Sum of Divergent Series

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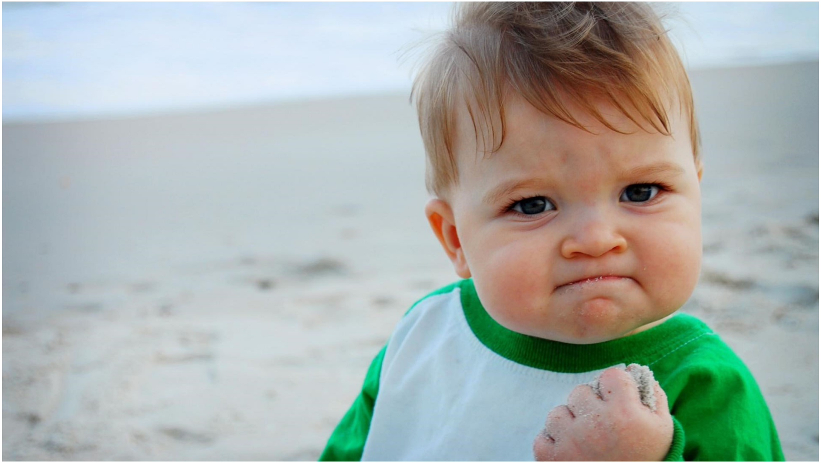
Uptake Math Club Lightning Talk

February 15 & 22, 2019

Outline

Motivating Examples



Convergence of Power Series

Last time, @willdiesel talked about the following Maclaurin series (i.e. Taylor series expanded at zero)

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

- @bernie asked about convergence (which is $|x| < 1$)
- But, what does it mean when we write $\sum_{n=0}^{\infty} x^n = \mathit{value}$?
  - Is "$=$" a mathematical equivalence symbol or an assignment operator?

Review of Summation of Real Numbers

Properties of Sum of Real Numbers

- **Commutativity**: For $\forall x, y \in \mathbb{R}$, $x + y = y + x$
- **Associativity**: For $\forall x, y, z \in \mathbb{R}$, $(x + y) + z = x + (y + z)$

Recursive Definition of $\sum$ Symbol

For $\forall \{a_i\}_{i \in \mathbb{N}} \subset \mathbb{R}$ and $\forall n \in \mathbb{N}$, define the summation symbol,

$$\sum_{i=1}^{n} a_i = a_n + \sum_{i=1}^{n-1} a_i \quad (\text{for } n > 1) \quad \text{and} \quad \sum_{i=1}^{1} a_1 = a_1$$

(Note: ∞ is not a real number nor a natural number.)

Sum of All Natural Numbers

Numberphile Video: ASTOUNDING! $1+2+3+\cdots=-\frac{1}{2}$

Consider the following sums

$$\begin{aligned} S &= 1+2+3+4+5+6+\cdots \\ S_1 &= 1-1+1-1+1-1+\cdots \\ S_2 &= 1-2+3-4+5-6+\cdots \end{aligned}$$

We want to find S and we establish S_1 and S_2 as intermediate steps.

Sum of All Natural Numbers (Cont.)

First, we find the values of "helper" sums S_1 and S_2.

Claim 1 and 2: $S_1 = \frac{1}{2}$ and $2S_2 = S_1 \implies S_2 = \frac{1}{4}$

Because the partial sums are $0, 1, 0, 1, \cdots$, so $\frac{1}{2}$ is sort of the average, hence $S_1 = \frac{1}{2}$. Furthermore,

$$\begin{aligned}
2S_2 &= S_2 + S_2 \\
&= 1 \; -2 + 3 - 4 + 5 - 6 + \cdots \\
&\qquad\;\; 1 - 2 + 3 - 4 + 5 - \cdots \\
&= 1 \; -1 + 1 - 1 + 1 - 1 + \cdots \\
&= S_1
\end{aligned}$$

Sum of All Natural Numbers (Cont.)

Next, we can fill the bridge to S.

