MA: ORDINARY DIFFERENTIAL EQUATIONS AND VECTOR CALCULUS

- UNIT-I: First Order ODE
- UNIT-II: Ordinary Differential Equations of Higher Order
- UNIT-III: Laplace transforms
- UNIT-IV: Vector Differentiation
- UNIT-V: Vector Integration

UNIT-I: First Order ODE

- Exact differential equations
- Equations reducible to exact differential equations
- Linear equations
- Bernoulli's equations
- Orthogonal trajectories (only in Cartesian Coordinates)
- Applications:
  - Newton's law of cooling
  - Law of natural growth and decay

Exact differential equations

- The equation should be in the form of $Mdx + Ndy = 0$
- Such that $\frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$
- Then, The general Solution is $\int_{(y\ constants)} Mdx + \int_{(eliminate\ x\ terms)} N\,dy = 0$

Equations reducible to exact differential equations

Method 1

- IF Equation $Mdx + Ndy = 0$ is **homogenous**
- Such that $\frac{\delta M}{\delta y} \neq \frac{\delta N}{\delta x}$
  - Find Integrating Factor $I_f = \frac{1}{Mx + Ny}$
  - Multiply $Mdx + Ndy = 0$ with I_f
  - Find the general solutions with the Exact Method for the resultant equation.

Method 2

- IF Equation $Mdx + Ndy = 0$ is **non-homogenous**
- Such that $\frac{\delta M}{\delta y} \neq \frac{\delta N}{\delta x}$ and $y\,f(x,y)dx + x\,g(x,y)dy = 0$
    - Find Integrating Factor $I_f = \frac{1}{Mx - Ny}$
    - Multiply $Mdx + Ndy = 0$ with I_f
    - Find the general solutions with the Exact Method for the resultant equation.

Method 3

- IF Equation $Mdx + Ndy = 0$ is **non-homogenous**
- Such that $\frac{\delta M}{\delta y} \neq \frac{\delta N}{\delta x}$ and $y\,f(x,y)dx + x\,g(x,y)dy \neq 0$
    - Find Integrating Factor $I_F = e^{\int f(x)}$ where $f(x) = \frac{1}{N}\left[\frac{\delta M}{\delta y} - \frac{\delta N}{\delta x}\right]$
    - Multiply $Mdx + Ndy = 0$ with I_f
    - Find the general solutions with the Exact Method for the resultant equation.

Method 4

- IF Equation $Mdx + Ndy = 0$ is **non-homogenous**
- Such that $\frac{\delta M}{\delta y} \neq \frac{\delta N}{\delta x}$ and $y\,f(x,y)dx + x\,g(x,y)dy \neq 0$
    - Find Integrating Factor $I_F = e^{\int g(x)}$ where $g(x) = \frac{1}{M}\left[\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right]$
    - Multiply $Mdx + Ndy = 0$ with I_f
    - Find the general solutions with the Exact Method for the resultant equation.

Linear equations

- IF Equation is in the form $\frac{\delta y}{\delta x} + y\,P(x) = Q(x)$
  - Then, Integrating Factor $I_F = e^{\int P(x)\,dx}$
  - And General Solution is $y\,.\,I_f = \int I_f\,Q(x)dx + c$

Bernoulli's equations

- IF Equation is in the form $\frac{\delta y}{\delta x} + y\,P(x) = y^n.\,Q(x)$
- Convert into the equation $\frac{1}{y^n}.\frac{\delta y}{\delta x} + y^{1-n}\,P(x) = Q(x)$
- Let $y^{n-1} = t$ and differentiate
- After solving $\frac{\delta t}{\delta x} + t\,P_1(x) = Q_1(x)$
- Solve using Linear equations method

Orthogonal trajectories (only in Cartesian Coordinates)

- If $f\,(x,\ y,\ c) = 0$
  - Differentiate with respect to x giving $f(x,\ y,\ \frac{\delta y}{\delta x}) = 0$
  - Replace $\frac{\delta y}{\delta x}$ with $\frac{-\,\delta x}{\delta y}$ giving $f(x,\ y,\ \frac{-\,\delta x}{\delta y}) = 0$ which is the DE of the family of curves.
  - Soive (**INTEGRATION**) $f(x,\ y,\ \frac{-\,\delta x}{\delta y}) = 0$

- If $f(x,\ y,\ \frac{-\,\delta x}{\delta y}) = 0$ is equal to $f(x,\ y,\ \frac{-\,\delta x}{\delta y}) = 0$ It is self Orthogonal.

Applications:

Newton's law of cooling

- $\frac{\delta\Theta}{\delta t} \alpha (\theta - \theta_0)$ or $\theta - \theta_0 = c\, e^{-kt}$
    - Where θ is temperature at tile "t" and θ_0 is the temperature of medium or room temperature.
- Case i: t = 0 to find "c"
- Case ii: t $\neq$ 0 to find "k"
- Case iii: Find the required value θ or t or θ_0

Law of natural growth and decay

- $\frac{\delta X}{\delta t} \alpha X$ **or** $\frac{\delta X}{\delta t} = \pm k X$ **or** $X = c\, e^{\pm kt}$
- Case i: t = 0 to find "c"
- Case ii: t $\neq$ 0 to find "k"
- Case iii: Find the required value X or "t"

UNIT-II: Ordinary Differential Equations of Higher Order

- Second order linear differential equations with constant coefficients
  - Non-homogeneous terms of the type:
    - e^{ax}
    - $sin\ ax$
    - $cos\ ax$
    - polynomials in x
    - $e^{ax}V(x)$ and $x\,V(x)$
- Method of variation of parameters
- Equations reducible to linear ODE with constant coefficients:
  - Legendre's equation
  - Cauchy-Euler equation
- Applications:
  - Electric circuits

Second order linear differential equations with constant coefficients

Given $f(D)y = 0$

General Solution is $Y = Y_c$

- Finding Y_c:
  - For Real and different
    - If $m_1, m_2, m_3, \ldots m_n$ are the roots for x then

      $$C.F = c_1e^{m_1x} + c_2e^{2x} + \ldots + c_ne^{m_nx}$$
  - For real and same
    - If $m_1, m_1, m_2, \ldots m_n$ are the roots for x then

      $$Constant\ coefficients = (c_1 + c_2x)e^{m_1x} + c_2e^{2x} + \ldots + c_ne^{m_nx}$$
  - For complex roots
    - If $\alpha \pm \beta i$ are the roots of x then

      $$Constant\ coefficients = (c_1cos\beta x + c_2sin\beta x)e^{\alpha x}$$

Non-homogeneous terms of the type:

Given $f(D)y = \phi(x)$

General Solution is $Y = Y_c + Y_p$

- Finding Y_c
  - USing pervious methods
- Finding Y_p
  - $Y_p = \frac{1}{f(D)} \phi(x)$
  - If $\phi(x) = e^{ax}$
    - Replace $f(D)$ with $f(a)$
    - simplify
      - If $f(a) = 0$
      - $Y_p = \frac{1}{f(a)} \frac{x^m}{m!} e^{ax}$ where $\frac{1}{f(a)}$ is non zero.
  - If $\phi(x) = sin\ ax$ or $\phi(x) = cos\ ax$
    - In $f(D)$ replace $D^2\ with\ -a^2$
    - Simplify (Only constants in the denominator)
      - If $f(-a^2) = 0$
      - For $\phi(x) = sin\ ax$ => $Y_p = -\frac{x}{2a} cos\ ax$
      - For $\phi(x) = cos\ ax$ => $Y_p = \frac{x}{2a} sin\ ax$
  - If $\phi(x) = x^m$
    - Redice $\frac{1}{f(D)} x^m$ => $\frac{1}{1 \pm \phi(D)} x^m$ => $(1 \pm \phi(D))^{-n} x^m$
      - Replace with $(1 + D)^{-1} = 1 - D + D^2 - D^3$...

        $(1 - D)^{-1} = 1 + D + D^2 + D^3$...

        $(1 + D)^{-2} = 1 - 2D + 3D^2 - 4D^3$...

        $(1 - D)^{-2} = 1 + 2D + 3D^2 + 4D^3$...

        $(1 + D)^{-3} = 1 - 3D + 6D^2 - 10D^3$...

        $(1 - D)^{-3} = 1 + 3D + 36 + 10D^3$...
      - Simplify
  - If $\phi(x) = e^{ax}.V(x)$
    - $Y_p = \frac{1}{f(D)} e^{ax}.V(x) = e^{ax}[\frac{1}{f(D+a)}. V(x)]$

        - Simplify $\frac{1}{f(D+a)}.\ V(x)$ using previous methods
        - Simplify (Only constants in the denominator)
    - If $\phi(x) = x.V(x)$
        - $Y_p = \frac{1}{f(D)}x.V(x) = [x - \frac{f^1(D)}{f(D)}]\frac{1}{f(D)}.\ V(x)$
        - Simplify $\frac{1}{f(D)}.\ V(x)$ using previous methods
        - Simplify (Only constants in the denominator)
- **General Solution is Y = Yc + Yp**

Method of variation of parameters

- Reduce the equation to $(D^2 + PD + Q)y = R$
- Find Y_c using previous methods
    - Where $Y_c = c_1 u(x) + c_2 v(x)$
- Find Y_p
    - $Y_p = Au + Bv$
    - Where $A = -\int \frac{v\,R}{w(u,v)}$ where $w(u, v) = uv' - vu'$
    - And $B = \int \frac{u\,R}{w(u,v)}$ where $w(u, v) = uv' - vu'$

Equations reducible to linear ODE with constant coefficients:

Cauchy-Euler equation

- Reduce equation to $\left(x^n \frac{d^n x}{dy^n} + p_1 x^{n-1} \frac{d^{n-1}x}{dy^{n-1}} + p_2 \frac{d^{n-2}x}{dy^{n-2}} + ... + p_n\right)y = Q$

  I.e, $\left(x^n D^n + p_1 x^{n-1} D^{n-1} + p_2 x^{n-2} D^{n-2} + ... + p_n\right)y = Q$

- Let $x = e^z,\ z = log\, x$ and

  $x\ D = \theta,\ x^2 D^2 = \theta(\theta - 1),\ x^3 D^3 = \theta(\theta - 1)(\theta - 2),\ ....$

- Find $Y_c\ and\ Y_p$ for the equation with z as variable and the general solution
- Replace z with x

Legendre’s equation

- Reduce the equation to

  $\left((ax + b)^n D^n + p_1 (ax + b)^{n-1} D^{n-1} + p_2 (ax + b)^{n-2} D^{n-2} + ... + p_n\right)y = Q$

- Let $ax + b = e^z,\ z = log\,(ax + b)$ and

  $(ax + b)\ D = \theta,\ (ax + b)^2 D^2 = \theta(\theta - 1),\ (ax + b)^3 D^3 = \theta(\theta - 1)(\theta - 2),\ ....$

- Find $Y_c\ and\ Y_p$ for the equation with z as variable and the general solution
- Replace z with (ax + b)

Applications:

Electric circuits

UNIT-III: Laplace transforms

- Laplace Transform of standard functions
- First shifting theorem
- Second shifting theorem
- Unit step function
- Dirac delta function
- Laplace transforms of functions when they are multiplied and divided by 't'
- Laplace transforms of derivatives and integrals of function
- Evaluation of integrals by Laplace transforms
- Laplace transform of periodic functions
- Inverse Laplace transform by different methods
- Convolution theorem (without proof)
- Applications:
    - Solving initial value problems by Laplace Transform method

Laplace Transform of standard functions

$f(t)$	$L\{f(t)\}$
1	$\frac{1}{s}$
K (constant)	$\frac{k}{s}$
t^n	$\frac{n!}{s^{n+1}}$
t^n (n is fraction or negative)	$\frac{s+n}{s^{n+1}}$
$Sin\ at$	$\frac{a}{s^2+a^2}$
$Cos\ at$	$\frac{s}{s^2+a^2}$
e^{at}	$\frac{1}{s-a}$
e^{-at}	$\frac{1}{s+a}$

$Sin\ hat$	$\frac{a}{s^2-a^2}$
$Cos\ hat$	$\frac{s}{s^2-a^2}$
t	$\frac{1}{a^2}$
$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$

First shifting theorem

If $L\{f(t)\} = \overline{f}(s)$ then $L\left\{e^{at}f(t)\right\} = \{\overline{f}(t)\}_{s \to s-a} = \{\overline{f}(s - a)\}$

Second shifting theorem

If $L\{f(t)\} = \overline{f}(s)$ and $g(t) = \left\{\frac{f(t-a),\, t>a}{0,\, t<a}\right\}$ then $L\{g(t)\} = e^{-as}\overline{f}(s)$ or

$$L\{f(t - a).g(t - a)\} = e^{-as}\overline{f}(s)$$

Unit step function

$$L\{u(t - a)\} = \frac{e^{-as}}{s}$$

Dirac delta function

$$L\{f(t - a)\} = e^{-as}$$

Change of Scale Property

$$L\{f(at)\} = \frac{1}{a}\overline{f}(\frac{s}{a})$$

Laplace transforms of functions when they are multiplied and divided by 't'

If $L\{f(t)\} = \overline{f}(s)$ then

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} \overline{f}(s)$$

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \overline{f}(s)$$

$$Sin\,A * Cos\,B = \frac{1}{2}(Sin(A+B) + Sin(A-B))$$

$$Cos\,A * Sin\,B = \frac{1}{2}(Sin(A+B) - Sin(A-B))$$

$$Cos\,A * Cos\,B = \frac{1}{2}(Cos(A+B) + Cos(A-B))$$

$$Sin\,A * Sin\,B = \frac{1}{2}(Cos(A-B) - Cos(A+B))$$

Laplace transforms of derivatives and integrals of function

If $L\{f(t)\} = \overline{f}(s)$ then

$$L\left\{\int_0^t f(t)dt\right\} = \frac{1}{s}\overline{f}(s)$$

Evaluation of integrals by Laplace transforms

If $L\{f(t)\} = \overline{f}(s)$ then

$$L\left\{\int_0^\infty e^{-at} f(t)dt\right\} = [\overline{f}(s)]_{s=a}$$

Laplace transform of periodic functions

If $L\{f(t)\} = \overline{f}(s)$ then

$$L\{f(t)\} = \frac{1}{1-e^{sT}}\int_0^T e^{-st}f(t)dt$$

Inverse Laplace transform by different methods

$\bar{f}(s)$	$L^{-1}\{\bar{f}(s)\} = f(t)$
$\frac{1}{s}$	1
$\frac{1}{s^{n+1}}$, n = +ve	$\frac{t^n}{n!}$
$\frac{1}{s^{n+1}}$, n < -1	$\frac{t^n}{\Gamma(n+1)}$
$\frac{1}{s-a}$	e^{at}
$\frac{1}{s+a}$	e^{-at}
$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sin at$
$\frac{s}{s^2+a^2}$	$\cos at$
$\frac{1}{s^2-a^2}$	$\frac{1}{a}\sin at$
$\frac{s}{s^2-a^2}$	$\cos at$

$\frac{1}{(s-a)^2+b^2}$	$\frac{1}{b}e^{at}\sin bt$
$\frac{s-a}{(s-a)^2+b^2}$	$e^{at}\cos bt$
$\frac{1}{(s-a)^2-b^2}$	$\frac{1}{b}e^{at}\sinh bt$
$\frac{s-a}{(s-a)^2-b^2}$	$e^{at}\cosh bt$
$\frac{2as}{(s^2+a^2)^2}$	$t\sin at$
$\frac{s^2-a^2}{(s^2+a^2)^2}$	$t\cos at$

