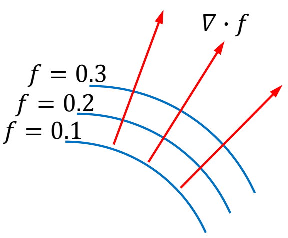
1 Fundamentals

Description	Symbol	Unit
Free electric charge in the system	Q	$C = As$
Volumetric free charge density	ρ	$\frac{C}{m^3}$
Individual electric current flowing through the surface S	I_i	A
Magnetic flux through the surface S	Φ	$Wb = Tm^2$
Induced voltage along the closed loop δS	u_i	V
Electric current density	$\vec{J} = \sigma \vec{E}$	$\frac{A}{m^2}$
Electric flux density	$\vec{D} = \epsilon \vec{E}$	$\frac{C}{m^2} = \frac{As}{m^2}$
Electric field	$\vec{E} = \frac{\vec{D}}{\epsilon}$	$\frac{V}{m}$
Magnetic flux density	$\vec{B} = \mu \vec{H}$	$T = \frac{Vs}{m^2}$
Magnetic field	$\vec{H} = \frac{\vec{B}}{\mu}$	$\frac{A}{m}$
Electric permittivity	$\epsilon = \epsilon_0 \epsilon_r$	$\epsilon_0 = 8.85 \cdot 10^{-12} \left[\frac{F}{m} = \frac{As}{Vm}\right]$
Magnetic permeability	$\mu = \mu_0 \mu_r$	$\mu_0 = 4\pi \cdot 10^{-7} \left[\frac{Tm}{A} = \frac{H}{m} = \frac{Vs}{Am}\right]$

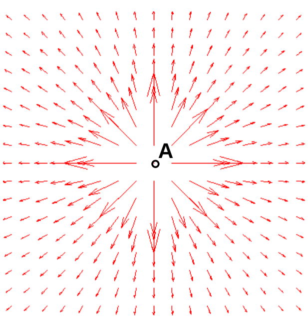
Gradient



The gradient of a scalar field reveals the direction of the steepest ascent of the scalar function (perpendicular to the corresponding isolines of the function).

Scalar field:	$f = f(\vec{r}) = f(x, y, z)$
Gradient is a vector:	$grad f = \lim_{diameter(D) \to 0} \frac{\oiint_{boundary(D)} f \cdot d\vec{A}}{measure(D)}, D \subseteq R^3$
Gradient in Cartesian coordinates:	$grad f = \frac{\partial f}{\partial x} \cdot \vec{e_x} + \frac{\partial f}{\partial y} \cdot \vec{e_y} + \frac{\partial f}{\partial z} \cdot \vec{e_z}$
∇-operator (Nabla):	$\nabla = \frac{\partial}{\partial x} \cdot \vec{e_x} + \frac{\partial}{\partial y} \cdot \vec{e_y} + \frac{\partial}{\partial z} \cdot \vec{e_z}$
Gradient and ∇-operator:	$grad f = \nabla \cdot f$

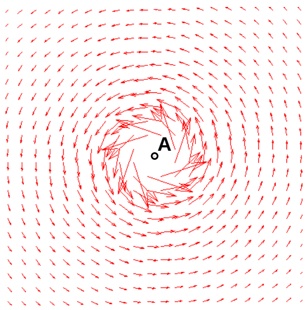
Divergence



In the central point A of this field distribution is: $div\vec{a} > 0$. Generally speaking, a positive or negative divergence reveals a field source or a field sink, respectively.

Vector field:	$\vec{a} = \vec{a}(\vec{r}) = \vec{a}(x, y, z)$
Divergence is a scalar:	$div\vec{a} = \lim_{diameter(D) \to 0} \frac{\oiint_{boundary(D)} \vec{a} \cdot d\vec{A}}{measure(D)}, D \subseteq R^3$
Divergence in Cartesian coordinates:	$div\vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$
Divergence and ∇-operator:	$div\vec{a} = \nabla \cdot \vec{a}$

Curl



In the central point A of this field distribution is: $curl\vec{a} > 0$. Generally speaking non-zero curl reveals a rotational field character. (Curl looks what stays in this area).

