Lecture Notes
Winter 2019

MATA37 - CALCULUS II FOR THE MATHEMATICAL SCIENCES

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UNIVERSITY OF TORONTO SCARBOROUGH

Instructor:

	Dr. Kathleen Smith
Email:	smithk@utsc.utoronto.ca
Office:	IC458
Office Hours:	TBA

1 Riemann Definition

Example 1 - Algebraic

$\int_0^1 (x^2 - 6x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} (\frac{i^3}{n^3} - \frac{6i}{n}) \cdot \frac{1}{n}$ by R. def of the definite integral.

$$\begin{aligned}
\sum_{i=1}^{n} (\frac{i^3}{n^3} - \frac{6i}{n}) \cdot \frac{1}{n} &= \frac{1}{n}\sum_{i=1}^{n} (\frac{i^3}{n^3} - \frac{6i}{n}) \\
&= \frac{1}{n}(\sum_{i=1}^{n} \frac{i^3}{n^3} - \sum_{i=1}^{n} \frac{6i}{n}) \\
\lim_{n\to\infty} \frac{1}{n}(\sum_{i=1}^{n} \frac{i^3}{n^3} - \sum_{i=1}^{n} \frac{6i}{n}) &= \lim_{n\to\infty} \frac{1}{n}(\frac{1}{n^3}\sum_{i=1}^{n} i^3 - \frac{6}{n} \cdot \sum_{i=1}^{n} i) \\
&= \lim_{x\to\infty} (\frac{1}{4} \cdot \frac{(n+1)^2}{n} - 3 \cdot \frac{n+1}{n}) = -\frac{11}{4}
\end{aligned}$$

Example 2 - Geometric

$\int_{-1}^{2} (1 + |x|)dx$

In theory, we can evaluate this as a Riemann sum limit, but it's far easier just to think of these as shapes.

We know what the graph of this looks like, simply draw out the graph and compute the area using elementary geometry.

Example 3 - Express as an Integral

$\lim_{x\to\infty} \frac{5}{n}(\sum_{i=1}^{n} \frac{6 + \frac{5i}{n}}{\sqrt{4 + \frac{5i}{n}}})$ we need a, b and $f(x)$

1. Decide what kind of Riemann Sum we are using.

Consider the i term, is there any (i-1)s or averages? Since there are only i's in the term by itself, this must be a right Riemann Sum.

2. Redistribute any leading coefficients.

$\lim_{x\to\infty} \frac{5}{n}(\sum_{i=1}^{n} \frac{6 + \frac{5i}{n}}{\sqrt{4 + \frac{5i}{n}}}) = \sum_{i=1}^{n} \frac{6 + \frac{5i}{n}}{\sqrt{4 + \frac{5i}{n}}}) \cdot \frac{5}{n}$

3. let $\Delta x = \frac{5}{n}$

$\sum_{i=1}^{n} \frac{6 + \frac{5i}{n}}{\sqrt{4 + \frac{5i}{n}}}) \cdot \frac{5}{n} = \sum_{i=1}^{n} \frac{2 + 4 + i\Delta x}{\sqrt{4 + i\Delta x}}) \cdot \Delta x$

we know now that $x_i = 4 + \Delta x$ which means $a = 4$, $b = 9$ and so,

we have: $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2 + x_i}{\sqrt{x_i}} \Delta x$

and so: $f(x_i) = \frac{2 + x_i}{\sqrt{x_i}} \Leftrightarrow f(x) = \frac{2 + x}{\sqrt{x}}$

Lastly, $\lim_{x\to\infty} \frac{5}{n}(\sum_{i=1}^{n} \frac{6 + \frac{5i}{n}}{\sqrt{4 + \frac{5i}{n}}}) = \int_4^9 (\frac{2 + x}{\sqrt{x}})dx$

Remark: The Riemann definition of the definite integral has serious problems in practical use, in theory it makes sense so long as the function is continuous across [a,b] or if function fhas a finite number of jump discontinuity.

Computing the riemann definition is also not very practical, as we would be stuck if we had a non polynomial integrand or a polynomial integrand of power greater than 3.

2 Theorem

Properties of the definite integral

Pages 344, 346, 347, and material from Kathleen Smith's brain.

Let $a, b \in \mathbb{R}, a < b$. Suppose that functions f and g are integrable.

1. $f(x) \geq 0$ on [a,b] $\Rightarrow \displaystyle\int_a^b f(x)dx \geq 0$

   $f(x) \leq 0$ on [a,b] $\Rightarrow \displaystyle\int_a^b f(x)dx \leq 0$

2. $\displaystyle\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

3. $\forall c \in \mathbb{R}, \displaystyle\int_a^b c \cdot f(x)dx = c \cdot \int_a^b f(x)dx$

4. $\displaystyle\int_a^a f(x)dx = 0$

5. $\displaystyle\int_a^b f(x)dx = \int_b^a -f(x)dx$

6. Union Integral Property $\forall c \in (a, b), \displaystyle\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

   Extremely useful for case defined functions.

7. $f(x) \leq g(x)$ on $[a, b] \Rightarrow \displaystyle\int_a^b f(x)dx \leq \int_a^b g(x)dx$

8. Integral Inequality $\exists m, M \in \mathbb{R}, \ni: m \leq f(x) \leq M, \forall x \in [a, b] \Rightarrow m(b-a) \leq \displaystyle\int_a^b f(x)dx \leq M(b-a)$

3 Proofs

Let $a, b \in \mathbb{R}, a < b$. Suppose that functions f and g are integrable on [a, b].

$\forall c \in \mathbb{R}$

Consider $\displaystyle\int_a^b [f(x) - c \cdot g(x)]dx$

let $P = \{x_1, x_2, x_3 \dots x_n\}$ be a Riemann Partition of $[a, b]$

$\displaystyle\int_a^b f(x) = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i)\frac{1}{n}$

$\displaystyle\int_a^b g(x) = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i)\frac{1}{n}$

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$\displaystyle\int_a^b [f(x) - c \cdot g(x)]dx = \int_a^b f(x)dx - c\int_a^b g(x)dx$

Consider RHS

$\displaystyle\int_a^b f(x)dx - c\int_a^b g(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i)\frac{1}{n} - c \cdot \lim_{n\to\infty} \sum_{i=0}^{n} g(x_i)\Delta x$ By Riemann Sums Definition

$\displaystyle\int_a^b f(x)dx - c\int_a^b g(x)dx = \lim_{n\to\infty} (\sum_{i=0}^{n} f(x_i) - c \cdot \sum_{i=0}^{n} g(x_i))\Delta x$ By Limit Laws

$\displaystyle\int_a^b f(x)dx - c\int_a^b g(x)dx = \lim_{n\to\infty} (\sum_{i=0}^{n} [f(x_i) - c \cdot g(x_i)])\Delta x$ By Summation Laws

$\displaystyle\int_a^b f(x)dx - c\int_a^b g(x)dx = \int_a^b [f(x) - c \cdot g(x)]dx$ By Riemann Sums Definition

End of lecture cliffhanger

Consider

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \in \mathbb{I} \end{cases}$$

Does the Definite Integral exist?