Course Notes

CSCA67 - Discrete Mathematics

UNIVERSITY OF TORONTO SCARBOROUGH

Instructors:

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1 Propositions, Implications

Definitions:

A **proposition** is a statement that evaluates to True or False. In computer science, its often referred to as a **Boolean expression**.

A **compound roposition** is a proposition statementt that involves multiple propositions joined by connectives. It takes multiple truth values as input and returns a single truth value as output.

A **connective** corresponds to English conjunctions such as "and", "or", "not" etc.

Basic connectives and truth tables:

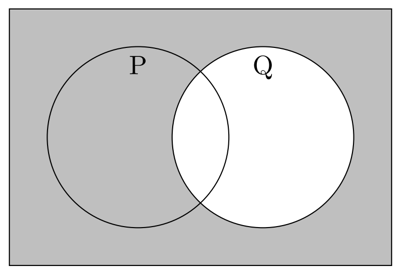
Symbol	Meaning
$\wedge$	"AND"
$\vee$	"OR"
$\rightarrow$	"IF...THEN"
$\leftrightarrow$	"IF AND ONLY IF"
$\neg$	"NOT"

P	Q	$P \wedge Q$	$P \vee Q$	$P \rightarrow Q$	$P \leftrightarrow Q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Implication:

Different ways of writing $P \rightarrow Q$:

1. If P then Q
2. If P, Q
3. Q, if P
4. P only if Q
5. P is sufficient for Q
6. Q is neccesary for P
7. If not Q, then not P
8. Not P or Q



Logical Equivalences:

Commutative	$p \wedge q \iff q \wedge p$	$p \vee q \iff q \vee p$
Associative	$(p \wedge q) \wedge r \iff p \wedge (q \wedge r)$	$(p \vee q) \vee r \iff p \vee (q \vee r)$
Distributive	$p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$
Identity	$p \wedge T \iff p$	$p \vee F \iff p$
Negation	$p \vee \neg p \iff T$	$p \wedge \neg p \iff F$
Double Negative	$\neg(\neg p) \iff p$	
Idempotent	$p \wedge p \iff p$	$p \vee p \iff p$
Universal Bound	$p \vee T \iff T$	$p \wedge F \iff F$
De Morgan's	$\neg(p \wedge q) \iff (\neg p) \vee (\neg q)$	$\neg(p \vee q) \iff (\neg p) \wedge (\neg q)$
Absorption	$p \vee (p \wedge q) \iff p$	$p \wedge (p \vee q) \iff p$
Conditional or ($\rightarrow$) Law	$(p \rightarrow q) \iff (\neg p \vee q)$	$\neg(p \rightarrow q) \iff (p \wedge \neg q)$
Biconditional	$(p \leftrightarrow q) \iff (p \rightarrow q) \wedge (q \rightarrow p)$	

Order of Operations:

1. NOT($\neg$) 2. AND($\wedge$) 3. OR($\vee$) 4. Quantifiers($\forall$/$\exists$) 5. ($\rightarrow$ / $\leftrightarrow$)

2 Predicates and Quantifiers

Forall:	$\forall$
There exists:	$\exists$
Negations:	
$\neg\forall = \exists$	$\neg\exists = \forall$

Prove statement in the form of $\exists x \in S, \ni: P(x)$
We simply need to find **one** value of x in the set S, that makes $P(x)$ true.
One value is enough.

Example:
There exists an integer n, such that n^2 is even.
$\exists n \in \mathbb{Z}, \ni: n^2 \in 2\mathbb{Z}$
Let $n = 2$, then $(2)^2 = 4$ which is an even number

Prove statemnet in the form of $\forall x \in S, \ni: P(x)$
This means we must use techniques such as algebraic manipulation to show that:
$P(x)$ holds for every arbitrary $x \in S$

Example:
Forall integers n, if n is odd, then n^2 is odd.
$\forall n \in \mathbb{Z}, n \in 2\mathbb{Z} \to n^2 \in 2\mathbb{Z}$
Let $n = 2k, k \in \mathbb{Z}$
then $n^2 = (2k)^2 = 4k^2$ which is an even number.
Therefore: Forall integers n, if n is odd, then n^2 is odd. QED

2.1 Modulus

$$10 \textbf{ mod } 3 = 1$$

The modulus or "mod" operator means the remainder when we divide two numbers.
Congruent mod means that two numbers have the same remainder when divided by one number.

