MATA31
Calculus 1, for Mathematical Sciences, Fall 2018

Various Core Proofs/Identities

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1 Identities

Basic Trigonometric Identities

Reciprocal Identities

$(\sin x)^{-1} = \csc x$
$(\cos x)^{-1} = \sec x$
$(\tan x)^{-1} = \cot x$

Pythagorean Identities

$\sin^2 x + \cos^x = 1$
$\tan^2 x + 1 = \sec^2 x$
$1 + \cot^2 x = \csc^2 x$

Angle Addition Identities

$\sin(x \pm y) = \sin x \cdot \cos y \pm \cos x \cdot \sin y$
$\cos(x + y) = \cos x \cdot \cos y - \sin x \cdot \sin y$
$\cos(x - y) = \cos x \cdot \cos y + \sin x \cdot \sin y$
$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \cdot \tan y}$$
$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \cdot \tan y}$$

Hyperbolic Trigonometric Identities

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{2}{e^x - e^{-x}}$$
$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Derivatives of Trigonometric Functions

Derivative of Trigonometric Functions

$(\sin x)' = \cos x \qquad (\csc x)' = -\csc x \cot x$
$(\cos x)' = -\sin x \qquad (\sec x)' = \sec x \tan x$
$(\tan x)' = \sec^2 x \qquad (\cot x)' = -\csc^2 x$

Derivative of Hyperbolic Trigonometric Functions

$(\sinh x)' = \cosh x \qquad (\operatorname{csch} x)' = -\operatorname{csch} x \cdot \coth x$
$(\cosh x)' = \sinh x \qquad (\operatorname{sech} x)' = -\operatorname{sech} x \cdot \tanh x$
$(\tanh x)' = \operatorname{sech}^2 x \qquad (\coth x)' = -\operatorname{csch}^2 x$

Derivative of Inverse Trigonometric Functions

$$(\sin^{-1} x)' = \frac{1}{\sqrt{1 - x^2}} \qquad (\csc^{-1} x)' = -\frac{1}{|x|\sqrt{x^2 - 1}}$$
$$(\cos^{-1} x)' = -\frac{1}{\sqrt{1 - x^2}} \qquad (\sec^{-1} x)' = -\frac{1}{|x|\sqrt{x^2 - 1}}$$
$$(\tan^{-1} x)' = \frac{1}{1 + x^2} \qquad (\cot^{-1} x)' = -\frac{1}{1 + x^2}$$

Derivatives of common functions

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}e^x = e^x \cdot \ln(e)$$

$$\frac{d}{dx}e^{2x} = e^x \cdot \ln(e) \cdot \frac{d}{dx}x = e^x \cdot 2x$$

$$\frac{d}{dx}x^n = nx^{(n-1)}$$

$$\frac{d}{dx}f + h = f' + h'$$

$$\frac{d}{dx}f(g) = f'(g) \cdot g'$$

$$\frac{d}{dx}f \cdot g = f' \cdot g + g' \cdot f$$

$$\frac{d}{dx}\frac{f}{g} = \frac{g \cdot f' - f \cdot g'}{g^2}$$

2 Definitions of Limits

Algebraic: $\lim_{x\to c} f(x) = L$
$\forall\epsilon > 0, \exists\delta > 0, \ni: 0 < |x-c| < \delta \Rightarrow |f(x) - L| < \epsilon$

Geometric: $\lim x \to cf(x) = L$
$\forall\epsilon > 0, \exists\delta > 0, \ni: x \in (c-\delta, c) \cup (c, c+\delta) \Rightarrow f(x) \in (L-\epsilon, L+\epsilon)$

