Trinity Centre for High Performance Computing

MSc in HPC course 5635b

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An Introduction to Mathematical Finance (5635b)

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Course Outline

Introduction

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Course Details

- I will be covering (primarily) Stochastic Calculus over 3-4 weeks.
- Donal Gallagher and then Roland Lichters will finish off the course.
- For my part of the course I will hand out one or more exercise sheets which will form part of the assessment for the course
- Rough solutions will be posted after the due date for the exercises

References / Acknowledgements

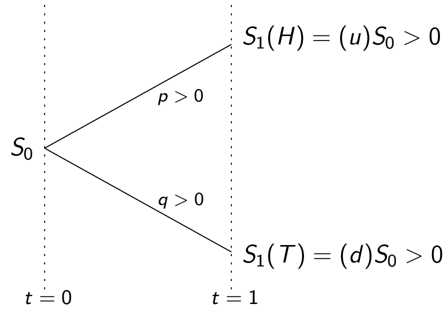
Much (not all) of the material in these slides was taken or adapted from

- Steven E. Shreve.
  Stochastic Calculus for Finance I: The Binomial Asset Pricing Model (Springer Finance).
  Springer, 1 edition, June 2005.
- Steven E. Shreve.
  Stochastic Calculus for Finance II: Continuous-Time Models (Springer Finance).
  Springer, 1st ed. 2004. corr. 2nd printing edition, June 2004.

These authors are not responsible for any errors in this course

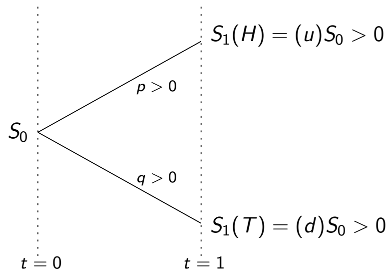
One Step Binomial Model (OSBM)

- Consider the value of one share of a stock at just two times, $t = 0$ and $t = 1$
- Has value S_0 at $t = 0$
- At $t = 1$ takes one of *only two*, positive, possible values:
  - $S_1(H)$ with probability $p > 0$
  - $S_1(T)$ with probability $q > 0$
- may be viewed as a weighted coin toss resulting in heads (H), or tails (T)
- We take $S_1(H) = uS_0$ and $S_1(T) = dS_0$, where $0 < d < u$
- if $S_1(H) = S_1(T)$ there is no uncertainty at $t = 1$



Expected Value of S_1

$$\mathbb{E}[S_1] = pS_1(H) + qS_1(T)\,.$$



Assumptions

- No taxes, transaction costs, margin costs
- stocks can be shorted at no additional cost
- shares or fractions of shares may be purchased without affecting price of share

Interest Rate r
Money Market

- In addition to the existence of the stock, there is also a *money market*
- Money may be invested or borrowed from the money market
- An amount X invested at time $t = 0$ yields $(1 + r)X$ at time $t = 1$
- An amount X borrowed at $t = 0$ results in a debt of $(1 + r)X$ at $t = 1$
- The rate r is usually assumed to be greater than 0, but is required only to be greater than -1

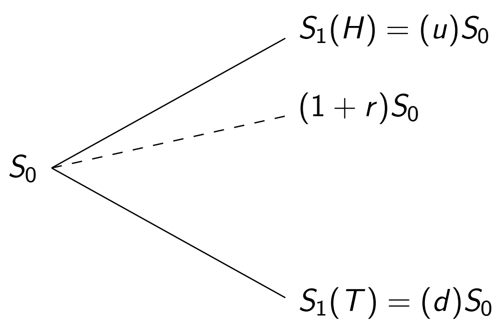
Arbitrage

Definition (Arbitrage)

Arbitrage may be defined as a trading strategy which begins with no money which has a zero probability of losing money and a positive probability of making money at some later time

- Steven E. Shreve.
Stochastic Calculus for Finance I: The Binomial Asset Pricing Model (Springer Finance).
Springer, 1 edition, June 2005.

No Arbitrage Condition for Binomial Model
$d < 1 + r < u$



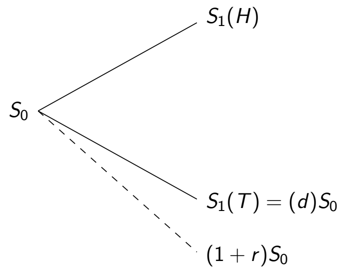
Suppose $d \geq 1 + r$

$t = 0$

- Borrow S_0 from money market
- Buy stock for S_0

$t = 1$

- Owe $(1 + r)S_0$
- Price of stock either $S_1(H)$ or $S_1(T)$
- But $S_1(H), S_1(T) \geq (1 + r)S_0$
- Since $u > d \geq 1 + r$, there is a positive probability of profit
- So there is an arbitrage



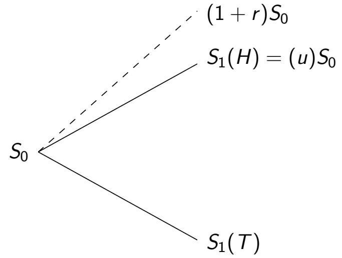
Suppose $u \leq 1 + r$

$t = 0$

- Short the stock for S_0
- Invest cash S_0 in money market

$t = 1$

- Receive $(1 + r)S_0$
- Price of stock either $S_1(H)$ or $S_1(T)$
- But $S_1(H), S_1(T) \leq (1 + r)S_0$
- Since $1 + r \geq u > d$, there is a positive probability of profit
- So there is an arbitrage



Derivatives in OSBM

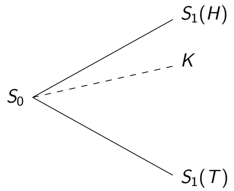
- Consider contracts which have payoffs at $t = 1$ which are contingent on the value of the stock
- So their value *derives* from the value of the *underlying* stock
- Examples
  - European call option
  - European put option
  - Forward contract

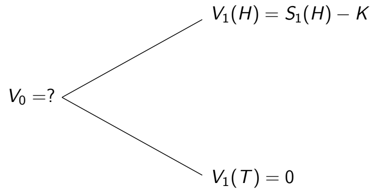
European Call Option in OSBM

Definition (European Call Option)

A contract entered into at time $t = 0$ which gives the holder the right but not the obligation to purchase the stock at time $t = 1$ for *strike price* K.

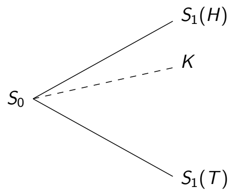
- Value at $t = 1$ is $S_1 = S_1(H)$ or $S_1(T)$
- Assume that $S_1(T) < K < S_1(H)$
- Value at time $t = 1$ known: $(S_1 - K)^+$
- What, if anything, is a fair value at $t = 0$?

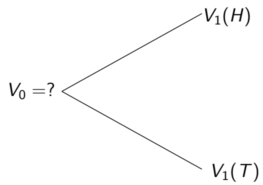




General Derivative in OSBM

- Value at $t = 0$ is V_0
- Value at $t = 1$ is $V_1 = V_1(H)$ or $V_1(T)$
- Payoff at time $t = 1$ is known in terms of S_1 – uncertainty occurs due to uncertainty as to which value S_1 will take
- What is V_0?





General Derivative in OSBM
Replicate the Option

- Create a “portfolio” of the stock and money market investment at $t = 0$
- Tune the relative amounts of stock and money market investment such that at $t = 1$ the portfolio takes the value of the derivative *no matter which value the stock takes* $t = 1$

Replicate the Option (ctd)

$t = 0$

- Start with cash X_0 at $t = 0$
- Purchase Δ_0 shares of the stock
- The cash position[a] is $(X_0 - \Delta_0 S_0)$

[a] If this is positive then $(1 + r)(X_0 - \Delta_0 S_0)$ will be obtained at $t = 1$. If it is negative, then the same amount will be owed at $t = 1$

$t = 1$

- The cash position is $\Delta_0 S_1 + (1 + r)(X_0 - \Delta_0 S_0)$
- As usual $S_1 = S_1(H)$ or $S_1(T)$, so

$$\Delta_0 S_1(H) + (1 + r)(X_0 - \Delta_0 S_0)\,,$$
$$\Delta_0 S_1(T) + (1 + r)(X_0 - \Delta_0 S_0)$$

...

Replicate the Option (ctd)

$t = 1$

So, in order that the portfolio replicates the value of the derivative at $t = 1$, set

$$\Delta_0 S_1(H) + (1 + r)(X_0 - \Delta_0 S_0) = V_1(H)\,,$$
$$\Delta_0 S_1(T) + (1 + r)(X_0 - \Delta_0 S_0) = V_1(T)\,;$$

two equations in two unknowns $(X_0\,, \Delta_0)$. Solving gives

$$X_0 = \frac{1}{1 + r}\left[\tilde{p} V_1(H) + \tilde{q} V_1(T)\right]\,,$$

and

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}\,.$$

Replicate the Option (ctd)
$\tilde{q}$ and $\tilde{p}$

where

$$\tilde{p} = \frac{1 + r - d}{u - d}\,,$$
$$\tilde{q} = \frac{u - (1 + r)}{u - d}\,,$$

$$\tilde{p} + \tilde{q} = 1\,.$$

and

$$\tilde{p}, \tilde{q} > 0 \quad \text{by assumption of no arbitrage – check this}$$

Example: European Call Option

Example

Suppose $S_0 = 4$, and $u = \frac{1}{d} = 2$. Also suppose $r = \frac{1}{4}$. Then $S_1(H) = 8$ and $S_1(T) = 2$.
Then

$$\frac{1}{1+r} = \frac{1}{1+\frac{1}{4}} = \frac{4}{5},$$

so

$$\begin{aligned} \tilde{p} &= \frac{1+r-d}{u-d} = \frac{1}{2}, \\ \tilde{q} &= \frac{u-(1+r)}{u-d} = \frac{1}{2}. \end{aligned}$$

Consider a European call option expiring at $t = 1$ with strike price $K = 5$. At $t = 1$, possible payoffs are:

$$\begin{aligned} V_1(H) &= (S_1(H) - K)^+ = (8-5)^+ = 3, \\ V_1(T) &= (S_1(T) - K)^+ = (2-5)^+ = 0. \end{aligned}$$

Example (ctd)

Example

So

$$\begin{aligned} X_0 &= \frac{1}{1+r}\left[\tilde{p}V_1(H) + \tilde{q}V_1(T)\right], \\ &= \frac{4}{5}\left[\frac{1}{2}\cdot 3 + \frac{1}{2}\cdot 0\right], \\ &= \frac{6}{5} = 1.20. \end{aligned}$$

Also,

$$\begin{aligned} \Delta_o &= \frac{V_1(H) - V_1(T)}{S_0(u-d)}, \\ &= \frac{3-0}{4\cdot\frac{3}{2}}, \\ &= \frac{3}{6} = \frac{1}{2}. \end{aligned}$$

Example (ctd)

Replication Process for writer of call option

$t = 0$

- Start with $X_0 = 1.2$
- purchase $\Delta_0 = \frac{1}{2}$ units of underlying asset for $\frac{1}{2} \cdot 4 = 2$ euro
- In order to do this, must borrow $2 - 1.2 = 0.8$ euro
- value of portfolio at $t = 0$ is

$$X_0 - \Delta_0 S_0 = 1.2 - 2 = -0.8$$

$t = 1$

- Owe $(1+r)\cdot 0.8 = \frac{5}{4}\cdot 0.8 = 1$ euro
- Value of portfolio

$$\begin{aligned} &\text{H: } \tfrac{1}{2}\cdot 8 + \tfrac{5}{4}(1.2 - \tfrac{1}{2}\cdot 4) = 3, \\ &\text{T: } \tfrac{1}{2}\cdot 2 + \tfrac{5}{4}(1.2 - \tfrac{1}{2}\cdot 4) = 0, \end{aligned}$$

- Payoff of derivative contract replicated by portfolio at $t = 1$ whether a head or a tail is tossed

No-Arbitrage Price?

- The "price" $X_0 = 1.2$ is the starting capital required by the seller to create a portfolio to hedge the payoff of the call option regardless of whether a head or a tail is tossed at $t = 1$
- It is also a no-arbitrage price:
  - Suppose the option seller could sell for a greater price, e.g., $C = 1.21$.
  - Then the seller could take the 0.01 cents and invest in a separate money market account at $t = 0$
  - Then the seller could use to remaining 1.20 to hedge the option as before
  - At $t = 1$ the portfolio created starting from $X_0 = 1.2$ would cover the option payoff no matter what
  - And there would be an additional $(1+r)(0.01)$ for the seller
  - So, positive probability of gain, and no possibility of loss – arbitrage

No-Arbitrage Price? (ctd)

Suppose the seller was selling for 1.19 euro

$t = 0$

- Then the *buyer* could buy for 1.19
- Reverse the portfolio strategy of the seller:
  - Sell short $\Delta_0 = \frac{1}{2}$ of the stock ($\frac{1}{2} \cdot 4 = 2$)
  - Use 1.19 of the 2 euro to buy the option
  - Invest 0.8 euro in one money market account
  - Invest remaining 0.01 euro in a separate money market account

No-Arbitrage Price? (ctd)

$t = 1$

- Must purchase $\Delta_0 = \frac{1}{2}$ stock on open market after selling short at $t = 0$
- If value of stock is $S_1(H) = 8$, then $\frac{1}{2}S_1 = 4$
  - receive $(1 + r) \cdot 0.8 = \frac{5}{4} \cdot 0.8 = 1$ euro from investment
  - Use call option to purchase asset for 5 euro
  - Sell half asset for 4 euro (so, cost is 1 euro).
  - But this matches the gain from the investment of 0.8 euro at $t = 0$
  - so break even
- If value of stock is $S_1(T) = 2$, then $\frac{1}{2}S_1 = 1$
  - Receive one euro from investment of 0.8 at $t = 0$
  - Call option worthless
  - But $\frac{1}{2} \cdot S_1(T) = 1$, which is the amount available
  - so break even

Separate from all of the above at $t = 1$ the investment of 0.01 returns $(1 + r)(0.01)$; implies arbitrage
So in this case $X_0 = 1.2$ is a no arbitrage price

Two Step Binomial

We assume now that the stock evolves in two steps from today to option expiry, taking three possible values at expiry, $S_2(HH) = S_0 u^2$, $S_2(HT) = S_2(TH) = S_0 ud$ and $S_2(TT) = S_0 d^2$, and two possible values after step one, $S_1(H) = S_0 u$ and $S_1(T) = S_0 d$. Following the arguments of the single step case, we have therefore the two possible option values after step one,

$$\begin{aligned} V_1(H) &= \frac{1}{1+r}\{\tilde{p}\, V_2(HH) + (1-\tilde{p})\, V_2(HT)\} \\ V_1(T) &= \frac{1}{1+r}\{\tilde{p}\, V_2(TH) + (1-\tilde{p})\, V_2(TT)\} \end{aligned}$$

where $V_2(HH)$, $V_2(TH) = V_2(HT)$ and $V_2(TT)$ at expiry are known and determined by the underlying stock values at expiry.
Now, given $V_1(H)$ and $V_1(T)$ we can determine the current option value using

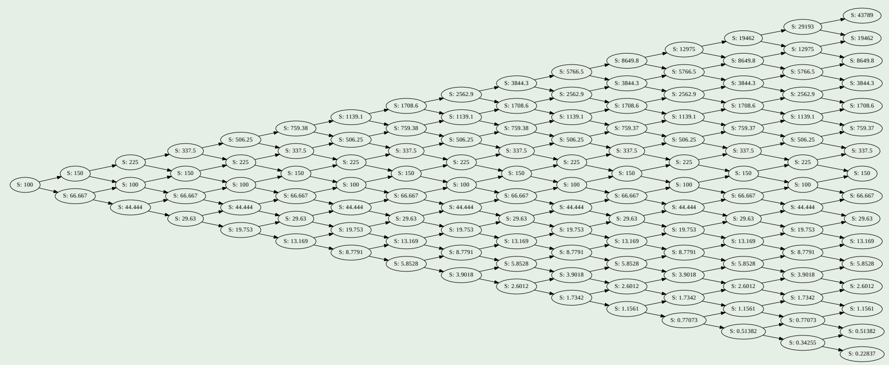
$$X_0 = \frac{1}{(1+r)^2}\left\{\tilde{p}^2\, V_2(HH) + 2\tilde{p}(1-\tilde{p})\, V_2(HT) + (1-\tilde{p})^2 V_2(TT)\right\},$$

where, for a European call option, $V_2(HH) = \max(S_0 u^2 - K, 0)$ etc.

15 step tree

You can keep going...

Example (15 steps; $S_0 = 100$, $u = \frac{1}{d} = 1.5$)



Many Step Binomial for European Call Option

We can now generalize the previous section's result for a European Call option to n binomial steps, where the stock evolves over n steps from today to option expiry, forming a recombining binomial tree so that the stock assumes values between $S_0 u^n$ and $S_0 d^n$. The call (put) option value C (P) is then given by:

$$
\begin{aligned}
X_0 = C &= \frac{1}{(1+r)^n} \sum_{i=0}^{n} \binom{n}{i} \tilde{p}^i (1-\tilde{p})^{n-i} \max\left\{S_0 u^i d^{n-i} - K, 0\right\} , \\
P &= \frac{1}{(1+r)^n} \sum_{i=0}^{n} \binom{n}{i} \tilde{p}^i (1-\tilde{p})^{n-i} \max\left\{K - S_0 u^i d^{n-i}, 0\right\} .
\end{aligned}
$$

The expected value of S_0 after n steps is

$$
\begin{aligned}
\tilde{\mu}_n &= S_0 \sum_{i=0}^{n} \binom{n}{i} \tilde{p}^i (1-\tilde{p})^{n-i} u^i d^{n-i} \\
&= S_0 \left(\tilde{p}\, u + (1-\tilde{p})\, d\right)^n \\
&= S_0 \,(1+r)^n
\end{aligned}
$$

Replication in Multi step Binomomial model

Consider an N-step binomial model with

$$\tilde{p} = \frac{1+r-d}{u-d} , \qquad \tilde{q} = \frac{u-(1+r)}{u-d} .$$

Let V_N represent the payoff of a derivative contract at step N. V_N is dependent on the first N coin tosses, $\omega_1\omega_2\cdots\omega_N$. Define random variables $V_{N-1},\ V_{N-2},\ \cdots\ ,\ V_0$ recursivly:

$$V_n(\omega_1\omega_2\cdots\omega_n) = \frac{1}{1+r}\left[\tilde{p} V_{n+1}(\omega_1\omega_2\cdots\omega_n H) + \tilde{q} V_{n+1}(\omega_1\omega_2\cdots\omega_n T)\right] ,$$

where $n = N-1, N-2, \cdots, 0$. Each V_n depends on the first n coin tosses.

