

Sample Midterm

CSCA67 - Discrete Mathematics

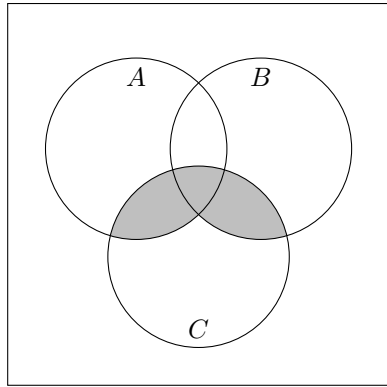


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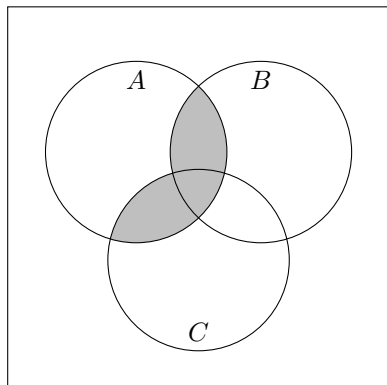
Question 1.

Part (a) Each option below is a predicate formula. Circle all (I bolded ones that i think are correct) answers equivalent to the given.



- (a) $a \wedge b \wedge c$
(b) $(a \wedge c) \vee (b \wedge c)$
(c) $(\neg b \rightarrow a) \wedge c$
(d) $c \wedge (b \vee a)$
 (e) None

Part (b) Write a propositional statement using \rightarrow and \neg if necessary equivalent to regions 3, 5, and 6. marks you can leave your solution using \wedge, \vee, \neg and \rightarrow . You do not need to specify which equivalence laws you use if you need them.



- $(a \wedge b) \vee (a \wedge c)$
 $a \wedge r(b \vee c)$
 $\neg(a \rightarrow \neg(\neg b \rightarrow c))$

Part (c) Circle (I just bolded.) every statement equivalent to $\neg a \rightarrow b$.

- | | |
|-----------------------------------|--|
| (a) $\neg b \rightarrow a$ | (f) a is necessary for $\neg b$ |
| (b) $a \rightarrow \neg b$ | (g) $\neg a$ is necessary for b |
| (c) a is sufficient for $\neg b$ | (h) b if $\neg a$ |
| (d) $\neg(a \wedge \neg b)$ | (i) $b \vee a$ |
| (e) b is sufficient for $\neg a$ | |

Question 2.

Determine whether \forall can be factored from an implication. In other words is

$$\forall x \in X, (p(x) \rightarrow q(x)) \iff \forall x \in X, p(x) \rightarrow \forall x \in X, q(x)$$

true? Explain your reasoning. Marks will only be given for your explanation.

No.

let $p(x) = x > 5$ and $q(x) = x^2 > 25$

$$\forall x \in \mathbb{Z}, (p(x) \rightarrow q(x)) \iff \forall x \in \mathbb{Z}, (x > 5 \rightarrow x^2 > 25)$$

which means: for all integers x , if x is greater than 5, then x^2 is greater than 25.

vs.

$$\forall x \in \mathbb{Z}, p(x) \rightarrow \forall x \in \mathbb{Z}, q(x) \iff \forall x \in \mathbb{Z}, x > 5 \rightarrow \forall x \in \mathbb{Z}, x^2 > 25$$

which means: if all integers are greater than 5, then all integers squared are greater than 25.

Sometimes, factoring in \forall can lead to a statement that is true, but never equivalent.

$\forall x \in X, (p(x) \rightarrow q(x))$ is saying for every x , if predicate p is true, then predicate q is true.

$\forall x \in X, p(x) \rightarrow \forall x \in X, q(x)$ is saying for every x , if predicate p is true, then for every x , predicate q is true, since are not equivalent, therefore \forall cannot be factored into or from an implication.

Question 3.

Answer the following questions to construct a direct proof that:

$$\forall n \in \mathbb{N}, n \geq 2, \forall a, b \in \mathbb{N}, a \equiv_n b \rightarrow a^2 \equiv_n b^2$$

Part (a)

Write a and b in terms of n using the division theorem and taking into consideration any common variables.

$$\begin{aligned} a &= k_1 \cdot n + r \\ b &= k_2 \cdot n + r \\ k_1, k_2 &\in \mathbb{Z}, r \in [0, (n-1)] \end{aligned}$$

Part (b)

Now complete the proof.

