RegT4 – Modern Control Engineering

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Contributors

The following students have either directly or indirectly contributed to this work: Naoki Pross.

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Contents

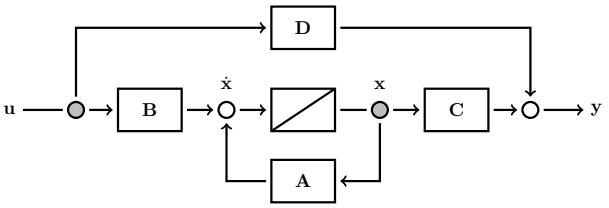


Figure 1: State space model of a LTI system.

1 State Space Models

1.1 Introduction

In modern control theory a *state* of a dynamic system is the smallest set of variables (called *state variables*) such that the knowledge of these variables at $t = t_0$, together with the knowledge of the input for $t \geq t_0$, completely determines the behavior of the system for any time $t \geq t_0$.

1.2 State Space Equations

The state space equations are, in their most general form, a set of first order differential equations and a set of outputs:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$$
$$\mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u}, t).$$

This can describe any system because higher order differential equations can (practically) always be rewritten as a set of first order differential equations. In the case where the system is *linear* they can be rewritten using linear algebra:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t),$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t),$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{B} \in \mathbb{R}^{n\times m}, \mathbf{C} \in \mathbb{R}^{p\times n}, \mathbf{D} \in \mathbb{R}^{p\times m}$ (see figure 1). In general in a time varying system all four matrices are actually functions of time ($\mathbf{A} = \mathbf{A}(t), \ldots$). Whereas for *linear time invariant* (LTI) systems they are constants.

1.3 Linearization of non LTI systems

In general dynamics are not LTI systems, however it is possible to create linear approximation of complex dynamics around a *set point* $(\mathbf{x}_0, \mathbf{u}_0)$, if it is also a fixed point (that means $\mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) = \mathbf{0}$ and $\mathbf{g}(\mathbf{x}_0, \mathbf{u}_0) = \mathbf{y}_0$), by computing the four Jacobian matrices:

$$\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_0, \mathbf{u}_0), \qquad \mathbf{B} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x}_0, \mathbf{u}_0),$$
$$\mathbf{C} = \frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\mathbf{x}_0, \mathbf{u}_0), \qquad \mathbf{D} = \frac{\partial \mathbf{g}}{\partial \mathbf{u}}(\mathbf{x}_0, \mathbf{u}_0).$$

Then the differences $\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}_0$, $\Delta\mathbf{u} = \mathbf{u} - \mathbf{u}_0$ are used in the model:

$$\Delta\dot{\mathbf{x}}(t) = \mathbf{A}\Delta\mathbf{x} + \mathbf{B}\Delta\mathbf{u},$$
$$\Delta\mathbf{y}(t) = \mathbf{C}\Delta\mathbf{x} + \mathbf{D}\Delta\mathbf{u}.$$

1.4 Time-Domain Solutions

1.4.1 Homogeneous Case

To solve for the solution of the system at rest (no inputs)

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x},$$

we assume that the solution is a power series in t, or

$$\mathbf{x}(t) = \mathbf{b}_0 + \mathbf{b}_1 t + \mathbf{b}_2 t^2 + \cdots + \mathbf{b}_k t^k + \ldots.$$

Substituting into the differential equation yields a solution

$$\mathbf{x}(t) = \underbrace{\left(\mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2 t^2 + \cdots + \frac{1}{k!}\mathbf{A}^k t^k + \ldots\right)}_{\text{Matrix exponential } e^{\mathbf{A}t}} \mathbf{x}(0),$$

which looks like the power series for the exponential function. Thus we define it to be the *matrix exponential* and write the solution as

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0).$$

The matrix exponential has the property that

$$e^{\mathbf{A}(t+s)} = e^{\mathbf{A}t}e^{\mathbf{A}s},$$

and in particular when $s = -t$ then the expression equals the identity matrix. Thus the inverse of $e^{\mathbf{A}t}$ is $e^{-\mathbf{A}t}$ and always exists. It is important to note that

$$e^{(\mathbf{A}+\mathbf{B})t} \neq e^{\mathbf{A}t}e^{\mathbf{B}t},$$

unless we are in the very special case when $\mathbf{AB} = \mathbf{BA}$.