- Steven E. Shreve.
  Stochastic Calculus for Finance I: The Binomial Asset Pricing Model (Springer Finance).
  Springer, 1 edition, June 2005.

Replication in Multi step Binomomial model

- Henceforth we'll write $V_{n+1}(\omega_1\omega_2\cdots\omega_n H) = V_{n+1}(H)$ and similarly for $V_{n+1}(T)$.
- The same style of notation will be used for the asset prices: $S_{n+1}(H)\,, S_{n+1}(T)$.
- Define
  $$\Delta_n = \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)} ,$$
  where $n = 0, 1, \cdots, N-1$.
- Finally define a portfolio value process $X_0, X_1, \cdots, X_N$ by
  $$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n) .$$
  If $X_0 = V_0$, then
  $$X_n(\omega_1\omega_2\cdots\omega_n) = V_n(\omega_1\omega_2\cdots\omega_n) ,$$
  for $0 \le n \le N$.

Replication in Multi step Binomomial model

Proof by induction. Let $X_0 = V_0$. Assume $X_n = V_n$. What is situation for $n+1$?
H : If a head is tossed for step $n+1$, then

$$X_{n+1}(H) = \Delta_n u S_n + (1+r)(X_n - \Delta_n S_n) .$$

Substiting for Δ_n gives

$$
\begin{aligned}
X_{n+1}(H) =& \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)} u S_n \\
&+ (1+r)\left(X_n - \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)} S_n\right) .
\end{aligned}
$$

$S_{n+1}(H) - S_{n+1}(T) = uS_n - dS_n = (u-d)S_n$, so

$$
\begin{aligned}
X_{n+1}(H) =& \frac{V_{n+1}(H) - V_{n+1}(T)}{u-d} u \\
&+ (1+r)\left(X_n - \frac{V_{n+1}(H) - V_{n+1}(T)}{u-d}\right) .
\end{aligned}
$$

By induction, $X_n = V_n$, and by the formula some slides above

$$V_n = \frac{1}{1+r}[\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)]\,.$$

So

$$\begin{aligned}
X_{n+1}(H) =& \frac{V_{n+1}(H) - V_{n+1}(T)}{u-d}u \\
& \quad + (1+r)V_n - (1+r)\frac{V_{n+1}(H) - V_{n+1}(T)}{u-d}\,, \\
=& \frac{V_{n+1}(H) - V_{n+1}(T)}{u-d}u + \underbrace{\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)}_{\text{substituted for } (1+r)V_n} \\
& \quad - (1+r)\frac{V_{n+1}(H) - V_{n+1}(T)}{u-d}\,, \\
=& \frac{1}{u-d}V_{n+1}(H)\,[u+1+r-d-1-r] \\
& \quad + \frac{1}{u-d}V_{n+1}(T)\,[-u+u-1-r+1+r]\,, \\
=& V_{n+1}(H)\,.
\end{aligned}$$

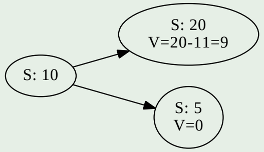
Similarly, in the case that a tail is tossed at step $n+1$, $X_{n+1}(T) = V_{n+1}(T)$.

Hedging a European call option

Mechanics of Hedging an Option

Example (Hedging a European call option)

- Suppose a European call option is sold at $t = 0$
- The value of the underlying asset at $t = 0$ is $S_0 = 10$
- The parameters of the tree are: $u = \frac{1}{d} = 2$
- The risk free interest rate between $t = 0$ and $t = 1$ is $r = 10\%$
- The strike price of the option is $K = 11$



The question is: If you are the seller of this European call option, how do you hedge your position so that you will suffer no loss regardless of the outcome at time $t = 1$

Hedging a European call option

Example (Hedging a European call option)

From the previous slide

$$\tilde{p} = \frac{1+r-d}{u-d} = \frac{1+0.1-\frac{1}{2}}{2-\frac{1}{2}} = 0.4$$

$$\tilde{q} = \frac{u-1-r}{u-d} = \frac{2-1-0.1}{2-\frac{1}{2}} = 0.6$$

Hedging a European call option

Example (Hedging a European call option)

Use the formula already derived to value V at $t = 0$:

$$\begin{aligned}
V = X_0 &= \frac{1}{1+r}\left[\tilde{p}V_1(H) + \tilde{q}V_1(T)\right] \\
&= \frac{1}{1.1}\Big[(0.4)(9) + (0.6)(0)\Big] \\
&= 3.2727
\end{aligned}$$

The call option is sold at $t = 0$ for this amount

Hedging a European call option

Example (Hedging a European call option)

- Having sold the call option, the seller is now exposed to variations in the price of the underlying asset in the next time step
- The derivation of the formula for the value of the call option indicates how the seller can hedge the option

Hedging a European call option

Example (Hedging a European call option)

At $t = 0$

- Sell option for $V = 3.2727$
- Calculate Δ based on the formula

$$\Delta = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{9 - 0}{20 - 5} = 0.6$$

- purchase 0.6 shares of underlying asset at 10 euros per share
- This requires $10 * 0.6 = 6$ euros. However have only charged 3.2727 euros for call option, so must borrow $6 - 3.2727 = 2.7273$ euros at $r = 0.1$
- Thus at $t = 0$
  - have portfolio of 0.6 units of underlying
  - have sold 2.7273 of bonds maturing at $t = 1$

Hedging a European call option

$t = 1$; $S_1 = 20$

Example (Hedging a European call option)

At $t = 1$:

If $S_1 = 20$

- The strike price of the option is 11 which is less than the current value, so the option will be exercised
- Therefore the option seller is obliged to sell 1 unit of underlying for price $K = 11$
- The value of the existing holding in underlying asset is now $(0.6)(20) = 12$
- Owe $2.7273 * 1.1 = 3.0$ euro
- Take payment from option holder of 11 euro
- Must purchase 0.4 units of underlying in order to have 1 unit of asset to deliver to option holder. This has cost $(0.4)(20) = 8$ euro
- Purchase 0.4 units; deliver 1 unit to holder. This leaves $11 - 8 = 3$ euro remaining, which serves to pay off the loan amount

Hedging a European call option

$t = 1$; $S_1 = 5$

Example (Hedging a European call option)

At $t = 1$:

If $S_1 = 5$

- The strike price of the option is 11 which is more than the current value of the underlying, so the option will not be exercised
- The value of the existing holding in underlying asset is now $(0.6)(5) = 3$ euro
- Owe $2.7273 * 1.1 = 3.0$ euro
- Sell the holding of the underlying and pay off the loan

Conditional Expected Values on a k-step Tree

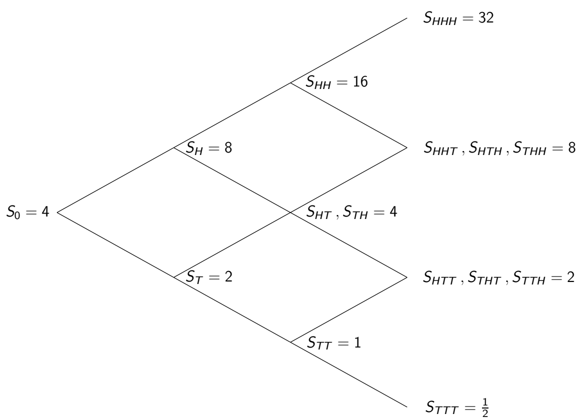
A k-Step Binomial Model

$k > n$

- In a k-step model we could have n coin tosses $\omega_1, \cdots \omega_n$ where $k > n$ and, $w_i = H$ or T
- At step n we will have $S_n(\omega_1 \cdots \omega_n)$, where the value of S_n will depend on the first n coin tosses
- And at step $n+1$ we will have $S_{n+1}(\omega_1 \cdots \omega_n H)$ (with probability p) **or** $S_{n+1}(\omega_1 \cdots \omega_n T)$ (with probability q)
- where

$$S_{n+1}(\omega_1 \cdots \omega_n H) = uS_n\,,$$
$$S_{n+1}(\omega_1 \cdots \omega_n T) = dS_n\,,$$

3 step binomial model



Conditional Expectation of S_{n+1}

Between step n and $n+1$

We call

$$\mathbb{E}_n[S_{n+1}] = [pS_{n+1}(H) + qS_{n+1}(T)]\,,$$

the conditional expectation of S_{n+1} based on the information available at step (time) n.

Since r is constant here, we also have

$$\mathbb{E}_n\left[\frac{S_{n+1}}{1+r}\right] = \frac{1}{1+r}[pS_{n+1}(H) + qS_{n+1}(T)]\,.$$

Examples
For $p = q = \frac{1}{2}$

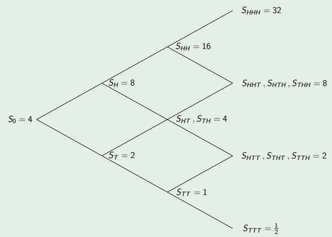
Example

Expected value of S_2 given that S_1 is a head

$$\mathbb{E}_1[S_2](H) = \frac{1}{2} \cdot 16 + \frac{1}{2} \cdot 4 = 10\,,$$

Expected value of S_2 given that S_1 is a tail

$$\mathbb{E}_1[S_2](T) = \frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 1 = \frac{5}{2}\,,$$



$\mathbb{E}_n[X](\omega_1 \cdots \omega_n)$

So far we've defined a conditional expectation between two consecutive steps. It can also be defined for steps separated by a larger gap. Let $0 \leq n \leq N$. Let X be a random variable depending on the first N coin tosses.

Definition ($\mathbb{E}_n(X)(\omega_1 \cdots \omega_n)$)

Let $0 \leq n \leq N$ and $\omega_1 \cdots \omega_n$ be a given sequence of coin tosses (assume we are at step n). There are 2^{N-n} possible sequences of coin tosses $\omega_{n+1} \cdots \omega_N$ between step n and step N. Let $\#H(\omega_{n+1} \cdots \omega_N)$ denote the number of heads in the sequences $\omega_{n+1} \cdots \omega_N$ and let $\#T(\omega_{n+1} \cdots \omega_N)$ denote the number of tails. Then we have the expected value of X (at step N) based on the information available at step n.

$$\mathbb{E}_n[X](\omega_1 \cdots \omega_n) = \sum_{\omega_{n+1} \cdots \omega_N} p^{\#H(\omega_{n+1} \cdots \omega_N)} q^{\#T(\omega_{n+1} \cdots \omega_N)} X(\omega_1 \cdots \omega_n \cdots \omega_N)$$

- Steven E. Shreve.
  Stochastic Calculus for Finance I: The Binomial Asset Pricing Model (Springer Finance).
  Springer, 1 edition, June 2005.

$\mathbb{E}_n[X](\omega_1 \cdots \omega_n)$
Special Cases

For a random variable $X(\omega_1 \cdots \omega_N)$ dependent on $N > n$ coin tosses

$$\mathbb{E}_0[X] = \mathbb{E}[X]\,,$$

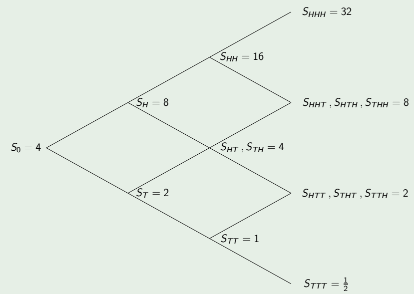
$$\mathbb{E}_N[X] = X\,.$$

Examples
For $p = q = \frac{1}{2}$

Example

Expected value of S_3 based on information available at step 1; i.e., given that S_1 is H

$$\mathbb{E}_1[S_3](H) = \left(\frac{1}{2}\right)^2 \cdot 32 + 2\left(\frac{1}{2}\right)^2 \cdot 8 + \left(\frac{1}{2}\right)^2 \cdot 2 = 12.5\,,$$



What is the value of $\mathbb{E}_1[S_3](T)$?

$\mathbb{E}_1[S_3]$ as a random variable

Note that $\mathbb{E}_1[S_3]$ can take two different values ($\mathbb{E}_1[S_3](H)$ or $\mathbb{E}_1[S_3](T)$) depending on whether a S_1 took the value H or T at step one. So $\mathbb{E}_1[S_3]$ is itself a random variable

Properties of Conditional Expectations

Discrete case

As before let $0 \leq n \leq N$ and let X, Y be random variables which are dependent on the first N coin tosses. Suppose that we know the first n coin tosses $\omega_1 \cdots \omega_n$, but not the remaining tosses $\omega_{n+1} \cdots \omega_N$. Then

Linearity of conditional expectations For constants c_1, c_2

$$\boxed{\mathbb{E}_n(c_1 X + c_2 Y) = c_1 \mathbb{E}_n(X) + c_2 \mathbb{E}_n(Y)\,.}$$

Taking out what is known If X only depends on the first n tosses (known),

$$\boxed{\mathbb{E}_n(XY) = X \cdot \mathbb{E}_n(Y)\,.}$$

Iterated conditioning if $0 \leq n \leq m \leq N$,

$$\boxed{\mathbb{E}_n(\mathbb{E}_m(X)) = \mathbb{E}_n(X)\,.}$$

This means in particular that $\mathbb{E}(\mathbb{E}_m(X)) = \mathbb{E}_0(\mathbb{E}_m(X)) = \mathbb{E}_0(X) = \mathbb{E}(X)$.

Independence If X depends *only* on coin tosses $n+1, \cdots, N$, then

$$\boxed{\mathbb{E}_n(X) = \mathbb{E}(X)\,.}$$

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Stochastic Process

Definition (Stochastic Process)

A *stochastic process* is a sequence of random variables indexed by time

In the context of a binomial model, the random variables of a stochastic process take values at each time step $(0, 1, \cdots)$.

Example

The stock process on a binomial tree is a stochastic process

Adapted Stochastic Process

Definition

Consider a binomial tree model. Let $M_0, M_1, \cdots, M_N$ be a sequence of random variables indexed by time step on the tree. The sequence M_i forms a stochastic process.

Suppose in addition that at each step n, M_n depends only on the first n coin tosses. Then this sequence is called an *adapted* stochastic process

Example

The stock process on a binomial tree is an adapted stochastic process because at each step n, the value of S_n is based on the first n coin tosses

Martingales

Definition (Martingale)

Let $M_0\,, M_1\,, \cdots\,, M_n\,, \cdots\,, M_N$ be an adapted stochastic process (each M_n is a random variable which is dependent on the first n coin tosses)

1. If, for each n
$$M_n = \mathbb{E}_n[M_{n+1}]\,, \quad 0, 1, \cdots, N-1$$
then this stochastic process is a *martingale*
2. If, for each n
$$M_n \leq \mathbb{E}_n[M_{n+1}]\,, \quad 0, 1, \cdots, N-1$$
then this stochastic process is a *submartingale* (tendancy to increase)
3. If, for each n
$$M_n \geq \mathbb{E}_n[M_{n+1}]\,, \quad 0, 1, \cdots, N-1$$
then this stochastic process is a *supermartingale* (tendancy to decrease)

We will be interested here in martingales only

Results following from Martingale definition

Let a stochastic process $\{M_i\}$ be a martingale. So $M_n = \mathbb{E}_n[M_{n+1}]\,, n = 0, 1, \cdots, N-1$. This means that

$$M_n = \mathbb{E}_n[M_{n+1}] = \mathbb{E}_n[\mathbb{E}_{n+1}[M_{n+2}]] = \underbrace{\mathbb{E}_n[M_{n+2}]}_{\text{by iterated conditioning}}$$

Extending this, we have for $m > n$, $M_n = \mathbb{E}_n[M_m]$.

Results following from Martingale definition

From the definition of a martingale

$$M_n = \mathbb{E}_n[M_{n+1}]\,.$$

Take $\mathbb{E}$ of both sides:

$$\mathbb{E}M_n = \mathbb{E}_0 M_n = \mathbb{E}_0[\mathbb{E}_n[M_{n+1}]] = \underbrace{\mathbb{E}_0 M_{n+1}}_{\text{iterated conditioning}} = \mathbb{E}[M_{n+1}]\,,$$

which becomes

$$\mathbb{E}M_0 = \mathbb{E}M_1 = \mathbb{E}M_2 = \cdots\,,$$

but $\mathbb{E}M_0 = M_0$, so

$$M_0 = \mathbb{E}M_1 = \mathbb{E}M_2 = \cdots = \mathbb{E}M_N\,.$$

$$M_0 = \mathbb{E}M_n\,, n = 0\,, 1\,, \cdots\,, N$$

Conditional Expectations in continuous time

$$\mathbb{E}_s[M_t] \approx \mathbb{E}[M(t)|\mathcal{F}(s)] \;\; (0 \leq s \leq t \leq T)$$

Properties of Conditional Expectations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G}$ be a sub σ-algebra of $\mathcal{F}$. Let X and Y be random variables.

