Course Notes
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# MATA22 - Linear Algebra I for Mathematical Sciences


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# 1 Vectors in Euclidean Spaces

Let $\vec{v} = [v_1, v_2, \ldots, v_n]$ and $\vec{w} = [w_1, w_2, \ldots, w_n]$ be vectors in $\mathbb{R}^n$.
**Vector Addition/Subtraction:** $\vec{v} \pm \vec{w} = [v_1 \pm w_1, v_2 \pm w_2, \ldots v_n \pm w_n]$
**Scalar Multiplication:** $r \cdot \vec{v} = [r \cdot v_1, r \cdot v_2, \ldots, r \cdot v_n], \forall r \in \mathbb{R}$

Let $\vec{u}, \vec{v}, \vec{w}$ be any vectors in $\mathbb{R}^n$, and let $r, s \in \mathbb{R}$
**Properties of Vector Addition**

| | |
|---|---|
| $(\vec{u} + \vec{v}) + \vec{w} = u + (\vec{v} + \vec{w})$ | An associative law |
| $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ | A commutative law |
| $0 + \vec{v} = \vec{v}$ | 0 as additive identity |
| $\vec{v} + (-\vec{v}) = 0$ | $-\vec{v}$ as additive inverse of $\vec{v}$ |

**Properties Involving Scalar Multiplication**

| | |
|---|---|
| $r \cdot (\vec{v} + \vec{w}) = r\vec{v} + r\vec{w}$ | A distributive law |
| $(r + s)\vec{v} = r\vec{v} + s\vec{v}$ | A distributive law |
| $r(s\vec{v}) = rs\vec{v}$ | An associative law |
| $1\vec{v} = \vec{v}$ | Preservation of scale |

These can all be proven easily with Vector Addition/Subtraction and Scalar Multiplication.

**Parallel Vectors**
Two nonzero vectors $\vec{v}$ and $\vec{w}$ are parallel if $\vec{v} = r\vec{w}$ with $r \neq 0$, if $r > 0$ then vectors have the same direction, and if $r < 0$, vectors then have opposite direction.

**Linear Combination**
Given vectors $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k \in \mathbb{R}^n$ and scalars $r_1, r_2, \ldots, r_k \in \mathbb{R}$
the vector $r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_k\vec{v}_k$ is a linear combination of the vectors $v$ with scalar coefficients $r$

**Linear Independence:** A set of vectors $\{v_1, \ldots v_k\}$ is linearly independent if $r_1\vec{v_1} + r_2\vec{v_2} \cdots + r_k\vec{v_k} = 0 \Rightarrow r_1 = r_2 \cdots = r_k = 0$

**Span**
Given vectors $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k \in \mathbb{R}^n$
The span of these vectors v is the set of all linear combinations of them and is denoted $sp(\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k)$
$sp(\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k) = \{r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_k\vec{v}_k | r_1, r_2, \ldots r_k \in \mathbb{R}\}$

# 2 The Norm & The Dot Product

The magnitude(norm) $\|\vec{v}\|$ of $\vec{v} = [v_1, v_2] = \sqrt{{v_1}^2 + {v_2}^2}$

**Properties of the Norm**
$\forall \vec{v}, \vec{w} \in \mathbb{R}^n$ and $\forall r \in \mathbb{R}$ we have the following:

| | |
|---|---|
| $\|\vec{v}\| \geq 0$ and $\|\vec{v}\| = 0 \Leftrightarrow \vec{v} = 0$ | Positivity |
| $\|r\vec{v}\| = \|r\|\|\vec{v}\|$ | Homogeneity |
| $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$ | Triangle inequality |

**Unit Vectors**
A vector in $\mathbb{R}^n$ is a unit vector if it has magnitude 1. Given any non zero $\vec{v}$, a unit vector in the same direction as $\vec{v}$ is given by $\dfrac{\vec{v}}{\|\vec{v}\|}$

**Dot Product**
Let $\vec{v} = [v_1, v_2, \ldots, v_n]$ and $\vec{w} = [w_1, w_2, \ldots, w_n]$ be vectors in $\mathbb{R}^n$. The dot product of $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ is defined to be the real number
$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \ldots v_n w_n$
$\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cdot \cos(\theta)$
$\|\vec{v}\|\|\vec{w}\| \cdot \cos(\theta) = v_1 w_1 + v_2 w_2 + \ldots v_n w_n$

The angle between non zero vectors $\vec{v}$ and $\vec{w}$ is $\arccos(\dfrac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|})$

