Week 9, LEC02 - November 16th
Discrete Mathematics, Fall 2018

CSCA67 - Lecture Notes

Current Instructor: Dr. Richard Pancer

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1 Pascal's Triangle

Blaise Pascal [1623-1662] was a French mathematician, physicist, inventor, writer and philosopher

- As a teenager, he invented the mechanical calculator.
- He collaborated with Pierre de Fermat in Probability Theory influencing modern economics and social sciences.
- Invented Pascals Triangle in his Treatise on the Arithmetic Triangle.

$$
\begin{array}{l ccccccccccccc}
n=0 & & & & & & & 1 & & & & & & \\
n=1 & & & & & & 1 & & 1 & & & & & \\
n=2 & & & & & 1 & & 2 & & 1 & & & & \\
n=3 & & & & 1 & & 3 & & 3 & & 1 & & & \\
n=4 & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
n=5 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
n=6 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
\hline
 & 0 & & 1 & & 2 & & 3 & & 4 & & 5 & & 6
\end{array}
$$

Q. What do you notice about each number? How is it related to pizza and toppings?
A. $(n,r)^{th}$ element = C(n,r)

Q. How is Pascals Triangle related to binomial expansion? I.e., how is it related to the coefficients of the polynomials found by expanding $(a+b)^n$?

A.
$$
\begin{array}{l}
(a+b)^1 = 1a + 1b \\
(a+b)^2 = 1a^2 + 2ab + 1b^2 \\
(a+b)^3 = 1a^3 + 3a^2b + 3b^2a + 1b^3 \\
\ldots
\end{array}
$$

Q. What do each line sum to?
A. $1(2^0), 2(2^1), 4(2^2), 8(2^3)\ldots$ powers of 2

Q. Read each line as a number (catenate). If a number has two digits carry the tens digit to the left and add. What do these numbers represent?
A. $1, 11, 121, 1331, 14541\ldots$ powers of 11.

Q. Colour all the odd numbers. What does this remind you of?
A. A well known repeating pattern, the Sierpinkski Triangle.

Q. What series of numbers do you get by adding up the numbers of the same colour which are on a stretched diagonal?
A. The Fibonacci sequence
Pascal was not the first to discover the triangle of binomial coefficients but was given credit because of how he related it to his work with probability and expectation.

2 Distributing Objects

Example. How many ways can 20 different diplomats be assigned to 5 different continents?
Solution. Rephrase the problem as an arrangement we already know.

$$5 \cdot 5 \cdot 5 \cdots 5 = (5)^20$$

Q. What if each continent needs to have 4 diplomats each?
A. $P(20,4,4,4,4,4) = \dfrac{20!}{(4!)^5}$

Example. How many ways are there to distribute 20 identical chocolate bars **and** 15 identical sticks of gum to 5 children?
Solution
First, distribute the 20 chocolate bars (indistinguishable).
4 partition: $C(20+(5-1),(5-1)) = C(24,4)$

Then, distribute the 15 sticks of gum to 5 children (indistinguishable).
4 partitions: $C(15 + (5 - 1), (5 - 1)) = C(19, 4)$
Total $= C(24, 4) \cdot C(19, 4)$

Example. How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 12$ with $x_i 0$?
Solution We must distribute 12 1s amongst 4 variables, and to distinguish the variables we need 3 partitions.
$C(12 + (4 - 1), (4 - 1)) = C(15, 3)$
However consider if we have a coefficient: $2x_1 + x_2 + x_3 + x_4 = 12$ with $x_i 0$?
Potential assignment 2/Exam question

Q. What if we require that each xi 1?
A. The only that changed is that x_i cannot be 0. We can simply put a 1 in each variable, except there are no longer 12 1s to distribute, but only 8.
This is called - Preloading, we preload the variables and it then essentially becomes the same problem.
$C([12 - 4] + (4 - 1), (4 - 1)) = C(11, 3)$

Theorem. The number of ways to distribute r identical objects into n distinct boxes with at least one object in each box is C(r-1, n-1). (Preload)
Proof. We need to place n of the r objects amongst the n boxes leaving us with r - n objects to distribute into the n boxes.
$C(r - 1, n - 1) \equiv C([r - n] + n - 1, n - 1) = C(r - 1, n - 1)$ *QED*

Q. How many ways can we do this?
A.