Quantum Chrono-Dynamics

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Preface

What is time? A deceptively simple question that has perplexed humanity for centuries, inviting contemplation and challenging our understanding of the world. We rely on time to measure motion, quantify velocities, and express durations. Its measurement, its steady tick-tock, provides a reference point for making sense of the world around us. Yet, what if time itself defies a universal definition? What if our perception of the past, present, and future is nothing more than a persistent illusion? In this thesis, we embark on the enigmatic nature of time. Driven by our insatiable curiosity and inspired by the timeless question, we delve into the realms of physics and philosophy to unravel the mysteries that surround this elusive concept. When we describe motion, we inevitably relate it to time. We assign numerical values, such as 10 m/s or 100 mph, to quantify velocities and understand the dynamics of objects in motion. However, the mathematical descriptions we use are rendered meaningless if we cannot first define time itself. Is time a universally applicable framework, synchronised like a grand metronome, counting the beats across the vastness of the galaxy? The resounding answer, my friends, is no. As living physicists, we come face to face with the unsettling truth that the past, present, and future are not as distinct as

they appear. Our intuitive perception suggests that time flows in a linear fashion, with moments slipping away into the abyss of the past. However, as we delve deeper into the intricacies of our universe, we confront the notion that this distinction is nothing more than a stubborn illusion, a construct of our human perception. In our quest to unravel the mysteries of time, we venture beyond the confines of classical physics and delve into the remarkable realm of quantum mechanics. Here, we encounter a world where particles can exist in multiple states simultaneously, defying our traditional understanding of cause and effect. Within this quantum realm, time assumes a new character, blurring the lines between the past, present, and future, and challenging the very fabric of our understanding. The exploration of time within the quantum realm opens up profound questions and sparks intense intellectual inquiry. How does time manifest within the quantum framework? Can we find a meaningful and consistent definition of time that encompasses the peculiarities of quantum phenomena? These questions propel us forward, urging us to uncover new insights and challenge established paradigms.

Dedication

In heartfelt gratitude, I dedicate this thesis to the individuals who have been instrumental in shaping my journey of exploration and discovery. It is through their unwavering support, inspiration, and guidance that I have grown as a researcher and as an individual. I am thankful for my beloved family, whose unconditional love and unwavering belief in my potential have been my anchor throughout this endeavor. Your constant encouragement and sacrifices have provided the foundation for my pursuits, and I am forever grateful for your presence in my life. To my respected

advisers, mentors, and colleagues, whose expertise, wisdom, and insightful guidance have shaped my thinking and propelled me to reach new heights. Your mentor-ship and collaboration have illuminated my path, and I am grateful for the opportunities to learn and grow under your tutelage. The scientific community, past and present, whose relentless pursuit of knowledge has expanded the horizons of human understanding. I stand in awe of the groundbreaking discoveries and contributions made by countless individuals who have dedicated their lives to unraveling the secrets of time and quantum mechanics. It is your collective efforts that inspire and challenge me to explore the frontiers of Quantum Chrono-Dynamics. To my cherished friends and loved ones, who have provided unwavering support, encouragement, and understanding throughout this journey. Your belief in me, your encouragement words and your presence have been a source of strength, joy, and balance in my life. Finally, I dedicate this thesis to the pursuit of knowledge itself, the timeless endeavor that fuels our curiosity and propels us forward. May our collective efforts contribute to the expanding tapestry of human understanding and inspire future generations to explore the mysteries that lie at the intersection of time and quantum mechanics.

1. The problem with defining time

I. Introduction

Time, a concept so deeply woven into the fabric of our existence, poses a remarkable challenge when it comes to defining it with precision. In this first chapter, we delve into the intricacies of time and explore the fundamental difficulties encountered in attempting to pin down its elusive nature. From philosophical debates to scientific inquiries, we embark on a journey to unravel the complexities that arise when grappling with the definition of time.

i. The Philosophical Conundrums

The subjective experience of time

Time, in its essence, is intimately related to our subjective experience of the world. However, attempting to define time solely on the basis of this subjective aspect presents a significant challenge. Individuals perceive time in different ways, influenced by factors such as culture, age, and personal circumstances. The subjective experience of time can be characterized by its fluidity, where moments of joy seem fleeting, while periods of monotony drag on indefinitely. This subjectivity makes it difficult to arrive at a

universally applicable definition of time. Philosophers throughout history have grappled with the philosophical puzzles arising from the subjective experience of time. One of the central questions revolves around the nature of temporal passage. Does time flow continuously, like a river steadily advancing, or is it a series of discrete moments, akin to the ticking of a clock? This debate highlights the intricate relationship between our perception of time and our understanding of its fundamental nature. The philosopher Henri Bergson argued that time is an inherently subjective experience, irreducible to simple measurements or mathematical models. He proposed the concept of "duration," emphasizing the continuous and evolving nature of time as it is experienced by individuals. According to Bergson, our subjective experience of time involves an interplay between memory and perception, where the past influences the present and the present shapes our anticipation of the future. This view challenges the notion of time as a mere measurement, urging us to consider the profound depth of our temporal experiences. Additionally, the philosophical concept of presentism raises further questions regarding the subjective experience of time. Presentism suggests that only the present moment is real, while past and future exist only as memories and anticipations. This perspective highlights the asymmetry of time, with the past becoming a repository of memories, and the future a realm of possibilities. However, presentism faces challenges in reconciling our subjective experience of the passage of time with the scientific understanding of temporality. The subjective nature of time necessitates a comprehensive understanding of the various dimensions that influence our experience of it. Philosophical inquiries into the subjectivity of time continue to challenge our conventional understanding, reminding us of the intricacies involved in attempting to define this

enigmatic phenomenon. By acknowledging the subjective experience of time and the philosophical debates surrounding it, we lay the groundwork for a deeper exploration of time within the framework of quantum mechanics. In subsequent sections, we will examine how these philosophical conundrums intersect with the challenges presented by quantum dynamics, further complicating the quest for a comprehensive definition of time.

The arrow of time

The arrow of time is a concept that delves into the directionality and asymmetry of temporal phenomena. It raises philosophical questions about the nature of time and plays a crucial role in our attempts to define it. The arrow of time suggests that time exhibits a preferred direction, distinguishing between the past and the future. One significant aspect of the arrow of time is the irreversibility of certain processes. Everyday experiences demonstrate that events unfold in a particular order, with causes preceding effects. This temporal asymmetry aligns with our intuitive understanding of time, where we remember the past but not the future. However, this poses a challenge when seeking a comprehensive definition of time. Entropy, a concept from thermodynamics, provides insight into the arrow of time. Entropy is a measure of the disorder or randomness within a system. The second law of thermodynamics states that in an isolated system, entropy tends to increase or remain constant, leading to the popular notion of "increasing entropy" or "the arrow of time." This asymmetry in entropy corresponds to our perception of the flow of time, where disorder tends to increase as events progress from the past to the future. The arrow of time raises profound questions about the fundamental nature of time. Does time inherently possess a directionality,

or is the arrow of time a consequence of the macroscopic nature of our observations? These questions continue to challenge our understanding of time and prompt philosophical debates about the nature of causality and the relationship between the past, present and future. The challenge in defining time arises from reconciling the intuitive understanding of the arrow of time with the symmetries inherent in fundamental physical laws. The laws of physics, including those described by classical mechanics and quantum mechanics, are typically time-reversible. They allow processes to occur both forward and backward in time, suggesting a symmetry that appears at odds with the arrow of time as we experience it. The philosophical conundrums surrounding the arrow of time remind us that time is not simply an abstract concept, but a complex phenomenon embedded in our daily experiences. The quest to define time requires careful consideration of its directionality, the relationship between cause and effect, and the intricate interplay between our subjective perception and the objective nature of temporal processes. In the subsequent sections, we will explore how the challenges posed by the arrow of time intertwine with the peculiarities of quantum mechanics, further deepening our understanding of the complex relationship between time and the quantum world.

ii. The Physics Perspective

Classical Definitions of Time

In classical physics, time is typically considered as an independent and absolute parameter, serving as a fundamental framework for describing the dynamics of the physical world. This classical notion of time stems from the work of Newton, who introduced the concept of absolute time—a uni-

versal, unchanging flow of time that exists independently of any physical processes or observers. In this classical framework, time is often treated as an external reference, providing a synchronised and consistent measure for comparing events. It serves as the backdrop against which the motions of objects and the progression of events are measured. Time is represented as a one-dimensional continuum, allowing for precise measurements of durations, intervals, and rates of change. The classical definitions of time are deeply intertwined with the concept of determinism, where the laws of physics dictate the evolution of a system based on initial conditions. According to this view, time acts as a fixed and linear progression, unfolding with strict causality and determinate outcomes. However, with the advent of modern physics, particularly the theory of relativity, the classical notions of time have been challenged. Einstein's theory of special relativity introduced the concept of "relativistic time," where the flow of time is influenced by the relative motion and gravitational fields of observers. Time dilation, the phenomenon in which time appears to pass differently for observers in relative motion or experiencing different gravitational forces, indicates the relativity of time. In the realm of general relativity, the interplay between time and gravity becomes even more intricate. Space-time, the four-dimensional framework that combines space and time, is now understood as a dynamic entity that can be influenced by the presence of mass and energy. This realisation leads to the concept of "curved space-time," where the geometry of space-time is affected by the distribution of matter and creates gravitational effects, including the bending of light and the warping of time. These revolutionary insights in physics highlight the need to reconsider classical definitions of time. Rather than an absolute and fixed entity, time is now viewed as a dynamic,

context-dependent quantity that can be influenced by various factors, including motion, gravity, and the geometry of space-time. While classical physics provides a useful framework for understanding the dynamics of macroscopic objects, it falls short when dealing with the quantum realm, where the classical definitions of time face further challenges. In the subsequent sections, we will explore the intersection between time and quantum mechanics, delving into the unique aspects of time dynamics within the quantum framework.

Time as a coordinate

In the frameworks of classical and relativistic physics, time is regarded as one of the coordinates in a multi-dimensional mathematical framework called space-time. Space-time unifies the three spatial dimensions with the dimension of time, providing a comprehensive framework for describing the positions and evolution of objects in the universe. Within this framework, time is treated as a coordinate on par with the spatial coordinates. It enables precise event specification and the measurement of temporal intervals, allowing physicists to describe the motion, interactions, and transformations of physical systems at both macroscopic and microscopic scales. Time as a coordinate in space-time has a crucial feature: the causal relationships between events. According to the theory of relativity, the space-time structure determines the causality between events. Causal influences propagate at or below the speed of light, connecting cause and effect through light-like or time-like intervals in space-time. Understanding the relationship between time and causality is essential for understanding the dynamics and evolution of physical systems. The notion of time as a coordinate is also fundamental in formulating the mathematical equations

governing physical phenomena. In classical mechanics, time serves as an independent variable, allowing for the formulation of differential equations that describe the dynamics of particles and systems. These equations involve time derivatives, which determine how physical quantities change with respect to time. Relativistic physics further intertwines the concept of time as a coordinate with the geometry of space-time. General relativity's equations describe the curvature of space-time caused by mass and energy, affecting both the spatial and temporal components. This leads to phenomena such as time dilation, where time appears to flow differently in regions of varying gravitational potential. However, in the quantum realm, treating time as a coordinate becomes more intricate. Quantum mechanics introduces the uncertainty principle, which limits the simultaneous precision with which certain pairs of physical quantities, including time and energy, can be known. This uncertainty imposes limitations on our ability to precisely measure and define time intervals in quantum systems. Furthermore, the fundamental equations of quantum mechanics, such as the Schrödinger equation, describe the time evolution of quantum states and the prediction of probabilities associated with measurement outcomes. However, interpreting time within the quantum framework remains an ongoing subject of investigation and debate.

iii. Quantum Challenges

The Time-Energy Uncertainty Principle

In the realm of quantum mechanics, the Time-Energy Uncertainty Principle arises as a consequence of the Heisenberg Uncertainty Principle, which states that certain pairs of physical observables cannot be pre-

cisely measured simultaneously. The Time-Energy Uncertainty Principle specifically deals with the relationship between the uncertainty in time measurements and the uncertainty in energy measurements. According to the Time-Energy Uncertainty Principle, there is a fundamental limitation to the simultaneous precision with which time and energy can be known. This means that as the uncertainty in the measurement of time decreases, the uncertainty in the measurement of energy increases, and vice versa. In other words, there exists a trade-off between the precision with which we can determine the timing of an event and the level of detail we can know about the energy associated with that event. This uncertainty principle has profound implications for our understanding of time in the quantum realm. It suggests that time cannot be precisely measured or defined on arbitrarily short timescales without introducing significant uncertainty in the associated energy. As a result, the concept of an instantaneous or perfectly defined moment in time becomes blurred, and our classical notion of time as a continuous and smoothly flowing entity is challenged. The Time-Energy Uncertainty Principle also has implications for the dynamics of quantum systems. It implies that a quantum system with a well-defined energy cannot simultaneously have a precisely defined time of occurrence. This has significant consequences for studying processes that unfold over short timescales, where the precise temporal ordering of events becomes uncertain due to the inherent uncertainty in energy. Additionally, the Time-Energy Uncertainty Principle has implications for the measurement of time intervals in quantum systems. The uncertainty in time measurements places a limit on the precision with which we can determine the duration of processes or the timing of events. This inherent limitation influences our ability to accurately measure and analyse the

temporal aspects of quantum phenomena. The Time-Energy Uncertainty Principle highlights the inherent uncertainty and indeterminacy associated with the concept of time in the quantum realm. It emphasizes that time in the quantum domain cannot be treated as an absolute, precisely measurable quantity but is subject to fundamental limits of uncertainty.

