

RegT4 – Modern Control Engineering

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Abstract

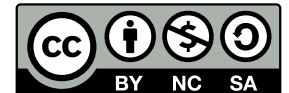
This document is “open source”, you can find the L^AT_EX sources at <https://github.com/NaoPross/RegT4>. All diagrams were made with TikZ. The content is based on the material of Prof. Dr. M. Kottman, from the course *Regelungstechnik 4* at the University of Applied Sciences Eastern Switzerland (OST). If you find typos or errors you can open an PR on Github or mail me at naoki.pross@ost.ch if I’m still around (until spring 2022) or np@ohm.ch.

Contributors

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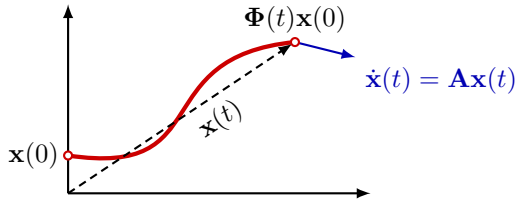


Figure 2: Interpretation of the \mathbf{A} and Φ matrices.

1.4.3 Nonhomogeneous Case

To expand the previous result, in the more general case when

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},$$

a few steps that are the same as in the scalar case yield the solution

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}(0) + \int_0^t \Phi(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau.$$

1.5 Laplace s -Domain Solution

1.5.1 Transfer Matrix

By taking the Laplace transform the state space equations of an LTI system they become:

$$\begin{aligned} s\mathbf{X}(s) - \mathbf{x}(0) &= \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s), \\ \mathbf{Y}(s) &= \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s). \end{aligned}$$

Similarly to the one dimensional case it is now possible to define the more general equivalent of a transfer function:

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D},$$

which is the *transfer matrix*.

1.5.2 Impulse Response

The inverse Laplace transform of the transfer matrix is the *impulse response*. However, setting $\mathbf{u}(t) = \delta(t)$ will also give the impulse response. By taking the result in section 1.4.3 and substituting into the output equation, we get that with a Dirac delta as input

$$\mathbf{g}(t) = \mathbf{C}\Phi(t)\mathbf{B} + \mathbf{D}\delta(t).$$

1.6 Controllability

A system is said to be *controllable* at time t_0 if it is possible by means of an unconstrained control vector to transfer the system from any initial state $\mathbf{x}(t_0)$ to any other state in a finite interval of time. To check if a system is completely controllable there is the $n \times nm$ *controllability matrix*

$$\mathbf{Q}_c = [\mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}].$$

If $\text{rank } \mathbf{Q}_c < n$ (has less than n linearly independent rows), or in the case when \mathbf{Q}_c is a square matrix $\det \mathbf{Q}_c = 0$, then the system is not totally state controllable.

It is also possible to determine the controllability from the transfer function (or matrix) of a system. It can be shown that if cancellation occurs (German: *Polkürzung*), then the system cannot be controlled in the direction of the canceled mode.

1.7 Observability

In the same (too) abstract spirit as in the controllability: a system is said to be *observable* at time t_0 if, with the system in state $\mathbf{x}(t_0)$, it is possible to determine this state from the observation of the output over a finite time interval. The observability can be determined through the $np \times n$ *observability matrix*

$$\mathbf{Q}_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix} = [\mathbf{C}^* \quad \mathbf{A}^*\mathbf{C}^* \quad \dots \quad (\mathbf{A}^*)^{n-1}\mathbf{C}^*],$$

where the asterisk is the conjugate transpose (for real matrices $\mathbf{A}^* = \mathbf{A}^T$). The system is observable only if $\text{rank } \mathbf{Q} \geq n$ (there are at least n linearly independent rows).

In terms of the transfer function: if cancellation occurs in the transfer function then the canceled mode cannot be observed for the output. In other words, no cancellation is a necessary and sufficient condition for total observability.

1.8 Stability

A state space model system is stable if the \mathbf{A} matrix is positive definite, which means that the real part of all eigenvalues $\text{Re}\{\lambda_i\} < 0$.

1.9 Equivalent Representations

In this section we assume without loss of generality that the system has a transfer function

$$\frac{Y(s)}{U(s)} = d + \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \dots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0}.$$

1.9.1 Controllable Canonical Form

This form (German: *Regelungsnormform*) is important in discussing the pole-placement approach. It is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u.$$

And the output is $y = [b_0 \quad \dots \quad b_{n-1}] \mathbf{x} + du$.

1.9.2 Observable Canonical Form

This mode is similar to previous form, in fact, the \mathbf{A} matrix is the transposed version of the controllable form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \dots & -a_n \\ 1 & 0 & 0 & \dots & -a_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 0 & \dots & -a_2 \\ 0 & 0 & 0 & \dots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-2} \\ b_{n-1} \end{bmatrix} u.$$

1.9.3 Diagonal Form (Decoupling)

An important form is when the \mathbf{A} matrix is diagonalized, i.e.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \lambda_{n-1} & \\ & & & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \tilde{\mathbf{B}}u.$$

In this representation the state variables are the *eigenstates* of the system, and they are decoupled from each other. Since the diagonal contains the eigenvalues, as noted in the stability section 1.8, they describe the poles of the transfer function $G(s)$, and therefore the stability of the system. In other words

$$\frac{Y(s)}{U(s)} = \frac{b_1 c_1}{s - \lambda_1} + \frac{b_2 c_2}{s - \lambda_2} + \cdots + \frac{b_n c_n}{s - \lambda_{n-1}}.$$

1.10 Similarity Transform

In the previous section there were many representation that are practical to use, but in fact there is nothing special about them. In fact, the state space can be transformed into an infinite number of other equivalent representations.

