Winter 2021

Course Summary

CSCC63 - Computability and Computational Complexity

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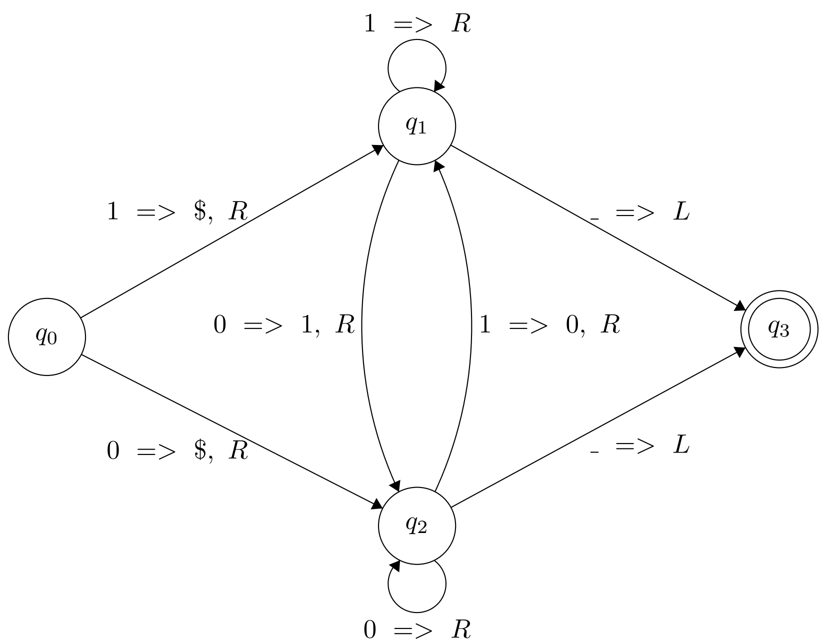
Tutorial One

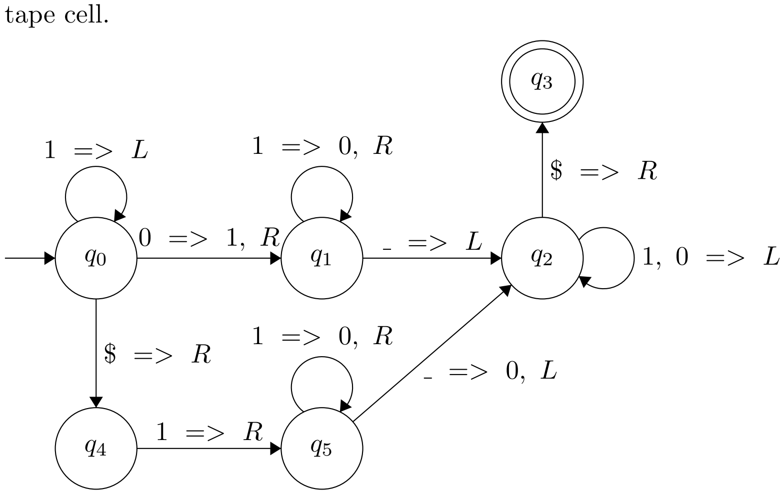
1. Skipping because question is bad

2. Let our input alphabet be $\{0, 1\}$. We consider our input to be a number in its binary representation. Design a TM that adds 2 to the input, then accepts with the head at the first tape cell.

   This first TM will append a \$ to the beginning this is important because we need to accept with the head at the first tape cell.



   The TM below will add 1 to the input, however the head must be at the end of the tape. We can easily compile these two to build a TM that adds 2 to the input and then accepts with the head at the first tape cell.



3. Post Correspondence Problem.

Tutorial Two

1. Number of effective enumerations to get to the string *aaabbc*:

   (a) $3^0 + 3^1 + 3^2 + 3^3 + 3^4 + 3^5 + 1$ to get to *aaaaaa*

   (b) $3^2 + 3^1 + 2 \cdot 3^0$ to get to *bbc* from *aaa*

   $3^0 + 3^1 + 3^2 + 3^3 + 3^4 + 3^5 + 1 + 3^2 + 3^1 + 2 \cdot 3^0 = 379$

2. $L = \{\langle M, w\rangle |\ M$ ever moves its head beyond the end of the input $w\}$

   Decidable because there are a total of $b = (|Q| \cdot |w|) * |\Gamma|^{|w|}$ total configurations of any given TM M in which the head doesn't move beyond the end of the input w.