2. Do particles tell time?

Spontaneous reactions, by their very nature, possess an intriguing characteristic: they cannot be predicted with certainty. Yet, despite this inherent unpredictability, these reactions do occur, each happening at a specific point in time. This intriguing phenomenon raises the question of whether particles, at some level, possess an intrinsic sense of time. In the realm of quantum mechanics, where particles exist as dynamic entities described by wave functions, the concept of time takes on a unique and nuanced form. While traditional classical mechanics relies on deterministic principles to precisely calculate the behaviour of particles over time, quantum mechanics introduces inherent uncertainties that challenge our traditional notions of predictability. Spontaneous reactions, such as radioactive decay or chemical reactions, exemplify the non-deterministic nature of the quantum world. They occur spontaneously, seemingly without external influence or intervention. Nevertheless, each spontaneous reaction unfolds at a specific moment, suggesting that particles possess a subtle but implicit awareness of time.

I. Extending the Wave function

In the realm of quantum mechanics, particles are described by wave functions, mathematical entities that encapsulate their probabilistic behaviour. Wave functions not only encompass the spatial dimensions of particles but also incorporate a temporal dimension. The inclusion of time within wave functions implies that particles possess an implicit understanding or sensitivity to temporal dynamics. Although the precise details of a particle's properties at a given moment may not be directly discernible from the wave function, the probabilistic nature of quantum mechanics allows for the assessment of probabilities associated with different time intervals. This suggests that particles, at a fundamental level, possess an innate connection to the temporal dimension.

Assume that we have a wave function, denoted as $\psi(x, t)$, which describes the probabilistic behaviour of a particle in quantum mechanics. With this wave function, one can investigate where the particle may be located at a given time, but inversely, given a position, we can describe the probability of *when* a particle can be at the given position. By analysing the wave function, one can extract valuable information about the probabilities associated with the particle's position for a specific time and the probabilities associated with the particle's presence at a particular position across various times. This seemingly brings back classical determinism as the particle somehow already knows its future and past. We could prove this concept by applying it directly to a system.

i. Dimension extraction

We start by considering the wave function of a freestanding electron confined within impenetrable walls. We define the wave function as $\psi(x, y, t)$, which satisfies the Schrödinger equation:

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + \hat{V}(x, y, t)\right]\psi(x, y, t) = E\psi(x, y, t)$$

where the potential $\hat{V}(x, y, t)$ is given by:

$$\hat{V}(x, y, t) = \begin{cases} 0 & 0 \leq x \leq a\,; 0 \leq y \leq b\,; 0 \leq c \\ \infty & a < x < 0 \\ \infty & b < y < 0 \\ \infty & c < z < 0 \end{cases}$$

Simplifying the equation, we expand the terms and obtain

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial t^2}\right) = E\psi$$

Assuming that the wave function can be expressed as $\psi(x, y, t) = X(x) \cdot Y(y) \cdot T(t)$ and the energy as $E = \varepsilon_x + \varepsilon_y + \varepsilon_t$, we can derive the following set of equations

$$\begin{cases} -\frac{\hbar^2}{2m}\left[\frac{\partial^2 X(x)}{\partial x^2}\right] & = \varepsilon_x \cdot X(x) \\ -\frac{\hbar^2}{2m}\left[\frac{\partial^2 Y(y)}{\partial y^2}\right] & = \varepsilon_y \cdot Y(y) \\ -\frac{\hbar^2}{2m}\left[\frac{\partial^2 T(t)}{\partial t^2}\right] & = \varepsilon_t \cdot T(t) \end{cases}$$

Then allowing $X(x), Y(y), T(t)$ to take the forms

$$X(x) = A_x \cdot \sin(k_x x) + B_x \cdot \sin(k_x x)$$

$$Y(y) = A_y \cdot \sin(k_y y) + B_y \cdot \sin(k_y y)$$

$$T(t) = A_t \cdot \sin(k_t t) + B_t \cdot \sin(k_t t)$$

By applying the boundary conditions, namely

$$\psi(0, y, t) = 0$$

$$\psi(x, 0, t) = 0$$

$$\psi(x, y, 0) = 0$$

we can deduce that

$$B_x = 0 \qquad \psi(L, y, t) = 0$$

$$B_y = 0 \qquad \psi(x, L, t) = 0$$

$$B_t = 0 \qquad \psi(x, y, L) = 0$$

With these conditions, we simplify the original set of equations to

$$\begin{cases} N \cdot \sin\left(\sqrt{\frac{2m\varepsilon_x}{\hbar^2}}L\right) \sin\left(\sqrt{\frac{2m\varepsilon_y}{\hbar^2}}y\right) \sin\left(\sqrt{\frac{2m\varepsilon_t}{\hbar^2}}t\right) & = 0 \\ N \cdot \sin\left(\sqrt{\frac{2m\varepsilon_x}{\hbar^2}}x\right) \sin\left(\sqrt{\frac{2m\varepsilon_y}{\hbar^2}}L\right) \sin\left(\sqrt{\frac{2m\varepsilon_t}{\hbar^2}}t\right) & = 0 \\ N \cdot \sin\left(\sqrt{\frac{2m\varepsilon_x}{\hbar^2}}x\right) \sin\left(\sqrt{\frac{2m\varepsilon_y}{\hbar^2}}y\right) \sin\left(\sqrt{\frac{2m\varepsilon_t}{\hbar^2}}L\right) & = 0 \end{cases}$$

Which is only true if the following expression is true

$$\sin\left(\sqrt{\frac{2m\varepsilon_x}{\hbar^2}}L\right) = \sin\left(\sqrt{\frac{2m\varepsilon_y}{\hbar^2}}L\right) = \sin\left(\sqrt{\frac{2m\varepsilon_t}{\hbar^2}}L\right) = 0$$

To solve this expression, we set $\sqrt{\frac{2m\varepsilon_\eta}{\hbar^2}}L = n_\eta \pi$, which allows us to find ε_x, ε_y, and ε_t as $\varepsilon_\eta = \frac{\hbar^2\pi^2}{2mL^2}n_\eta^2$. Meaning The energy levels for $E_{x,y,t}$ can be determined as follows

$$E_{x,y,t} = \frac{\hbar^2\pi^2}{2mL}\left(n_x^2 + n_y^2 + n_t^2\right)$$

Consequently, the wave function $\psi_{n_x,n_y,n_t}(x, y, t)$ can be expressed as the product

$$\psi_{n_x,n_y,n_t}(x, y, t) = \prod_k^{x,y,t} N \cdot \sin\left(\frac{n_k \pi k}{L}\right)$$

To determine the normalisation constant N, we can perform a triple integration

$$\int_0^L |\psi_{n_x,n_y,n_t}(x,y,t)|^2 \, dx \, dy \, dt = N^3 \prod_k^{x,y,t} \int_0^L \sin^2\left(\frac{n_k \pi k}{L}\right) dk$$
$$= 1$$

Solving this integration equation yields the final solution

$$\psi_{n_x,n_y,n_t}(x,y,t) = \frac{2}{L} \sin\left(\frac{n_x \pi x}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right) \sin\left(\frac{n_t \pi t}{L}\right)$$

When we obtain the solution to the wave function $\psi(x,y,t)$, we can begin the process of extracting time from it by isolating the temporal component from the equation. This allows us to disentangle the temporal dynamics of the particle from its spatial characteristics, enabling a focused analysis of its behaviour over time. By separating the time variable, we gain valuable insights into how the particle evolves and transforms as time progresses, further unravelling the intricate relationship between particles and the enigma of time within the quantum realm.

To isolate t from the given equation, we can start by rearranging the equation and then applying inverse trigonometric functions. Let's proceed step by step.

$$\psi(x,y,t) = \frac{2}{L} \sin\left(\frac{n_x \pi x}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right) \sin\left(\frac{n_t \pi t}{L}\right)$$
$$\frac{L}{2}\psi = \sin\left(\frac{n_x \pi x}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right) \sin\left(\frac{n_t \pi t}{L}\right)$$
$$\sin\left(\frac{n_t \pi t}{L}\right) = \frac{L}{2}\psi \csc\left(\frac{n_x \pi x}{L}\right) \csc\left(\frac{n_y \pi y}{L}\right)$$

Next, we can take the inverse sine of both sides of the equation

$$\frac{n_t \pi t}{L} = \arcsin\left(\frac{L}{2}\psi \csc\left(\frac{n_x \pi x}{L}\right) \csc\left(\frac{n_y \pi y}{L}\right)\right)$$

Therefore, the isolated value of t from the given equation is

$$t = \frac{L}{n_t \pi} \arcsin\left(\frac{L}{2}\psi \csc\left(\frac{n_x \pi x}{L}\right) \csc\left(\frac{n_y \pi y}{L}\right)\right)$$

The multiplicative nature of the wave solution allows us to obtain the isolation of even spacial coordinates using the same methods, to finally obtain the following relations:

$$\begin{cases} x &= \frac{L}{n_x \pi} \arcsin\left(\frac{L}{2}\psi \csc\left(\frac{n_y \pi y}{L}\right) \csc\left(\frac{n_t \pi t}{L}\right)\right) \\ y &= \frac{L}{n_y \pi} \arcsin\left(\frac{L}{2}\psi \csc\left(\frac{n_x \pi x}{L}\right) \csc\left(\frac{n_t \pi t}{L}\right)\right) \\ t &= \frac{L}{n_t \pi} \arcsin\left(\frac{L}{2}\psi \csc\left(\frac{n_x \pi x}{L}\right) \csc\left(\frac{n_y \pi y}{L}\right)\right) \end{cases}$$

Which for a generalised $\psi(\mathbf{r})$ gives

$$r_i = \frac{L}{n_t \pi} \arcsin\left(\frac{L}{2}\psi \prod_{j \neq i} \csc\left(\frac{n_{r_j} \pi r_j}{L}\right)\right)$$

The derived equation, which effectively describes the behavior of a freestanding electron within impenetrable walls, showcases its applicability and relevance in a specific scenario. However, to unlock its full potential and ensure its broader usefulness, generalization becomes imperative.