Claim 3: $S - S_2 = 4S \implies S = -\frac{1}{12}$

$$
\begin{aligned}
S - S_2 = &\ 1 + 2 + 3 + 4 + 5 + 6 + \cdots \\
&-(1 - 2 + 3 - 4 + 5 - 6 + \cdots) \\
= &\ 0 + 4 + 0 + 8 + 0 + 12 + \cdots \\
= &\ 4(1 + 2 + 3 + \cdots) \\
= &\ 4S
\end{aligned}
$$

Sum of All Natural Numbers (Cont.)

But, the result doesn't follow along with the intuition

$$\{a_i\} \subset \mathbb{R}_+ \implies \forall N \in \mathbb{N}, \sum_{i=1}^{N} a_i > 0$$

What may go wrong?

- What is the definition of sum?

  $\Sigma_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots$ is just a short hand notation

- Can we shift terms in a series under summation?
- Can we ignore all the zeros, despite the infinite amount?
- Is summation a linear operator?

Sum of All Natural Numbers (Cont.)

We will now form a contradiction. Consider

$$S_3 = 1 + 1 + 1 + 1 + 1 + \cdots$$
$$S_4 = 1 + 3 + 5 + 7 + 9 + \cdots$$

Claim 4, 5, and 6: $S_3 = \frac{1}{2}$, $S_4 = -\frac{2}{3}$, and $S = \frac{2}{3}$

$$S_3 - S_1 = 2S_3 \implies S_3 = \frac{1}{2}$$
$$S_4 = 2S - S_3 \implies S_4 = -\frac{2}{3}$$
$$S = S_4 + 2S \implies S = \frac{2}{3}$$

G.H.Hardy, Divergent Series (1949)

It is natural to suppose that the ... formulae will prove to be correct, and our transformations justifiable, if they are interpreted appropriately... This remark is trivial now: it does not occur to a modern mathematician that a collection of mathematical symbols should have a "meaning" until one has been assigned to it by definition. It was not a triviality even to the greatest mathematicians of the 18th century... mathematicians before Cauchy asked not "How shall we define $1-1+1-\cdots$*?" but "What is* $1-1+1-\cdots$*?", and that this habit of mind led them into unnecessary perplexities and controversies which were often really verbal.*

Outline

Summability

Cauchy's Definition for Sum of an Infinite Series

Limit of a Sequence

We define $\{s_i\}_i \to L \in \mathbb{R}$ if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \text{s.t. } \forall i > N, |s_i - L| < \epsilon$$

Convergent Series, i.e. Cauchy Summable

We define the sum of $\{a_i\}_i$ to be S if

$$\text{the sequence of partial sum } \left\{ s_n = \sum_{i=1}^{n} a_i \right\}_n \to S$$

Cauchy's Definition for Sum of an Infinite Series (Cont.)

Remarks

The $\sum$ operator should be think of as a partial function mapping a real sequence to a real number, if defined. ("partial" means the function can be undefined for some elements in its domain.)

Notation

We will write "sequence $\{a_i\}$ is Cauchy summable to S" as

- $\sum(\{a_i\}) = S$, or equivalently
- $a_1 + a_2 + a_3 + \cdots \longrightarrow S$

In this case, we also say the sequence is convergent.

Examples

- Geometric Series: $1 + x + x^2 + x^3 + \cdots \longrightarrow \frac{1}{1-x}$ for $\forall x \in (-1, 1)$
  - partial sum of first n terms:

$$1 + x + x^2 + x^3 + \cdots + x^{n-1} = \frac{1 - x^n}{1 - x}$$

  - as $n \to \infty$, the partial sum converges if $|x| < 1$
- Grandi's Series: $1 - 1 + 1 - 1 + \cdots \longrightarrow$ undefined
- Harmonic Series: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \longrightarrow$ undefined
- $1 + 2 + 3 + 4 + \cdots \longrightarrow$ undefined

Conditional Convergence

Absolutely Convergent

We say $\{a_i\}_i$ is *absolutely convergent* if $\{|a_i|\}_i$ is convergent

Conditionally Convergent

We say $\{a_i\}_i$ is *conditionally convergent* if

- $\{|a_i|\}_i$ is divergent, but
- $\{a_i\}_i$ is convergent

Remark

If $\{a_i\}_i$ is conditionally convergent, by arranging the order of terms, one can arrive at any sum.

