac{1}{2}\widetilde{q}^2\right) = 4.5$$

4 PROB4

4.1

FIRST:

$$E_1[S_2+S_3](H) = (p^2(16+32)+pq(16+8)+pq(4+8)+q^2(4+2)) = 30$$

$$E_1[S_2](H) = 12$$

$$E_1[S_3](H) = 32\left(\frac{2}{3}\right)^2 + 2\cdot 8\cdot (2/3)\cdot(1/3) + 2\cdot(1/3)^2 = 18$$

$$\Rightarrow E_1[S_2+S_3](H) = E_1[S_2](H) + E_1[S_3](H)$$

SECOND:

$$E_1[S_2+S_3](T) = p^2(4+8)+pq(4+2)+pq(1+2)+q^2(1+0.5) = 7.5$$

$$E_1[S_2](T) = 3$$

$$E_1[S_3](T) = 4.5$$

$$\Rightarrow E_1[S_2+S_3](T) = E_1[S_2](T) + E_1[S_3](T)$$

4.2

FIRST PART:

$$E_1[S_1S_2](H) = p(8\cdot 16) + q(8\cdot 4) = 94$$

$$E_1[S_2](H) = 12$$

$$S_1(H) = 8$$

$$E_1[S_2](H) \cdot S_1(H) = 96 = E_1[S_1 S_2](H)$$

SECOND PART:

$$E_1[S_1 S_2](T) = p(2 \cdot 4) + q(2 \cdot 1) = 6$$

$$E_1[S_2](T) = 3$$

$$S_1(T) = 2$$

$$E_1[S_2](T) \cdot S_1(T) = 6 = E_1[S_1 S_2](T)$$

4.3

First:

$$E_2[S_3](HH) = 32p + 8q = 24$$

$$E_2[S_3](HT) = 8p + 2q = 6$$

$$E_2[S_3](TH) = E_2[S_3](HT) = 6$$

$$E_2[S_3](TT) = 2p + 0.5q = 1.5$$

then:

$$E_1[E_2[S_3]](H) = pE_2[S_3](HH) + qE_2[S_3](HT) = 18$$

$$E_1[E_2[S_3]](H) = pE_2[S_3](TH) + qE_2[S_3](TT) = 4.5$$

finally:

$$E_1 S_3(H) = 18 = E_1[E_2[S_3]](H)$$

$$E_1 S_3(T) = 4.5 = E_1[E_2[S_3]](T)$$

5 PROB5

5.1

$$E_n[M_{n+1}] = E_n[S_{n+1}] = pS_{n+1}(H) + qS_{n+1}(T) = S_n(pu+qd) = S_n((1/3)\cdot 2 + (2/3)\cdot(1/2)) = S_n$$

Therefore, as $M_n = E_n[M_{n+1}]$, M_n is a martingale in this case.

5.2

$$E_n[M_{n+1}] = \frac{1}{(1+r)^{n+1}} E_n[S_{n+1}] = \frac{1}{(1+r)^{n+1}}(pS_{n+1}(H) + qS_{n+1}(T)) =$$

$$\frac{1}{(1+r)^n}\frac{1}{1+r}(S_n(pu+qd)) = M_n\frac{pu+qd}{1+r} = M_n\frac{(1/2)(2)+(1/2)(1/2)}{1+1/4} = M_n$$

Therefore, as $M_n = E_n[M_{n+1}]$, M_n is a martingale in this case.

5.3

$$E_n[M_{n+1}] = E_n[S_{n+1}] = pS_{n+1}(H) + qS_{n+1}(T) = S_n(pu+qd) = S_n((1/2)\cdot 2 + (1/2)\cdot(1/2)) = \frac{5}{4}S_n$$

Therefore, as $M_n \neq E_n[M_{n+1}]$, M_n is not a martingale in this case.

5.4

FIRST:

$$E_n[M_{n+1}] = E_n[S_{n+1}] = pS_{n+1}(H) + qS_{n+1}(T) = S_n(pu+qd) = S_n((2/3)\cdot 2 + (1/3)\cdot(1/2)) = \frac{3}{2}S_n$$

Therefore, as $M_n \neq E_n[M_{n+1}]$, M_n is not a martingale in this case.

SECOND:

$$E_n[M_{n+1}] = \frac{1}{(1+r)^{n+1}} E_n[S_{n+1}] = \frac{1}{(1+r)^{n+1}}(pS_{n+1}(H) + qS_{n+1}(T)) =$$

$$\frac{1}{(1+r)^n}\frac{1}{1+r}(S_n(pu+qd)) = M_n\frac{pu+qd}{1+r} = M_n\frac{(2/3)(2)+(1/3)(1/2)}{1+1/4} = 1.2\cdot M_n$$

Therefore, as $M_n \neq E_n[M_{n+1}]$, M_n is not a martingale in this case.

5.5

In this case, a risk-neutral measure pair (p, q) is one such that:

$$\frac{pu+qd}{1+r} = 1 \Rightarrow pu+qd = \frac{5}{4} \Rightarrow p = \frac{1}{2}\left(\frac{5}{4} - \frac{1}{2}q\right)$$

and in all those previous given cases, only $\left(\frac{1}{2}, \frac{1}{2}\right)$ was a risk-neutral measure.

6 Prob6

The risk-neutral probabilities given are:

$$\widetilde{p} = \frac{1+r-d}{u-d} = \frac{1+1/4-1/2}{2-1/2} = \frac{1}{2}$$

$$\widetilde{p} = \frac{u-(1+r)}{u-d} = \frac{2-(1+1/4)}{2-1/2} = \frac{1}{2}$$

Therefore, with:

$$M_n = \frac{S_n}{(1+r)^n}$$

then:

$$\widetilde{E}_n[M_{n+1}] = \frac{1}{(1+r)^n}\frac{1}{1+r}S_n(\widetilde{p}u+\widetilde{q}d) = \frac{S_n}{(1+r)^n}\frac{\widetilde{p}u+\widetilde{q}d}{1+r} = \frac{S_n}{(1+r)^n}\frac{(1/2)2+(1/2)(1/2)}{1+r} = \frac{S_n}{(1+r)^n} = M_n$$

Therefore, as $M_n = E_n[M_{n+1}]$, M_n is a martingale under those risk-neutral probabilities.

7 PROB7

Implemented using Python.
The code for part **a** can be found at ./prob7/program.py (IMPORTANT: the code for ALL parts of this assignment is in that file).
The plots resulting of the implementations for part **b** are in ./prob7/part_B/.
The plot for part **c** is in ./prob7/part_C/.

7.1 PART D

For part **d**, the resulting plot can be found in ./prob7/part_D/.
The distribution towards which these final values arrays should approximate, when plotted as a histogram, is a normal distribution with mean 0 and standard deviation t. As can be seen from the plots (both located at ./prob7/part_D/), the histogram resembles the distribution just mentioned (for both values of t used).