Given an arbitrary regular matrix \mathbf{P} , the state space in \mathbf{x} can be brought into a new set of state variables $\mathbf{x}' := \mathbf{P}\mathbf{x}$ by appropriately transforming the matrices as follows:

$$\begin{aligned} \mathbf{A}' &:= \mathbf{P}\mathbf{A}\mathbf{P}^{-1}, & \mathbf{B}' &:= \mathbf{P}\mathbf{B}, \\ \mathbf{C}' &:= \mathbf{C}\mathbf{P}^{-1}, & \mathbf{D}' &:= \mathbf{D}. \end{aligned}$$

1.11 Subsystem Aggregation

Series When after a subsystem 1 a subsystem 2 is connected in series the model of the entire path is

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} &= \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{B}_2 \mathbf{C}_1 & \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \mathbf{D}_1 \end{bmatrix} \mathbf{u}, \\ \mathbf{y} &= \begin{bmatrix} \mathbf{D}_2 \mathbf{C}_1 & \mathbf{C}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \mathbf{D}_2 \mathbf{D}_1 \mathbf{u}. \end{aligned}$$

Parallel When a subsystem 1 and a subsystem 2 are connected in parallel, the model of the entire system is

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} &= \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} \mathbf{u}, \\ \mathbf{y} &= \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + (\mathbf{D}_2 + \mathbf{D}_1) \mathbf{u}. \end{aligned}$$

Feedback When to a subsystem 1 a subsystem 2 is connected in the negative feedback path, the matrices of the entire system are

$$\begin{aligned} \mathbf{A}' &:= \begin{bmatrix} \mathbf{A}_1 - \mathbf{B}_1 \mathbf{K}_1 \mathbf{D}_2 \mathbf{C}_1 & -\mathbf{B}_1 \mathbf{K}_1 \mathbf{C}_2 \\ \mathbf{B}_2 \mathbf{K}_2 \mathbf{C}_1 & \mathbf{A}_2 - \mathbf{B}_2 \mathbf{K}_2 \mathbf{D}_1 \mathbf{C}_2 \end{bmatrix}, \\ \mathbf{B}' &:= \begin{bmatrix} \mathbf{B}_1 \mathbf{K}_1 \\ \mathbf{B}_2 \mathbf{K}_2 \mathbf{D}_1 \end{bmatrix}, & \mathbf{C}' &:= \begin{bmatrix} \mathbf{K}_2 \mathbf{C}_1 & -\mathbf{K} \mathbf{D}_1 \mathbf{C}_2 \end{bmatrix} \end{aligned}$$

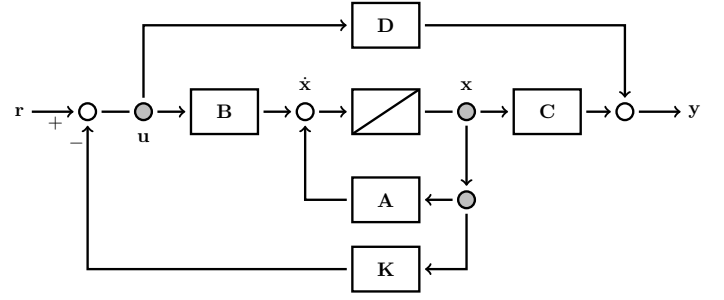


Figure 3: Pole placement through the gain matrix \mathbf{K} .

and $\mathbf{D}' := \mathbf{K}_2 \mathbf{D}_1$, where $\mathbf{K}_1 = (\mathbf{I} - \mathbf{D}_2 \mathbf{D}_1)^{-1}$ and $\mathbf{K}_2 = (\mathbf{I} - \mathbf{D}_1 \mathbf{D}_2)^{-1}$. In the case of a positive feedback the minus signs become pluses.

1.12 Create a State Space Model from a Transfer Function

[TODO: Ogata and skript p. 169, also from a diagram on p. 171.]

2 Control in State Space

2.1 Pole Placement

2.1.1 Gain Matrix

Consider the state space model

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t), \end{aligned}$$

and assume that we can somehow always know the state \mathbf{x} . Then we add a feedback loop from the state to the input with the *gain matrix* \mathbf{K} (see figure 3). This introduces a new term for the desired value \mathbf{r} and changes the state space equation to

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}\mathbf{r}.$$

The addition of \mathbf{K} allow to change the eigenvalues of the system matrix and thus move the poles. Therefore, the poles of the new matrix $\tilde{\mathbf{A}} := \mathbf{A} - \mathbf{B}\mathbf{K}$ are called the regulator poles. To allow an arbitrary placement of poles the system needs to be completely controllable. Without any additional construction, the gain matrix is basically a proportional controller (German: *P-Regler*).

2.1.2 Solve for the Gain Matrix

Now, if we wish to bring the poles of the closed loop system to the locations $\mu_1, \mu_2, \dots, \mu_n$, in the most general case we can use the fact that

$$\begin{aligned} \det(s\mathbf{I} - \tilde{\mathbf{A}}) &= \det(s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}) \\ &= (s - \mu_1)(s - \mu_2) \cdots (s - \mu_n) \\ &= s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n, \end{aligned}$$

and solve for the entries of \mathbf{K} by comparing coefficients.

