MAT B24 TUT0018, Tutorial 5 (Week 6)
Thursdays 3 - 5pm

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Theorem 3.10

Let V and V' be finite-dimensional vector spaces with basis B = (b1, b2 ... bn) and B' = (b'1, b'2, ... b'm) respectively.

Let T: V -> V'.
Let T': R^n -> R^m, T'([v]B) -> [T(v)]B'

Then the standard matrix representation T'

$$A = \begin{bmatrix} | & | & & | \\ [T(\vec{b}_1)]_{B'} & [T(\vec{b}_2)]_{B'} & \cdots & [T(\vec{b}_n)]_{B'} \\ | & | & & | \end{bmatrix}$$

24. Let $T: P_3 \to P_2$ be defined by $T(p(x)) = p'(2x + 1)$, where $p'(x) = D(p(x))$, and let $B = (x^3, x^2, x, 1)$ and $B' = (x^2, x, 1)$.

a. Find the matrix A.

b. Use A to compute $T(4x^3 - 5x^2 + 4x - 7)$.

T' ([v]B) = T(v)B'

T(x^3) = 3(2x + 1)^2 = 3(4x^2 + 4x + 1)

T(x^2) = 2(2x + 1) = 4x + 2

T(x) = 1

T(1) = 0

T(x^3)B' = [12, 12, 3]

T(x^2)B' = [0, 4, 2]

T(x)B' = [0, 0, 1]

T(1)B' = [0, 0, 0]

$$A = \begin{bmatrix} 12 & 0 & 0 & 0 \\ 12 & 4 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

$T(v) = T'([v]B) = A[v]B$

$[v]B = [4, -5, 4, -7] \Rightarrow T(v) = A\,[4, -5, 4, -7]$

26. Let $T: P_3 \to P_3$ be the linear transformation defined by $T(p(x)) = D^2(p(x)) - 4D(p(x)) + p(x)$. Find the matrix representation A of T, where $B = \{x, 1 + x, x + x^2, x^3\}$.

$B = B'$

$(x+1)^2 \quad (x+1)(x+1)$

$= x^2$

A = [[T(x)B], [T(1 + x)]B, [T(x + x^2)]B, [T(x^3)]B]

$$A = \begin{bmatrix} | & | & | & | \\ (-4+x)_B & (-3+x)_B & (-2-7x+x^2)_B & (6x+12x^2+x^3)_B \\ | & | & | & | \end{bmatrix}$$

Additional Problems

Problem one. Suppose that $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for the vector space V, that $C = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ is a basis for the vector space W, and that $T : V \to W$ is a linear transformation. Show that T is an invertible transformation if and only if the matrix representation $[T]_B^C$ (or denoted by $R_{B,C}$) of T with respect to the bases B and C is an invertible matrix.

WTP

R_B,C invertible <=> T is invertible

$R^{-1}(B,C)$

Assume that R_B,C is invertible,
this means there exists some matrix A such that R_B,C A = I.
Then let S: W -> V, and S(w) = Aw. Then S = T^(-1)
S(T (v)) = R_B,C * A * v = I * v

Assume that T is invertible, if T is invertible then there is some T^-1: W -> V such that
T o T^-1 is the identity. since this T^-1 exists, then there is some matrix representation of
T^-1, say A such that A T_A = I.

$[T^{-1}]^B_C$

$$C = \begin{bmatrix} | & | & \cdots & | \\ [Ab_1]_C & [Ab_2]_C & \cdots & [Ab_n]_C \\ | & | & & | \end{bmatrix} = R^{-1}(B,C)$$

$$R^{-1}(B,C) \circ R(B,C) = I$$

$$T : V \to W \qquad T(\vec{x}) = A\vec{x}$$

$$T^{-1} : \underset{C}{W} \to \underset{B}{V} \qquad T(\vec{x}) = A^{-1}\vec{x}$$

$$[T]_C^B = \begin{bmatrix} | & & | \\ [A\vec{b}_1]_C & \cdots & [A\vec{b}_n]_C \\ | & & | \end{bmatrix}$$

$$[T^{-1}]_B^C = \begin{bmatrix} | & & | \\ [A^{-1}\vec{c}]_B & \cdots & [A^{-1}\vec{c}]_B \\ | & & | \end{bmatrix}$$

Problem four. Let $T : P_4 \to P_4$ be the linear transformation defined by $T(p) = xp'(x) - p(x)$. Find a basis for the image of T and a basis for the kernel of T.

B = (1, x, x,^2, x^3, x^4)

$$A = \begin{bmatrix} | & & | \\ [T(1)]_B & \cdots & [T(x^4)]_B \\ | & & | \end{bmatrix} = \begin{bmatrix} -1 & & & & 0 \\ & 0 & & & \\ & & 1 & & \\ & & & 2 & \\ 0 & & & & 3 \end{bmatrix}$$

$$\operatorname{Ker}(T) = \{x\} \Rightarrow [x]_B = [0\ 1\ 0\ 0\ 0]$$

Img(T) = column space of A = sp([-1, 0, 0, 0, 0,], [0, 0, 1, 0, 0], [0, 0, 0, 2, 0], [0, 0, 0, 0, 3])

B = (1, x, x,^2, x^3, x^4)

basis for im(T) = { -1, x^2, 2x^3, 3x^4 }

Let T be a linear transformation with standard matrix representation A

Kernel(T) = Nullspace of A
Image(T) = Column space of A