

Sample Midterm

CSCA67 - Discrete Mathematics

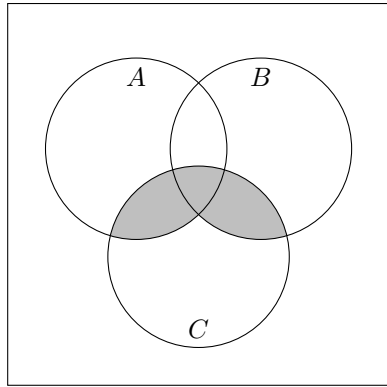


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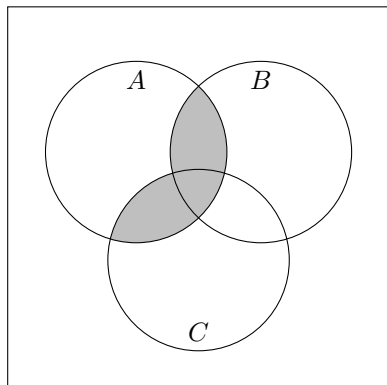
Question 1.

Part (a) Each option below is a predicate formula. Circle all (I bolded ones that i think are correct) answers equivalent to the given.



- (a) $a \wedge b \wedge c$
- (b) $(a \wedge c) \vee (b \wedge c)$**
- (c) $(\neg b \rightarrow a) \wedge c$**
- (d) $c \wedge (b \vee a)$**
- (e) None

Part (b) Write a propositional statement using \rightarrow and \neg if necessary equivalent to regions 3, 5, and 6. marks you can leave your solution using \wedge, \vee, \neg and \rightarrow . You do not need to specify which equivalence laws you use if you need them.



- $(a \wedge b) \vee (a \wedge c)$
- $a \wedge r(b \vee c)$
- $\neg(a \rightarrow \neg(\neg b \rightarrow c))$

Part (c) Circle (I just bolded.) every statement equivalent to $\neg a \rightarrow b$.

- (a) $\neg b \rightarrow a$**
- (f) a is necessary for $\neg b$**
- (b) $a \rightarrow \neg b$
- (g) $\neg a$ is necessary for b
- (c) a is sufficient for $\neg b$
- (h) b if $\neg a$**
- (d) $\neg(a \wedge \neg b)$
- (i) $b \vee a$**
- (e) b is sufficient for $\neg a$

Question 2.

Determine whether \forall can be factored from an implication. In other words is

$$\forall x \in X, (p(x) \rightarrow q(x)) \iff \forall x \in X, p(x) \rightarrow \forall x \in X, q(x)$$

true? Explain your reasoning. Marks will only be given for your explanation.

No.

let $p(x) = x > 5$ and $q(x) = x^2 > 25$

$$\forall x \in \mathbb{Z}, (p(x) \rightarrow q(x)) \iff \forall x \in \mathbb{Z}, (x > 5 \rightarrow x^2 > 25)$$

which means: for all integers x , if x is greater than 5, then x^2 is greater than 25.

vs.

$$\forall x \in \mathbb{Z}, p(x) \rightarrow \forall x \in \mathbb{Z}, q(x) \iff \forall x \in \mathbb{Z}, x > 5 \rightarrow \forall x \in \mathbb{Z}, x^2 > 25$$

which means: if all integers are greater than 5, then all integers squared are greater than 25.

Sometimes, factoring in \forall can lead to a statement that is true, but never equivalent.

$\forall x \in X, (p(x) \rightarrow q(x))$ is saying for every x , if predicate p is true, then predicate q is true.

$\forall x \in X, p(x) \rightarrow \forall x \in X, q(x)$ is saying for every x , if predicate p is true, then for every x , predicate q is true, since are not equivalent, therefore \forall cannot be factored into or from an implication.

Question 3.

Answer the following questions to construct a direct proof that:

$$\forall n \in \mathbb{N}, n \geq 2, \forall a, b \in \mathbb{N}, a \equiv_n b \rightarrow a^2 \equiv_n b^2$$

Part (a)

Write a and b in terms of n using the division theorem and taking into consideration any common variables.

$$\begin{aligned} a &= k_1 \cdot n + r \\ b &= k_2 \cdot n + r \\ k_1, k_2 &\in \mathbb{Z}, r \in [0, (n-1)] \end{aligned}$$

Part (b)

Now complete the proof.