If the state space representation is in the diagonal form (decoupled, eigenmodes), then finding \mathbf{K} is as simple as solving for the entries of \mathbf{K} :

$$\mathbf{A} - \mathbf{BK} = \begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{bmatrix} = \mathbf{M}.$$

Sometimes it is possible to do $\mathbf{K} = \mathbf{B}^{-1}(\mathbf{A} - \mathbf{M})$, but more often the system is overdetermined and we are free to choose some entries of \mathbf{K} (which is even easier).

If the state space representation is in the controllable canonical form, and

$$\mathbf{K} = \begin{bmatrix} k_1 & k_2 & \dots & k_n \end{bmatrix}$$

since $\tilde{\mathbf{A}} = \mathbf{A} - \mathbf{BK} =$

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n - k_1 & -a_{n-1} - k_2 & -a_{n-2} - k_3 & \dots & -a_1 - k_n \end{bmatrix},$$

which means that

$$G(s) = \frac{b_{n-1}s^{n-1} + \dots + b_1s + b_0}{s^n + (a_{n-1} + k_n)s^{n-1} + \dots + (a_0 + k_1)},$$

the poles are found by comparing the coefficients of the denominator:

$$\begin{aligned} s^n + (a_{n-1} + k_n)s^{n-1} + \dots + (a_0 + k_1) \\ = (s - \mu_1)(s - \mu_2) \dots (s - \mu_n). \end{aligned}$$

2.2 Linear Quadratic Regulator (LQR)

A problem of the pole placement approach is that it is not always obvious which poles should be moved and where. The *linear quadratic regulator* (LQR) also called *quadratic optimal regulator* solves this problem.

In the LQR design we define a performance index to minimize:

$$J = \int_0^\infty (\mathbf{x}^* \mathbf{Q} \mathbf{x} + \mathbf{u}^* \mathbf{R} \mathbf{u}) dt,$$

where \mathbf{Q} and \mathbf{R} are positive definite (or semidefinite) Hermitian or real symmetric matrices. The \mathbf{Q} matrix determines the importance of the relative error, whereas the \mathbf{R} matrix determines the cost (in energy) of modifying the state. Since in the pole placement $\mathbf{u} = -\mathbf{K}\mathbf{x}$ the problem becomes

$$\arg \min_{\mathbf{K}} \int_0^\infty \mathbf{x}^* (\mathbf{Q} + \mathbf{K}^* \mathbf{R} \mathbf{K}) \mathbf{x} dt.$$

Then, the expression can be further developed by using the fact that

$$\mathbf{x}(t) = e^{(\mathbf{A} - \mathbf{BK})t} \mathbf{x}(0),$$

which when substituted into the previous expression gives

$$\mathbf{x}(0)^* \underbrace{\int_0^\infty \left[e^{(\mathbf{A} - \mathbf{BK})t} \right]^* (\mathbf{Q} + \mathbf{K}^* \mathbf{R} \mathbf{K}) e^{(\mathbf{A} - \mathbf{BK})t} dt}_{\mathbf{S}(\mathbf{K})} \mathbf{x}(0).$$

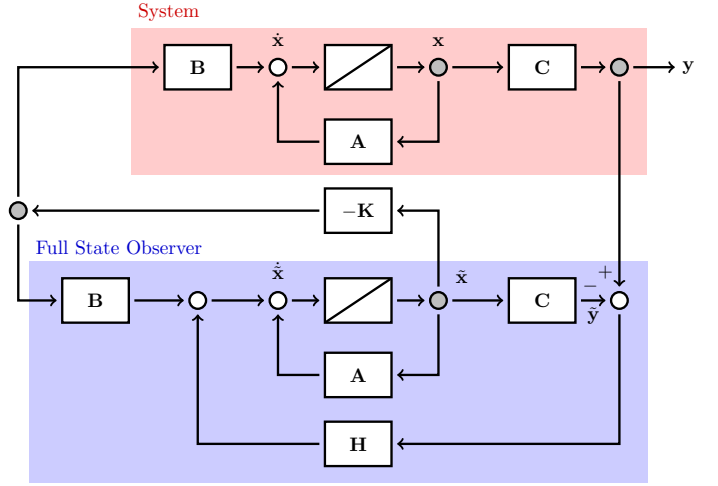


Figure 4: Full state observer to estimate \mathbf{x} for the pole placement matrix \mathbf{K} .

The minimization problem is completed by computing

$$\text{minimize } J \iff 0 = \frac{\partial J}{\partial K_{ij}} = \mathbf{x}^*(0) \frac{\partial \mathbf{S}}{\partial K_{ij}} \mathbf{x}(0).$$

It is possible to show that the resulting optimal gain matrix is $\mathbf{K} = \mathbf{R}^{-1} \mathbf{B}^* \mathbf{S}$. The LQR optimization problem can also be described in terms of the output

$$J = \int_0^\infty (\mathbf{y}^* \mathbf{V} \mathbf{y} + \mathbf{u}^* \mathbf{R} \mathbf{u}) dt,$$

and in a state space model $\mathbf{A}, \mathbf{B}, \mathbf{C}$ with $\mathbf{D} = \mathbf{0}$ can be rewritten using \mathbf{x} as follows: $\mathbf{y}^* \mathbf{V} \mathbf{y} = \mathbf{x}^* \mathbf{C}^* \mathbf{V} \mathbf{C} \mathbf{x}$.

2.3 PI-Control

[TODO: Skript p. 200.]

2.4 Observer

Sometimes it is not possible to know some states of \mathbf{x} of a plant (system is not totally observable). The idea of the *observer* is to create a new controller to estimate the state of the unknown variables from the output of the plant. [TODO: Skript p. 203.]