$$10 \equiv_3 7 \Leftrightarrow 10 \text{ mod } 3 = 7 \text{ mod } 3$$

2.2 Fundamental Theorem of Arithmetic

The **Fundamental Theorem of Arithmetic** states that any integer greater than 1 is either a **prime** number itself, or can be represented as the unique product of prime numbers.
For example:

$$\begin{aligned} 16 &= 2^4 \\ 18 &= 2^1 \cdot 3^2 \\ 21 &= 2^0 \cdot 3^1 \cdot 5^0 \cdot 7^1 \end{aligned}$$

Numbers that can be written as the unique product of primes are called **Composite Numbers**.
Reminder: a **prime number** is an number that can only be divided evenly by 1 and the number itself.

3 Basic Proof Strategies

To prove in the form of $P \to Q$:

Direct Proof: Assume P is true then prove Q

This form works because if we recall the truth table for $P \to Q$,
When P is true, Q must be true for the statement to evaluate to true.

Proof by Contrapositve: Assume $\neg Q$ is true then prove $\neg P$

This form works because the contrapositive is logically equivalent to the original,
$P \to Q \equiv \neg Q \to \neg P$

Proof by Contradiction: Assume $\neg(\neg(P \to Q)) \equiv P \wedge \neg Q$

Then we must derive some sort of contradiction.
Once we arrive at the contradiction, that means one of our assumptions cannot be correct.
for example if $\neg Q$ is false, that means Q is true.

Proof by Cases/Exhaustion: $X \vee Y \to Q$ Show $X \to Q \wedge Y \to Q$
Example:
$x \in \mathbb{Z} \to x^2 + x + 1 \in 2\mathbb{Z} + 1$ ($x^2 + x$ is odd)

Case 1: x is odd
$x = 2k + 1$
$(2k + 1)^2 + (2k + 1) + 1$
$= 4k^2 + 6k + 3$
$= 2(2k^2 + 3) + 3$ case holds when x is odd.

Case 2: x is even
$x = 2k$
$(2k^2) + 2k + 1$
$= 4k^2 + 2k + 1$
$= 2(2k^2 + k) + 1$ case holds when x is even.
Since we have proven both case are indepdently even, we can conclude $\forall x \in \mathbb{Z}, x^2 + x + 1 \in 2\mathbb{Z} + 1$

Some Definitions:

Theorem: A statement that has already been proved.

Axiom: A statement that is self evidently true.

Identiy: An equation that is true for all values of an arbitrary variable.

Proof: A mathematical argument demonstrating the truth of a proposition.

Tautology: A propositional logic formula that always evaluates to True. $(A \vee \neg A)$ - (I'm hungry or I'm not hungry)

Rational Number: A number that can be represented as the fraction of two relatively prime integers.

$$A \in \mathbb{Q} \to A = \frac{m}{n}, n \neq 0, m, n \in \mathbb{Z}, gcd(m, n) = 1$$

Logic in a nutshell

Statement	Ways to Prove it	Ways to Use it	How to Negate it
p	• Prove that p is true. • Assume p is false, and derive a contradiction.	• p is true. • If p is false, you have a contradiction.	not p
p and q	• Prove p, and then prove q.	• p is true. • q is true.	(not p) or (not q)
p or q	• Assume p is false, and deduce that q is true. • Assume q is false, and deduce that p is true. • Prove that p is true. • Prove that q is true.	• If $p \Rightarrow r$ and $q \Rightarrow r$ then r is true. • If p is false, then q is true. • If q is false, then p is true.	(not p) and (not q)
$p \Rightarrow q$	• Assume p is true, and deduce that q is true. • Assume q is false, and deduce that p is false.	• If p is true, then q is true. • If q is false, then p is false.	p and (not q)
$p \iff q$	• Prove $p \Rightarrow q$, and then prove $q \Rightarrow p$. • Prove p and q. • Prove (not p) and (not q).	• Statements p and q are interchangeable.	(p and (not q)) or ((not p) and q)
$(\exists x \in S)\ P(x)$	• Find an x in S for which $P(x)$ is true.	• Say "let x be an element of S such that $P(x)$ is true."	$(\forall x \in S)$ not $P(x)$
$(\forall x \in S)\ P(x)$	• Say "let x be any element of S." Prove that $P(x)$ is true.	• If $x \in S$, then $P(x)$ is true. • If $P(x)$ is false, then $x \notin S$.	$(\exists x \in S)$ not $P(x)$

Graph from Introduction to mathematical arguments - by Michael Hutchings

4 Proof of Irrationality

4.1 Approach 1 - Fundamental Theorem of Arithmetic

Prove that $\sqrt{2}$ is irrational.

