CSCB36

Summary Of Vassos Hadzilacos'

Course notes for CSC B36/236/240

INTRODUCTION TO THE THEORY OF COMPUTATION

Instructor: **Dr.Nick Cheng**
Email: nick@utsc.utoronto.ca
Office: IC348
Office Hours: TBA

1 PRELIMINARIES

Sets: A collection of objects **(Elements)**.
If an object a is an element of set A, we say that a is a **member of** A; denoted $a \in A$
The collection that contains no elements is called the **empty** or **null** set' denoted $\varnothing$
Cardinality/Size: Number of elements in a set. The **cardinality** of set A is denoted $|A|$, and is a non-negative integer. If A has a infinite number of elements, $|A| = \infty$, and if $A = \varnothing$, then $|A| = 0$.

Extensional Description: Describing a set by listing its elements explicitly, e.g. $A = \{1, 4, 5, 6\}$
Intentional Description: Describing a set by stating a property that characterizes its elements, e.g. $A = \{x | \text{x is a positive integer less than 5}\}$

Let A and B be sets.
If every element of A is also an element of B,
then A is a **subset** of B ($A \subseteq B$), and B is a **superset** of A ($B \supseteq A$).

If $A \subseteq B$ and $B \subseteq A$, then A is **equal** to B ($A = B$).
If $A \subseteq B$ and $A \neq B$, then A is a **proper subset** of B; ($A \subset B$ and B) is a **proper superset** of A ($B \supset A$).
Note the empty set is a subset of every set, and a proper subset of every set other than itself.

The **union** of A and B ($A \cup B$), is the set of elements that belong to A or B (or both).
The **intersection** of A and B ($A \cap B$), is the set of elements that belong to both A and B.
If no elements belongs to both A and B, their intersection is empty, and they are **disjoint** sets.
The **difference** of A and B, ($A - B$), is the set of elements that belong to A but do not belong to B.
Note that: $A - B = \varnothing \iff A \subseteq B$
The **union** and **intersection** can also be defined for an arbitrary (even infinite) number of sets.
let I be a set of indices, such that for each $i \in I$ there is a set A_i

$$\cup_{i \in I} A_i = \{x : \text{for some } i \in I, x \in A_i\}$$
$$\cap_{i \in I} A_i = \{x : \text{for each } i \in I, x \in A_i\}$$

The **powerset** is the set of subsets, e.g. $A = \{a, b, c\}, \mathcal{P}(A) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$
Partition of a set A, is pairwise disjoint subsets of A whose union is A. A partition of set A is a set $\mathcal{X} \subseteq \mathcal{P}(A)$, such that:

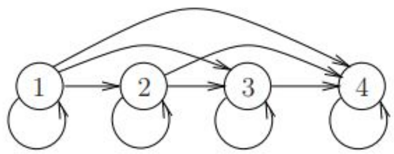
(i) for each $X \in \mathcal{X}, X \neq \varnothing$
(ii) for each $X, Y \in \mathcal{X}, X \neq Y$
(iii) $\cup_{X \in \mathcal{X}} X = A$

Ordered Pair: A mathematical construction that bundles two objects a, b together, in a particular order, denoted (a, b). By this definition, $(a, b) = (c, d) \iff a = c \wedge b = d$ and $(a, b) \neq (b, a)$ unless $a = b$.
We define an ordered pair (a, b) as the set $\{\{a\}, \{a, b\}\}$. We can also define ordered triples as ordered pairs, (a, b, c) can be defined as $(a, (b, c))$. This definition holds for ordered quadruples, quintuples, and ordered n-tuples for any integer $n > 1$.

