STAB57 Lecture 1

Class	STAB57
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Type	Lecture

`Population`: Every member of a group.
`Sample`: Subset of a population
`Parameter`: Some characteristic of the population
`Statistic`: Some characteristic of the Sample

`Probability` vs `Statistical` theories

In `Statistical` theories, often we don't know the mathematical model, instead we see the raw data. In `Probability` theories, we know the distribution.

We'll focus on the following topics:

- Sampling Distribution
- Convergence in Probability
- Convergence in Distribution
- Central Limit Theorem

Sampling Distribution

recall IID: Independent and identically distributed.

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables from some $f(x;\theta)$
Let $x_1, x_2, \ldots, x_n$ be realizations (samples) of these i.i.d. random variables

The sample average $\bar{x} = \displaystyle\sum_i \frac{x_i}{n}$, now is $\bar{x}$ also a random variable? absolutely.

In general, if $Y = h(x_1, \ldots, x_n)$ then Y is also a random variable, and has probability distribution.

Lastly, the probability distribution of Y is called a `Sampling Distribution`, sometimes this can be exactly identified, sometimes this is simply approximations.

Convergence in Probability

Let $a_n = 1 - 1/n$, we see that $\lim_{n\to\infty} a_n = 1$ this is called convergence.

For $n \to \infty$ the distribution of Y sometimes converges, but what does this mean?

When we say a random variable converges, we mean either convergence in probailities or convergence in distribution.

Definition

let $X_1, X_2, \ldots$ be an infinite sequence of random variables and let Y be another random variable, then we say $\{X_n\}$ converges in probability to Y if $\forall \epsilon > 0, \lim_{n\to\infty} P(|X_n - Y| > \epsilon) = 0$ this is to say $X_n \to^P Y$

Weak Law of Large Numbers

Let $M_n = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$, where each X are independent with population mean μ and a finite variance σ^2 Then the weak law states that $\forall \epsilon > 0, \lim_{n\to\infty} P(|M_n - \mu| > \epsilon) = 0$. That is, the average converge in probability to the common mean μ that is $M_n \to^P \mu$.

Excercise: 4.1.12, 4.2.2-4.2.6
Problems: 4.2.14-4.2.17

Convergence with Probability 1

$P(\lim_{n\to\infty} X_n = Y) = 1 \Leftrightarrow X_n \to^{a.s.} Y$ where `a.s.` means almost surely. Furthermore, $X_n \to^{a.s.} Y \Rightarrow X_n \to^P Y$ with the same limit, however the converse is not true.

Strong Law of Large Numbers

Let $M_n = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$, where each X are i.i.d. with finite mean μ, then $P(\lim_{n\to\infty} M_n = \mu) = 1$ that is $M_n \to^{a.s.} \mu$.

Note that being identically distributed allows us to make use of much stronger theorems (almost sure convergence).

Convergence in Distribution

Let $X_1, X_2, \ldots, X_n$ be a sequence of random variables. Let X be another random variable. Then the sequence $\{X_n\}$ converges in distribution to X, if $\forall x \in \mathbb{R}, P(X = x) = 0$, we have $\lim_{n\to\infty} P(X_n \leq x) = P(X \leq x)$. This is denoted by $X_n \to^D X$.

Central Limit Theorem

One of the most important theory in statistics, Average of any sequence of random variables with finite mean and variance is asymptotically normally distributed. Thus the sample averages follow normal distribution for a large enough sample size.

Once we figure out the distribution of a statistic, we can compute any probability corresponding to the variable.

$X_1, X_2, \ldots, X_n$ be i.i.d. sequence of random variables with mean μ and variance σ^2 we define $S_n = X_1 + X_2 + \cdots + X_n, M_n = \bar{X} = S_n/n$. This is to say that M_n is the mean variable.

Now we define a random variable:
$Z = \dfrac{S_n - n\mu}{\sqrt{n}\sigma} = \sqrt{n}\dfrac{M_n - \mu}{\sigma}$ we see that as $n \to \infty, Z_n \to^D Z, Z \sim N(0,1)$