Week 10, LEC02 - November 23th
Discrete Mathematics, Fall 2018

CSCA67 - Lecture Notes

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Q. Consider the experiment of rolling two dice and the probability of their sum being 7. What is the elementary sample space?

A.
$D_1 = \{1, 2, 3, 4, 5, 6\}$
$D_2 = \{1, 2, 3, 4, 5, 6\}$
$S = \{(i, j) : i, j \in [1, 6]\}$
$S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5)(1, 6), (2, 1), (2, 2)(2, 3)(2, 4) \ldots (6, 5), (6, 6)$
$|S| = 36$

Q. Would S = {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} also be an appropriate sample space?
A. No (maybe yes if you are careful).

Q. Why?
A. There are 36 elementary events, this sample space does not distinguish (1,2) and (2,1).
If we consider the sum 2, it can only happen one way where as the sum 3 can happen two ways.
Q. $Prob(sum = 7)$?
A. We can scan the sample space to see that there are only 6 rolls that result in the sum 7.
$P(sum = 7) = \dfrac{6}{36} = \dfrac{1}{6}$

Q. What is prob(sum=7) if we use S as the sample space?
A. $P(sum = 7) = \dfrac{1}{11}$

Q. Why doesnt this make sense?
A. Because there are multiple ways of rolling a sum of 7. as S is compound, not elemental.
We can use S as our sample space but need to be a little more careful.
Exercise. If we did use S = {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}
as our sample space what would each of the following be:

$$\begin{array}{lll} \text{P}(2) = \dfrac{1}{35} & \text{P}(3) = \dfrac{2}{36} & \text{P}(4) = \dfrac{3}{36} \\ \text{P}(5) = \dfrac{4}{36} & \text{P}(6) = \dfrac{5}{36} & \text{P}(7) = \dfrac{6}{36} \\ \text{P}(8) = \dfrac{5}{36} & \text{P}(9) = \dfrac{4}{36} & \text{P}(10) = \dfrac{3}{36} \\ \text{P}(11) = \dfrac{2}{36} & \text{P}(12) = \dfrac{1}{36} & \end{array}$$

Q. What do you notice about the following sum: $\displaystyle\sum_{i=2}^{12} P(i) = ??$

A. $\displaystyle\sum_{i=2}^{12} P(i) = 1$

Q. Would you expect this?
A. Because an outcome of this experiment must be in the set S.

Q. What is the probability of the sum of the die equalling 4 or 5?
A. $P(sum = 4) + P(sum = 5)$

Definition. The probability density p on a discrete sample space S is a function with domain S satisfying:

1. $0 \leq p(A) \leq 1$ for any $A \subset S$
2. $\sum_{s \in S} p(s) = 1 = p(S)$
2. $P(a \cup B) = p(A) + p(B)$ for disjoint (unrelated, independent) events $A, B \subset S$

Q. What does it mean for sets A and B to be disjoint?
A. Their intersection is empty, ie $A \cap B = \emptyset$

Example. A coin is tossed three times. What is the probability that a head turns (atleast) up two or three times?
Solution 1. Write out an appropriate sample space S and count the outcomes to determine the probability.
$S = \{(T,T,T),(T,T,H),(T,H,T),(T,H,H),(H,H,H),(H,H,T),(H,T,H),(H,T,T)\}$
$|S| = 8$
$P(\text{atleast 2 heads}) = \frac{|E|}{|S|} = \frac{4}{8} = \frac{1}{2}$

Solution 2. Use counting principles.
Q. How many ways are there to flip a coin three times?
A. Each flip can only be heads or tails, since there are two ways to flip a single coin, the total is $2 \cdot 2 \cdot 2 = 8 = |S|$

Q. How many ways are there to select two heads? How many ways are there to select three heads?
A. $\binom{3}{2} + \binom{3}{3} = 4$
So the probability of two or three heads is $\frac{4}{8}$

Example. What is the probability of rolling two die whose sum is larger than 3?
Solution. We could add up P(4) + P(5) + P(6) + P(7) + P(8) + P(9) + P(10) + P(11) + P(12). What is an easier way to compute this probability?
$\frac{3}{36} + \frac{4}{36} + \frac{5}{36} + \frac{6}{36} + \frac{5}{36} + \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{11}{12}$
$1 - (P(sum = 2) \text{ or } P(sum = 3)) = 1 - (\frac{1}{36} + \frac{2}{36}) = \frac{1}{12}$

Theorem. If two events E and F are complementary then

$$P(E) = 1 - P(F)$$
$$P(E) + P(F) = 1$$

The Birthday Problem. What is the probability that in a group of n people at least two have the same birthday?

365 days in a year

Solution. Lets represent the days of the year by the integers 1, 2, . . . , 365 and use E to represent the event that **at least** two people have the same birthday.

Example(one element in S) $\{1_i\}_{i=0}^{365}$

Q. What could we define our sample space S to be?
A. n - tuples of integers from 1-365 for example {1,1,1, . . . 365}.

Q. Lets assume all combinations or birthdays are equally likely. How many different ways are there for the n birthdays to fall?

A. $|S| = 365^n$

Q. Is it easy to count the number of tuples with 2 or more values the same?

A. No. Cannot be solved by scanning the sample space.

Q. How can we rephrase the problem to make it easier?

A. compute the complementary probability where everyone has a distinct birthday.