1.4.2 Transition Matrix

The result of the exponential matrix is of such relevance that it has a special name. The expression

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t}$$

is called the *transition matrix*, since it describes how the initial conditions change. The transition matrix has some important algebraic properties which are listed below:

$$\mathbf{\Phi}(0) = e^{\mathbf{A}0} = \mathbf{I},$$
$$\mathbf{\Phi}^{-1}(t) = \left(e^{\mathbf{A}t}\right)^{-1} = \mathbf{\Phi}(-t),$$
$$\mathbf{\Phi}(t)^n = \left(e^{\mathbf{A}t}\right)^n = \mathbf{\Phi}(nt),$$
$$\mathbf{\Phi}(t_1 + t_2) = e^{\mathbf{A}(t_1+t_2)} = \mathbf{\Phi}(t_1)\mathbf{\Phi}(t_2) = \mathbf{\Phi}(t_2)\mathbf{\Phi}(t_1),$$
$$\mathbf{\Phi}(t_2 - t_1)\mathbf{\Phi}(t_1 - t_0) = \mathbf{\Phi}(t_2 - t_0)$$
$$\dot{\mathbf{\Phi}}(t) = \mathbf{A}\mathbf{\Phi}(t).$$

A noteworthy result from the last property is that

$$\mathbf{A} = \dot{\mathbf{\Phi}}(t)\mathbf{\Phi}^{-1}(t) = \dot{\mathbf{\Phi}}(t)\mathbf{\Phi}(-t).$$

Finally, in the case when $\mathbf{A}$ is a diagonal matrix with entries $\lambda_1, \ldots, \lambda_n$ then

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}.$$

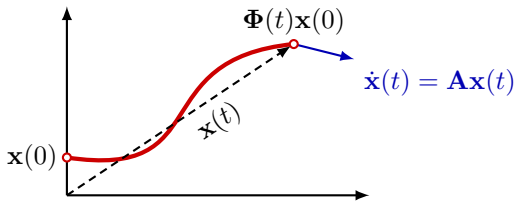


Figure 2: Interpretation of the $\mathbf{A}$ and $\mathbf{\Phi}$ matrices.

1.4.3 Nonhomogeneous Case

To expand the previous result, in the more general case when

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},$$

a few steps that are the same as in the scalar case give the solution

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau,$$

which can also be written using the transition matrix as:

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau.$$

1.5 Laplace s-Domain Solution

1.5.1 Transfer Matrix

By taking the Laplace transform the state space equations of an LTI system they become:

$$\begin{aligned} s\mathbf{X}(s) - \mathbf{x}(0) &= \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s), \\ \mathbf{Y}(s) &= \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s). \end{aligned}$$

Similarly to the one dimensional case it is now possible to define the more general equivalent of a transfer function:

$$\mathbf{G}(s) = \mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D},$$

which is the *transfer matrix*.

1.5.2 Impulse Response

The inverse Laplace transform of the transfer matrix is the *impulse response*. However, setting $\mathbf{u}(t) = \boldsymbol{\delta}(t)$ will also give the impulse response. By taking the result in section 1.4.3 and substituting into the output equation, we get that with a Dirac delta as input

$$\mathbf{g}(t) = \mathbf{C}\mathbf{\Phi}(t)\mathbf{B} + \mathbf{D}\boldsymbol{\delta}(t).$$

1.6 Controllability

A system is said to be *controllable* at time t_0 if it is possible my means of an unconstrained control vector to transfer the system from any initial state $\mathbf{x}(t_0)$ to any other state in a finite interval of time. To check if a system is completely controllable there is the $n \times nm$ *controllability matrix*

$$\mathbf{Q}_c = \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}.$$

If $\mathbf{Q}_c$ is singular (has linearly dependent rows), or equivalently its rank is not n, or (also equivalently) $\det \mathbf{Q}_c = 0$, then the system is not totally state controllable.

It is also possible to determine the controllability from the transfer function (or matrix) of a system. It can be shown that if cancellation occurs (German: *Polkürzung*), then the system cannot be controlled in the direction of the canceled mode.

1.7 Observability

In the same (too) abstract spirit as in the controllability: a system is said to be *observable* at time t_0 if, with the system in state $\mathbf{x}(t_0)$, it is possible to determine this state from the observation of the output over a finite time interval. The observability can be determined through the ?×? *observability matrix*

$$\mathbf{Q}_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix} = \begin{bmatrix} \mathbf{C}^* & \mathbf{A}^*\mathbf{C}^* & \cdots & (\mathbf{A}^*)^{n-1}\mathbf{C}^* \end{bmatrix},$$

where the asterisk is the conjugate transpose (for real matrices $\mathbf{A}^* = \mathbf{A}^\mathsf{T}$). The system is observable if $\mathbf{Q}$ is a regular matrix.

In terms of the transfer function: if cancellation occurs in the transfer function then the canceled mode cannot be observed for the output. In other words, no cancellation is a necessary and sufficient condition for total observability.

1.8 Stability

A state space model system is stable is the $\mathbf{A}$ matrix is positive definite, which means that the real part of all eigenvalues $\mathrm{Re}\{\lambda_i\} > 0$. Another way to determine if $\mathbf{A}$ is positive definite is to test whether the quadratic expression $\mathbf{x}^*\mathbf{A}\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$.