Vector field:	$\vec{a} = \vec{a}(\vec{r}) = \vec{a}(x, y, z)$
Curl is a vector:	$curl\vec{a} = \lim\limits_{diameter(D)\to 0} \frac{\oiint_{boundary(D)} d\vec{A}\times\vec{a}}{measure(D)}, D \subseteq R^3$
Curl in Cartesian coordinates:	$curl\vec{a} = \begin{vmatrix} \vec{e_x} & \vec{e_y} & \vec{e_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$ other coordinates in Bronstein: P.719ff
Curl and ∇-operator:	$curl\vec{a} = \nabla \times \vec{a}$

Theorems

Gauss theorem:	$\oiint\limits_{(\partial\Omega)} \vec{a}\cdot d\vec{S} = \iiint\limits_{(\Omega)} \nabla\cdot\vec{a}dV$
Stokes theorem:	$\int\limits_{(\partial S)} \vec{a}\cdot d\vec{l} = \iint\limits_{(S)} \nabla\times\vec{a}\cdot d\vec{S}$
Continuity equation:	$\oiint\limits_{(\partial\Omega)} \vec{J}(\vec{r})\cdot d\vec{S}(\vec{r}) = -\frac{\partial Q}{\partial t} = -\iiint\limits_{(\Omega)} \frac{\partial\rho(\vec{r})}{\partial t} dV(\vec{r})$

Rules

Curl of a gradient is always equal to zero:	$\nabla\times(\nabla a) \equiv 0$
Divergence of a curl si always equal to zero:	$\nabla\cdot(\nabla\times\vec{a}) \equiv 0$

2 Maxwell Equations

2.1 Integral form

Gauss's law, 1835
The electric flux density $\vec{D} = \epsilon\vec{E}$ through a closed oriented area (A) is equal to the total electric charge Q which is surrounded by this area.

$$\oiint\limits_{(\partial\Omega)} \vec{D}(\vec{r})\cdot d\vec{S}(\vec{r}) = Q = \iiint\limits_{(\Omega)} \rho(\vec{r})\cdot dV(\vec{r}) \tag{1}$$

Coulomb's law, 1785
The magnetic flux trough a closed oriented area is alway zero! This means there are no magnetic monopols.

$$\oiint\limits_{(\partial\Omega)} \vec{B}(\vec{r})\cdot d\vec{S}(\vec{r}) = 0 \tag{2}$$

Ampère's law, 1826
The sum of all currents through a closed oriented area can be computed as

$$\oint\limits_{(\partial S)} \vec{H}(\vec{r})\cdot d\vec{l}(\vec{r}) = \sum_{i=1}^{N} I_i = \iint\limits_{(S)} \vec{J}(\vec{r})\cdot d\vec{S}(\vec{r}) \tag{3}$$

Faraday's law, 1831
A time-dependent magnetic flux induces an electrical voltage.

$$u_i = \oint\limits_{(\partial S)} \vec{E}(\vec{r})\cdot d\vec{l}(\vec{r}) = -\frac{\partial\Phi}{\partial t} = -\frac{\partial}{\partial t}\iint\limits_{(S)} \vec{B}(\vec{r})\cdot d\vec{S}(\vec{r}) \tag{4}$$

2.2 Maxwell Equations in time domain

Using Gauss and Stokes Theorem on (1)-(4) the following Maxwell Equations in time domain can be obtained as

$$\nabla\cdot\vec{D} = \rho$$

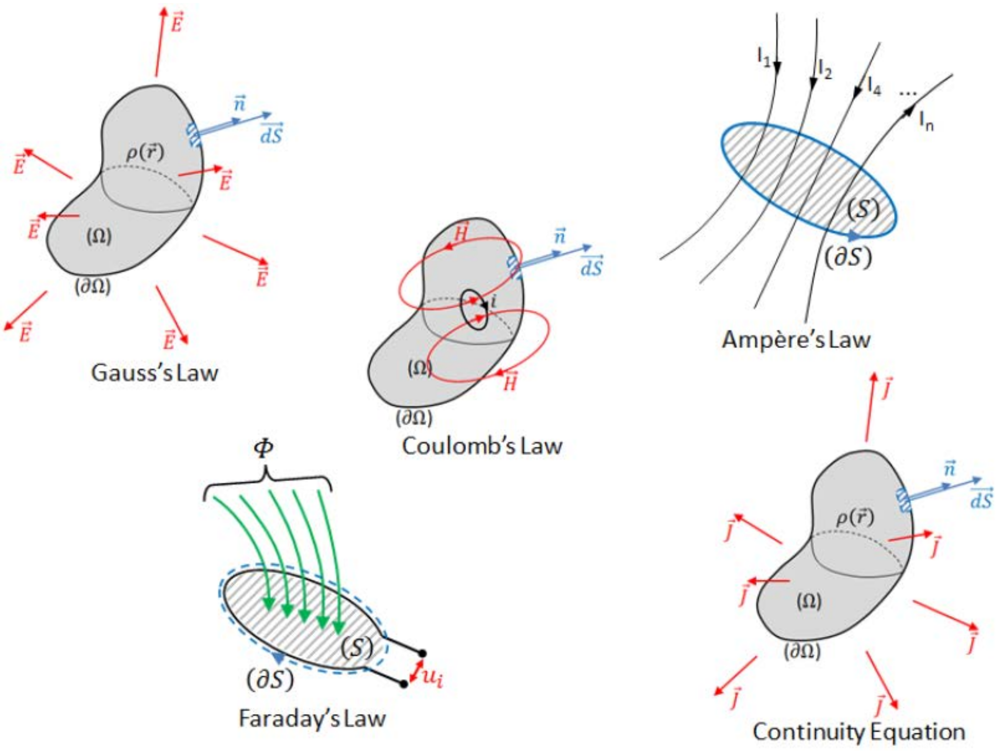
$$\nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} = \sigma\vec{E} + \epsilon\frac{\partial\vec{E}}{\partial t}$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon}$$

