

# Various Core Proofs/Identities

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# 1 Identities

## Basic Trigonometric Identities

### Reciprocal Identities

$$(\sin x)^{-1} = \csc x$$

$$(\cos x)^{-1} = \sec x$$

$$(\tan x)^{-1} = \cot x$$

### Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

### Angle Addition Identities

$$\sin(x \pm y) = \sin x \cdot \cos y \pm \cos x \cdot \sin y$$

$$\cos(x + y) = \cos x \cdot \cos y - \sin x \cdot \sin y$$

$$\cos(x - y) = \cos x \cdot \cos y + \sin x \cdot \sin y$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \cdot \tan y}$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \cdot \tan y}$$

## Hyperbolic Trigonometric Identities

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \operatorname{csch} x = \frac{2}{e^x - e^{-x}}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad \operatorname{coth} x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

## Derivatives of Trigonometric Functions

### Derivative of Trigonometric Functions

$$(\sin x)' = \cos x \quad (\csc x)' = -\csc x \cot x$$

$$(\cos x)' = -\sin x \quad (\sec x)' = \sec x \tan x$$

$$(\tan x)' = \sec^2 x \quad (\cot x)' = -\csc^2 x$$

### Derivative of Hyperbolic Trigonometric Functions

$$(\sinh x)' = \cosh x \quad (\operatorname{csch} x)' = -\operatorname{csch} x \cdot \coth x$$

$$(\cosh x)' = \sinh x \quad (\operatorname{sech} x)' = -\operatorname{sech} x \cdot \tanh x$$

$$(\tanh x)' = \operatorname{sech}^2 x \quad (\operatorname{coth} x)' = -\operatorname{csch}^2 x$$

### Derivative of Inverse Trigonometric Functions

$$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}} \quad (\csc^{-1} x)' = -\frac{1}{|x|\sqrt{x^2-1}}$$

$$(\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}} \quad (\sec^{-1} x)' = \frac{1}{|x|\sqrt{x^2-1}}$$

$$(\tan^{-1} x)' = \frac{1}{1+x^2} \quad (\cot^{-1} x)' = -\frac{1}{1+x^2}$$

### Derivatives of common functions

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

$$\frac{d}{dx} x^n = nx^{(n-1)}$$

$$\frac{d}{dx} f \cdot g = f' \cdot g + g' \cdot f$$

$$\frac{d}{dx} e^x = e^x \cdot \ln(e)$$

$$\frac{d}{dx} f + h = f' + h'$$

$$\frac{d}{dx} \frac{f}{g} = \frac{g \cdot f' - f \cdot g'}{g^2}$$

$$\frac{d}{dx} e^{2x} = e^x \cdot \ln(e) \cdot \frac{d}{dx} x = e^x \cdot 2x$$

$$\frac{d}{dx} f(g) = f'(g) \cdot g'$$

## 2 Uniqueness of Limits

**Theorem.**  $\lim_{x \rightarrow c} f(x) = L \wedge \lim_{x \rightarrow c} f(x) = M \implies L = M$

$$\lim_{x \rightarrow c} f(x) = L \wedge \lim_{x \rightarrow c} f(x) = M \implies L = M$$

**Proof:**

Suppose the contrary that:

$$\lim_{x \rightarrow c} f(x) = L \wedge \lim_{x \rightarrow c} f(x) = M, L \neq M$$

Assume that  $L > M$ ,  $L = M + K$ , WLOG

Let's choose  $\epsilon = \frac{k}{2}$ , this way the intervals do not overlap.

$$\lim_{x \rightarrow c} f(x) = L : \exists \delta_1 > 0, \exists : 0 < |x - c| < \delta_1 \rightarrow |f(x) - L| < \epsilon$$

$$\lim_{x \rightarrow c} f(x) = M : \exists \delta_2 > 0, \exists : 0 < |x - c| < \delta_2 \rightarrow |f(x) - M| < \epsilon$$

Let  $\delta = \min(\delta_1, \delta_2)$  so that for any  $\delta > 0$ ,  $f(x) \in (M - \epsilon, M + \epsilon)$  and  $f(x) \in (L - \epsilon, L + \epsilon)$ .

Contradiction: This is impossible since we set  $\epsilon = \frac{k}{2}$  to guarantee intervals do not overlap.

Therefore,  $\lim_{x \rightarrow c} f(x) = L \wedge \lim_{x \rightarrow c} f(x) = M \implies L = M$  by contradiction.