Linearity Let c_1 and c_2 be constants. Then

$$\mathbb{E}[c_1 X + c_2 Y|\mathcal{G}] = c_1\mathbb{E}[X|\mathcal{G}] + c_2\mathbb{E}[Y|\mathcal{G}]\,.$$

Taking out what is known Let X be $\mathcal{G}$-measurable, so X may be ascertained based on the information contained in $\mathcal{G}$. Then

$$\mathbb{E}[XY|\mathcal{G}] = X\mathbb{E}[Y|\mathcal{G}]\,.$$

Iterated conditioning If $\mathcal{H}$ is a sub-σ-algebra of $\mathcal{G}$ (i.e., H contains less information than $\mathcal{G}$). Then,

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]\,.$$

Independence Let X be independent of information available in $\mathcal{G}$. Then

$$\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]\,.$$

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  Stochastic Calculus for Finance II: Continuous-Time Models (Springer Finance).
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Binomial

Linearity For constants c_1, c_2

$$\mathbb{E}_n(c_1X+c_2Y) = c_1\mathbb{E}_n(X)+c_2\mathbb{E}_n(Y)\,.$$

Taking out what is known If X only depends on the first n tosses (known),

$$\mathbb{E}_n(XY) = X \cdot \mathbb{E}_n(Y)\,.$$

Iterated conditioning if $0 \le n \le m \le N$,

$$\mathbb{E}_n(\mathbb{E}_m(X)) = \mathbb{E}_n(X)\,.$$

Independence If X depends *only* on coin tosses $n+1, \cdots, N$, then

$$\mathbb{E}_n(X) = \mathbb{E}(X)\,.$$

Measure Theoretic

Linearity Let c_1 and c_2 be constants. Then

$$\mathbb{E}[c_1X + c_2Y|\mathcal{G}] = c_1\mathbb{E}[X|\mathcal{G}] + c_2\mathbb{E}[Y|\mathcal{G}]$$

Taking out what is known Let X be $\mathcal{G}$-measurable, so X may be ascertained based on the information contained in $\mathcal{G}$. Then

$$\mathbb{E}[XY|\mathcal{G}] = X\mathbb{E}[Y|\mathcal{G}]\,.$$

Iterated conditioning If $\mathcal{H}$ is a sub-σ-algebra of $\mathcal{G}$ (i.e., H contains less information than $\mathcal{G}$). Then,

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]\,.$$

Independence Let X be independent of information available in $\mathcal{G}$. Then

$$\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]\,.$$

Martingales Again

We've already had one definition of a martingale for a stochastic process M_n defined on a binomial tree using conditional expectations.

$$\mathbb{E}_n[M_{n+1}] = M_n\,.$$

An analgous definition using the measure theoretic conditional expectations that we have just encountered is

Definition

Let $M(t)$ be a stochastic process which is adapted to a filtration $\mathcal{F}(t)$. Then for $0 \le s \le t \le T$, $M(t)$ is a martingale if

$$\mathbb{E}[M(t)|\mathcal{F}(s)] = M(s)\,.$$

Discounted Asset Price

Risk-Neutral Measure

Let $\omega_1, \omega_2, \cdots, \omega_n$ be the first n coin tosses. So

$$S_n = S_n(\omega_1, \omega_2, \cdots, \omega_n)$$

$$\tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] =$$

$$\tilde{p} = \frac{1+r-d}{u-d}\,,\qquad \tilde{q} = \frac{u-(1+r)}{u-d}\,,$$

$$\begin{aligned}
\tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] &= \frac{1}{(1+r)^{n+1}}\left[\tilde{p}S_{n+1}(\omega_1,\omega_2,\cdots,\omega_n H) + \tilde{q}S_{n+1}(\omega_1,\omega_2,\cdots,\omega_n T)\right]\\
&= \frac{1}{(1+r)^{n+1}}\left[\tilde{p}uS_n(\omega_1,\omega_2,\cdots,\omega_n) + \tilde{q}dS_n(\omega_1,\omega_2,\cdots,\omega_n)\right]\\
&= \frac{S_n(\omega_1,\omega_2,\cdots,\omega_n)}{(1+r)^{n+1}}\left[\tilde{p}u+\tilde{q}d\right]\\
&= \frac{S_n(\omega_1,\omega_2,\cdots,\omega_n)}{(1+r)^{n}}\left[\frac{\tilde{p}u+\tilde{q}d}{1+r}\right]\\
&= \frac{S_n(\omega_1,\omega_2,\cdots,\omega_n)}{(1+r)^{n}} \implies \text{martingale}\,.
\end{aligned}$$

Therefore the discounted asset price process is a martingale under the risk-neutral measure.

Discounted Portfolio Value
Risk-Neutral Measure

From earlier on, the portfolio value process is defined by

$$X_{n+1}(H) = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)\,,$$

- where Δ_n is the amount of asset to be held on step n,
- the value S_{n+1} is one of two values $S_{n+1}(\omega_1\omega_2\cdots\omega_n H)$ or $S_{n+1}(\omega_1\omega_2\cdots\omega_n T)$
- These two values are written as $S_{n+1}(H)$, $S_{n+1}(T)$.

Then

$$\begin{aligned}
\tilde{\mathbb{E}}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right] &= \frac{1}{(1+r)^{n+1}}\tilde{\mathbb{E}}_n\left[\Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)\right]\,,\\
&= \underbrace{\frac{1}{(1+r)^{n+1}}\tilde{\mathbb{E}}_n\left[\Delta_n S_{n+1}\right] + \frac{1}{(1+r)^{n}}\tilde{\mathbb{E}}_n\left[(X_n - \Delta_n S_n)\right]}_{\text{linearity of conditional expectations}}\,,\\
&= \underbrace{\frac{1}{(1+r)^{n+1}}\Delta_n\tilde{\mathbb{E}}_n\left[S_{n+1}\right] + \frac{1}{(1+r)^{n}}(X_n - \Delta_n S_n)}_{\text{taking out what is known at step } n}\,,\\
&= \Delta_n\tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] + \frac{1}{(1+r)^{n}}(X_n - \Delta_n S_n)\,,\\
&= \Delta_n \underbrace{\frac{S_n}{(1+r)^n}}_{\frac{S_n}{(1+r)^n}\text{ is a martingale}} + \frac{1}{(1+r)^{n}}(X_n - \Delta_n S_n)\,,\\
&= \frac{X_n}{(1+r)^n} \quad\Longrightarrow\quad \text{martingale}\,.
\end{aligned}$$

Symmetric Random Walk

Symmetric Random Walk

- Start with a "fair" coin
- Result of a coin toss can be a head (H) or a tail (T)
- Since the coin is fair,

$$\begin{aligned}
P(H) &= p = \frac{1}{2}\,,\\
P(T) &= q = 1 - p = \frac{1}{2}\,.
\end{aligned}$$

- Take a sequence of coin tosses $\omega = \omega_1\omega_2\omega_3\omega_4$, where each ω_i is a coin toss
- Each coin toss is independent of the others

▸ Steven E. Shreve.
Stochastic Calculus for Finance II: Continuous-Time Models (Springer Finance).
Springer, 1st ed. 2004. corr. 2nd printing edition, June 2004.

Symmetric Random Walk

Define a random variable X_i

$$X_i = \begin{cases} +1, & \omega_i = H \\ -1, & \omega_i = T \end{cases}$$

$$\begin{aligned}
\mathbb{E}[X_i] &= p\cdot 1 + q\cdot(-1) = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot(-1) = 0\,,\\
\mathrm{Var}[X_i] &= \mathbb{E}[X_i^2] = p\cdot 1 + q\cdot(1) = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot(1) = 1\,,
\end{aligned}$$

where $\mathrm{Var}[X_i] = \mathbb{E}[X_i^2]$ because $\mathbb{E}[X_i] = 0$

Symmetric Random Walk

Define a process M_k, where $M_0 = 0$ and

$$M_k = \sum_{i=0}^{k} X_i \,.$$

The process $M_0, M_1, M_2, \cdots$ is called a (symmetric) random walk

Properties of the Random Walk

Increments

Let $0 = k_0 < k_1 < k_2 < \cdots < k_m$ be a set of integers

Example

$0 = k_0 < 5(k_1) < 9(k_2) < 15(k_3) < \cdots < 38(k_m)$.

Then the *increment*

$$\begin{aligned} M_{k_{i+1}} - M_{k_i} &= \sum_{j=1}^{k_{i+1}} X_j - \sum_{j=1}^{k_i} X_j \,, \\ &= \left(X_1 + X_2 + \cdots + X_{k_{i+1}}\right) - \left(X_1 + X_2 + \cdots X_{k_i}\right) \,, \\ &= X_{k_i+1} + X_{k_i+2} + \cdots + X_{k_{i+1}} \,, \\ &= \sum_{j=k_i+1}^{k_{i+1}} X_j \,. \end{aligned}$$

Properties of the Random Walk

Increments

Example

Letting $0 = k_0 < 5(k_1) < 9(k_2) < 15(k_3) < \cdots < 38(k_m)$,

$$\begin{aligned} M_{k_2} - M_{k_1} &= M_9 - M_5 = \sum_{j=1}^{9} X_j - \sum_{j=1}^{5} X_j \,, \\ &= (X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7 + X_8 + X_9) \\ &\quad - (X_1 + X_2 + X_3 + X_4 + X_5) \,, \\ &= (X_6 + X_7 + X_8 + X_9) \,, \\ &= \sum_{j=6}^{9} X_j = \sum_{j=5+1}^{9} X_j = \sum_{j=k_1+1}^{k_2} X_j \,. \end{aligned}$$

Independence of Increments

For $0 = k_0 < k_1 < k_2 < \cdots < k_m$, the increments

$$M_{k_1} - M_{k_0}, M_{k_2} - M_{k_1}, M_{k_3} - M_{k_2}, \cdots ,$$

are independent of each other.

$$\sum_{j=k_0+1}^{k_1} X_j, \sum_{j=k_1+1}^{k_2} X_j, \sum_{j=k_2+1}^{k_3} X_j, \cdots$$

This is because each increment is based on different groups of coin tosses and all the coin tosses are independent of each other

Example

Letting $0 = k_0 < 5(k_1) < 9(k_2) < 15(k_3) < \cdots < 38(k_m)$,
Then
$$M_{k_2} - M_{k_1} = M_9 - M_5 = X_6 + X_7 + X_8 + X_9 \,,$$
and
$$M_{k_3} - M_{k_2} = M_{15} - M_9 = X_{10} + X_{11} + X_{12} + X_{13} + X_{14} + X_{15} \,,$$
Since all the coin tosses are independent of each other, the increments are independent of each other

Expectation and Variance

$$\begin{aligned}
\mathbb{E}\left(M_{k_{i+1}} - M_{k_i}\right) &= \sum_{j=1}^{k_{i+1}} \mathbb{E}X_j - \sum_{j=1}^{k_i} \mathbb{E}X_j \,, \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}\left(M_{k_{i+1}} - M_{k_i}\right) &= \mathrm{Var}\left(\sum_{j=1}^{k_{i+1}} X_j - \sum_{j=1}^{k_i} X_j\right) \,, \\
&= \mathrm{Var}\left(\sum_{j=k_i+1}^{k_{i+1}} X_j\right) \,, \\
&= \sum_{j=k_i+1}^{k_{i+1}} \mathrm{Var}\,(X_j) \,, \\
&= \sum_{j=k_i+1}^{k_{i+1}} 1 = k_{i+1} - k_i \qquad \left(\text{because } \sum_{i=1}^{n} 1 = n\right)
\end{aligned}$$

Expectation and Variance

Example

Letting $0 = k_0 < 5(k_1) < 9(k_2) < 15(k_3) < \cdots < 38(k_m)$, Then
$$\begin{aligned}
\mathrm{Var}\,(M_{k_2} - M_{k_1}) &= \mathrm{Var}\,(M_9 - M_5) \,, \\
&= \mathrm{Var}\,(X_6 + X_7 + X_8 + X_9) \,, \\
&= (1 + 1 + 1 + 1) \,, \\
&= 4 = 9 - 5 = k_2 - k_1 \,.
\end{aligned}$$

Martingale Property for symmetric random walk

Let $0 \le k < l$ be integers (times). Then

$$\begin{aligned}
\mathbb{E}_k[M_l] &= \mathbb{E}_k[M_l - M_k + M_k] \,, \\
&= \mathbb{E}_k[M_l - M_k] + \mathbb{E}_k[M_k] \,,
\end{aligned}$$

At step k M_k is known, so $\mathbb{E}_k[M_k] = M_k$.
Also, the quantity $M_l - M_k$ is based only on coin tosses greater than k, so is independent of all coin tosses up to and including step k. So $\mathbb{E}_k[M_l - M_k] = \mathbb{E}[M_l - M_k]$.
Therefore,

$$\begin{aligned}
\mathbb{E}_k[M_l] &= \mathbb{E}_k[M_l - M_k] + \mathbb{E}_k[M_k] \,, \\
&= \mathbb{E}[M_l - M_k] + M_k \,, \\
&= 0 + M_k \,, \\
&= M_k \,.
\end{aligned}$$

So the symmetric random walk is a Martingale.

Martingale Property for symmetric random walk

Same calculation, Different notation

Let $0 \le k < l$ be integers (times). Then

$$\begin{aligned}
\mathbb{E}[M_l|\mathcal{F}_k] &= \mathbb{E}[M_l - M_k + M_k|\mathcal{F}_k]\,, \\
&= \mathbb{E}[M_l - M_k|\mathcal{F}_k] + \mathbb{E}[M_k|\mathcal{F}_k]\,, \\
&= \mathbb{E}[M_l - M_k|\mathcal{F}_k] + M_k\,, \\
&= \mathbb{E}[M_l - M_k] + M_k\,, \\
&= 0 + M_k\,, \\
&= M_k\,.
\end{aligned}$$

So the symmetric random walk is a Martingale.

Scaled Random Walk

Limiting Behaviour

- With the random walk defined in the previous slides there is no useful idea of limiting
- There is only one variable to limit: k, in M_k
- Will now define a *scaled* random walk

Scaled (Symmetric) Random Walk

Symmetric if $p = 1 - q = \frac{1}{2}$

Define

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt}\,.$$

- $W^{(n)}(t)$ is defined for n, t where nt is an integer
- For $n = 100$ and $t = 0.25$, $nt = 25$; an integer
- For $n = 100$ and $t = 0.00000001$, $nt = 0.000001$, not an integer
- Each unit interval in $[0, t]$ split into n parts of length $\frac{1}{n}$

Scaled (Symmetric) Random Walk

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt} = \sum_{j=1}^{nt} \frac{1}{\sqrt{n}} X_j$$

For each X_j term,

$$\frac{1}{\sqrt{n}} X_i = \begin{cases} +\frac{1}{\sqrt{n}}, & \omega_i = H \\ -\frac{1}{\sqrt{n}}, & \omega_i = T \end{cases}$$

So the step size is smaller as n gets larger

Independence of Increments of $W^{(n)}(t)$

$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt}$ is defined as a random walk, so its increments are independent from previous slides

Expectation and Variance of $\frac{1}{\sqrt{n}} X_j$

$$\mathbb{E}\left(\frac{1}{\sqrt{n}} X_j\right) = 0$$

$$\mathrm{Var}\left(\frac{1}{\sqrt{n}} X_j\right) = \mathbb{E}\left[\left(\frac{1}{\sqrt{n}} X_j\right)^2\right],$$

since $\mathbb{E}\left(\frac{1}{\sqrt{n}} X_j\right) = 0$.

$$\begin{aligned} \mathbb{E}\left[\left(\frac{1}{\sqrt{n}} X_j\right)^2\right] &= \frac{1}{n}\mathbb{E}\left(X_j^2\right), \\ &= \frac{1}{n} \cdot 1 = \frac{1}{n}. \end{aligned}$$

Expectation and Variance $W^{(n)}(t) - W^{(n)}(s)$

$s < t$

By definition of $W^{(n)}(t)$ as a symmetric random walk,

$$\mathbb{E}\left[W^{(n)}(t) - W^{(n)}(s)\right] = 0$$

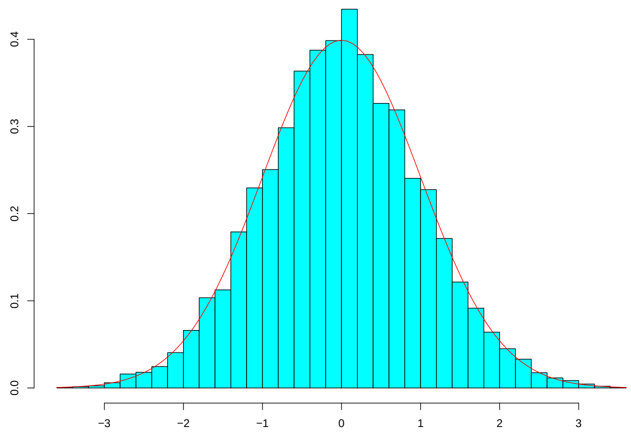
and

$$\begin{aligned} \mathrm{Var}\left[W^{(n)}(t) - W^{(n)}(s)\right] &= \mathrm{Var}\left[\frac{1}{\sqrt{n}}\left(M_{nt} - M_{ns}\right)\right], \\ &= \frac{1}{n}\mathrm{Var}\left[M_{nt} - M_{ns}\right], \\ &= \frac{1}{n}(nt - ns) \quad \text{(from earlier slides)}, \\ &= t - s. \end{aligned}$$

Limit of Scaled Symmetric Random Walk

In the limit as $n \to \infty$, $W^{(n)}(t)$ limits to *Brownian Motion*, $W(t)$.

Limit of Scaled Symmetric Random Walk to Normal distribution



Histogram of values at $t = 1$ of 10000 scaled random walks, each of length 5000. Red curve; density function of normal distribution $N(0,1)$. (Sample mean: $\mu = -0.00182$; sample variance: $\sigma = 1.00222$)

Brownian Motion

Definition of Brownian Motion

Wiener Process

Definition (Brownian Motion)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Then for each $\omega \in \Omega$ a Brownian motion is a *continuous* function $W(t)\,,\ t > 0$ which depends on ω, which has the properties that

1. $W(0) = 0\,,$
2. $W(t)$ is continuous (almost surely)
3. For $0 = t_0 < t_1 < t_2 < \cdots t_k < \cdots$, the variables

$$W(t_1) = (W(t_1) - W(t_0))\,, (W(t_2) - W(t_1))\,, \cdots, (W(t_{k+1}) - W(t_k)), \cdots,$$

are independent of each other. Thus $W(t)$ has *independent increments*. Moreover each increment $(W(t_{j+1}) - W(t_j))$ is normally distributed with

$$\mathbb{E}(W(t_{j+1}) - W(t_j)) = 0\,,$$

and

$$\mathrm{Var}(W(t_{j+1}) - W(t_j)) = t_{j+1} - t_j\,.$$

Martingale

Using the filtration notation, we give a definition of a martingale by analogy with the one we have already seen

Definition (Martingale)

A process $X(t)$ (e.g., Brownian Motion) is a martingale if, for $0 \leq s < t$,

$$\mathbb{E}\,[X(t)\,|\,\mathcal{F}(s)] = X(s)\,.$$

Such a process is drift free.