The norm of a vector can also be expressed in terms of its dot product with itself. Namely, for a vector $\vec{v} \in \mathbb{R}^n, \|\vec{v}\|^2 = v \cdot v$

**Perpendicular or Orthogonal Vectors**
Two vectors $\vec{v}$ and $\vec{w}$ are perpendicular or orthogonal if $\vec{v} \cdot \vec{w} = 0$, denoted as $\vec{v} \perp \vec{w}$

**Schwarz Inequality**
Let vectors $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^n$. Then $|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\|\|\vec{w}\|$

# 3 Matrices

**Notation**

A matrix $A$ of size $mxn$ has m rows and n columns.
The rows of matrix construct row vectors and the columns are its column vectors.

**Addition Subtraction and Scalar Multiplication**

**Definition**: Let A = $[a_{ij}]$ and $B = [b_{ij}]$ be $m \cdot n$ matrices and k be a scalar then:

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} B = \begin{bmatrix} 1 & 3 \\ -2 & -5 \\ 6 & -7 \end{bmatrix}$$

1. A + B is defined to be the matrix with the $ij^{th}$ entry $(A+B)_{ij} = a_{ij} + b_{ij}$

$$A + B = \begin{bmatrix} 3 & 2 \\ 1 & -3 \\ 4 & -3 \end{bmatrix}$$

2. A - B is defined to be the matrix with the $ij^{th}$ entry $(A-B)_{ij} = a_{ij} - b_{ij}$

$$A - B = \begin{bmatrix} 1 & -4 \\ 5 & 7 \\ -8 & 11 \end{bmatrix}$$

$A \pm B$ **is only defined if the matrix share the same size.**

3. kA is the matrix with the $ij^{th}$ entry $(kA)_{ij} = ka_{ij}$

$$2A = \begin{bmatrix} 4 & -2 \\ 6 & 4 \\ -4 & 8 \end{bmatrix}$$

**Definition** Let A be an n x m matrix and B be an m x k matrix, Then AB is defined to be the n x k matrix with the $ij^{th}$ entry given by

$$(AB)_{ij} = (i^{th} \text{ row of } A) \cdot (j^{th} \text{ column of } B)$$

$$A = \begin{bmatrix} 2 & 3 & 7 \\ 0 & 2 & -2 \end{bmatrix} B = \begin{bmatrix} 3 & 4 & 4 \\ -2 & 6 & 0 \\ 5 & -3 & -1 \end{bmatrix}$$

A is a 2X3 matrix where B is a 3X3 matrix, since 3 = 3, AB is defined.

$$AB = \begin{bmatrix} 35 & 5 & 1 \\ -9 & 15 & 1 \end{bmatrix}$$

BA is not defined because B has 3 columns where A only has 2 rows
Multiplication of Matrices is not commutative, even if both products are defined.

**Identity Matrix**
Let I be the $nxn$ matrix $[a_{ij}]$ such that $a_{ij} = 1 \forall i = j$ and $a_{ij} = 0, \forall i \neq j$

**Defintion** The transpose of n x m matrix A = $[a_{ij}]$ is denoted by $A^T$, and it has for an $ij^{th}$ entry $(A^T)_{ij} = a_{ji}$. In other words, the $i^{th}$ row of A is the $i^{th}$ column of $A^T$, OR the $j^{th}$ column of A is the $j^{th}$ row of $A^T$.

**Definition** Let A = $[a_{ij}]$ be a square matrix.

1. A is a symmetric matrix if $A = A^T$ (in other words, $a_{ij} = a_{ji}$)
2. A is a skew symmetric matrix if $A = -A^T$ (in other words, $a_{ij} = -a_{ji}$)

**Proof**
show that $A(B + C) = AB + AC$
let $A \in \mathbb{R}^{m,n}, B, C \in \mathbb{R}^{n,s}$

$$\begin{aligned}(A + (B + C))_{ij} &= \sum_{k=1}^{n} A_{ik}(B + C)_{kj} \\ &= \sum_{k=1}^{n} A_{ik}B_{k_j} + A_{ik}C_{kj} = (AB + BC)_{ij}\end{aligned}$$

show that $(AB)C = A(BC)$
let $A \in \mathbb{R}^{m,n}, B \in \mathbb{R}^{n,p}, C \in \mathbb{R}^{p,q}$

$$\begin{aligned}((AB)C)_{ij} &= \sum_{k=1}^{p} (AB)_{ik}C_{kj} \\ &= \sum_{k=1}^{p} (\sum_{l=1}^{n} A_{il}B_{lk})C_{kj} = \sum_{k=1}^{p} \sum_{l=1}^{n} A_{il}B_{lk}C_{kj} \\ (A(BC))_{ij} &= \sum_{l=1}^{n} A_{il}(BC)_{lj} \\ &= \sum_{l=1}^{n} A_{il}(\sum_{k=1}^{p} B_{lk}C_{lj}) = \sum_{k=1}^{p} \sum_{l=1}^{n} A_{il}B_{lk}C_{kj}\end{aligned}$$