3 Discrete-Time Modelling

3.1 Sampling

In discrete time the system works with sampled signals. The sampling time T must be chosen such that the sampling theorem (Nyquist Shannon) is satisfied, i.e.

$$\omega_s = \frac{2\pi}{T} \geq 2\omega_r,$$

where ω_r is the highest frequency component in the reference signal $r(t)$.

In practice sampling means using an analog to digital converter (ADC), but mathematically the sampling is done

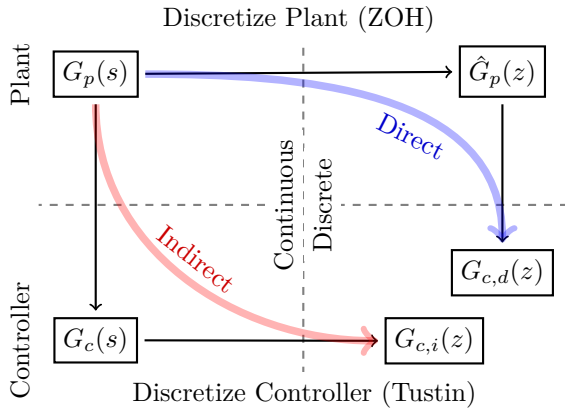


Figure 5: Direct vs indirect design. $G_{c,i}(z) \neq G_{c,d}(z)$.

through a multiplication with a Dirac comb $\sum_k \delta(t - kT)$. The converse operation, which is done with a digital to analog converter (DAC), is represented in many ways depending on the type of reconstruction. The simplest reconstruction is the zero order hold (ZOH) reconstruction, which keeps the signal fixed at the value of the last digital sample until the new one comes.

3.1.1 Choosing T Empirically

[TODO: See skript p. 220.]

3.2 Indirect Design

In the indirect design we create a controller in the continuous time domain and then discretize it (see figure 5). This is okay as long as the sampling theorem is satisfied.

Thus, given a plant with transfer function $G_p(s)$ we create a continuous time controller $G_c(s)$. Then to discretize it we can choose how to approximate the integrator element from one of the following 3 options:

$$\begin{aligned} s &= \frac{z-1}{T} & z &= 1 + sT & (\text{Euler}), \\ s &= \frac{z-1}{Tz} & z &= \frac{1}{1-sT} & (\text{Forward}), \\ s &= \frac{2}{T} \frac{z-1}{z+1} & z &= \frac{2+sT}{2-sT} & (\text{Tustin}). \end{aligned}$$

The last one (Tustin) is the most popular. To obtain the discretized controller transfer function $\hat{G}_c(z)$ we substitute s with the chosen approximation (for example Tustin):

$$\hat{G}_c(z) = G_c \left(s = \frac{2}{T} \frac{z-1}{z+1} \right).$$

3.3 ZOH Discretization (Direct Design)

3.3.1 Discretization of a Transfer Function

In the direct design the plant's continuous time model $G_p(s)$ is discretized. If we assume that the DAC is a ZOH, when the input is a discrete Dirac impulse in continuous time it becomes a rectangle of length T :

$$u(t) = \varepsilon(t) - \varepsilon(t - T) \rightsquigarrow U(s) = \frac{1 - e^{-sT}}{s}.$$

Then we compute the output of the system to this input in continuous time $U(s)G_p(s)$, inverse Laplace transform it, and then sample it in the time domain. By taking the z transform of the result we obtain a discretized transfer function

$$\hat{G}_p(z) = (1 + z^{-1}) \mathcal{Z} \left\{ \left(\mathcal{L}^{-1} \left\{ \frac{G(s)}{s} \right\} \right) \right\} \Big|_{t=kT}.$$

ZOH discretizations for a few common type of plants are listed in table 1.

Plant $G_p(s)$	ZOH $\hat{G}_p(z)$	Plant $G_p(s)$	ZOH $\hat{G}_p(z)$
K	$\frac{Kz}{z-1}$	$\frac{1}{s}$	$\frac{T}{z-1}$
$\frac{1}{s^2}$	$\frac{T}{2} \frac{z+1}{(z-1)^2}$	$\frac{K}{s\tau+1}$	$\frac{1 - e^{-T/\tau}}{z - e^{-T/\tau}}$

Table 1: ZOH discretization of a few plant types.

3.3.2 Discretization of a State Space Model

[TODO: Stuff from skript p. 250.]

3.3.3 Root Locus Method

[TODO: Stuff from skript p. 254.]

3.3.4 Pole Placement Design

Another way to design a controller is to simply state that the closed $G_f(z)$ has to have the poles at some determined locations. Then since in a closed feedback loop with a plant $G_p(z)$ and a controller $G_c(z)$:

$$G_f(z) = \frac{G_c(z)G_p(z)}{1 + G_p(z)G_c(z)},$$

by solving for $G_c(z)$:

$$G_c(z) = \frac{1}{G_p(z)} \frac{G_f(z)}{1 - G_f(z)}.$$

3.4 Discrete State Space

The discrete state space model equations are

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k), \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k), \end{aligned}$$

and similarly to the Laplace domain solution there is a *transfer matrix* in the z -domain:

$$\mathbf{G}(z) = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}.$$

Like in continuous time the eigenvalues of the \mathbf{A} matrix are the poles of the transfer function $G(z)$, and thus determine the stability of the system. The observability and controllability can be checked with the same matrices as in continuous time: \mathbf{Q}_o , \mathbf{Q}_c respectively. Because the system is a *difference*

equation (instead of *differential equation*) future states can be computed recursively:

$$\begin{aligned}\mathbf{x}(x) &= \mathbf{A}\mathbf{x}(n-1) + \mathbf{B}\mathbf{u}(n-1) \\ &= \mathbf{A}^n \mathbf{x}(0) + \sum_k^{n-1} \mathbf{A}^k \mathbf{B}\mathbf{u}(n-1-k).\end{aligned}$$

3.5 Deadbeat Controller Design

[TODO: Stuff from skript p. 256.]