Assume $a \equiv_n b$, then that means:

$$\begin{aligned} a &= k_1 \cdot n + r & a^2 &= k_1^2 \cdot n^2 + 2k_1 \cdot n \cdot r + r^2 \\ b &= k_2 \cdot n + r & b^2 &= k_2^2 \cdot n^2 + 2k_2 \cdot n \cdot r + r^2 \\ k_1, k_2 &\in \mathbb{Z}, r \in [0, (n-1)] \end{aligned}$$

$$a^2 = k_1^2 \cdot n^2 + 2k_1 \cdot n \cdot r + r^2 = n(k_1^2 \cdot n + 2k_1 \cdot r) + r^2$$

$$b^2 = k_2^2 \cdot n^2 + 2k_2 \cdot n \cdot r + r^2 = n(k_2^2 \cdot n + 2k_2 \cdot r) + r^2$$

Since both a^2 and b^2 share the same remainder r^2 when divided by n ,

$$\therefore \forall n \in \mathbb{N}, n \geq 2, \forall a, b \in \mathbb{N}, a \equiv_n b \rightarrow a^2 \equiv_n b^2.$$

QED

Question 4.

Prove that $\forall n \in \mathbb{N}, (n^2 - 1 \not\equiv_4 0) \rightarrow n$ is even.

Assume the contrapositive

$$n \in \{2k + 1\}, k \in \mathbb{Z} \rightarrow (n^2 - 1 \equiv_4 0)$$

$$\begin{aligned} n &= 2k + 1, k \in \mathbb{Z} \\ n^2 &= 4k^2 + 4k + 1 \\ n^2 - 1 &= 4k^2 + 4k = 4(k^2 + k) \\ n^2 - 1 &\equiv_4 0 \end{aligned}$$

Since we prove the contrapositive to be true, and the contrapositive is logically equivalent to the original,
 $\therefore n \in \{2k + 1\}, k \in \mathbb{Z} \rightarrow (n^2 - 1 \equiv_4 0)$

QED

Question 5.

Part (a) Write the following claim as an implication using quantifiers and \rightarrow . You may use $p(x, y)$ to denote that x and y are relatively prime.

a and n relatively prime is necessary for there to exist a unique natural number $b < n$ such that $a \cdot b \equiv_n 1$.

$$\exists n \in \mathbb{N}, \exists: b < n, a \cdot b \equiv_n 1 \rightarrow \gcd(a, n) = 1$$

Part (b) Prove the claim.

Assume: $\exists n \in \mathbb{N}, \exists: b < n, a \cdot b \equiv_n 1$

$$a \cdot b = n \cdot k + 1, k \in \mathbb{Z}$$

$$a \cdot b - n \cdot k = 1$$

Bezout's Identity:

if a and b are non zero integers, then there exists integers u and v such that:

$$\gcd(a, b) = au + bv$$

then by Bezout's Identity, since $a \cdot b - n \cdot k$ is some linear combination that equals 1,

1 must be $\gcd(a, n)$

$$a \cdot b - n \cdot k = 1 \rightarrow \gcd(a, n) = 1$$

$$\therefore \exists n \in \mathbb{N}, \exists: b < n, a \cdot b \equiv_n 1 \rightarrow \gcd(a, n) = 1$$

QED

Question 6.

Prove the statement $S(n) : \forall n \in \mathbb{N}, \exists a, b \in \mathbb{N}, \ni: 3a + 5b = n$

Predicate:

$$S(n) : \forall n \in \mathbb{N}, n \geq 8 \exists a, b \in \mathbb{N}, \ni: 3a + 5b = n$$

Base Case:

$$S(8) : 8 = 3a + 5b = 3(1) + 5(1)$$

$$S(9) : 9 = 3a + 5b = 3(3) + 5(0)$$

$$S(10) : 10 = 3a + 5b = 3(0) + 5(2)$$

Inductive Hypothesis:

Suppose $S(k)$, for some arbitrary $k \in [8, n)$

$$S(k) : k = 3a + 5b$$

Inductive Step:

$$S(n) \rightarrow S(n + 3)$$

$k = 3a + 5b$ Suppose $P(n)$ by **Inductive Hypothesis**

$$k + 3 = 3a + 5b + 3$$

$$k + 3 = 3(a + 1) + 5b$$

Since k is arbitrary, $\forall n \in \mathbb{N}, S(n)$

QED