Lecture Notes
Winter 2019

MATA37 - CALCULUS II FOR THE MATHEMATICAL SCIENCES

LEC03, Mar 11, 15, 1:00pm - 3:00pm, 2:00pm - 3:00pm

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1 Divergence of Sequences

let $\{a_n\}$ be a sequence of real numbers.

1. $\lim\limits_{a\to\infty} a_n = \infty \iff \forall M > 0, \exists N > 0, \ni: n > N \Rightarrow a_n > M$
2. $\lim\limits_{a\to\infty} a_n = -\infty \iff \forall M > 0, \exists N > 0, \ni: n > N \Rightarrow a_n < -M$

$a_n = \{8n^3 + n^2 - 2\}$
$\lim\limits_{a\to\infty} a_n = \infty \iff \forall M > 0, \exists N > 0, \ni: n > N \Rightarrow a_n > M$
Proof: Let $M > 0$ be arbitrary, Choose $N = M$
$8n^3 + n^2 - 2 > n$ and $n > n$

$$a_n = \{42 + (-1)^n\} = \begin{cases} 43 & n \in 2k, k \in \mathbb{Z} \\ 41 & n \in 2k+1, k \in \mathbb{Z} \end{cases}$$

Claim: a_n diverges.
Suppose that a_n converge to $l \in \mathbb{R}$

then $\forall \epsilon > 0, \exists N > 0, \ni: n > N \Rightarrow |a_n - l| < \epsilon$, which means $\exists N_1 > 0, \ni: n > N_1 \Rightarrow |a_n - L| < 1$

If $n \in 2k, k \in \mathbb{Z}, |43 - L| < 1$
If $n \in 2k+1, k \in \mathbb{Z}, |41 - L| < 1$

Now $2 = |43 - 41| = |43 - 41 + L - L| \leq |43 - L| + |41 - L| < 1 + 1 = 2$
but $2 \not< 2$, contradicting our original assumption. Thus a_n diverges.

$$a_n = (-2)^n + \pi = \begin{cases} 2^n + \pi & n \in 2k, k \in \mathbb{Z} \\ -2^n + \pi & n \in 2k+1, k \in \mathbb{Z} \end{cases}$$

Claim: a_n diverges

Case 1: $n \in 2k, k \in \mathbb{Z}$
let $M > 0$ be arbitrary, choose N
if $n > N \Rightarrow 2^n + \pi > n > N = M$

Case 2: $n \in 2k+1, k \in \mathbb{Z}$
let $M > 0$ be arbitrary, choose $N = N + \pi$
if $n > N \Rightarrow -2^n + \pi < -n + \pi < -M - \pi + \pi = M$

A sequence $\{a_n\}$ is a Cauchy sequence if $\forall \epsilon > 0, \exists N > 0, \ni: n, m > N \Rightarrow |a_n - a_m| < \epsilon$
Suppose a_n is convergent, then $\exists L \in \mathbb{R}, \ni: \lim\limits_{n\to\infty} a_n = L$
$\forall \epsilon > 0, \exists N > 0, \ni: n > N \Rightarrow |a_n - L| < \epsilon$
WTS $\forall \epsilon > 0, \exists N > 0, \ni: n, m > N \Rightarrow |a_n - a_m| < \epsilon$
let $\epsilon > 0$ be arbitrary
$|a_n - a_m| = |a_n - L + L - a_m| < |a_n - L| + |a_m - L| < \epsilon/2 + \epsilon/2 = \epsilon$

QED

Show that the sequence defined by $a_1 = \sqrt{6}, a_{n+1} = \sqrt{6 + a_n}, n \geq 1$ is bounded above.
Recall: bounded above means $\exists M \in \mathbb{R}, \ni$: $a_n \leq M, \forall n \in \mathbb{N}$

given $a_1 = \sqrt{6}, a_2 = \sqrt{6 + a_1} = \sqrt{6 + \sqrt{6}}, a_3 = \sqrt{6 + \sqrt{6 + \sqrt{6}}}$

we know $6 < 9 \iff \sqrt{6} < \sqrt{9}$

then $a_2 = \sqrt{6 + \sqrt{6}} < \sqrt{6 + \sqrt{9}} = \sqrt{6 + 3} = 3$

$a_3 = \sqrt{6 + \sqrt{6 + \sqrt{6}}} < \sqrt{6 + \sqrt{6 + \sqrt{9}}} = \sqrt{6 + \sqrt{9}} = \sqrt{6 + 3} = 3$

Choose $M = 3$

let $P(k) : a_k < M$
Base Case $n = 1$
$a_1 = \sqrt{6} < \sqrt{9} = 3$, thus base case holds.

