REVIEW OF ONE DIMENSION PROLATE SPHEROIDAL WAVE FUNCTIONS (PSWFS)

HAMED BAGHAL GHAFFARI

Abstract. In the present letter, I will review the one dimension prolate spheroidal wave functions (PSWFs) as the eigenfunctions of generalized Legendre differential equation operator. Thus, I will explain a famous precise method of their computations. Then, I will show that they are the eigenfunctions of finite Fourier transformations. Also, the most critical feature of PSWFs, spectral concentration, will be given.

1. Introduction

In 1961, the prolate spheroidal wave functions (known also as Slepian functions) has been introduced as the eigenfunction (bell labs paper) of the finite Fourier transformations. In that paper, also, some properties of PSWFs such as spectral concentration problem has been investigated. In this letter, I have tried to explain the construction of PSWFs and their properties in a simple language. A reference that I have used a lot is [3].

The orgnization of this letter is as follows. In section 2, I give the necessary preliminaries. The section 3 will be about the computation of PSWFs [1]. In section 4, I will give the proof saying that the PSWFs are the eigenfunctions of the finite Fourier transformation. Finally, in last section, we will prove the spectral concentration property[6].

2. Preliminaries

Let $L^2(\mathbb{R})$ be the class of integrable, complex-valued functions on the real line for which

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt < \infty. \tag{2.1}$$

In analogous manner, we denote by $L^2[-1,1]$ the class of all complex-valued functions $f(t)$ defined in $-1 \leq t \leq 1$ and integrable in absolute square in the interval $[-1,1]$.

Functions in $L^1(\mathbb{R})$ possess Fourier transforms. We write

$$\mathcal{F}^{-1}f(t) = \check{f}(t) = \int_{-\infty}^{\infty} f(\omega) e^{2\pi\omega t i} d\omega, \tag{2.2}$$

$$\mathcal{F}f(\omega) = \hat{f}(t) = \int_{-\infty}^{\infty} f(t) e^{-2\pi\omega t i} dt, \tag{2.3}$$

whenever the integrals exist.

Remark 2.1. The space $L^2(\mathbb{R})$ consists of those complex-valued functions that

$$\int_{\mathbb{R}} |f(x)|^2 dx < \infty,$$

is a Hilbert space with the definition of following inner-product

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}dx. \tag{2.4}$$

Proposition 2.2. *Let $f \in L^2(\mathbb{R})$. If the function is even and real, then its Fourier transformation will be even and real-valued. While if the function is odd and real, then its Fourier transformation will be odd and purely imaginary-valued.*

Definition 2.3. Let $T : \mathcal{H} \to \mathcal{H}$ be a linear bounded operator. Then we say that T is self-adjoint operator if $T = T^*$. That is,

$$\langle Tf, g\rangle = \langle f, Tg\rangle,$$

for any $f, g \in \mathcal{H}$.

Definition 2.4. Let $\mathcal{H}$ be a Hilbert space and also let B_1 is the closed unit ball of $\mathcal{H}$. A linear operator T on $\mathcal{H}$ is said to be compact if $T(B_1)$ is compact in the norm topology of $\mathcal{H}$.

Proposition 2.5. *If T is a compact, self-adjoint (Hermitian) operator on a separable Hilbert space H, then there exists a sequence of real numbers $\{\lambda_n\}$ tending to 0 and an orthonormal set $\{e_n\}$ in H such that*

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n\rangle e_n$$

for all $x \in H$.

Proof. For the proof see chapter one of [7]. □

Theorem 2.6. *An integral operator T on $L^2(X, d\mu)$ defined by $Tf(x) = \int_X K(x, y)f(y)d\mu(y)$, is compact if the function $K(x, y) \in L^2(X \times X, d\mu \times d\mu)$.*

Proof. For the proof see the chapter three of [7] □

Now we list some of the properties of the Fourier transform in the following proposition:

Proposition 2.7. *Let $f, g \in L^2(\mathbb{R})$. Then the Fourier transform has the following properties*

1 $\langle f, g\rangle = \langle \mathcal{F}f, \mathcal{F}g\rangle$ *Plancherl's identity*
2 $\|f\|^2 = \|\mathcal{F}f\|^2$ *Parseval's identity*
3 $\mathcal{F}(af + g) = a\mathcal{F}(f) + \mathcal{F}(g)$ *linearity*

Proof. For the proof see chapter nine of [4]. □

We may rewrite the first property as follows

$$\int_{-\infty}^{+\infty} f(t)\overline{g(t)}dt = \int_{-\infty}^{+\infty} \hat{f}(\omega)\overline{\hat{g}(\omega)}d\omega. \tag{2.5}$$

Definition 2.8. We denote by PW_c, the subclass of $L^2(\mathbb{R})$, consisting of those functions, $f(t)$, whose Fourier transforms, $\mathcal{F}f(\omega)$, vanish if $|\omega| > c$, where c is a positive constant. Every member, $f(t)$, of PW_c can be written as a finite Fourier transform of a function integrable in absolute square:

$$f(t) = \int_{-c}^{c} \mathcal{F}f(\omega)e^{2\pi\omega it}d\omega. \tag{2.6}$$

Functions in PW_c are called *bandlimited* and PW_c will be referred to as the class of bandlimited functions or the Paley-Wiener space. From (2.6), we can conclude that the functions of PW_c are entire functions of complex variable.

3. Construction of the Prolate Spheroidal Wave Function

We consider the following equation when c is a real number and for any $t \in [-1, 1]$

$$(1-t)^2\frac{d^2u}{dt^2} - 2t\frac{du}{dt} + (\chi_n(c) - 4\pi^2c^2t^2)u = 0, \tag{3.1}$$

where $\chi_n(c)$ is the real number and is the eigenvalue of the operator L_c, i.e.,

$$L_c(u) = \chi_n u; \quad L_c(u)(t) = (t^2-1)\frac{d^2u}{dt^2} + 2t\frac{du}{dt} + 4\pi^2c^2t^2u. \tag{3.2}$$

When $c = 0$ (3.1), becomes the classic Legendre differential equation.

Definition 3.1. The eigenfunctions of the operator L_c, defined at (3.2), are called **Prolate Spheroidal wave Functions.**

Definition 3.2. The nth Legendre polynomial $P_n(t)$ is a solution of (3.1) when $c = 0$:

$$\frac{d}{dt}(t^2-1)\frac{dP_n}{dt} = \chi_n(0)P_n(t) \quad \text{or} \quad (t^2-1)\frac{d^2P_n}{dt^2} + 2t\frac{dP_n}{dt} - \chi_n(0)P_n(t) = 0. \tag{3.3}$$

The constant $\chi_n(0) = n(n+1)$. Among the algebraic relationships satisfied by the Legendre functions, the most important for us will be Bonnet's recurrence formula

$$(n+1)P_{n+1}(t) - (2n+1)tP_n(t) + nP_{n-1}(t) = 0. \tag{3.4}$$

This formula is useful in computing the eigenvalues $\chi_n(c)$ in (3.1).