# 4 Systems of Linear Equations

The solution set of any system of equations is the intersection of the solution sets of the individual equations. That is, any solution of a system must be a solution of each equation in the system.

**Augmented matrix/Partitioned Matrix**

A be an $mxn$ matrix where $Ax = b$, the following is a summary of $Ax = b$

$$\text{denoted as } [A|b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

**Elementary row operations**

Interchanging two equations in a system

Multiplying an equation in a system by a nonzero constant

Replacing an equation in a system with the sum of itself and another equation from the system

**Row-Echelon Form**

A matrix is in this row-Echelon form if:

1. All rows containing only zeros appear below rows with nonzero entries.

This means that any row containing only zeros will sink to the bottom of the matrix.

2. The first nonzero entry in any row appears in a column to the right of the first nonzero entry in any preceding row.

$$Example = \begin{bmatrix} 3 & 2 & 0 & 9 & 2 \\ 0 & 3 & 2 & 4 & 1 \\ 0 & 0 & 3 & 2 & 4 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**Finding Solutions**

$$[H|c] = \left[\begin{array}{cccc|c} 3 & 2 & 0 & 9 & 2 \\ 0 & 3 & 2 & 4 & 1 \\ 0 & 0 & 3 & 2 & 4 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right]$$

In this case, Hx = c actually has no solutions, because of the last row.
$0x_1 + 0x_2 + 0x_3 + 0x_4 = 2$

$$[H|c] = \left[\begin{array}{ccccc|c} 1 & -3 & 0 & 5 & 0 & 4 \\ 0 & 0 & 1 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_5 = 1$
$x_3 + 2x_4 = -7$
$x_1 - 3x_2 + 5x_4 = 4$

If we solve each equation for the variable corresponding to the pivot in the matrix, we have:

$x_1 = 3x_2 - 5x_4 + 4$
$x_3 = -2x_4 - 7$
$x_5 = 1$

since $x_2$ and $x_4$ correspond to columns of H containing no pivot, we can assign any value $r$ and $s$ we please to $x_2, x_4$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3r - 5s + 4 \\ r \\ -2s - 7 \\ s \\ 1 \end{bmatrix} r, s \in \mathbb{R}$$

We call $x_2, x_4$ free variables, and our final answer for x is referred to as the **general solution** of the system. We can obtain particular solutions by setting r and s to specific values.

# 5 Inverse of Square Matrices

A system of n equations in n unknowns $x_1, x_2, \dots x_n$ can be expressed in matrix form as $Ax = b$, where A is the $n$ x $n$ coefficient matrix, $x$ is the n x 1 column vector with i-th entry $x_i$, and b is an $nx1$ column vector with constant entries.

If $CA = I = AC$ for square matrices A and C, then A is the inverse of C and C is the inverse of A.

It's true that $AC = I \Leftrightarrow CA = I$, Also $A$ must have a pivot in each column/row for it to have an inverse.

A matrix A is invertible if there exists an $n$ x $n$ matrix C such that $CA = AC = I_n$ if A is not invertible, it is **singular**.

$A^{-1} \neq \dfrac{1}{A}$ **Every Elementary Matrix is Invertible**

if $A$ and $B$ are invertible $n$ x $n$ matrices, then $AB$ is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$

**PROOF** By assumption, there exist matrices $A^{-1}$ and $B^{-1}$ where $AA^{-1} = A^{-1}A = I$ and $BB^{-1} = B^{-1}B = I$, making use of the associative law for matrix multiplication,

we have $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$

thus the inverse of $AB$ is $B^{-1}A^{-1}$

Computation of $A^{-1}$

To find $A^{-1}$, if it exists, proceed as follows:

1. Form the augmented matrix $[A|I]$

2. Apply the Gauss-Jordan method to attempt to reduce $[A|I]$ to $[I|C]$. If the reduction can be carried out, then $A^{-1} = C$ Otherwise, $A^{-1}$ does not exist.