Some Properties of Brownian Motion

Moments of $W(t)$

Let $0 \leq s < t$, then

- Moments, by definition:

$$W(t) - W(s) \sim N(0, t-s)\,.$$

  and, clearly $W(t) = W(t) - W(0) \sim N(0, t)$

$$\mathbb{E}[W(t)] = \mathbb{E}[W(s)] = 0, \qquad \mathbb{E}[(W(t) - W(s))^2] = t - s$$

- Covariance: $W(s)$ and $W(t) - W(s)$ are independent.
  So:

$$\begin{aligned} Cov[W(t), W(s)] &= \mathbb{E}[W(t)\,W(s)] - \mathbb{E}[W(t)]\,\mathbb{E}[W(s)] \\ &= \mathbb{E}[W(t)\,W(s)] \\ &= \mathbb{E}[(W(t) - W(s) + W(s))\,W(s)] \\ &= \mathbb{E}[(W(t) - W(s))\,W(s)] + \mathbb{E}[W^2(s)] \\ &= \mathbb{E}[W(t) - W(s)]\,\mathbb{E}[W(s)] + \mathbb{E}[W^2(s)] \\ &= s \end{aligned}$$

Brownian Motion is a Martingale

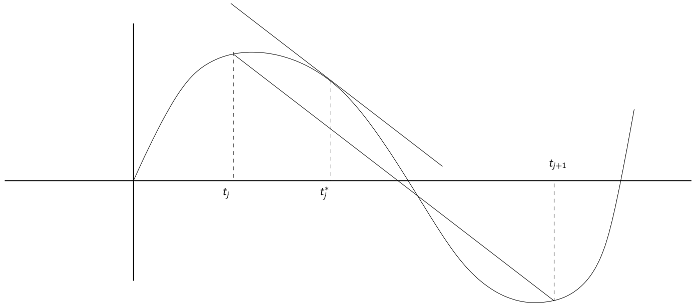
- $\mathbb{E}[W(t)] = W(0) = 0$.
- Likewise, conditional upon information up to time s ($0 < s < t$):

$$\begin{aligned} \mathbb{E}[W(t)|\mathcal{F}(s)] &= \mathbb{E}[W(t) - W(s) + W(s)|\mathcal{F}(s)] \\ &= \mathbb{E}[W(t) - W(s)|\mathcal{F}(s)] + \mathbb{E}[W(s)|\mathcal{F}(s)] \\ &= 0 + W(s) \\ &= W(s) \end{aligned}$$

The expected future value equals the current value, the process is *drift-free.*

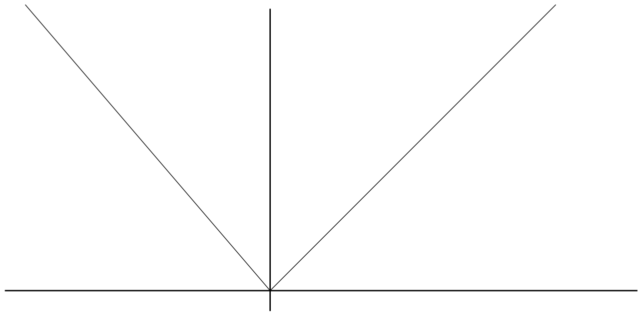
Variations of Brownian Motion

Differentiable everywhere



Not differentiable everywhere

$f(t) = |t|$



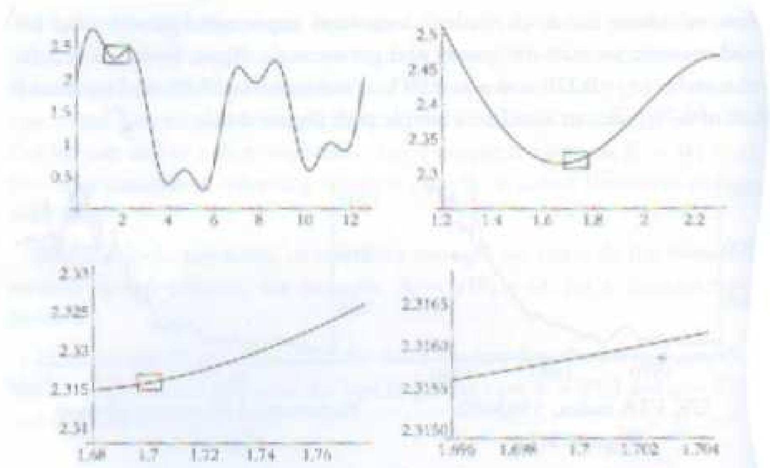


Figure 3.7 Progressive magnification around the point 1.7

- Martin Baxter and Andrew Rennie.
  Financial Calculus: an introduction to derivative pricing.
  CUP, 1996.

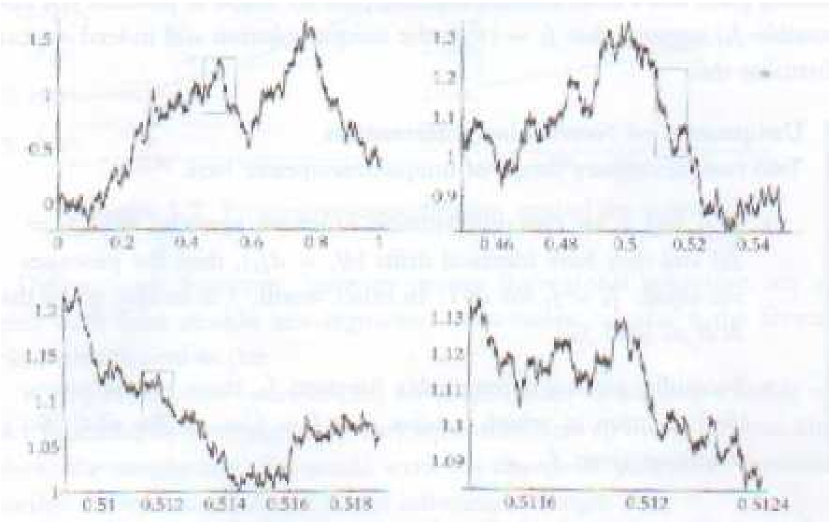


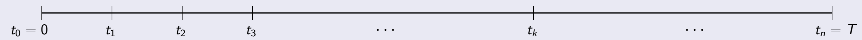
Figure 3.8 'Zooming in' on Brownian motion

- Martin Baxter and Andrew Rennie.
  Financial Calculus: an introduction to derivative pricing.
  CUP, 1996.

A Partition of the interval $[0, T]$

Definition (Partition)

A *partition* $\Pi = \{t_0\,, t_1\,, \cdots, t_n\}$ is a set of points in the interval $[0\,, T]$ such that $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$



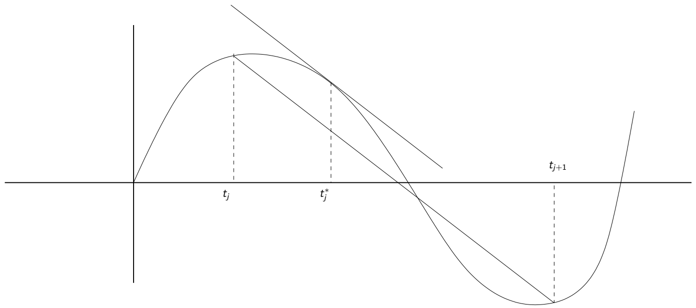
The *mesh* of the partition is defined as

$$||\Pi|| = \max_{k=0,\ldots,n-1}(t_{k+1} - t_k)$$

Mean Value Theorem

If $f(t)$ continuous on $[t_j, t_{j+1}]$ and differentiable on the interval (t_j, t_{j+1}), then there is some t_j^* in (t_j, t_{j+1}) such that

$$f'(t_j^*) = \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j}$$



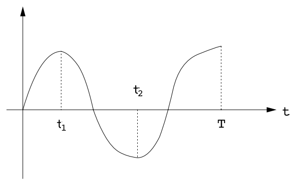
Note this does not in hold in the absence of differentiability e.g., $f(t) = |t|$.

First Variation of a differentiable function f

Definition

$$\begin{aligned} FV_f(T) &= \lim_{||\Pi||\to 0} \sum_{k=0}^{n-1} |f(t_{k+1}) - f(t_k)| \\ &= \lim_{||\Pi||\to 0} \sum_{k=0}^{n-1} |f'(t_k^*)|\,(t_{k+1} - t_k) \qquad \text{(mean value theorem)} \\ &= \int_0^T |f'(t)|dt. \end{aligned}$$

$FV_f(T)$ is a measure of up and down movement on the y axis (note the absolute value: $|f(t)|$). See also: `http://en.wikipedia.org/wiki/Total_variation`



Second (Quadratic) Variation of a differentiable function f

Definition

$$
\begin{aligned}
QV_f(T) &= \lim_{||\Pi||\to 0} \sum_{k=0}^{n-1} [f(t_{j+1}) - f(t_j)]^2 \\
&= \lim_{||\Pi||\to 0} \sum_{k=0}^{n-1} |f'(t_k^*)|^2 (t_{k+1} - t_k)^2 \\
&\leq \lim_{||\Pi||\to 0} \left(\max_{0\leq k<n} (t_{k+1} - t_k) \right) \sum_{k=0}^{n-1} |f'(t_k^*)|^2 (t_{k+1} - t_k) \\
&= \lim_{||\Pi||\to 0} ||\Pi|| \sum_{k=0}^{n-1} |f'(t_k^*)|^2 (t_{k+1} - t_k) \\
&= \lim_{||\Pi||\to 0} ||\Pi|| \int_0^T |f'(t)|^2 dt \\
&= 0\,, \text{ assuming } \int_0^T |f'(t)|^2 dt < \infty
\end{aligned}
$$

Quadratic Variation of Brownian Motion

Consider a partition $\Pi = \{t_0\,, t_1\,, \cdots, t_n\}$ of the interval $[0, t]$ such that $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$. The quadratic variation is defined to be

$$QV_W(t) = \lim_{||\Pi||\to 0} \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2 \,,$$

where $||\Pi|| = \max_{0\leq k<n}(t_{k+1} - t_k)$ is referred to as the mesh of the partition.

Quadratic Variation of Brownian Motion

We want to prove that, for $\Pi = \{t_0\,, t_1\,, \cdots, t_n\}$,

$$QV_W(t) = \lim_{||\Pi||\to 0} \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2 = t$$

Procedure:

1. Show that $\mathbb{E}\left[\sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2\right] = t$
2. Because $QV_W(t)$ itself is stochastic, it has a variance. We need to show this variance is zero (in the limit):

$$\text{Var}\left[\sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2\right] = 0 \text{ (as } ||\Pi|| \to 0)$$

Show that $\mathbb{E}\left[\sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2\right] = t$

$$\mathbb{E}\left[\sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2\right] = \sum_{j=0}^{n-1} \mathbb{E}\left[(W(t_{j+1}) - W(t_j))^2\right] .$$

Consider individual terms:

$$\mathbb{E}\left[(W(t_{j+1}) - W(t_j))^2\right] = \text{Var}\left[(W(t_{j+1}) - W(t_j))\right] = t_{j+1} - t_j \,.$$

Therefore

$$\mathbb{E}\left[\sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2\right] = \sum_{j=0}^{n-1} (t_{j+1} - t_j) = t \,.$$

Show that $\text{Var}\left[\sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))^2\right]=0$ (as $||\Pi||\to 0$)

We have

$$\text{Var}\left[\sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))^2\right].$$

Since individual terms of sum are independent of each other (independence of increments),

$$\text{Var}\left[\sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))^2\right]=\sum_{j=0}^{n-1}\text{Var}\left[(W(t_{j+1})-W(t_j))^2\right].$$

Individual terms of $\text{Var}\left[\sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))^2\right]$

Take individual terms and let $\Delta W_j=(W(t_{j+1})-W(t_j))$
Which means that $(W(t_{j+1})-W(t_j))^2$ is written as ΔW_j^2, so

$$\text{Var}\left[(W(t_{j+1})-W(t_j))^2\right]=\text{Var}\left[\Delta W_j^2\right]$$

$$\begin{aligned}
\text{Var}\left[\Delta W_j^2\right] &= \mathbb{E}\left[\left(\Delta W_j^2-\mathbb{E}\left[\Delta W_j^2\right]\right)^2\right],\\
&= \mathbb{E}\left[\left(\Delta W_j^2-(t_{j+1}-t_j)\right)^2\right],\\
&= \mathbb{E}\left[(\Delta W_j)^4-2\Delta W_j^2(t_{j+1}-t_j)+(t_{j+1}-t_j)^2\right],\\
&= \mathbb{E}[(\Delta W_j)^4]-2\mathbb{E}[\Delta W_j^2](t_{j+1}-t_j)+(t_{j+1}-t_j)^2,\\
&= \mathbb{E}[(\Delta W_j)^4]-2\underbrace{(t_{j+1}-t_j)}_{\mathbb{E}[\Delta W_j^2]}(t_{j+1}-t_j)+(t_{j+1}-t_j)^2,\\
&= \mathbb{E}[(\Delta W_j)^4]-(t_{j+1}-t_j)^2.
\end{aligned}$$

Individual terms of $\text{Var}\left[\sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))^2\right]$ (ctd)

We have

$$\mathbb{E}\left[(\Delta W_j)^4\right]=\mathbb{E}\left[(W(t_{j+1})-W(t_j))^4\right],$$

where $X=(W(t_{j+1})-W(t_j))$ is normally distributed with mean 0 and variance $(t_{j+1}-t_j)$, i.e.,

$$X\sim N(0,\sigma^2=(t_{j+1}-t_j)).$$

Based on the properties of the normal distribution,

$$\begin{aligned}
\mathbb{E}[X^4] &= 3\sigma^4,\ \text{(since the mean is zero)}\\
&= 3(t_{j+1}-t_j)^2.
\end{aligned}$$

So,

$$\mathbb{E}\left[(\Delta W_j)^4\right]=3(t_{j+1}-t_j)^2,$$

which means that

$$\begin{aligned}
\text{Var}\left[(W(t_{j+1})-W(t_j))^2\right] &= \mathbb{E}[(\Delta W_j)^4]-(t_{j+1}-t_j)^2,\\
&= 3(t_{j+1}-t_j)^2-(t_{j+1}-t_j)^2,\\
&= 2(t_{j+1}-t_j)^2.
\end{aligned}$$

Sum over all individual terms and take limit

So,

$$\begin{aligned}
\text{Var}\left[\sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))^2\right] &= 2\sum_{j=0}^{n-1}(t_{j+1}-t_j)^2,\\
&\le 2\max_{0\le k<n}(t_{k+1}-t_k)\sum_{j=0}^{n-1}(t_{j+1}-t_j),\\
&= 2||\Pi||\sum_{j=0}^{n-1}(t_{j+1}-t_j) = 2||\Pi||\cdot t.
\end{aligned}$$

In the limit as $||\Pi||\to 0$,

$$\lim_{||\Pi||\to 0}\text{Var}\left[\sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))^2\right]=0\cdot t=0.$$

Note that $\text{Var}\left[\sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))^2\right]$ is only zero in the limit

Therefore

$$\boxed{QV_W(t)=t.}$$

Recap

We wanted to prove that

$$QV_W(t) = \lim_{||\Pi||\to 0} \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2 = t$$

We showed that

1. Expected value is t;

$$\mathbb{E}\left[\sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2\right] = t$$

2. Variance is zero;

$$\text{Var}\left[\sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2\right] = 0 \text{ (as } ||\Pi|| \to 0)$$

Brownian motion accumulates 1 unit of quadratic variation per unit time

Differential Notation

The statement about the quadratic variation of Brownian motion

$$QV_W(T) = \lim_{||\Pi||\to 0} \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2 = T\,,$$

is informally referred to as

$$dW(t)dW(t) = dt\,.$$

This notation proves convenient later on as a shorthand
Other limits are referred to using a similar shorthand, and one which is also similar to the notation used in ordinary calculus;

The notation $dW(t)dt = 0$ is used to refer to the fact that the following limit vanishes

$$\lim_{||\Pi||\to 0} \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))\,(t_{j+1} - t_j) = 0\,, \tag{1}$$

and

$$\lim_{||\Pi||\to 0} \sum_{j=0}^{n-1} (t_{j+1} - t_j)^2 = 0\,, \tag{2}$$

has the notation $dtdt = 0$ assigned to it

Differential Notation (ctd)

$dW(t)dt = 0$

$$\begin{aligned}
\left|\sum_{j=0}^{n-1}\left(W(t_{j+1}) - W(t_j)\right)(t_{j+1} - t_j)\right| &\le \max_{0\le k<n}\left|\left(W(t_{k+1}) - W(t_k)\right)\right| \sum_{j=0}^{n-1}(t_{j+1} - t_j) \\
&= \max_{0\le k<n}\left|\left(W(t_{k+1}) - W(t_k)\right)\right| \cdot T \\
&\to 0\cdot T \quad (\text{as } ||\Pi|| \to 0)\,,
\end{aligned}$$

by continuity of $W(t)$ (which is continuous by definition).

$dtdt = 0$

$$\begin{aligned}
\left|\sum_{j=0}^{n-1}(t_{j+1} - t_j)^2\right| &\le ||\Pi|| \sum_{j=0}^{n-1}(t_{j+1} - t_j) \\
&= ||\Pi|| \cdot T \\
&\to 0\cdot T \quad (\text{as } ||\Pi|| \to 0)\,.
\end{aligned}$$

First Variation, Brownian Motion

Since the quadratic or second variation of a brownian motion process is finite- what does this imply for the first variation?

$$\begin{aligned}
FV_W(t) &= \lim_{||\Pi||\to 0} \sum_{j=0}^{n-1} |(W(t_{j+1}) - W(t_j))|\,, \\
&\ge \lim_{||\Pi||\to 0} \frac{\sum_{j=0}^{n-1}(W(t_{j+1}) - W(t_j))^2}{\max_{0\le k<n}|(W(t_{k+1}) - W(t_k))|}\,, \\
&= \lim_{||\Pi||\to 0} \frac{QV_W(t)}{\max_{0\le k<n}|(W(t_{k+1}) - W(t_k))|}\,, \\
&\to \infty \text{ (as } ||\Pi|| \to 0)\,,
\end{aligned}$$

The denominator goes to zero, because the Brownian motion is continuous almost surely.
This result indicates how strange a "function" Brownian motion is

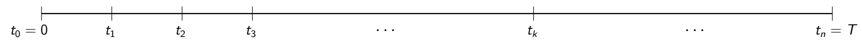
Itô Integral

Integration

For an ordinary function $f(x)$, we can define an integral as the limit of a sum:

$$\int_0^T f(t)dt = \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} f(t_j^*)(t_{j+1} - t_j),$$

where t_j^* is in $[t_i, t_{j+1}]$.



Remember:

$$||\Pi|| = \max_{k=0,\dots,n-1} (t_{k+1} - t_k)$$

http://en.wikipedia.org/wiki/Riemann_integral

Stochastic Integral

We want to define an integral where the integrator is a Wiener process,

$$I(t) = \int_0^t \Delta(s)\, dW(s)$$

where $\Delta(s)$ is square-integrable. $\Delta(t)$ is determined based on information collected up to time t and may be stochastic.

