III. Probability

III.02. Random Variables

Lecture based on

https://github.com/gwthomas/math4ml (Garrett Thomas, 2018)

Prof. Dr. Christoph Lippert

Digital Health & Machine Learning

Random Variables

A **random variable** is some uncertain quantity with an associated probability distribution over the values it can assume.

Formally, a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a function $X : \Omega \to \mathbb{R}$.

We denote the range of X by $X(\Omega) = \{X(\omega) : \omega \in \Omega\}$.

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For example, the event $X = 1$ is the set of outcomes $\{ht, th\}$.

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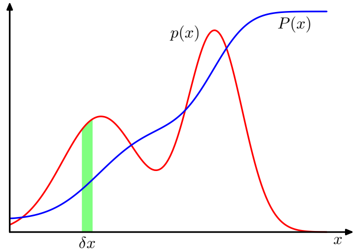
If p is parameterized by θ, we write $p(X; \theta)$ or $p(x; \theta)$

The **cumulative distribution function** (c.d.f.) gives the probability that a random variable is at most a certain value:

$$F(x) = \mathbb{P}(X \leq x)$$



(C.M. Bishop, Pattern Recognition and Machine Learning)

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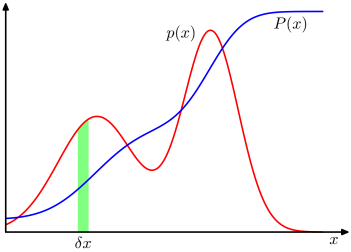
The c.d.f. can be used to give the probability that a variable lies within a certain range:

$$\mathbb{P}(a < X \leq b) = F(b) - F(a)$$



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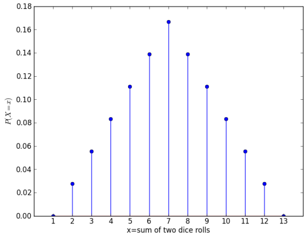
A **discrete random variable** is a random variable that has a countable range and assumes each value in this range with positive probability.
Discrete random variables are completely specified by their **probability mass function** (p.m.f.) $p : X(\Omega) \to [0,1]$ which satisfies

$$\sum_{x \in X(\Omega)} p(x) = 1$$



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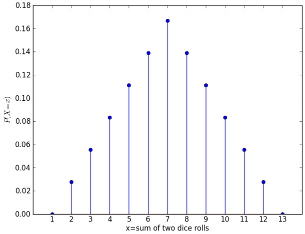
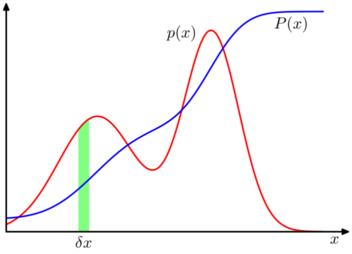
For a discrete X, the probability of a particular value is given exactly by its p.m.f.:

$$\mathbb{P}(X = x) = p(x)$$





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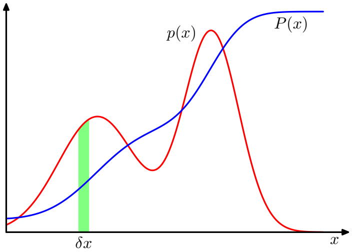
A **continuous random variable** is a random variable that has an uncountable range and assumes each value in this range with probability zero.



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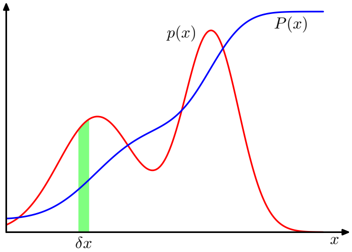
Usually there exists a function $p : \mathbb{R} \to [0, \infty)$ that satisfies

$$F(x) \equiv \int_{-\infty}^{x} p(z)\,\mathrm{d}z$$



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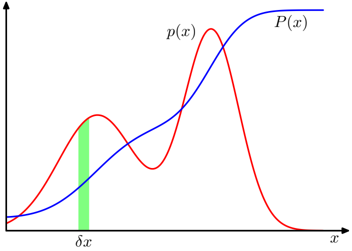
p is called a **probability density function** (p.d.f.).



Hence, the p.d.f. must satisfy

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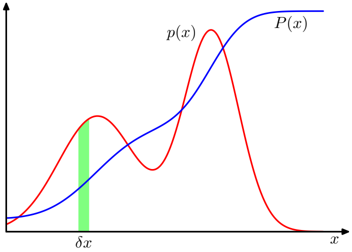

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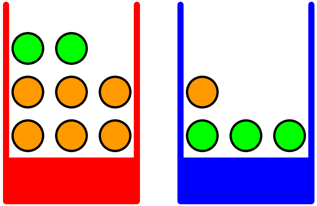
The values of this function are not themselves probabilities, since they could exceed 1.

