第 1 次作业

2022 年秋季学期

截止日期：2022-09-19 **允许讨论，禁止抄袭**

1. 针对以下矩阵求解 e^{At}.

   (1) $A=\begin{bmatrix}0 & 6\\ -1 & -5\end{bmatrix}$

   (2) $A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -6 & -3\end{bmatrix}$

2. 计算如下时间函数的拉普拉斯变换

   (1) $f(t)=\mathrm{e}^{-t}+2\mathrm{e}^{-2t}+t\mathrm{e}^{-3t}$;

   (2) $f(t)=t\sin(t)$.

3. 计算如下拉普拉斯变换变换式对应的时间函数

   (1) $F(s)=\frac{2}{s(s+2)}$;

   (2) $F(s)=\frac{10}{s(s+1)(s+10)}$;

   (3) $F(s)=\frac{2(s+2)}{(s+1)(s^2+4)}$.

4. 分别用状态空间方法和拉普拉斯变换求解如下常微分方程.

   (1) $\ddot{y}(t)+\dot{y}(t)+3y(t)=0$, $y(0)=1$, $\dot{y}(0)=2$;

   (2) $\ddot{y}(t)+\dot{y}(t)=\sin(t)$, $y(0)=1$, $\dot{y}(0)=2$;

   (3) $\ddot{y}(t)+2\dot{y}(t)=\mathrm{e}^{t}$, $y(0)=1$, $\dot{y}(0)=2$.

5. 考虑如下单输入单输出系统

$$
\begin{aligned}
\dot{x}(t)&=\begin{bmatrix}0 & 1\\ -3 & -5\end{bmatrix}x(t)+\begin{bmatrix}0\\ 1\end{bmatrix}u(t)\\
y(t)&=\begin{bmatrix}3 & 0\end{bmatrix}x(t)
\end{aligned}
$$

写出系统的传递函数 $G(s) = \frac{Y(s)}{U(s)}$.

6. 考虑如下带有一个共振状态的“四分之一车模型”的传递函数

$$G(s) = \frac{2s+4}{s^2(s^2+2s+4)}$$

写出该传递函数对应的状态空间模型.

7. 考虑如下 3 阶常微分方程描述的系统

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}x(t) + 3\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) + 3\frac{\mathrm{d}}{\mathrm{d}t}x(t) + x(t) = \frac{\mathrm{d}^3}{\mathrm{d}t^3}u(t) + 2\frac{\mathrm{d}^2}{\mathrm{d}t^2}u(t) + 4\frac{\mathrm{d}}{\mathrm{d}t}u(t) + u(t)$$

写出该系统的传递函数和状态空间模型.

8. 如图 1 所示, 质量为 M 的物体与弹簧和摩擦力构成的质量-弹簧-阻尼系统进行运动.

   (1) 推导由于外力 $r(t)$ 作用产生的运动方程, 即物体位移 $y(t)$ 如何变化. 此系统中, k 为弹簧系数, b 为墙摩擦力的摩擦系数.

   (2) 用状态空间模型描述该质量-弹簧-阻尼系统.

   (3) 写出该质量-弹簧-阻尼系统以外力 $r(t)$ 为输入、以位移 $y(t)$ 为输出的传递函数.

   (4) 当 $M=1$, $b=3$, $k=2$, $y(0)=1$, $\dot{y}(0)=0$, 以及 $r(t)=1$, $t \geq 0$ 时, 分别用状态空间方法和拉普拉斯变换方法求解物体位移 $y(t)$.

   (5) 用 matlab 或 python 等绘制 $y(t)$ 随时间演化的曲线.

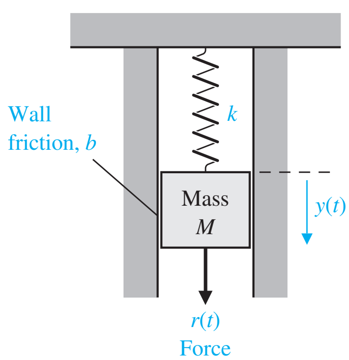


Figure 1: 质量-弹簧-阻尼系统

1、(1)

$$e^{At}=\mathcal{L}^{-1}\{(sI-A)^{-1}\}$$

$$=\mathcal{L}^{-1}\left\{\begin{pmatrix} s & -6 \\ 1 & s+5 \end{pmatrix}^{-1}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{1}{s^2+5s+6}\begin{pmatrix} s+5 & 6 \\ -1 & s \end{pmatrix}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{1}{s+2}\begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}+\frac{1}{s+3}\begin{pmatrix} -2 & -6 \\ 1 & 3 \end{pmatrix}\right\}$$

$$=\begin{pmatrix} 3e^{-2t}-2e^{-3t} & 6e^{-2t}-6e^{-3t} \\ -e^{-2t}+e^{-3t} & -2e^{-2t}+3e^{-3t} \end{pmatrix}$$