ii. The quest to generalization

To establish a more concise and generalized formulation, we begin with the assumption that the wave function $\Psi(\mathbf{r}, t)$ can be expanded using a complete orthonormal set of functions $\Phi_n(\mathbf{r})$. The expansion of $\Psi(\mathbf{r}, t)$ is given by:

$$\Psi(\mathbf{r}, t) = \sum_n C_n \Phi_n(\mathbf{r})$$

where C_n are complex coefficients. The time evolution of the wave function is described by the time-dependent Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \hat{H}\Psi(\mathbf{r},t)$$

By substituting the expansion of $\Psi(\mathbf{r},t)$ into the time-dependent Schrödinger equation and applying the orthonormality of the basis functions, we obtain:

$$i\hbar\frac{\partial C_m}{\partial t} = C_m E_m$$

where E_m is the eigenvalue corresponding to the term $C_m\Phi_m(\mathbf{r})$. This differential equation has a general solution:

$$C_m = C_{m_0}\exp\left(-\frac{iE_m t}{\hbar}\right)$$

where C_{m_0} is the initial value of C_m at $t = 0$. Substituting this solution back into the expansion of $\Psi(\mathbf{r},t)$, we obtain the generalized form:

$$\Psi(\mathbf{r},t) = \sum_n C_n\Phi_n(\mathbf{r})\exp\left(-\frac{iE_n t}{\hbar}\right)$$

To express it using a partial sum, we can write:

$$\Psi_N(\mathbf{r},t) = \exp\left(-\frac{iE_\iota t}{\hbar}\right)\sum_{n=1}^{N} C_n\Phi_n(\mathbf{r})$$

In this notation, $g(x_\iota)$ represents the value of the function $g(x)$ evaluated at $x = x_\iota$, and $f(x_n)$ represents the function $f(x)$ evaluated at $x = x_n$ within the summation. The sum $\sum_{n=1}^{N} f(x_n)$ is taken over the index n from 1 to N. This notation emphasizes that the function $g(x)$ is independent of the summation index n, and it can be factored out of the summation, in other words it obeys the expression:

$\sum_n f(x_n)g(x_n) = g(x_\iota)\sum_n f(x_n)$.To isolate t, we can take the natural logarithm of both sides:

$$\ln\left(\frac{\Psi_N(\mathbf{r},t)}{\sum_{n=1}^{N} C_n\Phi_n(\mathbf{r})}\right) = -\frac{iE_\iota t}{\hbar}$$

Finally, by multiplying both sides by $-\hbar/(iE_\iota)$, we obtain the isolated expression for t:

$$t = -\frac{i\hbar}{E_\iota}\ln\left(\frac{\Psi_N(\mathbf{r},t)}{\sum_{n=1}^{N} C_n\Phi_n(\mathbf{r})}\right) \tag{2.1}$$

This expression provides a means of ascertaining the time t associated with a specific state of the wave function, where the wave function is approximated using a finite number of terms. Interestingly, we observe a significant connection between time and energy manifested through the eigenstates, which exhibit an inverse relationship. Given that these values are influenced by probabilities, does not it become apparent that they embody a form of Time-Energy uncertainty?

iii. The emergence of decay

The fact that the quantities of time and energy are inversely proportional holds significant implications within the context of eq. (2.1). This inverse relationship signifies that as the energy increases, the corresponding time required for a particular process to unfold decreases, and vice versa. It suggests a fundamental connection between the temporal and energetic aspects of the system under consideration. In the realm of quantum mechanics, for instance, this relationship can be associated with the decay of a quantum state. Higher energies result in a faster decay or transition

to other states, leading to shorter time intervals for the process to unfold. Conversely, lower energies correspond to longer time durations.

An example that illustrates this inverse relationship is radioactive decay. In this phenomenon, the energy of an unstable nucleus decreases as it undergoes a transition to a more stable state. The decay process follows an exponential decay law, with the half-life ($t_{1/2}$) representing the time it takes for half of the radioactive substance to decay. In this scenario, the equation demonstrates that as the energy of the nucleus decreases, the corresponding time required for the decay process to occur increases. The logarithmic term in the equation captures this inverse proportionality, as a decrease in energy leads to a longer time interval for the decay to take place. For instance, consider two radioactive substances: Substance A with a higher initial energy and Substance B with a lower initial energy. Due to the inverse relationship between time and energy, Substance A will exhibit a shorter half-life ($t_{1/2A}$) compared to Substance B's half-life ($t_{1/2B}$). Consequently, Substance A will decay at a faster rate than Substance B, resulting in a shorter time interval for half of the substance to decay.

II. A closer look at uncertainty

In accordance with the discussion in §ii, eq. (2.1) reveals the manifestation of both time and energy within the framework of eigenstates. Considering the influence of probabilities on these values, it becomes reasonable to suggest the existence of a Time-Energy uncertainty. In order to prove this idea we are going to be relying on the Heisenberg uncertainty principle, which states that the product of the uncertainties in the measurements of two over-sables (A and B) is bounded by the commutator of the

corresponding operators, such as:

$$\Delta A \Delta B \geq \frac{1}{2} \left| [A, B] \right| \tag{2.2}$$

For the time-energy uncertainty, we consider the observables of time (Δt) and energy (ΔE). The corresponding operators for time (t) and energy (E) are time operator ($\hat{H}_t$) and energy operator ($\hat{H}_E$), respectively.

$$\Delta t \Delta E \geq \frac{1}{2} \left| [\hat{H}_t, \hat{H}_E] \right| \tag{2.3}$$

To derive the commutator $[\hat{H}_t, \hat{H}_E]$, we need to express the time and energy operators in terms of position and momentum operators. The time operator is given by:

$$\hat{H}_t = i\hbar \frac{\partial}{\partial E}$$

where $\frac{\partial}{\partial E}$ represents the derivative with respect to energy. The energy operator is given by:

$$\hat{H}_E = i\hbar \frac{\partial}{\partial t}$$

where $\frac{\partial}{\partial t}$ represents the derivative with respect to time. Using these expressions, we can calculate the commutator $[\hat{H}_t, \hat{H}_E]$:

$$\begin{aligned} [\hat{H}_t, \hat{H}_E] &= \left(i\hbar \frac{\partial}{\partial E} \right) \left(i\hbar \frac{\partial}{\partial t} \right) - \left(i\hbar \frac{\partial}{\partial t} \right) \left(i\hbar \frac{\partial}{\partial E} \right) \\ &= i\hbar^2 \left(\frac{\partial^2}{\partial E \partial t} - \frac{\partial^2}{\partial t \partial E} \right) \end{aligned}$$

Now, substituting eq. (2.3) back into eq. (2.3), we have:

$$\Delta t \Delta E \geq \frac{1}{2} \left| [\hat{H}_t, \hat{H}_E] \right| \tag{2.4}$$

The uncertainty in time (Δt) can be related to the uncertainty in energy (ΔE) by the reciprocal of the energy uncertainty:

$$\Delta t = \frac{1}{\Delta E}$$

Substituting this into eq. (2.4), we get:

$$\frac{1}{\Delta E}\Delta E \geq \frac{1}{2}\hbar^2 \left| \frac{\partial^2}{\partial E \partial t} - \frac{\partial^2}{\partial t \partial E} \right|$$

Simplifying, we obtain:

$$1 \geq \frac{1}{2}\hbar^2 \left| \frac{\partial^2}{\partial E \partial t} - \frac{\partial^2}{\partial t \partial E} \right|$$

The absolute value of the second partial derivatives represents the magnitude of the mixed second partial derivatives with respect to energy and time. Therefore, we have derived a time-energy uncertainty relation:

$$\Delta t \Delta E \geq \frac{\hbar}{2} \tag{2.5}$$

i. Uncertainty in Wave function analysis

Finding Energy domains

Consider a quantum harmonic oscillator, and let's explore the implications of the time-energy uncertainty principle on its wavefunction.

The wavefunction of the quantum harmonic oscillator can be expressed as:

$$\Psi(x,t) = \phi(x) \cdot \psi(t), \tag{2.6}$$

where $\phi(x)$ represents the spatial part of the wave function, and $\psi(t)$ represents the time-dependent part.

For the harmonic oscillator, the spatial wave function $\phi(x)$ is given by a Gaussian distribution centered at the equilibrium position, and $\psi(t)$ represents the time evolution of the system.

Now, let's examine the effects of the time-energy uncertainty principle on this system. According to the principle, the product of the uncertainties in energy (ΔE) and time (Δt) must satisfy:

$$\Delta E \cdot \Delta t \geq \frac{\hbar}{2}, \tag{2.7}$$

where $\hbar$ is the reduced Planck's constant.

The energy of the quantum harmonic oscillator is quantized and given by:

$$E = \left(n + \frac{1}{2}\right)\hbar\omega, \tag{2.8}$$

where n is a non-negative integer representing the energy level, and ω is the angular frequency of the oscillator.

To investigate the time-energy uncertainty, we consider the ground state ($n = 0$) of the harmonic oscillator, which has the lowest energy. In this state, the energy is precisely known as $E = \frac{1}{2}\hbar\omega$.

Assuming complete time certainty ($\Delta t = 0$) for the ground state implies that the wave function $\psi(t)$ becomes a Dirac delta function centered at a specific time, $t = t_0$.

$$\Psi(x, t) = \phi(x) \cdot \delta(t - t_0) \tag{2.9}$$

Here, the spatial part of the wavefunction $\phi(x)$ describes the probability distribution of finding the particle at different positions x, while the Dirac delta function $\delta(t - t_0)$ specifies the precise time at which the particle is observed.

By applying the Fourier transform, we can find the wavefunction in the energy domain, $\Psi(E)$. However, since the time part of the wave function is a Dirac delta function, the Fourier transform simplifies to:

$$\Psi(E) = \phi(x) \cdot e^{-i\frac{Et_0}{\hbar}} \tag{2.10}$$

In this case, the wave function in the energy domain, $\Psi(E)$, remains dependent on both the spatial part $\phi(x)$ and the specific time t_0. Therefore, even in the scenario of complete time certainty, the particle's wave function still exhibits a dependence on energy and space.

Time Probability Distribution from Energy Certainty

Consider a quantum system described by a wave function $\Psi(x,t)$, where x represents the position and t represents time. We want to explore the time probability distribution when the energy of the system is known with certainty.Let's assume that the energy of the system is precisely known ($\Delta E = 0$). In this case, the energy wave function $\Psi(E)$ becomes a Dirac delta function centered at the specific energy value E_0: $\Psi(E) = \delta(E - E_0)$

To obtain the time probability distribution, we need to find the wave function in the time domain using the Fourier transform. The Fourier transform pair between the energy wave function $\Psi(E)$ and the time wave function $\Psi(t)$ is given by:

$$\begin{aligned}\Psi(t) &= \frac{1}{\sqrt{2\pi\hbar}} \int \Psi(E) \cdot e^{-iHt/\hbar}\, dE \\ &= \frac{1}{\sqrt{2\pi\hbar}} \int \delta(E - E_0) \cdot e^{-iHt/\hbar}\, dE \\ &= \frac{1}{\sqrt{2\pi\hbar}} \cdot e^{-iE_0 t/\hbar}\end{aligned}$$

Next, we can express the energy E_0 in terms of the corresponding frequency ω_0 using the relation $E_0 = \hbar\omega_0$:

$$\Psi(t) = \frac{1}{\sqrt{2\pi}} \cdot e^{-i\omega_0 t/\hbar}$$

Taking the absolute square of the time wave function gives the probability density function, which represents the probability of finding the particle at a particular time.

$$\begin{aligned}|\Psi(t)|^2 &= \left| \frac{1}{\sqrt{2\pi}} \cdot e^{-i\omega_0 t/\hbar} \right|^2 \\ &= \frac{1}{2\pi}\end{aligned}$$

The probability density function $|\Psi(t)|^2$ is a constant value independent of time, indicating that the particle has a uniform probability distribution at all times. Therefore, in the case of complete energy certainty, the time probability distribution is uniform, and the particle is equally likely to be found at any time.