Assume the contrary that $\sqrt{2}$ is rational.
Then by the definition of rational numbers, $\sqrt{2} = \frac{m}{n}, \ni: m, n \in \mathbb{Z}, gcd(m, n) = 1, n \neq 0$

$$\sqrt{2} = \tfrac{m}{n}$$
$$n\sqrt{2} = m$$
$$2n^2 = m^2$$
$$m = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot x_3^{\alpha_3} \cdot \dots \cdot x_n^{\alpha_n} \qquad n = y_1{}_1^{\beta} \cdot y_2^{\beta_2} \cdot y_3^{\beta_3} \cdot \dots \cdot y_n^{\beta_n}$$

Each x, y are primes by the fundamental theorem of arithmetic.

$$m^2 = (x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot x_3^{\alpha_3} \cdot \dots \cdot x_n^{\alpha_n})(x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot x_3^{\alpha_3} \cdot \dots \cdot x_n^{\alpha_n})$$

This means that m^2 has $2n$ possible factors.

$$2n^2 = 2(y_1{}_1^{\beta} \cdot y_2^{\beta_2} \cdot y_3^{\beta_3} \cdot \dots \cdot y_n^{\beta_n})(y_1{}_1^{\beta} \cdot y_2^{\beta_2} \cdot y_3^{\beta_3} \cdot \dots \cdot y_n^{\beta_n})$$

This means that n^2 has $2n$ possible factors plus one factor 2.

as m^2 has an even number of prime factors, $2n^2$ will have an odd number of prime factors, contradicting the fundamental theorem.

$\therefore \sqrt{2} \in \mathbb{I}$ by contradiction. *QED*

4.2 Approach 2 - Definition of a Rational Number

Prove that $\sqrt{2}$ is irrational.

Assume the contrary that $\sqrt{2}$ is rational.
Then by the definition of rational numbers, $\sqrt{2} = \frac{m}{n}, \ni: m, n \in \mathbb{Z}, gcd(m, n) = 1, n \neq 0$

$gcd(m, n)$ means that m, n MUST be relative prime.

$$\sqrt{2} = \tfrac{m}{n}$$
$$n\sqrt{2} = m$$
$$2n^2 = m^2$$
$$2n^2 = m^2 \Rightarrow m^2 \in 2\mathbb{Z} \Rightarrow m \cdot m \in 2\mathbb{Z}$$

The previousline showed that m is even, so now we can substitute m with any arbitrary even number $2k$.

$$m = (2k), k \in \mathbb{Z}$$
$$2n^2 = (2k^2)$$
$$2n^2 = 4k^2$$
$$n^2 = 2k^2$$
$$n^2 \in 2\mathbb{Z} \Rightarrow n \in 2\mathbb{Z}$$
$$m, n \in 2\mathbb{Z} \Rightarrow gcd(m, n) \neq 1$$

Since m, n are both even, they cannot be relatively prime, $\therefore \sqrt{2} \in \mathbb{I}$ by contradiction. *QED*

5 Induction

Simple Induction Format:

Suppose we need to prove $P(n)$ for all natural numbers.

1. State the Predicates
$P(n) : \ldots$

2. Base case
Prove that $P(n)$ holds when n is the smallest possible natural number.
$P(0) : \ldots$ is True.

3. Inductive Hypothesis
Assume that $P(k)$ holds for any arbitrary k
$P(k) : \ldots$ is True.

4. Inductive Step
Prove that $P(k) \to P(k+1)$
Assume $P(k)$ then show $P(k+1)$

Example: Prove $\displaystyle\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

Stating the Predicate: $P(n) : \displaystyle\sum_{i=0}^{n} i = \frac{n(n+1)}{2}, \forall n \in \mathbb{N}$

Base case: $n = 0 : \displaystyle\sum_{i=0}^{0} i = 0 \quad \frac{0(0+1)}{2} = 0$

