CSCB36

Summary Of Vassos Hadzilacos'

Course notes for CSC B36/236/240

INTRODUCTION TO THE THEORY OF COMPUTATION

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1 PRELIMINARIES

Sets: A collection of objects **(Elements)**.
If an object a is an element of set A, we say that a is a **member of** A; denoted $a \in A$
The collection that contains no elements is called the **empty** or **null** set' denoted $\varnothing$
Cardinality/Size: Number of elements in a set. The **cardinality** of set A is denoted $|A|$, and is a non-negative integer. If A has a infinite number of elements, $|A| = \infty$, and if $A = \varnothing$, then $|A| = 0$.

Extensional Description: Describing a set by listing its elements explicitly, e.g. $A = \{1, 4, 5, 6\}$
Intentional Description: Describing a set by stating a property that characterizes its elements, e.g. $A = \{x|\text{x is a positive integer less than 5}\}$

Let A and B be sets.
If every element of A is also an element of B,
then A is a **subset** of B ($A \subseteq B$), and B is a **superset** of A ($B \supseteq A$).

If $A \subseteq B$ and $B \subseteq A$, then A is **equal** to B ($A = B$).
If $A \subseteq B$ and $A \neq B$, then A is a **proper subset** of B; ($A \subset B$ and B) is a **proper superset** of A ($B \supset A$).
Note the empty set is a subset of every set, and a proper subset of every set other than itself.

The **union** of A and B ($A \cup B$), is the set of elements that belong to A or B (or both).
The **intersection** of A and B ($A \cap B$), is the set of elements that belong to both A and B.
If no elements belongs to both A and B, their intersection is empty, and they are **disjoint** sets.
The **difference** of A and B, ($A - B$), is the set of elements that belong to A but do not belong to B.
Note that: $A - B = \varnothing \iff A \subseteq B$
The **union** and **intersection** can also be defined for an arbitrary (even infinite) number of sets.
let I be a set of indices, such that for each $i \in I$ there is a set A_i

$$\cup_{i \in I} A_i = \{x : \text{for some } i \in I, x \in A_i\}$$
$$\cap_{i \in I} A_i = \{x : \text{for each } i \in I, x \in A_i\}$$

The **powerset** is the set of subsets, e.g. $A = \{a, b, c\}, \mathcal{P}(A) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$
Partition of a set A, is pairwise disjoint subsets of A whose union is A. A partition of set A is a set $\mathcal{X} \subseteq \mathcal{P}(A)$, such that:

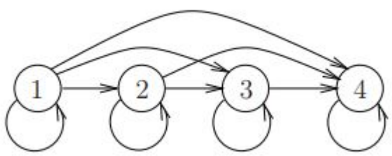
(i) for each $X \in \mathcal{X}, X \neq \varnothing$
(ii) for each $X, Y \in \mathcal{X}, X \neq Y$
(iii) $\cup_{X \in \mathcal{X}} X = A$

Ordered Pair: A mathematical construction that bundles two objects a, b together, in a particular order, denoted (a, b). By this definition, $(a, b) = (c, d) \iff a = c \wedge b = d$ and $(a, b) \neq (b, a)$ unless $a = b$.
We define an ordered pair (a, b) as the set $\{\{a\}, \{a, b\}\}$. We can also define ordered triples as ordered pairs, (a, b, c) can be defined as $(a, (b, c))$. This definition holds for ordered quadruples, quintuples, and ordered n-tuples for any integer $n > 1$.