Determine Whether the span of vectors is $\mathbb{R}^3$

If we can reduce a matrix containing the three vectors into the identity matrix, then the span of given vectors is all of $\mathbb{R}^3$

# 6

## 6.1 Homogeneous Systems

A linear system $A\vec{x} = \vec{b}$ is **homogeneous** if $b = 0$. A homogeneous linear system $A\vec{x} = 0$ is always consistent, because $x = 0$, the zero vector, is certainly a solution. The zero vector is called the **trivial solution**. Other solutions are **nontrivial solutions.** A homogeneous solution is special in that its solution set has self-contained algebraic structure of its own.

Consider these equations:

$5x + 4y = 0$

$2x - 2y = 0$

solution: $(0, 0)$ - Trivial Solution & $y = \dfrac{-5}{4}, x = y$ - Non Trivial Solution

Consider this Matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 6 & -3 & 0 \\ 1 & -2 & -1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

solution: $(0, 0, 0)$ - Trivial Solution, only trivial solution for this system.

Consider this system

$$\left[\begin{array}{cccc|c}1 & 2 & 3 & 2 & 0\\1 & 3 & 5 & 5 & 0\\2 & 4 & 7 & 1 & 0\\-1 & -2 & -6 & 7 & 0\end{array}\right] = \left[\begin{array}{cccc|c}1 & 2 & 3 & 2 & 0\\0 & 1 & 2 & 3 & 0\\0 & 0 & 1 & -3 & 0\\0 & 0 & -3 & 9 & 0\end{array}\right] = \left[\begin{array}{cccc|c}1 & 0 & 0 & -7 & 0\\0 & 1 & 0 & 9 & 0\\0 & 0 & 1 & -3 & 0\\0 & 0 & 0 & 0 & 0\end{array}\right]$$

Notice in the second matrix, we have the 4-th row as a scalar multiple of the third. And since the last row consists of only zeros, we are granted the free variable $x_4 = k$

$x_3 - 3k = 0 \Rightarrow x_3 = 3k$

$x_2 + 9k = 0 \Rightarrow x_2 = 9k$

$x_1 - 7k = 0 \Rightarrow x_1 = 7k$

solution: (7k, -9k, 3k ,k) $\forall k \in \mathbb{R}$

consider $A\vec{x} = \vec{0}$, this is a homogeneous linear system. If $h_1, h_2$ are solution to $A\vec{x} = \vec{0}$ then so is any linear combination of $h_1, h_2$, $rh_1 + sh_2$ is a solution of $A\vec{x} = 0, \forall r, s \in \mathbb{R}$

Proof: $A\vec{h_1} = \vec{0}, A\vec{h_2} = \vec{0}$

$rA\vec{h_1} + sA\vec{h_2} = r\vec{0} + s\vec{0} = \vec{0}$

Thus we conclude **every linear combination of solutions of a homogeneous system $A\vec{x} = 0$ is again the solution of the system.**

## 6.2 Subspaces

let $W \subset \mathbb{R}^n, W$ is a subspace of $\mathbb{R}^n$ if:

i) W is nonempty

ii) $\forall \vec{u}, \vec{v} \in W, \vec{u} + \vec{v} \in W$

iii) $\forall \vec{u} \in W, r\vec{u} \in W, r \in \mathbb{R}$

The solution set of a homogeneous system $A\vec{x} = 0$ in n unknowns is an example of a subset $W \in \mathbb{R}^n$ with the property that every linear combination of vectors in W is again in W. Note that W contains all linear combinations of its vectors if and only if it contains every sum of two of its vectors and every scalar multiple of each of its vectors.

Consider the following examples

1. $W = \{[x, 2x] | x \in \mathbb{R}\}$

   i) Is satisfied as it's obviously not empty.

   ii) Let $u = [a, 2a], v = [b, 2b] \implies u + v = [a + b, 2a + 2b] = [x, 2x]$ thus $W$ is closed under vector addition.

   iii) Let $u = [a, 2a] \Rightarrow cu = [ca, c2a], c \in \mathbb{R} \wedge cu \in W$ thus $W$ is closed under scalar multiplication.

   Since i), ii) and iii) are all satisfied, $W$ is a proper subspace.