(2)

$$e^{At}=\mathcal{L}^{-1}\left\{\begin{pmatrix} s & -1 & \\ & s & -1 \\ & 6 & s+3 \end{pmatrix}^{-1}\right\}$$

$$=\mathcal{L}^{-1}\left\{\begin{pmatrix} \frac{1}{s} & \frac{s+3}{s(s^2+3s+6)} & \frac{1}{s(s^2+3s+6)} \\ & \frac{s+3}{s^2+3s+6} & \frac{1}{s^2+3s+6} \\ & -\frac{6}{s^2+3s+6} & \frac{s}{s^2+3s+6} \end{pmatrix}\right\}$$

$$=\begin{pmatrix} 1 & e^{-\frac{3}{2}t}\left(\frac{\sqrt{15}}{30}\sin\frac{\sqrt{15}}{2}t-\frac{1}{2}\cos\frac{\sqrt{15}}{2}t\right)+\frac{1}{2} & e^{-\frac{3}{2}t}\left(-\frac{\sqrt{15}}{30}e^{-\frac{3}{2}t}\sin\frac{\sqrt{15}}{2}t-\frac{1}{6}e^{-\frac{3}{2}t}\cos\frac{\sqrt{15}}{2}t\right)+\frac{1}{6} \\ 0 & e^{-\frac{3}{2}t}\left(\cos\frac{\sqrt{15}}{2}t+\frac{\sqrt{15}}{5}\sin\frac{\sqrt{15}}{2}t\right) & \frac{2\sqrt{15}}{15}e^{-\frac{3}{2}t}\sin\frac{\sqrt{15}}{2}t \\ 0 & -\frac{4\sqrt{15}}{5}e^{-\frac{3}{2}t}\sin\frac{\sqrt{15}}{2}t & e^{-\frac{3}{2}t}\left(\cos\frac{\sqrt{15}}{2}t-\frac{\sqrt{15}}{5}\sin\frac{\sqrt{15}}{2}t\right) \end{pmatrix}$$

2.

(1) $\mathcal{L}\{e^{-t}+2e^{-2t}+te^{-3t}\}$

$= \frac{1}{s+1}+\frac{2}{s+2}-\left(\frac{1}{s+3}\right)'$

$= \frac{1}{s+1}+\frac{2}{s+2}+\frac{1}{(s+3)^2}$

(2) $\mathcal{L}\{t\sin t\}$

$= -\left(\frac{1}{s^2+1}\right)'$

$= \frac{2s}{(s^2+1)^2}$

3. (1) $\mathcal{L}^{-1}\{\frac{2}{s(s+2)}\}$

$= \mathcal{L}^{-1}\{\frac{1}{s}-\frac{1}{s+2}\}$

$= 1-e^{-2t}$

(2) $\mathcal{L}^{-1}\{\frac{10}{s(s+1)(s+10)}\}$

$= \mathcal{L}^{-1}\{\frac{1}{s}+\frac{-\frac{10}{9}}{s+1}+\frac{\frac{1}{9}}{s+10}\}$

$= 1-\frac{10}{9}e^{-t}+\frac{1}{9}e^{-10t}$

(3) $\mathcal{L}^{-1}\{\frac{2(s+2)}{(s+1)(s^2+4)}\}$

$= \mathcal{L}^{-1}\{\frac{\frac{2}{5}}{s+1}+\frac{-\frac{2}{5}s}{s^2+4}+\frac{\frac{12}{5}}{s^2+4}\}$

$= \frac{2}{5}e^{-t}-\frac{2}{5}\cos 2t+\frac{6}{5}\sin 2t$

4. (1) State space:

$\dot{x}(t)=\begin{pmatrix}0 & 1\\ -3 & -1\end{pmatrix}x(t), \quad x(0)=\begin{pmatrix}1\\2\end{pmatrix}$

$\Rightarrow x(t)=\exp\left\{\begin{pmatrix}0 & 1\\ -3 & -1\end{pmatrix}t\right\}\cdot\begin{pmatrix}1\\2\end{pmatrix}$