Cartesian Product of A and B is denoted $A \times B$ and is the set of ordered pairs (a, b) where $a \in A, b \in B$.
$|A \times B| = |A| \cdot |B|$, note that if A, B are distinct nonempty sets, $A \times B \neq B \times A$
The Cartesian product of $n > 1$ sets $A_1, A_2, \ldots, A_n$ denoted $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where $a_i \in A_i, i \in [1, n]$

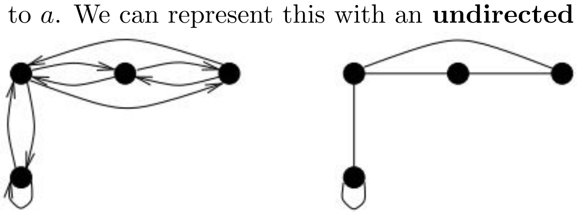
Relation R between sets A and B is a subset of the Cartesian product $A \times B$ ($R \subseteq A \times B$).
Arity: number of sets involved in the relation.
Two relations are **equal** if they contain exactly the same sets of ordered pairs. The two relations must refer to the exact same set of ordered pairs.

A binary relation (**arity** 2) between elements of the **same** set, can be represented graphically as a **directed graph**.
E.g. $R = \{(a, b)|a, b \in \{1, 2, 3, 4\}$ and $a \leq b\}$



R is a **reflexive** relation, if for each $a \in A, (a, a) \in R$, e.g. the relation $a \leq b$ between integers is reflexive, while $a < b$ is not.

R is a **symmetric** relation if for each $a, b \in R, (b, a) \in R$,
E.g. $R_1 = \{(a, b)|a$ and b are persons with atleast one parent in common$\}$ is symmetric. In the directed graph that represents a symmetric relation, whenever there is an arrow from a to b, there is an arrow from b to a. We can represent this with an **undirected graph**.



directed graph on the left, undirected graph on the right
R is a **transitive** relation if for each $a, b, c \in A, (a, b) \in R \land (b, c) \in R \longrightarrow (a, c) \in R$,
E.g. $R = \{(a, b)|a$ and b are persons and a is an ancestor of b$\}$
We see that if a is an ancestor of b, and b is an ancestor of c, then a is an ancestor of c.

R is an **equivalence relation** if it is reflexive, symmetric and transitive,
E.g. $R = \{(a, b)|a$ and b are persons with the same parents$\}$
a and a are the same person, thus have the same parents ($(a, a) \in R$), R is **reflexive**.
a and b share the same parents, b and a share the same parents ($(a, b) \in R, (b, a) \in R$), R is **symmetric**.
a and b share the same parents, b and c share the same parents, a and c must share the same parents. R is **transitive**.
thus we say R is an **equivalence relation**.

Let R be an equivalence relation and $a \in A$. The **equivalence class** of a under R is defined as the set $R_a = \{b|(a, b) \in R\}$, i.e., the set of all elements that are related to a by R.
If R is reflexive, then we know $\forall a \in A, R_a \neq \varnothing$
If R is transitive, then we know $\forall a, b \in R, R_a \neq R_b \longrightarrow R_a \cap R_b = \varnothing$

R is **partial order** if it is anti-symmetric and transitive.
R is **total order** if it is partial order and satisfies for each $a, b \in A$, either $(a, b) \in R$ or $(b, a) \in R$.

Let A and B be sets. A **function** f from A to B is a special kind of relation where each element $a \in A$ is associated with one element in B.
The relation $f \subseteq A \times B$ is a **function** if for each $a \in A$ there is exactly one $b \in B$ such that $(a, b) \in f$.

We write $f : A \to B$ to denote that f is a function from A to B, where A is the **domain** of the function, and B is the **range**.

The function $f : A \to B$ is:
onto/surjective if for every $b \in B$ there is at least one $a \in A$ such that $f(a) = b$
one-to-one/injective if for every element $b \in B$, there is at most one element $a \in A$ such that $f(a) = b$
bijective if it is **one-to-one** and **onto**, if $f : A \to B$ is a bijection, then $|A| = |B|$.
The **restriction** of a function $f : A \to B$ is to a subset A' of its domain, denoted $f|A'$, is a function $f' : A' \to B$ such that for every $a \in A, f'(a) = f(a)$.
An **initial segment** I of $\mathbb{N}$ is a subset of $\mathbb{N}$ with the following property: for any element $k \in I$, if $k > 0$ then $k - 1 \in I$. Thus, an initial segment of $\mathbb{N}$ is either the empty set, or the set $\{0, 1, 2, \ldots, k\}$ for some nonnegative integer k, or the entire set $\mathbb{N}$

let A be a set. A **sequence over** A is a function $\sigma : I \to A$, where I is an initial segment of $\mathbb{N}$.

