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CSCA67 - Lecture Notes

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Example. What is the probability of rolling two die whose sum is larger than 3?

Solution. We could add up P(4) + P(5) + P(6) + P(7) + P(8) + P(9) + P(10) + P(11) + P(12). What is an easier way to compute this probability?

1 - $P($ sum $= 2$ $) \vee P($ sum $= 3) = 1 - \frac{1}{36} - \frac{2}{36} = \frac{11}{12}$

The Birthday Problem. What is the probability that in a group of n people at least two have the same birthday?

Solution. Lets represent the days of the year by the integers 1, 2, . . . , 365 and use E to represent the event that at least two people have the same birthday.

Q. What could we define our sample space S to be?

A. n tuples of integers from 1 to 365

Q.Lets assume all combinations or birthdays are equally likely. How many different ways are there for the n birthdays to fall?

A. $|S| = 365^n$

Q. Is it easy to count the number of tuples with 2 or more values the same?

A. No. cannot be solved by directly scanning the sample space.

Q. How can we rephrase the problem to make it easier?

A. Compute the complementary probaility what is the event that everyone has a distinct(different) birthday?

Q. What if $n > 365$?

A. P(E) = 1, this is by Pigeon Hole Principle, if the birthdays are equally likely, by PHP ther will be atleast two people with the same birthday.

so this counting problem is only for $n \leq 365$

Q. Suppose n 365. What is P(E^*)?

$P(E) = 1 - P(E^*)$

A. There is only one way to count the number of ways to get distinct birthdays:

What is $|E^*|$?, $|E^*| = 365 \cdot 364 \cdot 363 \cdot \cdots \cdot (365 - n + 1) = P(365, n)$

Therefore P(E) = 1 - P(E^*) $= 1 - \frac{|E^*|}{|S|} = 1 - \frac{P(365, n)}{365^n}$

This question is on the exam

Q. What value of n would you guess would result in a probability of at least 50% that two people have the same birthday?

A. This equation is hard to solve....but if we simplify we get:

P(E) = 1 - P(E^*) $= 1 - \frac{P(365, n)}{365^n}$

P(E) ≥ 0.5

$1 - \frac{P(365, n)}{365^n} \geq 0.5 \ldots n \geq 23$

1 Sum and Product Rules

Exercise. Consider tossing a coin five times. What is the probability of getting the same result on the first two tosses or the last two tosses?

Solution. Let E be the event that the first two tosses are the same and F be the event that the last two tosses are the same.

Instead of 5 bins, we have 4 bins with the first one being big bin.
$|E| = 2^4$
$|F| = 2^4$
$|S| = 2^5$

Q. Is there any overlap in E and F? I.e., is $E \cap F = \emptyset$?

A. $E \cap F \neq \emptyset$ because we can potentially have an event where the first two flips are the same as the last two flips. eg(H,H,T,T,T)

$$|E \wedge F| = |E \cap F| = 2^3 = 8$$

3 bins, two on either side and one in the middle.

Now we can calculate the probability P(E or F):

$$\text{P(E or F)} = P(E) + P(F) - P(\text{E and F}) = \frac{16}{32} + \frac{16}{32} - \frac{8}{34} = \frac{24}{32} = \frac{3}{4}$$

Theorem (The Sum Rule) If E and F are events in an experiment then the probability that E or F occurs is given by: P(E or F) = P(E) + P(F) P(E and F)

Example. What is the probability when a pair of dice are rolled that at least one die shows a 5 or the dice sum to 8?

Solution.

E_5 - atleast one die shows a 5

E_8 - atleast two die sum to 8

$$|E_8| = 5 \qquad |E_5| = 11 \qquad |E_8 \cap E_5| = 2$$

$$\text{P}(E_6 \text{ or } E_8) = P(E_5) + P(E_8) - P(E_5 \cap E_8)$$

$$\text{P}(E_6 \text{ or } E_8) = \frac{11}{32} + \frac{5}{36} - \frac{2}{32} = \frac{14}{32}$$

Exercise. Given a bag of 3 red marbles, 5 black marbles and 8 green marbles select one marble and then a second. What is the probability that both are red?

We do not replace the first marble before selecting the second.

16 marbles Total

Solution.