2. $W = \{[x, y] \in \mathbb{R}^2 | xy \geq 0\}$

   i) Is satisfied as it's obviously not empty.

   ii) Let $u = [u_1, u_2], v = [v_1, v_2]$ for real numbers $u_1u_2, v_1v_2 \in \mathbb{R}^+$ then $u + v = [u_1 + v_1, u_2 + v_2]$ and $(u_1 + v_1)(u_2 + v_2) \geq 0$

   iii) Let $u = [u_1, u_2] \in W$ consider $cu = [cu_1, cu_2]$ now $cu_1 \cdot cu_2 \ngeq 0, \forall c \in \mathbb{R}$ thus condition 3 is not satisfied.

   Since ii) and iii) aren't satisfied W is not a proper subspace.

**Subspace Property of a Span**

if $W = sp(\vec{w_1}, \vec{w_2}, \ldots, \vec{w_k})$ of vectors $w$ where $k > 0$ then $W$ is a subspace of $\mathbb{R}^n$

It's not empty because $k > 0$.

$u = r_1\vec{w_1} + r_2\vec{w_2} + \cdots + r_k\vec{w_k}$

$v = s_1\vec{w_1} + s_2\vec{w_2} + \cdots + s_k\vec{w_k}$

$u + v = (r_1 + s_1)\vec{w_1} + (r_2 + s_2)\vec{w_2} + \cdots + (r_k + s_k)\vec{w_k}$

which again is a linear combination of $w_1, w_2, \ldots, w_k$ which of course is in the span, thus in $W$. which proves that W is closed under vector addition.

similarly if we consider $cu = (cr_1)w_1 + (cr_2)w_2 + \cdots + (cr_k)w_k$ which is in the span, thus in $W$ finally W is a subspace of $\mathbb{R}^n$

Consider $V$ as a subset of $\mathbb{R}^n$

**Row Space**: The span of row vectors of a given matrix.

Column Space: The span of column vectors of a given matrix.

**Nullspace**: The span of the solution set of $A\vec{x} = \vec{0}$

**Column Space Criterion**

A linear system $A\vec{x} = \vec{b}$ is only consistent(solution having) if $\vec{b}$ is in the column space of $A$

Also, the vectors in the nullspace of $\vec{A}$ are orthogonal to the row vectors of $\vec{A}$

## 6.3 Bases

let $W$ be a subspace of $\mathbb{R}^n$. A subset $\{\vec{w_1}.\vec{w_2}, \ldots \vec{w_k}\}$ of W is a basis for W if every vector in W can be expressed uniquely as a linear combination of $\vec{w_1}, \vec{w_2}, \ldots \vec{w_k}$ meaning vector $\vec{w_i}$ are linearly independent

**Proof** If $\{\vec{w_1}, \vec{w_2} \ldots, \vec{w_k}\}$ is a basis for W, then the expression for every vector in W as a linear combination of the $\vec{w_i}$ is unique, so, in particular, the linear combination that gives the zero vector must be unique. Because $0\vec{w_1} + 0\vec{w_2} \cdots + 0\vec{w_k} = \vec{0}$ it follows that $r_1\vec{w_1} + r_2\vec{w_2} + \cdots + r_k\vec{w_k} = \vec{0} \Rightarrow r_i = 0$

Conversely if we suppose that $0\vec{w_1} + 0\vec{w_2} + \cdots + 0\vec{w_k}$ is the only linear combination giving the zero vector. If we have two linear combinations

$\vec{u} = c_1\vec{w_1} + c_2\vec{w_2} + \cdots + c_k\vec{w_k}$

$\vec{v} = d_1\vec{w_1} + d_2\vec{w_2} + \cdots + d_k\vec{w_k}$

and we have

$\vec{u} - \vec{w} = (c_1 - d_1)\vec{w_1} + (c_2 - d_2)\vec{w_2} + \cdots + (c_k - d_k)\vec{w_k} = \vec{0}$

$c_1 - d_1 = c_2 - d_2 = \cdots = c_k - d_k = 0$

thus $c_i = d_i, \forall i \in [1, k]$ and our linear combination giving $u, v$ are unique

**Unique Solution Case** for $A\vec{x} = \vec{b}$

To determine whether a linear system $A\vec{x} = \vec{b}$ has a unique solution. **Theorem:** The system has exactly one solution for each $\vec{b}$ in the column space of $A$ if and only if the column vectors of A form a basis for this column space. As in if the column vectors of $A$ is linearly independent.

**Proof** let $A$ be a square $nxn$ matrix. Then each column vector $\vec{b} \in \mathbb{R}^n$ is a unique linear combination of the vectors of A if and only if $A\vec{x} = \vec{b}$ has a unique solution for each $\vec{b} \in \mathbb{R}^n$ Which is equivalent to saying $A$ is row reducible to the identity matrix.