Suppose that $\overline{\phi_n^c}$ is the nth $L^2[-1,1]$-normalized solution of the eigenvalue equation of (3.2). Since the Legendre polynomials $\{\overline{P_n}\}_{n=0}^{\infty}$ where $\overline{P_n} = \sqrt{n+\frac{1}{2}}P_n$ form an orthonormal basis for $L^2[-1,1]$, each $\overline{\phi_n^c}$ admits an expression of the form

$$\overline{\phi_n^c} = \sum_{m=0}^{\infty} \beta_{nm}\overline{P_m}. \tag{3.5}$$

By equation (3.3),

$$\chi_n\overline{\phi_n^c} = L_c(\overline{\phi_n^c}) = \sum_{m=0}^{\infty} \beta_{nm}L_c(\overline{P_m}) = \sum_{m=0}^{\infty} \beta_{nm}[m(m+1)\overline{P_m} + 4\pi^2c^2x^2\overline{P_m}] \tag{3.6}$$

However, two applications of the recurrence Bonnet's formula (3.4) will give

$$\begin{aligned}
x^2\overline{P_n(x)} &= (\frac{n+1}{2n+1})(\frac{\sqrt{n+\frac{1}{2}}}{\sqrt{n+\frac{5}{2}}})(\frac{n+2}{2n+3})\overline{P_{n+2}(x)} \\
&+ [(\frac{n+1}{2n+1})(\frac{n+1}{2n+3}) + (\frac{n}{2n+1})(\frac{n}{2n-1})]\overline{P_n(x)} \\
&+ (\frac{n}{2n+1})(\frac{\sqrt{n+\frac{1}{2}}}{\sqrt{n-\frac{3}{2}}})(\frac{n-1}{2n-1})\overline{P_{n-2}(x)}
\end{aligned}$$

Substituting this into (3.6) gives

$$\begin{aligned}
\chi_n \sum_{m=0}^{\infty} \beta_{nm}\overline{P_m} &= \sum_{m=0}^{\infty} \beta_{nm}[m(m+1)\overline{P_m} \\
&+ 4\pi^2c^2(\frac{m+1}{2m+1})(\frac{\sqrt{m+\frac{1}{2}}}{\sqrt{m+\frac{5}{2}}})(\frac{m+2}{2m+3})\overline{P_{m+2}(x)} \\
&+ 4\pi^2c^2[(\frac{m+1}{2m+1})(\frac{m+1}{2m+3}) + (\frac{m}{2m+1})(\frac{m}{2m-1})]\overline{P_m(x)} \\
&+ 4\pi^2c^2(\frac{m}{2m+1})(\frac{\sqrt{m+\frac{1}{2}}}{\sqrt{m-\frac{3}{2}}})(\frac{m-1}{2m-1})\overline{P_{m-2}(x)}]
\end{aligned}$$

so we have that

$$\begin{aligned}
\chi_n \sum_{m=0}^{\infty} \beta_{nm}\overline{P_m} &= \sum_{m=0}^{\infty} [\beta_{n(m-2)}4\pi^2c^2(\frac{m-1}{2m-3})(\frac{\sqrt{m-\frac{3}{2}}}{\sqrt{m+\frac{1}{2}}})(\frac{m}{2m-1}) \\
&+ \beta_{nm}(m(m+1) + 4\pi^2c^2[(\frac{m+1}{2m+1})(\frac{m+1}{2m+3}) + (\frac{m}{2m+1})(\frac{m}{2m-1})]) \\
&+ \beta_{n(m+2)}4\pi^2c^2(\frac{m+2}{2m+5})(\frac{\sqrt{m+\frac{5}{2}}}{\sqrt{m+\frac{1}{2}}})(\frac{m+1}{2m+3})]\overline{P_m(x)},
\end{aligned}$$

with the understanding that $\beta_{nm} = 0$ if $m < 0$. The orthogonality of the Legendre polynomials enables us to equate the coefficients on both sides of this equation to obtain the following recurrence formula for $\{\beta_{nm}\}_{m=0}^{\infty}$:

$$\begin{aligned}
&\beta_{n(m-2)}4\pi^2c^2(\frac{m-1}{2m-3})(\frac{\sqrt{m-\frac{3}{2}}}{\sqrt{m+\frac{1}{2}}})(\frac{m}{2m-1}) \\
+ \; &\beta_{nm}(m(m+1) + 4\pi^2c^2\frac{2m^2+2m-1}{(2m+3)(2m-1)} - \chi_n) \\
+ \; &\beta_{n(m+2)}4\pi^2c^2\frac{(m+1)(m+2)}{(2m+3)\sqrt{(2m+5)(2m+1)}} = 0.
\end{aligned}$$

Let $A = \{a_{mk}\}_{m,k=0}^{\infty}$, be the doubly infinite tridiagonal matrix with nonzero elements

$$a_{m,k} = \begin{cases} 4\pi^2 c^2 \frac{m(m-1)}{(2m-1)\sqrt{(2m-3)(2m+1)}}, & \text{if } m \geq 2,\ k = m-2 \\ m(m+1) + 4\pi^2 c^2 \frac{2m^2+2m-1}{(2m+3)(2m-1)}, & \text{if } m = k \geq 0 \\ 4\pi^2 c^2 \frac{(m+2)(m+1)}{(2m+3)\sqrt{(2m+5)(2m+1)}}, & \text{if } m \geq 0,\ k = m+2 \\ 0 \qquad else. \end{cases}$$

Therefore, $L_c\overline{P_m} = \sum_{k=0}^{\infty} a_{mk}\overline{P_m}$. As a result

$$\chi_n \sum_{m=0}^{\infty} \beta_{nm}\overline{P_m} = \chi_n\overline{\phi_n^c} = \sum_{k=0}^{\infty}(\sum_{m=0}^{\infty} \beta_{nm}a_{mk})\overline{P_k}. \tag{3.7}$$

For a fixed $n \geq 0$, let $b_n = (\beta_{n0}, \beta_{n1}, \cdots)^T$. Therefore,

$$A^T b_n = \chi_n b_n. \tag{3.8}$$

By truncating the sum at (3.5), after N_{tr} terms, we may write

$$(A^{tr})^T b_n^{tr} = \chi_n^{tr} b_n^{tr}. \tag{3.9}$$

So this is an eigenvalue problem which after solving, we will get the approximations of the β_{nm}, and as a result we will know the prolates $\overline{\phi_n^c}$ for $n = 0, 1, \cdots, N$. The graph of PSWFs have been plotted for $n = 0$, $n = 3$, $n = 10$, in Figure 1.

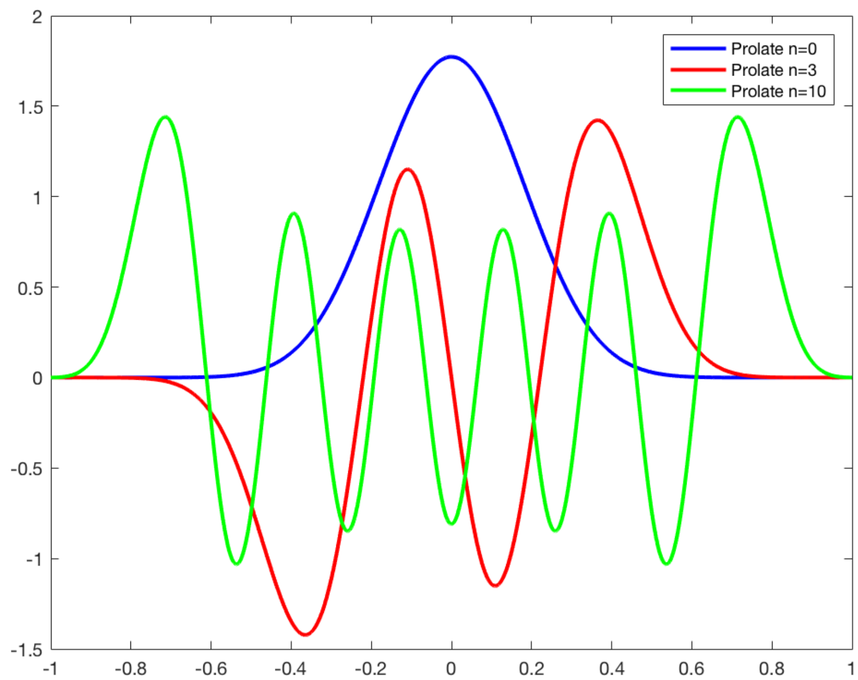

FIGURE 1. PSWFs for $c = 5$. Prolates on $[-1, 1]$ for $n = 0$, 3, 10 having n zeros in $[-1, 1]$.