Extracting time information

Understanding the application and proper utilization of equations is crucial in physics; merely deriving equations without acquiring the knowledge of their practical use renders the endeavor pointless. While it may be fascinating to discover an equation, it is the mastery of its application that truly propels the progress of ideas and physics. With this in mind, the following section is devoted to providing a comprehensive example, highlighting the significance of learning how to effectively employ equations.

We start by considering a real wave function in the position basis. Suppose we have a particle in a one-dimensional box of length L, and its wave function is given:

$$\psi(x) = A \sin\left(\frac{n\pi x}{L}\right)$$

where A is a normalization constant and n is a positive integer representing the quantum state, and x ranges from 0 to L. To ensure that he wave function is properly normalized, we need to determine the value of the normalization constant A. The normalization condition is given by:

$$\int |\psi(x)|^2 \, dx = 1$$

where $|\psi(x)|^2$ is the probabilty density function. To find the normalization constant A, we integrate the probability density function over the entire rnage of x (from 0 to L):

$$\int_0^L |\psi(x)|^2 \, dx = 1$$

For the given wave function, the probability density function is $|\psi(x)|^2 = A^2 \sin^2(\frac{n\pi x}{L})$. Integrating the probability density function over the en-

tire range yields an expression that helps us solve for the normalization constant.

$$
\begin{aligned}
1 &= \int_0^L A^2 \sin^2\left(\frac{n\pi x}{L}\right) dx \\
1 &= \int_0^L \left(\frac{1-\cos^2(\frac{2n\pi x}{L})}{2}\right) dx \\
1 &= \frac{A^2}{2}\int_0^L \left(1-\cos^2\left(\frac{2n\pi x}{L}\right)\right) dx \\
1 &= \frac{A^2}{2}\left[x - \frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right]_0^L \\
1 &= \frac{A^2}{2}\left[L - \frac{L}{2n\pi}\sin\left(2n\pi\right)\right] \\
1 &= \frac{A^2}{2}L
\end{aligned}
$$

And after some basic algebric simplifications, we find the normalization constant to be

$$A = \sqrt{\frac{2}{L}}$$

Now all we have to do is find the complex cofficients. We know that we can expand the wave function in terms of basis functions $\phi_n(x)$ defined by $\phi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$. To calculate the complex coefficients, we can solve the following integral.

$$
\begin{aligned}
C_n &= \int \psi(x)\phi_n^* \, dx \\
&= \int \left(\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\right)\left(\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\right) dx \\
&= \frac{2}{L}\int \left(1-\cos\left(\frac{2n\pi x}{L}\right)\right) dx \\
&= \frac{2}{L}\left[x - \frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right] + C
\end{aligned}
$$

Since we are integrating over the range $[0, L]$, we substitute $x = L$ and $x = 0$ into the integral

$$C_n(L) = \frac{2}{L}\left[L - \frac{L}{2n\pi}\sin\left(\frac{2n\pi L}{L}\right)\right] + C$$
$$C_n(0) = \frac{2}{L}\left[0 - \frac{L}{2n\pi}\sin\left(\frac{2n\pi 0}{L}\right)\right] + C$$

Since the wave function is normalized, the sum of the squares of the complex coefficients must equal 1. Squaring $C_n(L)$ and $C_n(0)$ and summing them up yields [1]:

$$\begin{aligned}|C_n(L)|^2 + |C_n(0)|^2 &= |\frac{2}{L}|^2\left[L^2\right] + |C|^2 \\ &= 1\end{aligned}$$

From the above expression, we can see that $|\frac{2}{L}|^2$ is a constant, let's denote it as B

$$B\left[L^2\right] + |C|^2 = 1$$
$$B = \frac{1}{L^2 + |C|^2}$$

Therefore, the complex coefficients are given by

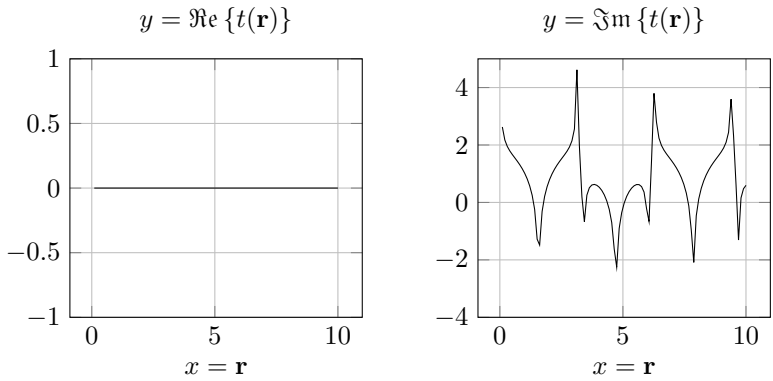
$$\begin{aligned}C_n &= \frac{2}{L}\left[x - \frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right] + C \\ &= BL + C\end{aligned}$$

After obtaining all the required parameters, we can now simply substitute them into 2.1.

ii. Time unlinearity

When visualized as graphs, the results would appear as follows

[1] To satisfy this condition, we choose that $|\frac{2}{L}|^2\left[L^2\right] + |C|^2 = 1$



The non-linearity of time implies that time does not follow a straightforward linear progression. Instead, it exhibits complex and varied patterns of movement. While the real part of time remains consistently zero, the non-zero values of the imaginary part indicate that time possesses intricate dynamics. This suggests that temporal movement involves more than a simple forward progression from past to future. Temporal trajectories, within this framework, encompass a range of patterns that deviate from linear paths. Some trajectories exhibit non-periodic and irregular movement, characterized by unique sequences of events. These trajectories often arise in chaotic systems or processes with high complexity, where small changes in initial conditions can lead to dramatically different temporal behaviors. The non-linearity of time also allows for the possibility of time-reversal trajectories, where the temporal order of events is reversed. While macroscopic time reversal is not commonly observed, it plays a crucial role in fundamental theories like quantum mechanics, where processes can be equally valid in both forward and backward temporal directions. Moreover, temporal trajectories can be influenced by external factors, driving forces, or changing conditions, resulting in time-dependent

movement. Such trajectories are not fixed or predetermined; instead, they evolve and adapt over time, responding to the dynamic nature of the system.

iii. Mutli-Particle systems

Utilizing this concept in practical applications within single-particle systems proves to be largely futile. Instead, our focus lies in comprehending the dynamics within multi-particle systems, particularly hydrogen systems. However, prior to advancing further, it is imperative to acquire the wave function pertaining to hydrogen.

The initial step is to formulate the Schrödinger equation in a format that is conducive to our objectives. Assuming we can express the wave function in spherical coordinates as $\psi(r,\theta,\phi) = R(r)\cdot Y(\theta,\phi)$, we must also establish the Laplacian operator in terms of spherical coordinates.

$$\nabla^2 = \frac{1}{r^2}\left(\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2}\right)$$

Consequently, the Schrödinger equation takes its initial form as follows

$$-\frac{\hbar^2}{2m}\left[\frac{Y^2}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{R}{r^2\sin^2\theta}\frac{\partial^2 Y}{\partial\phi}\right] + VRY = ERY \tag{2.11}$$

To facilitate the analysis, setting the Schrödinger equation to zero will be beneficial.

$$\left\{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}\left[V(r) - E\right]\right\} \tag{2.12}$$

$$\frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\theta^2}\right] = 0 \tag{2.13}$$

This allows us to split the equation into the following system of equations:

$$\begin{cases} \ell(\ell+1) = \dfrac{1}{R}\dfrac{d}{dr}\left(r^2\dfrac{dR}{dr}\right) - \dfrac{2mr^2}{\hbar^2}\left[V(r) - E\right] & (2.14) \\ -\ell(\ell+1) = \dfrac{1}{Y}\left[\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial Y}{\partial\theta}\right) + \dfrac{1}{\sin^2\theta}\dfrac{\partial^2 Y}{\partial\theta^2}\right] & (2.15) \end{cases}$$

To begin, let's focus on the first equation. Assuming we can express the angular wave function as the product of two separate functions, $\Theta(\theta)$ and $\Phi(\phi)$, we can rewrite the equation as follows:

$$\left\{\frac{1}{\Theta}\left[\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\right] + \ell\left(\ell+1\right)\sin^2\theta\right\} + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0$$

And assuming that,

- $\dfrac{1}{\Theta}\left[\sin\theta\dfrac{d}{d\theta}\left(\sin\theta\dfrac{d\Theta}{d\theta}\right)\right] + \ell(\ell+1)\sin^2\theta = m^2$
- $\dfrac{d^2\Phi}{d\Phi^2} = -m^2\Phi$

That allows it to rewrite it in the following form:

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[\ell\big(\ell+1\big)\sin^2\theta - m^2\right] = 0$$

We have one remaining task, which is to accurately define $\Theta(\theta)$. This can be accomplished as follows:

$$\Theta(\theta) = AP_\ell^m(\cos\theta)$$

where $P_\ell^m(\cos\theta)$ is the associated Legendre function defined as

$$\begin{cases} P_\ell^m(\cos\theta) & = (-1)^m(\sin\theta)^m\frac{d^m}{d(\cos\theta)^m}P_\ell(\cos\theta) \ \ \forall m \leq \ell \\ P_\ell(x) & \equiv \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell(x^2-1)^\ell \end{cases}$$

Upon replacing the variables ℓ and m, we can derive the following expressions:

$$\begin{cases} P_0^0(\cos\theta) & = 1 \\ P_1^0(\cos\theta) & = \cos\theta \\ P_1^1(\cos\theta) & = -\sin\theta \\ P_2^0(\cos\theta) & = \frac{1}{2}\left(3\cos^2\theta - 1\right) \end{cases}$$

And moving on to the radial part, if we define the derivative of the radial vector to be $dr^3 = r^2 \sin\theta dr d\theta d\phi$, we can then show that the spherical density is

$$\begin{aligned} \int |\psi|^2 r^2 \sin\theta dr d\theta d\phi &= \int |R|^2 r dr \int |Y|^2 \sin\theta d\theta d\phi \\ &= 1 \end{aligned}$$

Upon substituting the integration bounds into these expressions, we obtain:

$$\begin{aligned} 1 &= \int_0^\infty |R|^2 r^2 dr \\ &= \int_0^{2\pi} \int_0^\pi |Y|^2 \sin\theta d\theta d\phi \end{aligned}$$

Which finally implies that

$$Y_\ell^m(\theta,\phi) = \varepsilon \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}} e^{im\phi} P_\ell^m(\cos\theta) \tag{2.16}$$

$$\begin{aligned} \forall\ \varepsilon &= (-1)^m & m &\geq 0 \\ \varepsilon &= 1 & m &< 0 \end{aligned}$$

With the first equation solved, we can now proceed to the second equation.

