Lecture Notes
Winter 2019

MATA37 - CALCULUS II FOR THE MATHEMATICAL SCIENCES

LEC03, Jan 21st, 2:00pm - 3:00pm

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1 Recall

Darboux definition of the definite integral,

f is integrable on [a,b] $\iff \sup\{L(f,P)|\forall P \in [a,b]\} = \inf\{U(f,P)|\forall P \in [a,b]\} = \int_a^b f(x)dx$

2 Integrability reformulation

let $[a,b] \in \mathbb{R}, a < b$,

f is integrable on [a,b] $\iff \forall \epsilon > 0, \exists P$ partition of $[a,b]$ such that $U(f,p) - L(f,p) < \epsilon$

Note that we do not compute definite integrals with this definition, only if the function is integrable.

Example proof

Consider

$$g(x) = \begin{cases} -3 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Prove $\int_0^1 g(x)dx$ DNE by int reformulation.

Show $\neg(\forall \epsilon > 0, \exists P$ partition of $[a,b]$ such that $U(f,p) - L(f,p) < \epsilon)$

Show $\exists \epsilon > 0, \ni: \forall P$ partition of $[a,b], U(f,p) - L(f,p) \geq \epsilon$

Choose $\epsilon > 0$

Let P be an arbitrary partition of [0, 1]

for $i = 1, 2, 3, \dots n$, due to the density of $\mathbb{Q}, \mathbb{I}$ any interval will contain both.

$m_i = inf\{f(x)|x \in [x_{i-1}, x_i]\} = inf\{-3, 0\} = -3$

$M_i = sup\{f(x)|x \in [x_{i-1}, x_i]\} = sup\{-3, 0\} = 0$

So $U(g,p) - L(g,p) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) - \sum_{i=1}^{n} m_i(x_i - x_{i-1})$

*We have $0 - (-3) \cdot \sum_{i=0}(x_i - x_{i-1}) = 0 + 3\cdot$ length([0, 1]) $= 3(1-0) = 3$

This statement happens due to dramatic cancellations of the telescopic sum

We can simply pick $\epsilon = 3$ or any positive number below or equal to 3 to satisfy this proof. *QED*

3 Indefinite Integral

Recall from MATA31, suppose $f(x)$ is continuous on interval I, then the antiderivative of $f(x)$ on I is the is a function $F(x), \ni: F'(x) = f(x)$

Warmups

1. Consider $I = [0, 1]$ and $f(x) = e^x$
   Then $F(x) = e^x$; Let $x \in [0, 1]$, $F'(x) = e^x = f(x)$
2. Let $n \in \mathbb{Z}^+$ consider $f(x) = x^n, I \in \mathbb{R}$
   Them $F(x) = \dfrac{x^{n+1}}{n+1} + C$
   Let $x \in \mathbb{R}$, $F'(x) = x^n = f(x)$

antiderivatives are only unique up to addition of a constant.
ie $\displaystyle\int f(x)dx = F(x) + C$
Example
Evaluate $\displaystyle\int \frac{1}{4x+1}dx$, $F(x) = \dfrac{\ln(4x+1)}{4}$
Whenever we have a log/ln as an antiderivative, always enclose the argument inside abs brackets.
Sol'n = $\dfrac{\ln|4x+1|}{4} + C$

4 Properties of $\int f(x)dx$

If f, g are cont.
then

1. $\displaystyle\int (f(x) \pm g(x)) = \int f(x) \pm g(x)$
2. For any $r \in \mathbb{R}$, $\displaystyle\int kf(x)dx = k\int f(x)$

Proofs

1. Suppose f, g are cont. WTS $\displaystyle\int (f(x) \pm g(x)) = \int f(x) \pm \int g(x)$
   Consider the right side,
   Since $\displaystyle\int f(x) = F(x) + C$
   $\displaystyle\int g(x) = G(x) + C$
   $\displaystyle\int f(x) \pm \int g(x) = F(x) + C_1 \pm (G(x) + C_2)$ since both C_1, C_2 are arbitrary,
   $\displaystyle\int f(x) \pm \int g(x) = F(x) \pm G(x) + C = \int f(x) \pm g(x)$ *QED*

Exercise

Compute $\int (\frac{\sin(2x)}{\sin(x)} + 5^x)dx$

Start with this, and remember the $\sin(2x)$ identity.

Solution: $\int (\frac{\sin(2x)}{\sin(x)} + 5^x)dx = \int \frac{\sin(2x)}{\sin(x)}dx + \int 5^x dx$

5 FTOC - Part I

Pg 364

5.1 The Definite Integral

Let $a, b \in \mathbb{R}, a < b$.

IF f is continuous on $[a, b]$, and F is any antiderivative of f on $[a, b]$

THEN $\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$

ALSO $\int_a^b F'(x)dx = \int_a^b f(x)dx$

Example

Evaluate $\int_0^1 \frac{1}{1+x^2}dx$

Solution:

Note $f(x) = \frac{1}{1+x^2}$ is a rational function, thus it's continuous throughout it's domain.

In particular, f is cont on [0, 1]

Note $F(x) = \arctan(x), F'(x) = \frac{1}{1+x^2}$

$\therefore \int_0^1 \frac{1}{1+x^2}dx = \int f(x)|_0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4}$

NOTE in MATA37, we are granted permission to not check the hypothesis of **FTOC** to use it.

For Proof of FTOC I.

Suppose f is continuous on [a,b] and F is an antiderivative of f on [a,b]

given those, we WTS $\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$

Let $P = \{x_i\}_{i=0}^n$ be a Riemann Partition of $[a, b]$

ie Define $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$, i = 0,1,2,3 …n

So $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$, $x_i^* \in [x_{i-1}, x_i]$