**Lecture Notes**
Winter 2019

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# MATA37 - CALCULUS II FOR THE MATHEMATICAL SCIENCES

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**LEC03**, Feb 8th, 2:00pm - 3:00pm


**Instructor:**

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# 1 Polynom div

**Exercises**

1. Evaluate $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx$

   This cannot be solved with just the integration techniques, as the degree of the numerator > degree of the denominator.

   We must first use polynoimial division.

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\quad\quad x+1 \\ x^3 - x^2 - x + 1 & \; x^4 \quad\quad - 2x^2 + 4x + 1 \\ & -x^4 + x^3 \; + x^2 \; - x \\ \hline & \quad\quad x^3 \; - x^2 + 3x + 1 \\ & \quad -x^3 \; + x^2 \; + x - 1 \\ \hline & \quad\quad\quad\quad\quad 4x \end{array}$$

   Thus, by poly div,

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx = \int \frac{(x^2 - x^2 - x + 1) \cdot (x + 1) + 4x}{x^3 - x^2 - x + 1} dx = \int \frac{(x^2 - x^2 - x + 1) \cdot (x + 1)}{x^3 - x^2 - x + 1} + \frac{4x}{x^3 - x^2 - x + 1} dx$$

$$= \int x + 1 + \frac{4x}{x^3 - x^2 - x + 1} dx = \int (x + 1) dx + \int \frac{4x}{x^3 - x^2 - x + 1} dx$$

   Using PFD on $\int \frac{4x}{x^3 - x^2 - x + 1} dx = \int \frac{4x}{x^2(x - 1) - (x - 1)} dx = \int \frac{4x}{(x - 1)(x - 1)(x + 1)} dx = \int \frac{4x}{(x - 1)^2(x + 1)} dx$

   PFD: $\frac{4x}{(x - 1)^2(x + 1)} dx = \frac{A}{x + 1} + \frac{B}{(x - 1)} + \frac{C}{(x - 1)^2}$

   thus, $\frac{A(x - 1)^2 + B(x - 1)(x + 1) + C(x + 1)}{(x - 1)^2(x + 1)} = \frac{4x}{(x - 1)^2(x + 1)}$

   $\implies 4x = A(x^2 - 2x + 1) + B(x^2 - 1) + Cx + C = (A + B)x^2 + (-2A + C)x + (A - B + C)$

   | | |
   |---|---|
   | Coeff $x^2$: 0 = A + B | C = 2 |
   | Coeff $x^1$: 4 = C - 2A | A = -1 |
   | Coeff $x^0$: 0 = A - B + C | B = 1 |

   Finally, $\int x + 1 + \frac{4x}{x^3 - x^2 - x + 1} dx = \int x + 1 + \frac{-1}{x + 1} + \frac{1}{x - 1} + \frac{2}{(x - 1)^2} dx$

$$= \frac{x^2}{2} + x - \ln|x + 1| + \ln|x - 1| - 2(x - 1)^{-1} + C$$

# 2 Trig Integrals

**Definition**

Let $m, n, l, k \in \mathbb{Z}^{\geq 0}$

A **Trigonometric Integral** is an integral in one of the following forms:

1. $\int \sin^l(x)\cos^k(x)dx$ or $\int_a^b \sin^l(x)\cos^k(x)dx$
2. $\int \sec^m(x)\tan^n(x)dx$ or $\int_a^b \sec^m(x)\tan^n(x)dx$

**Consider the following:**

- $\int \sin^5(x)\cos^2(x)dx$
- $\int_0^\pi \sec^5(x)dx = \int_0^\pi \sec^5(x)\tan^0(x)dx$
- $\int \sin^2(t)\cot(t)dt = \int \sin^2(t)\frac{\cos(t)}{\sin(t)}dt = \int \sin(t)\cos(t)dt$

One way or another, these are all trigonometric integrals

- $\int \tan^{1/2}(x)\sec(x)dx$

However because this last one does not have an integer exponent, it's not a trigonometric integral.

**Example**

$$\int \cos^2(x)\sin^5(x)dx = \int \cos^2(x)(1-\cos^2(x))^2\sin(x)dx$$

whenever there is ODD/EVEN for sine-cosine

let $u = \cos(x), du = -\sin(x)$

thus $\int \cos^2(x)\sin^5(x)dx = -\int u^2(1-u^2)^2du = -\int u^2 - 2u^4 + u^6du = -\left(\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right)$

$$= -\left(\frac{\cos^3(x)}{3} - \frac{2\cos^5(x)}{5} + \frac{\cos^7(x)}{7}\right) + C$$

**Useful identities**

- $\sin^2(A) + \cos^2(A) = 1$
- $\tan^2(A) + 1 = \sec^2(A)$
- $\sin^2(A) = \frac{1-\cos(A)}{2}$
- $\cos^2(A) - \frac{1+\cos(2A)}{2}$
- $\sin(2A) = 2\sin(A)\cos(A)$

**Exercise**

1. Find $\int_0^{\pi/2} \cos^5(x)\sin^3(x)dx$

   whenever there is ODD/ODD case for sine-cosine, it does not matter which we pick for our u.

   let $u = \cos(x), du = -\sin(x)dx$

2. Find $\int \sin^2(3x)\cos^2(3x)dx$

   whenever there is EVEN/EVEN for sine-cosine, we must use our identities, to create an ODD/EVEN scenario.

   $\int \sin^2(3x)\cos^2(3x)dx = \int \frac{1-\cos(6x)}{2} \cdot \frac{1+\cos(6x)}{2}dx = \frac{1}{4}\int (1-\cos^2(6x))dx = \frac{1}{4}\int \sin^2(6x)dx$

   Again, we have an EVEN/EVEN sine-cosine integral, again we must use one of the identities

   $\frac{1}{4}\int \sin^2(6x)dx = \frac{1}{4}\int \frac{1-\cos(12x)}{2}dx = \frac{1}{8}\int 1-\cos(12x)dx = \frac{1}{8}\left(x - \frac{\sin(12x)}{12}\right) = C$

3. Find $\int \sec^3(x)\tan^3(x)dx$

   Note: Deciding on u-sub vs integration by parts

   If we cam wrote our integrand into the form of $\int \tan(x)$ **something** $\cdot \sec^2(x)dx$, then we u-sub

   Or

   If we cam wrote our integrand into the form of $\int \sec(x)$ **something** $\cdot \sec(x)\tan(x)dx$, then we u-sub

   Otherwise, integrate by parts

   $\int \sec^3(x)\tan^3(x)dx = \int \sec^2(x)\tan^2(x)\sec(x)\tan(x)dx = \int \sec^2(x)(\sec^2(x)-1)\sec(x)\tan(x)dx$

   let $u = \sec(x), du = \sec(x)\tan(x)dx$

   thus, $\int \sec^3(x)\tan^3(x)dx = \int u^2(u^2-1)du = \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\sec^5(x)}{5} - \frac{\sec^3(x)}{3} + C$

4. Find $\int \sec^3(x)dx = \int \sec(x)\sec^2(x)dx$

   Choose $u = \sec(x), du = \sec(x)\tan(x)$, and $v = \tan(x), dv = \sec^2(x)dx$

   Thus $\int \sec^3(x)dx = \sec(x)\tan(x) - \int \tan(x)\sec(x)\tan(x)dx$

   $\int \sec(x)\tan^2(x)dx = \int \sec(x)(1-\sec^2(x))dx = \int \sec^3(x) + \sec(x)dx$

   let $I = \int \sec^3(x)$

   $2I = \frac{\sec(x)\tan(x) + \ln|sec(x) + \tan(x)| + C}{2}$