$$\nabla\cdot\vec{H} = 0$$



$$\nabla \times \vec{B} = \mu \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \qquad \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

2.3 Maxwell Equations in frequency domain

If the field sources are harmonic sinusoidal time functions, the fields must have also this time dependence if the involved materials are linear. The fields can be represented as the following complex vectors

$$\vec{F}\left(\vec{r}, t\right) = \Re\{\underline{\vec{F}}\left(\vec{r}\right) \cdot e^{j\omega t}\}.$$

The main advantage of this approach is the following elimination of the time derivatives

$$\frac{\partial}{\partial t}\left(ej\omega t\right) = j\omega^{\cdot} e^{j\omega t}$$

applied to the Maxwell Equations in time domain the following is obtained

$$\nabla \cdot \underline{\vec{D}}\left(\vec{r}\right) = \underline{\rho}\left(\vec{r}\right)$$

$$\nabla \cdot \underline{\vec{B}}\left(\vec{r}\right) = 0$$

$$\nabla \times \underline{\vec{H}}\left(\vec{r}\right) = \underline{\vec{J}}\left(\vec{r}\right) + j\omega \cdot \underline{\vec{D}}\left(\vec{r}\right)$$

$$\nabla \times \underline{\vec{E}}\left(\vec{r}\right) = -j\omega \underline{\vec{B}}\left(\vec{r}\right)$$

2.4 Potentials

2.4.1 Electric scalar potential

In case when the electric charge is present and the magnetic field does not depend on time the following can be written

$$\nabla \cdot \vec{D} = \nabla \cdot \left(\epsilon \vec{E} \right) = \rho$$

$$\nabla \times \vec{E} = 0$$

Due to the fact that a curl of a gradient is always equal to zero, it is possible in this case to represent the electric field as a gradient of a scalar function as

$$\vec{E} = -\nabla\Phi$$

which is called the electric scalar potential. It must fulfil the following partial differential equation

$$-\nabla \cdot (\epsilon\nabla\Phi) = \rho$$

2.4.2 magnetic vector potential

In case when the electric field does not depend on time the following can be written

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{H} = \vec{J}$$

Due to the fact that a divergence of a curl is always equal to zero, it is possible in this case to represent the magnetic flux density as a curl of a vector function

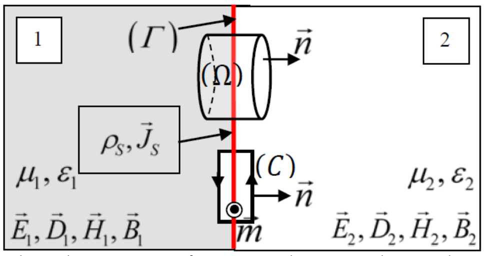
$$\vec{B} = \nabla \times \vec{A}$$

which is called the magnetic vector potential. It must fulfil the following partial differential equation

$$\underbrace{\underbrace{\nabla \times \left(\frac{1}{\mu}\underbrace{\nabla \times \vec{A}}_{\perp}\right)}_{\perp} = \vec{J}}_{||}$$

3 Boundary Conditions (BC)

3.1 Interface Conditions



$$\left(\vec{D}_1 - \vec{D}_2\right) \cdot \vec{n} = \rho_S$$
$$\left(\vec{B}_1 - \vec{B}_2\right) \cdot \vec{n} = 0$$
$$\left(\vec{H}_2 - \vec{H}_1\right) \times \vec{n} = \vec{J}_S$$
$$\left(\vec{E}_2 - \vec{E}_1\right) \times \vec{n} = 0$$

The above interface conditions show the field behavior over the border between two different materials.
The normal flux density continuity conditions can be derived by integrating Equations (1) and (2) over the cylinder Ω depicted above.
The tangential field continuity conditions can be proven by integrating Equations (3) and (4) along the contour C shown above.

