**MATA31**

Calculus 1, for Mathematical Sciences, Fall 2018

# Various Core Proofs/Identities


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# 1 Identities

**Basic Trigonometric Identities**

Reciprocal Identities

$(\sin x)^{-1} = \csc x$
$(\cos x)^{-1} = \sec x$
$(\tan x)^{-1} = \cot x$

Pythagorean Identities

$\sin^2 x + \cos^x = 1$
$\tan^2 x + 1 = \sec^2 x$
$1 + \cot^2 x = \csc^2 x$

Angle Addition Identities

$\sin(x \pm y) = \sin x \cdot \cos y \pm \cos x \cdot \sin y$
$\cos(x + y) = \cos x \cdot \cos y - \sin x \cdot \sin y$
$\cos(x - y) = \cos x \cdot \cos y + \sin x \cdot \sin y$
$\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \cdot \tan y}$
$\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \cdot \tan y}$

**Hyperbolic Trigonometric Identities**

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{2}{e^x - e^{-x}}$$
$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

**Derivatives of Trigonometric Functions**

Derivative of Trigonometric Functions

$(\sin x)' = \cos x \qquad (\csc x)' = -\csc x \cot x$
$(\cos x)' = -\sin x \qquad (\sec x)' = \sec x \tan x$
$(\tan x)' = \sec^2 x \qquad (\cot x)' = -\csc^2 x$

Derivative of Hyperbolic Trigonometric Functions

$(\sinh x)' = \cosh x \qquad (\operatorname{csch} x)' = -\operatorname{csch} x \cdot \coth x$
$(\cosh x)' = \sinh x \qquad (\operatorname{sech} x)' = -\operatorname{sech} x \cdot \tanh x$
$(\tanh x)' = \operatorname{sech}^2 x \qquad (\coth x)' = -\operatorname{csch}^2 x$

Derivative of Inverse Trigonometric Functions

$$(\sin^{-1} x)' = \frac{1}{\sqrt{1 - x^2}} \qquad (\csc^{-1} x)' = -\frac{1}{|x|\sqrt{x^2 - 1}}$$
$$(\cos^{-1} x)' = -\frac{1}{\sqrt{1 - x^2}} \qquad (\sec^{-1} x)' = -\frac{1}{|x|\sqrt{x^2 - 1}}$$
$$(\tan^{-1} x)' = \frac{1}{1 + x^2} \qquad (\cot^{-1} x)' = -\frac{1}{1 + x^2}$$

**Derivatives of common functions**

$\frac{d}{dx}\ln x = \frac{1}{x}$ $\frac{d}{dx}x^n = nx^{(n-1)}$ $\frac{d}{dx}f \cdot g = f' \cdot g + g' \cdot f$

$\frac{d}{dx}e^x = e^x \cdot \ln(e)$ $\frac{d}{dx}f + h = f' + h'$ $\frac{d}{dx}\frac{f}{g} = \frac{g \cdot f' - f \cdot g'}{g^2}$

$\frac{d}{dx}e^{2x} = e^x \cdot \ln(e) \cdot \frac{d}{dx}x = e^x \cdot 2x$ $\frac{d}{dx}f(g) = f'(g) \cdot g'$

# 2 Uniqueness of Limits

$\lim_{x\to c} f(x) = L \land \lim_{x\to c} f(x) = M \implies L = M$

**Proof:**

Suppose the contrary that:

$\lim_{x\to c} f(x) = L \land \lim_{x\to c} f(x) = M, L \neq M$

Assume that $L > M, L = M + K$, WLOG

Let's choose $\epsilon = \frac{k}{2}$, this way the intervals do not overlap.

$\lim_{x\to c} f(x) = L : \exists\delta_1 > 0, \ni: 0 < |x - c| < \delta_1 \to |f(x) - L| < \epsilon$

$\lim_{x\to c} f(x) = M : \exists\delta_2 > 0, \ni: 0 < |x - c| < \delta_2 \to |f(x) - M| < \epsilon$

Let $\delta = min(\delta_1, \delta_2)$ so that for any $\delta > 0$, $f(x) \in (M - \epsilon, M + \epsilon)$ and $f(x) \in (L - \epsilon, L + \epsilon)$.

Contradiction: This is impossible since we set $\epsilon = \frac{k}{2}$ to guarantee intervals do not overlap.

Therefore, $\lim_{x\to c} f(x) = L \land \lim_{x\to c} f(x) = M \implies L = M$ by contradiction.

*QED*

# 3 Mean Value Theorem

if function f is continuous across $[a, b]$ and differentiable across$(a, b)$
then, there exists atleast one $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$
This means if the conditions satisfy, there will be atleast one point c where the instataneous rate of change is the same as the average rate of change.

