Week 7, LEC02 - November 1st
Discrete Mathematics, Fall 2018

CSCA67 - Lecture Notes

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1 Introduction to Counting

Counting Pizza Toppings*

The commercial's deal was:

- 2 pizzas
- up to 5 toppings on each
- 11 toppings to choose from
- all for $7.98 (back in 1997).

Important: duplicate toppings not allowed, order of topppings does not matter, and CHEESE IS NOT A TOPPING.

The commercial's math kid claimed there are $1,048,576$ possibilities.

Let's do the calculation ourselves.

Q. How many ways can we order a pizza with 0 toppings?

A. 1

Q. How many ways can we order a pizza with 1 topping?

A. 11

Q. How many ways can we order a pizza with 2 topping?

A. $\dfrac{11 \cdot 10}{2} = 55$

Q. How many ways can we order a pizza with 3 toppings?

A. $\dfrac{11 \cdot 10 \cdot 9}{3!}$

Combinations of $x, y, z = 6 = 3! = 3 \cdot 2 \cdot 1$

Q. How many ways can we order a pizza with 4 toppings?

A. $\dfrac{11 \cdot 10 \cdot 9 \cdot 8}{4!}$

5 toppings: $\dfrac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{5!}$, 6 toppings: $\dfrac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{6!}$

Therefore, the toatl number of ways to order a single pizza with up to 5 toppings when choosing from 11 toppings is:

$$1 + 11 + \frac{11 \cdot 10}{2} + \frac{11 \cdot 10 \cdot 9}{3!} + \frac{11 \cdot 10 \cdot 9 \cdot 8}{4!} + \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{5!} = 1024$$

Note: On the final, you will not be asked to resolve this calculation.

Q. How did they get 1,048,576 in the commercial?

A. $1024 \cdot 1024 = 1,048,576$, This is not the correct answer!

Q. What was their mistake? and how do we correct for it?

A. The order of the pizza toppings does not matter, (pizza A, pizza B) is the same thing as (pizza B, pizza A) so, $\dfrac{1024 \cdot 1024}{2!} = 524,288$

Q. This is still not quite correct. Why?

A. We don't have 2 orders of (pizza A, pizza A), but we still have to divide by 2.

How many ways can we order two identical pizzas? 1024. So, $\frac{1024 \cdot 1024}{2!} + \frac{1024}{2!} = 524,800$

Q. How do we know that 524,800 is the correct answer?

A. One way to convince ourselves is to try and find another way to count he same problem.

we counted all the possible orders and then removed duplicates.

Exercise. Recount using a different method.

The 2 pizzas are either different or the same.

Different: $\frac{1024 \cdot 1023}{2!}$

Same: 1024

Total: $\frac{1024 \cdot 1023}{2!} + 1024 = 524,800$

Potentially on E6:

Exercise: What if double-toppings were allowed? or triple, or quadruple... (ie., two of the toppings are the same)?

Create a secure password

Google forces you to select a password combined of 8 or more alphanumeric charcates.

Q. How many alphanumeric characters are there?

A. a-z, A-Z, 0-9 = 62 toatl characters

Q. How many passwords are there of length 4?

A. $62^4 > 1.4 \cdot 10^7$

Q. How many passwords are there of length 8?

A. $62^8 > 2.1 \cdot 10^14$

So there are 14 million times as many passwords of length 8 than of length 4.

2 The Sum Rule

When counting, we need to be able to determine whether to sum or multiply the number of objects.

The Sum Rule. If an operation can be perforned in n different ways, each having x_1 possible outcomes, then the total number of outcomes possible is:

$$\sum_{i=1}^{n} x_i = x_1 + x_2 \cdots + x_n$$

Example. Ordering pizza. Suppose a pizza shop offers 5 types of toppings and one has the choice of 3 topping, 2 topping or 1 topping.

Duplicate toppings are not allowed.

Order of toppings does not matter.

Determine how many different pizzas can be orderd.

Note: On the final, if a problem feels like it's unspecified, you are allowed to make your own assumptions, and answer them according to assumptions.

If you are right, you will not be penalized.

Number of Toppings	Number of Pizza Choices (x_i)
1	5
2	$\frac{5 \cdot 4}{2!}$
3	$\frac{5 \cdot 4 \cdot 3}{3!}$

3 The Product Rule

The Product Rule. Suppose an operation takes k steps and that:

- The first step can be performed x_1 ways.
- The second step can be performed in x_2 ways.

Then the Whole operation can be performed in:

$$\prod_{i=1}^{k} x_1 = x_1 \cdot x_2 \cdot \cdots \cdot x_k$$

Q. We have seen two different counting scenarios - creating a Google password and ordering apair of pizzas. Which one involved the sum rule and which one involved the product rule?

A. The password problem involved the product rule, the pizza problem used both.

4 Arrangement

An arrangement is a grouping of objects. There are two types of arrangements:

Definition. A **Permutation** is an arrangement in which order matters.

Definition. A **Combination** is an arrangement in which order does not matter.

5 Permutations

Definition. An r-permutation of n distincts objects is an orderd arrangement of r of the n objects. We use the notation $P(n, r)$

The formula is derived as $n \cdot (n-1)_1 \cdot (n-2)_2 \cdot \cdots \cdot (n-r+1)_r$

$P(11, 3) = 11 \cdot 10 \cdot 9$, $11 - 3 + 1$, $n = 11, r = 3$

Q. In terms of factorials, how can we rewrite this formula?

A. $P(n, r) = \dfrac{n!}{(n-r)!}$

6 Combinations

Defintion. An r-cominbation of n distinct objects is an unorderd selection, or a subset of r of the n objects.

We can think of combinations in terms of permutations.

Q. Given $P(n.r)$, the number of r-permutations of n objects, how can we derive the number of $C(n, r)$ of r-combinations of n objects?

A. $C(n, r) = \dfrac{P(n, r)}{r!} = \dfrac{n!}{(n-r)! \cdot sr!}$

$C(n, r) = {C_r}^n \equiv n(C_r) \equiv \binom{n}{r}$

Example. In Poker each player has 5 cards. A standard deck has 52 cards. How many 5-card hands are possible?

$C(52,5) = \binom{52}{5} = \dfrac{52!}{47! \cdot 5!} = 2,598,960$

A flush is when all 5 cards have the same suit. If there are 4 suits, i.e., 13 cards per suit, how many ways are there to obtain a flush?

$4 \cdot C(13,5) = 4 \cdot \dfrac{13!}{8! \cdot 5!} = 5148$

Possiblity of a flush $= \dfrac{5148}{2,598,960} \approx 0.2\%$

Q. Should flushes happen very often?
A. No.

Exercise. How many different 8-digit binary sequences are there with six 1s and two 0s?
for example: 11011101
8 bins must be filled in with six 1s and two 0s.

We should consider only filling in the 0s, as the 1s should fill in the rest of the space accordingly.
This is a combination:
$C(8,2) = C(8,6) = \dfrac{8!}{2! \cdot 6!}$

Bin technique - Pancer