In ordinary calculus, with differentiable function $f(t)$ instead of $W(t)$, we could define

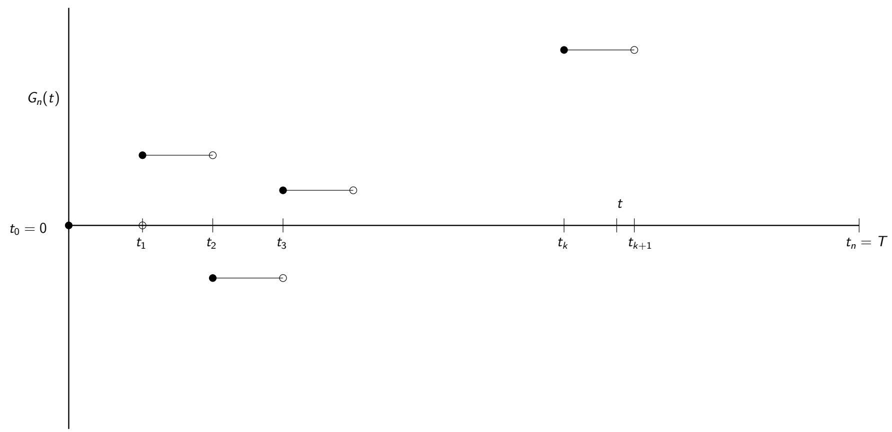
$$\int_0^t \Delta(s)\, df(s) = \int_0^t \Delta(s)\, f'(s)\, ds.$$

This does not work here, because W is not differentiable.

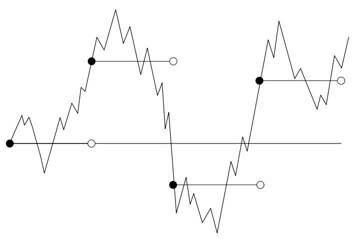
Instead we discretize, choose a partition first, define what we mean, and then shrink the mesh.

Step Function $\Delta(t)$

For a partition $\Pi = \{t_0, t_1, \cdots, t_n\}$ of the interval $[0, T]$, where $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$, define a "step function" $\Delta_n(t)$, on Π to be a function which holds a constant value in each interval $[t_j, t_{j+1})$.



Step function approximating general function



Stochastic Integral, Definition

We choose a partition $\Pi = t_0, t_1, ..., t_n$ of the time interval $[0, T]$,

$$0 = t_0 \leq t_1 \leq \cdots \leq t_n = T, \qquad ||\Pi|| = \max_{k=0,\ldots,n-1} (t_{k+1} - t_k).$$

We then define the stochastic integral of a step function $\Delta_\Pi(t)$ as

$$I_\Pi(t) = \sum_{j=0}^{n-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) = \int_0^T \Delta_\Pi(t)\, dW(t)\,,$$

and an integral for a general function $\Delta(t)$,

$$I(T) = \int_0^T \Delta(t)\, dW(t) = \lim_{||\Pi||\to 0} I_\Pi(T),$$

where

$$\lim_{||\Pi||\to 0, n\to\infty} \Delta_\Pi(T) = \Delta(T)\,.$$

$$\left[\text{Actually:} \qquad \lim_{n\to\infty} \mathbb{E} \int_0^T |\Delta_\Pi(t) - \Delta(t)|^2 dt = 0\,.\right]$$

Itô and Stratonovich

The position in time inteval $[t_k, t_{k+1}]$ where we evaluate $\Delta(t)$ is crucial, we obtain different values of $I(t)$ in the limit depending on this choice:

- Left point: popular in Finance (think of Δ as asset holdings chosen due information up to time t_k and then exposed to random movements of the price W per unit holding over the next time period). The resulting integral is called *Itô integral*, to be used in the following.
- Mid point: popular in Physics, the resulting integral is called Stratonovich integral

$\int_0^T W(t)\, dW(t)$

In ordinary calculus we have for $f(0) = 0$

$$\int_0^T f(t)\, df(t) = \int_0^T f(t)\, f'(t)\, dt = \frac{1}{2}\int_0^T \frac{d}{dt}(f^2(t))dt = \frac{1}{2} f^2(T)$$

For the Itô integral we will show that

$$I(T) = \int_0^T \Delta(t)\, dW(t) = \int_0^T W(t)\, dW(t) = \frac{1}{2}\,(W^2(T) - T)$$

Approximate $W(t)$ with a step function

$$
\Delta_\Pi(t) = W_\Pi(t) = \begin{cases} W(0) = 0 & \text{if } 0 \le t < \frac{T}{n}, \\ W(\frac{T}{n}) & \text{if } \frac{T}{n} \le t < \frac{2T}{n}, \\ W(\frac{2T}{n}) & \text{if } \frac{2T}{n} \le t < \frac{3T}{n}, \\ \vdots & \\ W(\frac{(n-1)T}{n}) & \text{if } \frac{(n-1)T}{n} \le t < T, \end{cases}
$$

So,

$$
\begin{aligned}
\int_0^T W(t)dW(t) &= \lim_{n\to\infty} \int_0^T \Delta_\Pi(t)dW(t) \\
&= \lim_{n\to\infty} \sum_{j=0}^{n-1} W\left(\frac{jT}{n}\right)\left[W\left(\frac{(j+1)T}{n}\right) - W\left(\frac{jT}{n}\right)\right].
\end{aligned}
$$

$\int_0^T W(t)\,dW(t)$

Letting $W_j = W\left(\frac{jT}{n}\right)$, consider the sum:

$$
\begin{aligned}
\frac{1}{2}\sum_{j=0}^{n-1}(W_{j+1} - W_j)^2 &= \frac{1}{2}\sum_{j=0}^{n-1} W_{j+1}^2 - \sum_{j=0}^{n-1} W_{j+1}\,W_j + \frac{1}{2}\sum_{j=0}^{n-1} W_j^2 \\
&= \frac{1}{2}\,W_n^2 + \frac{1}{2}\sum_{j=0}^{n-1} W_j^2 - \sum_{j=0}^{n-1} W_{j+1}\,W_j + \frac{1}{2}\sum_{j=0}^{n-1} W_j^2 \\
&= \frac{1}{2}\,W_n^2 + \sum_{j=0}^{n-1} W_j^2 - \sum_{j=0}^{n-1} W_{j+1}\,W_j \\
&= \frac{1}{2}\,W_n^2 - \sum_{j=0}^{n-1} W_j(W_{j+1} - W_j)
\end{aligned}
$$

$\int_0^T W(t)\,dW(t)$

Take the limit as $||\Pi|| \to 0$ gives:

$$
\frac{1}{2}T = \frac{1}{2}\,W^2(T) - \int_0^T W(t)dW(t),
$$

so,

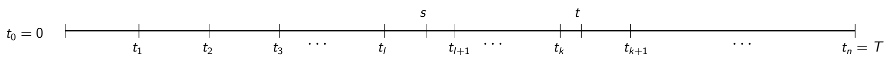
$$
\int_0^T W(t)dW(t) = \frac{1}{2}\,W^2(T) - \frac{1}{2}T.
$$

$I_\Pi(t)$ is a Martingale

In order to show $I_\Pi(t)$ is a Martingale, we need to show that for $0 \le s \le t \le T$,

$$
\mathbb{E}\left[I_\Pi(t)\,|\,\mathcal{F}(s)\right] = I_\Pi(s).
$$

Set up a partition as follows



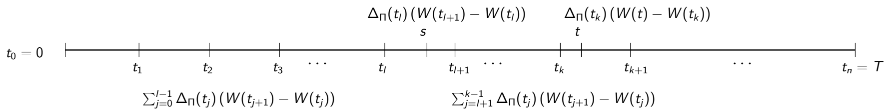
where we have $0 \le s < t \le T$, such that for $l < k$ (i.e., $t_l < t_k$), $s \in [t_l, t_{l+1})$ and $t \in [t_k, t_{k+1})$. As before, we have

$$
I_\Pi(t) = \sum_{j=0}^{k-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) + \Delta_\Pi(t_k)\,(W(t) - W(t_k))
$$

$I_\Pi(t)$ is a Martingale (ctd)

We can split $I_\Pi(t)$ up into four parts:

$$
\begin{aligned}
I_\Pi(t) &= \sum_{j=0}^{k-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) + \Delta_\Pi(t_k)\,(W(t) - W(t_k))\ , \\
&= \sum_{j=0}^{l-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) + \Delta_\Pi(t_l)\,(W(t_{l+1}) - W(t_l)) \\
&\quad + \sum_{j=l+1}^{k-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) + \Delta_\Pi(t_k)\,(W(t) - W(t_k))
\end{aligned}
$$



$I_\Pi(t)$ is a Martingale (ctd)

So $\mathbb{E}\,[\, I_\Pi(t) \mid \mathcal{F}(s)]$ becomes

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{j=0}^{l-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) \,\middle|\, \mathcal{F}(s)\right] \\
&+ \mathbb{E}\,[\, \Delta_\Pi(t_l)\,(W(t_{l+1}) - W(t_l)) \mid \mathcal{F}(s)] \\
&+ \mathbb{E}\left[\sum_{j=l+1}^{k-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) \,\middle|\, \mathcal{F}(s)\right] \\
&+ \mathbb{E}\,[\, \Delta_\Pi(t_k)\,(W(t) - W(t_k)) \mid \mathcal{F}(s)]
\end{aligned}
$$

By taking out what is known, this becomes:

$$
\begin{aligned}
&\sum_{j=0}^{l-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) \\
&+ \Delta_\Pi(t_l)\,(\mathbb{E}\,[\, W(t_{l+1}) \mid \mathcal{F}(s)] - W(t_l)) \\
&+ \cdots
\end{aligned}
$$

Using the fact that $W(t)$ is a martingale ($\mathbb{E}\,[\, W(t) \mid \mathcal{F}(s)] = W(s)$) gives

$$
\sum_{j=0}^{l-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) + \Delta_\Pi(t_l)\,(W(s) - W(t_l)) + \cdots
$$

$I_\Pi(t)$ is a Martingale (ctd)

So far we have

$$
\begin{aligned}
\mathbb{E}\,[\, I_\Pi(t) \mid \mathcal{F}(s)] = I_\Pi(s) \\
&+ \mathbb{E}\left[\sum_{j=l+1}^{k-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) \,\middle|\, \mathcal{F}(s)\right] \\
&+ \mathbb{E}\,[\, \Delta_\Pi(t_k)\,(W(t) - W(t_k)) \mid \mathcal{F}(s)]
\end{aligned}
$$

$I_\Pi(t)$ is a Martingale (ctd)

What is

$$
\mathbb{E}\left[\sum_{j=l+1}^{k-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) \,\middle|\, \mathcal{F}(s)\right] ?
$$

Looking at terms individually

$$
\begin{aligned}
\mathbb{E}\,[\, \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) \mid \mathcal{F}(s)] &= \mathbb{E}\,[\, \mathbb{E}\,[\, \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) \mid \mathcal{F}(t_j)] \mid \mathcal{F}(s)] \\
&= \mathbb{E}\,[\, \Delta_\Pi(t_j)\,(\mathbb{E}\,[\, W(t_{j+1}) \mid \mathcal{F}(t_j)] - W(t_j)) \mid \mathcal{F}(s)] \\
&= \mathbb{E}\,[\, \Delta_\Pi(t_j)\,(W(t_j) - W(t_j)) \mid \mathcal{F}(s)] \\
&= 0\,,
\end{aligned}
$$

where we used the iterated conditioning rule along with the fact that $s < t_j$.
So,

$$
\mathbb{E}\left[\sum_{j=l+1}^{k-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) \,\middle|\, \mathcal{F}(s)\right] = 0\,.
$$

$I_\Pi(t)$ is a Martingale (ctd)

Now we have

$$\begin{aligned}\mathbb{E}\left[\, I_\Pi(t) \,|\, \mathcal{F}(s)\right] &= I_\Pi(s) \\ &\quad + \mathbb{E}\left[\, \Delta_\Pi(t_k)\,(W(t) - W(t_k)) \,|\, \mathcal{F}(s)\right]\end{aligned}$$

Using a similar iterated conditioning argument to the one used on the previous slide,

$$\mathbb{E}\left[\, \Delta_\Pi(t_k)\,(W(t) - W(t_k)) \,|\, \mathcal{F}(s)\right] = 0$$

Therefore

$$\mathbb{E}\left[\, I_\Pi(t) \,|\, \mathcal{F}(s)\right] = I_\Pi(s)\,,$$

So $I_\Pi(t)$ is a Martingale.

Itô Isometry

Since $I_\Pi(t)$ is a Martingale,

$$\mathbb{E}(I_\Pi(t)) = I(0) = 0\,,$$

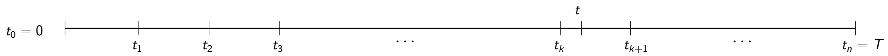
So

$$\mathrm{Var}(I_\Pi(t)) = \mathbb{E}(I_\Pi(t)^2)\,.$$

We will show that

$$\mathbb{E}((I_\Pi(t))^2) = \mathbb{E}\int_0^t (\Delta_\Pi(u))^2 du$$

Use a similar partition to the one used before,



So,

$$I_\Pi(t) = \sum_{j=0}^{k-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) + \Delta_\Pi(t_k)(W(t) - W(t_k))$$

Itô Isometry (ctd)

$$I_\Pi(t) = \sum_{j=0}^{k-1} \Delta_\Pi(t_j)\,(W(t_{j+1}) - W(t_j)) + \Delta_\Pi(t_k)(W(t) - W(t_k))\,.$$

Let $\Delta W_j = (W(t_{j+1}) - W(t_j))\,, 0 \le j < k$, and let $\Delta W_k = (W(t) - W(t_k))$. Then rewrite the Itô integral as,

$$I_\Pi(t) = \sum_{j=0}^{k} \Delta_\Pi(t_j)\Delta W_j\,.$$

So,

$$\begin{aligned}\mathbb{E}((I_\Pi(t))^2) &= \mathbb{E}\left[\left(\sum_{j=0}^{k}\Delta_\Pi(t_j)\Delta W_j\right)\left(\sum_{i=0}^{k}\Delta_\Pi(t_i)\Delta W_i\right)\right]\,, \\ &= \mathbb{E}\left[\sum_{j=0}^{k}(\Delta_\Pi(t))^2\Delta W_j^2 + 2\sum_{0\le i<j\le k}\Delta_\Pi(t_i)\Delta_\Pi(t_j)\Delta W_i \Delta W_j\right]\,,\end{aligned}$$

Itô Isometry (ctd)

Taking the second term first

$$\begin{aligned}&\mathbb{E}\left[2\sum_{0\le i<j\le k}\Delta_\Pi(t_i)\Delta_\Pi(t_j)\Delta W_i \Delta W_j\right] \\ &= 2\sum_{0\le i<j\le k}\mathbb{E}\left[\Delta_\Pi(t_i)\Delta_\Pi(t_j)\Delta W_i \Delta W_j\right] \\ &= 2\sum_{0\le i<j\le k}\mathbb{E}[\Delta_\Pi(t_i)\Delta_\Pi(t_j)\Delta W_i]\,\underbrace{\mathbb{E}[\Delta W_j]}_{=0} \\ &= 0\,.\end{aligned}$$

Because

- $\Delta_\Pi(t_i)\Delta_\Pi(t_j)\Delta W_i$ $\mathcal{F}(t_j)$-measurable
- ΔW_j independent of $\mathcal{F}(t_j)$

Itô Isometry (ctd)

So,

$$\mathbb{E}((I_\Pi(t))^2) = \mathbb{E}\left[\sum_{j=0}^{k}(\Delta_\Pi(t))^2 \Delta W_j^2\right] + 0$$

and

$$\mathbb{E}\left[\sum_{j=0}^{k}(\Delta_\Pi(t_j))^2 \Delta W_j^2\right] = \sum_{j=0}^{k}\mathbb{E}\left[(\Delta_\Pi(t_j))^2 \Delta W_j^2\right] = \sum_{j=0}^{k}\mathbb{E}\left[(\Delta_\Pi(t_j))^2\right]\mathbb{E}\left[\Delta W_j^2\right]$$
$$= \sum_{j=0}^{k-1}\mathbb{E}\left[(\Delta_\Pi(t_j))^2\right](t_{j+1}-t_j) + \mathbb{E}\left[(\Delta_\Pi(t_k))^2\right](t-t_j)$$
$$= \mathbb{E}\left[\sum_{j=0}^{k-1}(\Delta_\Pi(t_j))^2(t_{j+1}-t_j)\right] + \mathbb{E}[(\Delta_\Pi(t_k))^2](t-t_j)$$
$$= \mathbb{E}\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}(\Delta_\Pi(u))^2 du + \mathbb{E}\int_{t_k}^{t}(\Delta_\Pi(u))^2 du$$
$$= \boxed{\mathbb{E}\int_0^t (\Delta_\Pi(u))^2 du}$$

Quadratic variation of Itô integral

Since the Itô integral is written as

$$I(t) = \int_0^t G(u)dW(u),$$

In informal notation, this can be written as

$$dI(t) = G(u)dW(t)$$

Again informally, the quadratic variation is written,

$$dIdI = G(t)dW(t)G(t)dW(t) = (G(t))^2 dt$$

So, the quadratic variation of the Itô integral is

$$QV_I(t) = \int_0^t (G(u))^2 du$$

Summary: Properties of Itô Integral

For an Itô integral

$$I(T) = \int_0^T G(t)dW(t),$$

- Expected Value:
$$\mathbb{E}[I(T)] = 0$$
,
- Variance: (Itô Isometry):
$$\text{Var}[I(T)] = \int_0^T \mathbb{E}[G^2(t)]dt,$$
- Quadratic variation:
$$\text{QV}_I(T) = \int_0^T [G^2(t)]dt,$$
- Martingale: for $0 \le s < t$,
$$\mathbb{E}[I(t)|\mathcal{F}(s)] = I(s)$$

Chain Rule
$f(W(t))$

For an expression of the form $f(W(t))$, if asked for

$$\frac{d}{dt}f(W(t)),$$

would normally write

$$\frac{d}{dt}f(W(t)) = \frac{df(W(t))}{dW}\frac{dW(t)}{dt}$$

or

$$df(W(t)) = \frac{df(W(t))}{dW}\frac{dW(t)}{dt}dt$$

or

$$df(W(t)) = \frac{df(W(t))}{dW}dW(t)$$

But $\frac{dW(t)}{dt}$ does not exist

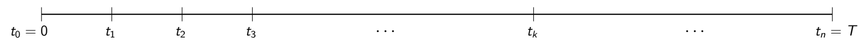
df(t,W(t))

For a function $f(t, W(t))$ of time and $W(t)$, we would write

$$df(t, W(t)) = \frac{\partial f(t, W(t))}{\partial t} dt + \frac{\partial f(t, W(t))}{\partial W(t)} dW(t),$$

But $\frac{dW(t)}{dt}$ does not exist

Partition of interval $[0, T]$



Taylor Series

Given a differentiable function $f(x)$ and two points x_j and x_{j+1}, then

$$f(x_{j+1}) = f(x_j) + f'(x_j)(x_{j+1} - x_j) + \frac{1}{2} f''(x_j)(x_{j+1} - x_j)^2 + \cdots ,$$

where $x_{j+1} = x_j + (x_{j+1} - x_j)$.

