MAT B24 TUT0018, Tutorial 1 (Week 2)
Thursdays 3 - 5pm

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Vector Space

A vector space is a set V (over a field F), together with two operations "vector addition" and "scalar multiplication" that satisfies the following 8 axioms

vector addition: $V \times V \to V$
scalar multiplication: $F \times V \to V$

Additive Axioms (forall u, v, w in V)
A1: $u + (v + w) = (u + v) + w$ (associativity)
A2: $u + v = v + u$ (commutativity)
A3: exists 0 in V, such that $v + 0 = v$ (identity), forall v in V
A4: exists $-v$ such that $v + (-v) = 0$ (inverse)

Scalar Multiplicative Axioms (forall r, s in F, u, v in V)
S1: $r(u + v) = ru + rv$
S2: $(r + s)v = rv + sv$
S3: $r(sv) = (rs)v$
S4: $1v = v$

A3: exists 0 in V, such that $v + 0 = v$ (identity)
Identity Uniqueness Lemma: The identity described as above, 0 is unique.

Assume for a contradiction that the identity element is not unique,
This is to say that there are two elements i, j in V such that $i \neq j$
forall v in V, $v + i = v$, $v + j = v$

$v + i = v$, take $v = j \Rightarrow j + i = j$
$v + j = v$, take $v = i \Rightarrow i + j = i$

$(i + j) = (j + i) \Rightarrow i = j$ thus elements i, j cannot exist and the identity is unique

Exercises

2. set R^2 with the usual scalar multiplication but addition is defined as follows: [x, y] + [r, s] = [y + s, x + r]

let u, v, w = [0, 1], [1, 2], [2, 0]
([0, 1] + [1, 2]) + [2, 0] = [3, 1] + [2, 0] = [1, 5]
[0, 1] + ([1, 2] + [2, 0]) = [0, 1] + [2, 3] = [4, 2]

Since this space does not satisfy associativity of addition, it is not a vector space.

3. The set R^2 with addition defined as $[x, y] + [a, b] = [x + a + 1, y + b]$

Show A2 holds, let $v1 = [x, y]$, $v2 = [a, b]$ be arbitrary vectors in V
WTS $v1 + v2 = v2 + v1$
$v1 + v2 = [x + a + 1, y + b]$ by definition of + over V
$v2 + v1 = [a + x + 1, b + y]$ by definition of + over V
$= [x + a + 1, y + b]$ by definition of + over R
$= v1 + v2$, and thus A2 holds.

Show A3 holds, let $0 = [-1, 0]$, let $v = [x, y]$ be arbitrary vector in V
WTS $v + 0 = v$
$v + 0 = [x + (-1) + 1, y + 0]$ by definition of + over V
$= [x, y] = v$

6. The set F of all functions mapping R to R, where addition is defined as follows: $f(x) + g(x) = \max(f(x), g(x))$

S1: $r(u + v) = ru + rv$
let $u = 1$, $v = 2$, let $r = -1$

$-1 * \max(1, 2) = -2$
$\max(-1, -2) = -1$