1.9 Equivalent Representations

In this section we assume without loss of generality that the system has a transfer function

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}.$$

1.9.1 Controllable Canonical Form

This form (German: *Regelungsnormalform*) is important in discussing the pole-placement approach. It is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u.$$

1.9.2 Observable Canonical Form

This mode is similar to previous form, in fact, the $\mathbf{A}$ matrix is the transposed version of the controllable form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \dots & -a_n \\ 1 & 0 & 0 & \dots & -a_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 0 & \dots & -a_2 \\ 0 & 0 & 0 & \dots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix} u.$$

1.9.3 Diagonal Form (Decoupling)

An important form is when the $\mathbf{A}$ matrix is diagonalized, i.e.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \lambda_{n-1} & \\ & & & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \tilde{\mathbf{B}} u.$$

In this representation the state variables are the *eigenstates* of the system, and they are decoupled from each other. Since the diagonal contains the eigenvalues, as noted in the stability section 1.8, they describe the poles of the transfer function $G(s)$, and therefore the stability of the system.

1.9.4 Normal Jordan Form

1.10 Similarity Transform

In the previous section there were many representation that are practical to use, but in fact there is nothing special about them. In fact, the state space can be transformed into an infinite number of other equivalent representations.

Given an arbitrary regular matrix $\mathbf{P}$, the state space in $\mathbf{x}$ can be brought into a new set of state variables $\mathbf{x}' := \mathbf{P}\mathbf{x}$ by appropriately transforming the matrices as follows:

$$\mathbf{A}' := \mathbf{P}\mathbf{A}\mathbf{P}^{-1}, \qquad \mathbf{B}' := \mathbf{P}\mathbf{B},$$
$$\mathbf{C}' := \mathbf{C}\mathbf{P}^{-1}, \qquad \mathbf{D}' := \mathbf{D}.$$

1.11 Subsystem Aggregation

1.11.1 Series

When after a subsystem 1 a subsystem 2 is connected in series the model of the entire path is

$$\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{B}_2\mathbf{C}_1 & \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2\mathbf{D}_1 \end{bmatrix} \mathbf{u},$$
$$\mathbf{y} = \begin{bmatrix} \mathbf{D}_2\mathbf{C}_1 & \mathbf{C}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \mathbf{D}_2\mathbf{D}_1\mathbf{u}.$$

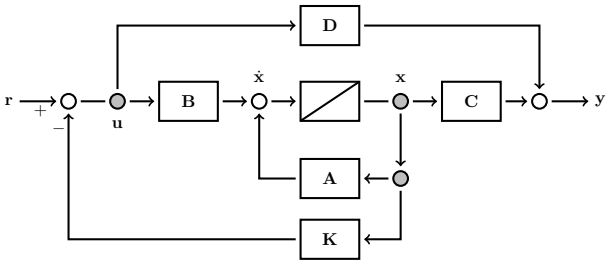


Figure 3: Pole placement through the gain matrix $\mathbf{K}$.

1.11.2 Parallel

When a subsystem 1 and a subsystem 2 are connected in parallel, the model of the entire system is

$$\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} \mathbf{u},$$
$$\mathbf{y} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + (\mathbf{D}_2 + \mathbf{D}_1)\mathbf{u}.$$

1.11.3 Feedback

When to a subsystem 1 a subsystem 2 is connected in the negative feedback path, the matrices of the entire system are

$$\mathbf{A}' := \begin{bmatrix} \mathbf{A}_1 - \mathbf{B}_1\mathbf{K}_1\mathbf{D}_2\mathbf{C}_1 & -\mathbf{B}_1\mathbf{K}_1\mathbf{C}_2 \\ \mathbf{B}_2\mathbf{K}_2\mathbf{C}_1 & \mathbf{A}_2 - \mathbf{B}_2\mathbf{K}_2\mathbf{D}_1\mathbf{C}_2 \end{bmatrix},$$
$$\mathbf{B}' := \begin{bmatrix} \mathbf{B}_1\mathbf{K}_1 \\ \mathbf{B}_2\mathbf{K}_2\mathbf{D}_1 \end{bmatrix},$$
$$\mathbf{C}' := \begin{bmatrix} \mathbf{K}_2\mathbf{C}_1 & -\mathbf{K}\mathbf{D}_1\mathbf{C}_2 \end{bmatrix}$$
$$\mathbf{D}' := \mathbf{K}_2\mathbf{D}_1,$$

where $\mathbf{K}_1 = (\mathbf{I} - \mathbf{D}_2\mathbf{D}_1)^{-1}$ and $\mathbf{K}_2 = (\mathbf{I} - \mathbf{D}_1\mathbf{D}_2)^{-1}$. In the case of a positive feedback the minus signs become pluses.