*QED*

## 3 One Sided Limits

**Theorem.**  $\lim_{x \rightarrow c} f(x) = L \iff \lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = L$

We need to prove that:

1.  $\lim_{x \rightarrow c} f(x) = L \implies \lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = L$
2.  $\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = L \implies \lim_{x \rightarrow c} f(x) = L$

1. Assume  $\lim_{x \rightarrow c} f(x) = L$

$$\forall \epsilon > 0, \exists \delta > 0, \exists : 0 < |x - c| < \delta \implies |f(x) - L| < \epsilon$$

$$0 < |x - c| < \delta \equiv x \in (c - \delta, c) \cup (c, c + \delta)$$

$$x \in (c - \delta, c) \implies |f(x) - L| < \epsilon$$

$$x \in (c, c + \delta) \implies |f(x) - L| < \epsilon$$

$$\therefore \lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = L$$

2. Assume  $\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = L$

$$\forall \epsilon > 0, \exists \delta_1, \delta_2 > 0, \exists :$$

$$x \in (c - \delta, c) \vee x \in (c, c + \delta) \implies |f(x) - L| < \epsilon$$

$$\text{Let } \delta = \min(\delta_1, \delta_2)$$

$$0 < |x - c| < \delta \implies |f(x) - L| < \epsilon$$

$$\text{By definition: } \lim_{x \rightarrow c} f(x) = L$$

*QED*

## 4 Mean Value Theorem

if function  $f$  is continuous across  $[a, b]$  and differentiable across  $(a, b)$

then, there exists atleast one  $c \in (a, b)$  such that  $f'(c) = \frac{f(b) - f(a)}{b - a}$

This means if the conditions satisfy, there will be atleast one point  $c$  where the instantaneous rate of change is the same as the average rate of change.

## 5 Applications of MVT

Suppose  $f(x)$  is continuous on  $[1, 5]$  and differentiable on  $(1, 5)$  and  $f'(x) < \frac{3}{8}, \forall x \in (1, 5)$ . If  $f(1) = 1$ , show that  $f(x) < \frac{5}{2}, \forall x \in [1, 5]$

Since  $f$  is continuous and differentiable...by Mean Value Theorem,  $\exists c \in (1, 5), \ni: f'(c) = \frac{f(5) - f(1)}{5 - 1}$

$$\text{let } x \in (1, 5), f'(x) = \frac{f(x) - f(1)}{x - 1} < \frac{3}{8} \iff \frac{f(x) - 1}{x - 1} < \frac{3}{8}$$

$$f(x) - 1 < \frac{3}{8}(x - 1), f(x) < \frac{3}{8}(x - 1) + 1$$

$$f(x) < \frac{3}{8}(5 - 1) + 1 \rightarrow f(x) < \frac{5}{2}$$

*QED*

Suppose  $f(x)$  is odd for all  $x$  and diff across every real number. Prove that for every positive number  $b$ , there exists a positive number  $c$ , in  $(-b, b)$  such that  $f'(c) = \frac{f(b)}{b}$

Given  $b > 0, f(-x) = -f(x)$

since  $f$  is differentiable on  $(-\infty, \infty) \implies f$  is continuous on  $(-\infty, \infty)$

$f$  is cont on  $[-b, b]$  and differentiable on  $(-b, b)$

because  $(-b, b) \subset (-\infty, \infty)$ ,

$$\text{By MVT, } \exists c \in (-b, b) \ni: f'(c) = \frac{f(b) - f(-b)}{b - (-b)} = \frac{f(b) + f(b)}{b + b} = \frac{f(b)}{b}$$

## 6 Rolle's Theorem

Suppose  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$  and  $f(a) = f(b)$  then there exists atleast a  $c \in (a, b)$  such that  $f'(c) = 0$

## 7 Applications R'T

Show that the function  $2x + \cos x$  has exactly one real root.

$$\text{Let } f(x) = 2x + \cos x$$

$$f(-\pi) = -2\pi + \cos(-\pi) = -2\pi - 1 < 0$$

$$f(0) = 2 + \cos(0) = 1 > 0$$

Since  $f(x)$  is a sum of a polynomial and periodic trigonometric function,  $f$  is continuous and differentiable for all  $x$ , By IVT,  $\exists c \in (-\pi, 0) \ni: f(c) = 0$

Suppose  $f(x)$  has two roots on  $a, b$  a  $b$ , then  $f(a) = f(b) = 0$ , Since  $f$  is continuous on  $[a, b]$  and differentiable on open interval  $(a, b)$ .