4. PSWFs as a finite Fourier transformation

Definition 4.1. Let $f \in L^2(\mathbb{R})$, then we define the bandlimiting operator $P_c : L^2(\mathbb{R}) \to PW_c$ as

$$P_c f(t) = \int_{-c}^{c} \hat{f}(\omega) e^{2\pi \omega i t} d\omega = (\chi_{[-c,c]}(\omega)\hat{f}(\omega))^{\vee}(t). \tag{4.1}$$

In fact, $P_c f(t)$ is the bandlimited version of $f(t)$. A useful representation of P_c may be obtained as follows

$$\begin{aligned} P_c f(t) = \int_{-c}^{c} \hat{f}(\xi) e^{2\pi i t \xi} d\xi &= \int_{-c}^{c} \int_{-\infty}^{\infty} f(s) e^{-2\pi i s \xi} ds\, e^{2\pi i t \xi} d\xi \\ &= \int_{-\infty}^{\infty} f(s) \frac{\sin(2\pi c(t-s))}{\pi(t-s)} ds. \end{aligned}$$

We denote by $\mathcal{D}_T$ the subclass of functions f of $L^2(\mathbb{R})$ each of which vanishes for $|t| > T$, where T is a positive constant. Members of $\mathcal{D}_T$ are called T-timelimited and $\mathcal{D}_1 = \mathcal{D}$ will be referred to as the class of timelimited functions.

Definition 4.2. Let $f \in L^2(\mathbb{R})$, then we define the timelimiting operator $Q_T : L^2(\mathbb{R}) \to L^2[-T, +T]$ as

$$Q_T f(t) = \begin{cases} f(t) & |t| \leq T \\ 0 & |t| > T \end{cases} \tag{4.2}$$

When $T = 1$, then $Q_T f(t) = Qf(t)$. By the definitions of operators P_c and Q, we may write

$$P_c Q f(t) = \int_{-1}^{1} f(s) \frac{\sin 2\pi c(t-s)}{\pi(t-s)} ds.$$

Definition 4.3. Let $f \in L^2[-1, 1]$. Then we define the operator $\mathcal{G}_c : L^2[-1, 1] \to L^2[-1, 1]$, as follows

$$\mathcal{G}_c f(t) = \chi_{[-1,1]}(t) \int_{-1}^{1} f(s) e^{2\pi i c s t} ds = Q \delta_c \mathcal{F} Q f(t), \tag{4.3}$$

where δ_c is dilation operator i.e., $\delta_c f(x) = f(cx)$. Later, we will give a direct calculation of the $\mathcal{G}_c(\overline{\phi_n^c})$. Also, the (4) and (4) prove that the (PSWFs) are the eigenfunctions of the
By this, we have that

$$\mathcal{G}_c^* : L^2[-1, 1] \to L^2[-1, 1].$$

For the calculation of $\mathcal{G}_c^*$, we have that if $g \in PW_c,\ f \in L^2[-1, 1]$ we have

$$\mathcal{G}_c^* f(t) = \chi_{[-1,1]}(t) \int_{-1}^{1} f(s) e^{-2\pi i c s t} ds. \tag{4.4}$$

Now we are calculating the combination of Q and P_c so we have

$$Q P_c f(t) = \chi_{[-1,1]}(t) \int_{-1}^{1} f(u) \frac{\sin(2\pi c(u-t))}{\pi(u-t)} du. \tag{4.5}$$

Also we have that,

$$\begin{aligned}
\mathcal{G}_c^*\mathcal{G}_c f(t) &= \chi_{[-1,1]}(t)\int_{-1}^{1}\mathcal{G}_c f(s)e^{-2\pi icst}ds = \chi_{[-1,1]}(t)\int_{-1}^{1}\left(\chi_{[-1,1]}(s)\int_{-1}^{1}f(u)e^{2\pi icus}du\right)e^{-2\pi icst}ds \\
&= \chi_{[-1,1]}(t)\int_{-1}^{1}f(u)\left(\int_{-1}^{1}e^{2\pi ics(u-t)}ds\right)du = \chi_{[-1,1]}(t)\int_{-1}^{1}f(u)\frac{\sin(2\pi c(u-t))}{\pi c(u-t)}du.
\end{aligned}$$

Therefore, if $f \in L^2[-1,1]$, then

$$\mathcal{G}_c^*\mathcal{G}_c f(t) = \frac{1}{c}QP_c f(t). \tag{4.6}$$

Now we prove that prolate spheroidal wave functions are bandlimited functions. In order to do that firstly, we will prove the commutativity of the L_c and F_c operators.

Theorem 4.4. *(Lucky Accident) The operators L_c and $\mathcal{G}_c$ on $L^2[-1,1]$, commute.*

Proof.

$$\begin{aligned}
\mathcal{G}_c L_c f(t) &= \chi_{[-1,1]}(t)\int_{-1}^{1}L_c f(s)e^{2\pi icst}ds \\
&= \chi_{[-1,1]}(t)\int_{-1}^{1}L_c(e^{2\pi icst})f(s)ds \\
&= \chi_{[-1,1]}(t)\int_{-1}^{1}[(s^2-1)\frac{d^2}{ds^2}(e^{2\pi icst}) + 2s\frac{d}{ds}(e^{2\pi icst}) + (4\pi^2c^2s^2)(e^{2\pi icst})]f(s)ds \\
&= \chi_{[-1,1]}(t)\int_{-1}^{1}[(s^2-1)(-4\pi^2c^2t^2)e^{2\pi icst} + 2t\frac{d}{dt}(e^{2\pi icst}) - \frac{d^2}{dt^2}(e^{2\pi icst})]f(s)ds \\
&= \chi_{[-1,1]}(t)\int_{-1}^{1}[(-4\pi^2c^2s^2)(-e^{2\pi icst}) + t^2\frac{d^2}{dt^2}(e^{2\pi icst}) \\
&+ 2t\frac{d}{dt}(e^{2\pi icst}) - \frac{d^2}{dt^2}(e^{2\pi icst})]f(s)ds \\
&= \chi_{[-1,1]}(t)\int_{-1}^{1}[(t^2-1)\frac{d^2}{dt^2}(e^{2\pi icst}) + 2t\frac{d}{dt}(e^{2\pi icst}) + 4\pi^2c^2t^2(e^{2\pi icst})]f(s)ds \\
&= (t^2-1)\frac{d^2}{dt^2}\chi_{[-1,1]}(t)\int_{-1}^{1}e^{2\pi icst}f(s)ds + 2t\frac{d}{dt}\chi_{[-1,1]}(t)\int_{-1}^{1}e^{2\pi icst}f(s)ds \\
&+ 4\pi^2c^2t^2\chi_{[-1,1]}(t)\int_{-1}^{1}e^{2\pi icst}f(s)ds \\
&= L_c\mathcal{G}_c f(t)
\end{aligned}$$

□

Since L_c commutes with both $\mathcal{G}_c$ and $\mathcal{G}_c^*$ and L_c is self-adjoint on $L^2[-1,1]$. Therefore, $\mathcal{G}\mathcal{G}_c^*\frac{1}{c}QP_c$ is also self-adjoint.