1.12 Create a State Space Model from a Transfer Function

1.13 MATLAB Reference

1.13.1 State Space Models

1.13.2 Padé Approximation

2 Control in State Space

2.1 Pole Placement

2.1.1 Gain Matrix

Consider the state space model

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t),$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t),$$

and assume that we can somehow always know the state $\mathbf{x}$. Then we add a feedback loop from the state to the input with the *gain matrix* $\mathbf{K}$ (see figure 3). This introduces a new term for the desired value $\mathbf{r}$ and changes the state space equation to

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\,\mathbf{x} + \mathbf{Br}.$$

The addition of $\mathbf{K}$ allow to change the eigenvalues of the system matrix and thus move the poles. Therefore, the poles of the new matrix $\tilde{\mathbf{A}} := \mathbf{A} - \mathbf{BK}$ are called the regulator poles. To allow an arbitrary placement of poles the system needs to be completely controllable. Without any additional construction, the gain matrix is basically a proportional controller (German: *P-Regler*).

2.1.2 Solve for the Gain Matrix

Now, if we wish to bring the poles of the closed loop system to the locations $\mu_1, \mu_2, \ldots, \mu_n$, in the most general case we can use the fact that

$$\begin{aligned}\det\left(s\mathbf{I} - \tilde{\mathbf{A}}\right) &= \det\left(s\mathbf{I} - \mathbf{A} + \mathbf{BK}\right)\\ &= (s-\mu_1)(s-\mu_2)\cdots(s-\mu_n)\\ &= s^n + \alpha_1 s_{n-1} + \cdots + \alpha_{n-1}s + \alpha_n,\end{aligned}$$

and solve for the entries of $\mathbf{K}$ by comparing coefficients.

If the state space representation is in the diagonal form (decoupled, eigenmodes), then finding $\mathbf{K}$ is as simple as solving for the entries of $\mathbf{K}$:

$$\mathbf{A} - \mathbf{BK} = \underbrace{\begin{bmatrix}\mu_1 & & \\ & \ddots & \\ & & \mu_n\end{bmatrix}}_{\mathbf{M}}.$$

Sometimes it is possible to do $\mathbf{K} = \mathbf{B}^{-1}(\mathbf{A} - \mathbf{K})$, but more often the system is overdetermined and we are free to choose some entries of $\mathbf{K}$ (which is even easier).

If the state space representation is in the controllable canonical form, and

$$\mathbf{K} = \begin{bmatrix}k_1 & k_2 & \ldots & k_n\end{bmatrix}$$

since $\tilde{\mathbf{A}} = \mathbf{A} - \mathbf{BK} =$

$$\begin{bmatrix}0 & 1 & 0 & \ldots & 0\\ 0 & 0 & 1 & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \ldots & 1\\ -a_n - k_1 & -a_{n-1} - k_2 & -a_{n-2} - k_3 & \ldots & -a_1 - k_n\end{bmatrix},$$

which means that

$$G(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + (a_{n-1} + k_n)s^{n-1} + \cdots + (a_0 + k_1)},$$

the poles are found by comparing the coefficients of the denominator:

$$\begin{aligned}s^n + (a_{n-1} + k_n)s^{n-1} + \cdots + (a_0 + k_1)\\ = (s-\mu_1)(s-\mu_2)\cdots(s-\mu_n).\end{aligned}$$

2.2 Linear Quadratic Regulator (LQR)

A problem of the pole placement approach is that it is not always obvious which poles should be moved and where. The *linear quadratic regulator* (LQR) also called *quadratic optimal regulator* solves this problem.

In the LQR design we define a performance index to minimize:

$$J = \int_0^\infty \left(\mathbf{x}^*\mathbf{Q}\mathbf{x} + \mathbf{u}^*\mathbf{R}\mathbf{u}\right)\,dt,$$

where $\mathbf{Q}$ and $\mathbf{R}$ are positive definite (or semidefinite) Hermitian or real symmetric matrices. The $\mathbf{Q}$ matrix determines the importance of the relative error, whereas the $\mathbf{R}$ matrix determines the cost (in energy) of modifying the state. Since in the pole placement $\mathbf{u} = -\mathbf{Kx}$ the problem becomes

$$\arg\min_{\mathbf{K}} \int_0^\infty \mathbf{x}^*\left(\mathbf{Q} + \mathbf{K}^*\mathbf{R}\mathbf{K}\right)\mathbf{x}\,dt.$$

Then, the expression can be further developed by using the fact that

$$\mathbf{x}(t) = e^{(\mathbf{A}-\mathbf{BK})t}\mathbf{x}(0),$$

which when substituted into the previous expression gives

$$\mathbf{x}(0)^* \underbrace{\int_0^\infty \left[e^{(\mathbf{A}-\mathbf{BK})t}\right]^* \left(\mathbf{Q} + \mathbf{K}^*\mathbf{R}\mathbf{K}\right) e^{(\mathbf{A}-\mathbf{BK})t}\,dt}_{\mathbf{S}(\mathbf{K})}\,\mathbf{x}(0).$$