Induction Step
$P(k) \Rightarrow P(k+1)$
let $k \in \mathbb{N}$ be arbitrary

Inductive Hypothesis
Assume that $P(k)$ holds, such that $a_k < M$ show $a_{k+1} < M$
$a_{k+1} = \sqrt{6 + a_k} < \sqrt{6 + 3} < \sqrt{9} < 3$ By Inductive Hypothesis
$\therefore \forall k \in \mathbb{N}, a_k < 3 \Rightarrow a_{k+1} < 3$ Since k is arbitrary
$\therefore a_n < M, M = 3, \forall n \in \mathbb{N}$ by PMI *QED*

2 Monotone Convergence

Theorem: Bounded Monotone Convergence Theorem **(BMCT)**
If $\{a_n\}$ is bounded, and $\{a_n\}$ is monotone (strictly) increasing/decreasing,
$\{a_n\}$ is bounded above and increasing, or $\{a_n\}$ is bounded below and decreasing.
Then $\{a_n\}$ converges
Note: This is very good to use for recursively defined series.

Example
Prove $a_1 = \sqrt{6}, a_{n+1} = \sqrt{6 + a_n}$ is convergent
Let's try to use **BMCT**

CLAIM: $\{a_n\}$ is bounded above and increasing.
We've done the proof for this sequence having an upper bound, thus we must prove monotonicity.
WTS: $a_n < a_{n+1}, \forall n \in \mathbb{N}$

let $n \in \mathbb{N}$ be arbitrary, consider $(a_n)^2 - (a_{n+1})^2 = (a_n)^2 - (6 + (a_n)) = (a_n)^2 - a_n - 6 = (a_n - 3)(a_n + 2)$
$(a_n - 3) < 0$ when $a_n < 3$, then by bounded proof, this must be negative. but $(a_n + 2) > 0$
this negative multiplied by positive is negative. this means $(a_n)^2 - (a_{n+1})^2 < 0 \Rightarrow a_n - a_{n+1} < \sqrt{0}$
thus $a_n < a_{n+1}, \forall n \in \mathbb{N}$
$\therefore$ By **BMCT**, our sequence $\{a_n\}$ is convergent.

Proof of **BMCT**
Suppose $\{a_n\}$ is increasing and $\{a_n\}$ is bounded above.
WTS: $\{a_n\}$ Converges.

WTS: $\exists L \in \mathbb{R}, \ni: \forall \epsilon > 0, \exists N > 0, \ni: \forall n \in \mathbb{N}, n > N \Rightarrow |a_n - l| < \epsilon$

Consider $A = \{a_n | n \in \mathbb{R}\} \subset \mathbb{R}$
Observation: $A \neq \varnothing$, and A must be bounded above.
By the **Completeness Axiom**,
Every non empty set that is bounded above has least upper bound, $\alpha = \sup(A)$
Choose $L = \alpha$, let $\epsilon > 0$ be arbitrary, choose $N \in \mathbb{N}, \wedge \alpha - \epsilon < a_N$

Suppose $n > N$

$$a_n - \alpha \leq 0 \text{ because } \alpha = \sup(A)$$
$$a_n \leq \alpha < \alpha + \epsilon, \epsilon > 0$$
$$\alpha - \epsilon < a_N \leq a_n \leq \alpha \leq \alpha + \epsilon \text{ by choice of } N$$
$$\Rightarrow \alpha - \epsilon < a_n < \alpha + \epsilon$$
$$\Rightarrow -\epsilon < a_n - \alpha < \epsilon$$
$$\Rightarrow |a_n - \alpha| < \epsilon, \text{ as wanted}$$

QED

3 Series

Definition: A series is a complication of a sequence, Let $\{a_n\}$ be a sequence,

the formal sum $a_1 + a_2 + a_3 + a_4 + \cdots + a_n = \displaystyle\sum_{i=1}^{\infty} a_n$ is called an infinite series.

a_n is the general form, for each $n \in \mathbb{N}$ the finite sum $S_n = a_1 + a_2 + a_3 + \cdots + a_n =$ n-th partial sum

If $\{S_n\} = \{s_1, s_2, s_3, s_4 \dots\} = \{a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots\}$ converges

ie. $\displaystyle\lim_{n\to\infty} S_n = s \in \mathbb{R}$, we say $\displaystyle\sum_{n=1}^{\infty} a_n$ converges with sum s, we write $\displaystyle\lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} a_n = s$

IF $\displaystyle\sum_{n=1}^{\infty} a_n$ does not converge, then it diverges and we don't have a finite sum.

eg. $1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{5} \cdots = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

$S_4 = a_1 + a_2 + a_3 + a_4 = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4}$

Example

$\displaystyle\sum_{n=1}^{\infty} \ln(\frac{n+1}{n}) = \sum_{n=1}^{\infty} \ln(n+1) - \ln(n)$

Note that this is a telescoping sum!

$S_n = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$

$S_n = \ln(2) - \ln(1) + \ln(3) - \ln(2) + \ln(4) - \ln(3) \cdots + \ln(n) - \ln(n-1) + \ln(n+1) - \ln(n) = \ln(1) - \ln(n+1)$

Consider $\displaystyle\lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln(n+1) = \infty$ Thus this partial sum is divergent.