For a function $f(t, x(t))$ and points $(t_j , x(t_j))$ and $(t_{j+1} , x(t_{j+1}))$

$$\begin{aligned} f(t_{j+1}, x(t_{j+1})) = {} & f(t_j, x(t_j)) \\ & + f_t(t_j, x(t_j))(t_{j+1} - t_j) + f_x(t_j, x(t_j))(x(t_{j+1}) - x(t_j)) \\ & + \frac{1}{2} f_{tt}(t_j, x(t_j))(t_{j+1} - t_j)^2 \\ & + f_{tx}(t_j, x(t_j))(t_{j+1} - t_j)(x(t_{j+1}) - x(t_j)) \\ & + \frac{1}{2} f_{xx}(t_j, x(t_j))(x(t_{j+1}) - x(t_j))^2 \\ & + \text{higher order terms} \cdots , \end{aligned}$$

where

$$f_t = \frac{\partial f(t,x)}{\partial t}, \qquad f_x = \frac{\partial f(t,x)}{\partial x},$$
$$f_{tt} = \frac{\partial^2 f(t,x)}{\partial t^2}, \qquad f_{tx} = \frac{\partial^2 f(t,x)}{\partial t \partial x},$$
$$f_{xx} = \frac{\partial^2 f(t,x)}{\partial x^2}.$$

Itô's Formula for $f(t, W(t))$

The function $f(x)$ is differentiable, so we can expand it as before

$$\begin{aligned} f(t_{j+1}, W(t_{j+1})) = {} & f(t_j, W(t_j)) \\ & + f_t(t_j, W(t_j))(t_{j+1} - t_j) + f_x(t_j, W(t_j))(W(t_{j+1}) - W(t_j)) \\ & + \frac{1}{2} f_{tt}(t_j, W(t_j))(t_{j+1} - t_j)^2 \\ & + f_{tx}(t_j, W(t_j))(t_{j+1} - t_j)(W(t_{j+1}) - W(t_j)) \\ & + \frac{1}{2} f_{xx}(t_j, W(t_j))(W(t_{j+1}) - W(t_j))^2 \\ & + \text{higher order terms} \cdots , \end{aligned}$$

where

$$f_t = \frac{\partial f(t,x)}{\partial t}, \qquad f_x = \frac{\partial f(t,x)}{\partial W(t)},$$
$$f_{tt} = \frac{\partial^2 f(t,x)}{\partial t^2}, \qquad f_{tx} = \frac{\partial^2 f(t,x)}{\partial t \partial W(t)},$$
$$f_{xx} = \frac{\partial^2 f(t,x)}{\partial W^2(t)}.$$

Itô's Formula for $f(t, W(t))$ (ctd)

Summing, we have

$$
\begin{aligned}
f(T, W(T)) - f(0, W(0)) = & \sum_{j=0}^{n-1}\left[f(t_{j+1}, W(t_{j+1})) - f(t_j, W(t_j))\right] \\
= & \sum_{j=0}^{n-1} f_t(t_j, W(t_j))(t_{j+1} - t_j) \\
& + \sum_{j=0}^{n-1} f_x(t_j, W(t_j))(W(t_{j+1}) - W(t_j)) \\
& + \sum_{j=0}^{n-1} \frac{1}{2} f_{xx}(t_j, W(t_j))(W(t_{j+1}) - W(t_j))^2 \\
& + \sum_{j=0}^{n-1} \frac{1}{2} f_{tt}(t_j, W(t_j))(t_{j+1} - t_j)^2 \\
& + \sum_{j=0}^{n-1} f_{tx}(t_j, W(t_j))(t_{j+1} - t_j)(W(t_{j+1}) - W(t_j)) \\
& + \text{higher order terms} \cdots .
\end{aligned}
$$

Itô's Formula for $f(t, W(t))$ (ctd)

$\lim_{||\Pi|| \to 0}$

In the limit as $||\Pi|| \to 0$, this becomes,

$$
\begin{aligned}
& f(T, W(T)) - f(0, W(0)) = \\
& = \int_0^T f_t(t, W(t))dt \leftarrow \left(\lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} f_t(t_j, W(t_j))(t_{j+1} - t_j)\right) \\
& + \int_0^T f_x(t, W(t))dW(t) \leftarrow \left(\lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} f_x(t_j, W(t_j))(W(t_{j+1}) - W(t_j))\right) \\
& + \frac{1}{2}\int_0^T f_{xx}(t, W(t))dt \leftarrow \left(\lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} \frac{1}{2} f_{xx}(t_j, W(t_j))(W(t_{j+1}) - W(t_j))^2\right) \\
& + 0 \leftarrow \left(\lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} \frac{1}{2} f_{tt}(t_j, W(t_j))(t_{j+1} - t_j)^2\right) \\
& + 0 \leftarrow \left(\lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} f_{tx}(t_j, W(t_j))(W(t_{j+1}) - W(t_j))(t_{j+1} - t_j)\right) ,
\end{aligned}
$$

using arguments very like ones we have seen before.

Itô's Formula for $f(t, W(t))$

So we have,

$$f(T, W(T)) - f(0, W(0)) = \int_0^T df(t, W(t)) = \int_0^T f_t(t, W(t))dt + \int_0^T f_W(t, W(t))dW(t) + \frac{1}{2}\int_0^T f_{WW}(t, W(t))dt$$

In informal differential notation,

$$\boxed{df(t, W(t)) = f_t dt + f_W dW(t) + \frac{1}{2} f_{WW} dt ,}$$

or

$$\boxed{df(t, W(t)) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial W} dW(t) + \frac{1}{2}\frac{\partial^2 f}{\partial W^2} dt .}$$

or, if you like,

$$\boxed{df(t, W(t)) = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}\right) dt + \frac{\partial f}{\partial W} dW(t) .}$$

$\int_0^T WdW$ again

The integral can be quickly evaluated using Itô's formula
Let $f(x) = \frac{1}{2}x^2$. Then

$$
\begin{aligned}
\frac{\partial f(x)}{\partial x} &= f_x(x) = x , \\
\frac{\partial^2 f(x)}{\partial x^2} &= f_{xx}(x) = 1 .
\end{aligned}
$$

If we replace x by W, the Itô formula gives

$$
\begin{aligned}
df(W) &= \underbrace{f_t}_{f_t = 0} dt + f_W dW + \frac{1}{2} f_{WW} dt , \\
&= WdW + \frac{1}{2} \cdot 1 \cdot dt ,
\end{aligned}
$$

So

$$
\begin{aligned}
\int_0^T df(W) = f(W(T)) - f(W(0)) &= \frac{1}{2}(W(T))^2 + \underbrace{0}_{W(0)=0} \\
&= \int_0^T WdW + \int_0^T \frac{1}{2} \cdot dt = \int_0^T WdW + \frac{1}{2}T ,
\end{aligned}
$$

Therefore

$$\int_0^T WdW = \frac{1}{2}W^2(T) - \frac{1}{2}T .$$

Product Rule

Let $X(t, W(t))$ and $Y(t, W(t))$, so that $X \cdot Y$ is a function of t and $W(t)$, too.

$$
\begin{aligned}
d[XY] &= \left(\frac{\partial XY}{\partial t} + \frac{1}{2} \frac{\partial^2 XY}{\partial W^2} \right) dt + \frac{\partial XY}{\partial W} dW \\
&= \ldots \\
&= X\, dY + Y\, dX + \frac{\partial X}{\partial W} \frac{\partial Y}{\partial W} dt \\
&= X\, dY + Y\, dX + dX\, dY
\end{aligned}
$$

Itô Process

An Itô process $X(t)$ is defined

$$X(t) = X(0) + \int_0^t A(t)\, dt + \int_0^t B(t)\, dW\,,$$

or, informally,

$$dX(t) = A(t)\, dt + B(t)\, dW\,.$$

Conditions are imposed on the functions $A(t)$ and $B(t)$

$$
\begin{aligned}
\mathbb{E} \int_0^t B^2(u) du &< \infty\,, \\
\int_0^t |A(u)| du &< \infty\,.
\end{aligned}
$$

Quadratic Variation for $X(t)$

informally

Using the rules we have already described, the quadratic variation can be obtained informally as follows

$$
\begin{aligned}
QV_X(t) &= dX(t) dX(t) \\
&= (A(t)\, dt + B(t)\, dW)^2 \\
&= A^2(t) dtdt + 2A(t)B(t)dWdt + B^2(t)dWdW \\
&= 0 + 0 + B^2(t)dWdW \\
&= B^2(t)dt\,.
\end{aligned}
$$

Integral with respect to Itô Process

We've seen Itô integrals with respect to Brownian Motion:

$$\int_0^t G(u)dW(u)\,.$$

We can also define an integral with respect to an Itô process by splitting up the $A(t)$ and $B(t)$ terms

$$\int_0^t G(u)dX(u) = \int_0^t G(u)A(u)du + \int_0^t G(u)B(u)dW(u)\,.$$

Itô's Formula for $f(t, X(t))$

instead of $f(t, W(t))$

Proceeding as before, we have (replace $W(t)$ by $X(t)$):

$$
\begin{aligned}
f(t_{j+1}, X(t_{j+1})) = f(t_j, X(t_{j+1})) \\
&+ f_t(t_j, X(t_{j+1}))(t_{j+1} - t_j) + f_x(t_j, X(t_{j+1}))(X(t_{j+1}) - X(t_{j+1})) \\
&+ \frac{1}{2} f_{tt}(t_j, X(t_{j+1}))(t_{j+1} - t_j)^2 \\
&+ f_{tx}(t_j, X(t_{j+1}))(t_{j+1} - t_j)(X(t_{j+1}) - X(t_{j+1})) \\
&+ \frac{1}{2} f_{xx}(t_j, X(t_{j+1}))(X(t_{j+1}) - X(t_{j+1}))^2 \\
&+ \text{higher order terms} \cdots ,
\end{aligned}
$$

where

$$
f_t = \frac{\partial f(t,x)}{\partial t}, \qquad f_x = \frac{\partial f(t,x)}{\partial X(t)},
$$

$$
f_{tt} = \frac{\partial^2 f(t,x)}{\partial t^2}, \qquad f_{tx} = \frac{\partial^2 f(t,x)}{\partial t \partial X(t)},
$$

$$
f_{xx} = \frac{\partial^2 f(t,x)}{\partial X(t)^2}.
$$

Itô's Formula for $f(t, X(t))$

So, for $f(t, X(t))$,

$$
\boxed{df(t, X(t)) = f_t(t, X(t))dt + f_x(t, X(t))dX + \frac{1}{2} f_{xx}(t, X(t))dXdX ,}
$$

where

$$
dX(t)dX(t) = B^2(t)dt .
$$

Summary

Itô's Formula for $f(t, X(t))$

For $dX(t) = A(t)\,dt + B(t)\,dW$ and a function $f(t, X(t))$

$$
\boxed{df(t, X) = f_t(t, X)dt + f_x(t, X)dX + \frac{1}{2} f_{xx}(t, X)dXdX ,}
$$

$$
\boxed{df(t, X) = f_t dt + f_x dX + \frac{1}{2} f_{xx} B^2(t)dt ,}
$$

$$
\boxed{df(t, X) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial X} dX + \frac{1}{2} \frac{\partial^2 f}{\partial X^2} B^2(t)dt .}
$$

$$
\boxed{df(t, X) = \left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial X^2} B^2(t) \right) dt + \frac{\partial f}{\partial X} dX .}
$$

Itô's Formula for $f(t, X(t))$

In terms of $W(t)$...

$$
\begin{aligned}
df(t, X) =& f_t(t, X(t))dt + f_x(t, X(t))dX + \frac{1}{2} f_{xx}(t, X(t))B^2(t)dt , \\
=& f_t(t, X(t))dt + f_x(t, X(t))(A(t)\,dt + B(t)\,dW) + \frac{1}{2} f_{xx}(t, X(t))B^2(t)dt , \\
=& f_t(t, X(t))dt + f_x(t, X(t))A(t)\,dt + f_x(t, X(t))B(t)\,dW \\
&+ \frac{1}{2} f_{xx}(t, X(t))B^2(t)dt .
\end{aligned}
$$

Stochastic Differential Equations

Basic Differential Equations
Reminder

Example ($df(t) = \mu f(t)dt$)

Consider the differential equation

$$df(t) = \mu f(t)dt \,,$$

where μ is a constant.
We can rewrite this as

$$\frac{1}{f(t)} df(t) = \mu dt \,,$$

Let $f(t=0) = f_0$. We want a formula for the value of $f(T)$, for $T > 0$, i.e., on the interval $[0, T]$. So again we integrate the equation:

$$\int_0^T \frac{df(t)}{f(t)} = \mu \int_0^T dt = \mu(T-0) \,,$$

$$\begin{aligned} \int_0^T \frac{df(t)}{f(t)} &= \log f(t)\Big|_0^T \\ &= \log f(T) - \log f(0) \\ &= \log \frac{f(T)}{f(0)} = \log \frac{f(T)}{f_0} \text{ (using the properties of } \log x) \end{aligned}$$

Basic Differential Equations

Example ($df(t) = \mu f(t)dt$ (ctd.))

From the previous slide

$$\log \frac{f(T)}{f_0} = \mu T \,,$$

Raising both sides to the power of e (remember e^x and $\exp(x)$ are the same thing)

$$\exp\left(\log \frac{f(T)}{f_0}\right) = \exp(\mu T) \,,$$

Solution:

$$\boxed{f(T) = f_0 e^{\mu T} \,,}$$

because $\exp\left(\log \frac{f(T)}{f_0}\right) = \frac{f(T)}{f_0}$.

Brownian Motion

First regular Brownian Motion, or SDE's of the form

$$dS = \mu(t)dt + \sigma(t)dW \,.$$

In fact we know the solution already, we just integrate to get

$$S(t) - S(0) = \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s)$$

When $\mu(t), \sigma(t)$ are constant, we have

$$S(t) - S(0) = \mu t + \sigma W(t)$$

Geometric Brownian Motion
A special case of general SDE

Consider

$$dS = \mu(t)S(t)dt + \sigma(t)S(t)dW.$$

which is special case of

$$dS = (f(t) + \mu(t)S(t))dt + (g(t) + \sigma(t)S(t))dW\,.$$

with $f(t) = g(t) = 0$
This is known as a homogeneous equation (recall from ODE's) because all the terms depend on S.

Geometric Brownian Motion

From the previous slide

$$dS = \mu(t)S(t)dt + \sigma(t)S(t)dW(t)\,.$$

Need to solve to obtain $S(t)$. Try solution of form $S = e^f$. Then $f = \ln S$. $df(S(t)) = ?$.
Apply Itô:

$$\begin{aligned} f_t &= 0\,, \\ f_S &= \frac{1}{S}\,, \\ f_{SS} &= \frac{-1}{S^2}\,. \end{aligned}$$

Therefore

$$\begin{aligned} df(S) &= f_S dS + \frac{1}{2} f_{SS} dS\, dS\,, \\ &= \frac{1}{S} dS - \frac{1}{2}\frac{1}{S^2} dS\, dS\,, \\ &= \frac{1}{S}\left(\mu(t)S(t)dt + \sigma(t)S(t)dW(t)\right) - \frac{1}{2}\frac{1}{S^2}\sigma^2(t)S^2 dt\,, \\ &= \left(\mu(t) - \frac{1}{2}\sigma(t)^2\right) dt + \sigma(t)dW(t)\,. \end{aligned}$$

Geometric Brownian Motion (ctd)

From the previous slide

$$\begin{aligned} \int_0^T df(S(t)) &= f(T) - f(0) = \log(S(T)) - \log(S(0)) \\ &= \int_0^T \left(\mu(t) - \frac{1}{2}\sigma(t)^2\right) dt + \int_0^T \sigma(t)dW(t)\,. \end{aligned}$$

Therefore

$$S(T) = S(0) \exp\left\{\int_0^T \left(\mu(t) - \frac{1}{2}\sigma(t)^2\right) dt + \int_0^T \sigma(t)dW(t)\right\}$$

When μ, σ are constant, this becomes

$$S(T) = S(0) \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t)\right\}$$

An Interesting Martingale

Consider the Generalised Geometric Brownian Motion with $\mu(t) = 0$

$$S(t) = S(0)\, \exp\left(-\frac{1}{2}\int_0^t \sigma^2(t')dt' + \int_0^t \sigma(t')\, dW(t')\right)\,.$$

Use Ito's formula to derive the SDE for $S(t)$:

$$dS(t) = \sigma(t)\, S(t)\, dW(t)$$

so that

$$S(t) - S(t_0) = \int_{t_0}^t \sigma(t')\, S(t')\, dW(t')$$

and the expectation conditional on information up to t_0 is

$$\mathbb{E}[S(t)|\mathcal{F}(t_0)] = \mathbb{E}[S(t_0)|\mathcal{F}(t_0)] + \mathbb{E}\left[\int_{t_0}^t \sigma(t')\, S(t')\, dW(t')|\mathcal{F}(t_0)\right] = S(t_0).$$

$\Rightarrow$ $S(t)$ is a Martingale – i.e., drift-free (generalised) Geometric Brownian Motion is a Martingale

Distribution of Itô integral with deterministic integrand

An Itô integral

$$I(t) = \int_0^t G(s)dW(s)\,,$$

where $G(t)$ is deterministic is normally distributed with mean 0 and variance $\int_0^t G^2(s)ds$.

Mean $I(t)$ is a martingale, so $EI(t) = I(0) = 0$

Variance $\mathbb{E}I(t) = 0$, so

$$\text{Var}\,[I(t)] = \mathbb{E}I^2(t) = \mathbb{E}\int_0^t G^2(s)ds$$

Distribution If $I(t)$ is normally distributed, then its moment generating function (MGF) will be of the same form as that of a normally distributed random variable
The MGF of a normally distributed random variable, X, with mean 0 and variance σ^2 is

$$\text{MGF}(X) = \mathbb{E}\left[e^{uX}\right] = e^{\frac{1}{2}u^2\sigma^2}$$

If $I(t)$ is normally distributed with mean 0 and variance $\int_0^t I^2(s)ds$, then it should have

$$\text{MGF}(I(t)) = \mathbb{E}\left[e^{uI(t)}\right] = e^{\frac{1}{2}u^2\int_0^t G^2(s)ds}$$

Distribution of Itô integral with deterministic integrand

If

$$\mathbb{E}\left[e^{uI(t)}\right] = e^{\frac{1}{2}u^2\int_0^t G^2(s)ds}\,,$$

then

$$\mathbb{E}\left[e^{uI(t)}\right] e^{-\frac{1}{2}u^2\int_0^t G^2(s)ds} = 1\,.$$

Since $G(t)$ is deterministic, this can be written as

$$\mathbb{E}\left[e^{uI(t)}e^{-\frac{1}{2}u^2\int_0^t G^2(s)ds}\right] = 1\,,$$

or

$$\mathbb{E}\left[e^{uI(t)-\frac{1}{2}u^2\int_0^t G^2(s)ds}\right] = 1\,.$$

Writing $\sigma(t) = uG(t)$, and $I(t) = \int_0^t G(s)dW(s)$,

$$\begin{aligned}
\mathbb{E}\left[e^{u\int_0^t G(s)dW(s)-\frac{1}{2}u^2\int_0^t G^2(s)ds}\right] &= 1\,,\\
= \quad \mathbb{E}\left[e^{\int_0^t uG(s)dW(s)-\frac{1}{2}\int_0^t u^2G^2(s)ds}\right] &,\\
= \quad \mathbb{E}\left[e^{\int_0^t \sigma(s)dW(s)-\frac{1}{2}\int_0^t \sigma^2 ds}\right] &= 1\,,
\end{aligned}$$

Distribution of Itô integral with deterministic integrand

But

$$S(t) = e^{\int_0^t \sigma(s)dW(s)-\frac{1}{2}\int_0^t \sigma^2 ds}\,,$$

is a martingale. Also $S(0) = 1$. Therefore

$$\mathbb{E}(S(t)) = 1\,,$$

as required. So $I(t)$ is normally distributed with mean 0 and variance $\int_0^t G^2(s)ds$.

