Lecture Notes
Winter 2019

MATA37 - CALCULUS II FOR THE MATHEMATICAL SCIENCES

LEC03, Mar 11, 15, 1:00pm - 3:00pm, 2:00pm - 3:00pm

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Exercises

Prove $a_n = \dfrac{n^2-2}{n^2+2n+2}$ converges

WTS: $\forall l \in \mathbb{R} \epsilon > 0, \exists N > 0, \ni: n > N \Rightarrow |a_n - l| < \epsilon$
given $n > N$
Choose $l = 1 \ni$:
$|\dfrac{n^2-2}{n^2+2n+2} - 1| = |\dfrac{-4-2n}{n^2+2n+2}| = 2\dfrac{n+2}{n^2+2n+2} \leq 2\dfrac{n+2}{n^2+2n} = \dfrac{2}{n} = \dfrac{2}{N} = \epsilon$

Theorem

If $\{a_n\}$ converges to a and $\{b_n\}$ converges to b
Then $\{a_n + b_n\}$ converges to $a + b$

Proof

Suppose $\{a_n\}$ converges to a and $\{b_n\}$ converges to b
$\forall \epsilon > 0, \exists N \geq 0, \ni: \forall n \in \mathbb{N}, n > 0 \Rightarrow |a_n + b_n - (a+b)| < \epsilon$
Let $\epsilon > 0$ be arbitrary, if $\{a_n\}$ converges to a and $\{b_n\}$ converges to b
Choose $N = \max\{N_1, N_2\}$
1. Then $\exists N_1 > 0, \ni: n > N_2 \Rightarrow |a_n - a| < \epsilon/2$
2. Then $\exists N_2 > 0, \ni: n > N_2 \Rightarrow |b_n - b| < \epsilon/2$

Recall Triangle Inequality: $|a \pm b| \leq |a| + |b|$
1. $\wedge$ 2. $\implies |a_n + n_n - (a+b)| = |a_n - a + b_n - b| \leq |a_n - a| + |b_n - b| < \epsilon/2 + \epsilon/2 = \epsilon$ *QED*

Theorem

IF $\{a_n\}$ Converges, THEN it's limit is unique.

Proof
Suppose $\{a_n\}$ converges, Suppose $\{a_n\}$ converges to $l_1, l_2, \in \mathbb{R}, \ni: l_1 \neq l_2$
Show $\forall \epsilon > 0, |l_1 - l_2| < \epsilon$, It will follow that $l_1 - l_2 = 0 \Rightarrow l_1 = l_2$
Let $\epsilon > 0$ be arbitrary
1. Then $\exists N_1 > 0, \ni: n > N_2 \Rightarrow |a_n - k_1| < \epsilon/2$
2. Then $\exists N_2 > 0, \ni: n > N_2 \Rightarrow |a_n - l_2| < \epsilon/2$
If $n > \max\{N_1, N_2\}$ Then:
$|l_1 - l_2| = |l_1 - l_2 + a_n - a_n| = |a_n - l_1 - a_n + l_2| = |a_n - l_2 - (a_n - l_1)| \leq |a_n - l_2| + |-(a_n - l_1)| = |a_n - l_2| + |a_n - l_1| < \epsilon/2 + \epsilon/2 = \epsilon$
Thus as $|l_1 - l_2| < \epsilon, l_1 = l_2$ *QED*

Recall
IF $\{a_n\}$ converges, THEN $\exists c \in \mathbb{R}, \ni: |\{a_n\}| < c, \forall n \in \mathbb{N}$
This is to be proven for an exercise.