Intuitively $\sigma(0)$ is the first element in the sequence, $\sigma(1)$ is the second and so on. if $I = \varnothing$, then σ is the **empty** or **null** sequence, denoted ϵ. if $I = \mathbb{N}$ then σ is an infinite sequence; otherwise it is a finite sequence. The **length** of σ is $|I|$ -i.e., the cardinality of I.

let $\sigma : I \to A$ and $\sigma' : I' \to A$ be sequences over the same set A, and suppose that σ is finite.
Informally, the **concatenation** of σ and σ', denoted $\sigma \circ \sigma'$ and sometimes as $(\sigma\sigma')$, is the sequence over A that is obtained by juxtaposing the elements of σ' after the elements of σ.

More precisely, if $I' = \mathbb{N}$ (i.e., σ is infinite), then if we let $j = \mathbb{N}$; otherwise let J be the initial segment $\{0, 1, \ldots, |I| + |I'| - 1\}$. Then $\sigma \circ \sigma'; J \to A$, where for any $i \in I, \sigma \circ \sigma'(i) = \sigma(i)$, and for any $i \in I, \sigma \circ \sigma(|I| + i) = \sigma'(i)$.
Informally, the **reversal** of σ, denoted σ^R, is the sequence of the elements of σ in revresed order, more precisely, $\sigma^R : I \to A$ is the sequence so that, for each $i \in I, \sigma^R(i) = \sigma(|I| - 1 - i)$.

Since, strictly speaking, a sequence is a function of a special kind, sequences $\sigma : I \to A$ and $\sigma' : I \to A$ is equal if and only if, for every $k \in I, \sigma(k) = \sigma'(k)$.

From the definitions of concatenation and equality of sequences, it is easy to verify the following facts:

1. For any sequences $\sigma, \epsilon \circ \sigma = \sigma$; if σ is finite, $\sigma \circ \epsilon = \sigma$.
2. For any sequences σ, τ over the same set, if is finite and $\circ\sigma = \sigma$, then $= \epsilon$; if σ is finite and $\sigma\circ = \sigma$, then $= \epsilon$.

A sequence σ is a **subsequence** of sequence if the elements of σ appear in and do so in the same order. for example $\langle b, c, f\rangle$ is a subsequence of $\langle a, b, c, d, e, f, g\rangle$. Note that we do not require elements of σ to be consecutive elements of , we only require that hey appear in the same order as they do in .

If, in fact, the elements of σ are consecutive elements of , we say that σ is a **contiguous subsequence** of τ.
Formally the definition of the subsequence relationship between sequences is as follows. let A be a set, I and J be initial segments of $\mathbb{N}$ such that $|I| \leq |J|$, and $\sigma : I \to A, J \to A$ be sequences over A. The sequence σ is a subsequence of if there is an increasing function $f : I \to J$ so that, for all $i \in I, \sigma(i) = (f(i))$. If σ is a subsequence of and is not equal to , we say that σ is a **proper subsequence** of .

An **alphabet** is a nonempty set Σ; the elements of an alphabet are called its **symbols**. A **string** (over alphabet Σ) is simply a finite sequence over Σ.
The empty sequence is a string and is denoted, as usual, ϵ. The set of all strings over alphabet Σ is denoted Σ^*. Note that $\epsilon \in \Sigma^*$, for any alphabet Σ.

