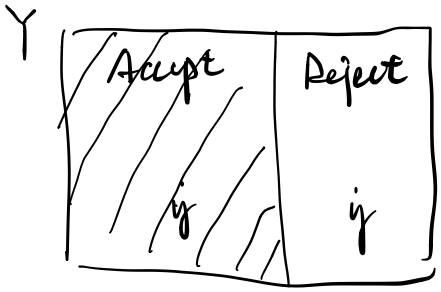
- Inverse transform
- Acceptance-Rejection
- Transformation (today)
- sums and mixtures

Remark:

$P(X=y)$ $\qquad Y \sim g(t)$

$P(\text{Accept} \mid Y=y)$ $\qquad \text{Accept} = \begin{cases} 1, & \text{acc} \\ 0, & \text{rej} \end{cases}$

$P(Y=y \mid \text{Accept})$ $\qquad$ by-product



Q: What does accept mean.

A: Decision Rule $\quad u < \dfrac{f(y)}{c\,g(y)}$

△ Transformation

1° $Z \sim N(0,1)$, then $V = Z^2 \sim \chi^2(1)$

2° $U \sim \chi^2(m)$, $V \sim \chi^2(n)$, and U, V ind.

then $\quad F = \dfrac{U/m}{V/n} \sim F_{m,n}$ $\qquad$ two degree freedom

3° $Z \sim N(0,1)$, $V \sim \chi^2(n)$, Z and V ind.

then $\quad T = \dfrac{Z}{\sqrt{V/n}} \sim t_n$

4° $U, V \overset{\text{ind}}{\sim}$ Unif $(0,1)$ then $Z_1 = \sqrt{-2\log U}\,\cos(2\pi V)$,

$$Z_2 = \sqrt{-2\log U}\,\sin(2\pi V)$$

are ind. $N(0,1)$'s

"Box-Muller Transformation"

5° $\left.\begin{array}{l} U \sim \text{Gamma}(r, \lambda) \\ V \sim \text{Gamma}(s, \lambda) \end{array}\right\}$ ind.

then $\frac{U}{U+V} \sim \text{Beta}(r, s)$

Keep these in mind, Let's see some examples.

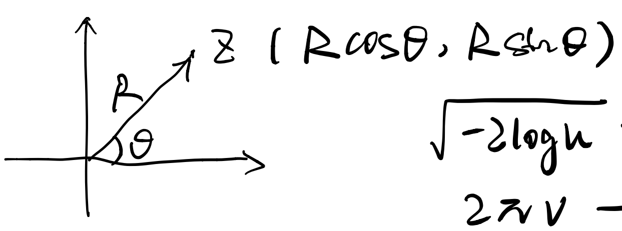
4° $Z_1^2 + Z_2^2$ $\qquad -2\log U$

$\sim \chi^2(2)$

Quantile - quantile plot "qqplot"

B-M : $Z_1 = \sqrt{-2\log U}\cdot\cos(2\pi V)$

$Z_2 = \sqrt{-2\log U}\cdot\sin(2\pi V)$



$\sqrt{-2\log U} \to R$ proved by qqplot

$2\pi V \to \theta$ $V \in [0,1]$ easy understood

5° also using qqplot to prove that.

△ Sums and mixtures

1° Convolution (between variables):

Sum of independent r.v.'s

$Z = X + Y$, X, Y ind.

eg. $\chi^2(\nu)$: convolution of ν iid $N(0,1)^2$'s

i.i.d.: independently identically distributed

$Z_1^2 + \cdots + Z_\nu^2 \sim \chi^2(\nu)$

eg. Gamma(r, λ): convolution of r iid Exp(λ)'s

eg. NegBin(r, p): convolution of r iid Geom(p)'s

$\mathbb{N}^*$, prob.

negative binomial: number of failures before the r-th success of Bern(p) trials.

geometric: number of failures before the first success of Bern(p) trials.

$Z = X + Y$

Discrete: $P(Z=z) = \sum_{k=-\infty}^{+\infty} P(X=k)\,P(Y=z-k)$

Continuous: if $X \sim f$, $Y \sim g$, then $z \sim f * g$

where $(f*g)(z) = \int_{-\infty}^{\infty} f(t)\,g(z-t)\,dt$

More about qqplot : Quantile vs. Quantile

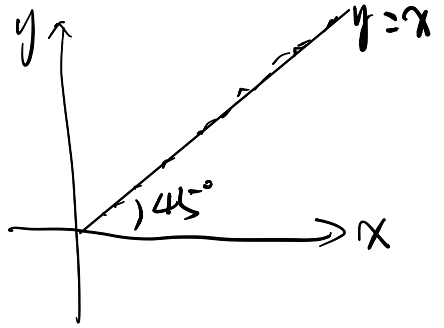
compare two dist. whether they're equal or not

whether $Z_1^2+Z_2^2$ & $-2\log u$ follow the same dist.

empirical dist.

theoretical dist.