Assume $a \equiv_n b$, then that means:

$$\begin{aligned} a &= k_1 \cdot n + r & a^2 &= k_1^2 \cdot n^2 + 2k_1 \cdot n + r^2 \\ b &= k_2 \cdot n + r & b^2 &= k_2^2 \cdot n^2 + 2k_2 \cdot n + r^2 \\ k_1, k_2 &\in \mathbb{Z}, r \in [0, (n-1)] \end{aligned}$$

$$\begin{aligned} a^2 &= k_1^2 \cdot n^2 + 2k_1 \cdot n + r^2 = n(k_1^2 \cdot n + 2k_1) + r^2 \\ b^2 &= k_2^2 \cdot n^2 + 2k_2 \cdot n + r^2 = n(k_2^2 \cdot n + 2k_2) + r^2 \end{aligned}$$

Since both a^2 and b^2 share the same remainder r^2 when divided by n ,
 $\therefore \forall n \in \mathbb{N}, n \geq 2, \forall a, b \in \mathbb{N}, a \equiv_n b \rightarrow a^2 \equiv_n b^2$.

QED

Question 4.

Prove that $\forall n \in \mathbb{N}, (n^2 - 1 \not\equiv_4 0) \rightarrow n$ is even.

Assume the contrapositive

$$n \in \{2k + 1\}, k \in \mathbb{Z} \rightarrow (n^2 - 1 \equiv_4 0)$$

$$\begin{aligned} n &= 2k + 1, k \in \mathbb{Z} \\ n^2 &= 4k^2 + 4k + 1 \\ n^2 - 1 &= 4k^2 + 4k = 4(k^2 + k) \\ n^2 - 1 &\equiv_4 0 \end{aligned}$$

Since we prove the contrapositive to be true, and the contrapositive is logically equivalent to the original,
 $\therefore n \in \{2k + 1\}, k \in \mathbb{Z} \rightarrow (n^2 - 1 \equiv_4 0)$

QED

Question 5.

Part (a) Write the following claim as an implication using quantifiers and \rightarrow . You may use $p(x, y)$ to denote that x and y are relatively prime.

a and n relatively prime is necessary for there to exist a unique natural number $b < n$ such that $a \cdot b \equiv_n 1$.

$$\exists n \in \mathbb{N}, \exists: b < n, a \cdot b \equiv_n 1 \rightarrow \gcd(a, n) = 1$$

Part (b) Prove the claim.

Assume: $\exists n \in \mathbb{N}, \exists: b < n, a \cdot b \equiv_n 1$

$$a \cdot b = n \cdot k + 1, k \in \mathbb{Z}$$

$$a \cdot b - n \cdot k = 1$$

Bezout's Identity:

if a and b are non zero integers, then there exists integers u and v such that:

$$\gcd(a, b) = au + bv$$

then by Bezout's Identity, since $a \cdot b - n \cdot k$ is some linear combination that equals 1,

1 must be $\gcd(a, n)$

$$a \cdot b - n \cdot k = 1 \rightarrow \gcd(a, n) = 1$$

$$\therefore \exists n \in \mathbb{N}, \exists: b < n, a \cdot b \equiv_n 1 \rightarrow \gcd(a, n) = 1$$

QED

Question 6.

Prove the statement $S(n) : \forall n \in \mathbb{N}, \exists a, b \in \mathbb{N}, \ni: 3a + 5b = n$

Predicate:

$$S(n) : \forall n \in \mathbb{N}, \exists a, b \in \mathbb{N}, \ni: 3a + 5b = n$$

Base Case:

$$S(18) : 18 = 4a + 7b = 4(1) + 7(2)$$

$$S(19) : 19 = 4a + 7b = 4(3) + 7(1)$$

$$S(20) : 20 = 4a + 7b = 4(5) + 7(0)$$

$$S(21) : 21 = 4a + 7b = 4(0) + 7(3)$$

Inductive Hypothesis:

Suppose $P(k)$, for some arbitrary $k \in [18, n)$

$$P(k) : k = 4a + 7b$$

Inductive Step:

$$p(k) \rightarrow p(k + 4)$$

$k = 4a + 7b$ Suppose $p(k)$ by **Inductive Hypothesis**

$$k + 4 = 4a + 7b + 4$$

$$k + 4 = 4(a + 1) + 7b$$

Since k is arbitrary, $\forall n \in \mathbb{N}, S(k)$

QED