I. Linear Algebra

I.1. Vector Spaces

Lecture based on

https://github.com/gwthomas/math4ml (Garrett Thomas, 2018)

https://mml-book.github.io/ (Deisenroth et al. 2020, Mathematics for Machine Learning)

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Important classes of spaces in which our **data** will live and our **operations** will take place:

- vector spaces
- metric spaces
- normed spaces
- inner product spaces

They capture important properties of Euclidean space but in a more general way.

A **vector space** V is a set of **vectors** on which two operations are defined:

- addition of two vectors
- multiplication of a vector by a real valued **scalar**[1]

V must satisfy

(i) There exists an **additive identity** (written $\mathbf{0}$) in V such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for all $\mathbf{x} \in V$

(ii) For each $\mathbf{x} \in V$, there exists an **additive inverse** (written $-\mathbf{x}$) such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$

(iii) There exists a **multiplicative identity** (written 1) in $\mathbb{R}$ such that $1\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in V$

(iv) **Commutativity**: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for all $\mathbf{x}, \mathbf{y} \in V$

(v) **Associativity**: $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ and $\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\alpha, \beta \in \mathbb{R}$

(vi) **Distributivity**: $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$ and $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$ for all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha, \beta \in \mathbb{R}$

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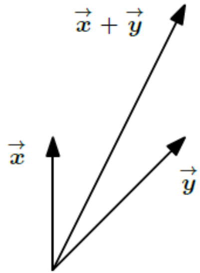
[1] More generally, vector spaces can be defined over any **field** $\mathbb{F}$. We take $\mathbb{F} = \mathbb{R}$ in this document to avoid an unnecessary diversion into abstract algebra.

The **Euclidean space** denoted $\mathbb{R}^N$ is a vector space.
The vectors in this space consist of N-tuples of real numbers:

$$\mathbf{x} = (x_1, x_2, \dots, x_N)$$

We usually treat them as $N \times 1$ matrices, or **column vectors**:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$



(Deisenroth et al., 2020)

Addition and scalar multiplication are defined component-wise on vectors in $\mathbb{R}^N$:

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}, \quad \alpha\mathbf{x} = \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

Euclidean space is used to mathematically represent physical space, with notions such as distance, length, and angles.

Although it becomes hard to visualize for $N > 3$, these concepts generalize mathematically in obvious ways.

Even when you're working in more general settings than $\mathbb{R}^N$, it is often useful to visualize vector addition and scalar multiplication in terms of 2D vectors in the plane or 3D vectors in space.

Rules for calculations with vectors

1. $\mathbf{v} + \mathbf{x} = \mathbf{w} \Leftrightarrow \mathbf{x} = \mathbf{w} - \mathbf{v}$
2. $\lambda \mathbf{v} = \mathbf{0} \Leftrightarrow \lambda = 0$ or $\mathbf{v} = 0$
3. $(-\lambda)\mathbf{v} = -(\lambda \mathbf{v})$
4. $-(\mathbf{v} + \mathbf{w}) = -\mathbf{v} - \mathbf{w}$

Note:
Despite being natural and intuitive, we have to prove that they hold for a vector space.

Example

Polynomials in $\mathbb{R}$ are a vector space.

$$a_N X^N + \cdots + a_1 X^1 + a_0 = \sum_{k=0}^{N} a_k X^k$$

where the **coefficients** $a_k \in \mathbb{R}$.
Two polynomials $\mathbf{p}$ and $\mathbf{v}$ are equal, if all their coeficients are equal

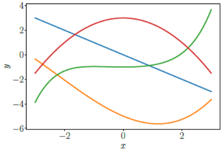
$$\sum_{i=0}^{K} a_i X^i = \sum_{i=0}^{L} b_i X^i \Leftrightarrow K = L \text{ and } a_i = b_i \forall i$$

We can add polynomials by adding all their coefficients.

$$\mathbf{p} + \mathbf{v} = \sum_{i=0}^{\max(K,L)} (a_i + b_i) X^i$$

We can multiply by a scalar $\lambda \in \mathbb{R}$

$$\lambda \mathbf{p} = \sum_{i=0}^{K} (\lambda a_i) X^i$$



(Deisenroth et al., 2020)

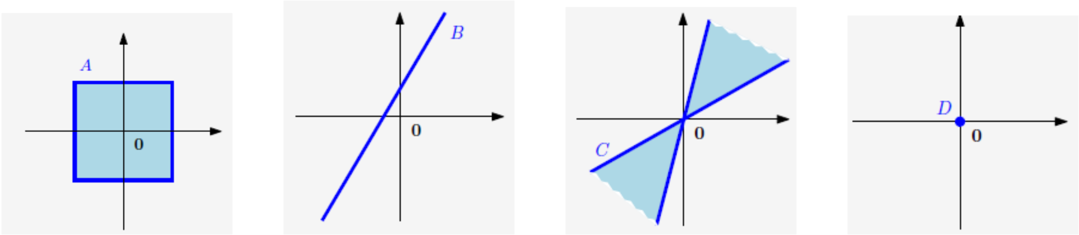
Vector spaces can contain other vector spaces.

If V is a vector space, then $S \subseteq V$ is a **subspace** of V if

1. $\mathbf{0} \in S$
2. S is closed under addition: $\mathbf{x}, \mathbf{y} \in S$ implies $\mathbf{x} + \mathbf{y} \in S$
3. S is closed under scalar multiplication: $\mathbf{x} \in S, \alpha \in \mathbb{R}$ implies $\alpha\mathbf{x} \in S$

Note that V is always a subspace of V, as is the trivial vector space which contains only $\mathbf{0}$.

As a concrete example, a line passing through the origin is a subspace of Euclidean space.



no no no yes

(Deisenroth et al., 2020)