By Rolle's Theorem  $\exists r \in (a, b) \ni: f'(r) = 0$

$f'(x) = 2 - \sin(x) > 0$ , Contradiction, rolles theorem fails and therefore there must be exactly one root and one root only.

*QED*

## 8 Fermat's Theorem

If  $f$  has a local min/max at  $x = c$  and  $f'(c)$  exists, then  $f'(c) = 0$ .

$f'(x) = 0$  or undefined for  $c$ -pts:

$f'(x) > 0 \Rightarrow f(x)$  is increasing

$f'(x) < 0 \Rightarrow f(x)$  is decreasing

## 9 Proof of Sum Law for Limits

Prove  $\lim_{x \rightarrow c} f(x) = L \wedge \lim_{x \rightarrow c} g(x) = M \Rightarrow \lim_{x \rightarrow c} [f(x) \pm g(x)] = L \pm M$

Suppose  $\lim_{x \rightarrow c} f(x) = L \wedge \lim_{x \rightarrow c} g(x) = M$

$\lim_{x \rightarrow c} f(x) = L$

$\forall \epsilon_1 > 0, \exists \delta_1 > 0, \ni: 0 < |x - c| < \delta_1 \Rightarrow |f(x) - L| < \epsilon_1$

$\lim_{x \rightarrow c} g(x) = M$

$\forall \epsilon_2 > 0, \exists \delta_2 > 0, \ni: 0 < |x - c| < \delta_2 \Rightarrow |g(x) - M| < \epsilon_2$

Consider  $\epsilon_1, \epsilon_2 = \frac{\epsilon}{2}$

There must consequently be a  $\delta_1 > 0$  and a  $\delta_2 > 0$  such that:

$0 < |x - c| < \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{2}$

$0 < |x - c| < \delta_2 \Rightarrow |g(x) - M| < \frac{\epsilon}{2}$

let  $\delta = \min(\delta_1, \delta_2)$  such that  $0 < |x - c| < \delta \Rightarrow |f(x) - L| < \frac{\epsilon}{2} \wedge |g(x) - M| < \frac{\epsilon}{2}$

$0 < |x - c| < \delta \Rightarrow |f(x) - L| + |g(x) - M| < \epsilon$

$0 < |x - c| < \delta \Rightarrow |f(x) - L + (g(x) - M)| \leq |f(x) - L| + |g(x) - M| < \epsilon$

$0 < |x - c| < \delta \Rightarrow |f(x) + g(x) - (L + M)| < \epsilon$

Thus:  $\lim_{x \rightarrow c} f(x) \pm g(x) = L \pm M$

and:  $\lim_{x \rightarrow c} f(x) = L \wedge \lim_{x \rightarrow c} g(x) = M \Rightarrow \lim_{x \rightarrow c} [f(x) \pm g(x)] = L \pm M$

*QED*

## 10 Proof of Limits and Derivatives

Prove  $\frac{d}{dx} [\ln x] = \frac{1}{x}$

Prove  $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$

$\forall M > 0, \exists \delta > 0, \ni: 0 < |x| < \delta \Rightarrow \frac{1}{x^2} > M$

$\frac{1}{x^2} > M \rightarrow x^2 < \frac{1}{M} \rightarrow x < \sqrt{\frac{1}{M}}$

Proof:

Let  $\delta = \sqrt{\frac{1}{M}}$

$|x| < \delta \rightarrow \frac{1}{|x|} > M \rightarrow \frac{1}{|x|^2} > M \rightarrow \frac{1}{x^2} > M$

$\frac{d}{dx} [\ln x] = \lim_{h \rightarrow 0} \frac{\ln(x+h) - \ln(x)}{h} = \lim_{h \rightarrow 0} \frac{\ln(\frac{x+h}{x})}{h}$

$= \lim_{h \rightarrow 0} \frac{1 + \frac{h}{x}}{h} = \lim_{h \rightarrow 0} (\frac{1}{h}) [\ln(1 + \frac{h}{x})] = \lim_{h \rightarrow 0} [\ln(1 + \frac{h}{x}) \frac{1}{h}]$

let  $n = \frac{h}{x}; h = nx; \frac{1}{h} = \frac{1}{nx}$

$\lim_{x \rightarrow 0} [(\ln(1+n) \frac{1}{n}) \frac{1}{x}] = \lim_{x \rightarrow 0} \frac{1}{x} \cdot [\ln(1+n) \frac{1}{n}] = \frac{1}{x} \cdot \ln e$

$= \frac{1}{x}$

*QED*

