Course Notes
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MATA22 - Linear Algebra I for Mathematical Sciences

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1 Vectors in Euclidean Spaces

Let $\vec{v} = [v_1, v_2, \ldots, v_n]$ and $\vec{w} = [w_1, w_2, \ldots, w_n]$ be vectors in $\mathbb{R}^n$.
Vector Addition/Subtraction: $\vec{v} \pm \vec{w} = [v_1 \pm w_1, v_2 \pm w_2, \ldots v_n \pm w_n]$
Scalar Multiplication: $r \cdot \vec{v} = [r \cdot v_1, r \cdot v_2, \ldots, r \cdot v_n], \forall r \in \mathbb{R}$

Let $\vec{u}, \vec{v}, \vec{w}$ be any vectors in $\mathbb{R}^n$, and let $r, s \in \mathbb{R}$
Properties of Vector Addition

$(\vec{u} + \vec{v}) + \vec{w} = u + (\vec{v} + \vec{w})$	An associative law
$\vec{v} + \vec{w} = \vec{w} + \vec{v}$	A commutative law
$0 + \vec{v} = \vec{v}$	0 as additive identity
$\vec{v} + (-\vec{v}) = 0$	$-\vec{v}$ as additive inverse of $\vec{v}$

Properties Involving Scalar Multiplication

$r \cdot (\vec{v} + \vec{w}) = r\vec{v} + r\vec{w}$	A distributive law
$(r + s)\vec{v} = r\vec{v} + s\vec{v}$	A distributive law
$r(s\vec{v}) = rs\vec{v}$	An associative law
$1\vec{v} = \vec{v}$	Preservation of scale

These can all be proven easily with Vector Addition/Subtraction and Scalar Multiplication.

Parallel Vectors

Two nonzero vectors $\vec{v}$ and $\vec{w}$ are parallel if $\vec{v} = r\vec{w}$ with $r \neq 0$, if $r > 0$ then vectors have the same direction, and if $r < 0$, vectors then have opposite direction.

Linear Combination

Given vectors $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k \in \mathbb{R}^n$ and scalars $r_1, r_2, \ldots, r_k \in \mathbb{R}$
the vector $r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_k\vec{v}_k$ is a linear combination of the vectors v with scalar coefficients r

Span

Given vectors $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k \in \mathbb{R}^n$
The span of these vectors v is the set of all linear combinations of them and is denoted $sp(\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k)$
$sp(\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k) = \{r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_k\vec{v}_k | r_1, r_2, \ldots r_k \in \mathbb{R}\}$

2 The Norm & The Dot Product

The magnitude(norm) $\|\vec{v}\|$ of $\vec{v} = [v_1, v_2] = \sqrt{{v_1}^2 + {v_2}^2}$

Properties of the Norm
$\forall \vec{v}, \vec{w} \in \mathbb{R}^n$ and $\forall r \in \mathbb{R}$ we have the following:

$\|\vec{v}\| \geq 0$ and $\|\vec{v}\| = 0 \Leftrightarrow \vec{v} = 0$	Positivity
$\|r\vec{v}\| = \|r\|\|\vec{v}\|$	Homogeneity
$\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$	Triangle inequality

Unit Vectors
A vector in $\mathbb{R}^n$ is a unit vector if it has magnitude 1. Given any non zero $\vec{v}$, a unit vector in the same direction as $\vec{v}$ is given by $\dfrac{\vec{v}}{\|\vec{v}\|}$

Dot Product
Let $\vec{v} = [v_1, v_2, \ldots, v_n]$ and $\vec{w} = [w_1, w_2, \ldots, w_n]$ be vectors in $\mathbb{R}^n$. The dot product of $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ is defined to be the real number
$\vec{v} \cdot \vec{w} = v_1w_1 + v_2w_2 + \ldots v_nw_n$
$\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cdot \cos(\theta)$
$\|\vec{v}\|\|\vec{w}\| \cdot \cos(\theta) = v_1w_1 + v_2w_2 + \ldots v_nw_n$

The angle between non zero vectors $\vec{v}$ and $\vec{w}$ is $\arccos(\dfrac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|})$

The norm of a vector can also be expressed in terms of its dot product with itself. Namely, for a vector $\vec{v} \in \mathbb{R}^n, \|\vec{v}\|^2 = v \cdot v$

Perpendicular or Orthogonal Vectors
Two vectors $\vec{v}$ and $\vec{w}$ are perpendicular or orthogonal if $\vec{v} \cdot \vec{w} = 0$, denoted as $\vec{v} \perp \vec{w}$

Schwarz Inequality
Let vectors $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^n$. Then $|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\|\|\vec{w}\|$

3 Matrices

Notation

A matrix A of size mxn has m rows and n columns.
The rows of matrix construct row vectors and the columns are its column vectors.
Addition Subtraction and Scalar Multiplication
Definition: Let A = $[a_{ij}]$ and $B = [b_{ij}]$ be $m \cdot n$ matrices and k be a scalar then:

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} B = \begin{bmatrix} 1 & 3 \\ -2 & -5 \\ 6 & -7 \end{bmatrix}$$

1. A + B is defined to be the matrix with the ij^{th} entry $(A+B)_{ij} = a_{ij} + b_{ij}$

$$A + B = \begin{bmatrix} 3 & 2 \\ 1 & -3 \\ 4 & -3 \end{bmatrix}$$

2. A - B is defined to be the matrix with the ij^{th} entry $(A-B)_{ij} = a_{ij} - b_{ij}$

$$A - B = \begin{bmatrix} 1 & -4 \\ 5 & 7 \\ -8 & 11 \end{bmatrix}$$

$A \pm B$ **is only defined if the matrix share the same size.**

3. kA is the matrix with the ij^{th} entry $(kA)_{ij} = ka_{ij}$

$$2A = \begin{bmatrix} 4 & -2 \\ 6 & 4 \\ -4 & 8 \end{bmatrix}$$

Definition Let A be an n x m matrix and B be an m x k matrix, Then AB is defined to be the n x k matrix with the ij^{th} entry given by

$$(AB)_{ij} = (i^{th} \text{ row of } A) \cdot (j^{th} \text{ column of } B)$$

$$A = \begin{bmatrix} 2 & 3 & 7 \\ 0 & 2 & -2 \end{bmatrix} B = \begin{bmatrix} 3 & 4 & 4 \\ -2 & 6 & 0 \\ 5 & -3 & -1 \end{bmatrix}$$

A is a 2X3 matrix where B is a 3X3 matrix, since 3 = 3, AB is defined.

$$AB = \begin{bmatrix} 35 & 5 & 1 \\ -9 & 15 & 1 \end{bmatrix}$$

BA is not defined because B has 3 columns where A only has 2 rows
Multiplication of Matrices is not commutative, even if both products are defined.

Identity Matrix
Let I be the nxn matrix $[a_{ij}]$ such that $a_{ij} = 1 \forall i = j$ and $a_{ij} = 0, \forall i \neq j$

Defintion The transpose of n x m matrix A = $[a_{ij}]$ is denoted by A^T, and it has for an ij^{th} entry $(A^T)_{ij} = a_{ji}$. In other words, the i^{th} row of A is the i^{th} column of A^T, OR the j^{th} column of A is the j^{th} row of A^T.

Definition Let A = $[a_{ij}]$ be a square matrix.

1. A is a symmetric matrix if $A = A^T$ (in other words, $a_{ij} = a_{ji}$)
2. A is a skew symmetric matrix if $A = -A^T$ (in other words, $a_{ij} = -a_{ji}$)

Proof
show that $A(B + C) = AB + AC$
let $A \in \mathbb{R}^{m,n}, B, C \in \mathbb{R}^{n,s}$

$$\begin{aligned}(A + (B + C))_{ij} &= \sum_{k=1}^{n} A_{ik}(B + C)_{kj} \\ &= \sum_{k=1}^{n} A_{ik}B_{k_j} + A_{ik}C_{kj} = (AB + BC)_{ij}\end{aligned}$$

show that $(AB)C = A(BC)$
let $A \in \mathbb{R}^{m,n}, B \in \mathbb{R}^{n,p}, C \in \mathbb{R}^{p,q}$

$$\begin{aligned}((AB)C)_{ij} &= \sum_{k=1}^{p}(AB)_{ik}C_{kj} \\ &= \sum_{k=1}^{p}(\sum_{l=1}^{n} A_{il}B_{lk})C_{kj} = \sum_{k=1}^{p}\sum_{l=1}^{n} A_{il}B_{lk}C_{kj} \\ (A(BC))_{ij} &= \sum_{l=1}^{n} A_{il}(BC)_{lj} \\ &= \sum_{l=1}^{n} A_{il}(\sum_{k=1}^{p} B_{lk}C_{lj}) = \sum_{k=1}^{p}\sum_{l=1}^{n} A_{il}B_{lk}C_{kj}\end{aligned}$$

4 Systems of Linear Equations

The solution set of any system of equations is the intersection of the solution sets of the individual equations. That is, any solution of a system must be a solution of each equation in the system.

Augmented matrix/Partitioned Matrix

A be an mxn matrix where $Ax = b$, the following is a summary of $Ax = b$

$$\text{denoted as } [A|b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

Elementary row operations

Interchanging two equations in a system

Multiplying an equation in a system by a nonzero constant

Replacing an equation in a system with the sum of itself and another equation from the system

Row-Echelon Form

A matrix is in this row-Echelon form if:

1. All rows containing only zeros appear below rows with nonzero entries.

This means that any row containing only zeros will sink to the bottom of the matrix.