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}\left[V(r) - E\right] R = \ell(\ell+1) R$$

$$\text{let } u(r) \equiv rR(r) \implies R = \frac{u}{r}$$
$$\frac{dR}{dr} = \left(r - \frac{du}{dr} - u\right) r^{-2}$$
$$\frac{d}{dr}\left[r^2 \frac{dR}{dr}\right] = r\frac{d^2u}{dr^2}$$

Once all the variables and expressions have been defined, we can substitute them back into eq. (2.11) to obtain,

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(x) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right]u = Eu$$

Considering $V(x)$ as the Coulomb force, we can set it equal to $V(r) = -\frac{e^2}{4\pi\varepsilon_0}\frac{1}{r}$. Furthermore, we introduce two new variables: $k = \sqrt{-2mE}/\hbar$ and $\rho \equiv kr$. We define ρ_0 as $\rho_0 \equiv \frac{me^2}{4\pi\varepsilon_0\hbar^2 k}$.

$$\implies -\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[-\frac{e^2}{4\pi\varepsilon_0}\frac{1}{r} + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right] = Eu$$
$$\implies \frac{1}{k^2}\frac{du^2}{dr^2} = \left[1 - \frac{me^2}{2\pi\varepsilon_0\hbar k}\frac{1}{kr} + \frac{\ell(\ell+1)}{(kr)^2}\right]u$$
$$\implies \frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{\ell(\ell+1)}{\rho^2}\right]$$

Applying the boundary conditions, we obtain two sets of limits corresponding to each bound:

$$\begin{cases} \lim_{\rho\to\infty} & \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right]u = u \\ \lim_{\rho\to\infty} & \frac{du^2}{d\rho^2} = u \end{cases}$$

$$\begin{cases} \lim_{\rho\to 0} & \left[1 - \frac{\rho_0}{\rho} + \frac{\ell(\ell+1)}{\rho^2}\right]u = u\frac{\ell(\ell+1)}{\rho} \\ \lim_{\rho\to 0} & \frac{du^2}{d\rho^2} = u\frac{\ell(\ell+1)}{\rho} \end{cases}$$

We could then redefine and approximate $u(\rho)$ such as,

$$\begin{aligned} u(\rho) &= Ae^{-\rho} + Be^{-\rho} \\ &= C\rho^{\ell+1} + D\rho^{-}1 \\ &\approx Ae^{-\rho}\ (Be^{-\rho} = 0) \\ &\approx C\rho^{\ell+1}\ (D\rho^{-}1 = 0) \end{aligned}$$

Introducing a new function $\nu(\rho)$ in our definition of $u(\rho)$ would be highly beneficial.

$$u(\rho) = \rho^{\ell+1}e^{-\rho}\nu(\rho)$$

As a result, the first and second derivatives of u would be modified as follows:

$$\dot{u} = \rho^{\ell}e^{-\rho}\left[(\ell+1-\rho)\nu + \rho\frac{d\nu}{d\rho}\right]$$

$$\ddot{u} = \rho^{\ell}e^{-\rho}[-2\ell - 2 + \rho + \frac{\ell(\ell+1)}{\rho}\nu + 2(\ell+1-\rho)\frac{d\nu}{d\rho} + \rho\frac{d^2\nu}{d\rho^2}]$$

This indicates that the equation represented by eq. (2.15) can be reformulated as:

$$\rho\frac{d^2\nu}{d\rho^2} + 2(l+1-\rho)\frac{d\nu}{d\rho} + [\rho_0 - 2(l+1)]\,\nu = 0 \tag{2.17}$$

If we do not define ν, our progress will be hindered. To address this, we can define ν in the following manner:

$$\nu(\rho) = \sum_{j\geq 0} c_j\rho^j$$

This indicates that the first and second derivatives of ν can be outlined

as:

$$\begin{aligned}\frac{d\nu}{d\rho} &= \sum_{j\geq 0} j c_j \rho^{j-1} \\ &= \sum_{j\geq 0} (j+1) c_{j+1} \rho^{j} \\ \frac{d^2\nu}{d\rho^2} &= \sum_{j\geq 0} j(j+1) c_{j+1} \rho^{j-1}\end{aligned}$$

Upon substitution of ν into eq. (2.17), it undergoes a transformation resulting in the following lengthy expression:

$$\begin{aligned}\sum_{j\geq 0} j(j+1)c_{j+1}\rho^j + 2(\ell+1)\sum_{j\geq 0}(j+1)c_j\rho^j \\ -2\sum_{j\geq 0} jc_j\rho^j + \big[\rho_0 - 2(\ell+1)\big]\sum_{j\geq 0} c_j\rho_j = 0\end{aligned} \tag{2.18}$$

$$\begin{aligned}j(j+1)c_{j+1} + 2(\ell+1)(j+1)c_j \\ - 2jc_j + \big[\rho_0 - 2(\ell+1)\big]c_j = 0\end{aligned} \tag{2.19}$$

Given that $c_{j+1} = \frac{2(j+\ell+1)-\rho_0}{(j+1)(j+2\ell+2)}c_j$ and for a significantly large value of j, we can make an approximate numerical estimation for both c_{j+1} and c_j as follows:

$$\begin{aligned}c_{j+1} &\approx \frac{2j}{j(j+1)}c_j \\ &= \frac{2}{j+1}c_j \\ c_j &= \frac{2^j}{j!}c_0\end{aligned}$$

The missing part of our analysis pertains to the determination of the energy. Based on our defined potential energy, we can express the equation

for energy as follows:

$$\begin{aligned}
E &= -\frac{\hbar^2}{2m}k^2 \\
&= -\frac{me^4}{8\pi^2\varepsilon_0\hbar^2\rho_0^2} \\
E_n &= -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2\right]\frac{1}{n} \\
&= \frac{E_1}{n} \quad \forall\ n \in \mathbb{Z}^* \\
E_1 &= -13.6 \text{ eV}
\end{aligned}$$

Now that we have introduced the new parameter m, we can redefine the function $u(r)$ as $u(r) = rR(r)$ such as

$$R_{ml} = \frac{1}{r}\rho^{l+1}e^{-\rho}\nu(\rho) \tag{2.20}$$

When the values of n and ℓ are 1 and 0 respectively, the wavefunction $\psi_{1,0,0}$ can be expressed as the product of the radial function $R_{1,0}(r)$ and the spherical harmonic function $Y_0^0(\theta,\phi)$, such as

$$\psi_{1,0,0}(r,\theta,\phi) = R_{1,0}(r)\cdot Y_0^0(\theta,\phi)$$

And by applying the appropriate boundary conditions,

$$\begin{aligned}
\int_0^\infty |R_{1,0}|^2 r^2\,dr &= \frac{|c_0|^2}{a^2}\int_0^\infty e^{-2r/a}r^2\,dr \\
&= 1
\end{aligned}$$

From this we can infer from $|c_0|^2\frac{a}{4} = 1$ that $|c_0| = \frac{2}{\sqrt{a}}$, and since $Y_0^0 = \frac{1}{\sqrt{4\pi}}$, then the $1,0,0$ state of the wave function can be then written as

$$\psi_{1,0,0}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a^3}}e^{-r/a}$$

Let us examine now the scenario where $n = 2$, denoting E_2 as

$$E_2 = \frac{-13.6}{4} = -3.4 \text{ eV}$$

In this situation, we find that there exist four potential states for m, depending on the value of l. When $\ell = 0$, it indicates that $m = 0$. Furthermore, when $\ell = 1$, the possible values of m are 0, $+1$ and -1. When ℓ is equal to 0, the coefficients c_1 correspond to c_0 with j being 0, while the coefficients c_2 with j being 1 are equal to 0. In this case, we can represent $\nu(\rho)$ as c_0 multiplied by the quantity $(1 - \rho)$. Contrary to the previous state observed for $n = 1$, the notation $R_{2,1}$ is depicted as

$$R_{2,1} = \frac{c_0}{4a^2} r e^{-r/2a}$$

This notation implies that

$$\psi_{2,1,0} = \frac{1}{\sqrt{6a}} \cdot \frac{1}{2a^2} r^{-r/2} \left(\sqrt{\frac{3}{4\pi}} \cos\theta \right)$$

So now, as we reach the final steps, we can solve for Ψ assuming that $\Psi(\mathbf{r}, t) = \psi_n(\mathbf{r}) \cdot e^{iE_n t/\hbar}$ such as

$$\begin{aligned} \psi(\mathbf{r}, t) &= \frac{\psi_{1,0,0} + \psi_{2,1,0}}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2\pi a_0^3}} \cdot e^{-iE_{1,0,0}t\hbar} \left(e^{-r/a_0} + e^{-i\omega t} \frac{Z}{a_0} \frac{e^{-r/2a_0}}{4\sqrt{2}} \right) \end{aligned}$$

And thus, by employing the given expression, we can determine the initial state of the wave function.

$$\psi_0 = \frac{1}{\sqrt{2\pi a_0^3}} \left(1 + \frac{Z}{a_0 4\sqrt{2}} \right)$$

Finally, we arrive at the complete expression for the Hydrogen wave function as follows:

$$\Psi(r,\theta,\phi)_{n,\ell,m} = \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-\ell-1)!}{2n[(n+1)!]^3}} e^{-r/na} \left(\frac{2r}{na}\right)^{\ell} \cdot \left[L_{n-\ell-1}^{2\ell+1}\left(\frac{2r}{na}\right)\right] \varepsilon \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}} P_\ell^m(\cos\theta) e^{im\phi} \tag{2.21}$$

The figures presented reveal the profound insights gained from our unorthodox approach of graphing spatial quantities for specific sections of space. Despite the limitations of our relatively small sample size of 6 slices, these graphical representations establish a clear correlation, providing a compelling understanding of the interplay between temporal dynamics and particle behavior. Through the observation of particle behavior within localized spatial contexts, we acquire a distinct perspective on how temporal information manifests within the spatial domain. These graphical depictions serve as a novel lens through which we can explore the intricate relationship between space and time. The figures underscore the significance of graphing spatial quantities, as they offer fresh avenues for comprehending the fundamental nature of temporal information and its impact on particle behavior. By employing this unconventional methodology, we gain the ability to discern hidden temporal patterns embedded within the fabric of our physical reality.

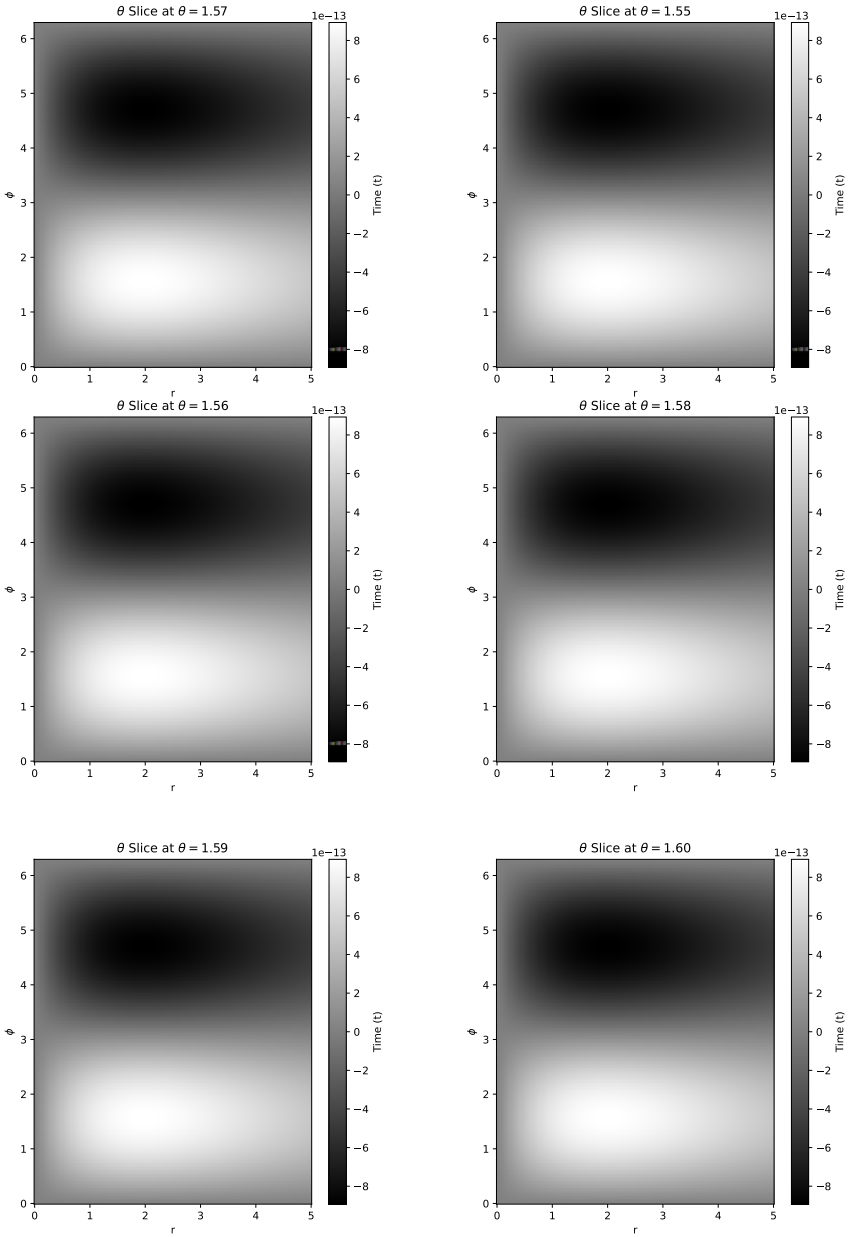


Figure 2.2: Time extractions from a Hydrogen atom

3. Relativistic Chronology

I. Quantum Relativism

For a long time, marrying quantum mechanics and general relativity has been a formidable challenge, often regarded as a perplexing conundrum in the realm of physics. The two theories, each successful in their respective domains, appear to be fundamentally incompatible when attempts are made to merge them into a unified framework. Quantum mechanics, with its probabilistic nature and discrete energy levels, clashes with general relativity's smooth and continuous space-time fabric governed by Einstein's field equations. The discrepancies between the two theories have led to a myriad of theoretical difficulties, including infinities and inconsistencies that have proven difficult to resolve. This longstanding puzzle has motivated physicists to explore various approaches and frameworks, such as string theory, loop quantum gravity, and quantum field theory in curved space-time, in hopes of finding a coherent and all-encompassing theory that can harmoniously incorporate both quantum mechanics and general relativity. Despite the challenges, the quest for unifying these theories remains a tantalizing pursuit, holding the promise of unlocking profound insights into the nature of the universe and its fundamental laws. But

what does this mean for us? Let us endeavor to ascertain the ramifications of integrating our conceptual framework while accounting for relativistic effects.