$x_{(1)} < \cdots < x_{(1000)}$ $y_{(1)} < \cdots < y_{(1000)}$



$(x_{(1)}, y_{(1)})$

$\vdots$

$(x_{(1000)}, y_{(1000)})$

2° Mixtures

$$F_X(x) = \sum_{k=1}^{K} \theta_k F_{X_k}(x)$$

then X is the mixture of $X_1, X_2, \cdots X_K$

with mixing weights $(\theta_1, \theta_2, \cdots, \theta_K)$

$\theta_k \geq 0$, and $\sum_k \theta_k = 1$

eg. $X_1 \sim N(0,1)$ $X_2 \sim N(10,1)$, ind.

sum: $S = X_1 + X_2 \sim N(10, 2)$

mixture: $F_X(x) = \frac{1}{4} F_1(x) + \frac{3}{4} F_2(x)$ (weights $\theta_1 = \frac{1}{4}$, $\theta_2 = \frac{3}{4}$)

$X \sim N(0,1)$ with prob. $\frac{1}{4}$

$X \sim N(10,1)$ with prob. $\frac{3}{4}$

① Generate u from Unif $(0,1)$

② If $u \le \frac{1}{4}$, generate x from $N(0,1)$
otherwise, generate x from $N(10,1)$

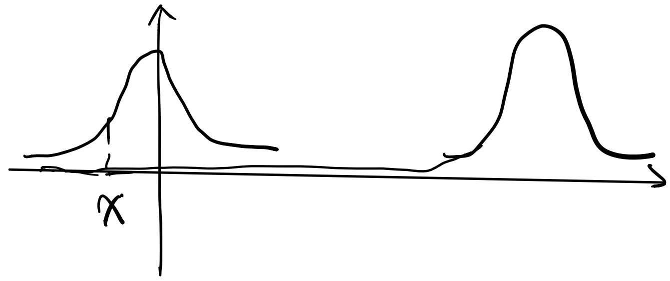
$F_X(x) = P(X \le x)$

$\|$

$\frac{1}{4} F_1(x) + \frac{3}{4} F_2(x)$

$\|$

$\frac{1}{4} P(X_1 \le x) + \frac{3}{4} P(X_2 \le x)$



Something in the code

y <- apply (X, MARGIN = 1 , FUN = sum)

1 means apply the func on the row
2 means — — — — column

△ Generating random vectors
from multivariate normal dist.

Some useful properties:

Def: $x \in \mathbb{R}^d$ with density

$$f(x_1, \cdots, x_d) = \frac{1}{(2\pi)^{\frac{d}{2}} |\Sigma|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right\}$$

where mean para $\mu = E(x) \in \mathbb{R}^d$

covariance $\Sigma = Cov(X)$ is $d \times d$ positive definite matrix

Denote $X \sim N_d(\mu, \Sigma)$

Redefine of mixtures:

In general, If we have $X_1, \cdots, X_k$, with mixing weight $(\theta_1, \cdots \theta_k)$ satisfy $\theta_i \geq 0$ and $\sum_{i=1}^{k} \theta_k = 1$, which means X has probability θ_i to be r.v. X_i, then we will have

$F_X(x) = \sum_{k=1}^{K} \theta_k F_{X_k}(x)$, which is a property of the mixture

Remark: ① Marginals conditionals are Gaussian

② $X = (X_1, \cdots, X_d)^T$

then $X_1, \cdots, X_d$ ind $\Leftrightarrow$ Σ diag

③ Linear transformations are normal

$Y = CX + b \sim N(C\mu + b, C \Sigma C^T)$

Suppose we can generate from $N(0,1)$

how to generate $X \sim N_d(\mu, \Sigma)$?

Steps: ① Generate $Z_1, \cdots, Z_d \overset{i.i.d.}{\sim} N(0,1)$

$Z = (Z_1, \cdots, Z_d)^T \sim N_d(0, I_d)$

② Transformation $X = CZ + \mu$, then $X \sim N_d(\mu, \Sigma)$

The question becomes how to find C so that $\Sigma = CC^T$

1° Eigen-decomposition

2° Singular value decomposition

3° Choleski factorization

1° $\Sigma = U \Lambda U^T$

Λ: diagonal matrix with eigenvalues of Σ non-increasingly ordered

U: orthogonal matrix of eigenvectors of Σ

def: $C = \Sigma^{\frac{1}{2}} = U \Lambda^{\frac{1}{2}} U^T$

2° SVD: a generalization of eigen-decomposition for square matrix to the rectangle matrix

$\Sigma = U \Lambda V^T$

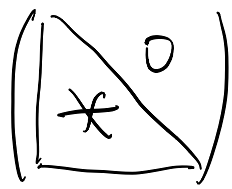
U: matrix with orthonormal column, collecting left singular vectors

V: matrix with orthonormal column, collecting right singular vectors

Λ: diag. matrix with singular values

3° Choleski factorization: $\Sigma = LL^T$

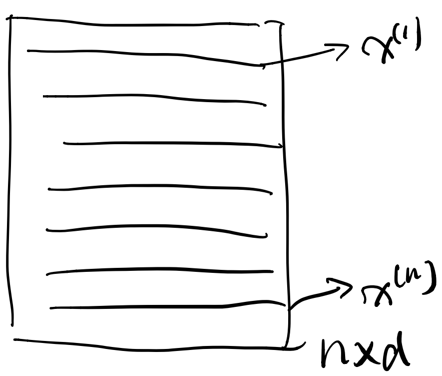
L: lower-triangular matrix →



eg. $\mu = 0$

$\Sigma = \begin{pmatrix} 1 & 0.9 \\ 0.9 & 1 \end{pmatrix}$ correlation matrix

```
ev = eigen(Sigma, symmetric = TRUE)
lambda <- ev$values
V <- ev$vectors
```




```
R <- . ~

X <- Z %*% R
n×d  n×d   d×d
```