Multiple Random Variables



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Example

We repeat ten times:

1. We pick one of the jars with equal probabilities.
2. We sample (with replacement) a fruit from the jar.

We define the random variables

- X: Number of times we chose the red jar (R).
- Y: Number of times we picked an orange (O).

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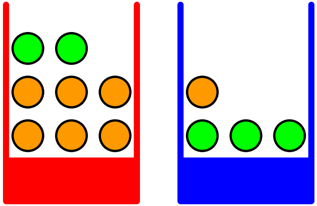

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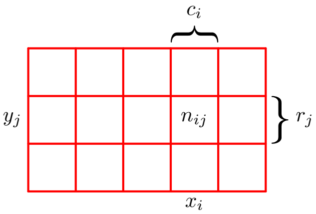
For some random variables $X_1, \ldots, X_n$, the **joint distribution** is written $p(X_1, \ldots, X_n)$ and gives probabilities over entire assignments to all the X_i simultaneously.



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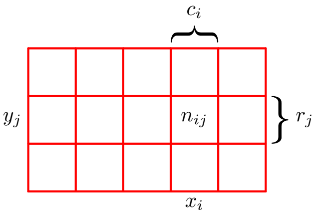
For $N \to \infty$:

- $p(x_i, y_j) = \frac{n_{ij}}{N}$

Example

Random variables $X \in [1, ..., M]$, $Y \in [1, ..., L]$.
After N draws, we define:



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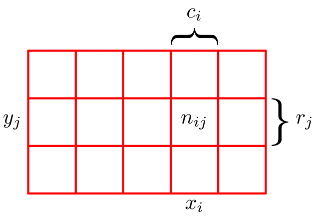

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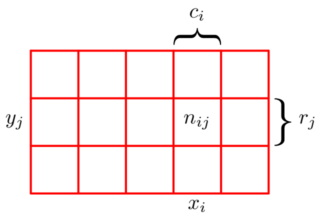

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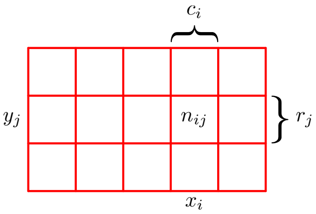
Joint distributions
Sum Rule



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For $N \to \infty$:

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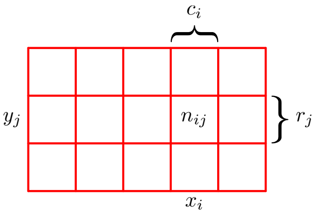

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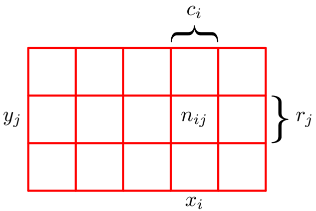
If we have a joint distribution over some set of random variables, it is possible to obtain a distribution for a subset of them by "summing out" (or "integrating out" in the continuous case) the variables we don't care about:

$$p(X) = \sum_{y} p(X, y)$$

Product Rule



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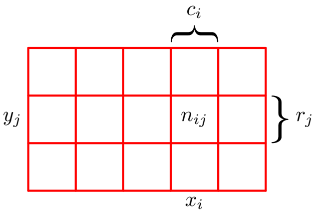

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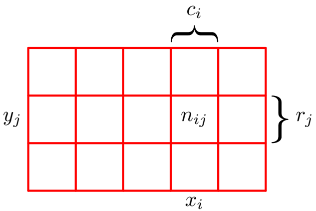

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It follows **Bayes' theorem**:

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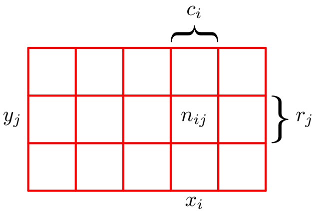

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Or, using the sum rule,

$$p(Y|X) = \frac{p(X|Y)p(Y)}{\sum_y p(X, y)}$$

We say that two variables X and Y are **independent** if their joint distribution factors into their respective distributions, i.e.

$$p(X, Y) = p(X)p(Y)$$

It is often convenient to assume that several random variables are **independent and identically distributed** (i.i.d.), so that their joint distribution can be factored entirely:

$$p(X_1, \ldots, X_n) = \prod_{i=1}^{n} p(X_i)$$

where $X_1, \ldots, X_n$ all share the same p.m.f./p.d.f.