Lemma 4.5. *The operator QP_c is compact.*

Proof. According to (2.6) we just need to check that $\chi_{[-1,1]}(t)\frac{\sin(2\pi(u-t))}{\pi c(u-t)} \in L^2(\mathbb{R}\times\mathbb{R}, du\times dt)$. So we have that

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\chi_{[-1,1]}(t)\frac{\sin^2(2\pi c(u-t))}{\pi^2(u-t)^2}dt\,du &= \int_{-\infty}^{\infty}\chi_{[-1,1]}(t)(\int_{-\infty}^{\infty}\frac{\sin^2(2\pi c(u-t))}{\pi^2(u-t)^2}dt)du \\
&= \int_{-\infty}^{\infty}\chi_{[-1,1]}(t)dt\int_{-\infty}^{\infty}|\hat{\chi}_{[-c,c]}(u)|^2du \\
&= \int_{-\infty}^{\infty}\chi_{[-1,1]}(t)dt\int_{-\infty}^{\infty}|\chi_{[-c,c]}(u)|^2du \\
&= 4c<\infty.
\end{aligned}$$

□

Now, we are stating a fundamental theorem about commutative operators,

Theorem 4.6. *If two self-adjoint (Hermitian) operators, and one of them is compact operator, acting on a Hilbert space to the same Hilbert space, commute, then they share common eigenbasis.*

Remark 4.7. According to the (4.6), L_c and $\mathcal{G}_c^*\mathcal{G}_c$ have the same eigenfunctions. Therefore, L_c and QP_c have the same eigenfunctions.

Now let ϕ be an eigenfunction of QP_c with eigenvalue λ, i.e.,

$$QP_c\phi=\lambda\phi.$$

Let's take $\varphi=P_c\phi$. Then $P_cQ\varphi=P_cQP_c\phi=P_c(\lambda\phi)=\lambda\varphi$, i.e., φ is an eigenfunction of P_cQ with the same eigenvalue. Now, we will prove that ϕ is the eigenfunction $\mathcal{G}_c$. But before proceeding further we need to present an operator $F_c: PW_c\to PW_c$ as follows,

$$F_c(f(t))=\int_{-1}^{1}f(s)e^{2\pi icst}ds. \tag{4.7}$$

Assume that ψ_j is the eigenfunction P_cQ, i.e., $P_cQ\psi_j=\lambda_j\psi_j$ then we have that

$$\begin{aligned}
\lambda_j\hat{\psi}_j(c\xi) &= (P_cQ\psi_j)^\wedge(c\xi) \\
&= \int_{-\infty}^{\infty}e^{-2\pi ixc\xi}(P_cQ\psi_j)(x)dx \\
&= \int_{-\infty}^{\infty}e^{-2\pi ixc\xi}(\int_{-1}^{1}\psi_j(t)\frac{\sin 2\pi c(x-t)}{\pi(x-t)}dt)dx \\
&= \int_{-1}^{1}\psi_j(t)(\int_{-\infty}^{\infty}e^{-2\pi ixc\xi}\frac{\sin 2\pi c(x-t)}{\pi(x-t)}dx)dt \\
&= \int_{-1}^{1}\psi_j(t)e^{-2\pi itc\xi}(\int_{-\infty}^{\infty}e^{-2\pi ixc\xi}\frac{\sin 2\pi cx}{\pi x}dx)dt \\
&= \chi_{[-1,1]}(\xi)\int_{-1}^{1}\psi_j(t)e^{-2\pi itc\xi}dt \\
&= \chi_{[-1,1]}(\xi)\int_{-\infty}^{\infty}\psi_j(t)\overline{\chi_{[-1,1]}(t)e^{2\pi itc\xi}}dt
\end{aligned}$$

$$\begin{aligned}
&= \chi_{[-1,1]}(\xi) \int_{-\infty}^{\infty} \hat{\psi}_j(t) \frac{\sin 2\pi(c\xi - t)}{\pi(c\xi - t)} dt \\
&= \chi_{[-1,1]}(\xi) \int_{-\infty}^{\infty} \hat{\psi}_j(ct) \frac{\sin 2\pi c(\xi - t)}{\pi(\xi - t)} dt.
\end{aligned}$$

Therefore,

$$QP_c(\delta_c \hat{\psi}_j)(\xi) = \lambda_j(\delta_c \hat{\psi}_j)(\xi), \tag{4.8}$$

where $\delta_c f(x) = f(cx)$. Hence $\delta_c \hat{\psi}_j$ and $Q\psi_j$ are the eigenfunctions of QP_c, with the same eigenvalue λ_i. But according to the [6] the eigenvalues of the QP_c are non-degenerate, as a result we have that

$$\delta_c \hat{\psi}_j = \alpha_j Q\psi_j \quad \text{for some}\, \alpha_j \in \mathbb{C}.$$

Now, we want to calculate the α_j. From [6], and (2.2) it can be concluded that every ψ_j can be chosen to be real-valued, and if j is even then ψ_j is even and $\hat{\psi}_j$ is real-valued, while if j is odd then ψ_j is odd and $\hat{\psi}_j$ is purely imaginary. Hence we may write $\hat{\psi}_j(c\xi) = i^j A_j Q\psi_j(\xi)$, where A_j is a real number. Now by having normalized ψ_j as an eigenfunctions of the operator $P_c Q$

$$1 = \int_{-1}^{1} |\hat{\psi}_j(\xi)|^2 d\xi = c \int_{-1}^{1} |\hat{\psi}_j(c\xi)|^2 d\xi = A_j^2 c \int_{-1}^{1} |\psi_j(c\xi)|^2 d\xi = A_j^2 c\lambda_j.$$

$$\hat{\psi}_j(ct) = \frac{\epsilon_j (i)^j Q\psi_j(t)}{\sqrt{c\lambda_j}}, \tag{4.9}$$

where ψ_j is the eigenfunction of the QP_c and $\epsilon_j = \pm 1$. By (4.9)

$$\frac{\epsilon_j i^j}{\sqrt{c\lambda_j}} \int_{-1}^{1} \psi_j(\xi) e^{2\pi i \xi t} d\xi = \frac{1}{c} \psi_j(\frac{t}{c}).$$

Therefore,

$$F_c \psi_j = \sqrt{\frac{\lambda_j}{c}} i^{-j} \epsilon_j \psi_j,$$

i.e., ψ_j is also the eigenfunction of F_c. Now, we know that ϕ is an eigenfunction of QP_c with the eigenvalue λ and since $\lambda\phi = Q\varphi$. So we can write

$$\begin{aligned}
\mathcal{G}_c \phi(t) &= \chi_{[-1,1]}(t) \int_{-1}^{1} \phi(s) e^{2\pi cst} ds \\
&= \frac{\chi_{[-1,1]}(t)}{\lambda} \int_{-1}^{1} \varphi(s) e^{2\pi icst} ds \\
&= \frac{\chi_{[-1,1]}(t)}{\lambda} F_c \varphi(t) \\
&= \frac{\alpha}{\lambda} \chi_{[-1,1]}(t) \varphi(t) \quad \alpha\, \text{is the associated eigenvalue for}\, \varphi. \\
&= \frac{\alpha}{\lambda} \lambda \phi(t) = \alpha \phi(t).
\end{aligned}$$