The minimization problem is completed by computing

$$\text{minimize } J \iff 0 = \frac{\partial J}{\partial K_{ij}} = \mathbf{x}^*(0)\frac{\partial \mathbf{S}}{\partial K_{ij}}\mathbf{x}(0).$$

It is possible to show that the resulting optimal gain matrix is

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^*\mathbf{S}.$$

The determinant is a measure of how the unit square ($n = 2$), cube ($n = 2$) or hypercube ($n > 3$) is scaled by the linear transformation. The determinant of a 2×2 matrix is computed using the "fish rule":

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

By hand, for a 3×3 matrix, Sarrus' rule is probably the fastest method:

$$\det \underbrace{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}}_{\mathbf{A}} \rightsquigarrow \begin{bmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{bmatrix},$$

the red lines are added, while the blue ones are subtracted:

$$\det \mathbf{A} = aei + bfg + cdh - ceg - afh - bdi.$$

A.2 Inverse Matrix

The inverse of a 2×2 matrix is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

I remember it as "transposed on the wrong diagonal and add minuses (and divide by the determinant)". For bigger matrices, there are the method of Gaussian elimination:

$$\left[\mathbf{A} \,|\, \mathbf{I}\right] \xrightarrow{\text{Gauss}} \left[\mathbf{I} \,|\, \mathbf{A}^{-1}\right],$$

or Cramers' rule $\mathbf{A}^{-1} = \operatorname{adj} \mathbf{A} / \det \mathbf{A}$.

A.3 Eigenvalues and Eigenvectors

The eigenvalue problem is to find the vectors that are unaffected by a matrix A (up to a scalar factor):

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v} \iff (\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \mathbf{0}.$$

There are n numbers λ, the *eigenvalues*, and equally many *eigenvectors* $\mathbf{v}$. This problem is solved by first finding the λ's and then later the eigenvectors.

In the right expression, we need a matrix $\mathbf{A} - \lambda \mathbf{I}$ to move any vector to the origin, this is accomplished by forcing

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0,$$

and solving for the λ's. Then to find the eigenvectors we put back each eigenvalue λ_i into the original equation to obtain a system of equations

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{v}_i = \mathbf{0}.$$

This system of equations is solved for the components of $\mathbf{v}_i$, and it is overdetermined. This is because there are infinitely many eigenvectors each eigenspace of $\mathbf{A}$.

For a 2×2 matrix the eigenvalue problem is a parabola, so there is a shortcut:

$$m = \frac{1}{2} \operatorname{tr} \mathbf{A}, \qquad p = \det \mathbf{A},$$

then $\lambda_{1,2} = m \pm \sqrt{m^2 - p}$.

A.4 Diagonalization

If a $n \times n$ matrix $\mathbf{A}$ has n distinct eigenvalues, it is *diagonalizable*. To diagonalize $\mathbf{A}$, first we compute the eigenvalues $\lambda_1, \ldots, \lambda_n$ and the eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, then a matrix is formed by using the eigenvectors as rows:

$$\mathbf{T} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix}.$$

This matrix is then used for a change of bases into the eigenbasis of $\mathbf{A}$, where it is a diagonal matrix:

$$\mathbf{\Lambda} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

A.5 Jordan Normal Form

A.6 Caley-Hamilton Theorem

A.7 Useful Matrices and Products

Quadratic Form The quadratic product is an expression of the form $\mathbf{x}^* \mathbf{Q} \mathbf{x}$. Let $\mathbf{x} \in \mathbb{C}^{2 \times 1}$ be a complex column vector and a, b, q be complex numbers:

$$\mathbf{x}^* \underbrace{\begin{bmatrix} a & q \\ q & b \end{bmatrix}}_{\mathbf{Q}} \mathbf{x} = a|x_1|^2 + q(x_1^* x_2 + x_1 x_2^*) + b|x_2|^2.$$

If working only with real numbers:

$$\mathbf{x}^\mathsf{T} \mathbf{Q} \mathbf{x} = a x_1^2 + 2q x_1 x_2 + b x_2^2.$$

For bigger (real) n dimensional vectors:

$$\mathbf{x}^\mathsf{T}\mathbf{Q}\mathbf{x} = \sum_{i,j}^{n^2} Q_{ij}\, x_i x_j.$$

Column Row Product For two complex numbers a and b:

$$\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \rightsquigarrow \quad \mathbf{x}\mathbf{x}^* = \begin{bmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{bmatrix}.$$

Rotation Matrices In 2 dimensions:

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Rotation matrices belong to the special orthogonal group, therefore

$$\mathbf{R}\mathbf{R}^\mathsf{T} = \mathbf{I} \iff \mathbf{R}^{-1} = \mathbf{R}^\mathsf{T} \text{ and } \det\mathbf{R} = 1.$$