$= \begin{pmatrix} e^{-\frac{t}{2}}(\cos\frac{\sqrt{11}}{2}t+\frac{\sqrt{11}}{11}\sin\frac{\sqrt{11}}{2}t) & \frac{2\sqrt{11}}{11}e^{-\frac{t}{2}}\sin\frac{\sqrt{11}}{2}t \\ -\frac{6\sqrt{11}}{11}e^{-\frac{t}{2}}\sin\frac{\sqrt{11}}{2}t & e^{-\frac{t}{2}}(\cos\frac{\sqrt{11}}{2}t-\frac{\sqrt{11}}{11}\sin\frac{\sqrt{11}}{2}t \end{pmatrix}$

$\cdot\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix} e^{-\frac{t}{2}}(\cos\frac{\sqrt{11}}{2}t+\frac{5\sqrt{11}}{11}\sin\frac{\sqrt{11}}{2}t) \\ e^{-\frac{t}{2}}(2\cos\frac{\sqrt{11}}{2}t-\frac{8\sqrt{11}}{11}\sin\frac{\sqrt{11}}{2}t) \end{pmatrix}$

$\Rightarrow y(t)=(1\ \ 0)\,x(t)=e^{-\frac{t}{2}}(\cos\frac{\sqrt{11}}{2}t+\frac{5\sqrt{11}}{11}\sin\frac{\sqrt{11}}{2}t)$

Laplace transform:

$s^2Y(s)-s-2+sY(s)-1+3Y(s)=0$

$\Rightarrow Y(s)=\frac{s+3}{s^2+s+3}$

$\therefore y(t)=\mathcal{L}^{-1}\{\frac{s+3}{s^2+s+3}\}$

$= \mathcal{L}^{-1}\{\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{11}{4}}+\frac{\frac{5}{2}}{(s+\frac{1}{2})^2+\frac{11}{4}}\}$

$= e^{-\frac{t}{2}}\left(\cos\frac{\sqrt{11}}{2}t+\frac{5\sqrt{11}}{11}\sin\frac{\sqrt{11}}{2}t\right)$

(2) State space :

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \sin t \end{pmatrix}$$

$$x(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad y(t) = (1 \ 0)\, x(t)$$

$$\Rightarrow x(t) = \exp\left\{\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} t\right\} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \int_0^t \exp\left\{\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}(t-\tau)\right\} \cdot \begin{pmatrix} 0 \\ \sin\tau \end{pmatrix} d\tau$$

$$= \begin{pmatrix} 1 & 1-e^{-t} \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \int_0^t \begin{pmatrix} 1 & 1-e^{\tau-t} \\ 0 & e^{\tau-t} \end{pmatrix}\begin{pmatrix} 0 \\ \sin\tau \end{pmatrix} d\tau$$

$$= \begin{pmatrix} 3-2e^{-t} \\ 2e^{-t} \end{pmatrix} + \int_0^t \begin{pmatrix} (1-e^{\tau-t})\sin\tau \\ e^{\tau-t}\sin\tau \end{pmatrix} d\tau = \begin{pmatrix} -\frac{5}{2}e^{-t} - \frac{1}{2}\sin t - \frac{1}{2}\cos t + 4 \\ \frac{5}{2}e^{-t} + \frac{1}{2}\sin t - \frac{1}{2}\cos t \end{pmatrix}$$

$$\therefore y(t) = -\frac{5}{2}e^{-t} - \frac{1}{2}\sin t - \frac{1}{2}\cos t + 4$$

Laplace transform :

$$s^2 Y(s) - s - 2 + sY(s) - 1 = \frac{1}{s^2+1}$$

$$\Rightarrow Y(s) = \frac{s+3+\frac{1}{s^2+1}}{s(s+1)}$$

$$\therefore y(t) = \mathcal{L}^{-1}\left\{\frac{-\frac{5}{2}}{s+1} + \frac{-\frac{1}{2}}{s^2+1} + \frac{-\frac{1}{2}s}{s^2+1} + \frac{4}{s}\right\}$$

$$= -\frac{5}{2}e^{-t} - \frac{1}{2}\sin t - \frac{1}{2}\cos t + 4$$

(3) State space :

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ e^t \end{pmatrix}, \quad x(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\Rightarrow x(t) = \exp\left\{\begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} t\right\} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \int_0^t \exp\left\{\begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix}(t-\tau)\right\} \cdot \begin{pmatrix} 0 \\ e^\tau \end{pmatrix} d\tau$$

$$= \begin{pmatrix} 1 & \frac{1}{2}-\frac{1}{2}e^{-2t} \\ 0 & e^{-2t} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \int_0^t \begin{pmatrix} 1 & \frac{1}{2}-\frac{1}{2}e^{2\tau-2t} \\ 0 & e^{2\tau-2t} \end{pmatrix}\begin{pmatrix} 0 \\ e^\tau \end{pmatrix} d\tau$$