**The Square Case, m = n**

let A be an $n$ x $n$ matrix. the following are equivalent:

1. The linear system $A\vec{x} = \vec{b}$ has a unique solution for each $\vec{b} \in \mathbb{R}^n$.
2. The matrix $A$ is row equivalent to the identity matrix $I$.
3. The matrix $A$ is invertible.
4. The column vectors of $A$ form a basis for $\mathbb{R}^n$.

**To determine** whether vectors $\vec{v_1} = [1, 1, 3], \vec{v_2} = [3, 0, 4], \vec{3} = [1, 4, -1]$ form a basis for $\mathbb{R}^n$

We must see if the matrix $A$ having $\vec{v_1}, \vec{v_2}, \vec{v_3}$ is capable of being reduced to the identity matrix. such that so long as we can reduce it to echelon-form.

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 4 \\ 3 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -3 & 3 \\ 0 & -5 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -9 \end{bmatrix}$$

As we are able to reduce this to the identity matrix, $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is a basis for $\mathbb{R}^3$

**Recall** A linear system having the same number of equations as unknowns is called a square matrix. When a square matrix is reduced to echelon form, the result is a square matrix having only zero entries below the main diagonal.

**The Non-Square Case, m $\neq$ n**

For a non square matrix, if we want to determine whether it has unique solutions, we see if the row-echelon form of it contains a pivot in its n columns. Consequently, the reduced row echelon form of this matrix must contain the identity matrix.

let A be an $m$ x $n$ matrix. the following are equivalent:

1. The linear system $A\vec{x} = \vec{b}$ has a unique solution for each $\vec{b} \in \mathbb{R}^n$.
2. The reduced row-echelon form of $A$ consists of the $n$ x $n$ identity matrix followed by $m - n$ rows of zeros.
3. The column vectors of $A$ form a basis for $\mathbb{R}^n$.

**Corollary** If a linear system $A\vec{x} = \vec{b}$ is consistent and has fewer equations than unknowns, then it has an infinite number of solutions.

# 7 Independence & Dimension

**Finding a Basis** for a span of vectors

let $\vec{w_1}, \vec{w_2}, \dots \vec{w_k}$ be vectors in $\mathbb{R}6n$ and let $W = sp(\vec{w_1}, \vec{w_2}, \dots \vec{w_k})$ Now $W$ can be charcaterized as the smallest subspace of $\mathbb{R}^n$ containing all of the vectors $\vec{w_1}, \vec{w_2}, \dots \vec{w_k}$, because every subspace containing these vectors must contain all linear combinations of them, and consequently must include every vector in $W$.

let us assume that $\vec{w_1}, \vec{w_2}, \dots \vec{w_k}$ is not a basis for $W$, then we must be able to express the zero vector as a linear combination of $\vec{w_j}$ in some non trivial way.

For illustration purposes, suppose that: $2\vec{w_2} - 5\vec{w_6} + \dfrac{1}{3}\vec{w_7} = 0$

With this, we see that $w_2.w_6, w_7$ can be expressed as a linear combination of the other two.

We claim that we can delete $\vec{w_7}$ from our list $\vec{w_1}, \vec{w_2}, \dots \vec{w_k}$ and the remaining $\vec{w_j}$ will still span $W$. The space spanned by the remainig $\vec{w_j}$ will still contain $\vec{w_7}$, and we continue until no such nontrivial linear combination giving the zero vector exists. The final list of remaining vectors will still span $W$ and be a basis for $W$.

**Example**

Find a basis for $W = sp([2, 3], [0, 1], [4, -6]) \in \mathbb{R}^2$

We see that $2[3, 2] - 12[0, 1] - [4, -6] = [0, 0]$ Thus we can delete any of the three vectors, and the remaining will still span $W$

**Linear Dependence/Independence**

let $\{\vec{w_1}, \vec{w_2}, \dots, \vec{w_k}\}$ be a set of vectors in $\mathbb{R}^n$.

A **Dependence Relation** in this set is if $r_1\vec{w_1} + r_2\vec{w_2} + \cdots + r_k\vec{w_k} = \vec{0}$ with atleast one $r_j \neq 0$

If this exists, the set of vectors is linearly dependent, otherwise the set of vectors is **Linearly Independent**

**Recall** that two nonzero vectors are linearly independent if one is not a scalar multiple of the other, such that they will span the entirety of the $\mathbb{R}^n$

Now we can redefine basis such that $\vec{w_1}, \vec{w_2}, \ldots, \vec{w_k}$ form a basis for a subspace $W \in \mathbb{R}^n$ if and only if the vectors $\vec{w_1}, \vec{w_2}, \ldots, \vec{w_k}$ span $W$ and are linearly independent.