2. The first nonzero entry in any row appears in a column to the right of the first nonzero entry in any preceding row.

$$Example = \begin{bmatrix} 3 & 2 & 0 & 9 & 2 \\ 0 & 3 & 2 & 4 & 1 \\ 0 & 0 & 3 & 2 & 4 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Finding Solutions

$$[H|c] = \left[\begin{array}{cccc|c} 3 & 2 & 0 & 9 & 2 \\ 0 & 3 & 2 & 4 & 1 \\ 0 & 0 & 3 & 2 & 4 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right]$$

In this case, Hx = c actually has no solutions, because of the last row.
$0x_1 + 0x_2 + 0x_3 + 0x_4 = 2$

$$[H|c] = \left[\begin{array}{ccccc|c} 1 & -3 & 0 & 5 & 0 & 4 \\ 0 & 0 & 1 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_5 = 1$
$x_3 + 2x_4 = -7$
$x_1 - 3x_2 + 5x_4 = 4$

If we solve each equation for the variable corresponding to the pivot in the matrix, we have:

$x_1 = 3x_2 - 5x_4 + 4$
$x_3 = -2x_4 - 7$
$x_5 = 1$

since x_2 and x_4 correspond to columns of H containing no pivot, we can assign any value r and s we please to x_2, x_4

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3r - 5s + 4 \\ r \\ -2s - 7 \\ s \\ 1 \end{bmatrix} r, s \in \mathbb{R}$$

We call x_2, x_4 free variables, and our final answer for x is referred to as the **general solution** of the system. We can obtain particular solutions by setting r and s to specific values.

5 Inverse of Square Matrices

A system of n equations in n unknowns $x_1, x_2, \ldots x_n$ can be expressed in matrix form as $Ax = b$, where A is the n x n coefficient matrix, x is the n x 1 column vector with i-th entry x_i, and b is an $nx1$ column vector with constant entries.

If $CA = I = AC$ for square matrices A and C, then A is the inverse of C and C is the inverse of A.

It's true that $AC = I \Leftrightarrow CA = I$, Also A must have a pivot in each column/row for it to have an inverse.

A matrix A is invertible if there exists an n x n matrix C such that $CA = AC = I_n$ if A is not invertible, it is **singular**.

$A^{-1} \neq \dfrac{1}{A}$ **Every Elementary Matrix is Invertible**

if A and B are invertible n x n matrices, then AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$

PROOF By assumption, there exist matrices A^{-1} and B^{-1} where $AA^{-1} = A^{-1}A = I$ and $BB^{-1} = B^{-1}B = I$, making use of the associative law for matrix multiplication,

we have $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$

thus the inverse of AB is $B^{-1}A^{-1}$

Computation of A^{-1}

To find A^{-1}, if it exists, proceed as follows:

1. Form the augmented matrix $[A|I]$

2. Apply the Gauss-Jordan method to attempt to reduce $[A|I]$ to $[I|C]$. If the reduction can be carried out, then $A^{-1} = C$ Otherwise, A^{-1} does not exist.

Determine Whether the span of vectors is $\mathbb{R}^3$

If we can reduce a matrix containing the three vectors into the identity matrix, then the span of given vectors is all of $\mathbb{R}^3$

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6.1 Homogeneous Systems

A linear system $A\vec{x} = \vec{b}$ is **homogeneous** if $b = 0$. A homogeneous linear system $A\vec{x} = 0$ is always consistent, because $x = 0$, the zero vector, is certainly a solution. The zero vector is called the **trivial solution**. Other solutions are **nontrivial solutions.** A homogeneous solution is special in that its solution set has self-contained algebraic structure of its own.

Consider these equations:

$5x + 4y = 0$

$2x - 2y = 0$

solution: $(0, 0)$ - Trivial Solution & $y = \dfrac{-5}{4}, x = y$ - Non Trivial Solution

Consider this Matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 6 & -3 & 0 \\ 1 & -2 & -1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

solution: $(0, 0, 0)$ - Trivial Solution, only trivial solution for this system.

Consider this system

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 0 \\ 1 & 3 & 5 & 5 & 0 \\ 2 & 4 & 7 & 1 & 0 \\ -1 & -2 & -6 & 7 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & -3 & 9 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & -7 & 0 \\ 0 & 1 & 0 & 9 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Notice in the second matrix, we have the 4-th row as a scalar multiple of the third. And since the last row consists of only zeros, we are granted the free variable $x_4 = k$

$x_3 - 3k = 0 \Rightarrow x_3 = 3k$

$x_2 + 9k = 0 \Rightarrow x_2 = 9k$

$x_1 - 7k = 0 \Rightarrow x_1 = 7k$

solution: (7k, -9k, 3k ,k) $\forall k \in \mathbb{R}$

6.2 Subspaces

The solution set of a homogeneous system $A\vec{x} = 0$ in n unknowns is an example of a subset $W \in \mathbb{R}^n$ with the property that every linear combination of vectors in W is again in W. Note that W contains all linear combinations of its vectors if and only if it contains every sum of two of its vectors

7 Review