We consider the Dirac equation for an electron in the presence of an electromagnetic field. In this simplified scenario, the Coulomb potential is zero, and the vector potential is given by a constant magnetic field. The specific details of the electromagnetic field are as follows: The Dirac equation for the unbounded electron in this electromagnetic field is given by:

$$(i\boldsymbol{\gamma} \cdot \boldsymbol{D} - m)\psi = 0$$

where $\boldsymbol{\gamma}$ represents the gamma matrices, $\boldsymbol{D} = \boldsymbol{\nabla} + ie\mathbf{A}$ is the covariant derivative with the electron charge e, and m is the electron mass.

To find the wave function solutions, we separate the four-component spinor ψ as:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

where ψ_1, ψ_2, ψ_3, and ψ_4 are functions representing the different components of the electron wave function.

Next, we substitute the explicit forms of the scalar and vector potentials into the Dirac equation:

$$(i\gamma^0\partial_t + i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + ie\gamma^0\mathbf{A} - m)\psi = 0$$

Using the chosen representation for the gamma matrices:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$
$$\boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}$$

where I is the 2×2 identity matrix and $\boldsymbol{\sigma}$ are the Pauli matrices, the Dirac equation becomes:

$$\begin{pmatrix} m & \boldsymbol{\sigma} \cdot (-i\boldsymbol{\nabla} + e\mathbf{A}) \\ \boldsymbol{\sigma} \cdot (-i\boldsymbol{\nabla} + e\mathbf{A}) & -m \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = i\partial_t \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

Expanding the matrix equation component-wise, we obtain the following equations:

$$(m\psi_1 - i(\boldsymbol{\sigma} \cdot (-i\boldsymbol{\nabla} + e\mathbf{A}))\psi_4) = i\partial_t\psi_1$$
$$(m\psi_2 - i(\boldsymbol{\sigma} \cdot (-i\boldsymbol{\nabla} + e\mathbf{A}))\psi_3) = i\partial_t\psi_2$$
$$(m\psi_3 - i(\boldsymbol{\sigma} \cdot (-i\boldsymbol{\nabla} + e\mathbf{A}))\psi_2) = i\partial_t\psi_3$$
$$(m\psi_4 - i(\boldsymbol{\sigma} \cdot (-i\boldsymbol{\nabla} + e\mathbf{A}))\psi_1) = i\partial_t\psi_4$$

Now, let's consider the particle solution. We assume a plane wave form for the wave function:

$$\psi = u_p e^{i(\mathbf{p}\cdot\mathbf{r} - Et)}$$

where $\mathbf{p} = (p_x, p_y, p_z)$ is the momentum vector and $E = \sqrt{\mathbf{p}^2 + m^2}$ is the energy.

Substituting this into the Dirac equation, we obtain the following equations:

$$(m\psi_1 - i(\boldsymbol{\sigma} \cdot (-i\boldsymbol{\nabla} + e\mathbf{A}))\psi_4) = E\psi_1$$
$$(m\psi_2 - i(\boldsymbol{\sigma} \cdot (-i\boldsymbol{\nabla} + e\mathbf{A}))\psi_3) = E\psi_2$$
$$(m\psi_3 - i(\boldsymbol{\sigma} \cdot (-i\boldsymbol{\nabla} + e\mathbf{A}))\psi_2) = -E\psi_3$$
$$(m\psi_4 - i(\boldsymbol{\sigma} \cdot (-i\boldsymbol{\nabla} + e\mathbf{A}))\psi_1) = -E\psi_4$$

Simplifying these equations further, we obtain:

$$(m - E)\psi_1 - i(\boldsymbol{\sigma} \cdot (-i\boldsymbol{\nabla} + e\mathbf{A}))\psi_4 = 0$$
$$(m - E)\psi_2 - i(\boldsymbol{\sigma} \cdot (-i\boldsymbol{\nabla} + e\mathbf{A}))\psi_3 = 0$$
$$(m + E)\psi_3 - i(\boldsymbol{\sigma} \cdot (-i\boldsymbol{\nabla} + e\mathbf{A}))\psi_2 = 0$$
$$(m + E)\psi_4 - i(\boldsymbol{\sigma} \cdot (-i\boldsymbol{\nabla} + e\mathbf{A}))\psi_1 = 0$$

Solving these equations, we obtain the following expressions for the particle wave function components:

$$\psi_1 = \frac{1}{\sqrt{2m(E+m)}}(\boldsymbol{\sigma} \cdot (\mathbf{p} - ie\mathbf{A}))\psi_4$$
$$\psi_2 = \frac{1}{\sqrt{2m(E+m)}}(\boldsymbol{\sigma} \cdot (\mathbf{p} - ie\mathbf{A}))\psi_3$$
$$\psi_3 = \frac{1}{\sqrt{2m(E+m)}}(\boldsymbol{\sigma} \cdot (\mathbf{p} - ie\mathbf{A}))\psi_2$$
$$\psi_4 = \frac{1}{\sqrt{2m(E+m)}}(\boldsymbol{\sigma} \cdot (\mathbf{p} - ie\mathbf{A}))\psi_1$$

Substituting these expressions into the plane wave form of the wave function, we obtain the complete particle wave function solution:

$$\psi_p = u_p e^{i(\mathbf{p}\cdot\mathbf{r} - Et)} \tag{3.1}$$

where u_p is given by:

$$u_p = \sqrt{2m(E+m)} \begin{pmatrix} 1 \\ \frac{p_x + ip_y}{E+m} \\ \frac{p_z}{E+m} \\ 0 \end{pmatrix}$$

Similarly, we can find the antiparticle solution by assuming a plane wave form for the wave function:

$$\psi = u_a e^{i(\mathbf{p}\cdot\mathbf{r} - Et)}$$

where $\mathbf{p} = (p_x, p_y, p_z)$ is the momentum vector and $E = -\sqrt{\mathbf{p}^2 + m^2}$ is the energy.

Substituting this into the Dirac equation and solving, we obtain the following expressions for the antiparticle wave function components:

$$\begin{aligned}
\psi_1 &= \frac{1}{\sqrt{2m(-E+m)}}(\boldsymbol{\sigma}\cdot(\mathbf{p} - ie\mathbf{A}))\psi_4 \\
\psi_2 &= \frac{1}{\sqrt{2m(-E+m)}}(\boldsymbol{\sigma}\cdot(\mathbf{p} - ie\mathbf{A}))\psi_3 \\
\psi_3 &= \frac{1}{\sqrt{2m(-E+m)}}(\boldsymbol{\sigma}\cdot(\mathbf{p} - ie\mathbf{A}))\psi_2 \\
\psi_4 &= \frac{1}{\sqrt{2m(-E+m)}}(\boldsymbol{\sigma}\cdot(\mathbf{p} - ie\mathbf{A}))\psi_1
\end{aligned}$$

Substituting these expressions into the plane wave form of the wave function, we obtain the complete antiparticle wave function solution:

$$\psi_a = u_a e^{i(\mathbf{p}\cdot\mathbf{r} - Et)}$$

where u_a is given by:

$$u_a = \sqrt{2m(-E+m)} \begin{pmatrix} 0 \\ \frac{p_z}{-E+m} \\ \frac{p_x - ip_y}{-E+m} \\ 1 \end{pmatrix}$$

By employing the identical methodology utilized for eq. (2.1), we can extract the time component from eq. (3.1), resulting in an equation representing t as follows:

$$\begin{aligned} \frac{\psi}{u_a} &= e^{i(\mathbf{p}\cdot\mathbf{r} - Et)} \\ \ln \frac{\psi}{u_a} &= i(\mathbf{p}\cdot\mathbf{r} - Et) \\ \frac{1}{i} \ln \frac{\psi}{u_a} &= \mathbf{p}\cdot\mathbf{r} - Et \end{aligned}$$

To finally obtain

$$t = \frac{1}{E}\left(\mathbf{p}\cdot\mathbf{r} + i \ln \frac{\psi}{u_a}\right) \tag{3.2}$$

By graphing the time values in a manner comparable to our previous approach, we can discern analogous patterns, suggesting that the tidal shifts are not attributable to relativistic effects.

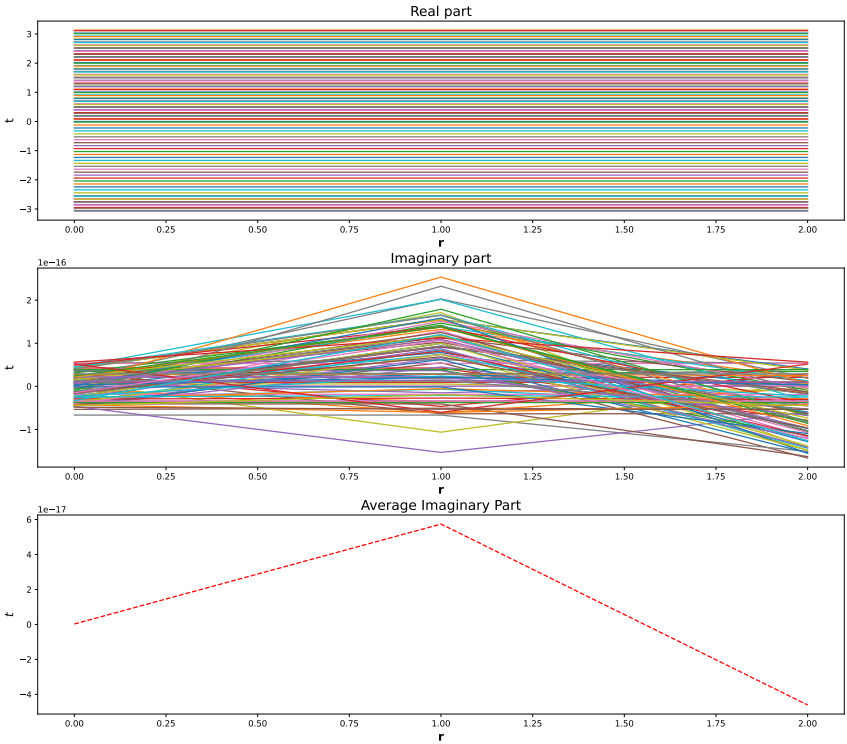
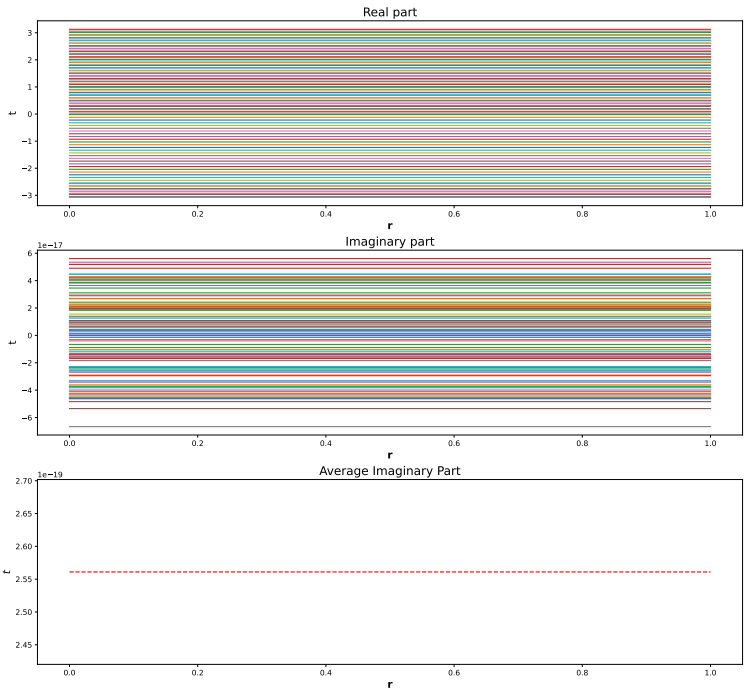