A Linear Algebra

A.1 Determinant

The determinant is a measure of how the unit square ($n = 2$), cube ($n = 3$) or hypercube ($n > 3$) is scaled by the linear transformation. The determinant of a 2×2 matrix is computed using the “fish rule”:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

By hand, for a 3×3 matrix, Sarrus’ rule is probably the fastest method:

$$\det \underbrace{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}}_{\mathbf{A}} \rightsquigarrow \begin{bmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{bmatrix},$$

the red lines are added, while the blue ones are subtracted:

$$\det \mathbf{A} = aei + bfg + cdh - ceg - afh - bdi.$$

A.2 Inverse Matrix

The inverse of a 2×2 matrix is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

I remember it as “transposed on the wrong diagonal and add minuses (and divide by the determinant)”. For bigger matrices, there are the method of Gaussian elimination:

$$\left[\mathbf{A} \mid \mathbf{I} \right] \xrightarrow{\text{Gauss}} \left[\mathbf{I} \mid \mathbf{A}^{-1} \right],$$

or Cramers’ rule $\mathbf{A}^{-1} = \text{adj } \mathbf{A} / \det \mathbf{A}$.

A.3 Eigenvalues and Eigenvectors

The eigenvalue problem is to find the vectors that are unaffected by a matrix \mathbf{A} (up to a scalar factor):

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \iff (\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}.$$

There are n numbers λ , the *eigenvalues*, and equally many *eigenvectors* \mathbf{v} . This problem is solved by first finding the λ ’s and then later the eigenvectors.

In the right expression, we need a matrix $\mathbf{A} - \lambda\mathbf{I}$ to move any vector to the origin, this is accomplished by forcing

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0,$$

and solving for the λ ’s. Then to find the eigenvectors we put back each eigenvalue λ_i into the original equation to obtain a system of equations

$$(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{v}_i = \mathbf{0}.$$

This system of equations is solved for the components of \mathbf{v}_i , and it is overdetermined. This is because there are infinitely many eigenvectors each eigenspace of \mathbf{A} .

For a 2×2 matrix the eigenvalue problem is a parabola, so there is a shortcut:

$$m = \frac{1}{2} \text{tr } \mathbf{A}, \quad p = \det \mathbf{A},$$

then $\lambda_{1,2} = m \pm \sqrt{m^2 - p}$.

A.4 Diagonalization

If a $n \times n$ matrix \mathbf{A} has n distinct eigenvectors (linearly independent), it is *diagonalizable*. To diagonalize \mathbf{A} , first we compute the eigenvalues $\lambda_1, \dots, \lambda_n$ and the eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, then a matrix is formed by using the eigenvectors as columns:

$$\mathbf{T} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}.$$

This matrix is then used for a change of bases into the eigenbasis of \mathbf{A} , where it is a diagonal matrix:

$$\mathbf{\Lambda} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

A.5 Positive Definite

A Hermitian complex (or real symmetric) matrix \mathbf{A} is said to be *positive definite* if the quadratic form $\mathbf{x}^* \mathbf{A} \mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$. If $\mathbf{x}^* \mathbf{A} \mathbf{x} \leq 0$ then it is *semi-definite*. If the quadratic form is smaller than 0 it is *negative definite* or (semi-definite). Matrix definiteness can be controlled by using the eigenvalues λ_i because:

- if all $\lambda_i > 0$, then the matrix is positive definite,
- if all $\lambda_i \geq 0$ the matrix is positive semi-definite.

And the converse for negative definiteness. Further, definiteness can be checked using the determinants of the submatrices $\mathbf{A}_{[ij,kl]}$ as follows: given a matrix

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

If $\det \mathbf{A}_{[11,11]} > 0$ (red), $\det \mathbf{A}_{[11,22]} > 0$ (orange), $\det \mathbf{A}_{[11,33]} > 0$ (yellow), etc. the matrix is positive definite.

The quadratic form $\mathbf{x}^* \mathbf{Q} \mathbf{x}$ for $\mathbf{x} \in \mathbb{C}^{2 \times 1}$ a, b, q as complex numbers is

$$\mathbf{x}^* \underbrace{\begin{bmatrix} a & q \\ q & b \end{bmatrix}}_{\mathbf{Q}} \mathbf{x} = a|x_1|^2 + q(x_1^* x_2 + x_1 x_2^*) + b|x_2|^2.$$

If working only with real numbers:

$$\mathbf{x}^T \mathbf{Q} \mathbf{x} = ax_1^2 + 2qx_1x_2 + bx_2^2 = \sum_{i,j} Q_{ij} x_i x_j.$$

A.6 Jordan Normal Form

A.7 Caley-Hamilton Theorem

A.8 Useful Matrices and Products

Column Row Product For two complex numbers a and b :

$$\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} \rightsquigarrow \mathbf{x} \mathbf{x}^* = \begin{bmatrix} |a|^2 & ab^* \\ a^* b & |b|^2 \end{bmatrix}.$$

Rotation Matrices In 2 dimensions:

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \quad \mathbf{R}(-\theta) = \mathbf{R}(\theta)^T.$$

Rotation matrices belong to the special orthogonal group, therefore $\mathbf{R} \mathbf{R}^T = \mathbf{I} \iff \mathbf{R}^{-1} = \mathbf{R}^T$ and $\det \mathbf{R} = 1$.

