Week 8, LEC02 - November 9th

Discrete Mathematics, Fall 2018

CSCA67 - Lecture Notes

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$S(n)$: Given n lines in the plane such that no two are parallel, and no three intersect at a single point, we have $\binom{n}{3}$ Triangles.

Claim: $\forall n \in \mathbb{N}, n \geq 3, s(n)$

Base $n = 3$, by observation(inspection), given three lines only one triangle can be formed.

$\binom{3}{3} = 1$, $S(n)$ for n= 3.

Inductive Hypothesis: $k \in \mathbb{N}, k \geq 3, S(k)$

Inductive Step: $S(k) \to S(k+1)$

With k lines in the plane, we have $\binom{k}{3}$ triangles, by Inductive Hypothesis.

When the $(k+1)^{st}$ line is added, it must cross every existing pair of lines in the plane.

When this happens, a new triangle is formed.

we have $\binom{k}{2}$ ways of selecting 2 lines from k lines.

$\therefore$ Total number of Triangles after the $(k+1)^{st}$ line is added is $\binom{k}{3}$(IH) $+\binom{k}{2}$(New triangle from $(k+1)^{st}$ line).

$= \dfrac{k!}{(k-3)!3!} + \dfrac{k!}{(k-2)!2!} = \dfrac{(k+1)!}{(k+1-3)!3!} = \binom{k+1}{3}$. $S(k+1-)$

Conclusion: Since k is arbitrary, $\forall n \in \mathbb{N}, n \geq 3, S(n)$

This question will re-appear on assignment 2!

1 Counting with Repetitions

The genetic code of an organism stored in DNA molecules consist of 4 nucleotides:

Adenine, Cytosine, Guanine and Thymine

- It is possible to sequece short strings of molecules.

- One way to sequence the nucleotides of a longer string of DNA is to split the string into shorter sequences.

- A C-enzyme will split a DNA-sequence at each C. This means that each fragment will end at a C except possibly the last fragment.

- Similarly for A-enzymes, G-enzymes and T-enzymes.

- If the original nucleotide is split on each of C, A, G and T then it can be sequenced as it is most likely a unique sequence that can be constructed by each of the four sets of fragments.

Example. Given a 20-nucleotide string split at the Cs, one might have the fragments:

AC, AC, AAATC, C, C, C, TATA, TGGC

Q. How many different 20-nucleotide strings could have given rise to the above set of fragments? In other words, how many different arrangements are there of these fragments?

A. There are 8 bins with TATA at the end as it doesn't end with a C.

So we have 7 fragments to place into the 7 bins.

first, let's place the three C's first in $C(7,3)$ ways

Next, let's place the ACs next in $C(4,2)$ ways

then, let's place the AAATC next in $C(2,1)$ ways

lastly, we have TGGC in $C(1,1)$ ways

Total $= C(7,3) \cdot C(4,2) \cdot C(2,1) \cdot C(1,1)$
Total $= \frac{7!}{3!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{1!1!} \cdot \frac{1!}{1!}$
Total $= \frac{7!}{3!2!1!1!}$

Given n objects, with r_1 of type 1, r_2 of type 2, . . . , r_m of type m where

$$r_1 + r_2 + r_3 \cdots + r_m = n$$

then the number of arrangements of the n objects, denoted by $P(n; r_1, r_2, \ldots r_m)$ is:

$$\binom{n}{r_1}\binom{n-r_1}{r_2}\binom{n-r_1-r_2}{r_3}\cdots\binom{n-r_1-r_2\cdots-r_{m-1}}{r_m}$$

Q. What does this formula simplify to?

A. $\frac{n!}{r_1!r_2!\ldots r_m!} = P(n; r_1, r_2, \ldots r_m)$

2 Selections With Repetitions

Example. While shopping at the St. Lawrence market, you decide to buy half a dozen bagels. There are three flavours to choose from.

Q. How many different ways can you select your 6 bagels?

A. Rephrase as an arrangement problem, such as 2 Sesame, 3 Poppy Seed and 1 plain.

Q. How is this an arrangement problem?

A. 8 bins, and two partitions. Everything between the start and first partition is a sesame, everything between the first and second partition is a poppy seed, and the last partition to the right will be plain bagels.

sesame	sesame	**partition**	poppy seed	poppy seed	poppy seed	**partition**	plain

Q. How can we also think of this as a selection problem?

A. $C(8,2) = C(8,6)$.

$C(8,2) = C(6+(3-1), 2)$, (bagels + partitions)

Let n be the number of choices, r the number of items selected. Then $8 = r + (n-1)$ and we are choosing r = 6.

This results in: $\dfrac{\text{items} + (\text{choices} - 1)}{(\text{choices} - 1)}$

Example How many ways are there to select a committee of 15 politicians from a room full of:

indistinguishable Democrats,

indistinguishable Republicans

indistinguishable Independents

if every party must have at least two members on the committee?

Solution

First choose two members from each party, this leaves us with 9 members to select. $15 - (3 \cdot 2) = 9$

This is just the bagel problem!

$C(9(3-1), (3-1))$