Order Matters

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\cdots\longrightarrow \ln(2), \text{ but}$$

- $1-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}+\frac{1}{3}-\frac{1}{10}-\frac{1}{12}-\frac{1}{14}-\frac{1}{16}+\frac{1}{5}-\cdots\longrightarrow 0$
- $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\frac{1}{13}+\frac{1}{15}-\cdots\longrightarrow \frac{3}{2}\ln(2)$
- $1+\frac{1}{3}-\frac{1}{2}-\frac{1}{4}+\frac{1}{5}+\frac{1}{7}-\frac{1}{6}-\frac{1}{8}+\frac{1}{9}+\frac{1}{11}-\frac{1}{10}-\cdots\longrightarrow \ln(2)$
- $1+\frac{1}{3}+\frac{1}{5}-\frac{1}{2}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{4}+\frac{1}{13}+\frac{1}{15}+\frac{1}{17}-\cdots\longrightarrow \frac{1}{2}\ln(12)$

Proof is left as an exercise to the readers. Hint:

$$\frac{1}{1+n}=\int_0^1 x^n dx \qquad \underbrace{\sum_{n=0}^{\infty}\int_0^1=\int_0^1\sum_{n=0}^{\infty}}_{\text{under some technical conditions}} \qquad \underbrace{\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}}_{\text{for } |x|<1}$$

Solution to the 2nd Example

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\frac{1}{13}+\frac{1}{15}-\frac{1}{8}+\cdots\longrightarrow\quad ?$$

First, we recognize that the above sum can be written as

$$\sum_{k=0}^{\infty}\left(\frac{1}{4k+1}+\frac{1}{4k+3}-\frac{2}{4k+4}\right)$$

Then, we apply hint $\frac{1}{1+n}=\int_0^1 x^n dx$, e.g., $4k+3=1+(4k+2)$

$$\sum_{k=0}^{\infty}\left(\int_0^1 x^{4k}dx+\int_0^1 x^{4k+2}dx-\int_0^1 2x^{4k+3}dx\right)$$

Notice the linearity of integration

$$\sum_{k=0}^{\infty}\int_0^1 (x^{4k}+x^{4k+2}-2x^{4k+3})dx$$

Solution to the 2nd Example (Cont.)

We can exchange the order of summation and integration (with some caution)

$$\int_0^1 \big(\sum_{k=0}^{\infty} x^{4k} + x^{4k+2} - 2x^{4k+3}\big)\,dx$$

Since the integration restricts $x \in (0,1)$, then the inner summation converges for each component, which is (close to) a geometric series

$$\int_0^1 \Big(\big(\sum_{k=0}^{\infty} x^{4k}\big) + \big(\sum_{k=0}^{\infty} x^{4k+2}\big) - \big(\sum_{k=0}^{\infty} 2x^{4k+3}\big)\Big)\,dx$$

Solution to the 2nd Example (Cont.)

Notice that

$$\sum_{k=0}^{\infty} x^{4k+2} = x^2 \cdot \sum_{k=0}^{\infty} (x^4)^k \qquad \sum_{k=0}^{\infty} 2x^{4k+3} = 2x^3 \cdot \sum_{k=0}^{\infty} (x^4)^k$$

Thus, we can apply the formula for geometric series with $|x| < 1$

$$\int_0^1 \left(\frac{1}{1-x^4} + \frac{x^2}{1-x^4} - \frac{2x^3}{1-x^4}\right) dx = \frac{3}{2}\ln(2)$$

The above integral can be evaluated through partial fraction decomposition or by simply invoking Mathematica.

