DATA130004: Homework 3

Due in class on October 23, 2019

1. Exercises 5.12, and 5.14.
2. Given two random variables X and Y, prove the law of total variance
$$\mathrm{var}(Y) = \mathrm{E}\{\mathrm{var}(Y|X)\} + \mathrm{var}\{\mathrm{E}(Y|X)\}.$$
Be explicit at every step of your proof.
3. Define $\theta = \int_A g(x)\,dx$, where A is a bounded set and $g \in \mathcal{L}_2(A)$. Let f be an importance function which is a density function supported on the set A.
   (a) Describe the steps to obtain the importance sampling estimator $\hat{\theta}_n$, where n is the number of random samples generated during the process.
   (b) Show that the Monte Carlo variance of $\hat{\theta}_n$ is
   $$\mathrm{var}(\hat{\theta}_n)x = \frac{1}{n}\left\{\int_A \frac{g^2(x)}{f(x)}\,dx - \theta^2\right\}$$
   (c) Show that the *optimal* importance function f^*, i.e., the minimizer of $\mathrm{var}(\hat{\theta}_n)$, is
   $$f^*(x) = \frac{|g(x)|}{\int_A |g(x)|\,dx},$$
   and derive the theoretical lower bound of $\mathrm{var}(\hat{\theta}_n)$.