Lecture Notes

Winter 2019

MATA37 - CALCULUS II FOR THE MATHEMATICAL SCIENCES

LEC03, Jan 18th, 2:00pm - 3:00pm

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1 Darboux Def of $\int_a^b f(x)dx$

Setup

first, suppose that f is bounded on [a,b] ie. $\exists c \in \mathbb{R}^+, \ni: |f(x)| < c, \forall x \in [a, b]$
Let $a, b \in \mathbb{R}, a < b$
Suppose f is bdd(bounded), on $[a, b]$ Let $P = \{x_i\}_{i=0}^n$ be any partition of [a,b].

for $i = 1 \ldots n$
define $m_i = inf\{f(x)|x \in [x_{i-1}, x_i]\}$
define $M_i = sup\{f(x)|x \in [x_{i-1}, x_i]\}$

then the Uppersum: $U(f,p) = \sum_{i=1}^n M_i(x_i - x_{i-1})$
and the lowersum: $L(f,p) = \sum_{i=1}^n m_i(x_i - x_{i-1})$

Now let's consider this again

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \in \mathbb{I} \end{cases}$$

Compute $L(f, P)$ where P is any partition of [0,3]
Sol'n
recall: $m_i = inf\{f(x)|x \in [x_{i-1}, x_i]\}$
The possible values of $f(x)$ are simply 1 or 0. Because of the density of $\mathbb{Q} and \mathbb{I}$.
$m_i = inf\{0, 1\} = 0$
$\therefore L(f,p) = \sum_{i=1}^n m_i(x_i - x_{i-1}) = \sum_{i=1}^n 0 \cdot (x_i - x_{i-1}) = 0$

Definition
Let $a, b \in \mathbb{R}, a < b$ suppose that f is bdd on $[a, b]$ let P be any partition of $[a, b]$
We say that our function is integrable on $[a, b]$ if and only if:
$\sup\{L(f,P)|\forall P \in [a,b]\} = \inf\{U(f,P)|\forall P \in [a,b]\} = \int_a^b f(x)dx$
This definition is only used to prove that certain integrals don't exist.

2 Proof

Prove that the following is not integrable on [0,3]

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \in \mathbb{I} \end{cases}$$

Let's compute the lower sums and the upper sums.
We know that from $L(f,P) = 0$
and for the $U(f,P)$
we know that $M_i = sup\{f(x)|x \in [x_{i-1}, x_i]\}$
$f(x)$ can only be 0 or 1, because $\mathbb{Q}$ and $\mathbb{I}$ are dense.
therefore, $M_i = 1$ and $U(f,p) = \sum_{i=1}^{n} 1 \cdot (x_i - x_{i-1}) = (x_1 - x_0) + (x_2 - x_1) + (x_3 - x_2) \cdots + (x_n - x_{n-1}) = -x_0 + x_n = -0 + 3 = 3$

Because P is arbitrary, $L(f,P) = 0, \forall P$ and $U(f,P) = 3, \forall P$
$\therefore sup\{L(f,P) = 0\}$ and $inf(U(f,P) = 3)$
as $0 \neq 3$, thus $\sup\{L(f,P)|\forall P \in [0,3]\} \neq \inf\{U(f,P)|\forall P \in [0,3]\} \implies \int_0^3 f(x)dx$ Does not exist