B Vector Analysis

B.1 Gradient Vector

The *gradient* of a function $f(\mathbf{x}), \mathbf{x} \in \mathbb{R}^m$ is a column vector containing the partial derivatives in each direction.

$$\boldsymbol{\nabla} f(\mathbf{x}) = \sum_{i=1}^{m} \partial_{x_i} f(\mathbf{x}) \mathbf{e}_i = \begin{pmatrix} \partial_{x_1} f(\mathbf{x}) \\ \vdots \\ \partial_{x_m} f(\mathbf{x}) \end{pmatrix}$$

The gradient vector always points towards *the direction of steepest ascent*, and thus is always perpendicular to contour lines.

B.2 Jacobian Matrix

The *Jacobian* $\mathbf{J}_f$ (sometimes written as $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$) of a function $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}^n$ is a matrix $\in \mathbb{R}^{m\times n}$ whose entry at the i-th row and j-th column is given by $(\mathbf{J}_f)_{i,j} = \partial_{x_j} f_i$, so

$$\mathbf{J}_f = \begin{pmatrix} \partial_{x_1} f_1 & \cdots & \partial_{x_m} f_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} f_n & \cdots & \partial_{x_m} f_n \end{pmatrix} = \begin{pmatrix} (\boldsymbol{\nabla} f_1)^t \\ \vdots \\ (\boldsymbol{\nabla} f_m)^t \end{pmatrix}$$

C Laplace Transform

The Laplace transform of a function f is

$$F(s) = \mathcal{L}\, f(t) = \int_0^\infty f(T)\, e^{-st}\, dt,$$

and in the other direction the inverse Laplace transform is

$$f(t) = \mathcal{L}^{-1}\, F(s) = \frac{1}{2\pi i} \lim_{T\to\infty} \oint_{\gamma - iT}^{\gamma + iT} F(s)\, e^{st}\, ds.$$

The Laplace transform converts integration and differentiation problems into algebraic problems. Some important properties and transform pairs are listed in table 1.

D z-Transform

D.1 Definition

The z-transform of a sequence (f_n) is defined to be

$$F(z) = \sum_{k=0}^{\infty} f_k z^{-k},$$

where z is a complex number. The inverse transform is

$$f_n = \frac{1}{2\pi i} \oint_C F(z) z^{n-1}\, dz,$$

where C is a counterclockwise path encircling the origin and entirely in the region of convergence (ROC).

D.2 Relation to the Laplace Transform

The z-transform is related to the Laplace transform, when a sequence is generated by sampling a function. In other words with a sampling time T we can construct a continuous time function

$$f(t) = \sum_k f_k \delta(t - kT)$$

of which is then possible to take the Laplace transform resulting in

$$\mathcal{L}\left\{\sum_k f_n \delta(t - kT)\right\} = \sum_{k=0}^{\infty} f_k e^{-kTs} = \sum_{k=0}^{\infty} f_k z^{-k},$$

where in the last step the substitution $z = e^{sT}$ was made. Therefore the multiplication by z^{-1} can interpreted as adding a delay of T.

E Idiot's section

$$\int \ln x\, dx = x\ln x - x + C$$

$$\int \sin^2 ax\, dx = \frac{x}{2} - \frac{\sin 2ax}{4a} + C$$

$$\int x e^{ax}\, dx = \frac{e^{ax}}{a^2}(ax - 1) + C$$

$$\int x^2 e^{ax}\, dx = e^{ax}\left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right) + C$$

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx) + C$$

Property	**Time Domain**	**s-Domain**	**Function**	**Laplace Tr.**
Linearity	$\sum_k a_k f_k(t)$	$\sum_k a_k F_k(s)$	$\delta(t)$	1
Time shift ($\tau > 0$)	$f(t-\tau)$	$e^{-s\tau}F(s)$	$\varepsilon(t)$	$\frac{1}{s}$
Damping	$e^{at}f(t)$	$F(s-a)$	t^n	$\frac{n!}{s^{n+1}}$
Stretching ($a > 0$)	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
Differentiation	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0^+) - \ldots - f^{(n-1)}(0^+)$	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
Integration	$\int_0^t f(t)\,dt$	$\frac{1}{s}F(s)$		

Table 1: Useful Laplace transform rules and pairs. All functions are assumed to be causal (multiplied with the step response $\varepsilon(t)$).

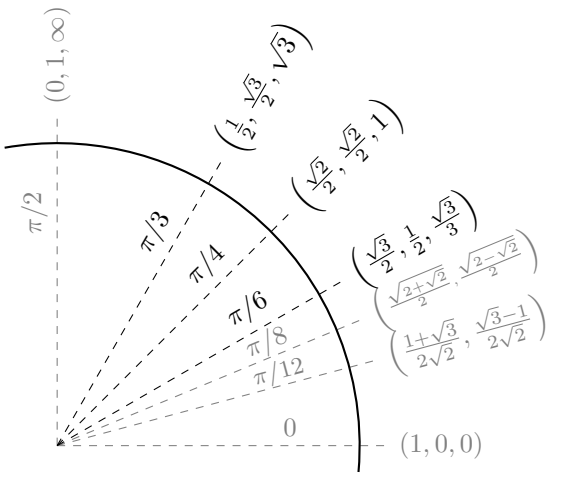


Figure 4: Useful angles.

$$\cos^2(x) + \sin^2(x) = 1 \qquad \cosh^2(x) - \sinh^2(x) = 1$$

$\cos(\alpha + 2\pi)$	$=$	$\cos(\alpha)$	$\sin(\alpha + 2\pi)$	$=$	$\sin(\alpha)$
$\cos(-\alpha)$	$=$	$\cos(\alpha)$	$\sin(-\alpha)$	$=$	$-\sin(\alpha)$
$\cos(\pi - \alpha)$	$=$	$-\cos(\alpha)$	$\sin(\pi - \alpha)$	$=$	$\sin(\alpha)$
$\cos(\frac{\pi}{2} - \alpha)$	$=$	$\sin(\alpha)$	$\sin(\frac{\pi}{2} - \alpha)$	$=$	$\cos(\alpha)$
$\cos(\alpha + \beta)$	$=$	$\cos\alpha\cos\beta - \sin\alpha\sin\beta$			
$\sin(\alpha + \beta)$	$=$	$\sin\alpha\cos\beta - \cos\alpha\sin\beta$			
$\cos(2\alpha)$	$=$	$\cos^2\alpha - \sin^2\alpha$			
	$=$	$1 - 2\sin^2\alpha$			
	$=$	$2\cos^2\alpha - 1$			
$\sin(2\alpha)$	$=$	$2\sin\alpha\cos\alpha$			
$\tan(2\alpha)$	$=$	$(2\tan\alpha)(1 + \tan^2\alpha)^{-1}$			

Table 2: A few useful trigonometric identities.

$$(af)' = af' \qquad (u(v))' = u'(v)v'$$
$$(uv)' = u'v + uv' \qquad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$
$$\left(\sum u_i\right)' = \sum u_i' \qquad (\ln u)' = \frac{u'}{u}$$
$$(f^{-1})' = \frac{1}{f'(f^{-1}(x))}$$

Table 3: Rules for differentiation, f, u, v are differentiable functions of x.

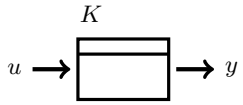
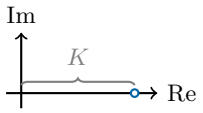
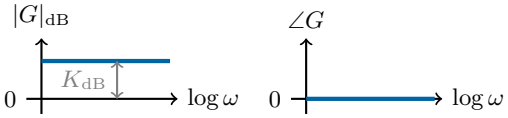
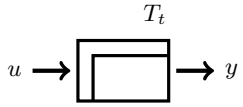
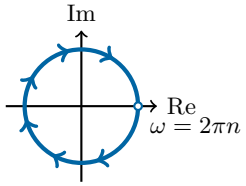
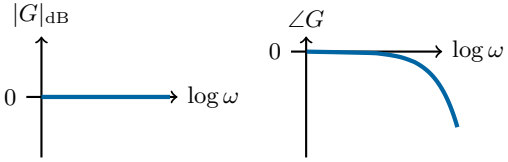
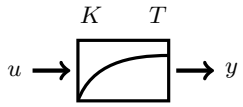
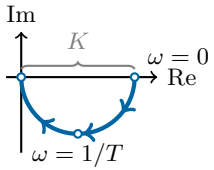
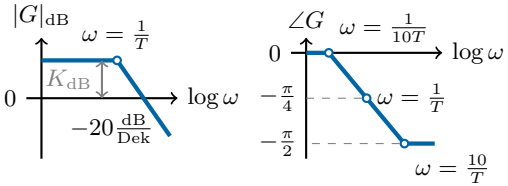
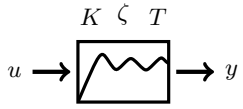
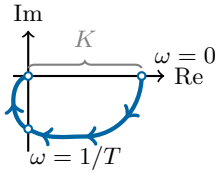
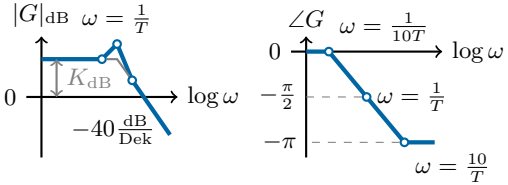
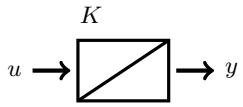
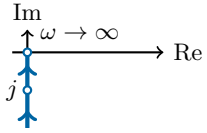
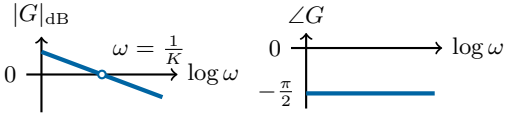
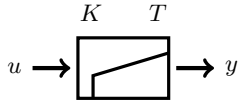
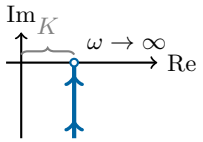
Linearity	$\int k(u+v) = k\left(\int u + \int v\right)$
Partial fraction decomposition	$\int \frac{Q}{P_n}\,dx = \sum_{k=1}^{n} \int \frac{A_k}{x - r_k}\,dx$
Affine transformation	$\int f(\lambda x + \ell)\,dx = \frac{1}{\lambda} F(\lambda x + \ell) + C$
Integration by parts	$\int u\,dv = uv - \int v\,du$
Power rule $n \neq -1$	$\int u^n \cdot u' = \frac{u^{n+1}}{n+1} + C$
Logarithm rule	$\int \frac{u'}{u} = \ln\lvert u\rvert + C$
General substitution $x = g(u)$	$\int f(x)\,dx = \int (f \circ g)\ g'\,du$ $= \int \frac{f \circ g}{(g^{-1})' \circ g}\,du$
Universal substitution	$t = \tan(x/2), dx = 2/(1+t^2)dt$ $\sin(x) = \frac{2t}{1+t^2},\ \cos(t) = \frac{1-t^2}{1+t^2}$