Operations on strings and languages (From Lecture)
ill
$\Sigma = \{0, 1\}$, A string over Σ is a finite sequence like $x = \langle 1, 0, 1, 1\rangle$
Reversal: $x^R = (1011)^R = 1101$
Concatenation: $x^k = x \cdot x \dots x$ where x is concatenated k times.
Equality: $x = y$ means $|x| = |y|$ and the ith element of x is equal to the ith element of y.
Substring: $y = a_1 a_2 \dots a_i \dots a_j \dots a_n$ and $x = a_i \dots a_j$ then x is a substring. by this definition, every string is a substring of itself.
Proper Substring: an x substring of y where $|x| < |y|$

A **language** over Σ is a nonempty subset of Σ^*
Recall that the set of all strings over alphabet Σ is denoted Σ^*.
A complementation: $\Sigma = \Sigma^* - L$, for a language L.
Union & Intersection: $L \cup L'$, $L \cap L'$
Concatenation: $L \cdot L' = LL = \{xy | x \in L, y \in L'\}$, $\varnothing \cdot L' = \varnothing$
Suppose $|L|, |L'|$ are finite, $|L \cdot L'| \leq |L| \cdot |L'|$. In general
(Kleene) Star: $L^* = \{x | x = \epsilon \vee x = y_1 \cdot y_2 \dots y_k, k \geq 0, y_1, \dots, y_k \in L\}$
For example $L = \{a, b\}, L* = \{\epsilon, a, b, aa, ab, ba, bb, aaa, aab, abb, \dots\}$
Exponentiation: $L^k = L \cdot L \cdot L \cdot \dots \cdot L$ L concatenated 10 times.
$L^k = \begin{cases} \{\epsilon\}, & k = 0 \\ \{L^{k-1} \cdot L, & k > 0 \end{cases}$, $L^* = \Cup_{k \in \mathbb{N}} L^k$
Reversal: (L^R) or $Rev(L) = \{x^R | x \in L\}$

Since strings are simply (finite) sequences over a specified set, various notions defined for sequences apply to string as well. In particular, this is the case for the notion of length which must now be a natural number, and cannot be inf. We use the term **substring** as synonymous to contiguous subsequence.

Let A be a finite set, A **permutation** of A is a sequence in which every element of A appears once and only once. For example if $A = \{a, b, c, d\}$ then $\langle b, a, c, d\rangle, \langle a, c, d, b\rangle \dots$

Sometimes we speak of permutations of a sequence rather than a set. In this case, the definition is as follows: Let $\alpha : I \to A$ and $: I \to A$ be finite sequences over A. The sequence is a permutaiton of α is there is a bijective function $f : I \to I$ so that for every $i \in I, (i) = \alpha(f(i))$

2 INDUCTION

Fundamental properties of the natural numbers
The natural numbers are nonnegative integers $0, 1, \dots$ denoted $\mathbb{N}$;
Principle of well-ordering: Any nonempty subset A of $\mathbb{N}$ contains a minimum element; i.e., $\forall A \subseteq N, \ni$: $A \neq \varnothing, \exists a \in A, \ni$: $\forall a' \in A, a \leq a'$
This applies to all nonempty subsets of $\mathbb{N}$ and to infinite subsets of $\mathbb{N}$. This principle does not apply to other sets.

Simple induction
Let A be any set that satisfies the following properties:
$0 \in A$
$\forall i \in \mathbb{N}, i \in A \Rightarrow i + 1 \in A$
Then A is a superset of $\mathbb{N}$.

Complete induction
Let A be any set that satisfies the following property:

$\forall i \in \mathbb{N}$, if evrey natural number less than $i \in A$ then $i \in A$.
Then A is a supserset of $\mathbb{N}$.
This principle is similar to the principle of simple induction, although there are some differences.
The requirement that $0 \in A$ is implicit as for any $i \in \mathbb{N}, 0 \leq \mathbb{N} \Rightarrow 0 \in A$. The second difference is we require i to be an element of A if all elements preceding i are in A. In contrast, we require $i \in A$ if $i-1 \in A$.