Computing a basis for $W = sp(\vec{w_1}, \vec{w_2}, \ldots, \vec{w_k}) \in \mathbb{R}^n$. Determinig whether there is a nontrivial dependence relation.

$x_1\vec{w_1} + x_2\vec{w_2} + \cdots + x_k\vec{w_k}$ some $x_i \neq 0$ we simply need to determine whether the linear system $A\vec{x} = \vec{0}$ has a nontrivial solution, where $A$ is the matrix with column vectors $w_i$

Now **Finding Basis of $\mathbf{W = sp(\vec{w_1}\vec{w_2}, \ldots \vec{w_k})}$**

1. Form the matrix A whose i-th column vector is $\vec{w_i}$
2. Row reduce A to row-echelon form H
3. The set of all $\vec{w_i}$ such that the jth column of $H$ is a pivot basis for $W$

Note that we can test whether vectors in $\mathbb{R}^n$ are independent by reducing a matrix having them as column vectors. If the row reduction of the matrix having them as column vectors yields $n$ x $n$ identity matrix $I$. On the other hand, more than $n$ vectors in $\mathbb{R}^n$ must be dependent because with $m$ x $n$ matrix with $m < n$ cannot have a pivot in every column.

let $W$ be a subspace of $\mathbb{R}^n$. The number of elements in a basis in $W$ is dimension of $W$, and is denoted dim(W).

Now the dimension of $\mathbb{R}^n$ is n, because we have the standard basis $\{e_1, e_2, \ldots, e_n\}$. Now $R^n$ cannot be spanned by fewer $n$ vectors, because spanning set can always be cut down.

Find the dimension of the subspace $W = sp(\vec{w_1}, \vec{w_2}, \vec{w_3}, \vec{w_4}) \in \mathbb{R}^3$ where $\vec{w_1} = [1, -3, 1], \vec{w_2} = [-2, 6, -2], \vec{w_3} = [2, 1, -4], \vec{w_4} = [-1.10 - 7]$

we can form a matrix $A = \begin{bmatrix} 1 & -2 & 2 & -1 \\ -3 & 6 & 1 & 10 \\ 1 & -2 & -4 & -7 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

As we can see there are pivots in the 1st and 3rd column, thus the column vectors $\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$

form a basis for $W$, the column space of $A$, and so $\dim(W) = 2$.

1. Every subspace $W$ of $\mathbb{R}^n$ has a basis and $\dim(W) \leq n$.
2. Every independent set of vectors in $\mathbb{R}^n$ can be enlarged, if necessary, to become basis for $\mathbb{R}^n$.
3. If $W$ is a subspace of $\mathbb{R}^n$ and $\dim(W) =$ k, then
   (a) every independent set of $k$ vectors in $W$ is a basis for $W$
   (b) and every set of $k$ vectors in $W$ that spans $W$ is a basis for $W$.

**Example** Enlarge the independent set $\{[1, 1, -1], [1, 2, -2]\}$ to a basis for $\mathbb{R}^3$

Let $\vec{v_1} = [1, 1, -1], \vec{v_2} = [1, 2, -2]$. We know a spanning set of $\mathbb{R}^3 = \{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$. We write $\mathbb{R}^3 = sp(\vec{v_1}, \vec{v_2}, \vec{v_1}, \vec{e_2}, \vec{e_3})$ If we create a matrix with these vectors as the column vectors we have

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ -1 & -2 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

We can see pivots occur in columns 1, 2, 4, thus a basis containing $\vec{v_1}, \vec{v_2}$ is $\{[1, 1, -1], [1, 2, -2], [0, 1, 0]\}$

# 8 The Rank of A Matrix

We discussed the three subspaces associated with an $m$ x $n$ matrix A:
its column space, its row space and its null space(solution space of $A\vec{x} = \vec{0}$).

row space($A$) = column space ($A$) = number of pivots in row-echelon form of A = number of non zero rows in row-echelon form of A = rank of A

We can find the dimension of the **column space** of $A$ by row reducing A to row-echelon form. This dimension is the number of columns of H having pivots.
We can find the dimension of the row space, note that row operations does not change the row space.
**Finding Bases** for Spaces Associated with a Matrix
Let $A$ be an $m$ x $n$ matrix with row-echelon form H.