General Patterns

Note that terms with odd denominators should be negative, and the opposite applies to even denominators. Also notice that the multipliers for k must be the same within parentheses.

$$\sum_{k=0}^{\infty}\left(\frac{4}{8k+4}-\frac{1}{8k+2}-\frac{1}{8k+4}-\frac{1}{8k+6}-\frac{1}{8k+8}\right)$$

$$\sum_{k=0}^{\infty}\left(\frac{1}{4k+1}+\frac{1}{4k+3}-\frac{2}{4k+4}\right)$$

$$\sum_{k=0}^{\infty}\left(\frac{1}{4k+1}+\frac{1}{4k+3}-\frac{1}{4k+2}-\frac{1}{4k+4}\right)$$

$$\sum_{k=0}^{\infty}\left(\frac{1}{6k+1}+\frac{1}{6k+3}+\frac{1}{6k+5}-\frac{3}{6k+6}\right)$$

Caveat: Fubini's Theorem is Not Trivial

Consider

$$a_{ij} = \begin{cases} +1 & \text{, if } i = j+1 \\ -1 & \text{, if } i = j-1 \\ 0 & \text{, otherwise} \end{cases}$$

Then we can get

$$\sum_i \sum_j a_{ij} = -1$$

$$\sum_j \sum_i a_{ij} = +1$$

Easier to digest in matrix form

$$(a_{ij}) = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & -1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & -1 & 0 & \cdots \\ 0 & 0 & 1 & 0 & -1 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

where for each row i, sum $\sum_j a_{ij}$ is

$$-1, 0, 0, 0, 0, 0, \cdots$$

while for each column j, sum $\sum_i a_{ij}$ is

$$+1, 0, 0, 0, 0, 0, \cdots$$

From Partial Sum to Partial Mean

Cesaro Summable

We define $a_1 + a_2 + a_3 + \cdots \xrightarrow{\mathfrak{C}} S$, if
the sequence of Cesaro mean

$$\left\{\sigma_n = \frac{\sum_{i=1}^{n} s_i}{n}\right\}_n \to S$$

Now, we have a precise definition for

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots \xrightarrow{\mathfrak{C}} \frac{1}{2}$$

This summation was implicitly used by Frobenius in 1880, prior to Cesaro (1890). Cesaro's key contribution was not the discovery of this method, but his idea that *one should give an explicit definition of the sum of a divergent series.*

From Partial Series to the Entire Series at Once

Abel Summable

We define $a_0 + a_1 + a_2 + a_3 + \cdots \xrightarrow{\mathfrak{A}} S$, if
for $\forall r \in [0,1)$, Abel mean

$$A(r) = \sum_{k=0}^{\infty} a_k r^k$$

exists in Cauchy sense, and it has finite limit at $r = 1$

$$\lim_{r \to 1} A(r) = S$$

Example

$$1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + \cdots \xrightarrow{\mathfrak{A}} \frac{1}{4}$$

- General term: $(-1)^k(k+1)$ for $k = 0, 1, 2, 3, \cdots$
- Abel mean: for $\forall r \in [0, 1)$,

$$A(r) = \sum_{k=0}^{\infty}(-1)^k(k+1)r^k = \frac{1}{(1+r)^2}$$

- Its limit exists due to continuity,

$$\lim_{r\to 1} A(r) = \lim_{r\to 1} \frac{1}{(1+r)^2} = \frac{1}{4}$$

Summability: Cauchy $\implies$ Cesaro $\implies$ Abel

Recap on definitions: we say $\{a_i\}_i$ is (...) summable to S if

- Cauchy:

$$\lim_{n\to\infty} \underbrace{\sum_{i=0}^{n} a_i}_{s_n} = S$$

- Cesaro:

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \underbrace{\left(\sum_{i=0}^{k} a_i\right)}_{s_k} = S$$

- Abel:

$$\lim_{r\to 1} \underbrace{\sum_{k=0}^{\infty} a_k r^k}_{A(r)} = S$$

See visual demonstration.