So ϕ is the eigenfunction of $\mathcal{G}_c$. So we summarize all things that we have obtained as follows,

Theorem 4.8. *The eigenbasis that we found for L_c operator acting on $L^2[-1,1]$, are also the eigenbasis of the operator $\mathcal{G}_c$ which has been defined at (4.3). Therefore, we proved that the prolate spheroidal wave functions are time restrictions of bandlimited signals.*

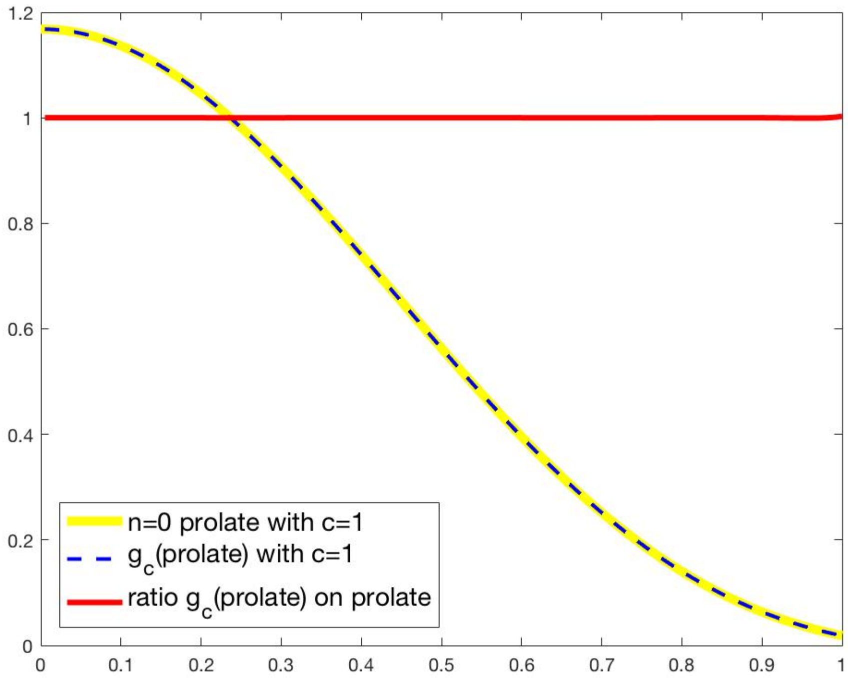

FIGURE 2. PSWFs for $c = 1.$ on $[-1,1]$ for $n = 0$ and the $\mathcal{G}_c$ of the PSWFs on $[-1,1]$.

By this calculation, we are able to see the eigenvalue of $\mathcal{G}_c$ numerically. Let's take the $\overline{P_n(x)}$ as the Legendre polynomials that we calculated the PSWFS in terms of them. So here we will calculate the $\mathcal{G}_c\overline{P_n(t)}$, and then after, we will use our coefficient data base and thus we will be able to calculate $\mathcal{G}_c\psi_n^c(t)$. We use the equation 7.321 of page 797 from [5], so we have that

$$\mathcal{G}_c\overline{P_n(t)} = \sqrt{(n+\frac{1}{2})}\; i^n \frac{J_{n+\frac{1}{2}}(2\pi ct)}{\sqrt{ct}}. \tag{4.10}$$

But, it seems that we might have some problems in calculations eigenvalues at 0. We need to separate the even and odd calculations. So for even part we have that

$$\mu_{2m}^c = \frac{\mathcal{G}_c\overline{\psi_{2m}^c(x)}}{\overline{\psi_{2m}^c(x)}} = \frac{\sum\limits_{n=0}^{\infty} a_{n(2m)}\mathcal{G}_c(\overline{P_{2n}(x)})}{\sum\limits_{n=0}^{\infty} a_{n(2m)}\overline{P_{2n}(x)}} = \frac{\sum\limits_{n=0}^{\infty} a_{n(2m)}\left[\sqrt{(2n+\frac{1}{2})}i^{2n}\frac{J_{2n+\frac{1}{2}}(2\pi ct)}{\sqrt{ct}}\right]}{\sum\limits_{n=0}^{\infty} a_{n(2m)}\overline{P_{2n}(x)}}.$$

Now we use the following expansion series for Bessel functions [2]

$$J_\nu(z) = \sum_{k=0}^{\infty}\frac{(-1)^k z^{\nu+2k}}{2^{\nu+2k}k!\Gamma(\nu+k+1)}.$$

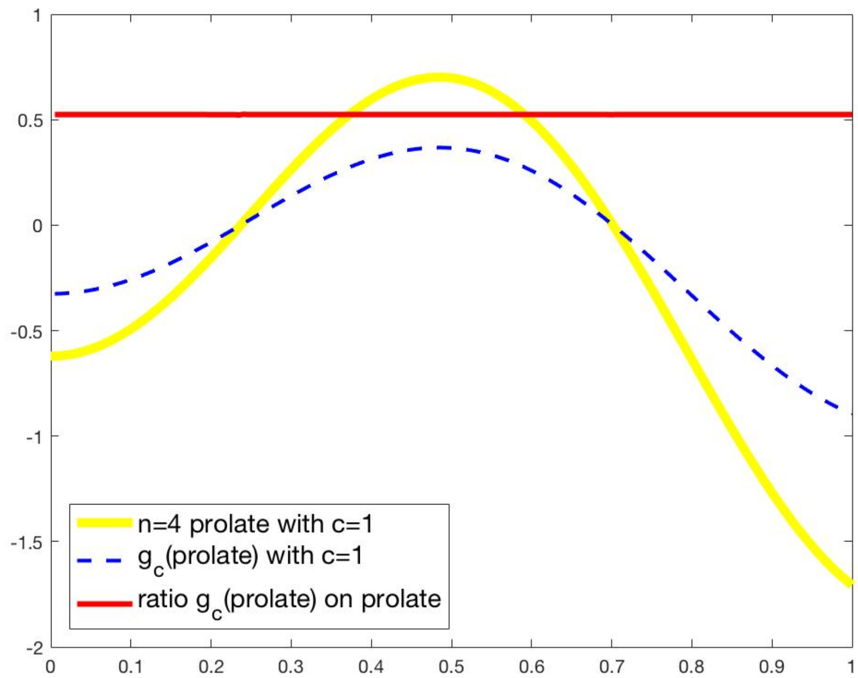

FIGURE 3. PSWFs for $c = 1.$ on $[-1, 1]$ for $n = 4$ and the $\mathcal{G}_c$ of the PSWFs on $[-1, 1]$.

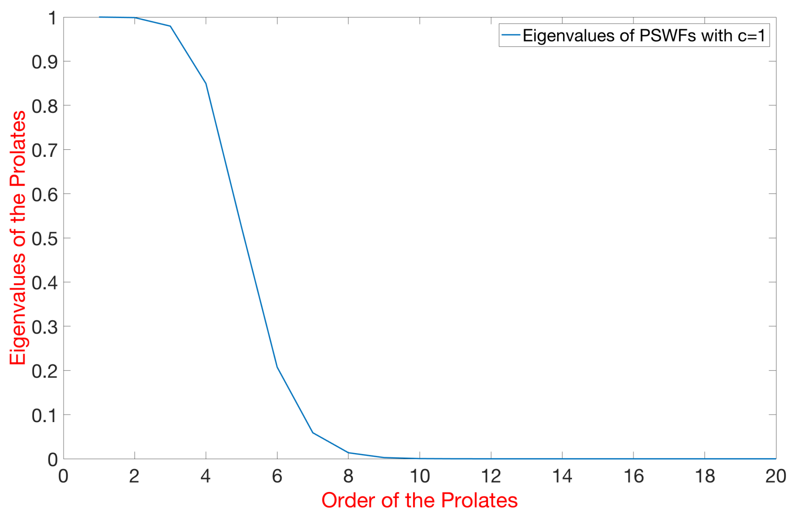

FIGURE 4. PSWFs for $c = 1.$ on $[-1, 1]$ for $n = 4$ and the $\mathcal{G}_c$ of the PSWFs on $[-1, 1]$.