B Vector Analysis

B.1 Gradient Vector

The *gradient* of a function $f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^m$ is a column vector containing the partial derivatives in each direction.

$$\nabla f(\mathbf{x}) = \sum_{i=1}^m \partial_{x_i} f(\mathbf{x}) \mathbf{e}_i = \begin{pmatrix} \partial_{x_1} f(\mathbf{x}) & \cdots & \partial_{x_m} f(\mathbf{x}) \end{pmatrix}^T.$$

The gradient vector always points towards the *direction of steepest ascent*, and thus is always perpendicular to contour lines.

B.2 Jacobian Matrix

The *Jacobian* \mathbf{J}_f (sometimes written as $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$) of a function $\mathbf{f} : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a matrix $\in \mathbb{R}^{n \times m}$ whose entry at the i -th row and j -th column is given by $(\mathbf{J}_f)_{i,j} = \partial_{x_j} f_i$, so

$$\mathbf{J}_f = \begin{pmatrix} \partial_{x_1} f_1 & \cdots & \partial_{x_m} f_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} f_n & \cdots & \partial_{x_m} f_n \end{pmatrix} = \begin{pmatrix} (\nabla f_1)^T \\ \vdots \\ (\nabla f_n)^T \end{pmatrix}$$

C Laplace Transform

The Laplace transform of a function f and its inverse transform are

$$F(s) = \mathcal{L} f(t) = \int_0^\infty f(T) e^{-st} dt,$$

$$f(t) = \mathcal{L}^{-1} F(s) = \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \oint_{\gamma-iT}^{\gamma+iT} F(s) e^{st} ds.$$

The Laplace transform converts integration and differentiation problems into algebraic problems. Some important properties and transform pairs are listed in table 2.

D z-Transform

D.1 Definition

The z -transform of a sequence (f_n) and its inverse transform (sometime written as $\mathcal{Z}, \mathcal{Z}^{-1}$) are defined to be

$$F(z) = \sum_{k=0}^\infty f_k z^{-k}, \quad f_n = \frac{1}{2\pi i} \oint_C F(z) z^{n-1} dz,$$

where z is a complex number and C is a counterclockwise path encircling the origin and entirely in the region of convergence.

D.2 Relation to the Laplace Transform

The z -transform is related to the Laplace transform, when a sequence is generated by sampling a function. In other words with a sampling time T we can construct a continuous time function

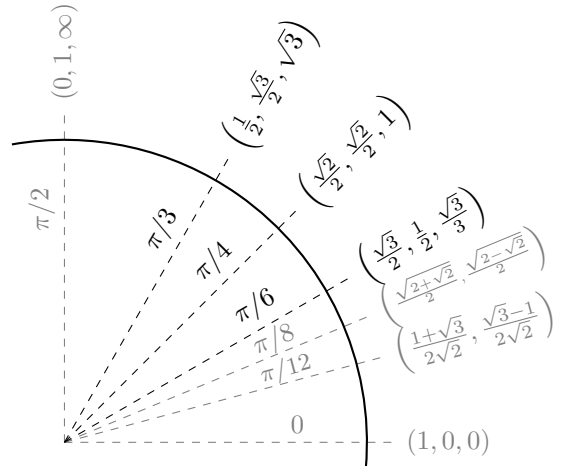
$$f(t) = \sum_k f_k \delta(t - kT)$$

of which is then possible to take the Laplace transform resulting in

$$\mathcal{L} \left\{ \sum_k f_k \delta(t - kT) \right\} = \sum_{k=0}^\infty f_k e^{-kTs} = \sum_{k=0}^\infty f_k z^{-k},$$

where in the last step the substitution $z = e^{sT}$ was made. Therefore the multiplication by z^{-1} can interpreted as adding a delay of T .

E Idiot's section



Property	Time Domain	s-Domain	Function	Laplace Tr.
Linearity	$\sum_i a_i f_i(t)$	$\sum_i a_i F_i(s)$	$\delta(t)$	1
Time shift ($\tau > 0$)	$f(t - \tau)$	$e^{-s\tau} F(s)$	$\varepsilon(t)$	$\frac{1}{s}$
Damping	$e^{at} f(t)$	$F(s - a)$	t^n	$\frac{n!}{s^{n+1}}$
Stretching ($a > 0$)	$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$	$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
Differentiation	$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0^+) - \dots - f^{(n-1)}(0^+)$	$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
Integration	$\int_0^t f(t) dt$	$\frac{1}{s} F(s)$		

Table 2: Useful Laplace transform rules and pairs. All functions are assumed to be causal (multiplied with $\varepsilon(t)$).

Property	Time Domain	z-Domain	Sequence	z-Tr.
Linearity	$\sum_i a_i f_k(k)$	$\sum_i a_i F_i(z)$	$\delta(k)$	1
Forward time shift	$f(k + n)$	$z^n F(z) - z^n f(0) - \dots - z f(n - 1)$	$\varepsilon(k)$	$\frac{z}{z - 1}$
Backwards time shift	$f(k - n)$	$z^{-n} F(z)$	k	$\frac{z}{(z - 1)^2}$
Convolution	$f_1 * f_2 = \sum_{i=0}^k f_1(k) f_2(k - i)$	$F_1(z) F_2(z)$	a^k	$\frac{z}{z - a}$

Table 3: Useful rules and transform pairs of the z -transform. All sequences can be converted into time-domain functions by using a Dirac comb. Some are easier for example $k \rightsquigarrow kT$.