Cartesian Product of A and B is denoted $A \times B$ and is the set of ordered pairs (a, b) where $a \in A, b \in B$.
$|A \times B| = |A| \cdot |B|$, note that if A, B are distinct nonempty sets, $A \times B \neq B \times A$
The Cartesian product of $n > 1$ sets $A_1, A_2, \ldots, A_n$ denoted $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where $a_i \in A_i, i \in [1, n]$

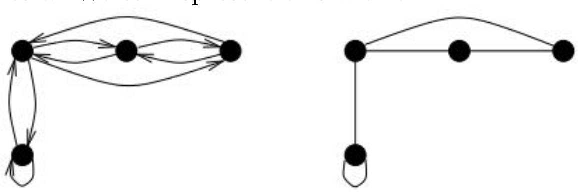
Relation R between sets A and B is a subset of the Cartesian product $A \times B$ ($R \subseteq A \times B$).
Arity: number of sets involved in the relation.
Two relations are **equal** if they contain exactly the same sets of ordered pairs. The two relations must refer to the exact same set of ordered pairs.

A binary relation (**arity** 2) between elements of the **same** set, can be represented graphically as a **directed graph**.
E.g. $R = \{(a, b) | a, b \in \{1, 2, 3, 4\} \text{ and } a \leq b\}$



R is a **reflexive** relation, if for each $a \in A, (a, a) \in R$, e.g. the relation $a \leq b$ between integers is reflexive, while $a < b$ is not.

R is a **symmetric** relation if for each $a, b \in R, (b, a) \in R$,
E.g. $R_1 = \{(a, b) | a \text{ and } b \text{ are persons with atleast one parent in common}\}$ is symmetric. In the directed graph that represents a symmetric relation, whenever there is an arrow from a to b, there is an arrow from b to a. We can represent this with an **undirected graph**.



directed graph on the left, undirected graph on the right

R is a **transitive** relation if for each $a, b, c \in A, (a, b) \in R \wedge (b, c) \in R \longrightarrow (a, c) \in R$,
E.g. $R = \{(a, b) | a \text{ and } b \text{ are persons and a is an ancestor of b}\}$
We see that if a is an ancestor of b, and b is an ancestor of c, then a is an ancestor of c.

R is an **equivalence relation** if it is reflexive, symmetric and transitive,
E.g. $R = \{(a, b) | a \text{ and } b \text{ are persons with the same parents}\}$
a and a are the same person, thus have the same parents ($(a, a) \in R$), R is **reflexive**.
a and b share the same parents, b and a share the same parents ($(a, b) \in R, (b, a) \in R$), R is **symmetric**.
a and b share the same parents, b and c share the same parents, a and c must share the same parents. R is **transitive**.
thus we say R is an **equivalence relation**.

Let R be an equivalence relation and $a \in A$. The **equivalence class** of a under R is defined as the set $R_a = \{b | (a, b) \in R\}$, i.e., the set of all elements that are related to a by R.
If R is reflexive, then we know $\forall a \in A, R_a \neq \varnothing$
If R is transitive, then we know $\forall a, b \in R, R_a \neq R_b \longrightarrow R_a \cap R_b = \varnothing$

R is **partial order** if it is anti-symmetric and transitive.
R is **total order** if it is partial order and satisfies for each $a, b \in A$, either $(a, b) \in R$ or $(b, a) \in R$.

Let A and B be sets. A **function** f from A to B is a special kind of relation where each element $a \in A$ is associated with one element in B.
The relation $f \subseteq A \times B$ is a **function** if for each $a \in A$ there is exactly one $b \in B$ such that $(a, b) \in f$.

We write $f : A \to B$ to denote that f is a function from A to B, where A is the **domain** of the function, and B is the **range**.

The function $f : A \to B$ is:
onto/surjective if for every $b \in B$ there is at least one $a \in A$ such that $f(a) = b$
one-to-one/injective if for every element $b \in B$, there is at most one element $a \in A$ such that $f(a) = b$
bijective if it is **one-to-one** and **onto**, if $f : A \to B$ is a bijection, then $|A| = |B|$.
The **restriction** of a function $f : A \to B$ is to a subset A' of its domain, denoted $f|A'$, is a function $f' : A' \to B$ such that for every $a \in A, f'(a) = f(a)$.