Figure 3.1: Tidal extractions from Relativistic Particles

i. 4 Component Momentum

We can clearly see that we got similar results to our previous graph, as shown in the figure chapter II, §ii, where the real part of time remains constant (or zero) and the imaginary part exhibits erratic movement. Specifically, we notice an increase at the center of the particle itself, which aligns really well with the characteristics of a relativistic particle. From the analysis of the average imaginary graph, we can observe that we obtained rather different results in terms of the form of the function

compared to the previous graph. The first graph resembled a sinusoidal-like wave, whereas the current graph shows a simple increase followed by a decrease. This indicates that the backward/forward tidal movement is not caused by relativistic effects but rather by something very different. Surprisingly, when the momentum vector travels uni-directionally, no tidal motion appears to take place. This phenomenon can be observed in the accompanying figure below.



This suggests that momentum expands from three dimensions to four, with the fourth component being time. The fourth component, time, is dependent on the values of the first three components. For instance, if two of the three components are zero, there will be no movement in time.

This could be expressed mathematically as

$$\textbf{IF } (\mathbf{p}_x \wedge \mathbf{p}_y) \vee (\mathbf{p}_x \wedge \mathbf{p}_z) \vee (\mathbf{p}_y \wedge \mathbf{p}_z) \textbf{ THEN } p_t = 0 \tag{3.3}$$

We can define this concept as 4-momentum. To begin, let's consider the classical expression for energy, which is given by the famous equation $E = mc^2$, where E represents the energy, m is the mass of the particle, and c is the speed of light in vacuum. On the other hand, momentum is classically defined as $\mathbf{p} = m\mathbf{v}$, where $\mathbf{p}$ denotes momentum, m is mass, and $\mathbf{v}$ is velocity.

However, classical mechanics fails to account for the behavior of particles moving at speeds close to that of light. To address this limitation and develop a more comprehensive description, we turn to the principles of special relativity. According to special relativity, the energy and momentum of a particle are intrinsically connected and cannot be treated independently. They are related by a fundamental equation known as the relativistic energy-momentum relation:

$$E^2 = (\mathbf{p}c)^2 + (mc^2)^2 \tag{3.4}$$

From this expression we can express 4-momentum in the following manner

$$\begin{aligned}
\frac{E^2}{(mc^2)^2} &= \frac{(\mathbf{p}c)^2 + (mc^2)^2}{(mc^2)^2} \\
\frac{E^2}{(mc^2)^2} &= \frac{\mathbf{p}^2}{m^2} + 1 \\
\frac{E}{mc^2} &= \sqrt{\frac{\mathbf{p}^2}{m^2} + 1} \\
\frac{E}{c} &= \left(\sqrt{\frac{\mathbf{p}^2}{m^2} + 1}\right) \cdot (mc^2) \\
\frac{E}{c} &= \left(\sqrt{\frac{m^2\mathbf{v}^2c^2}{m^2c^2} + \frac{c^2}{c^2}}\right) \cdot (mc^2) \\
\frac{E}{c} &= \left(\sqrt{1 + \frac{\mathbf{v}^2}{c^2}}\right) \cdot (mc^2)
\end{aligned}$$

To finally obtain the expression

$$\mathbf{p} = \left(\frac{E}{c}, \mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_z\right)$$

II. Relativity in the electron-cloud

In continuation of the themes explored in the preceding section, we now progress further by extending our investigation to multiparticle systems. Just as in Chapter II, where we began with a single particle system confined within an unbounded box, our focus now shifts towards a more intricate setup involving multiple particles. For this intriguing exploration, we have chosen hydrogen as our model system.

Unlike the context of Chapter II, which focused on foundational concepts, the present section places a special emphasis on the relativistic equation of hydrogen. In particular, we shall dedicate our efforts to unraveling the solution to the Dirac equation for hydrogen. However, we consider the

hydrogen atom as a one-particle problem with a particle of reduced mass m in an external Coulomb potential $-\alpha/r$. The Dirac equation for this problem reads

$$H\Psi(x) = 0, \quad H = -i\partial + m - \gamma^0 \frac{\alpha}{r} \tag{3.5}$$

Next, we propose the following assumptions for the wave function:

$$\Psi(x) = e^{-iEt}\Psi(\mathbf{x}), \quad \Psi(\mathbf{x}) = \begin{pmatrix} \varphi(\mathbf{x}) \\ \chi(\mathbf{x}) \end{pmatrix}$$

Substituting this assumption into the Dirac equation (3.5), we obtain the coupled equations for the spinor components:

$$\begin{aligned} \left(E + \frac{\alpha}{r} - m\right)\varphi &= -i\boldsymbol{\sigma}\boldsymbol{\nabla}\chi \\ \left(E + \frac{\alpha}{r} + m\right)\chi &= -i\boldsymbol{\sigma}\boldsymbol{\nabla}\varphi \end{aligned} \tag{3.6}$$

To analyze the angular momentum properties, we introduce the angular momentum operator $\mathbf{J} = \mathbf{L}+\mathbf{S}$, where $\mathbf{L}$ is the orbital momentum operator and $\mathbf{S}$ is the electron spin operator:

$$\mathbf{L} = -i\mathbf{x} \times \boldsymbol{\nabla}, \quad \mathbf{S} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}$$

Taking the commutator with the Hamiltonian, we find:

$$[\mathbf{J}, H] = [\mathbf{L}, H] + [\mathbf{S}, H] = \boldsymbol{\gamma} \times \boldsymbol{\nabla} - \boldsymbol{\gamma} \times \boldsymbol{\nabla} = 0$$

This implies that there exists a common basis that simultaneously diagonalizes both the Hamiltonian and the angular-momentum operator. We can utilize this property to propose the following assumption for the spinor components:

$$\varphi_{nk}(\mathbf{x}) = \frac{g_{nk}(r)}{r}\Omega_{km}(\theta, \varphi), \quad \chi_{nk}(\mathbf{x}) = \frac{if_{nk}(r)}{r}\Omega_{-km}(\theta, \varphi)$$

Here, the Weyl spinor Ω is defined as follows:

$$\begin{aligned}\Omega_{km} &= \sum_{m_l=-l}^{l} \sum_{m_s=\pm 1/2} \left\langle \theta, \varphi \mid l, m_l; \frac{1}{2}, m_s \right\rangle \left\langle l, m_l; \frac{1}{2}, m_s \mid j, m \right\rangle \\ &= \frac{1}{\sqrt{2k+1}} \begin{pmatrix} \sqrt{k+\frac{1}{2}-m} Y_{k,m-1/2}(\theta,\varphi) \\ -\operatorname{sgn} k \sqrt{k+\frac{1}{2}+m} Y_{k,m+1/2}(\theta,\varphi) \end{pmatrix}\end{aligned}$$

where Y_{lm} denotes the spherical harmonics. The quantum number k is defined as:

$$k = \mp\left(j + \frac{1}{2}\right) = \begin{cases} -l-1, & j = l + 1/2 \\ l, & j = l - 1/2 \end{cases}$$

where $j = l \pm 1/2$ represents the total momentum quantum number. By utilizing the identity:

$$\boldsymbol{\sigma}\mathbf{a}\boldsymbol{\sigma}\mathbf{b} = \mathbf{ab} + i\boldsymbol{\sigma}\mathbf{a} \times \mathbf{b}$$

we can insert unity in the form $1 = (\boldsymbol{\sigma}\mathbf{e}_r)^2$ into the spin-momentum equations (3.6) to rewrite them as:

$$\boldsymbol{\sigma}\boldsymbol{\nabla}\boldsymbol{\sigma}\mathbf{e}_r = \partial_r + \frac{2}{r} - i\boldsymbol{\sigma}\mathbf{e}_r \times \boldsymbol{\nabla} = \frac{1}{r}\left(\partial_r r + K\right)$$

Here, we have defined the operator:

$$K = 1 + 2\mathbf{SL} = 1 + \mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2$$

Finally, the spin-momentum equations can be expressed in terms of the new basis as:

$$\begin{aligned}\left(E_{nk} + \frac{\alpha}{r} - m\right) g_{nk}\Omega_{km} &= \left(\partial_r f_{nk} + f_{nk}K\right)\boldsymbol{\sigma}\mathbf{e}r\Omega{-}km \\ \left(E_{nk} + \frac{\alpha}{r} + m\right) f_{nk}\Omega_{-km} &= -\left(\partial_r g_{nk} + g_{nk}K\right)\boldsymbol{\sigma}\mathbf{e}r\Omega km\end{aligned} \tag{3.7}$$

Moreover, it's noteworthy that Ω is an eigenstate of the operator K:

$$K\Omega_{km} = -k\Omega_{km}$$

$$\boldsymbol{\sigma}\mathbf{e}_r\Omega_{km} = \Omega_{-km}$$

Thus, we may rewrite the equations in eq. (3.6) as follows:

$$\left(\left(\partial_r - \frac{k}{r}\right) f - \left(E + \frac{\alpha}{r} - m\right) g\right)\Omega_{km} = 0$$
$$\left(\left(\partial_r + \frac{k}{r}\right) g + \left(E + \frac{\alpha}{r} + m\right) f\right)\Omega_{-km} = 0$$

Or equivalently:

$$\partial_r \begin{pmatrix} g \\ f \end{pmatrix} = \underbrace{\begin{pmatrix} -\frac{k}{r} & -E - \frac{\alpha}{r} - m \\ E + \frac{\alpha}{r} - m & \frac{k}{r} \end{pmatrix}}_{=:M} \begin{pmatrix} g \\ f \end{pmatrix} \tag{3.8}$$

However, we can simplify the solution of eq. (3.8) by converting it into a Schrödinger-type form through another derivative:

$$\begin{aligned} \partial_r^2 \begin{pmatrix} g \\ f \end{pmatrix} &= \left[M^2 + \partial_r M\right] \begin{pmatrix} g \\ f \end{pmatrix} \\ &= \left[\left(\frac{k^2 - \alpha^2}{r^2} - \frac{2E\alpha}{r} + m^2 - E^2\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right. \\ &\qquad \left. + \frac{1}{r^2} \begin{pmatrix} k & \alpha \\ -\alpha & -k \end{pmatrix}\right] \begin{pmatrix} g \\ f \end{pmatrix} \end{aligned} \tag{3.9}$$

The second matrix in the square parenthesis has the eigenvalues:

$$\lambda_\pm = \pm\gamma, \quad \gamma = \sqrt{k^2 - \alpha^2}$$

Taking into account that k can be both positive and negative, we can

diagonalize it by the transformation $B' = A^{-1}BA$, where:

$$A = \frac{1}{\sqrt{2k(k-\gamma)}} \begin{pmatrix} \alpha & -(k-\gamma) \\ -(k-\gamma) & \alpha \end{pmatrix}$$

$$A^{-1} = \frac{k}{\gamma} \frac{1}{\sqrt{2k(k-\gamma)}} \begin{pmatrix} \alpha & k-\gamma \\ k-\gamma & \alpha \end{pmatrix}$$