A few useful trigonometric identities:

of x :

$$\cos^2(x) + \sin^2(x) = 1 \quad \cosh^2(x) - \sinh^2(x) = 1$$

$\cos(\alpha + 2\pi) =$	$\cos(\alpha)$	$\sin(\alpha + 2\pi) =$	$\sin(\alpha)$
$\cos(-\alpha) =$	$\cos(\alpha)$	$\sin(-\alpha) =$	$-\sin(\alpha)$
$\cos(\pi - \alpha) =$	$-\cos(\alpha)$	$\sin(\pi - \alpha) =$	$\sin(\alpha)$
$\cos(\frac{\pi}{2} - \alpha) =$	$\sin(\alpha)$	$\sin(\frac{\pi}{2} - \alpha) =$	$\cos(\alpha)$
$\cos(\alpha + \beta) =$	$\cos \alpha \cos \beta - \sin \alpha \sin \beta$		
$\sin(\alpha + \beta) =$	$\sin \alpha \cos \beta + \cos \alpha \sin \beta$		
$\cos(2\alpha) =$	$\cos^2 \alpha - \sin^2 \alpha$		
	$= 1 - 2 \sin^2 \alpha$		
	$= 2 \cos^2 \alpha - 1$		
$\sin(2\alpha) =$	$2 \sin \alpha \cos \alpha$		
$\tan(2\alpha) =$	$(2 \tan \alpha)(1 + \tan^2 \alpha)^{-1}$		

The geometric series has a closed form formula for $|q| < 1$:

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1 - q}.$$

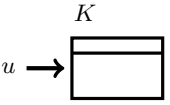
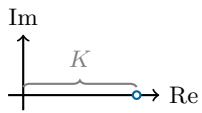
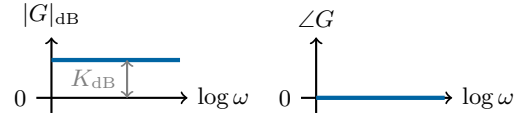
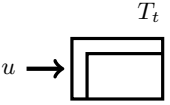
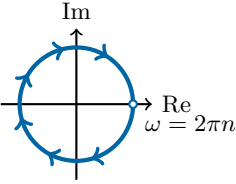
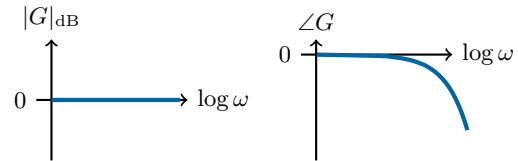
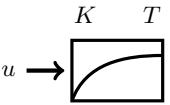
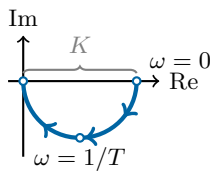
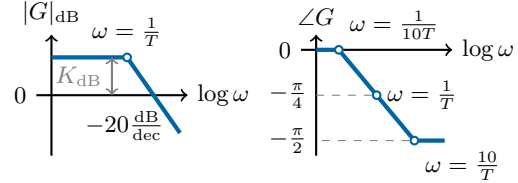
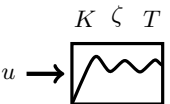
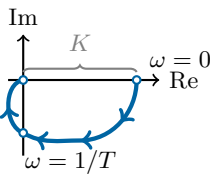
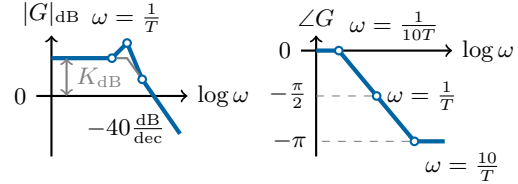
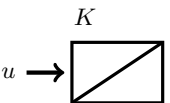
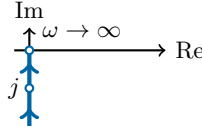
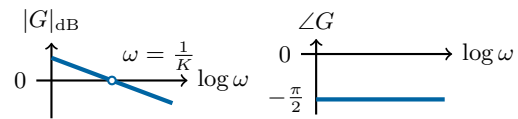
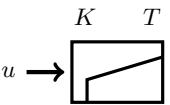
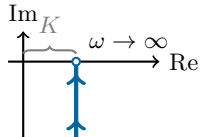

Rules for differentiation, f, u, v are differentiable functions

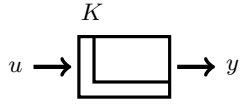
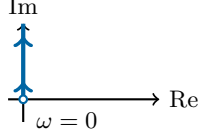
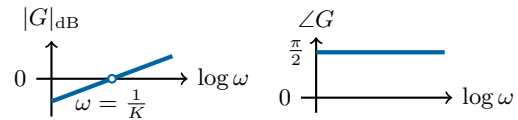
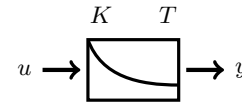
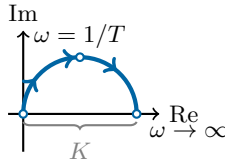
$$\begin{aligned} (af)' &= af' & (u(v))' &= u'(v)v' \\ (uv)' &= u'v + uv' & \left(\frac{u}{v}\right)' &= \frac{u'v - uv'}{v^2} \\ \left(\sum u_i\right)' &= \sum u_i' & (\ln u)' &= \frac{u'}{u} \\ (f^{-1})' &= \frac{1}{f'(f^{-1}(x))} \end{aligned}$$