   With b many iterations, we look for two things: either M at some point goes beyond the end of the input w, or a loop occurs when a configuration shows up two or more times. If the loop occurs and nothing goes beyond the end of the input w, we can say with certainty that $\langle M, w\rangle$ belongs to our set L.

3. All finite languages are decidable because there will be an upper bound for the string lengths, say n. Then there will only be possible strings of n symbols, we can easily construct a TM with a state for each of those strings, corresponding to accepts/rejects.

4. $L = \{\langle M, w\rangle | M$ tries to move its head off the left side of the input $w\}$

   Consider the following mapping reduction: $HALT \leq_m L$

```
0    P on input <M, w>:
1        define M_0 on input <x>:
2            run M on w
3            move the head to the left x infinitely
4        return M_0
```

   Assume that $\langle M, w\rangle \in HALT$

   Then line 2 halts, we then infinitely move the head to the left of x. $M_0 \in L$

   Assume that $\langle M, w\rangle \notin doesHALT$

   Then line 2 loops, M_0 does nothing, $M_0 \notin L$

   Since $HALT$ is not decidable, L mustn't be decidable.

Tutorial Three

1. Prove whether or not these are recognizable.

(a) $X_1 = \{\langle M \rangle | M \text{ is a TM, } M \text{ accepts some string that ends with } 101\}$
Recognizable via the following:
Let $w_0, w_1, \ldots$ be an effective enumeration over the input

```
M' on input <M>:
    for i = 0 to infinity:
        for k = 0 to i:
            if w_i ends with 101:
                run M on w_i for k steps
                accept if M accepts
```

This recognizer works because if:

- $M \in X_1$, eventually M accepts some w_i and M' accepts
- $M \notin X_1$, M will never accept that w_i and M' will thus loop forever

(b) $X_2 = \{\langle M \rangle | M \text{ is a TM, } M \text{ loops on some string that ends with } 101\}$
Consider the following mapping reduction, $\overline{HALT} \leq_m X_2$

```
0 P on input <M, w>:
1     define M_0 on input x:
2         pass
3         run M on w
4         accept x
5     return M_0
```

Suppose that $\langle M, w \rangle \in \overline{HALT}$
Then M loops on w, thus M_0 infinitely loops on line 3, then M_0 loops on all strings, including those that end with 101. $M_0 \in X_2$
Suppose that $\langle M, w \rangle \notin \overline{HALT}$
Then M halts on w, and M_0 accepts all input x on line 4, this includes all inputs that end with 101. $M_0 \notin X_2$
since $\overline{HALT} \leq_m X_2$ and $\overline{HALT}$ is not recognizable, M_0 must not be recognizable.

(c) $X_3 = \{\langle M \rangle | M \text{ is a TM, } M \text{ accepts all strings that ends with } 101\}$
Consider the following mapping reduction, $ALL_{TM} \leq_m X_3$

```
P on input <M>:
    define M_0 on input x:
        pass
        run M on x
        if M accepts, accept
    return M_0
```

Suppose that $\langle M \rangle \in ALL_{TM}$
Then M accepts all inputs x and thus M_0 accepts all inputs x. Then M_0 accepts all inputs including all strings that end with 101. $M_0 \in X_3$
Suppose that $\langle M \rangle \notin ALL_{TM}$

2. Suppose that L is a recognizable language. show that $L \leq_m HALT$

   Use the following reduction:

```
0   P on input <M, w>:
1       define M_0:
2           run M on w
3           if M accepts, halt
4           else loop
5       return <M, w>
```

   Suppose that $\langle M, w \rangle \in L$, then M accepts w, line 3 halts M_0 and thus $\langle M_0, w \rangle \in \overline{HALT}$

   Suppose that $\langle M, w \rangle \notin L$, then M either rejects w or loops,

   in either case line 3 will never run and thus $\langle M_0, w \rangle \notin HALT$

Tutorial Four

1. Consider the following language
   $X = \{\langle M\rangle |$ if M accepts any string, then M loops on some other string of the same length$\}$

   (a) Prove that X is co-recognizable
   $\overline{X} = \{\langle M\rangle | M$ accepts some string and M halts on all other strings of the same length$\}$
   We can use the following mapping reduction X: