Course Notes
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MATA37 - CALCULUS II FOR THE MATHEMATICAL SCIENCES

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1 Cheat Sheet

Properties of $\sum$ - Notation

(i) $\sum_{k=1}^{m}(a_k+b_k)=\sum_{k=1}^{m}a_k+\sum_{k=1}^{m}b_k$

(ii) $\sum_{k=1}^{m}c\cdot a_k=c\cdot\sum_{k=1}^{m}a_k$

(iii) $\sum_{k=1}^{m}a_k=\sum_{k=1}^{l-1}a_k+\sum_{k=l}^{m}a_k$

Riemann Definition of the $\int_a^b f(x)dx$

Let $a,b\in\mathbb{R}, a<b$. Suppose that f is integrable on $[a,b]$.
let $P=\{x_1,x_2,x_3\dots x_n\}$ be a Riemann Partition of $[a,b]$

$$\int_a^b f(x)=\lim_{n\to\infty}\sum_{i=0}^{n}f(x_i^*)\Delta x, \Delta x=\frac{b-a}{n},\ x_i=a+i\Delta x$$
$$x_i^*=x_{i-1},x_i,\frac{x_{i-1}+x_i}{2}$$

Darboux integrability Def of $\int_a^b f(x)dx$

Let $a,b\in\mathbb{R},a<b$
first, suppose that f is bounded on [a,b] ie. $\exists c\in\mathbb{R}^+,\ni$: $|f(x)|<c,\forall x\in[a,b]$
Suppose f is bounded, on $[a,b]$ Let $P=\{x_i\}_{i=0}^{n}$ be any partition of [a,b].

for $i=1\dots n$; define $m_i=inf\{f(x)|x\in[x_{i-1},x_i]\}$, $M_i=sup\{f(x)|x\in[x_{i-1},x_i]\}$

Uppersum: $U(f,p)=\sum_{i=1}^{n}M_i(x_i-x_{i-1})$, Lowersum: $L(f,p)=\sum_{i=1}^{n}m_i(x_i-x_{i-1})$

$$\int_a^b f(x)dx \text{ exists} \Leftrightarrow sup\{L(f,p)|P\in[a,b]\}=inf\{U(f,p)|P\in[a,b]\}=\int_a^b f(x)dx$$

Integrability Reformulation
let $a,b\in\mathbb{R},a<b$,
f is integrable on $[a,b]\iff\forall\epsilon>0,\exists P$ partition of $[a,b]$ such that $U(f,p)-L(f,p)<\epsilon$

Let $a,b\in\mathbb{R},a<b$
FTOC I:
IF f is continuous on $[a,b]$, and F is any antiderivative of f on $[a,b]$
THEN $\int_a^b f(x)dx=F(x)|_a^b=F(b)-F(a)$
ALSO $\int_a^b F'(x)dx=\int_a^b f(x)dx$

FTOC II:
IF f is cont on $[a,b]$, define $F(x)=\int_a^x f(t)dt, x\in[a,b]$
THEN F is cont on $[a,b]$ and F is differentiable on (a,b), Moreover $F'(x)=f(x)x',\forall x\in[a,b]$

This means F is an antiderivative of F on $[a,b]$
$\frac{dF}{dx}=\frac{d}{dx}(\int_a^x f(t)dt)=f(x)$

2 Chapter 4; Sigma Notation

Definition.
$\sum$ - Notation:
Sigma notation is convenient way to express a sum of a collection of objects that take on a common form.
If we have a_k and b_k that are real-valued fcns(functions), then $a_m + a_{m+1} \cdots + a_k \cdots + a_n = \sum_{k=m}^{n} a_k$
a_k - General Term
k - index(letter doesn't matter)
m, n - inital index value and final index value

Example.
Express $-1+2-3+4-5+\cdots-21$ in $\sum$ - notation.
$\sum_{k=1}^{21}(-1)^k$ will produce the alternating signs.
$\sum_{k=1}^{21}(-1)^k \cdot k = -1+2-3+4-5+\cdots-21$

Sums are NOT unique, and there will be multiple ways of expressing the same sum.
$\sum_{k=1}^{21}(-1)^k \cdot k = \sum_{k=0}^{20}(-1)^{k+1} \cdot (k+1)$

Theorem. Pg(319 - 320)
Properties of $\sum$ - Notation
consider $m, l, k \in \mathbb{Z}^+, \ni: 1 \leq k \leq m, l < m.$ and let $c \in \mathbb{R}$
If we have a_k and b_k that are real-valued fcns(functions), then:

$$\text{(i) } \sum_{k=1}^{m}(a_k + b_k) = \sum_{k=1}^{m} a_k + \sum_{k=1}^{m} b_k$$
$$\text{(ii) } \sum_{k=1}^{m} c \cdot a_k = c \cdot \sum_{k=1}^{m} a_k$$
$$\text{(iii) } \sum_{k=1}^{m} a_k = \sum_{k=1}^{l-1} a_k + \sum_{k=l}^{m} a_k$$

Proofs.

(i) Suppose that a_k and b_k are real valued functions ($a_k, b_k \in \mathbb{R}$), let $m, k \in \mathbb{Z}^+$

WTS $\displaystyle\sum_{k=1}^{m}(a_k + b_k) = \sum_{k=1}^{m} a_k + \sum_{k=1}^{m} b_k$

Consider $\displaystyle\sum_{k=1}^{m}(a_k + b_k)$

$$\sum_{k=1}^{m}(a_k + b_k) = a_1 + a_2 \cdots + a_m + b_1 + b_2 \cdots + b_m$$

$$\sum_{k=1}^{m}(a_k + b_k) = a_1 + b_1 + a_2 + b_2 \cdots + a_m + b_m = \sum_{k=1}^{m} a_k + \sum_{k=1}^{m} b_k$$

(ii) Suppose that a_k is a real valued function ($a_k \in \mathbb{R}$), let $c \in \mathbb{R}, m, k \in \mathbb{Z}^+$

WTS $\displaystyle\sum_{k=1}^{m} c \cdot a_k = c \cdot \sum_{k=1}^{m} a_k$

Consider $\displaystyle\sum_{k=1}^{m} c \cdot a_k$

$$\sum_{k=1}^{m} c \cdot a_k = c \cdot a_1 + c \cdot a_2 \cdots + c \cdot a_m = c(a_1 + a_2 \cdots + a_m) = c \cdot \sum_{k=1}^{m} a_k$$

(iii) Suppose that a_k is a real valued function ($a_k \in \mathbb{R}$), let $l, m, k \in \mathbb{Z}^+, \ni: 1 \leq l < m < k$

WTS $\displaystyle\sum_{k=1}^{m} a_k = \sum_{k=1}^{l-1} a_k + \sum_{k=l}^{m} a_k$

Since $1 \leq l < m < k$,

$$\sum_{k=1}^{m} a_k = a_1 + a_2 + \cdots + a_{l-1} + a_l + a_{l+1} \cdots + a_m$$

$$\sum_{k=1}^{m} a_k = \sum_{k=1}^{l-1} a_k + \sum_{k=l}^{m} a_k$$

Example

1. evaluate $\sum_{i=0}^{203}(2i-1)$.

$$\begin{aligned}\sum_{i=0}^{203}(2_{i-1}) &= \sum_{i=1}^{203}2i-\sum_{i=1}^{203}1 \text{ by } \sum \text{ - property (i)}\\ &= 2\cdot\sum_{i=1}^{203}i-\sum_{i=1}^{203}1 \text{ by } \sum \text{ - property (ii)}\\ &= \frac{2(203)(204)}{2}-203 \text{ By definition of geometric series}\\ &= 41209\end{aligned}$$

2. evaluate $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{5}{n^4}(k^3+1)$ using Sigma notation properties

Sol'n

$\sum_{k=1}^{n}\frac{5}{n^4}(k^3+1)=\frac{5}{n^4}\cdot\sum_{k=1}^{n}(k^3+1)=\frac{5}{n^4}\cdot(\sum_{k=1}^{n}k^3+\sum_{k=1}^{n}1)$

$$\begin{aligned}\sum_{k=1}^{n}k^3+\sum_{k=1}^{n}1 &= \frac{n^2(n+1)^2}{4}+n\\ \sum_{k=1}^{n}k^3+\sum_{k=1}^{n}1 &= \frac{n^2(n+1)^2+4n}{4}\\ \frac{5}{n^4}\cdot(\sum_{k=1}^{n}k^3+\sum_{k=1}^{n}1) &= \frac{5}{n^4}\cdot(\frac{n^2(n+1)^2+4n}{4})\\ \lim_{n\to\infty}\sum_{k=1}^{n}\frac{5}{n^4}(k^3+1) &= \lim_{n\to\infty}\frac{5}{n^4}\cdot(\frac{n^2(n+1)^2+4n}{4})\end{aligned}$$