Some useful derivatives and integrals:

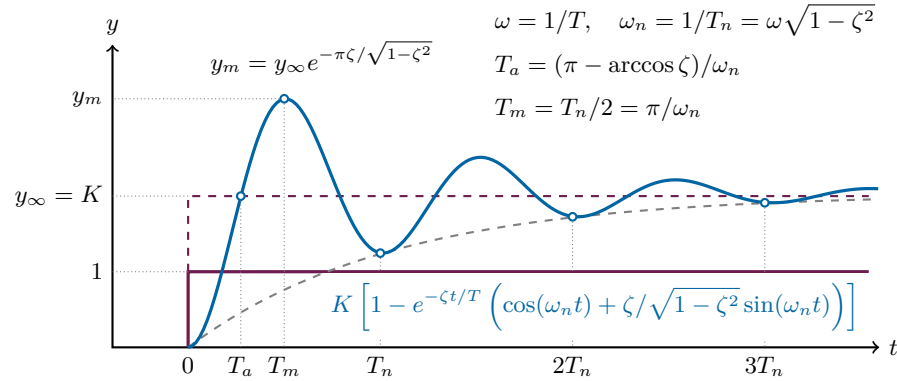
f	f'	f	f'
x^n	nx^{n-1}	a^x	$a^x \ln a$
$\sqrt[n]{x}$	$1/\left(x^n \sqrt[n]{x^{n-1}}\right)$	$\ln x$	$1/x$
$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\tan x$	$1/\cos^2 x$	$1/\tan x$	$-1/\sin^2 x$
$\arcsin x$	$1/\sqrt{1 - x^2}$	$\arccos x$	$-1/\sqrt{1 - x^2}$
$\arctan x$	$1/(1 + x^2)$		
$\sinh x$	$\cosh x$	$\tanh x$	$1/\cosh^2 x$
$\operatorname{arcsinh} x$	$1/\sqrt{1 + x^2}$	$\operatorname{arccosh} x$	$1/\sqrt{x^2 - 1}$

F Block Diagram Elements

Type	Symbol	Differential Equation	Frequency Response	Nyquist Diagram	Bode Plot
P		$y = Ku$	K		
T		$y(t) = \varepsilon(t)u(t - T_t)$	$e^{-j\omega T_t}$		
PT ₁		$T\dot{y} + y = Ku$	$\frac{K}{j\omega T + 1}$		
PT ₂		$T^2\ddot{y} + 2\zeta T\dot{y} + y = Ku$	$\frac{K}{T^2(j\omega)^2 + 2\zeta T(j\omega) + 1}$		
I		$\dot{y} = Ku$	$\frac{K}{j\omega}$		
PI		$y = K \left(u + \int_0^t \frac{u}{T} d\tau \right)$	$K \left(1 + \frac{1}{j\omega T} \right)$		

Type	Symbol	Differential Equation	Frequency Response	Nyquist Diagram	Bode Plot
D	 <p>Not realizable!</p>	$y = K\dot{u}$	$j\omega K$		
DT ₁		$T\dot{y} + y = K\dot{u}$	$\frac{j\omega K}{1 + j\omega T}$		

F.1 PT₂-Element



G Lead-Lag Element

$$K \frac{1 + j\omega T_1}{1 + j\omega T_2} = \begin{cases} \text{Lead} & T_1 > T_2 \\ \text{Lag} & T_1 < T_2 \end{cases}$$

H MATLAB Reference

Command	Description
$G = \text{tf}(N,D);$ $G = \text{tf}(N,D,"InputDelay",T);$ $G = \text{tf}(N,D,TS);$	Create a continuous time transfer function with denominator N and denominator D (both arrays). Add a delay of e^{-sT} . Create a discrete time transfer function with sampling time TS .
$SYS = \text{ss}(A,B,C,D);$ $SYS = \text{ss}(A,B,C,D,"InputDelay",T);$ $SYS = \text{ss}(A,B,C,D,TS);$	Create a continuous time state space model from the matrices A B C and D . Add a delay of e^{-sT} . Create a discrete time model with sampling time TS .
$CO = \text{ctrb}(A,B);$ $CO = \text{ctrb}(SYS);$	Compute the controllability matrix of a system from the matrices A and B or from the system object.
$OB = \text{obsv}(A,C);$ $OB = \text{obsv}(SYS);$	Compute the observability matrix of a system from the matrices A and C or from the system object.
$SYS = \text{tf2ss}(G);$ $G = \text{ss2tf}(SYS);$ $K = \text{dcgain}(SYS);$ $K = \text{dcgain}(G);$	Conversion between state space and transfer function. Compute the DC gain of a system.
$K = \text{place}(A,B,P);$	Compute a state feedback matrix such that the eigenvalues of $A - BK$ are those specified in the vector P .
$[K,S,CLP] = \text{lqr}(SYS,Q,R,N);$ $[K,S,CLP] = \text{lqr}(A,B,C,D,Q,R,N);$ $[K,S,CLP] = \text{dlqr}(A,B,Q,R,N)$	Calculates the optimal gain matrix K for the continuous or discrete state-space model SYS .
$[N,D] = \text{pade}(T,N);$ $SYSX = \text{pade}(SYS,N);$	Returns a N th order Padé approximation of the continuous time delay e^{-sT} . Returns delay free approximation of a system SYS with a delay.