Lecture Notes
Winter 2019

MATA37 - CALCULUS II FOR THE MATHEMATICAL SCIENCES

LEC03, Jan 28th, 2:00pm - 3:00pm

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Recall, **FTOC - Part II** Let $a, b \in \mathbb{R}, a < b$

IF f is cont on $[a, b]$, define $F(x) = \displaystyle\int_a^x f(t)dt, x \in [a, b]$

THEN F is cont on $[a, b]$ and F is differentiable on (a, b), Moreover $F'(x) = f(x), \forall x \in [a, b]$

This means F is an antiderivative of F on $[a, b]$

$\frac{dF}{dx} = \frac{d}{dx}(\int_a^x f(t)dt) = F(x)$

1 Proof

Suppose f is cont on $[a.b]$

Define $F(x) = \displaystyle\int_a^x f(t)dt, x \in [a, b]$

WTS that $F'(x) = f(x), \forall x \in [a, b]$ as differentiability implies continuity.

1. Case I: let $x \in (a, b)$ be arbitrary

   Consider $F'(x) = \displaystyle\lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$ By Definition of the derivative.

   $F'(x) = \displaystyle\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h}$ By definition of F.

   now if $h > 0$, $\displaystyle\int_a^{x+h} f(t)dt - \int_a^x f(t)dt = \int_x^{x+h} f(t)dt$

   and if $h < 0$, $\displaystyle\int_a^{x+h} f(t)dt - \int_a^x f(t)dt = -\int_{x+h}^x f(t)dt = \int_x^{x+h} f(t)dt$

   So $F'(x) = \displaystyle\lim_{h \to 0} \frac{\int_x^{x+h} f(t)dt}{h} = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)dt = \lim_{h \to 0} \frac{1}{(x+h) - x} \int_x^{x+h} f(t)dt$

By MVT for integrals, $F'(x) = \displaystyle\lim_{h \to 0} \frac{\int_x^{x+h} f(t)dt}{h} = f(c)$, for some $c \in (x, x+h)$	MVT for integrals **Note** Let $a, b \in \mathbb{R}$ **IF** f is cont. on [a,b], **THEN** $\exists c \in (a, b), \ni$: $\displaystyle\int_a^b f(t)dt = f(c) \cdot (b - a)$

   Since $c \in (x, x+h)$,

   $h \to 0 \Leftrightarrow |c - x| \to 0 = c \to x$

   Because as h approaches 0, from either side we collapse the interval and forces c to become x.

   $\displaystyle\lim_{h \to 0} f(c) = \lim_{c \to x} f(c) = f(x)$

2. Case 2: let $x = a \vee x = b$

   Consider $F'(x) = \displaystyle\lim_{h \to 0^+} \frac{F(x+h) - F(x)}{h}$

   Analogous proof of $F'_+(a) = f(a)$ and $F'_-(b) = f(b)$

   Just replace the 2 - sided limit with the appropriate 1 sided limit.

   so F'(x) exist for all $x \in (a, b)$

   $\therefore F$ is diff on (a,b)

   By Case II, F is diff on [a, b] but as Diff. $\Rightarrow$ Cont.

   $\therefore F$ is cont. on [a,b] *QED*

Find $\dfrac{d}{dx}\displaystyle\int_{\sin(x)}^{\cos(x)} \arctan(t)dt$ Using FTOC II

Solution: $f(t) = \arctan(t)$ is cont on $\mathbb{R} \subset dom(f)$, Let $c \in (\sin(x), \cos(x))$
$\therefore \dfrac{d}{dx}(\displaystyle\int_{\sin(x)}^{\cos(x)} f(t)dt) = \dfrac{d}{dx}\left(\int_{\sin(x)}^{c} f(t)dt + \int_{c}^{\cos(x)} f(t)dt\right) = \dfrac{d}{dx}\left(-\int_{c}^{\sin(x)} f(t)dt + \int_{c}^{\cos(x)} f(t)dt\right)$
Note that $\int_c^{\sin(x)} f(t)dt = F(\sin(x))$ and $\int_c^{\cos(x)} f(t) = F(\cos(x))$
Thus: $\dfrac{d}{dx}(-F(\sin(x)) + F(\cos(x))) = -F'(\sin(x)) \cdot \cos(x) + F'(\cos(x)) \cdot (-\sin(x))$ By Diff. Rules
$= -f(\sin(x)) \cdot \cos(x) - f(\cos(x)) \cdot \sin(x)$ By FTOC II, $F' = f$
$= -\arctan(\sin(x)) \cdot \cos(x) - \arctan(\cos(x)) \cdot \sin(x)$

2 Integration Techniques

1. Substitution Rule Let $a, b \in \mathbb{R}, a < b$
   if f and g' are cont. on $[a, b]$
   then
   (a) $\displaystyle\int f(g(x)) \cdot g'(x)dx = \int f(u)du.$ let $u = g(x), du = g'(x)$
   (b) $\displaystyle\int_a^b f(g(x)) \cdot g'(x) = \int_{g(a)}^{g(b)} f(u)du$

   Examples:

   (a) Evaluate $\displaystyle\int \frac{\ln^2(x)}{x}dx = \int (\ln^2 x) \cdot \frac{1}{x}dx$
   Let $u = \ln x, du = \frac{1}{x}dx$
   Now we have $\displaystyle\int \frac{\ln^2(x)}{x}dx = \int u^2 du = \frac{u^3}{3} + C = \frac{\ln^3 x}{3} + C$
   (b) Find $\displaystyle\int_0^1 \frac{e^{\tan^{-1}(x)}}{1+x^2}dx$
   Let $u = \tan^{-1} x, du = \frac{1}{1+x^2}dx$
   Now we have $\displaystyle\int_0^{\frac{\pi}{4}} e^u du = e^{\frac{\pi}{4}} - e^0 = e^{\frac{\pi}{4}}$
   (c) Evaluate $\displaystyle\int \cos(x) \cdot \sin^5(x)dx$
   let $u = \sin(x), du = \cos(x)dx$
   Now we have $\displaystyle\int \cos(x) \cdot \sin^5(x)dx = \int u^5 = \frac{u^6}{6} + c = \frac{\sin^6(x)}{6} + C$
   (d) Find $\displaystyle\int \frac{1}{(3x-5)^{2000}}$
   let $u = (3x-5), du = 3dx, \frac{du}{3} = dx$
   Now we have $\displaystyle\int \frac{1}{3u^{2000}} = \frac{1}{3} \cdot \frac{u^{-1999}}{-1999} = \frac{-1}{3(1999)}(3x-5)^{-1999} + C$