$$= \begin{pmatrix} 2-e^{-2t} \\ 2e^{-2t} \end{pmatrix} + \int_0^t \begin{pmatrix} \frac{1}{2}e^\tau - \frac{1}{2}e^{3\tau-2t} \\ e^{3\tau-2t} \end{pmatrix} d\tau = \begin{pmatrix} -\frac{5}{6}e^{-2t} + \frac{1}{3}e^t + \frac{3}{2} \\ \frac{5}{3}e^{-2t} + \frac{1}{3}e^t \end{pmatrix}$$

$$\therefore y(t) = -\frac{5}{6}e^{-2t} + \frac{1}{3}e^t + \frac{3}{2}$$

Laplace transform:

$$s^2Y(s)-s-2+2(sY(s)-1)=\frac{1}{s-1}$$

$$\Rightarrow Y(s)=\frac{-\frac{5}{6}}{s+2}+\frac{\frac{1}{3}}{s-1}+\frac{\frac{3}{2}}{s}$$

$$\Rightarrow y(t)=\mathcal{L}^{-1}\{Y(s)\}$$

$$=-\frac{5}{6}e^{-2t}+\frac{1}{3}e^{t}+\frac{3}{2}$$

5. $$\begin{cases} sX(s)=\begin{pmatrix}0 & 1\\ -3 & -5\end{pmatrix}X(s)+\begin{pmatrix}0\\1\end{pmatrix}U(s) \\ Y(s)=\begin{pmatrix}3 & 0\end{pmatrix}X(s)\end{cases}$$

$$\Rightarrow G(s)=\frac{Y(s)}{U(s)}=\begin{pmatrix}3 & 0\end{pmatrix}\begin{pmatrix}s & -1\\ 3 & s+5\end{pmatrix}^{-1}\begin{pmatrix}0\\1\end{pmatrix}=\frac{1}{s^2+5s+3}\begin{pmatrix}3 & 0\end{pmatrix}\begin{pmatrix}s+5 & 1\\ -3 & s\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$=\frac{3}{s^2+5s+3}$$

6. $$G(s)=\frac{2s+4}{s^4+2s^3+4s^2}$$

$$\Rightarrow y^{(4)}(t)+2y^{(3)}(t)+4\ddot{y}(t)=2\dot{u}(t)+4u(t)$$

$$\Rightarrow \dot{x}(t)=\begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -4 & -2\end{pmatrix}x(t)+\begin{pmatrix}0\\0\\0\\1\end{pmatrix}u(t)$$

$$y(t)=\begin{pmatrix}4 & 2 & 0 & 0\end{pmatrix}x(t)$$

7. $$G(s)=\frac{s^3+2s^2+4s+1}{s^3+3s^2+3s+1}$$

$$\dot{x}(t)=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -3 & -3\end{pmatrix}x(t)+\begin{pmatrix}0\\0\\0\\1\end{pmatrix}u(t)$$

$$y(t)=\begin{pmatrix}1 & 4 & 2 & 1\end{pmatrix}x(t)$$

8. (1) $-ky(t) - b\dot{y}(t) + r(t) = M\ddot{y}(t)$

$$\Rightarrow \ddot{y}(t) + \frac{b}{M}\dot{y}(t) + \frac{k}{M}y(t) = \frac{1}{M}r(t)$$

(2) $$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r(t)$$

$$y(t) = \begin{pmatrix} \frac{1}{M} & 0 \end{pmatrix} x(t)$$

(3) $$G(s) = \frac{Y(s)}{R(s)} = \frac{\frac{1}{M}}{s^2 + \frac{b}{M}s + \frac{k}{M}} = \frac{1}{Ms^2 + bs + k}$$

(4) State space:

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow x(t) = \exp\left\{\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} t\right\} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \exp\left\{\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}(t-\tau)\right\} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} d\tau$$

$$= \begin{pmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} 2e^{-\tau} - e^{-2\tau} & e^{-\tau} - e^{-2\tau} \\ -2e^{-\tau} + 2e^{-2\tau} & -e^{-\tau} + 2e^{-2\tau} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} d\tau$$

$$= \begin{pmatrix} 2e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} \end{pmatrix} + \int_0^t \begin{pmatrix} e^{-\tau} - e^{-2\tau} \\ -e^{-\tau} + 2e^{-2\tau} \end{pmatrix} d\tau = \begin{pmatrix} \frac{1}{2} + e^{-t} - \frac{1}{2}e^{-2t} \\ -e^{-t} + e^{-2t} \end{pmatrix}$$

$$\therefore y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t) = \frac{1}{2} + e^{-t} - \frac{1}{2}e^{-2t}$$

