MATA31

Calculus 1, for Mathematical Sciences, Fall 2018

Various Core Proofs/Identities

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1 Identities

Basic Trigonometric Identiteis

Reciprocal Identities

$(\sin x)^{-1} = \csc x$
$(\cos x)^{-1} = \sec x$
$(\tan x)^{-1} = \cot x$

Pythagorean Identities

$\sin^2 x + \cos^x = 1$
$\tan^2 x + 1 = \sec^2 x$
$1 + \cot^x = \csc^2 x$

Angle Addition Identities

$\sin(x \pm y) = \sin x \cdot \cos y \pm \cos x \cdot \sin y$
$\cos(x + y) = \cos x \cdot \cos y - \sin x \cdot \sin y$
$\cos(x - y) = \cos x \cdot \cos y + \sin x \cdot \sin y$
$\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \cdot \tan y}$
$\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \cdot \tan y}$

Hyperbolic Trigonometric Identities

$$\sin h = \frac{e^x + e^{-x}}{2} \qquad \cos h = \frac{e^x - e^{-x}}{2}$$

Derivatives of Trigonometric Functions

Derivative of Trigonometric Functions

$(\sin x)' = \cos x$ $\quad (\csc x)' = -\csc x \cot x$
$(\cos x)' = -\sin x$ $\quad (\sec x)' = \sec x \tan x$
$(\tan x)' = \sec^2 x$ $\quad (\cot x)' = -\csc^2 x$

Derivative of Hyperbolic Trigonometric Functions

$(\sinh x)' = \cosh x$ $\quad (\operatorname{csch} x)' = -\operatorname{csch} x \cdot \coth x$
$(\cosh x)' = \sinh x$ $\quad (\operatorname{sech} x)' = -\operatorname{sech} x \cdot \tanh x$
$(\tanh x)' = \operatorname{sech}^2 x$ $\quad (\coth x)' = -\operatorname{csch}^2 x$

Derivative of Inverse Trigonometric Functions

$(\sin^{-1} x)' = \dfrac{1}{\sqrt{1 - x^2}}$ $\quad (\csc^{-1} x)' = -\dfrac{1}{|x|\sqrt{x^2 - 1}}$
$(\cos^{-1} x)' = -\dfrac{1}{\sqrt{1 - x^2}}$ $\quad (\sec^{-1} x)' = -\dfrac{1}{|x|\sqrt{x^2 - 1}}$
$(\tan^{-1} x)' = \dfrac{1}{1 + x^2}$ $\quad (\cot^{-1} x)' = -\dfrac{1}{1 + x^2}$

2 Uniqueness of Limits

$\lim\limits_{x \to c} f(x) = L \wedge \lim\limits_{x \to c} f(x) = M \implies L = M$

Proof:

Suppose the contrary that:

$\lim\limits_{x \to c} f(x) = L \wedge \lim\limits_{x \to c} f(x) = M, L \neq M$

Assume that $L > M, L = M + K$, WLOG

Let's choose $\epsilon = \dfrac{k}{2}$, this way the intervals do not overlap.

$\lim\limits_{x \to c} f(x) = L : \exists \delta_1 > 0, \ni: 0 < |x - c| < \delta_1 \to |f(x) - L| < \epsilon$

$\lim\limits_{x \to c} f(x) = M : \exists \delta_2 > 0, \ni: 0 < |x - c| < \delta_2 \to |f(x) - M| < \epsilon$

Let $\delta = min(\delta_1, \delta_2)$ so that for any $\delta > 0$, $f(x) \in (M - \epsilon, M + \epsilon)$ and $f(x) \in (L - \epsilon, L + \epsilon)$.

Contradiction: This is impossible since we set $\epsilon = \dfrac{k}{2}$ to guarantee intervals do not overlap.

Therefore, $\lim\limits_{x \to c} f(x) = L \wedge \lim\limits_{x \to c} f(x) = M \implies L = M$ by contradiction.

QED