Desired Properties of Summation Methods

Generally speaking, it is incorrect to manipulate infinite series as if they were finite sums. But there are some properties we still hope to maintain.

- **Regularity**: agree with Cauchy sum, if exists
- **Linearity**: $\Sigma(\{ka_i + b_i\}_i) = k\Sigma(\{a_i\}_i) + \Sigma(\{b_i\}_i)$
- **Stability** (a.k.a. Translativity): $\Sigma(\{a_i\}_{i=0}^{\infty}) = a_0 + \Sigma(\{a_i\}_{i=1}^{\infty})$
    - **Finite Re-indexability**: $\Sigma(\{a_i\}_i) = \Sigma(\{a_{\pi(i)}\}_i)$ where π is any permutation on a finite subset of indices
    - Stability $\implies$ Finite re-indexability

Note that not all conditions are equally important in summability theory and it is quite restrictive to require all properties.

Examples: Linear and Stable Summation Method

Geometric Series: $\Sigma(c, r) = \frac{c}{1-r}$, for $\forall c \in \mathbb{R}$ and $r \neq 1$

Let $\Sigma(c, r)$ denote the sum of geometric series $\{cr^k\}_{k=0}^{\infty}$ under any linear and stable summation method. Then

$$\begin{aligned} \Sigma(c, r) = \sum_{k=0}^{\infty} c \cdot r^k &= c + \sum_{k=0}^{\infty} c \cdot r^{k+1} && \text{(stability)} \\ &= c + r \cdot \sum_{k=0}^{\infty} c \cdot r^k && \text{(linearity)} \\ &= c + r\Sigma(c, r) \end{aligned}$$

Therefore, $1 + 2 + 4 + 8 + 16 + \cdots = \frac{1}{1-2} = -1$

Examples: Linear and Stable Summation Method (Cont.)

$1+2+3+\cdots$ is NOT Summable Under Any Linear Stable Method

Suppose there exists linear and stable Σ s.t. $\Sigma(\{n\}_{n=1}^{\infty}) = S \in \mathbb{R}$,

$$\begin{aligned} S &= 1+2+3+4+5+\cdots \\ S &= 0+1+2+3+4+\cdots && \text{(stability)} \\ \implies & 1+1+1+\cdots = 0 && \text{(linearity)} \end{aligned}$$

Apply the same trick (shifting and subtraction) on $1+1+1+\cdots$, one can arrive at

$$0+0+0+0+\cdots = -1$$

which is not compatible with linearity.

$1+2+3+4+\cdots$ is Ramanujan Summable

Ramanujan Summation

For function f with no divergence at zero

$$C(a)=\int_0^a f(t)dt-\frac{1}{2}f(0)-\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}f^{(2k-1)}(0)$$

where B_{2k} is the $2k$-th Bernoulli number and $C(0)$ is used as the sum of the divergent sequence. Consider $f(x)=x$ and $B_2=\frac{1}{6}$,

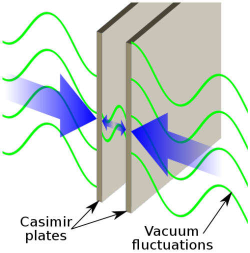
$$f(1)+f(2)+f(3)+\cdots=-\frac{1}{2}f(0)-\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}f^{(2k-1)}(0)=-\frac{1}{12}$$

Why Bother?

$$1 + 2 + 3 + 4 + 5 + 6 + \cdots \longrightarrow \text{undefined}$$

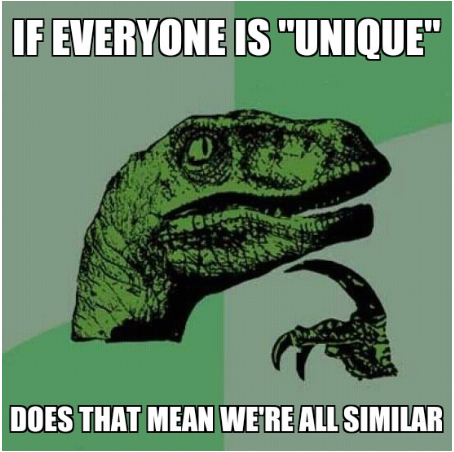
$$1 + 2 + 3 + 4 + 5 + 6 + \cdots \xrightarrow{\mathfrak{R}} -\frac{1}{12}$$

- Mathematically, it is interesting
  - Any consistently defined mathematical object deserves study
  - Classification of "convergence"
- There are many use cases in physics
  - String theory
  - Quantum Mechanics: Casimir Effect
    - Attractive force between uncharged parallel metallic plates in a vacuum



Outline

Uniqueness



Going Above and Beyond

Analytic Function

An analytic function is an infinitely differentiable function such that its Taylor series converges pointwise in the domain.

Analytic Continuation

Analytic continuation is a technique to extend the domain of a given analytic function while remain analytic. If an analytic continuation exists, it is guaranteed to be unique. Therefore, surprisingly, knowing the value of a complex function in some finite complex domain uniquely determines the value of the function at every other point.

Analytic Continuation of Euler Zeta Function

Amazing video from 3Blue1Brown: Visualizing the Riemann hypothesis and analytic continuation. (9 min 40 s – 17 min 20 s)

- Euler established that

$$E(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} n^{-s}$$

  converges in Cauchy sense for $\forall s > 1$
  (Hint: easy check using integral test.)

- Riemann treated it as a complex function and found its analytic continuation ζ which satisfies the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$$

Sum of All Natural Numbers

To find the sum of all natural numbers, it remains to find $\zeta(-1)$,

$$\zeta(-1) = 2^{-1}\pi^{-2}\sin\left(-\frac{\pi}{2}\right)\Gamma(2)\zeta(2) = -\frac{1}{2\pi^2}\zeta(2)$$

where

- $\Gamma(n) = (n-1)!$ for $\forall n \in \mathbb{N}$, hence $\Gamma(2) = 1$

We will now compute $\zeta(2)$, i.e. $E(2)$, which is convergent in Cauchy sense

$$\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

Basel Problem: $\zeta(2) = E(2) = \frac{\pi^2}{6}$

Consider $p(x) = \frac{\sin(x)}{x}$,

- Power Series Expansion

$$p(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}$$

- Factorization (Formally, Weierstrass Factorization Theorem)

$$p(x) = (1 - \frac{x}{\pi})(1 + \frac{x}{\pi})(1 - \frac{x}{2\pi})(1 + \frac{x}{2\pi})(1 - \frac{x}{3\pi})(1 + \frac{x}{3\pi}) \cdots$$
$$= \prod_{n=1}^{\infty} (1 - \frac{x^2}{n^2\pi^2}) = 1 - (\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots)x^2 + \cdots$$

Basel Problem: $\zeta(2) = E(2) = \frac{\pi^2}{6}$ (Cont.)

The two expression of $p(x)$ must match, i.e. coefficients for each x^k power mush agree. In particular, let's look at coefficients for x^2 term. (This method works for any even order. Euler obtained up to 26.)

$$-\frac{1}{3!} = -\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots\right)$$

$$-\frac{1}{6} = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\implies \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Consequently, we have $\zeta(-1) = -\frac{1}{12}$

Bonus: Riemann Hypothesis

Euler showed that (now known as Euler Product Formula)

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} = \frac{1}{1-2^{-s}} \cdot \frac{1}{1-3^{-s}} \cdot \frac{1}{1-5^{-s}} \cdot \cdots$$

In regards to zeros, Riemann hypothesized, for $\forall s \in \mathbb{C}$ s.t. $\zeta(s) = 0$

- Either, $s = -2n$ for some $n \in \mathbb{N}$, known as *trivial zeros*
- Or, $\mathrm{Re}(s) = \frac{1}{2}$, forming a *critical line*

Questions?