Could use Partial fraction when denominator has

S.No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}$, $a \neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$
	• where $x^2 + bx + c$ cannot be factorised further	

Convolution theorem (without proof)

$$L^{-1}\{\overline{f}(s) * \overline{g}(s)\} = f(u) * g(u) = \int_{o}^{u} f(t) * g(t - u)\, dt$$

$$\overline{f^{n}}(s) = \frac{(-1)^{n}\, \overline{f^{n}}(s)}{t}$$

Applications:

Solving initial value problems by Laplace Transform method

$$y^{i} = s.L(y) - y(0)$$

$$y^{ii} = s^{2}.L(y) - s.y(0) - y^{i}(0)$$

$$y^{iii} = s^{3}.L(y) - s^{2}.y(0) - s.y(0) - y^{i}(0)$$

UNIT-IV: Vector Differentiation

- Vector point functions and scalar point functions
- Gradient
- Divergence and Curl
- Directional derivatives
- Tangent plane and normal line
- Vector identities
- Scalar potential functions
- Solenoidal and Irrotational vectors

Vector point functions and scalar point functions

$$\frac{\delta}{\delta t}(\overline{a} \pm \overline{b}) = \frac{\delta}{\delta t}\overline{a} \pm \frac{\delta}{\delta t}\overline{b}$$

$$\frac{\delta}{\delta t}(\overline{a} \,.\, \overline{b}) = \frac{\delta}{\delta t}\overline{a} \,.\, \overline{b} + \overline{a} \,.\, \frac{\delta}{\delta t}\overline{b}$$

$$\frac{\delta}{\delta t}(\overline{a}\, x\, \overline{b}) = \frac{\delta}{\delta t}\overline{a}\, x\, \overline{b} + \overline{a}\, x\, \frac{\delta}{\delta t}\overline{b}$$

Gradient

$$\nabla = \overline{i}\frac{\delta}{\delta x} + \overline{j}\frac{\delta}{\delta y} + \overline{k}\frac{\delta}{\delta z}$$

$$\nabla\phi = \overline{i}\frac{\delta\phi}{\delta x} + \overline{j}\frac{\delta\phi}{\delta y} + \overline{k}\frac{\delta\phi}{\delta z}$$

Directional derivatives

Directional derivatives ϕ in direction of $\overline{e} = \frac{\nabla f}{|\overline{\nabla f}|}$

Find $\overline{e} \,.\, \nabla\phi$

Unit Vector:

For Vector: $\bar{e} = \frac{\bar{a}}{|\bar{a}|}$,

For Scalar: $\bar{e} = \frac{\nabla f}{|\overline{\nabla f}|}$

In x-y Plane, $\bar{e} = -\bar{k}$

In y-z Plane, $\bar{e} = -\bar{i}$

In z-x Plane, $\bar{e} = -\bar{j}$

Angle between 2 surfaces:

$$Cos\,\Theta = \frac{\nabla f \,.\, \nabla g}{|\nabla f|\,|\nabla g|}$$

Generally,

$$\bar{r} = x\,\bar{i} + y\,\bar{j} + z\,\bar{k}$$

$$|\bar{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

$$\frac{\delta r}{\delta x} = \frac{x}{r}, \frac{\delta r}{\delta y} = \frac{y}{r}, \frac{\delta r}{\delta z} = \frac{z}{r}$$

Tangent plane and normal line

Normal Line: $\overline{e} = \frac{\nabla f}{|\overline{\nabla f}|}$ at (x, y, z)

Internally Orthogonal: $\nabla f . \nabla g = 0$

Vector identities

Triple Products

$$A \bullet (B \times C) = B \bullet (C \times A) = C \bullet (A \times B)$$

$$A \times (B \times C) = (C \times B) \times A = B(A \bullet C) - C(A \bullet B)$$

Product Rules

$$\nabla (fg) = f\nabla g + g\nabla f$$

$$\nabla(A \bullet B) = A \times (\nabla \times B) + (A \bullet \nabla)B + B \times (\nabla \times A) + (A \bullet \nabla)B$$

$$\nabla \bullet (fA) = f(\nabla \bullet A) + A \bullet (\nabla f)$$

$$\nabla \times (fA) = f(\nabla \times A) - A \times (\nabla f)$$

$$\nabla \bullet (A \times B) = B \bullet (\nabla \times A) + A \bullet (\nabla \times B)$$

$$\nabla \times (A \times B) = (B \bullet \nabla)A - (A \bullet \nabla)B + A(\nabla \bullet B) - B(\nabla \bullet A)$$

Scalar potential functions

$$\overline{F} = \nabla\phi$$

Solenoidal and Irrotational vectors

Divergence of Vector

$$div\,\overline{f} = \overline{i} \bullet \frac{\delta\overline{f}}{\delta x} + \overline{j} \bullet \frac{\delta\overline{f}}{\delta y} + \overline{k} \bullet \frac{\delta\overline{f}}{\delta z}$$

Solenoidal Vector: $div\,\overline{f} = 0$

Curl of Vector

$$Curl\,\overline{f} = \overline{i} \times \frac{\delta\overline{f}}{\delta x} + \overline{j} \times \frac{\delta\overline{f}}{\delta y} + \overline{k} \times \frac{\delta\overline{f}}{\delta z}$$

$$Curl\,\overline{f} = \sum \overline{i} \times \frac{\delta\overline{f}}{\delta x}$$

Irrotational of Vector: $curl\,\overline{f} = 0$

UNIT-V: Vector Integration

- Line integrals
- Surface integrals
- Volume integrals
- Theorems of Green, Gauss and Stokes (without proofs) and their applications

Line integrals

Work Done

$$\int_{A}^{B} \overline{F} \bullet d\overline{r}$$

Where, $\overline{F}$ *is a vector*, $\overline{r} = x\,\overline{i} + y\,\overline{j} + z\,\overline{k}$, A to B is displacement

Line integrals

$$\int_{C} \overline{F} \bullet d\overline{r}$$

Where, $\overline{F}$ *is a vector*, $\overline{r} = x\,\overline{i} + y\,\overline{j} + z\,\overline{k}$, C is a curve

Surface integrals

$$\int_{S} \overline{F} \bullet \overline{n}\, ds = \int\int_{R} \frac{\overline{F} \bullet \overline{n}}{|\overline{F} \bullet \overline{k}|} dx\, dy$$, For x-y Plane

$$\int_{S} \overline{F} \bullet \overline{n}\, ds = \int\int_{R} \frac{\overline{F} \bullet \overline{n}}{|\overline{F} \bullet \overline{i}|} dy\, dz$$, For y-z Plane

$$\int_{S} \overline{F} \bullet \overline{n}\, ds = \int\int_{R} \frac{\overline{F} \bullet \overline{n}}{|\overline{F} \bullet \overline{j}|} dz\, dx$$, For z-x Plane

Volume integrals

$$\overline{F} = F_1\overline{i} + F_2\overline{j} + F_3\overline{k}$$

$$\int_V \overline{F} dv = \int_x \int_y \int_z (F_1\overline{i} + F_2\overline{j} + F_3\overline{k}) dx\, dy\, dz$$

Theorems of Green, Gauss and Stokes (without proofs) and their applications

Green Theorem

$$\oint_c M dx + N dy = \int\int_R \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right) dx\, dy$$

Gauss Theorem

$$\oint_s F \bullet \overline{n}\, ds = \int_v \nabla \bullet \overline{F}\, dv$$

Stokes Theorem

$$\oint F \bullet \overline{n}\, dr = \int\int curl(F) \bullet n\, ds$$