Sample Midterm

CSCA67 - Discrete Mathematics

Instructors:

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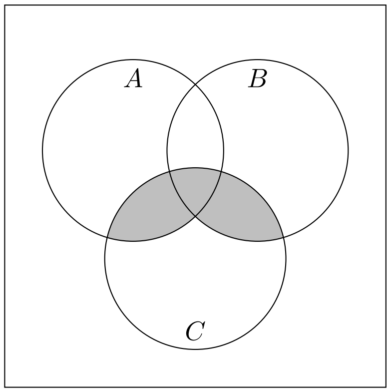
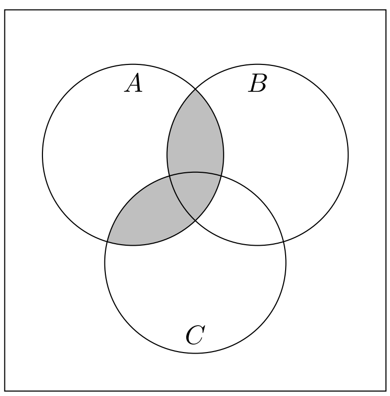
Question 1.

Part (a) Each option below is a predicate formula. Circle all (I bolded ones that i think are correct) answers equivalent to the given.



(a) $a \wedge b \wedge c$
(b) $(a \wedge c) \vee (b \wedge c)$
(c) $(\neg b \rightarrow a) \wedge c$
(d) $c \wedge (b \vee a)$
(e) None

Part (b) Write a propositional statement using $\rightarrow$ and $\neg$ if neccesary equivalent to regions 3, 5, and 6. marks you can leave your solution using $\wedge, \vee, \neg$ and $\rightarrow$. You do not need to specify which equivalence laws you use if you need them.



$(a \wedge b) \vee (a \wedge c)$

$a \wedge r(b \vee c)$

$\neg(a \rightarrow \neg(\neg b \rightarrow c))$

Part (c) Circle (I just bolded.) every statement equivalent to $\neg a \rightarrow b$.

(a) $\neg b \rightarrow a$

(b) $a \rightarrow \neg b$

(c) a is sufficient for $\neg b$

(d) $\neg(a \wedge \neg b)$

(e) b is sufficient for $\neg a$

(f) a is neccesary for $\neg b$

(g) $\neg a$ is neccesary for b

(h) b if $\neg a$

(i) $b \vee a$

Question 2.

Determine whether $\forall$ can be factored from an implication. In other words is

$$\forall x \in X, (p(x) \rightarrow q(x)) \iff \forall x \in X, p(x) \rightarrow \forall x \in X, q(x)$$

true? Explain your reasoning. Marks will only be given for your explanation.

No.

let $p(x) = x > 5$ and $q(x) = x^2 > 25$

$$\forall x \in \mathbb{Z}, (p(x) \rightarrow q(x)) \iff \forall x \in \mathbb{Z}, (x > 5 \rightarrow x^2 > 25)$$

which means: for all integers x, if x is greater than 5, then x^2 is greater than 5.

vs.

$$\forall x \in \mathbb{Z}, p(x) \rightarrow \forall x \in \mathbb{Z}, q(x) \iff \forall x \in \mathbb{Z}, x > 5 \rightarrow \forall x \in \mathbb{Z}, x^2 > 25$$

which means: if all integers are greater than 5, then all integers squared are greater than 25.

Sometimes, factoring in $\forall$ can lead to a statment that is true, but never equivalent.

$\forall x \in X, (p(x) \rightarrow q(x))$ is saying for every x, if predicate p is true, then predicate q is true.

$\forall x \in X, p(x) \rightarrow \forall x \in X, q(x)$ is saying for every x, if predicate p is true, then for every x, predicate q is true, since are not equivalent, therefore $\forall$ cannot be factored into or from an implication.

Question 3.

Answer the following questions to construct a direct proof that:

$$\forall n \in \mathbb{N}, n \geq 2, \forall a, b \in \mathbb{N}, a \equiv_n b \rightarrow a^2 \equiv_n b^2$$

Part (a)

Write a and b in terms of n using the division theorem and taking into consideration any common variables.

$$a = k_1 \cdot n + r$$
$$b = k_2 \cdot n + r$$
$$k_1, k_2 \in \mathbb{Z}, r \in [0, (n-1)]$$

Part (b)

Now complete the proof.