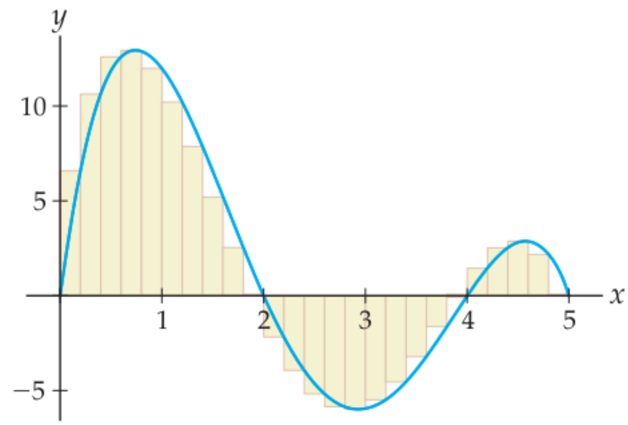
Skipping over some algebra...

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{5}{n^4}(k^3+1) = \frac{5}{4}$$

Summation Formulas

$$\begin{aligned}\sum_{i=1}^{n}i &= \frac{n(n+1)}{2}\\ \sum_{i=1}^{n}i^2 &= \frac{n(n+1)(2n+1)}{6}\\ \sum_{i=1}^{n}i^3 &= \frac{n^2(n+1)^2}{4}\\ \sum_{i=1}^{n}r^i &= \frac{1-r^{n+1}}{1-4}, r\neq 1\end{aligned}$$

3 Chapter 4.2; Riemann Sum



Definition.
Let $a, b \in \mathbb{R}, a < b$ Suppose $[a, b] \in Dom(f)$
A partition P is a collection of n + 1 points,
P = $\{x_0, x_1, x_2 \ldots x_n\}$, $\ni$: $x_0 = a < x_1 < x_2 \ldots x_n = b$

Example. Consider I = [0,1],
then P = $\{x_0 = 0, x_1 = 0.1, x_2 = 0.5, x_3 = 1\}$

Example. Consider [a,b], $a, b \in \mathbb{R}, a < b$,
then P = $\{x_0, x_1, x_2 \ldots x_n\}$, $\ni$: $x_i = a + i\triangle x, i \in [0, n]$
The sum of the areas of these partitions is called a **Riemann Sum**

4 Riemann Definition

Example 1 - Algebraic

$\int_0^1 (x^2 - 6x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} (\frac{i^3}{n^3} - \frac{6i}{n}) \cdot \frac{1}{n}$ by R. def of the definite integral.

$$\begin{aligned} \sum_{i=1}^{n} (\frac{i^3}{n^3} - \frac{6i}{n}) \cdot \frac{1}{n} &= \frac{1}{n} \sum_{i=1}^{n} (\frac{i^3}{n^3} - \frac{6i}{n}) \\ &= \frac{1}{n} (\sum_{i=1}^{n} \frac{i^3}{n^3} - \sum_{i=1}^{n} \frac{6i}{n}) \\ \lim_{n\to\infty} \frac{1}{n} (\sum_{i=1}^{n} \frac{i^3}{n^3} - \sum_{i=1}^{n} \frac{6i}{n}) &= \lim_{n\to\infty} \frac{1}{n} (\frac{1}{n^3} \sum_{i=1}^{n} i^3 - \frac{6}{n} \cdot \sum_{i=1}^{n} i) \\ &= \lim_{x\to\infty} (\frac{1}{4} \cdot \frac{(n+1)^2}{n} - 3 \cdot \frac{n+1}{n}) = -\frac{11}{4} \end{aligned}$$

Example 2 - Geometric

$\int_{-1}^{2} (1 + |x|)dx$

In theory, we can evaluate this as a Riemann sum limit, but it's far easier just to think of these as shapes.

We know what the graph of this looks like, simply draw out the graph and compute the area using elementary geometry.

Example 3 - Express as an Integral

$\lim_{x\to\infty} \frac{5}{n} (\sum_{i=1}^{n} \frac{6 + \frac{5i}{n}}{\sqrt{4 + \frac{5i}{n}}})$ we need a, b and $f(x)$

1. Decide what kind of Riemann Sum we are using.

Consider the i term, is there any (i-1)s or averages? Since there are only i's in the term by itself, this must be a right Riemann Sum.

2. Redistribute any leading coefficients.

$\lim_{x\to\infty} \frac{5}{n} (\sum_{i=1}^{n} \frac{6 + \frac{5i}{n}}{\sqrt{4 + \frac{5i}{n}}}) = \sum_{i=1}^{n} \frac{6 + \frac{5i}{n}}{\sqrt{4 + \frac{5i}{n}}}) \cdot \frac{5}{n}$

3. let $\Delta x = \frac{5}{n}$

$\sum_{i=1}^{n} \frac{6 + \frac{5i}{n}}{\sqrt{4 + \frac{5i}{n}}}) \cdot \frac{5}{n} = \sum_{i=1}^{n} \frac{2 + 4 + i\Delta x}{\sqrt{4 + i\Delta x}}) \cdot \Delta x$

we know now that $x_i = 4 + \Delta x$ which means $a = 4$, $b = 9$ and so,

we have: $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2 + x_i}{\sqrt{x_i}} \Delta x$

and so: $f(x_i) = \frac{2 + x_i}{\sqrt{x_i}} \Leftrightarrow f(x) = \frac{2 + x}{\sqrt{x}}$

Lastly, $\lim_{x\to\infty} \frac{5}{n} (\sum_{i=1}^{n} \frac{6 + \frac{5i}{n}}{\sqrt{4 + \frac{5i}{n}}}) = \int_4^9 (\frac{2 + x}{\sqrt{x}})dx$

Remark: The Riemann definition of the definite integral has serious problems in practical use, in theory it makes sense so long as the function is continuous across [a,b] or if function fhas a finite number of jump discontinuity.

Computing the riemann definition is also not very practical, as we would be stuck if we had a non polynomial integrand or a polynomial integrand of power greater than 3.

5 Theorem

Properties of the definite integral

Pages 344, 346, 347, and material from Kathleen Smith's brain.

Let $a, b \in \mathbb{R}, a < b$. Suppose that functions f and g are integrable.

1. $f(x) \geq 0$ on [a,b] $\Rightarrow \displaystyle\int_a^b f(x)dx \geq 0$

   $f(x) \leq 0$ on [a,b] $\Rightarrow \displaystyle\int_a^b f(x)dx \leq 0$

2. $\displaystyle\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

3. $\forall c \in \mathbb{R}, \displaystyle\int_a^b c \cdot f(x)dx = c \cdot \int_a^b f(x)dx$

4. $\displaystyle\int_a^a f(x)dx = 0$

5. $\displaystyle\int_a^b f(x)dx = \int_b^a -f(x)dx$

6. Union Integral Property $\forall c \in (a, b), \displaystyle\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

   Extremely useful for case defined functions.

7. $f(x) \leq g(x)$ on $[a, b] \Rightarrow \displaystyle\int_a^b f(x)dx \leq \int_a^b g(x)dx$

8. Integral Inequality $\exists m, M \in \mathbb{R}, \ni: m \leq f(x) \leq M, \forall x \in [a, b] \Rightarrow m(b-a) \leq \displaystyle\int_a^b f(x)dx \leq M(b-a)$

6 Proofs

Let $a, b \in \mathbb{R}, a < b$. Suppose that functions f and g are integrable on [a, b].

$\forall c \in \mathbb{R}$

Consider $\displaystyle\int_a^b [f(x) - c \cdot g(x)]dx$

let $P = \{x_1, x_2, x_3 \ldots x_n\}$ be a Riemann Partition of $[a, b]$

$\displaystyle\int_a^b f(x) = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i)\frac{1}{n}$

$\displaystyle\int_a^b g(x) = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i)\frac{1}{n}$

WTS

$\displaystyle\int_a^b [f(x) - c \cdot g(x)]dx = \int_a^b f(x)dx - c\int_a^b g(x)dx$

Consider RHS

$\displaystyle\int_a^b f(x)dx - c\int_a^b g(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i)\frac{1}{n} - c \cdot \lim_{n\to\infty} \sum_{i=0}^{n} g(x_i)\Delta x$ By Riemann Sums Definition

$\displaystyle\int_a^b f(x)dx - c\int_a^b g(x)dx = \lim_{n\to\infty} (\sum_{i=0}^{n} f(x_i) - c \cdot \sum_{i=0}^{n} g(x_i))\Delta x$ By Limit Laws

$\displaystyle\int_a^b f(x)dx - c\int_a^b g(x)dx = \lim_{n\to\infty} (\sum_{i=0}^{n} [f(x_i) - c \cdot g(x_i)])\Delta x$ By Summation Laws

$\displaystyle\int_a^b f(x)dx - c\int_a^b g(x)dx = \int_a^b [f(x) - c \cdot g(x)]dx$ By Riemann Sums Definition

Consider

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \in \mathbb{I} \end{cases}$$

Does the Definite Integral exist?

7 Darboux Definition of $\int_a^b f(x)dx$

Setup f is bounded on [a,b] ie. $\exists c \in \mathbb{R}^+, \ni: |f(x)| < c, \forall x \in [a,b]$

Let $a, b \in \mathbb{R}, a < b$

Suppose f is bdd(bounded), on $[a,b]$ Let $P = \{x_i\}_{i=0}^n$ be any partition of [a,b].

for $i = 1 \ldots n$

define $m_i = inf\{f(x)|x \in [x_{i-1}, x_i]\}$

define $M_i = sup\{f(x)|x \in [x_{i-1}, x_i]\}$

then the Uppersum: $U(f,p) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$

and the lowersum: $L(f,p) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$

Now let's look at Consider this again

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \in \mathbb{I} \end{cases}$$

Compute $L(f,P)$ where P is any partition of [0,3]

Sol'n:

recall: $m_i = inf\{f(x)|x \in [x_{i-1}, x_i]\}$

The possible values of $f(x)$ are simply 1 or 0. Because of the density of $\mathbb{Q} and \mathbb{I}$.

$m_i = inf\{0,1\} = 0$

$\therefore L(f,p) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i=1}^{n} 0 \cdot (x_i - x_{i-1}) = 0$

Definition

Let $a, b \in \mathbb{R}, a < b$ suppose that f is bdd on $[a,b]$ let P be any partition of $[a,b]$

We say that our function is integrable on $[a,b]$ if and only if:

$\sup\{L(f,P)|\forall P \in [a,b]\} = \inf\{U(f,P)|\forall P \in [a,b]\} = \int_a^b f(x)dx$

This definition is only used to prove that certain integrals don't exist.

8 Proof

Prove that the following is not integrable on [2,3]

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \in \mathbb{I} \end{cases}$$

Let's compute the lower sums and the upper sums.
We know that from $L(f,P) = 0$
and for the $U(f,P)$
we know that $M_i = sup\{f(x)|x \in [x_{i-1}, x_i]\}$
$f(x)$ can only be 0 or 1, because $\mathbb{Q}$ and $\mathbb{I}$ are dense.
therefore, $M_i = 1$ and $U(f,p) = \displaystyle\sum_{i=1}^{n} 1 \cdot (x_i - x_{i-1})$
$= (x_1 - x_0) + (x_2 - x_1) + (x_3 - x_2) \cdots + (x_n - x_{n-1})$ dramatic cancellations.
$= -x_0 + x_n = -0 + 3 = 3$
Because P is arbitrary, $L(f,P) = 0 \forall P$ and $U(f,P) = 3 \forall P$
$\therefore sup\{L(f,P) = 0\}$ and $inf(U(f,P) = 3)$
as $0 \neq 3$, thus $\sup\{L(f,P)|\forall P \in [a,b]\} \neq \inf\{U(f,P)|\forall P \in [a,b]\} = \displaystyle\int_a^b f(x)dx$