Euler Method for Ordinary Differential Equations

Consider a (deterministic) initial value problem:

$$\frac{dx(t)}{dt} = a(t,x)\,, \qquad x(t_0) = x_0\,.$$

- It is not possible in general to find explicit solutions to such an equation
- Numerical approximations to the solution are often required
- one type of numerical approximation is a discrete time approximation
- the continuous time differential equation is replaced by a discrete time difference equation which generates values $y_1, y_2, \cdots, y_n, \cdots$ which are approximations to $x(t_1; t_0, x_0), x(t_2; t_0, x_0), \cdots, x(t_n; t_0, x_0), \cdots$, at the time points $t_0 < t_1 < t_2 < \cdots < t_n < \cdots$
- if the differences between the time points, $\Delta_n = t_{n+1} - t_n$ are small enough then the method should be reasonably accurate

Euler Method:

$$y_{n+1} = y_n + a(t_n, y_n)\Delta_n\,,$$

where $y_0 = x_0$.

Euler-Maruyama Method for Stochastic Differential Equations

Consider a SDE

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW\,, \qquad X(t_0) = X_0\,.$$

For $t_0 < t_1 < t_2 < \cdots < t_n < \cdots$, the Euler-Maruyama (discrete-time) approximation to the solution to this SDE is:

$$Y_{n+1} = Y_n + a(t_n, Y_n)(t_{n+1} - t_n) + b(t_n, Y_n)(W(t_{n+1}) - W(t_n))\,,$$

with $Y_0 = X_0$.
Writing $(t_{n+1} - t_n)$ as Δ_n and $(W(t_{n+1}) - W(t_n))$ as ΔW_n, this becomes

$$Y_{n+1} = Y_n + a(t_n, Y_n)\Delta_n + b(t_n, Y_n)\Delta W_n\,,$$

Strong Convergence

For the SDE

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW\,, \qquad X(t_0) = X_0\,.$$

Let $X(T)$ be the solution of the SDE at some time $T > 0$. Let $Y(T)$ be the solution at time T obtained using the Euler-Maruyama method. Additionally assume that $X(T)$ and $Y(T)$ are evaluated using the same underlying Brownian Motion path. We can obtain an estimate of pathwise closeness between the actual solution and the Euler-Maruyama approximation by calculating

$$\epsilon = \mathbb{E}(|X(T) - Y(T)|)\,.$$

The expected value is required by the presence of the Brownian motion term

Weak Convergence

For the SDE

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW\,, \qquad X(t_0) = X_0\,.$$

Let $X(T)$ be the solution of the SDE at some time $T > 0$. Let $Y(T)$ be the solution at time T obtained using the Euler-Maruyama method.
We can obtain an estimate of closeness of moments of the actual solution to those of the Euler-Maruyama approximation may be obtained by calculating

$$|\mathbb{E}[X(T)] - \mathbb{E}[Y(T)]|\,.$$

For the Euler-Maruyama method, the order of strong convergence is generally different from the order of weak convergence

Order of Convergence

Suppose an error ϵ has the following form in terms of a time interval size Δ (assuming equal interval sizes)

$$\epsilon(\Delta) = C\Delta^\gamma\,.$$

Then

$$\log \epsilon = \gamma \log \Delta + \log C\,.$$

Remember Portfolio from Binomial Model

- In the One step Binomial Model, we started at time $t = 0$ with an amount of cash X_0 and purchased Δ_0 units of the underlying asset (borrowing money if necessary to do so).
- This resulted in a portfolio of value $\Delta_0 S_0 + (X_0 - \Delta_0 S_0)$ (at time $t = 0$)
- at time $t = 1$, the value of the portfolio becomes

$$X_1 = \Delta_0 S_1 + (1 + r)(X_0 - \Delta_0 S_0)\,,$$

  where S_1 can be either $S_1(H)$ or $S_1(T)$
- The intent was to choose X_0, Δ_0 such as to replicate the value of the option at $t = 1$, regardless of the value of that option

Rewriting the formula for X_0

From the previous slide, changing the formula so that we move from step $n \to n + 1$ rather than from step $0 \to 1$:

$$\begin{aligned} X_{n+1} &= \Delta_n S_{n+1} + (1 + r)(X_n - \Delta_n S_n)\,, \\ &= \Delta_n S_{n+1} + X_n - \Delta_n S_n + r(X_n - \Delta_n S_n)\,, \end{aligned}$$

so,

$$\begin{aligned} X_{n+1} - X_n &= \Delta_n(S_{n+1} - S_n) + r(X_n - \Delta_n S_n)\,, \\ &= \Delta_n(S_{n+1} - S_n) + 1 \cdot r(X_n - \Delta_n S_n)\,, \end{aligned}$$

where the 1 refers to the fact that interest is being calculated over 1 timestep
We could rewrite this as

$$\Delta(X_n) = \Delta_n \Delta(S_n) + r(X_n - \Delta_n S_n)\underbrace{\Delta t}_{=1}\,,$$

A Portfolio in Continuous Time

- Consider an asset S, which has value $S(t)$ at each time t between 0 and $T > 0$
- Can form a portfolio X of that asset as well as money market investment. The value of the portfolio at time t will be $X(t)$
- At each point in time, t, the investor will hold $\Delta(t)$ units of the asset (where $\Delta(t) \geq 0$ but may be less than 1)
- The remainder of the portfolio, $(X(t) - \Delta(t)S(t))$, is invested in the money market

A Portfolio in Continuous Time

Differential of Portfolio Value

The differential of the portfolio value, $dX(t)$ is

$$dX(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt$$

Compare this with

$$X_{n+1} - X_n = \Delta_n(S_{n+1} - S_n) + 1 \cdot r(X_n - \Delta_n S_n)\,,$$

A Portfolio in Continuous Time
What is $dS(t)$

So, we have,

$$dX(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt\,.$$

What is $dS(t)$?

- Just like in the binomial model, we need some expression for the change in the asset price between "time steps"
- In our discrete time model, we used a binomial tree
- That won't do for continuous time
- We will use Geometric Brownian Motion:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)\,,$$

  where μ and σ are constants in this model

A Portfolio in Continuous Time
Substitute for $dS(t)$

$$\begin{aligned} dX(t) &= \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt\,, \\ &= rX(t)dt + \Delta(t)(\mu - r)S(t)dt + \Delta(t)\sigma S(t)dW(t)\,. \end{aligned}$$

A European Call Option

Consider a European call option with strike price $K > 0$ that pays $(S(T) - K)^+$ at time $T > 0$. We want to find a value for this call option at time $t = 0$. In fact we'll want an expression for the value of the option at time t.Let that value be $c(\cdots)$.

- What should the value $c(\cdots)$ depend on?
- It should depend on K, the strike price (which is known)
- It should probably depend on σ in the asset price model
- It should depend on the risk free rate r
- Finally, it should depend on time t and the price of the option $S(t)$
- The only quantities listed above that vary in interval $[0, T]$ are t and $S(t)$
- So we'll write the option value at time t as $c(t, S(t))$: a formula for option value at time t in terms of asset prices at time t

Probability Measures

- It is sometimes possible to "switch between different probability measures" (remember $(p, q), (\tilde{p}, \tilde{q})$)
- Two "Equivalent" measures agree on events that have probability 0 (and therefore on events that have probability 1)
- Here we will only switch between measures that are equivalent

Changing Probability Measures

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let Z be an almost surely non-negative random variable with $\mathbb{E}Z = 1$. For any $A \in \mathcal{F}$, define

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega) .$$

Then $\widetilde{\mathbb{P}}$ is a probability measure, and X is a nonnegative random variable

$$\tilde{\mathbb{E}}X = \mathbb{E}[ZX]$$

If Z is almost surely positive, and Y is a nonnegative random variable, then

$$\mathbb{E}[Y] = \tilde{\mathbb{E}}[\frac{Y}{Z}]$$

In this notation,

$$\mathbb{E}X = \int_\Omega X(\omega) d\mathbb{P}(\omega) ,$$

and

$$\tilde{\mathbb{E}}X = \int_\Omega X(\omega) d\widetilde{\mathbb{P}}(\omega) ,$$

Equivalent Probability Measures

Definition (Equivalent Probability Measures)

Let Ω be a nonempty set and $\mathcal{F}$ be a σ-algebra of subsets of Ω. The probability measures $\mathbb{P}$, $\widetilde{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ are equivalent if they agree on which sets in $\mathcal{F}$ have probability 0. They will therefore also agree on all sets which have probability 1

Radon-Nikodým

Theorem (Radon-Nikodým)

Let $\mathbb{P}$, $\widetilde{\mathbb{P}}$ be equivalent probability measures defined on $(\Omega, \mathcal{F})$. Then there exists an almost surely positive random variable Z such that $\mathbb{E}Z = 1$ and

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega) \text{ for every } A \in \mathcal{F} .$$

Girsanov

- $\mathbb{P}$ is the "actual" probability measure. $W(t)$ is a Brownian motion under $\mathbb{P}$
- Choose a $\Theta(t)$ and define $d\widetilde{W}(t) = dW(t) + \Theta(t)dt$
- Then you can define a $\widetilde{\mathbb{P}}$
- $d\widetilde{W}(t)$ is a Brownian motion under $\widetilde{\mathbb{P}}$

Girsanov

Theorem (Girsanov)

Let $W(t), 0 \le t \le T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t), 0 \le t \le T$ be a filtration for the Brownian motion. Let $\Theta(t), 0 \le t \le T$ be an adapted process. Define

$$Z(t) = e^{-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)du},$$

and

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)du,$$

where $\mathbb{E}\int_0^t \Theta^2(u)Z^2(u)du < \infty$.
Set $Z = Z(T)$. Then $\mathbb{E}Z = 1$ and under the probability measure $\widetilde{\mathbb{P}}$ given by

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega)d\mathbb{P}(\omega),$$

$\widetilde{W}(t), 0 \le t \le T$ is itself a Brownian motion
Finally, $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent because $Z > 0$, so they agree on sets with probability 0 and 1
$\widetilde{\mathbb{P}}$ will be referred to as the risk-neutral measure

Stock Process under Risk Neutral Measure

Let

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW, \; 0 \le t \le T$$

- Generalised geometric brownian motion
- $\mu(t)$ (mean rate of return), $\sigma(t)$(*volatility*) allowed to be adapted processes
- expression for stock price process

$$S(t) = S(0)e^{\int_0^t \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right)ds + \int_0^t \sigma(s)dW(s)}$$

Discount Process
$D(t)$

Suppose we have an interest rate process $r(t)$. Then the discount process is

$$D(t) = e^{-\int_0^t R(s)ds}.$$

Also letting $D = e^{-I(t)} = f(I(t))$, where $I(t) = \int_0^t R(s)ds$. This means that $dI(t) = r(t)dt$ and $dI(t)dI(t) = 0$.

$$\begin{aligned} dD(t) &= df(I(t)), \\ &= f_t dt + f_I dI + \frac{1}{2}f_{II}dIdI, \\ &= f_I dI, \\ &= -e^{-I(t)}dI = -D(t)dI = -D(t)r(t)dt \end{aligned}$$

- Although $D(t)$ can be stochastic, it has zero quadratic variation
- It has a derivative $(-D(t)r(t))$
- the stock price above with non-zero quadratic variation is "more" random
- the interest rate is reasonably predicatable over short times
- the asset isn't in this model

Differential of Discounted Stock Price

$$\begin{aligned} d(D(t)S(t)) &= dD(t)S(t) + D(t)dS(t) + dD(t)dS(t), \\ &= -D(t)r(t)S(t)dt + D(t)(\mu(t)S(t)dt + \sigma(t)S(t)dW) + 0, \\ &= D(t)S(t)(\mu(t) - r(t))dt + D(t)\sigma(t)S(t)dW, \end{aligned}$$

which we can rewrite is

$$D(t)S(t)\sigma(t)\left(\frac{\mu(t) - r(t)}{\sigma(t)} + dW(t)\right).$$

Rewrite this as

$$D(t)S(t)\sigma(t)(\Theta(t) + dW(t)),$$

where

$$\Theta(t) = \frac{\mu(t) - r(t)}{\sigma(t)}.$$

Discounted Stock Process
Apply Girsanov

Using $\Theta(t)$ defined on the previous slide, we define

$$Z(t) = e^{-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)du},$$

and

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega)d\mathbb{P}(\omega).$$

By Girsanov, we know that

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)du,$$

is a Brownian motion.
Since, $\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)du$,

$$d\widetilde{W} = dW + \Theta(u)dt,$$

so the discounted stock process

$$D(t)S(t)\sigma(t)(\Theta(t) + dW(t)),$$

becomes

$$D(t)S(t)\sigma(t)d\widetilde{W}.$$

Discounted Stock Process under $\widetilde{\mathbb{P}}$
Risk-Neutral Measure

Since $\widetilde{W}$ is a Brownian motion the differential of the discounted stock process is

$$d(D(t)S(t)) = D(t)S(t)\sigma(t)d\widetilde{W},$$

so $D(t)S(t)$ is a martingale under $\widetilde{\mathbb{P}}$
$\widetilde{\mathbb{P}}$ called the risk neutral measure because

- it is equivalent to the original measure $\mathbb{P}$
- and it makes the discounted stock price process $D(t)S(t)$ into a martingale

Undiscounted Stock Process under $\widetilde{\mathbb{P}}$

Stock process is

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW, \; 0 \le t \le T$$

Since

$$d\widetilde{W} = dW + \Theta(u)dt,$$

$$dW = d\widetilde{W} - \Theta(u)dt,$$

So, under $\widetilde{\mathbb{P}}$, $dS(t)$ becomes

$$\begin{aligned} dS(t) &= \mu(t)S(t)dt + \sigma(t)S(t)(d\widetilde{W} - \Theta(u)dt), \\ &= \mu(t)S(t)dt + \sigma(t)S(t)\left(d\widetilde{W} - \frac{\mu(t) - r(t)}{\sigma(t)}dt\right), \\ &= \mu(t)S(t)dt - S(t)\mu(t)dt + S(t)r(t)dt + \sigma(t)S(t)d\widetilde{W}, \\ &= S(t)r(t)dt + \sigma(t)S(t)d\widetilde{W}. \end{aligned}$$

Risk Neutral Measure

The Risk Neutral measure is a measure under which the discounted asset price is a martingale

- Under $\widetilde{\mathbb{P}}$, $S(t)$ is expected to have a return equal to the money market rate
- volatility unchanged
- change of measure has not changed price paths
- change of measure has shifted probabilities of price paths and (usually) lowered the expected rate of return
- similarly in binomial tree, under risk neutral measure, asset values were not changed, but probabilities of taking those values were

Portfolio Process under Risk Neutral Measure
Discounted Portfolio

As before,

$$dX(t) = \Delta(t)dS(t) + r\Big(X(t) - \Delta(t)S(t)\Big)dt\,.$$

The discounted Portfolio is $D(t)X(t)$ which has a differential

$$\begin{aligned}
d\left(D(t)X(t)\right) &= D(t)dX(t) + X(t)dD(t) + dD(t)dX(t)\,,\\
&= D(t)\Big(\Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt\Big) - r(t)X(t)D(t)dt + 0\\
&= D(t)\Delta(t)dS(t) + D(t)r(t)X(t)dt - D(t)r(t)\Delta(t)S(t)dt - D(t)r(t)X(t)dt\\
&= D(t)\Delta(t)dS(t) - D(t)r(t)\Delta(t)S(t)dt\\
&= D(t)\Delta(t)\left[\mu(t)S(t)dt + \sigma(t)S(t)dW(t) - r(t)S(t)dt\right]\\
&= D(t)\Delta(t)\left[(\mu(t) - r(t))S(t)dt + \sigma(t)S(t)dW(t)\right]\\
&= \Delta(t)d\left(D(t)S(t)\right)\,,\\
&= \Delta(t)D(t)S(t)\sigma(t)d\widetilde{W}\,.
\end{aligned}$$

Therefore the discounted portfolio process is a martingale under the risk neutral measure

Zero term on second line arises since $dD(t) = -r(t)D(t)dt$ contains only a dt term, $dD(t)dX(t)$ must be 0.