Inductive Hypothesis: Assume for any arbitrary $k \geq 0$, $P(k)$ holds.
$P(k) = \displaystyle\sum_{i=0}^{k} k = \frac{k(k+1)}{2}$
Inductive Step: Prove $P(k) \to P(k+1)$

$P(k+1) = \displaystyle\sum_{i=0}^{k+1} i = 1 + 2 + 3 \cdots + k + (k+1)$

$P(k+1) = \frac{k(k+1)}{2} + (k+1)$ **by Inductive Hypothesis**

$P(k+1) = \frac{k(k+1)+2(k+1)}{2}$
$P(k+1) = \frac{(k+1)(k+2)}{2}$
Conclusion:
$\therefore P(k) \to P(k+1)$
$\displaystyle\sum_{i=0}^{n} i = \frac{n(n+1)}{2}, \forall n \in \mathbb{N}$

6 Pigeonhole Principle

Core Principle: There exists n pigeons and m pigeonholes, if $n > m$, there must be atleast one pigeonhole with atleast two pigeons.

Example: Prove that if 7 distinct numbers are selected from $\{1, 2, \ldots 11\}$, then some two will add to 12.

Pigeons: 7 distinct numbers

Pigeonholes: 6 **sets** of numbers that add up to 12.

$$\{1, 11\}, \{2, 10\}, \{3, 9\}, \{4, 8\}, \{5, 7\}, \{6\}$$

Note: If we select 7 numbers from a set of 6, we will be forced to select atleast 2 of the numbers from the same set.

$\therefore$ if 7 distinct numbers are selected from $\{1, 2, \ldots 11\}$, then atleast two will add up to 12. *QED*

Prove that for any 3 integers we pick, the sum of 2 of their squares is be even.

$\forall x \in \mathbb{Z}, x^2 \bmod 2 = r, r \in [0, 1]$

let the 3 integers be pigeons, and let the 2 possible remainders be holes.

$3 > 2$ implies that in any scenario, if we choose 3 integers squared, there will be atleast 2 with the remainder 0 or atleast 2 with the remainder 1.

Case 1. 2 or more of the integers squared divided by 2 has the remainder 1.

$$\begin{aligned} {x_1}^2 + {x_2}^2 &= 2k + 1 + 1 \\ {x_1}^2 + {x_2}^2 &= 2k + 2 = 2(k + 1) \end{aligned}$$

Case 2. 2 or more of the integers squared divided by 2 has the remainder 0.

$${x_1}^2 + {x_2}^2 = 2k$$

QED

7 Proof Samples

7.1 Euclid's Proof for Infinite Primes

Assume to the contrary that there are a finite number of primes,
then let this be the complete set of primes: $p_1, p_2, p_3 \dots p_n$

let $A = (p_1 \cdot p_2 \cdot p_3 \cdot p_4 \dots, \cdot p_n) + 1$
A is not divisible by any known primes as it always leaves a remainder of 1.
so either A is a prime number itself, or A has a unique prime factor that is not in the existing list.

Contradictions:
if A is a prime number, then p_n is not the greatest prime.
if A is a composite number, then $p_1, p_2, p_3 \dots p_n$ does not contain all the primes.
Therefore, there must be an infinite number of primes.

QED

7.2 Arithmetic mean and Geometric mean

Defintion. The arithmetic mean of a_1, a_2:

$$\frac{a_1 + a_2}{2}$$

Definition. The geometric mean of a_1, a_2:

$$\sqrt{a_1 \cdot a_2}$$

Prove that: $\forall a_1, a_2 \in \mathbb{Z}^+, \frac{a_1 + a_2}{2} \geq \sqrt{a_1 \cdot a_2}$

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 \cdot a_2}$$

$$(\frac{a_1 + a_2}{2})^2 \geq a_1 \cdot a_2$$

$$\frac{{a_1}^2 + 2(a_1 \cdot a_2) + {a_2}^2}{4} \geq a_1 \cdot a_2$$

$${a_1}^2 + 2(a_1 \cdot a_2) + {a_2}^2 \geq 4(a_1 \cdot a_2)$$

$${a_1}^2 + 2(a_1 \cdot a_2) + {a_2}^2 - 4(a_1 \cdot a_2) \geq 0$$

$${a_1}^2 - 2(a_1 \cdot a_2) + {a_2}^2 \geq 0$$

$$(a_1 - a_2)^2 \geq 0$$

QED