Discrete Mathematics, Fall 2015

CSCA67 - FINAL EXAM & ANSWERS

GOOD LUCK EVERYONE!
-Richard

Instructors:

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1. Your local grocery store just received a large shipment of apples, oranges, pears, and bananasmore than 100 pieces each. You are shopping at the store and will purchase your fruit for the week.

   (a) How many ways can you select 10 pieces of fruit from your stores supply of apples, oranges, pears, and bananas?
   let f_i denote a kind of fruit
   $f_1 + f_2 + f_3 + f_4 = 10, f_i \geq 0, i \in [1, 4]$
   Total $= \binom{10 + (4-1)}{4-1}$

   (b) How many ways can you select 10 pieces of fruit from your stores supply of apples, oranges, pears, and bananas if you need at least one piece of each kind of fruit?
   let f_i denote a kind of fruit
   $f_1 + f_2 + f_3 + f_4 = 10, f_i \geq 1, i \in [1, 4] \equiv f_1 + f_2 + f_3 + f_4 = 6, f_i \geq 0, i \in [1, 4]$
   Therefore, Total $= \binom{6 + (4-1)}{4-1}$

2. In how many ways can six 3s and four 2s be arranged in a row so that the 2s are always apart?

   Let's write this as □3□3□3□3□3□3□3□

   We see that we have 7 □s to fill our four 2s in, so that the 2s are always apart.

   $\therefore$ ways to arrange six 3s and four 2s be arranged in a row so that the 2s are always apart is $\binom{7}{4}$

3. Your Toronto Maple Leafs won 30 of 82 games last season (i.e., the 2014-2015 season), giving them a winning percentage of 37%. If we assume this means the probability of the Leafs winning any given game is 0.37, then we can predict how they would have done in a playoff series.

   Answer the following questions to determine the probability that the Leafs would have won a best of 7 playoff series (i.e., won 4 games) had they made the playoffs last season.

   (a) Rephrase this question in terms of sequences of 0s and 1s.
   What is the shortest length of a sequence? 4. (4 wins)
   What is the longest length of a sequence? 7. (7 combination of wins and loses)

   (b) Calculate the number of sequences which correspond to the Leafs winning the series. (Note that the answer is not C(7, 4).)
   Consider this: The last game cannot result in a loss.
   so we have:
   First 4 games $= \binom{4}{4}$
   First 5 games: □□□□1 $= \binom{4}{3}$
   First 6 games: □□□□□1 $= \binom{5}{3}$
   First 7 games: □□□□□□1 $= \binom{6}{3}$
   Total $= 1 + \binom{4}{3} + \binom{5}{3} + \binom{6}{3} = 35$

(c) Calculate the number of sequences as they relate to this problem. (Note that the answer is not 2^7 as not all series would last 7 games.)

$2 \cdot 35 = 70$

(d) Calculate the probability that the Leafs would win the series.

Realistically, none. The leafs were trash in 2014.

Winrate = 37%

Probability of winning in 4 = $(0.37)^4$

Probability of winning in 5 = $(0.37)^{\binom{4}{3}} \cdot (1-0.37)^1 \cdot \binom{4}{3}$

Probability of winning in 6 = $(0.37)^{\binom{5}{3}} \cdot (1-0.37)^2 \cdot \binom{5}{3}$

Probability of winning in 7 = $(0.37)^{\binom{6}{3}} \cdot (1-0.37)^3 \cdot \binom{6}{3}$

Therefore, the total = $(0.37)^4 + \sum_{i=4}^{6} (0.37)^{\binom{i}{3}} \cdot (1-0.37)^{i-3} \cdot \binom{i}{3}$

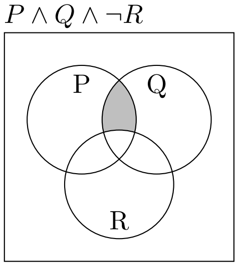
(e) What is your best guess for the probability that the Leafs will ever win the Stanley Cup again (the ultimate prize in the NHL)?

To be completely honest, never.

Assuming that the NHL goes on for eternity, the chances of the Leafs winning the Stanley Cup again is 100%.

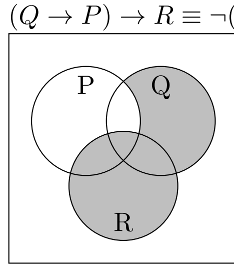
4. For each of the following statements, list the numbered region(s) of the venn diagram that represent(s) when the statement is true.

(a) $P \wedge Q \wedge \neg R$



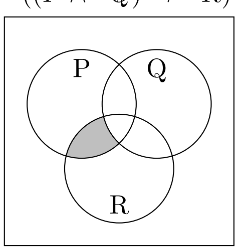
(b) $(Q \to P) \to R$

$(Q \to P) \to R \equiv \neg(\neg Q \vee P) \vee R \equiv (Q \wedge \neg P) \vee R$



(c) $\neg((P \wedge \neg Q) \to \neg R)$

$\neg((P \wedge \neg Q) \to \neg R) \equiv \neg(\neg(P \wedge \neg Q) \vee \neg R) \equiv \neg(\neg P \vee Q \vee \neg R) \equiv P \wedge \neg Q \wedge R$



5. Logical equivalence and contrapositive.

   (a) Circle the statements below that are equivalent to $(a \wedge b) \rightarrow c$.
   2. $\neg c \rightarrow (\neg a \vee \neg b)$ ✓. $(a \wedge b) \rightarrow c \equiv \neg(\neg a \vee \neg b) \rightarrow c \equiv \neg c \rightarrow (\neg a \vee \neg b)$
   3. c is neccesary for a and b ✓. $a \rightarrow b \equiv$ b is neccesary for a
   5. a and b are sufficient for c ✓. $a \rightarrow b \equiv$ a is sufficient for b