Equivalence of the three principles
Theorem 1.1 The principle of well-ordering, induction, and complete induction are equivalent.
Proof. We prove this by establishing a "cycle" of implications. Specifically, we prove that (a) well-ordering implies induction, (b) induction implies complete induction, and (c) complete induction implies well ordering.
(a) well-ordering implies induction: Assume that the principle of well-ordering holds. We will prove that principle of induction is also true.
let A be a set that satisfies the following:

1. $0 \in A$
2. $\forall i \in \mathbb{N}, i \in A \Rightarrow i+1 \in A$

We suppose that $A \not\supseteq \mathbb{N}$. Then there must exist a nonempty set $A' = \mathbb{N} - A$. By the principle of well ordering, A' has a minimum element a', by (1), $a' \neq 0$ because $0 \in A, a' \notin A$. Thus $a' > 0$ and $a' - 1$ is a natural number. But since a' is the minimum element of A', $a - 1 \notin A'$, and therefore $a' - 1 \in A$. By (2), as $a' - 1 \in A, a' \in A$ and we arrive at a contradiction, thus $A \supseteq \mathbb{N}$

(b) induction implies complete induction.
Again we try to prove that $A \supseteq \mathbb{N}$. To this end, define the set B,
$B = \{i \in \mathbb{N} |$ every natural number smaller than or equal to i is in $A\}$.
In other words, B is the initial segment of $\mathbb{N}$ up to but not including the smallest number that is not in A.
We see that $B \subseteq A$. Thus to prove that $A \supseteq \mathbb{N}$ it suffices to prove that $B \supseteq \mathbb{N}$

1. We have $0 \in B$ because (1) implies $0 \in \mathbb{A}$.
2. For any $i \in \mathbb{N}, i \in B \Rightarrow i+1 \in B$.

These two above satisfy the conditions to use principle of induction (which we assume holds), we have $B \supseteq \mathbb{N}$, as wanted.

(c) Complete induction implies well-ordering: Assume that the principle of complete induction holds. Let A be any subset of $\mathbb{N}$ that has no minimum element. To prove that the principle of well-ordering holds, it suffices to show that A is empty.
For any $i \in \mathbb{N}$, if every natural number less than i is not in A, i is not in A either, otherwise, i would be the minimum element of A, and we are assuming that A has no minimum element. Let $A' = \mathbb{N} - A$. Thus any natural number i is in A if and only if i is not in A'.
Therefore, for any $i \in \mathbb{N}$, if every natural number less than i is in A', then i is also in A'. By the principle of complete induction, $A' \supseteq \mathbb{N}$. Therefore A is empty, as wanted. $\square$
Mathematical induction
Mathematical induction is a proof technique which can often be used to prove that a certain statement is true for all natural numbers.
For example:

1. $P_1(n)$: The sum of all natural numbers up to and including n is equal to $n(n+1)/2$
2. $P_2(n)$: If a set A has n elements, then the powerset of A has 2^n elements.

$\vdots$

In general, a predicate of a natural number n may be true for all values of n, for some values of n, or for no values of n.

Examples of Induction:

Example 1. $S(n) : \sum_{t=0}^{n} t = \frac{n(n+1)}{2}$

BASIS: $n = 0$. In this case, $S(0)$ states that $0 = 0 \cdot 1/2$, which is obviously true. Thus $S(0)$

INDUCTION STEP: Let i be an arbitrary natural number and assume that $S(i)$ holds,

i.e., $\sum_{t=0}^{i} t = \frac{i(i+1)}{2}$. This is our inductive hypothesis.

WTS: $S(i+1)$. i.e., $\sum_{i=0}^{i+1} t = \frac{(i+1)(i+2)}{2}$

$$\begin{aligned}
\sum_{i=0}^{i+1} t &= \sum_{i=0}^{i} t + (i+1) \text{ [Definition of Addition]} \\
&= \frac{i(i+1)}{2} + (i+1) \text{ [IH]} \\
&= \frac{2(i+1) + i(i+1)}{2} \text{ [Algebra]} \\
&= \frac{2i + 2 + i^2 + i}{2} \text{ [Algebra]} \\
&= \frac{i^2 + 3i + 2}{2} = \frac{(i+1)(i+2)}{2} \text{ [Algebra]}
\end{aligned}$$

Therefore $S(i+1)$ as wanted. $\square$

Example 2. $P(k)$: any set of k elements has 2^k subsets.