So we have that

$$\mu_{2m}^c = \frac{\sum_{n=0}^{\infty} a_{n(2m)} \lim_{t\to 0} \Big[\sqrt{(2n+\frac{1}{2})}\frac{i^{2n}}{\sqrt{ct}}\Big[\sum_{k=0}^{\infty} \frac{(-1)^k(2\pi ct)^{2k+(2n+\frac{1}{2})}}{k!\Gamma(k+(2n+\frac{1}{2})+1)2^{2k+(2n+\frac{1}{2})}}\Big]\Big]}{\sum_{n=0}^{\infty} a_{n(2m)}\overline{P_{2n}(0)}}$$

$$= \frac{a_{0(2m)}\Big[\frac{\sqrt{\frac{1}{2}}(\pi c)^{\frac{1}{2}}}{\sqrt{c}\Gamma(\frac{3}{2})}\Big]}{\sum_{n=0}^{\infty} a_{n(2m)}\overline{P_{2n}(0)}} = \frac{a_{0(2m)}\sqrt{2}}{\sum_{n=0}^{\infty} a_{n(2m)}\overline{P_{2n}(0)}}.$$

Now we can do the same calculation for the odd PSWFs.

$$\begin{aligned}
\mu^c_{2m+1} &= \frac{\mathcal{G}_c\overline{\psi^c_{2m+1}(x)}}{\overline{\psi^c_{2m+1}(x)}} = \frac{\sum\limits_{n=0}^{\infty} a_{(2n+1)(2m+1)}\mathcal{G}_c(\overline{P_{2n+1}(x)})}{\sum\limits_{n=0}^{\infty} a_{(2n+1)(2m+1)}\overline{P_{2n+1}(x)}} \\
&= \frac{\sum\limits_{n=0}^{\infty} a_{(2n+1)(2m+1)}\big[\sqrt{(2n+\frac{1}{2})}i^{2n+1}\frac{J_{2n+\frac{1}{2}}(2\pi ct)}{\sqrt{ct}}\big]}{\sum\limits_{n=0}^{\infty} a_{(2n+1)(2m+1)}\overline{P_{2n+1}(x)}} \\
&= \frac{\sum\limits_{n=0}^{\infty} a_{(2n+1)(2m+1)}\lim\limits_{t\to 0}\big[\sqrt{(2n+\frac{3}{2})}\frac{i^{2n+1}}{\sqrt{ct}}\big[\sum\limits_{k=0}^{\infty}\frac{(-1)^k(2\pi ct)^{2k+(2n+\frac{3}{2})}}{k!\Gamma(k+(2n+\frac{3}{2})+1)2^{2k+(2n+\frac{3}{2})}}\big]\big]}{\sum\limits_{n=0}^{\infty} a_{(2n+1)(2m+1)}\overline{P_{2n+1}(0)}} \\
&= \frac{\sum\limits_{n=0}^{\infty} a_{(2n+1)(2m+1)}\lim\limits_{t\to 0}\big[\sqrt{(2n+\frac{3}{2})}\frac{i^{2n+1}t}{\sqrt{c}}\big[\sum\limits_{k=0}^{\infty}\frac{(-1)^k(\pi c)^{2k+(2n+\frac{3}{2})}t^{2k+2n}}{k!\Gamma(k+(2n+\frac{3}{2})+1)}\big]\big]}{\sum\limits_{n=0}^{\infty} a_{(2n+1)(2m+1)}\overline{P_{2n+1}(0)}} \\
&= \frac{a_{1(2m+1)}\lim\limits_{t\to 0}\big[\frac{\sqrt{\frac{3}{2}}i2\pi c}{\frac{3}{2}}t\big]}{\sum\limits_{n=0}^{\infty} a_{(2n+1)(2m+1)}\overline{P_{2n+1}(0)}} = \frac{i\,a_{1(2m+1)}\sqrt{\frac{2}{3}}2\pi c}{\sum\limits_{n=0}^{\infty} a_{(2n+1)(2m+1)}\overline{\tilde{P}_{2n+1}(0)}},
\end{aligned}$$

where $\tilde{P}_{2n+1}(x) = \frac{P_{2n+1}(x)}{x}$.

Now here we close this section with a simple result.

Theorem 4.9. *Let μ^c_n and λ^c_n be the eigenvalues of the operators $\mathcal{G}_c$, and, QP_c, repectively, i.e., $\mathcal{G}_c\psi^c_n(x) = \mu^c_n\psi^c_n(x)$, and $QP_c\psi^c_n(x) = \lambda^c_n\psi^c_n(x)$. Then,*

$$|\mu^c_n|^2 = \frac{1}{c}\lambda^c_n. \tag{4.11}$$

Proof. Since $\langle\psi^c_n,\psi^c_n\rangle_{L^2([-1,1])} = 1$, and using (4.6) we have that

$$\begin{aligned}
|\mu^c_n|^2 &= |\mu^c_n|^2\langle\psi^c_n,\psi^c_n\rangle \\
&= \langle\mathcal{G}_c\psi_j,\mathcal{G}_c\psi_j\rangle \\
&= \langle\mathcal{G}^*_c\mathcal{G}_c\psi^c_n,\psi^c_n\rangle \\
&= \langle\frac{1}{c}QP_c\psi^c_n,\psi^c_n\rangle \\
&= \frac{1}{c}\lambda^c_n.
\end{aligned}$$

□

5. Unique Features PSWFs

In this section, we review three essential feature of PSWFs which makes them special among other special functions.

5.1. **Dual Orthogonality.** At this part, we will see the amazing feature of the PSWFs which is dual orthogonality. Let's assume that $\psi_n^c(x)$ is a time-limited which we constructed in section 3. The way that we constructed shows that $\psi_n^c(x)$s are orthonormal at $L^2[-1,1]$. So we have that $\langle\psi_n^c, \psi_m^c\rangle = \delta_{mn}$. We define the following

$$\phi_n^c(x) = \frac{1}{\sqrt{\lambda_n^c}} P_c \psi_n^c(x). \tag{5.1}$$

$$\begin{aligned}
\langle\phi_n^c, \phi_m^c\rangle_{L^2(\mathbb{R})} &= \langle\frac{1}{\sqrt{\lambda_n^c}} P_c\psi_n^c, \frac{1}{\sqrt{\lambda_m^c}} P_c\psi_m^c\rangle_{L^2(\mathbb{R})} \\
&= \frac{1}{\sqrt{\lambda_n^c}}\frac{1}{\sqrt{\lambda_m^c}}\langle P_c\psi_n^c, \psi_m^c\rangle_{L^2([-1,1])} \\
&= \frac{1}{\sqrt{\lambda_n^c}}\frac{1}{\sqrt{\lambda_m^c}}\langle P_c\psi_n^c, Q\psi_m^c\rangle_{L^2([-1,1])} \\
&= \frac{1}{\sqrt{\lambda_n^c}}\frac{1}{\sqrt{\lambda_m^c}}\langle QP_c\psi_n^c, \psi_m^c\rangle_{L^2([-1,1])} \\
&= \frac{1}{\sqrt{\lambda_n^c}}\frac{1}{\sqrt{\lambda_m^c}}\langle \lambda_n^c\psi_n^c, \psi_m^c\rangle_{L^2([-1,1])} \\
&= \delta_{mn},
\end{aligned}$$