   (b) If the implication $a \rightarrow b$ is difficult to prove directly, we can attempt a proof by contrapositive. Use a truth table to show that the contrapositive of $a \rightarrow b$ is equivalent to $a \rightarrow b$.
   Show $a \rightarrow b \equiv \neg b \rightarrow \neg a$

a	b	$a \rightarrow b$	$\neg b \rightarrow \neg a$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

   $\therefore a \rightarrow b \equiv \neg b \rightarrow \neg a$

   (c)

$a \rightarrow b$	Given
$\neg a \vee b$	Conditional or $(\rightarrow)$ Law
$b \vee \neg a$	Commutative Property
$\neg b \rightarrow \neg a$	Conditional or $(\rightarrow)$ Law

6. Modular arithmetic.

   (a) State the Division Theorem. (Hint: This is the theorem we often use when writing proofs involving the mod operator.)
   $\forall n \in \mathbb{N}, n = kp + r, k \in \mathbb{N}, p \in \mathbb{N}, r \in [0, n)$

   (b) Prove that $\forall a, b, c \in \mathbb{N}, a \equiv_n b \rightarrow ca \equiv_n cb$
   Assume $a \equiv_n b$
   By the Division Theorem:
   $a = k_1 \cdot n + r, b = k_2 \cdot n + r, k_1, k_2 \in \mathbb{Z}$
   $c \cdot a = c \cdot k_1 \cdot n + c \cdot r$
   $c \cdot b = c \cdot k_2 \cdot n + c \cdot r$ *QED*
   $c \cdot a \bmod n = c \cdot r$
   $c \cdot b \bmod n = c \cdot r$
   $\therefore ca \equiv_n cb$

7. Consider the following statement S: The negative of any irrational number is irrational.

   (a) Write S as an implication
   $\forall n \in \mathbb{R}, n \in \mathbb{I} \rightarrow -n \in \mathbb{I}$

   (b) Proof:
   Suppose the contrapositive:
   $\exists n \in R, -n \in \mathbb{Q} \rightarrow n \in \mathbb{Q}$
   Assume $-n \in \mathbb{Q}, \ni: -n = \dfrac{m}{n}, n \neq 0, gcd(m, n) = 1, m, n \in \mathbb{Z}$
   $-n = \dfrac{m}{n}$
   $n = -\dfrac{m}{n} \rightarrow n \in \mathbb{Q}$
   $\therefore \exists n \in R, -n \in \mathbb{Q} \rightarrow n \in \mathbb{Q}$
   $\therefore \forall n \in \mathbb{R}, n \in \mathbb{I} \rightarrow -n \in \mathbb{I}$ *QED*

8. Prime numbers and the Fundamental Theorem of Arithmetic (FTA).

   (a) State the FTA

   Every integer greater than 1 is either a prime number or a composite number. A composite number is a number that can be written unique multiple of primes.

   (b) Prove or disprove that for all prime $p \in \mathbb{N}$, $\sqrt{p}$ is irrational. (Recall that a prime number is a natural number greater than 1 that has no positive divisors other than 1 and itself.)

   Assume the contrary:

   there exists a prime $p \in \mathbb{N}, \sqrt{p} \in \mathbb{Q}, \ni$: $\sqrt{p} = \dfrac{m}{n}, n \neq 0, gcd(m, n) = 1, m, n \in \mathbb{Z}$

   This would mean that:

   $\sqrt{p} = \dfrac{m}{n}$, $p = \dfrac{m^2}{n^2}$, $n^2 \cdot p = m^2$
   This implies that p must be a factor of m
   By the division theorem, $m = k \cdot p, k \in \mathbb{Z}$
   $n^2 \cdot p = (k \cdot p)^2$, $n^2 \cdot p = k^2 \cdot p^2$, $n^2 = k^2 \cdot p$
   This implies that p must be a factor of n
   Contradiction, if p is a factor of both m and n, then they cannot be co-prime,
   $\therefore$ forall prime numbers $p \in \mathbb{N}$, $\sqrt{p} \in \mathbb{I}$

   QED

9. We wish to prove that for all natural numbers $n \geq 0$, $10^n - 1$ is an integer multiple of 9. Fill in the missing steps of the proof:

   Define S(n): $9|10^n - 1$

   Prove $\forall n \in \mathbb{N}, S(n)$

   Base case: n $= 0$

   $10^0 - 1 = 0$ and $9|0 \therefore S(n), n = 0$

   Inductive Hypothesis: Assume that for some arbitrary $k \in \mathbb{N}, k \geq 0, S(k)$

   $S(k) : 9|10^k - 1 \equiv 10^k - 1 = 9 \cdot z, z \in \mathbb{Z}$

   Inductive Step: Prove $S(k) \rightarrow S(k+1)$

   Consider S(k + 1): $10^{k+1} - 1$
   $= 10^k + 10^1 - 1$
   $= 9 \cdot z + 10^1 - 1$ **By Inductive Hypothesis**
   $= 9 \cdot z + 9$
   $= 9(z + 1)$

   $\therefore \forall n \in \mathbb{N}, S(n)$

   QED

10. Define the function a(n) as follows:

$$a(n) = \begin{cases} 5 & a = 1 \\ 10 & a = 2 \\ 2a(n-1) + a(n-2) & n \geq 3 \end{cases}$$