# 4 Applications of MVT

Suppose $f(x)$ is continuous on $[1,5]$ and differentiable on (1,5) and $f'(x) < \frac{3}{8}, \forall x \in (1,5)$. If $f(1) = 1$, show that $f(x) < \frac{5}{2}, \forall x \in [1.5]$

Since f is continuous and differentiable...by Mean Value Theorem, $\exists c \in (1,5), \ni\colon f'(c) = \frac{f(5) - f(1)}{5-1}$

let $x \in (1,5), f'(x) = \frac{f(x) - f(1)}{x-1} < \frac{3}{8} \iff \frac{f(x)-1}{x-1} < \frac{3}{8}$

$f(x) - 1 < \frac{3}{8}(x-1),\ f(x) < \frac{3}{8}(x-1) + 1$

$f(x) < \frac{3}{8}(5-1) + 1 \rightarrow f(x) < \frac{5}{2}$ *QED*

Suppose $f(x)$ is odd for all x and diff across every real number. Prove that for every positive number b, there exists a positive number c, in $(-b, b)$ such that $f'(c) = \frac{f(b)}{b}$

Given $b > 0, f(-x) = -f(x)$

since $f$ is differentiable on $(-\infty, \infty) \implies f$ is continuous on $(-\infty, \infty)$

$f$ is cont on [-b,b] and differentiable on (-b,b)

becuase (-b,b)$\subset (-\infty, \infty)$,

By MVT, $\exists c \in (-b, b) \ni\colon f'(c) = \frac{f(b) - f(-b)}{b - (-b)} = \frac{f(b) + f(b)}{b + b} = \frac{f(b)}{b}$

# 5 Rolle's Theorem

Suppose f is continuous on [a,b] and differentiable on (a,b) and $f(a) = f(b)$ then there exists atleast a $c \in (a,b)$ such that $f'(c) = 0$

# 6 Applications R'T

Show that the function $2x + \cos x$ has exactly one real root.

Let $f(x) = 2x + \cos x$

$f(-\pi) = -2\pi + \cos(-\pi) = -2pi - 1 < 0$

$f(0) = 2 + \cos(0) = 1 > 0$

Since $f(x)$ is a sum of a polynomial and periodic trigonometric function, f is continuous and differentiable for all x, By IVT, $\exists c \in (-\pi, 0) \ni\colon f(c) = 0$

Suppose $f(x)$ ias two roots on a,b a ¡ b, then $f(a) = f(b) = 0$, Since f is continuous on [a,b] and differentiable on open interval (a,b).

By Rolle's Theorem $\exists r \in (a,b) \ni\colon f'(r) = 0$

$f'(x) = 2 - \sin(x) > 0$, Contradiction, rolles theorem fails and therefore there must be exactly one root and one root only. *QED*

# 7 Fermat's Theorem

If $f$ has a local min/max at x = c and $f'(c)$ exists, then f'(c) = 0.

$f'(x) = 0$ or undefined for c-pts:
$f'(x) > 0 \Rightarrow f(x)$ is increasing
$f'(x) < 0 \Rightarrow f(x)$ is decreasing

# 8 Proof of Sum Law for Limits

Prove $\lim\limits_{x \to c} f(x) = L \wedge \lim\limits_{x \to c} g(x) = M \implies \lim\limits_{x \to c}[f(x) \pm g(x)] = L \pm M$

Suppose $\lim\limits_{x \to c} f(x) = L \wedge \lim\limits_{x \to c} g(x) = M$
$\lim\limits_{x \to c} f(x) = L$
$\forall \epsilon_1 > 0, \exists \delta_1 > 0, \ni: 0 < |x - c| < \delta_1 \implies |f(x) - L| < \epsilon_1$

$\lim\limits_{x \to c} g(x) = M$
$\forall \epsilon_2 > 0, \exists \delta_2 > 0, \ni: 0 < |x - c| < \delta_1 \implies |g(x) - M| < \epsilon_2$
Consider $\epsilon_1, \epsilon_2 = \dfrac{\epsilon}{2}$
There must consequently be a $\delta_1 > 0$ and a $\delta_2 > 0$ such that:
$0 < |x - c| < \delta_1 \implies |f(x) - L| < \dfrac{\epsilon}{2}$
$0 < |x - c| < \delta_1 \implies |g(x) - M| < \dfrac{\epsilon}{2}$
let $\delta = \min(\delta_1, \delta_2)$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \dfrac{\epsilon}{2} \wedge |g(x) - M| < \dfrac{\epsilon}{2}$
$0 < |x - c| < \delta \implies |f(x) - L| + |g(x) - M| < \epsilon$
$0 < |x - c| < \delta \implies |f(x) - L + (g(x) - M)| \leq |f(x) - L| + |g(x) - M| < \epsilon$
$0 < |x - c| < \delta \implies |f(x) + g(x) - (L + M)| < \epsilon$