On the contrary, we have that

$$\begin{aligned}
\langle Q\phi_n^c, Q\phi_m^c\rangle_{L^2([-1,+1])} &= \frac{1}{\lambda_n^c}\langle QP_c\psi_n^c, QP_c\psi_m^c\rangle_{L^2([-1,+1])} \\
&= \lambda_n^c\langle\psi_n^c, \psi_m^c\rangle_{L^2([-1,+1])} = \lambda_n^c\delta_{mn}.
\end{aligned}$$

Now let's assume that $\phi_n^c(x)$ is a bandlimited PSWFs so we define the following

$$\psi_n^c(x) = \frac{1}{\sqrt{\lambda_n^c}} Q\phi_n^c(x). \tag{5.2}$$

So we see that

$$\begin{aligned}
\langle\psi_n^c, \psi_m^c\rangle_{L^2([-1,+1])} &= \langle\frac{1}{\sqrt{\lambda_n^c}} Q\phi_n^c, \frac{1}{\sqrt{\lambda_m^c}} Q\phi_m^c\rangle_{L^2([-1,+1])} \\
&= \frac{1}{\sqrt{\lambda_n^c}}\frac{1}{\sqrt{\lambda_m^c}}\langle Q\phi_n^c, Q\phi_m^c\rangle_{L^2([-1,+1])} \\
&= \frac{1}{\sqrt{\lambda_n^c}}\frac{1}{\sqrt{\lambda_m^c}}\lambda_n^c\delta_{mn} \\
&= \delta_{mn}.
\end{aligned}$$

5.2. **Spectral Concentration Problem.**

Theorem 5.1. *Let $f \in L^2(\mathbb{R})$ and ψ_j are PSWFs that we just constructed. Then*

$$\frac{\int_{-1}^{1}|\psi_0(x)|^2dx}{\int_{-\infty}^{\infty}|\psi_0(x)|^2dx} \geq \frac{\int_{-1}^{1}|f(x)|^2dx}{\int_{-\infty}^{\infty}|f(x)|^2dx}.$$

Proof. Let's assume that $\psi_n^c(x)$ is a time-limited version of PSWFs and it is the eigenfunction of the QP_c, i.e.,

$$QP_c\psi_n^c(x) = \lambda_n^c\psi_n^c(x).$$

Now, we write $P_cQP_c\psi_n^c(x)=\lambda_n^cP_c\psi_n^c(x)$, we write, $\varphi_n^c(x)=P_c\psi_n^c(x)$, so $P_cQ\varphi_n^c(x)=\lambda_n^c\varphi_n^c(x)$. This means the eigenvalues of the QP_c and P_cQ are the same. Having this, we may write

$$\begin{aligned}
\|P_c\psi_n^c\|_{[-1,1]}^2 &= \int_{-1}^1 P_c\psi_n^c(x)\overline{P_c\psi_n^c(x)}dx\\
&= \int_{\mathbb{R}} QP_c\psi_n^c(x)\overline{P_c\psi_n^c(x)}dx=\lambda_n^c\int_{\mathbb{R}}\psi_n^c(x)\overline{P_c\psi_n^c(x)}dx\\
&= \lambda_n^c\int_{\mathbb{R}}\psi_n^c(x)\overline{P_c^2\psi_n^c(x)}dx=\lambda_n^c\int_{\mathbb{R}}P_c\psi_n^c(x)\overline{P_c\psi_n^c(x)}dx\\
&= \lambda_n^c\|P_c\psi_n^c\|_{\mathbb{R}}^2,
\end{aligned}$$

and,

$$\begin{aligned}
\|P_c\psi_0^c\|_{[-1,1]}^2 &= \int_{-1}^1 |P_c\psi_0^c(x)|^2dx=\int_{-1}^1|QP_c\psi_0^c(x)|^2dx\\
&= \|P_c\psi_0^c\|_{[-1,1]}^2=(\lambda_0^c)^2\|\psi_0^c(x)\|_{\mathbb{R}}^2
\end{aligned}$$

Also we have that

$$\begin{aligned}
\|P_c\psi_0^c(x)\|_{\mathbb{R}}^2 &= \langle P_c\psi_0^c,P_c\psi_0^c\rangle_{L^2(\mathbb{R})}=\langle\psi_0^c,P_c\psi_0^c\rangle_{L^2(\mathbb{R})}=\langle Q\psi_0^c,P_c\psi_0^c\rangle_{L^2(\mathbb{R})},\\
&= \langle\psi_0^c,QP_c\psi_0^c\rangle_{L^2(\mathbb{R})}=\lambda_0^c\|\psi_0^c(x)\|_{\mathbb{R}}^2
\end{aligned}$$

Let $f\in PW_c$, and we define $\phi_n^c(x)=\frac{\varphi_n^c(x)}{\sqrt{\lambda_n^c}}$. We will prove that $\phi_n^c(x)$ are basis for PW_c. So we will prove the following

i $\phi_n^c(x)\in PW_c$

ii $\|\phi_n^c(x)\|_2=1$

iii if $f\in PW_c$, and $\langle f,\phi_n^c\rangle=0\,\forall n,$ then $f=0$.

(the proof of these items are easy). Now let $f\in PW_c$ be an arbitrary element. So there exist $\{c_n\}_{n=0}^\infty$, such that $f(x)=\sum\limits_{n=0}^\infty c_n\phi_n^c$. So we have that

$$\begin{aligned}
\|f\|_{[-1,1]}^2 &= \int_{-1}^1|f(x)|^2dx=\langle Qf,f\rangle_{L^2[-1,1]}=\langle Q\big(\sum_{n=0}^\infty c_n\phi_n^c\big),\sum_{m=0}^\infty c_m\phi_m^c\rangle_{L^2[-1,1]}\\
&= \sum_{n,m=0}^\infty c_n\overline{c_m}\langle Q\phi_n^c,\phi_m^c\rangle=\sum_{n,m=0}^\infty c_n\overline{c_m}\langle Q\phi_n^c,P_c\phi_m^c\rangle\\
&= \sum_{n,m=0}^\infty c_n\overline{c_m}\langle P_cQ\phi_n^c,\phi_m^c\rangle=\sum_{n,m=0}^\infty c_n\overline{c_m}\langle\lambda_n^c\phi_n^c,\phi_m^c\rangle\\
&= \sum_{n=0}^\infty|c_n|^2\lambda_n^c\le\lambda_0<1=\sum_{n=0}^\infty|c_n|^2=\|f\|_{L^2(\mathbb{R})}.
\end{aligned}$$