Pricing Under a Risk Neutral Measure

- The Black-Scholes-Merton equation was derived by asking what inital capital $X(0)$ and portfolio process $\Delta(t)$ was needed for a European call option in order that at final time T,

$$X(T) = (S(T) - K)^+$$

- We can generalize to any derivative product $V(T)$ who's payoff is known at some final time T – i.e., $V(T)$ is a random variable which is $\mathcal{F}(T)$ measurable and represents the derivative payoff at T
- The payoff $V(T)$ may be complex – it may depend on anything that occurs between $t = 0$ and $t = T$
- *Suppose* that we can start with some $X(0)$ and find a $\Delta(t)$ such that $X(T) = V(T)$
- Then from a previous slide $D(t)X(t)$ is a martingale under the risk neutral measure $\widetilde{\mathbb{P}}$, so

$$D(t)X(t) = \tilde{\mathbb{E}}\left[\,D(T)X(T) \mid \mathcal{F}(t)\right]$$

  and since we have $X(T) = V(T)$

$$D(t)X(t) = \tilde{\mathbb{E}}\left[\,D(T)X(T) \mid \mathcal{F}(t)\right] = \tilde{\mathbb{E}}\left[\,D(T)V(T) \mid \mathcal{F}(t)\right]$$

- We define the price of the derivative at time t to be $V(t) = X(t)$, so the above formula can be written as:

$$D(t)V(t) = \tilde{\mathbb{E}}\left[\,D(T)V(T) \mid \mathcal{F}(t)\right], 0 \le t \le T\,.$$

  This is the *risk-neutral* pricing formula
- In particular

$$V(0) = \tilde{\mathbb{E}}[D(T)V(T)]\,,$$

  since $D(0) = e^0 = 1$
- Since $D(t) = e^{-\int_0^t r(u)du}$ is known at time t the risk-neutral pricing formula can be rewritten as

$$V(t) = \tilde{\mathbb{E}}\left[e^{-\int_t^T r(u)du}V(T)\,\middle|\,\mathcal{F}(t)\right], 0 \le t \le T\,.$$

Summary: Risk-Neutral Pricing Formula

$$D(t)V(t) = \tilde{\mathbb{E}}\left[\,D(T)V(T) \mid \mathcal{F}(t)\right], 0 \le t \le T\,,$$

or

$$V(t) = \tilde{\mathbb{E}}\left[e^{-\int_t^T r(u)du}V(T)\,\middle|\,\mathcal{F}(t)\right], 0 \le t \le T\,.$$

At time $t = 0$

$$V(0) = \tilde{\mathbb{E}}[D(T)V(T)]\,,$$

Deriving the Black-Scholes-Merton Formula
Using the risk neutral pricing formula

Consider a European call option on an asset whose dynamics are modelled by Geometric Brownian motion with the volatility σ constant. Let the interest rate r also be constant.
Let $0 \le t \le T$. The payoff of the option at time T is

$$V(T) = (S(T) - K)^+\,,$$

According to the risk neutral pricing formula, the price of the call option at some time $t \le T$ is

$$V(t) = \tilde{\mathbb{E}}\left[e^{-r(T-t)}(S(T) - K)^+\,\middle|\,\mathcal{F}(t)\right], 0 \le t \le T\,.$$

Reminder: Undiscounted Stock Price under $\widetilde{\mathbb{P}}$

Under the risk neutral measure $\widetilde{\mathbb{P}}$,

$$dS(t) = S(t)r(t)dt + \sigma(t)S(t)d\widetilde{W}(t)\,,$$

where $r(t)$ is the money market rate.
Therefore

$$S(t) = S(0)e^{\int_0^t \left(r(s)-\frac{1}{2}\sigma^2(s)\right)ds+\int_0^t \sigma(s)d\widetilde{W}(s)}\,,$$

Finding $S(T)$

Under the risk neutral measure $\widetilde{\mathbb{P}}$, with constant money market rate and constant σ

$$dS(t) = rS(t)dt + \sigma S(t)d\widetilde{W}(t)$$

$$S(t) = S(0)e^{\sigma\widetilde{W}(t)+(r-\frac{1}{2}\sigma^2)t}\,,$$

We can write

$$\begin{aligned} S(T) &= \frac{S(T)}{S(t)}S(t)\,, \\ &= S(t)e^{\sigma(\widetilde{W}(T)-\widetilde{W}(t))+(r-\frac{1}{2}\sigma^2)(T-t)}\,, \\ &= S(t)e^{-\sigma\sqrt{\tau}Y+(r-\frac{1}{2}\sigma^2)\tau}\,, \end{aligned}$$

where $\tau = T - t$, and

$$Y = -\frac{\widetilde{W}(T)-\widetilde{W}(t)}{\sqrt{\tau}} \sim N(0,1)$$

Finding $\tilde{\mathbb{E}}\left[e^{-r(T-t)}(S(T)-K)^+ \,\middle|\, \mathcal{F}(t)\right]$,

$$S(T) = S(t)e^{-\sigma\sqrt{\tau}Y+(r-\frac{1}{2}\sigma^2)\tau}\,,$$

Therefore $S(T)$ depends on two random variables:

- $S(t)$ The asset price which is known at time t
- Y A random variable

$$Y = -\frac{\widetilde{W}(T)-\widetilde{W}(t)}{\sqrt{\tau}}\,,$$

which is independent[1] of all information obtained up to time t

Therefore

$$\tilde{\mathbb{E}}\left[e^{-r(T-t)}(S(T)-K)^+ \,\middle|\, \mathcal{F}(t)\right]\,,$$

becomes

$$\tilde{\mathbb{E}}\left[e^{-r\tau}\left(S(t)e^{-\sigma\sqrt{\tau}Y+(r-\frac{1}{2}\sigma^2)\tau}-K\right)^+\right]\,,$$

Replacing $S(t)$ by x for now, this becomes

$$\tilde{\mathbb{E}}\left[e^{-r\tau}\left(xe^{-\sigma\sqrt{\tau}Y+(r-\frac{1}{2}\sigma^2)\tau}-K\right)^+\right]\,,$$

[1] $\widetilde{W}$ (T) - $\widetilde{W}$ (t) is idependent of $\mathcal{F}(t)$

Calculating the expectation

$$\tilde{\mathbb{E}}\left[e^{-r\tau}\left(xe^{-\sigma\sqrt{\tau}Y+(r-\frac{1}{2}\sigma^2)\tau}-K\right)^+\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-r\tau}\left(xe^{-\sigma\sqrt{\tau}y+(r-\frac{1}{2}\sigma^2)\tau}-K\right)^+ e^{-\frac{1}{2}y^2}dy\,,$$

The quantity

$$\left(xe^{-\sigma\sqrt{\tau}y+(r-\frac{1}{2}\sigma^2)\tau}-K\right)^+ = \left(xe^{-\sigma\sqrt{\tau}y+(r-\frac{1}{2}\sigma^2)\tau}-K\right)\,,$$

when

$$\left(xe^{-\sigma\sqrt{\tau}y+(r-\frac{1}{2}\sigma^2)\tau}-K\right) \geq 0\,,$$

which happens when

$$y \leq \frac{\log\frac{x}{K}+(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} = d_-(x)$$

Therefore

$$\tilde{\mathbb{E}}\left[e^{-r\tau}\left(xe^{-\sigma\sqrt{\tau}Y+(r-\frac{1}{2}\sigma^2)\tau}-K\right)^+\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(x)} e^{-r\tau}\left(xe^{-\sigma\sqrt{\tau}y+(r-\frac{1}{2}\sigma^2)\tau}-K\right)e^{-\frac{1}{2}y^2}dy\,,$$

Calculating the Expectation

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(x)} e^{-r\tau}\left(xe^{-\sigma\sqrt{\tau}y+(r-\frac{1}{2}\sigma^2)\tau}-K\right)e^{-\frac{1}{2}y^2}\,dy =$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(x)} e^{-r\tau}xe^{-\sigma\sqrt{\tau}y+(r-\frac{1}{2}\sigma^2)\tau}e^{-\frac{1}{2}y^2}\,dy - K\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(x)} e^{-r\tau}e^{-\frac{1}{2}y^2}\,dy$$

Taking the first term above,

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(x)} e^{-r\tau}xe^{-\sigma\sqrt{\tau}y+(r-\frac{1}{2}\sigma^2)\tau}e^{-\frac{1}{2}y^2}\,dy &= \frac{x}{\sqrt{2\pi}}\int_{-\infty}^{d_-(x)} e^{-r\tau-\sigma\sqrt{\tau}y+(r-\frac{1}{2}\sigma^2)\tau-\frac{1}{2}y^2}\,dy\\
&= \frac{x}{\sqrt{2\pi}}\int_{-\infty}^{d_-(x)} e^{-\sigma\sqrt{\tau}y-\frac{1}{2}\sigma^2\tau-\frac{1}{2}y^2}\,dy\\
&= \frac{x}{\sqrt{2\pi}}\int_{-\infty}^{d_-(x)} e^{-\frac{1}{2}(2\sigma\sqrt{\tau}y+\sigma^2\tau+y^2})dy\\
&= \frac{x}{\sqrt{2\pi}}\int_{-\infty}^{d_-(x)} e^{-\frac{1}{2}(y+\sigma\sqrt{\tau})^2})dy
\end{aligned}$$

Calculating the Expectation
First term (ctd)

Let

$$I = \frac{x}{\sqrt{2\pi}}\int_{-\infty}^{d_-(x)} e^{-\frac{1}{2}(y+\sigma\sqrt{\tau})^2}\,dy$$

Let $u = y+\sigma\sqrt{\tau}$. Then $du = dy$. Also, when $y\to-\infty$, $u\to-\infty$. When $y = d_-(x)$, $u = d_-(x)+\sigma\sqrt{\tau}$.
With this change of variable,

$$I = \frac{x}{\sqrt{2\pi}}\int_{-\infty}^{d_-(x)+\sigma\sqrt{\tau}} e^{-\frac{1}{2}u^2}\,du = N(d_-(x)+\sigma\sqrt{\tau}),$$

Where $N(z)$ is the cumulative distribution function of the standard normal distribution

$$\begin{aligned}
d_-(x)+\sigma\sqrt{\tau} &= \frac{\log\frac{x}{K}+(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}+\sigma\sqrt{\tau},\\
&= \frac{\log\frac{x}{K}+(r-\frac{1}{2}\sigma^2)\tau+\sigma^2\tau}{\sigma\sqrt{\tau}},\\
&= \frac{\log\frac{x}{K}+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \equiv d_+(x)
\end{aligned}$$

Calculating the Expection
Second term

$$\begin{aligned}
K\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(x)} e^{-r\tau}e^{-\frac{1}{2}y^2}\,dy &= Ke^{-r\tau}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(x)} e^{-\frac{1}{2}y^2}\,dy,\\
&= Ke^{-r\tau}N(d_-(x))
\end{aligned}$$

Calculating the Expection

Combining the terms from the previous slides:

$$\tilde{\mathbb{E}}\left[e^{-r\tau}\left(xe^{-\sigma\sqrt{\tau}Y+(r-\frac{1}{2}\sigma^2)\tau}-K\right)^+\right] = xN(d_+(\tau,x)) - K\exp^{-r\tau}N(d_-(\tau,x))$$

Replacing x by $S(t)$ this becomes:

$$S(t)N(d_+(\tau,S(t))) - K\exp^{-r\tau}N(d_-(\tau,S(t))),$$

where

$$d_\pm(\tau,S(t)) = \frac{\log\frac{S(t)}{K}+(r\pm\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}.$$

Existence of Portfolio Process

- The risk neutral pricing formula is derived under the assumption that a hedging portfolio exists.
- i.e., that a $\Delta(t)$ exists so that, starting with $X(0)$ an investor can invest in the underlying and the money market such that $X(T) = V(T)$
- does such $\Delta(t)$ exist?
- Start with "Martingale Representation Theorem": it says that (under certain assumptions) all martingales look like:

$$M(t) = M(0) + \int_0^t \Gamma(u)dW(u)\,,\ 0 \le t \le T\,,$$

for some $\Gamma(t)$

- Next (under certain assumptions), all martingales under $\widetilde{\mathbb{P}}$ look like

$$\widetilde{M}(t) = \widetilde{M}(0) + \int_0^t \widetilde{\Gamma}(u)d\widetilde{W}(u)\,,$$

for some $\widetilde{\Gamma}(t)$

Martingale Representation Theorem

Theorem (Martingale Representation Theorem)

Let $W(t), 0 \le t \le T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t), 0 \le t \le T$ be a filtration generated by the Brownian motion. Let $M(t), 0 \le t \le T$ be a martingale with respect to this filtration, i.e., for $0 \le s \le t \le T$, $\mathbb{E}[M(t) \mid \mathcal{F}(s)] = M(s)$.
Then there is a process $\Gamma(u)$ such that

$$M(t) = M(0) + \int_0^t \Gamma(u)dW(u)\,,\ 0 \le t \le T\,.$$

Extension to martingale representation theorem

With the additional assumption on the previous slide that the filtration is the one generated by the Brownian Motion, an extension to the Girsanov theorm can be made

Theorem (Girsanov)

Let $W(t), 0 \le t \le T$, be a Brownian motion on a probability space $\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t), 0 \le t \le T$ be a filtration for the Brownian motion. Let $\Theta(t), 0 \le t \le T$ be an adapted process. Define

$$Z(t) = e^{-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)du}\,,$$

and

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)du\,,$$

where $\mathbb{E}\int_0^t \Theta^2(u)Z^2(u)du < \infty$.
Set $Z = Z(T)$. Then $\mathbb{E}Z = 1$ and under the probability measure $\widetilde{\mathbb{P}}$ given by

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega)d\mathbb{P}(\omega)\,,$$

$\widetilde{W}(t), 0 \le t \le T$ is itself a Brownian motion
If $\widetilde{M}(t), 0 \le t \le T$ is a martingale under $\widetilde{\mathbb{P}}$, then there is a process $\widetilde{\Gamma}(u)$ such that

$$\widetilde{M}(t) = \widetilde{M}(0) + \int_0^t \widetilde{\Gamma}(u)d\widetilde{W}(u)$$

▸ Steven E. Shreve.
Stochastic Calculus for Finance II: Continuous-Time Models (Springer Finance).

Existence of Portfolio Process

Assume that the extended form of the Martingale Representation Theorem applies.

- Let $V(T)$ be a random variable, known at time T ($\mathcal{F}$ (T) measurable) which represents the payoff at time T of a derivative product
- Define $V(t)$ by risk-neutral pricing formula

$$D(t)V(t) = \tilde{\mathbb{E}}[D(T)V(T) \mid \mathcal{F}(t)]\,,$$

for $0 \le t \le T$. More generally $D(t)V(t)$ is a Martingale under $\widetilde{\mathbb{P}}$; for $0 \le s \le t \le T$:

$$\begin{aligned}\tilde{\mathbb{E}}[D(t)V(t) \mid \mathcal{F}(s)] &= \tilde{\mathbb{E}}\left[\tilde{\mathbb{E}}[D(T)V(T) \mid \mathcal{F}(t)] \,\middle|\, \mathcal{F}(s)\right]\,,\\ &= \tilde{\mathbb{E}}[D(T)V(T) \mid \mathcal{F}(s)]\,,\\ &= D(s)V(s)\,.\end{aligned}$$

So $D(t)V(t)$ must look like

$$D(t)V(t) = V(0) + \int_0^t \widetilde{\Gamma}(u)d\widetilde{W}(u)$$

Existence of Portfolio Process

Consider a portfolio $X(t)$, with $dX(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)s(t))dt$.
Then from before

$$\begin{aligned} d(D(t)X(t)) &= D(t)dX(t) + X(t)dD(t) + 0\,, \\ &= \Delta(t)d(D(t)S(t))\,, \\ &= \Delta(t)D(t)S(t)\sigma(t)d\widetilde{W}(t)\,. \end{aligned}$$

Integration of this equation gives

$$\int_0^t d(D(t)X(t)) = \int_0^t \Delta(u)D(u)S(u)\sigma(u)d\widetilde{W}(u)\,,$$

which means that

$$\begin{aligned} D(t)X(t) &= D(0)X(0) + \int_0^t \Delta(u)D(u)S(u)\sigma(u)d\widetilde{W}(u)\,, \\ &= X(0) + \int_0^t \Delta(u)D(u)S(u)\sigma(u)d\widetilde{W}(u)\,, \end{aligned}$$

since $D(0) = 1$.

Existence of Portfolio Process

This expression on the previous slide has the same form as the one dictated by the Martingale Representation theorem. In order to make $X(t) = V(t)$, let $X(0) = V(0)$, and let

$$\widetilde{\Gamma}(t) = \Delta(t)D(t)S(t)\sigma(t)$$

or

$$\Delta(t) = \frac{\widetilde{\Gamma}(t)}{D(t)S(t)\sigma(t)}$$

means that $X(t) = V(t)\,,\ 0 \le t \le T$.

Risk-Neutral Measure
Recap

Definition (Risk-Neutral Measure)

A probability measure $\widetilde{\mathbb{P}}$ is said to be risk neutral if

1. $\widetilde{\mathbb{P}}$ an $\mathbb{P}$ are equivalent
2. under $\widetilde{\mathbb{P}}$, the discounted asset price, $D(t)S(t)$, is a martingale

Note the under $\widetilde{\mathbb{P}}$, the discounted portfolio value, $D(t)X(t)$ is also a martingale

- Steven E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models (Springer Finance).* Springer, 1st ed. 2004. corr. 2nd printing edition, June 2004.

Arbitrage

Definition (Arbitrage)

An arbitrage is a portfolio value process $X(t)$ with the property that, for $T > 0$

- $X(0) = 0$
- $\mathbb{P}(X(T) \ge 0) = 1\,,$
- $\mathbb{P}(X(T) > 0) > 0\,.$

- Steven E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models (Springer Finance).* Springer, 1st ed. 2004. corr. 2nd printing edition, June 2004.

First theorem of asset pricing

Theorem (First theorem of asset pricing)

If a market model has a risk-neutral probability measure than it does not allow arbitrage

The existence of a risk neutral measure $\widetilde{\mathbb{P}}$ implies that for any portfolio value process, $X(t)$

$$D(t)X(t) = \tilde{\mathbb{E}}\left[D(T)X(T) \mid \mathcal{F}(t)\right];$$

in particular, $X(0) = \tilde{\mathbb{E}}[D(T)X(T)]$.
Suppose $X(0) = 0$ and $\mathbb{P}(X(T) \geq 0) = 1$. This means that $\mathbb{P}(X(T) < 0) = 0$. There exists a risk-neutral measure $\widetilde{\mathbb{P}}$. Therefore $\widetilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$. Therefore $\widetilde{\mathbb{P}}(X(T) < 0) = 0$. And, under $\widetilde{\mathbb{P}}$, $\tilde{\mathbb{E}}[D(T)X(T)] = X(0) = 0$. This means that $\widetilde{\mathbb{P}}(X(T) > 0) = 0$ since otherwise this would mean $\tilde{\mathbb{E}}[D(T)X(T)] > 0$. Since $\widetilde{\mathbb{P}}(X(T) > 0) = 0$, $\mathbb{P}(X(T) > 0) = 0$. So, $X(t)$ is not an arbitrage. □

- Steven E. Shreve.
  Stochastic Calculus for Finance II: Continuous-Time Models (Springer Finance).
  Springer, 1st ed. 2004. corr. 2nd printing edition, June 2004.

Complete Market Model

Definition (Complete Market Model)

A market model is *complete* if every derivative product can be hedged

- Steven E. Shreve.
  Stochastic Calculus for Finance II: Continuous-Time Models (Springer Finance).
  Springer, 1st ed. 2004. corr. 2nd printing edition, June 2004.

Second theorem of asset pricing

Theorem (Second theorem of asset pricing)

If a market model has a risk-neutral measure, then the market model is complete if and only if the risk-neutral measure is unique

Review of Course Topics