Q. What is the probability of the first marble being red?

A. $\frac{3}{16}$

Q. What is the probability of the second marble being red?

A. $\frac{2}{15}$

Q. Probability of both marbles being red?

A. $\frac{3}{16} \cdot \frac{2}{15}$

When the probability of an event E depends on a previous event F happening we denote this probability as $P(E|F)$.

Theorem (The Product Rule). If E and F are two events in an experiment then the probability that both E and F occur is:

$$P(E \wedge F) = P(E) \cdot P(E|F) = P(F) \cdot P(F|E)$$

Q. What does it mean if P(E—F) = P(E)?

A. Two events are independent

Q. Therefore, if E and F are two independent events, what is (*)?

A. $P(E \wedge F) = P(E) \cdot P(F)$

Example. Suppose there is a noisy communication channel in which either a 0 or a 1 is sent with the following probabilities:

- Probability a 0 is sent is 0.4.
- Probability a 1 is sent is 0.6.
- Probability that due to noise, a 0 is changed to a 1 during transmission is 0.2.
- Probability that due to noise, a 1 is changed to a 0 during transmission is 0.1.

Suppose that a 1 is received. What is the probability that a 1 was sent?

Let A denote that a 1 was received and B denote the event that a 1 was sent.

Q. What is the probability that we are solving for?

A. $P(B|A)$ B - 1 was sent, A - 1 was received.

How can we solve for this? The Product Rule says that:

$$P(A \wedge B) = P(B) \cdot P(A|B) = P(A) \cdot P(B|A)$$

Q. What is P(B)? **A.** P(B) = 0.6 (given)

Q. What is $P(A|B)$? **A.**$P(A|B) = 1 - (0.1) = 0.9$

Observation. There are only two possible events that result in a 1 being received. Either a 1 is sent and received or a 0 is sent and due to noise a 1 is received.

The probability that A happens is the sum:

P(A) = P(0 is sent)·P(A—0 is sent) + P(1 is sent)·P(A—1 is sent)

therefore:

$$\text{P(A)} = 0.4 \cdot 0.2 + 0.6 \cdot (1 - 0.1) = 0.62$$

Now we can compute P(B—A):

$$\text{P(B—A)} = \frac{P(B) \cdot P(A|B)}{P(A)} = 0.87$$

We used two important relationships in the example. The first, is

Bayes Rule.

Theorem (Bayes Rule). Let A and B be events in the same sample space. If neither P(A) nor P(B) are zero, then:

$$\frac{P(B) \cdot P(A|B)}{P(A)}$$

The second is the concept of total probability:

Theorem (Total Probability). Let a sample space S be a disjoint union of events $E + 1, E + 2, ..., E_n$ with positive probabilities, and let $A \subset S$. Then:

$$P(A) = \sum_{i=1}^{n} P(A|E_i) \cdot P(E_i)$$

2 Probability In Complexity Analysis

When we talk about how good an algorithm is, we often use the terms worst case complexity. This means, the number of steps the algorithm takes for the worst possible input.

Sometimes we care more about the average or expected number of steps.

Example. Searching a list L of integers from left to right. In the following list:

L = 3, 5, 23, 6, 4, 1, 7, 10, 26, 8, 9, 11, 15

if we search by visiting each integer from left to right we will perform 6 visits if we are looking for the number 1 and only 2 visits if we are searching for 5.

Q. How many integers do we visit in the worst case?

A. In general, for a list of length n, n comparisons. (len(L)) Happens when the item we are looking for is on the far right of the list.

Q. If we assume that we are searching for an integer k in the list and that all positions of the list are equally likely to hold the integer we are looking for, what is the probability of finding the integer in the ith position of a list of length n?

Expected Average it's in the list and everything is equally likely.

A. $\frac{1}{n}$

Q. How many steps(comparisons) does it take to find the integer if it is in the ith position?

A. Call this $N(i)$

We compute the expected E(n) number of steps of the algorithm by computing for each of the i positions:

P(k in the i th position) N(i)

and then taking the sum over all n possible values for i

$$\text{E(n)} = \sum_{i=1}^{n} P(\text{k in the ith position}) \cdot N(i)$$

$$= \sum_{i=1}^{n} \frac{1}{n} \cdot i$$

$$= \frac{1}{n} \cdot \frac{n(n+1)}{2} = \frac{n(n+1)}{2}$$