3 Uniqueness of Limits

Theorem. $\lim_{x\to c} f(x) = L \wedge \lim_{x\to c} f(x) = M \implies L = M$

$\lim_{x\to c} f(x) = L \wedge \lim_{x\to c} f(x) = M \implies L = M$
Proof:
Suppose the contrary that:
$\lim_{x\to c} f(x) = L \wedge \lim_{x\to c} f(x) = M, L \neq M$
Assume that $L > M, L = M + K$, WLOG
Let's choose $\epsilon = \frac{k}{2}$, this way the intervals do not overlap.
$\lim_{x\to c} f(x) = L : \exists\delta_1 > 0, \ni: 0 < |x-c| < \delta_1 \to |f(x) - L| < \epsilon$
$\lim_{x\to c} f(x) = M : \exists\delta_2 > 0, \ni: 0 < |x-c| < \delta_2 \to |f(x) - M| < \epsilon$
Let $\delta = min(\delta_1, \delta_2)$ so that for any $\delta > 0$, $f(x) \in (M-\epsilon, M+\epsilon)$ and $f(x) \in (L-\epsilon, L+\epsilon)$.
Contradiction: This is impossible since we set $\epsilon = \frac{k}{2}$ to guarantee intervals do not overlap.
Therefore, $\lim_{x\to c} f(x) = L \wedge \lim_{x\to c} f(x) = M \implies L = M$ by contradiction.

QED

4 One Sided Limits

Theorem. $\lim_{x\to c} f(x) = L \iff \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$

We need to prove that:
1. $\lim_{x\to c} f(x) = L \Rightarrow \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$
2. $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L \Rightarrow \lim_{X\to c} f(x) = L$

1. Assume $\lim_{x\to c} f(x) = L$

$\forall\epsilon > 0, \exists\delta > 0, \ni: 0 < |x-c| < \delta \Rightarrow |f(x) - L| < \epsilon$
$0 < |x-c| < \delta \equiv x \in (c-\delta, c) \cup (c, c+\delta)$
$x \in (c-\delta, c) \Rightarrow |f(x) - L| < \epsilon$
$x \in (c, c+\delta) \Rightarrow |f(x) - L| < \epsilon$
$\therefore \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$

2. Assume $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$

$\forall\epsilon > 0, \exists\delta_1, \delta_2 > 0, \ni:$
$x \in (c-\delta, c) \vee x \in (c, c+\delta) \Rightarrow |f(x) - L| < \epsilon$
Let $\delta = \min(\delta_1, \delta_2)$
$0 < |x-c| < \delta \Rightarrow |f(x) - L| < \epsilon$
By definition: $\lim_{x\to c} f(x) = L$

QED

5 Supremum, Infimum

The Least Upper Bound Axiom: Every none-empty set of real numbers that has an upper bound has a least upper bound.

Theorem. $lub(S) = M \wedge \epsilon > 0 \Rightarrow \exists s \in S, \ni: M - \epsilon < s \leq M$

Proof: Let $\epsilon > 0$, since M is an upper bound for S, the condition $s \leq M$ is satisfied by all numbers $s \in S$, therefore there must exist a number in S where $M - \epsilon < s$.

Assume the contrary: $\exists! s \in S, \ni: M - \epsilon < s \Rightarrow \forall x \in S, x \leq M - \epsilon$

Contradiction: this means that $M - \epsilon$ is the upper bound for set S, but this cannot be as M is our least upper bound and $M - \epsilon < M, \epsilon > 0$

Therefore, there must exist $s \in S, \ni: M - \epsilon < s \leq M$ QED

The Greatest Lower Bound Axiom: Every none-empty set of real numbers that has a lower bound has a greatest lower bound.

Theorem. $glb(S) = m \wedge \epsilon > 0 \Rightarrow \exists s \in S, \ni: m \leq s < m + \epsilon$

Proof: Let $\epsilon > 0$ be given, since m is a lower bound for S, the condition $m \leq s$ is satisfied by all numbers $s \in S$, therefore there must exist a number in S where $s < m + \epsilon$

Assume the contrary: $\exists! s \in S, \ni: s < m + \epsilon \Rightarrow \forall s \in S, s \geq m + \epsilon$

Contradiction: this means that $m + \epsilon$ is the lower bound for set S, but this cannot be because m is our greatest lower bound and $m < m + \epsilon, \epsilon > 0$

Therefore, there must exist $s \in S, \ni: m \leq s < m + \epsilon$ QED

6 Proofs of continuity

$$\lim_{x \to c} f(x) = f(c)$$

Constant Functions

$f(x) = C$

Show $\lim_{x \to c} f(x) = C$

$\forall \epsilon > 0, \exists \delta > 0, \ni: |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$

$|f(x) - f(c)| = |C - C| < \epsilon$

$|C - C| < \epsilon$ is True $\forall \epsilon > 0$.

if we have $P \to Q$, and Q is always true

then $P \to Q$ is always true.

$\therefore \lim_{x \to c} f(x) = f(c)$

Power Functions

$f(x) = x^2$

Show $\lim_{x \to c} f(x) = c^2$

$\forall \epsilon > 0, \exists \delta > 0, \ni: |x - c| < \delta \Rightarrow |x^2 - c^2| < \epsilon$

$|x^2 - c^2| = |x - c||x + c|$

lets set $\delta \leq 1, |x - c| < 1 \Longrightarrow |x| - |c| < 1 \Longrightarrow |x| < 1 + |c|$

Proof:

let $\delta = \min(1, \dfrac{\epsilon}{1 + 2|c|})$

$|x^2 - c^2| = |x - c||x + c| < (1 + 2|c|) \cdot |x - c|$

$< (1 + 2|c|) \cdot \dfrac{\epsilon}{1 + 2|c|} = \epsilon$

$\therefore \lim_{x \to c} x^2 = c^2$

Exponential Functions

$f(x) = e^x$

Show $\lim_{x \to c} f(x) = e^c$

$\forall \epsilon > 0, \exists \delta > 0, \ni: |x - c| < \delta \Rightarrow |e^x - e^c| < \epsilon$

$|x - c| < \delta, x - c < \delta \equiv x < c + \delta \equiv e^x < e^{x+\delta}$

$0 \leq e^x - e^c < e^c(e^\delta - 1)$

7 Mean Value Theorem

if function f is continuous across $[a, b]$ and differentiable across(a, b)

then, there exists atleast one $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

This means if the conditions satisfy, there will be atleast one point c where the instataneous rate of change is the same as the average rate of change.

8 Applications of MVT

Suppose $f(x)$ is continuous on $[1,5]$ and differentiable on (1,5) and $f'(x) < \frac{3}{8}, \forall x \in (1,5)$. If $f(1) = 1$, show that $f(x) < \frac{5}{2}, \forall x \in [1.5]$

Since f is continuous and differentiable...by Mean Value Theorem, $\exists c \in (1,5), \ni: f'(c) = \frac{f(5)-f(1)}{5-1}$

let $x \in (1,5), f'(x) = \frac{f(x)-f(1)}{x-1} < \frac{3}{8} \iff \frac{f(x)-1}{x-1} < \frac{3}{8}$

$f(x) - 1 < \frac{3}{8}(x-1)$, $f(x) < \frac{3}{8}(x-1)+1$

$f(x) < \frac{3}{8}(5-1)+1 \to f(x) < \frac{5}{2}$ *QED*

Suppose $f(x)$ is odd for all x and diff across every real number. Prove that for every positive number b, there exists a positive number c, in $(-b,b)$ such that $f'(c) = \frac{f(b)}{b}$

Given $b > 0, f(-x) = -f(x)$

since f is differentiable on $(-\infty,\infty) \Longrightarrow f$ is continuous on $(-\infty,\infty)$

f is cont on [-b,b] and differentiable on (-b,b)

becuase (-b,b)$\subset (-\infty,\infty)$,

By MVT, $\exists c \in (-b,b) \ni: f'(c) = \frac{f(b)-f(-b)}{b-(-b)} = \frac{f(b)+f(b)}{b+b} = \frac{f(b)}{b}$

9 Rolle's Theorem

Suppose f is continuous on [a,b] and differentiable on (a,b) and $f(a) = f(b)$ then there exists atleast a $c \in (a,b)$ such that $f'(c) = 0$

10 Applications R'T

Show that the function $2x + \cos x$ has exactly one real root.

Let $f(x) = 2x + \cos x$

$f(-\pi) = -2\pi + \cos(-\pi) = -2pi - 1 < 0$

$f(0) = 2 + \cos(0) = 1 > 0$

Since $f(x)$ is a sum of a polynomial and periodic trigonometric function, f is continuous and differentiable for all x, By IVT, $\exists c \in (-\pi,0) \ni: f(c) = 0$

Suppose $f(x)$ ias two roots on a,b a ¡ b, then $f(a) = f(b) = 0$, Since f is continuous on [a,b] and differentiable on open interval (a,b).

