

Various Core Proofs/Identities

Current Instructor: Natalia Breuss



UNIVERSITY OF
TORONTO
SCARBOROUGH

Instructor(s):

	Dr. Natalia Breuss
Email:	n.breuss@utoronto.ca
Office:	IC484
Office Hours:	Wednesday 11:00 - 13:00

1 Identities

Basic Trigonometric Identities

Reciprocal Identities

$$(\sin x)^{-1} = \csc x$$

$$(\cos x)^{-1} = \sec x$$

$$(\tan x)^{-1} = \cot x$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

Angle Addition Identities

$$\sin(x \pm y) = \sin x \cdot \cos y \pm \cos x \cdot \sin y$$

$$\cos(x + y) = \cos x \cdot \cos y - \sin x \cdot \sin y$$

$$\cos(x - y) = \cos x \cdot \cos y + \sin x \cdot \sin y$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \cdot \tan y}$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \cdot \tan y}$$

Hyperbolic Trigonometric Identities

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \operatorname{csch} x = \frac{2}{e^x - e^{-x}}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad \operatorname{coth} x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Derivatives of Trigonometric Functions

Derivative of Trigonometric Functions

$$(\sin x)' = \cos x \quad (\csc x)' = -\csc x \cot x$$

$$(\cos x)' = -\sin x \quad (\sec x)' = \sec x \tan x$$

$$(\tan x)' = \sec^2 x \quad (\cot x)' = -\csc^2 x$$

Derivative of Hyperbolic Trigonometric Functions

$$(\sinh x)' = \cosh x \quad (\operatorname{csch} x)' = -\operatorname{csch} x \cdot \coth x$$

$$(\cosh x)' = \sinh x \quad (\operatorname{sech} x)' = -\operatorname{sech} x \cdot \tanh x$$

$$(\tanh x)' = \operatorname{sech}^2 x \quad (\operatorname{coth} x)' = -\operatorname{csch}^2 x$$

Derivative of Inverse Trigonometric Functions

$$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}} \quad (\csc^{-1} x)' = -\frac{1}{|x|\sqrt{x^2-1}}$$

$$(\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}} \quad (\sec^{-1} x)' = -\frac{1}{|x|\sqrt{x^2-1}}$$

$$(\tan^{-1} x)' = \frac{1}{1+x^2} \quad (\cot^{-1} x)' = -\frac{1}{1+x^2}$$

Derivatives of common functions

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

$$\frac{d}{dx} x^n = nx^{(n-1)}$$

$$\frac{d}{dx} f \cdot g = f' \cdot g + g' \cdot f$$

$$\frac{d}{dx} e^x = e^x \cdot \ln(e)$$

$$\frac{d}{dx} f + h = f' + h'$$

$$\frac{d}{dx} \frac{f}{g} = \frac{g \cdot f' - f \cdot g'}{g^2}$$

$$\frac{d}{dx} e^{2x} = e^x \cdot \ln(e) \cdot \frac{d}{dx} x = e^x \cdot 2x$$

$$\frac{d}{dx} f(g) = f'(g) \cdot g'$$

2 Derivative of Trigonometric functions

$$y = \sin^{-1}(x) \iff \sin(y) = x$$

$$\frac{d}{dx} \sin(y) = \frac{d}{dx} x$$

$$\cos y \cdot y' = 1$$

$$y' = \frac{1}{\cos y}$$

$$\text{Identity: } \cos^2 y = 1 - \sin^2 y$$

$$y' = \frac{1}{1 - \sin^2 y}, \sin y = x$$

$$y' = \frac{1}{1 - x^2}$$

$$y = \sinh^{-1}(x) \iff \sinh y = x$$

$$x = \frac{e^y - e^{-y}}{2} \quad 2x = e^y - \frac{1}{e^y}$$

$$2x \cdot e^y = e^{2y} - 1 \quad e^{2y} - (2x)e^y - 1 = 0$$

$$\text{Let } a = e^y$$

$$a^2 - (2x)a - 1 = 0$$

$$a = \frac{2x \pm \sqrt{(2x)^2 - 4(1)(-1)}}{2}$$

$$a = x \pm \sqrt{x^2 + 1}$$

$$e^y > 0 \rightarrow a = x + \sqrt{x^2 + 1}$$

$$\therefore y' = \ln(x + \sqrt{x^2 + 1})$$

3 Uniqueness of Limits

$$\lim_{x \rightarrow c} f(x) = L \wedge \lim_{x \rightarrow c} f(x) = M \implies L = M$$

Proof:

Suppose the contrary that:

$$\lim_{x \rightarrow c} f(x) = L \wedge \lim_{x \rightarrow c} f(x) = M, L \neq M$$

Assume that $L > M, L = M + K$, WLOG

Let's choose $\epsilon = \frac{k}{2}$, this way the intervals do not overlap.

$$\lim_{x \rightarrow c} f(x) = L : \exists \delta_1 > 0, \exists: 0 < |x - c| < \delta_1 \rightarrow |f(x) - L| < \epsilon$$

$$\lim_{x \rightarrow c} f(x) = M : \exists \delta_2 > 0, \exists: 0 < |x - c| < \delta_2 \rightarrow |f(x) - M| < \epsilon$$

Let $\delta = \min(\delta_1, \delta_2)$ so that for any $\delta > 0$, $f(x) \in (M - \epsilon, M + \epsilon)$ and $f(x) \in (L - \epsilon, L + \epsilon)$.

Contradiction: This is impossible since we set $\epsilon = \frac{k}{2}$ to guarantee intervals do not overlap.

Therefore, $\lim_{x \rightarrow c} f(x) = L \wedge \lim_{x \rightarrow c} f(x) = M \implies L = M$ by contradiction.

QED

4 Mean Value Theorem

if function f is continuous across $[a, b]$ and differentiable across (a, b)

then, there exists atleast one $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

This means if the conditions satisfy, there will be atleast one point c where the instantaneous rate of change is the same as the average rate of change.

5 Applications of MVT

Suppose $f(x)$ is continuous on $[1, 5]$ and differentiable on $(1, 5)$ and $f'(x) < \frac{3}{8}, \forall x \in (1, 5)$. If $f(1) = 1$, show that $f(x) < \frac{5}{2}, \forall x \in [1, 5]$

Since f is continuous and differentiable...by Mean Value Theorem, $\exists c \in (1, 5), \ni: f'(c) = \frac{f(5) - f(1)}{5 - 1}$

$$\text{let } x \in (1, 5), f'(x) = \frac{f(x) - f(1)}{x - 1} < \frac{3}{8} \iff \frac{f(x) - 1}{x - 1} < \frac{3}{8}$$

$$f(x) - 1 < \frac{3}{8}(x - 1), f(x) < \frac{3}{8}(x - 1) + 1$$

$$f(x) < \frac{3}{8}(5 - 1) + 1 \rightarrow f(x) < \frac{5}{2}$$

QED

Suppose $f(x)$ is odd for all x and diff across every real number. Prove that for every positive number b , there exists a positive number c , in $(-b, b)$ such that $f'(c) = \frac{f(b)}{b}$

Given $b > 0, f(-x) = -f(x)$

since f is differentiable on $(-\infty, \infty) \implies f$ is continuous on $(-\infty, \infty)$

f is cont on $[-b, b]$ and differentiable on $(-b, b)$

because $(-b, b) \subset (-\infty, \infty)$,

$$\text{By MVT, } \exists c \in (-b, b) \ni: f'(c) = \frac{f(b) - f(-b)}{b - (-b)} = \frac{f(b) + f(b)}{b + b} = \frac{f(b)}{b}$$

6 Rolle's Theorem

Suppose f is continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = f(b)$ then there exists atleast a $c \in (a, b)$ such that $f'(c) = 0$

7 Applications of Rolle's Theorem

Show that the function $2x + \cos x$ has exactly one real root.

$$\text{Let } f(x) = 2x + \cos x$$

$$f(-\pi) = -2\pi + \cos(-\pi) = -2\pi - 1 < 0$$

$$f(0) = 2 + \cos(0) = 1 > 0$$

Since $f(x)$ is a sum of a polynomial and periodic trigonometric function, f is continuous and differentiable for all x . By IVT, $\exists c \in (-\pi, 0) \ni: f(c) = 0$

Suppose $f(x)$ has two roots on a, b a i b , then $f(a) = f(b) = 0$. Since f is continuous on $[a, b]$ and differentiable on open interval (a, b) .

$f'(x) = 2 - \sin(x) > 0$, Contradiction, rolles theorem fails and therefore there must be exactly one root and one root only.

QED

8 Fermat's Theorem

If f has a local min/max at $x = c$ and $f'(c)$ exists, then $f'(c) = 0$.

$f'(x) = 0$ or undefined for c-pts:

$f'(x) > 0 \Rightarrow f(x)$ is increasing

$f'(x) < 0 \Rightarrow f(x)$ is decreasing