1. For a basis of the row space of $A$, use the nonzero rows of H.
2. For a basis of the column space of $A$, use the columns of $A$ corresponding to the columns of $H$ containing pivots.
3. For a basis of the nullspace of $A$, use $H$ and back substitution to solve $H\vec{x} = 0$

**Example** Find the rank, basis for the row space, a basis for the column space, and a basis for the nullspace.

# 9 Review

Let $A_1, A_2 \dots A_n$ be $nxn$ symmetric matrices, Let $r_1, r_2, \dots r_n \in \mathbb{R}$.

Prove that $r_1A_1 + r_2A_2 \cdots + r_kA_k$ is symmetric

Consider $(r_1A_1 + r_2A_2 \cdots + r_kA_k)^T = (r_1A_1)^T + (r_2A_2)^T \cdots + (r_kA_k)^T$

$= r_1(A_1)^T + r_2(A_2)^T \cdots + r_k(A_k)^T$

But since $\forall i \in [i,k], (A_i)^T = (A_i)$

$r_1(A_1)^T + r_2(A_2)^T \cdots + r_k(A_k)^T = (r_1A_1 + r_2A_2 \cdots + r_kA_k) = (r_1A_1 + r_2A_2 \cdots + r_kA_k)^T$

**Basis:** A set $B = \{\vec{b_1}, \vec{b_2} \dots \vec{b_k}\}$ is a basis of subspace if:

1. The vectors are linearly independent
2. They span the subspace

Meaning we can generate all the vectors in a subspace by taking linear combinations of them.

Determine if:

$$\begin{bmatrix}1\\3\\5\end{bmatrix} \in span\left(\begin{bmatrix}1\\5\\5\end{bmatrix}, \begin{bmatrix}0\\3\\2\end{bmatrix}, \begin{bmatrix}6\\1\\2\end{bmatrix}\right)$$

Check of $\exists r_1, r_2, r_3 \ni$:

$$\begin{bmatrix}1\\3\\5\end{bmatrix} = \left(r_1\begin{bmatrix}1\\5\\5\end{bmatrix} + r_2\begin{bmatrix}0\\3\\2\end{bmatrix} + r_3\begin{bmatrix}6\\1\\2\end{bmatrix}\right) = \left[\begin{array}{ccc|c}1 & 0 & 6 & 1\\5 & 3 & 1 & 3\\5 & 2 & 2 & 5\end{array}\right] = \left[\begin{array}{ccc|c}1 & 0 & 0 & 25/13\\0 & 1 & 0 & -25/13\\0 & 0 & 1 & -2/13\end{array}\right]$$

Since we have solutions for $r_1, r_2, r_3$, thus $[1, 3, 5]$ is in the span of those 3 vectors

Equation of line through point $(5,7)$ that's parallel to $[10,1]$

$[x,y]=[5,7]+t[10,1]$

**Norm** of a plane is a vector that is perpendicular to the plane.

Consider plane with point $(1,1,1)$ vector $[5,7,9]$ and $[2,6,0]$ find equation of the plane

we have $Ax+By+Cz=D$ where $[A,B,C]$ is a vector perpendicular to $[5,7,9]$ and $[2,6,0]$. and we solve for D by plugging in any point on the plane.

we have $[A,B,C]\cdot[5,6,7]=0$

$5A+7B+9C=0$

we have $[A,B,C]\cdot[2,6,0]=0$

$2A+6B=0 \Rightarrow A=-3B$

$$Thus, \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} -3B \\ B \\ 8 \end{bmatrix} = B\begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix}, B \in \mathbb{R}$$

Now we take $\vec{n}$ (normal) $=[-3.1,8]$

-3x + y + 8z = D $\Longrightarrow$ D = 6

Equation of the plane $=-3x+y+8z=6$

Let $A=\begin{bmatrix} 1 & 0 & 6 \\ 5 & 3 & 1 \\ 5 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**Column space** are columns corresponding to the pivoted columns in RREF of A

$C(A)=\left(\begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix}\right)$

Nullspace of $A$ is set of vectors satisfying $A\vec{x}=0$ but since $A$ RREF is the Identity matrix, the solution is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\vec{0}$ is not a basis vector. as it is not linearly independent.

Verify the rank-nullity equation

$n=$ rank(A) + nullity(A)

$3=3+0$

use the dimensions of the nullspace to determine if $A$ is invertible. since dimension of (nullspace) is 0, it's invertible.