I. Linear Algebra

I.2. Subspaces and Linear Maps

Lecture based on

https://github.com/gwthomas/math4ml (Garrett Thomas, 2018)

https://mml-book.github.io/ (Deisenroth et al., 2020, Mathematics for Machine Learning)

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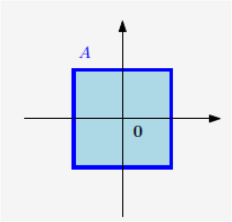
Vector spaces can contain other vector spaces.

If V is a vector space, then $S \subseteq V$ is a **subspace** of V if

- i) $\mathbf{0} \in S$
- ii) S is closed under addition: $\mathbf{x}, \mathbf{y} \in S$ implies $\mathbf{x} + \mathbf{y} \in S$
- iii) S is closed under scalar multiplication: $\mathbf{x} \in S, \alpha \in \mathbb{R}$ implies $\alpha\mathbf{x} \in S$

Note that V is always a subspace of V, as is the trivial vector space which contains only $\mathbf{0}$.

As a concrete example, a line passing through the origin is a subspace of Euclidean space.



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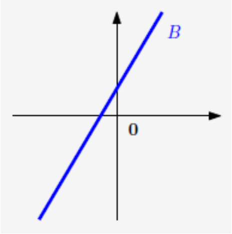

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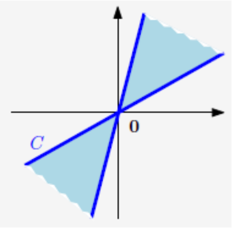

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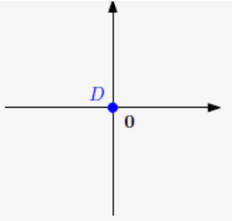


yes

(Deisenroth et al., 2020)

$T : V \to W$ is called a **linear map**, if

$$\forall \mathbf{x}, \mathbf{y} \in V \;\; \forall \lambda, \kappa \in \mathbb{R} : T(\lambda \mathbf{x} + \kappa \mathbf{y}) = \lambda T(\mathbf{x}) + \kappa T(\mathbf{y})$$

.

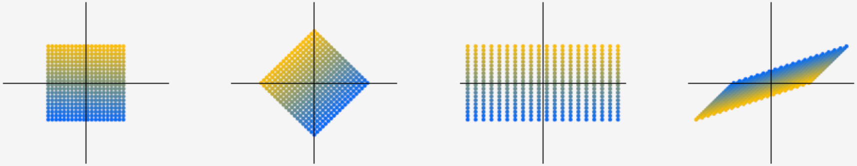
If $T : V \to W$ is a linear map, then

- the **nullspace**[a] of T as
$$\operatorname{null}(T) = \{\mathbf{x} \in V \mid T\mathbf{x} = \mathbf{0}\}$$
- the **range** of T is
$$\operatorname{range}(T) = \{\mathbf{y} \in W \mid \exists \mathbf{x} \in V \text{ such that } T\mathbf{x} = \mathbf{y}\}$$

[a] It is sometimes called the **kernel** in algebra, but we avoid this terminology because the word "kernel" has another meaning in machine learning.

Example

A matrix $\mathbf{A}$ defines a linear map.



(a) Original data. (b) Rotation by 45°. (c) Stretch along the horizontal axis. (d) General linear mapping.

We consider three linear transformations of a set of vectors in $\mathbb{R}^2$ with the transformation matrices

$$\boldsymbol{A}_1 = \begin{bmatrix} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{bmatrix}, \quad \boldsymbol{A}_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{A}_3 = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix}. \quad (2.97)$$

(Deisenroth et al., 2020)

The nullspace of the map is the **columnspace** of $\mathbf{A}$

The vectors $\mathbf{v}_1, \dots, \mathbf{v}_R \in V$ are **linearly independent** if and only if

$$\sum_{i=1}^{R} \lambda_i \mathbf{v}_i = \mathbf{0} \Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_R = 0$$

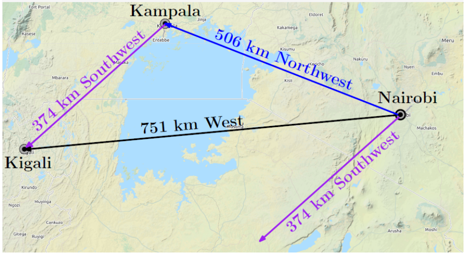
Example

A person in Nairobi (Kenya) describes the way to Kigali (Rwanda):

"You can get to Kigali by first going 506km NW to Kampala (Uganda) and then 374km SW."

She adds "It is about 751km W of here."

Although the last statement is true, it is not necessary to find Kigali given the previous information.



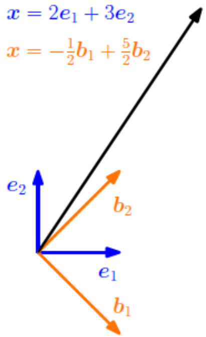
(Deisenroth et al., 2020)

Definition

A subset B of a vector-space V is called **basis** of V, if

- $\text{span}(B) = V$
- B is linearly independent

Different coordinate representations of a vector $\boldsymbol{x}$, depending on the choice of basis.



(Deisenroth et al., 2020)

Definition

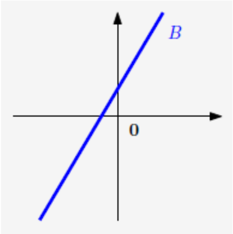
Let V be a vector space, $\mathbf{x}_0 \in V$ and $U \subseteq V$ a subspace.

Then the subset

$$\begin{aligned} L = \mathbf{x}_0 + U :&= \{\mathbf{x}_0 + \mathbf{u} : \mathbf{U} \in U\} \\ &= \{\mathbf{v} \in V | \exists \mathbf{u} \in U : \mathbf{v} = \mathbf{x}_0 + \mathbf{u}\} \subseteq V \end{aligned}$$

is called **affine subspace** or **linear manifold** of V.
U is called **direction** or **direction space** and $\mathbf{x}_0$ is called **support point**.

Example



(Deisenroth et al., 2020)

Note that the definition excludes $\mathbf{0}$ if $\mathbf{x}_0 \notin U$.

- In ML, affine subspaces define **hyperplanes** and are often described by **parameters**.
- If $(b_1, \dots, \mathbf{b}_K)$ is an ordered basis of U, then every element $\mathbf{x} \in L$ can be uniquely described as

$$\mathbf{x} = \mathbf{x}_0 + \lambda_1 \mathbf{b}_1 + \cdots + \lambda_K \mathbf{b}_K$$

Summary

- A subspace $S \subseteq V$
  - $\mathbf{0} \in S$
  -
  - S is closed under addition: $\mathbf{x}, \mathbf{y} \in S$ implies $\mathbf{x} + \mathbf{y} \in S$
  - S is closed under scalar multiplication: $\mathbf{x} \in S, \alpha \in \mathbb{R}$ implies $\alpha\mathbf{x} \in S$
- Linear maps induce subspaces
- $B \subset V$ is a **basis** of V, if
  - $\mathrm{span}(B) = V$
  - B is linearly independent
- An affine subspace V is a subspace U shifted by a support point x_0

$$L = x_o + U$$