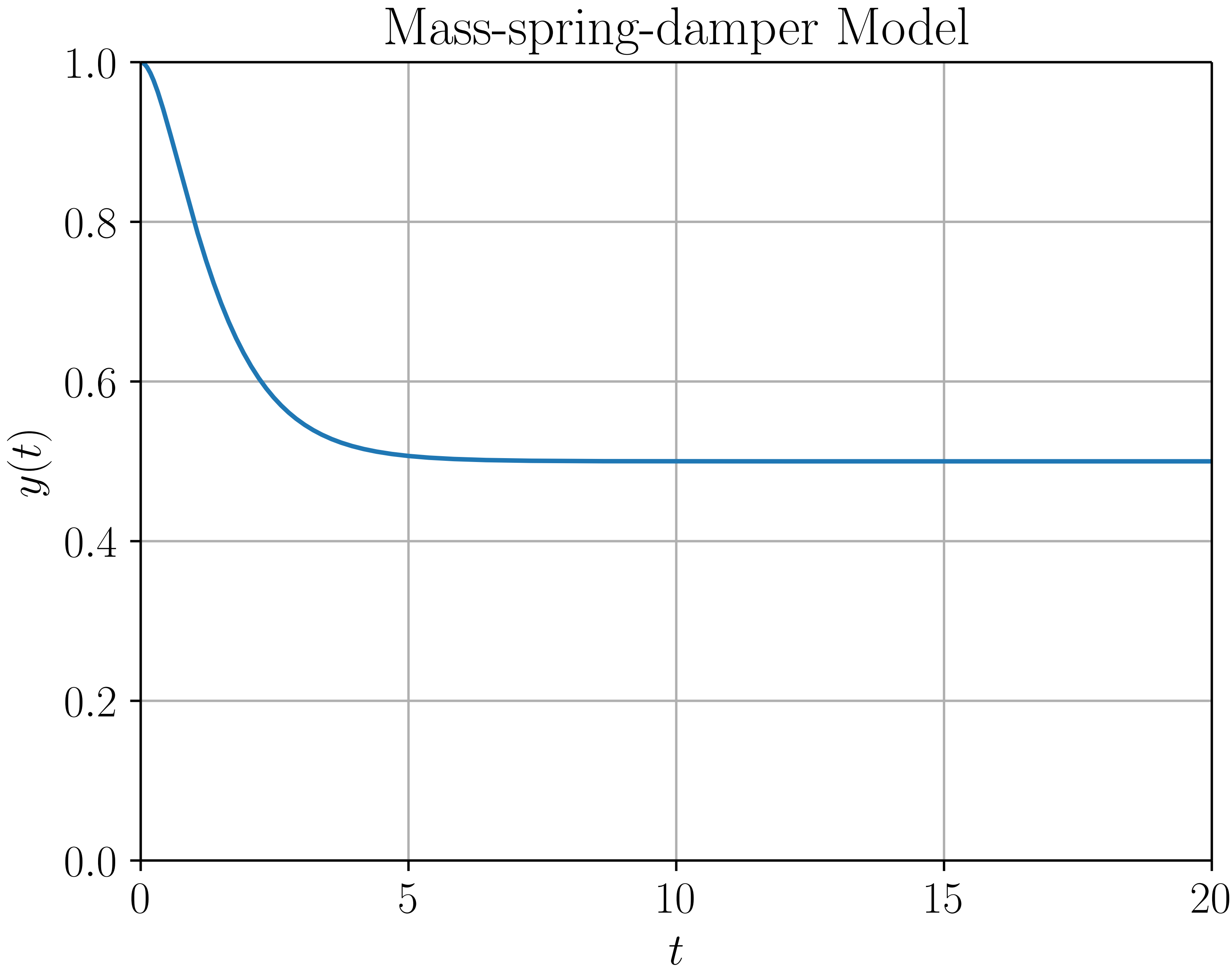
Laplace transform:

$$s^2 Y(s) - s + 3(sY(s) - 1) + 2Y(s) = \frac{1}{s}$$

$$\rightarrow Y(s) = \frac{s^2 + 3s + 1}{s(s^2 + 3s + 2)}$$

$$\therefore y(t) = \mathcal{L}^{-1}\left\{\frac{\frac{1}{2}}{s} + \frac{1}{s+1} + \frac{-\frac{1}{2}}{s+2}\right\}$$

$$= \frac{1}{2} + e^{-t} - \frac{1}{2}e^{-2t}$$


Mass-spring-damper Model
1.0
0.8
0.6
0.4
0.2
0.0
$y(t)$
0
5
10
15
20
t