By defining:

$$\begin{pmatrix} g \\ f \end{pmatrix} = A \begin{pmatrix} u^+ \\ u^- \end{pmatrix}$$

Additionally, we introduce the parameter:

$$x = 2Cr, \quad C = \sqrt{m^2 - E^2}$$

With this transformation, eq. (3.9) becomes:

$$\partial_x^2 u^\pm = \left[\frac{\gamma(\gamma \pm 1)}{x^2} - \frac{E\alpha}{C}\frac{1}{x} + \frac{1}{4} \right] u^\pm \tag{3.10}$$

Remarkably, this equation bears resemblance to the non-relativistic, radial Schrödinger equation eq. (2.13). As a consequence, we make the assumption:

$$\begin{pmatrix} u^+ \\ u^- \end{pmatrix} = x^\gamma e^{-x/2} \begin{pmatrix} \mathcal{A}_+ x L^+ \\ \mathcal{A}_- L^- \end{pmatrix}$$

Substituting this into eq. (3.10), we obtain:

$$x\partial_x^2 L^\pm + (2\gamma + 1 \pm 1 - x)\partial_x L^\pm + (\underbrace{\frac{E\alpha}{C} - \gamma}_{=:nr} - \frac{1}{2} \mp \frac{1}{2})L^\pm = 0 \tag{3.11}$$

For bound state solutions that are normalizable, we require:

$$0 \leq n_r - \frac{1}{2} \mp \frac{1}{2}, \quad n_r \in \mathbb{N} \tag{3.12}$$

As a result, if $n_r = 0$, the only solution for L^+ in eq. (3.11) is the trivial one, $L^+|_{nr=0} = 0$.

The two solutions to eq. (3.11) can be expressed in terms of the Laguerre polynomials L_n^α. By using the normalization condition, we can determine the energy levels:

$$C = \frac{\alpha m}{\sqrt{\alpha^2 + (n_r + \gamma)^2}}, \quad E = \frac{m}{\sqrt{1 + \frac{\alpha^2}{(n_r+\gamma)^2}}}$$

Expanding this expression up to order α^4 yields:

$$\frac{E_{nk}}{m} \approx 1 - \frac{\alpha^2}{2n^2} + \frac{\alpha^4}{2n^3}\left(\frac{3}{4n} - \frac{1}{|k|}\right) \pm O\left(\alpha^6\right) \tag{3.13}$$

where we have reintroduced the principal quantum number $n = n_r + |k|$. The energy difference between a state (nk) and the ground state is given by:

$$\begin{aligned}\Delta E_{nk} &= E_{nk} - E_{1,-1} \\ &= \text{Ryd}\left(1 - \frac{1}{n^2} + \alpha^2\left(\frac{1}{4} + \frac{3}{4n^4} - \frac{1}{|k|n^3}\right)\right) \pm O\left(\alpha^6\right)\end{aligned}$$

Continuing with the calculation, we can write the solutions for $u^\pm$ as follows:

$$\begin{pmatrix} u_{nk}^+ \\ u_{nk}^- \end{pmatrix} = x^\gamma e^{-x/2} \begin{pmatrix} \mathcal{A}_+ x L_{n-|k|-1}^{2\gamma+1} \\ \mathcal{A}_- L_{n-|k|}^{2\gamma-1} \end{pmatrix} \tag{3.14}$$

However, since we obtained this solution by differentiation of eq. (3.8), it is too general. This is evident from the presence of two unknowns, $\mathcal{A}_+$ and $\mathcal{A}_-$. Note that in the special case $n = |k|$, the upper component in eq. (3.14) vanishes, $u^+ = 0$, and only $\mathcal{A}_-$ remains as the only unknown variable. To fix the ratio $\mathcal{F} = \mathcal{A}_-/\mathcal{A}_+$ for $n > |k|$, we plug our solution into eq. (3.8):

$$\partial_x \begin{pmatrix} u^+ \\ u^- \end{pmatrix} = A^{-1} M A \begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \Bigg[-\frac{\gamma}{x} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2C\gamma} \begin{pmatrix} E_\alpha & -(E_k + \gamma m) \\ E_k - \gamma m & -E_\alpha \end{pmatrix} \Bigg] \begin{pmatrix} u^+ \\ u^- \end{pmatrix} \tag{3.15}$$

Considering the case $n = |k|$, we have $u^+ = 0$, so the upper component of eq. (3.15) yields:

$$Ek + \gamma m = 0 \quad \text{for} \quad n_r = 0$$

This result shows that $n_r = 0$ enforces negative values of k, and thus $n > l$, as we had argued above. For $n > |k|$, we may derive the ratio $\mathcal{F}$ from eq. (3.15). Since the individual orders in x have to cancel each other, we consider only terms of the order x^γ in the upper row of eq. (3.15):

$$(\gamma + 1) \begin{pmatrix} n_r + 2\gamma \\ n_r - 1 \end{pmatrix} = -\gamma \begin{pmatrix} n_r + 2\gamma \\ n_r - 1 \end{pmatrix} - \frac{Ek + \gamma m}{2C\gamma} \mathcal{F} \begin{pmatrix} n_r + 2\gamma - 1 \\ n_r \end{pmatrix}$$

$$\implies \mathcal{F} = -\frac{C n_r (n_r + 2\gamma)}{Ek + \gamma m} = \frac{\gamma m - kE}{C} = \frac{1}{\alpha} \left(\gamma \sqrt{\alpha^2 + (n_r + \gamma)^2} - k (n_r + \gamma) \right)$$

Thus, we have derived the relationship between u^+ and u^-, revealing that u^-/u^+ is of the order of α for $k > 0$ and $1/\alpha$ for $k < 0$. In the ladder case, the factor $1/\alpha$ is nullified by a corresponding factor in $\mathcal{A}_+$. To determine

the constants $\mathcal{A}\pm$, we turn to the normalization condition:

$$\begin{aligned}
1 &\stackrel{!}{=} \int d^3x \Psi^\dagger_{nkm} \Psi_{nkm} \\
&= \int_0^\infty dr \left(g_{nk}^2 \int d\Omega \left|\Omega_{km}\right|^2 + f_{nk}^2 \int d\Omega \left|\Omega_{-km}\right|^2 \right) \\
&= \int_0^\infty dr \left(g_{nk}^2 + f_{nk}^2 \right) \\
&= \int_0^\infty dr \left(u_{nk}^{+2} + u_{nk}^{-2} - 2\frac{\alpha}{k} u_{nk}^+ u_{nk}^- \right)
\end{aligned} \tag{3.16}$$

To compute the individual terms, we utilize the generating function of the Laguerre polynomials:

$$\begin{aligned}
\int_0^\infty dr u_{nk}^{+2} &= \frac{\mathcal{A}_+^2}{2C} \int dx x^{2\gamma+2} e^{-x} \left(L_{n_r-1}^{2\gamma+1}(x) \right)^2 \\
&= \frac{\mathcal{A}_+^2}{2C} \int dx x^{2\gamma+2} e^{-x} \cdot \frac{\partial_y^{n_r-1} \partial_z^{n_r-1}}{\left((n_r-1)!\right)^2} \\
&\qquad \cdot \frac{\exp\left[-\frac{xy}{1-y} - \frac{xz}{1-z} \right]}{(1-y)^{2\gamma+2}(1-z)^{2\gamma+2}} \Bigg|_{y=z=0} \\
&= \frac{\mathcal{A}_+^2}{2C} \frac{\Gamma(2\gamma+3)}{\left((n_r-1)!\right)^2} \partial_y^{n_r-1} \partial_z^{n_r-1} \frac{1-y-z+yz}{(1-yz)^{2\gamma+3}} \Bigg|_{y=z=0} \\
&= \frac{\mathcal{A}_+^2}{C} \frac{(n_r+2\gamma+2) + (n_r-1)\,\Gamma\left(n_r+2\gamma+1\right)}{(n_r-1)!} \\
\int_0^\infty dr u_{nk}^{-2} &= \frac{\mathcal{A}_-^2}{C} \frac{(n_r+\gamma)\,\Gamma\left(n_r+2\gamma+1\right)}{n_r!} \\
-2\frac{\alpha}{k} \int_0^\infty dr u_{nk}^+ u_{nk}^- &= \frac{2\alpha}{k} \frac{\mathcal{A}_+ \mathcal{A} + 2\gamma)}{C} \frac{\Gamma\left(n_r+2\gamma+1\right)}{(n_r-1)!}
\end{aligned} \tag{3.17}$$

Using these results in eq. (3.16) implies for $n_r > 0$

$$\begin{aligned}
1 \stackrel{!}{=} \frac{\mathcal{A}_+^2}{C} \frac{(n_r+\gamma)\,\Gamma\left(n_r+2\gamma+1\right)}{(n_r-1)!} \\
\cdot \left(1 + \frac{\mathcal{F}^2}{n_r\left(n_r+2\gamma\right)} + 2\frac{\alpha}{k} \frac{\mathcal{F}}{n_r+\gamma} \right)
\end{aligned}$$

so that we finally obtain the normalization constants as

$$\mathcal{A}_+ = \sqrt{\frac{C}{n_r+\gamma}\frac{(n_r-1)!}{\Gamma(n_r+2\gamma+1)}\frac{1}{2}\left(\left(\frac{Ek}{\gamma m}\right)^2+\frac{Ek}{\gamma m}\right)}$$

$$= \sqrt{\frac{C(n_r-1)!}{\Gamma(n_r+2\gamma+1)}\frac{n_r+\gamma+\frac{\gamma}{k}\sqrt{\alpha^2+(n_r+\gamma)^2}}{2\frac{\gamma^2}{k^2}\left(\alpha^2+(n_r+\gamma)^2\right)}} \tag{3.18}$$

$$\mathcal{A}_- = \sqrt{\frac{Cn_r!}{\Gamma(n_r+2\gamma)}\frac{n_r+\gamma-\frac{\gamma}{k}\sqrt{\alpha^2+(n_r+\gamma)^2}}{2\frac{\gamma^2}{k^2}\left(\alpha^2+(n_r+\gamma)^2\right)}} \tag{3.19}$$

For $n = |k|$ we simply set $\mathcal{A}_+ = 0$ while the expression for $\mathcal{A}_-$remains valid with $n_r = 0$.

To finally relativistic wave function of Hydrogen which reads as:

$$\Psi_{nkm}(r,\theta,\varphi) = \frac{1}{r}\begin{pmatrix} g_{nk}(r)\Omega_{km}(\theta,\varphi) \\ if_{nk}(r)\Omega_{-km}(\theta,\varphi) \end{pmatrix} \tag{3.20}$$

i. Chronological Consequences in G.R.

The integration of General Relativity and quantum mechanics has yielded profound revelations regarding the nature of time. Quantum mechanics has uncovered a method to extract time using the wave function, thereby revealing the curvature of time itself. This discovery challenges the conventional linear and continuous perception of time, prompting further investigation into the complex relationship between these foundational theories.

The implications of time's curvature have significant consequences for string theory, which seeks to unify all fundamental forces, including gravity. The non-linear behavior of time presents a challenge to the assumptions within string theory about the progression of time and its role in the

unification of forces. Consequently, researchers are prompted to reevaluate how time is incorporated into the framework of string theory, potentially leading to a more comprehensive understanding of temporal dynamics within the theory.

The non-linear behavior of time also raises intriguing questions about temporal symmetry and its potential breaking. Symmetry is a fundamental concept in modern theoretical physics, governing the formulation and comprehension of the laws that govern the universe. However, the revelation of time's curvature challenges the notion that temporal symmetry holds universally and at all scales. This departure from traditional symmetrical concepts of time may shed light on various asymmetries observed in physical processes, including the arrow of time, which denotes the direction of increasing entropy.

4. From Equilibrium to Asymmetry