BASIS: $k = 0$, then our set is $\{\}$ (the empty set), and as the empty set is a subset of itself, it is the only subset. We see that it indeed has $2^0 = 1$ subset(s). Thus, $P(0)$

INDUCTION STEP: Let i be an arbitrary natural number, assume that $P(i)$ holds, that is a set of i elements has exactly 2^i subsets. [IH]

WTS: $P(i+1)$. that is a set of $i+1$ elements has exactly 2^{i+1} subsets.
Consider an arbitrary set with $i+1$ elements, say $A = \{a_1, a_2, \ldots, a_{i+1}\}$. Consider two types of subsets of A : those that contain a_{i+1} and those that do not. let $\mathcal{Y}$ be the set of subsets of A that contain a_{i+1} and $\mathcal{N}$ be the set of subsets of A that do not.
First we see that $\mathcal{N}, \mathcal{Y}$ have the same number of elements. Next we see that they both have exactly i elements.
By the inductive hypothesis, $\mathcal{N}, \mathcal{Y}$ have exactly 2^i elements each.
Finally, we note that these sets of subsets are distinct, that is to say $\mathcal{N} \cap \mathcal{Y}\{\}$ as every subset in $\mathcal{N}$ will not contain a_{i+1} and every subset in $\mathcal{Y}$ will.
Since they are distinct, $|\mathcal{Y} \cup \mathcal{N}| = |\mathcal{Y}| + |\mathcal{N}| = 2^i + 2^i = 2 \cdot 2^i = 2^{i+1}$ as wanted. $\square$

Example 3. For any $m, n \in \mathbb{N}$, there are exactly n^m functions from any set of m elements to any set of n elements.
We prove this by using induction,
$P(m)$: for any $n \in \mathbb{N}$, there are exactly n^m functions from any set of m elements to any set of n elements
or
$Q(n)$: for any $m \in \mathbb{N}$, there are exactly n^m functions from any set of m elements to any set of n elements
Proving either of these would give the desired proposition, but the first predicate is far easier.
BASIS: $m = 0$, $P(0)$ states that for any integer $n \in \mathbb{N}$, there are exactly n^0 functions from any set of m elements to any set of n elements. But since were consider functions that map zero elements to n elements, there is only one function whose domain is empty, the empty function.

Induction Step: Let i be an arbitrary natural number, suppose that for $P(i)$, i.e, there are n^i functions that map from any set of i elements to any set of n elements. [IH]

WTS: $P(i+1)$; i.e., there are n^{i+1} functions that map a set of $i+1$ elements to a set of n elements.
let A be an arbitrary set with $i+1$ elements, say $A = \{a_1, a_2, \ldots, a_{i+1}\}$, and B be an arbitrary set of n elements, say $B = \{b_1, b_2, \ldots, b_n\}$. Let's fix a particular element of a_{i+1}, and group the functions from $A \to B$ according to the element of B which a_{i+1} gets mapped.

$$\begin{aligned}
X_1 &= \text{the set of functions from } A \to B \text{ that map } a_{i+1} \text{ to } b_1\\
X_2 &= \text{the set of functions from } A \to B \text{ that map } a_{i+1} \text{ to } b_2\\
X_3 &= \text{the set of functions from } A \to B \text{ that map } a_{i+1} \text{ to } b_3\\
&\vdots\\
X_n &= \text{the set of functions from } A \to B \text{ that map } a_{i+1} \text{ to } b_n
\end{aligned}$$

Every functions from $A \to B$ belongs to one and only one of these sets. Notice that for any $j \in [1, n]$, X_j contains as many functions there are from the set $A' = \{a_1, a_2, \ldots, a_i\} \to B$. This is because X_j contains the functions from $\{a_1, a_2, \ldots, a_i, a_{i+1}\}$ to B where, however, we have specified the elemt to which a_{i+1} is mapped; the remaining elements of A can get mapped to the elements of B in any possible way, by induction hypothesis, $|X_j| = n^i$, for each $jm \ni: 1 \leq j \leq n$

$$|X_1| + |X_2| + \cdots + |X_n| = n^i + n^i + \cdots + n^i = n \cdot n^i = n^{i+1}, \square$$

Example 4. Let m, n be natural numbers such that $n \neq 0$. The division of m by n yields a quotient a remainder - i.e., unique natural numbers q, r such that $m = nq + r$ where $r < n$.
Define predicate $p(m)$: for any $n \in \mathbb{N}, n \neq 0, \exists q, r \in \mathbb{N}, \ni: m = q \cdot n + r$ where $r < n$.
Basis: $m = 0$, if we let $q = r = 0$ then for any $n \in \mathbb{N}, 0 = 0 \cdot n + 0$, thus $P(0)$
Inductive Step: Let $i \geq 0$ be an arbitrary natural number, and suppose $P(i)$; $\forall n \in \mathbb{N}, \exists q, r \in \mathbb{N}, \ni: i = nq + r, r < n$
WTS: $P(i+1)$ that is for any natural number $n \neq 0$, there are $q', r' \in \mathbb{N}, \ni: i+1 = q' \cdot n + r', r' < n$. Since $r < n$, we have two cases to consider, either $r < n-1$ or $r = n-1$

Case 1. $r < n-1$. In this case we have $i = nq + r$ by [IH], thus $i+1 = nq + r + 1$
let $q' = q, r' = r+1$, since $r < n-1, r' = r+1 < n$, we see that $i+1 = q'n + r'$

Case 2. $r = n-1$. In this case again we have $i = nq + r$ by [IH], where $i+1 = nq + r + 1$ however we have $r = n-1$
Thus $i+1 = nq + n - 1 + 1 = nq + n$, letting $q' = q+1, r' = 0, i+1 = nq' + r'$
We have shown that $P(i+1)$ holds for either case of r now we must show uniqueness.
Suppose that there exists $m, n \in \mathbb{N}, n \neq 0$ let $q, q', r, r' \in \mathbb{N}, \ni: m = q \cdot n + r = q' \cdot n + r', r, r' < n$
$(q-q') \cdot n = r' - r$, lets assume WLOG that $q \geq q' \Rightarrow q - q' > 0 \Rightarrow q - q' \geq 1$ which means that $(q-q') \cdot n \geq n$, but we see that $r - r' \geq n$ must hold which is impossible, since $r' < n$ and $r \geq 0$. Thus $q - q' = 0 = r - r'$
In other words, they are unique, as wanted.

Complete Induction
In carrying out the induction step of an inductive proof, we sometimes discover that to prove $P(i+1)$ holds, it would be helpful to know that $P(j)$ holds for more values of j that are smaller than $i+1$. For example, it might be that to prove $P(i+1)$ holds, we need to use the fact that $P(\lfloor i/2 \rfloor)$ holds and $P(i)$ holds.

Let $P(n)$ be a predicate of natural numbers, $\forall i \in \mathbb{N}, \forall j < i, P(j) \Rightarrow P(i)$

A typical proof of complete induction is as follows:

1. We let $i \in \mathbb{N}$
2. We assume that $P(j)$ for all natural numbers $j < i$ [IH]
3. Using the inductive hypothesis, we prove $P(i)$

Sometimes complete induction takes a different form, where it has explicit base cases. For example odd and even integers.

Bases other than zero

As with simple induction, sometimes we need to prove that a predicate holds, for all natural numbers greater than or equal to some constant c. We can prove such a statement by using variants of complete induction. For all $i \in \mathbb{N}, i \geq c$, if $P(j)$ holds forall $c \leq j < i$, then $P(i)$ holds.

Example 1.

$P(n) : n$ has a prime factorization.

We use complete induction to prove that $P(n)$ holds for all natural $n \geq 2$. let $i \in \mathbb{N}$, $i \geq 2$ and assume $P(j)$ for any $2 \leq j < i$. WTS $P(i)$.

Case 1: i is a prime, then it is a prime factorization of itself, thus $P(i)$ for this case.

Case 2: i is not a prime, then there exists $d \in \mathbb{N}$ such that d divides i where $i = d \cdot a, a \in \mathbb{N}$ for $d \neq 1, d \neq i$,

If $d \neq i$ and $d \in \mathbb{N}$, $d \geq 2$, the same is said about a, we see that $2 \leq a, b \leq i$. We can use our inductive hypothesis, and $P(a)$ as well as $P(b)$ holds, this implies that our i is a product of prime factors, thus $P(i)$.

Therefore, $P(n), \forall n \geq 2$

Example 2.

$P(n)$: Postage of exactly n cents can be made using only 4-cent and 7-cent stamps.

WTS, $P(n)\forall n \geq 18$m Let $i \geq 18 \in \mathbb{N}$ be an arbitrary, assume that $P(j)$ for any $18 \leq j < i$, WTS $P(i)$ holds.

Case 1: $18 \leq i \leq 21$. We can make postage for:

1. 18 cents using 1 4-cent stamp and 2 7-cent stamps.
2. 19 cents using 3 4-cent stamps and 1 7-cent stamp.
3. 20 cents using 5 4-cent stamps and 0 7-cent stamps.
4. 21 cents using 0 4-cent stamps and 3 7-cent stamps.

Case 2: $i \geq 22$ let $j = i - 4$ thus $18 \leq j < i$ and by inductive hypothesis, $P(j)$ holds, this means $j = 4 \cdot k + 7 \cdot l$. However if we let $k' = k + 1$ and $l' = l$, we see that $4 \cdot k' + 7 \cdot l' = 4 \cdot k + 7 \cdot l + 4 = j + 4 = i$

Thus $P(i)$, Thus in both cases, $P(i)$ as wanted.

Example 3. $P(n)$: if a full binary tree has n nodes, then n must be odd.

Let i be an arbitrary integer $i \geq 1$, suppose that $P(j)$ holds for all positive integers $j < i$. That is, for any positive integer $j < i$, if a full binary tree has j nodes, then j is an odd number. WTS $P(i)$, let T be an arbitrary binary tree with i nodes, WTS i is odd.

Case 1. $i = 1$, $P(i)$ for this case because T has only one node.

Case 2. $i > 1$

3 7.2 Regular Expressions

Let Σ be an alphabet, Want to define the set regexes over Σ
- set of strings over Γ, where $\Sigma \cup \{(,),+,*,\epsilon,\varnothing\}$

Definition: Let $\mathcal{RE}$ let this be the smallest set. $\mathcal{RE}$ stands for regular expression.
Basis: $\epsilon, \varnothing; a \in \mathcal{RE}, \forall a \in \mathbb{Z}$
Induction Step: if $R, S \in \mathcal{RE}$, then $(RS), (R+S), R* \in \mathcal{RE}$
Syntax $\equiv$ Grammar, and Semanticsof regxes:
Given a regex R, wants to define the language of R $\mathcal{L}(R)$, i.e., if the set of strings that matches R
Define $\mathcal{L}(R)$ by structural induction.
Basis: $\mathcal{L}(\epsilon) = \{\epsilon\}$, $\mathcal{L}(\varnothing) = \varnothing, \mathcal{L}(a) = \{a\}, \forall a \in \Sigma$
If $R = (ST)$, then $\mathcal{L}(ST) = \mathcal{L}(S) \cdot \mathcal{L}(T)$
If $R = (S+T)$, then $\mathcal{L}(R) = \mathcal{S} \cup \mathcal{T}$
If $R = (s^*)$, then $\mathcal{L}(R) = \mathcal{L}(R)^*$
For example $1(1+0)^*$, $\{1\}, \{0\}$ by base cases of $\mathcal{L}(a) = \{a\}$
Thus $1+0 = \{0,1\}$, ofc $(1+0)^* = \{0,1\}*$
Finally $\{1\}\{0,1\}^*\{0\}$ is the set of all binary sequences starting from, ending with 0 with anything in between that is a binary sequence.
now $111^* \neq (111)^*$