Note that the max $\sum\limits_{n=0}^{\infty} |c_n|^2\lambda_n^c = \lambda_0^c$. In fact, this is because $\sum\limits_{n=0}^{\infty} |c_n|^2 = 1$, and, $\cdots < \lambda_n < \cdots < \lambda_1 < \lambda_0$. Hence,

$$\frac{\int_{-1}^{1}|f(x)|^2dx}{\int_{-\infty}^{\infty}|f(x)|^2dx} \leq \lambda_0^c = \frac{\|P_c\psi_0^c\|^2_{[-1,1]}}{\|P_c\psi_0^c(x)\|^2_{\mathbb{R}}} = \frac{(\lambda_0^c)^2\|\psi_0^c(x)\|^2_{\mathbb{R}}}{\lambda_0^c\|\psi_0^c(x)\|^2_{\mathbb{R}}} = \lambda_0^c \leq \frac{\int_{-1}^{1}|\psi_0(x)|^2dx}{\int_{-\infty}^{\infty}|\psi_0(x)|^2dx}.$$

□

5.3. **Approximation by PSWFs.** In this section, we will see the approximation of the timelimited functions by PSWFs. Let's assume that $f(x) \in L^2[-1,1]$. Since the PSWFs constitutes a basis for $L^2[-1,1]$, so

$$f(x) = \sum_{i=0}^{\infty} \alpha_i\psi_i^c(x),$$

where $\alpha_i = \int_{-1}^{1} f(x)\overline{\psi_i^c(x)}dx$. Let's try to approximate the characteristic function, i.e.,

$$\chi_{[-1,1]}(x) = \begin{cases} 1 & x \in [-1,1], \\ 0 & x \notin [-1,1]. \end{cases} \tag{5.3}$$

Then, $\alpha_i = \int_{-1}^{1}\overline{\psi_i^c(x)}dx = \int_{-1}^{1}\overline{\psi_i^c(x)}e^{2\pi ic\langle x,0\rangle}dx = \mu_i^c\psi_i^c(0)$. But we know that just we see that the eigenvalues of the PSWFs decreases drastically after first few (See Figure **??**). Therefore, in fact we just need just few PSWFs in order to do the approximations (See Figure 5.3).

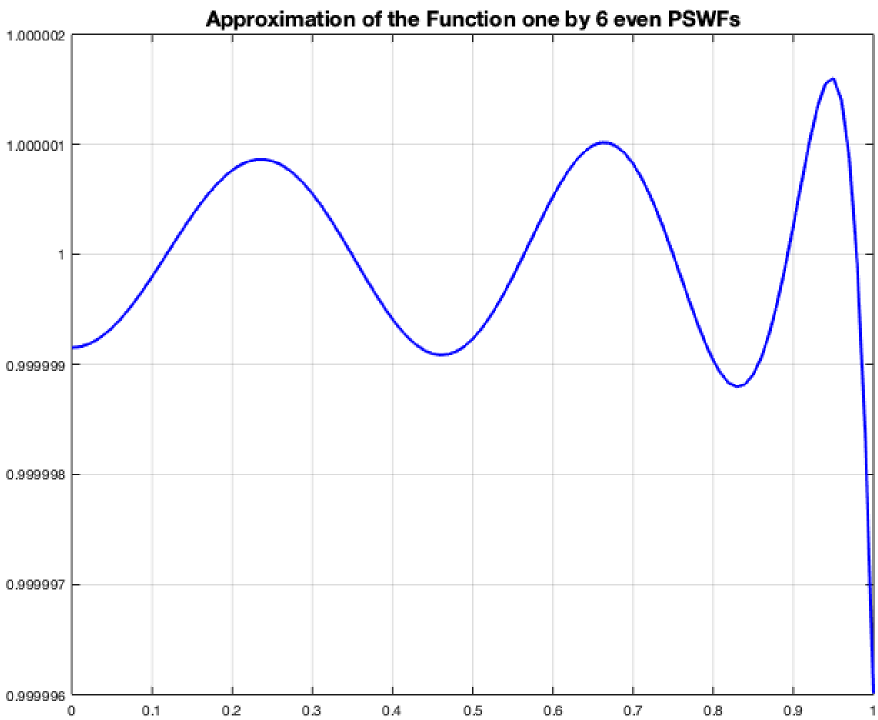

FIGURE 5. Approximation of one by only six even PSWFs

5.4. **Spectrum Accumulation.**

Theorem 5.2. *Let $\psi_n^c(x)$ is a time-limited version of PSWFs and let $QP_c\psi_n^c(x) = \lambda_n^c\psi_n^c(x)$. Then we have that*

$$\sum_{n=0}^{\infty}|\lambda_n^c||\psi_n^c(x)|^2 = 2c. \tag{5.4}$$

Proof. By (4.5) we have that

$$\begin{aligned}
\sum_{n=0}^{\infty}|\lambda_n^c||\psi_n^c(x)|^2 &= \sum_{n=0}^{\infty}\lambda_n^c\psi_n^c(x)\psi_n^c(x)\\
&= \sum_{n=0}^{\infty}QP_c\psi_n^c(x)\psi_n^c(x)\\
&= \sum_{n=0}^{\infty}\chi_{[-1,1]}(x)\int_{-1}^{1}\psi_n^c(y)\frac{\sin(2\pi c(y-x))}{\pi(y-x)}dy\psi_n^c(x)\\
&= \sum_{n=0}^{\infty}\langle\chi_{[-1,1]}(x)\frac{\sin(2\pi c(\cdot-x))}{\pi(\cdot-x)},\psi_n^c(\cdot)\rangle\psi_n^c(x)\\
&= \chi_{[-1,1]}(x)\frac{\sin(2\pi c(s-x))}{\pi(s-x)}|_{s=x} = 2c.
\end{aligned}$$

□

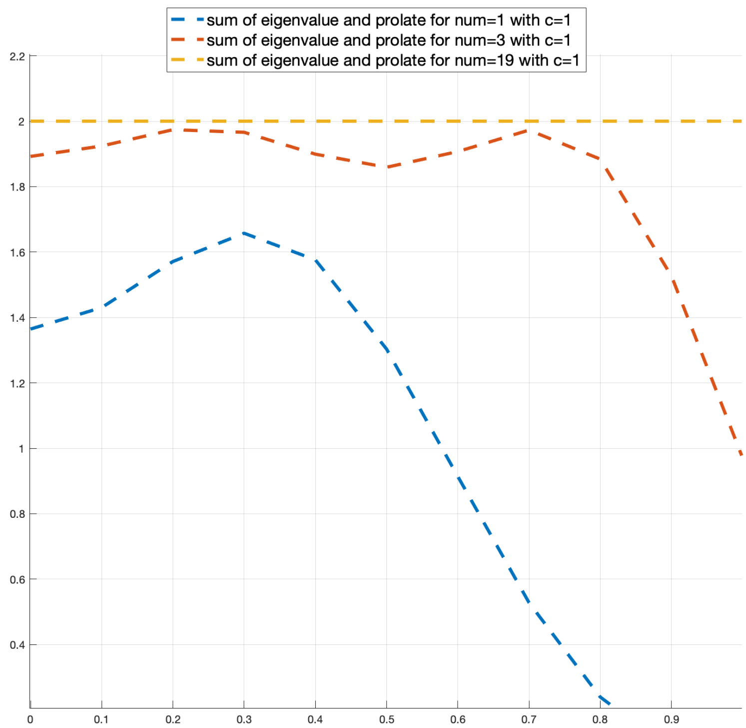

FIGURE 6. Spectrum Accumulation of one dimension PSWFs

Remark 5.3. The algorithm and the MATLAB, Maple, Mathematica, Python, sagemath, Julia including computations of prolates and their Fourier transformations will be uploaded in Git-hub community by the account of **Hamed Baghal Ghaffari**.

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