By Rolle's Theorem $\exists r \in (a,b) \ni: f'(r) = 0$

$f'(x) = 2 - \sin(x) > 0$, Contradiction, rolles theorem fails and therefore there must be exactly one root and one root only. *QED*

11 Fermat's Theorem

If f has a local min/max at x = c and $f'(c)$ exists, then f'(c) = 0.

$f'(x) = 0$ or undefined for c-pts:

$f'(x) > 0 \Rightarrow f(x)$ is increasing

$f'(x) < 0 \Rightarrow f(x)$ is decreasing

12 Proof of Sum Law for Limits

Prove $\lim_{x\to c} f(x) = L \wedge \lim_{x\to c} g(x) = M \Longrightarrow \lim_{x\to c}[f(x) \pm g(x)] = L \pm M$

Suppose $\lim_{x\to c} f(x) = L \wedge \lim_{x\to c} g(x) = M$

$\lim_{x\to c} f(x) = L$

$\forall \epsilon_1 > 0, \exists \delta_1 > 0, \ni: 0 < |x - c| < \delta_1 \Longrightarrow |f(x) - L| < \epsilon_1$

$\lim_{x\to c} g(x) = M$

$\forall \epsilon_2 > 0, \exists \delta_2 > 0, \ni: 0 < |x - c| < \delta_1 \Longrightarrow |g(x) - M| < \epsilon_2$

Consider $\epsilon_1, \epsilon_2 = \dfrac{\epsilon}{2}$

There must consequently be a $\delta_1 > 0$ and a $\delta_2 > 0$ such that:

$0 < |x - c| < \delta_1 \Longrightarrow |f(x) - L| < \dfrac{\epsilon}{2}$

$0 < |x - c| < \delta_1 \Longrightarrow |g(x) - M| < \dfrac{\epsilon}{2}$

let $\delta = \min(\delta_1, \delta_2)$ such that $0 < |x - c| < \delta \Longrightarrow |f(x) - L| < \dfrac{\epsilon}{2} \wedge |g(x) - M| < \dfrac{\epsilon}{2}$

$0 < |x - c| < \delta \Longrightarrow |f(x) - L| + |g(x) - M| < \epsilon$

$0 < |x - c| < \delta \Longrightarrow |f(x) - L + (g(x) - M)| \leq |f(x) - L| + |g(x) - M| < \epsilon$

$0 < |x - c| < \delta \Longrightarrow |f(x) + g(x) - (L + M)| < \epsilon$

Thus: $\lim_{x\to c} f(x) \pm g(x) = L \pm M$

and: $\lim_{x\to c} f(x) = L \wedge \lim_{x\to c} g(x) = M \Longrightarrow \lim_{x\to c}[f(x) \pm g(x)] = L \pm M$ QED

13 Proof of Limits and Derivatives

Prove $\lim_{x \to 0} \frac{1}{x^2} = \infty$

$\forall M > 0, \exists \delta > 0, \ni: 0 < |x| < \delta \implies \frac{1}{x^2} > M$

$\frac{1}{x^2} > M \to x^2 < \frac{1}{M} \to x < \sqrt{\frac{1}{M}}$

Proof:

Let $\delta = \sqrt{1}M$

$|x| < \delta \to \frac{1}{|x|} > M \to \frac{1}{|x|^2} > M \to \frac{1}{x^2} > M$

Prove $\frac{d}{dx}[\ln x] = \frac{1}{x}$

$\frac{d}{dx}[\ln x] = \lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h} = \lim_{h \to 0} \frac{\ln(\frac{x+h}{x})}{h}$

$= \lim_{h \to 0} \frac{1 + \frac{h}{x}}{h} = \lim_{h \to 0} (\frac{1}{h})[\ln(1 + \frac{h}{x})] = \lim_{h \to 0} [\ln(1 + \frac{h}{x})^{\frac{1}{h}}]$

let $n = \frac{h}{x}; h = nx; \frac{1}{h} = \frac{1}{nx}$

$\lim_{x \to 0} [(\ln(1+n)^{\frac{1}{n}})^{\frac{1}{x}}] = \lim_{x \to 0} \frac{1}{x} \cdot [\ln(1+n)^{\frac{1}{n}}] = \frac{1}{x} \cdot \ln e$
$= \frac{1}{x}$

QED