Assume $a \equiv_n b$, then that means:

$$\begin{array}{ll} a = k_1 \cdot n + r & a^2 = {k_1}^2 \cdot n^2 + 2k_1 \cdot n + r^2 \\ b = k_2 \cdot n + r & b^2 = {k_2}^2 \cdot n^2 + 2k_2 \cdot n + r^2 \\ k_1, k_2 \in \mathbb{Z}, r \in [0, (n-1)] & \\ \hline \end{array}$$

$$a^2 = {k_1}^2 \cdot n^2 + 2k_1 \cdot n + r^2 = n({k_1}^2 \cdot n + 2k_1) + r^2$$

$$b^2 = {k_2}^2 \cdot n^2 + 2k_2 \cdot n + r^2 = n({k_2}^2 \cdot n + 2k_2) + r^2$$

Since both a^2 and b^2 share the same remainder r^2 when divided by n,

$\therefore \forall n \in \mathbb{N}, n \geq 2, \forall a, b \in \mathbb{N}, a \equiv_n b \rightarrow a^2 \equiv_n b^2$. *QED*

Question 4.

Prove that $\forall n \in \mathbb{N}, (n^2 - 1 \not\equiv_4 0) \rightarrow n$ is even.

Assume the contrapositive
$n \in \{2k+1\}, k \in \mathbb{Z} \rightarrow (n^2 - 1 \equiv_4 0)$

$$\begin{aligned}
&n = 2k+1, k \in \mathbb{Z} \\
&n^2 = 4k^2 + 4k + 1 \\
&n^2 - 1 = 4k^2 + 4k = 4(k^2 + k) \\
&n^2 - 1 \equiv_4 0
\end{aligned}$$

Since we prove the contrapositive to be true, and the contrapositive is logically equivalent to the original,
$\therefore n \in \{2k+1\}, k \in \mathbb{Z} \rightarrow (n^2 - 1 \equiv_4 0)$ QED

Question 5.

Part (a) Write the following claim as an implication using quantifiers and $\rightarrow$. You may use p(x, y) to denote that x and y are relatively prime.

a and n relatively prime is necessary for there to exist a unique natural number $b < n$ such that $a \cdot b \equiv_n 1$.

$$\exists n \in \mathbb{N}, \ni: b < n, a \cdot b \equiv_n 1 \rightarrow gcd(a, n) = 1$$

Part (b) Prove the claim.
Assume: $\exists n \in \mathbb{N}, \ni: b < n, a \cdot b \equiv_n 1$

$a \cdot b = n \cdot k + 1, k \in \mathbb{Z}$
$a \cdot b - n \cdot k = 1$

Bezout's Identity:
if a and b are non zero integers, then there exists integers u and v such that:
$gcd(a, b) = au + bv$

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then by Bezout's Identity, since $a \cdot b - n \cdot k$ is some linear combination that equals 1, 1 must be $gcd(a, n)$
$a \cdot b - n \cdot k = 1 \longrightarrow gcd(a, n) = 1$

$\therefore \exists n \in \mathbb{N}, \ni: b < n, a \cdot b \equiv_n 1 \rightarrow gcd(a, n) = 1$ QED

Question 6.

Prove the statement $S(n) : \forall n \in \mathbb{N}, \exists a, b \in \mathbb{N}, \ni: 3a + 5b = n$

Predicate:

$S(n) : \forall n \in \mathbb{N}, n \geq 8 \exists a, b \in \mathbb{N}, \ni: 3a + 5b = n$

Base Case:

$S(8) : 8 = 3a + 5b = 3(1) + 5(1)$

$S(9) : 9 = 3a + 5b = 3(3) + 5(0)$

$S(10) : 10 = 3a + 5b = 3(0) + 5(2)$

Inductive Hypothesis:

Suppose $S(k)$, for some arbitrary $k \in [8, n)$

$S(k) : k = 3a + 5b$

Inductive Step:

$S(k) \to S(n + 3)$

$k = 3a + 5b$ Suppose $P(n)$ by **Inductive Hypothesis**

$k + 3 = 3a + 5b + 3$

$k + 3 = 3(a + 1) + 5b$

Since k is arbitrary, $\forall n \in \mathbb{N}, S(k)$

QED