$\triangle$ Method of for generating r.v.'s

Goal : $X \sim f(x)$, where f is known

{ discrete r.v. "p.m.f." prob. mass function
{ continuous r.v. "p.d.f." prob. density function

Method: 1° Inverse transform

2° Acceptance - Rejection Algorithms

3° Transformation method

$\triangle$ Inverse transform :

we have $F(x) = P(X \le x)$ c.d.f. cummulative distribution function

note that X : random , x : fixed

Remark : 1° F is non-decreasing

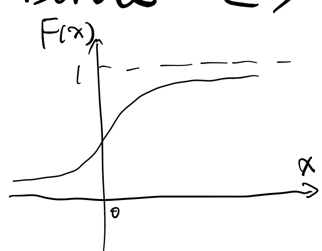
2° F is right - continuous

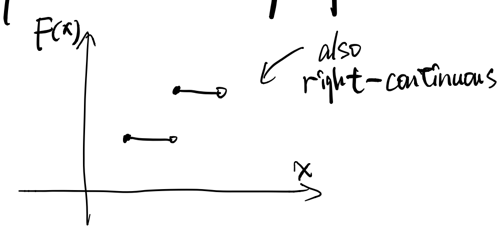
$$\lim_{\varepsilon \to 0^+} F(x+\varepsilon) = F(x) \quad \forall x \in R$$

3° $\lim_{x \to -\infty} F(x) = 0 \qquad \lim_{x \to +\infty} F(x) = 1$

r.v. X is continuous $\Leftrightarrow$ F is cont.

discrete $\Leftrightarrow$ F is a step func.





Thm 3.1 If X is a continuous r.v. with cdf F

then $U = F(X) \sim \text{Unif}(0,1)$

(continuous uniform dist. $f_U(u) = 1_{[0,1]}(u) = \begin{cases} 1 & 0 \le u \le 1 \\ 0 & \text{otherwise} \end{cases}$

then $F_U(u) = \begin{cases} 1 & u > 1 \\ u & 0 \le u \le 1 \\ 0 & u < 0 \end{cases}$)

pf: Define the quantile function which is the inverse of F, $F^{-1}(u) = \inf\{x : F(x) \ge u\}$ $(0 \le u \le 1)$

$$P(U \le u) = P(F(X) \le u) = P(X \le F^{-1}(u))$$

$$= F(F^{-1}(u)) = u \quad (0 \le u \le 1)$$

$\therefore U \sim \text{Unif}(0,1)$ #

Steps: ① Derive $F^{-1}(u)$

② Generate u_i from $\text{Unif}(0,1)$

③ Obtain $x_i = F^{-1}(u_i)$

then x_i's can be seen as random numbers from X

Requirement: F^{-1} is easy to calculate

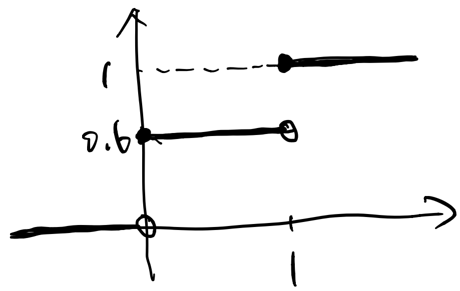
eg. $f(x) = 3x^2 \, 1_{\{0 \le x \le 1\}}$

$F(x) = x^3 \, 1_{\{0 \le x \le 1\}}$? what if $x > 1$.

See the code now

eg. $X \sim Bern(p)$ $P(X=1) = p = 1 - P(X=0)$ $p=0.4$

$$F(x) = \begin{cases} 1-p = 0.6 & x=0 \\ 1 & x=1 \end{cases}$$



$$F^{-1}(u) = \begin{cases} 0 & u \le 0.6 \\ 1 & u > 0.6 \end{cases}$$

△ Acceptance - Rejection

Goal: generate from $X \sim f(t)$

Tool: we can generate easily from another r.v. $Y \sim g(t)$

and $\frac{f(t)}{g(t)} \le c \quad \forall t$

Steps for AR:

① Generate y from $Y \sim g(t)$

② Generate u from $Unif(0,1)$

③ If $u < \frac{f(y)}{cg(y)}$, Accept. $x=y$

otherwise, reject y and continue

Pf: Consider the discrete case   continue is hw.

"Generate → Decide accept or not"

$$P(\text{accept} \mid Y) = P\left(u < \frac{f(Y)}{cg(Y)} \,\middle|\, Y\right) = \frac{f(Y)}{cg(Y)}$$

$$P(\text{accept}) = \sum_y P(\text{accept} \mid Y=y) \cdot P(Y=y)$$

$$= \sum_y \frac{f(y)}{c\,g(y)} \cdot g(y) = \frac{1}{c} \sum_y f(y) = \frac{1}{c}$$

$$P(X=y) = P(Y=y \mid \text{accept})$$

$$= \frac{P(\text{accept} \mid Y=y) \cdot P(Y=y)}{P(\text{accept})}$$

$$= \frac{\frac{f(y)}{c\,g(y)} \cdot g(y)}{\frac{1}{c}} = f(y)$$

<u>Remark</u>: ① depends on c only

② small c is better

③ On average, accepting one sample needs c runs

eg. 3.7: Beta (3, 2) $\begin{cases} f(x) = 6x(1-x), & 0 < x < 1 \\ g(x) = 1, & 0 \le x \le 1 \end{cases}$

Note that: the Domain of g should include the Domain of f

$$\frac{f(x)}{g(x)} = 6x(1-x) \le \frac{3}{2}$$, it is the optimal bound

$$\le 6$$ a loose bound

then let's see the code