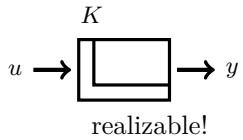
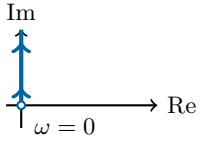
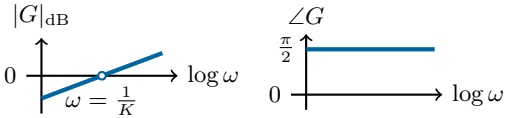
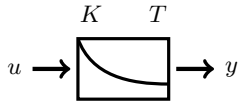
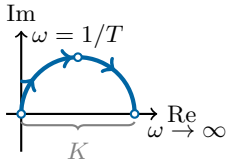
Table 4: Integration rules, f, u, v are integrable functions of x.

f	f'	f	f'
x^n	nx^{n-1}	a^x	$a^x \ln a$
$\sqrt[n]{x}$	$1/\left(x^n \sqrt[n]{x^{n-1}}\right)$	$\ln x$	$1/x$
$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\tan x$	$1/\cos^2 x$	$1/\tan x$	$-1/\sin^2 x$
$\arcsin x$	$1/\sqrt{1-x^2}$	$\arccos x$	$-1/\sqrt{1-x^2}$
$\arctan x$	$1/\left(1+x^2\right)$		
$\sinh x$	$\cosh x$	$\tanh x$	$1/\cosh^2 x$
$\operatorname{arcsinh} x$	$1/\sqrt{1+x^2}$	$\operatorname{arccosh} x$	$1/\sqrt{x^2-1}$

Table 5: Some useful derivatives and integrals.

F Block Diagram Elements

Type	Symbol	Differential Equation	Frequency Response	Nyquist Diagram	Bode Plot
P		$y = Ku$	K		
T		$y(t) = \varepsilon(t)u(t - T_t)$	$e^{-j\omega T_t}$		
PT_1		$T\dot{y} + y = Ku$	$\dfrac{K}{j\omega T + 1}$		
PT_2		$T^2\ddot{y} + 2\zeta T\dot{y} + y = Ku$	$\dfrac{K}{T^2(j\omega)^2 + 2\zeta T(j\omega) + 1}$		
I		$\dot{y} = Ku$	$\dfrac{K}{j\omega}$		
PI		$y = K\left(u + \int_0^t \dfrac{u\,d\tau}{T}\right)$	$K\left(1 + \dfrac{1}{j\omega T}\right)$		

Type	Symbol	Differential Equation	Frequency Response	Nyquist Diagram	Bode Plot
D	 realizable! Not	$y = K\dot{u}$	$j\omega K$		
DT_1		$T\dot{y} + y = K\dot{u}$	$\frac{j\omega K}{1 + j\omega T}$		

F.1 PT_2–Element

F.1.1 Damping

$$K/G_{\mathrm{PT}_2} = T^2(j\omega)^2 + 2\zeta T(j\omega) + 1 = \begin{cases} \text{aperiodic} & \zeta = 1 \\ \text{stable} & 0 < \zeta < 1 \\ \text{metastable} & \zeta = 0 \\ \text{unstable} & \zeta < 0 \end{cases}$$

F.1.2 Step Response