

# I. Linear Algebra

## I.10. Pseudoinverse

Lecture based on

**<https://github.com/gwthomas/math4ml>** (Garrett Thomas, 2018)

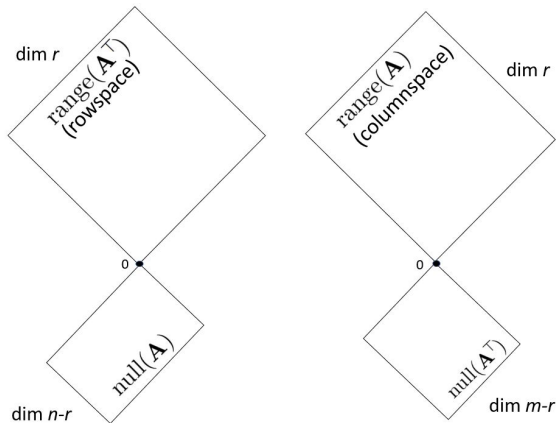
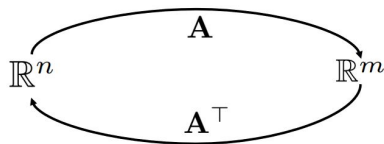
**<https://mml-book.github.io/>** (Deisenroth et al., 2020, Mathematics for Machine Learning)

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$$\min_{\mathbf{x}} \frac{1}{2} (\mathbf{b} - \mathbf{Ax})^\top (\mathbf{b} - \mathbf{Ax})$$

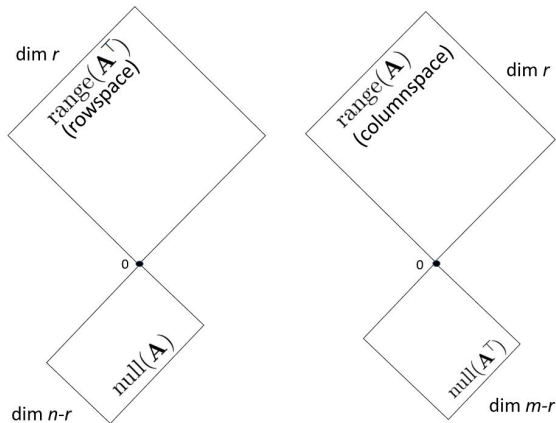
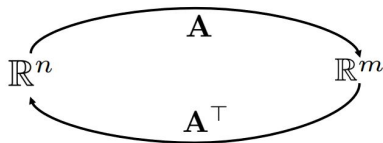
$$\nabla (\mathbf{b} - \mathbf{Ax})^\top (\mathbf{b} - \mathbf{Ax}) = \mathbf{A}^\top (\mathbf{b} - \mathbf{Ax})$$



# Moore-Penrose Pseudoinverse

The pseudo-inverse has the following properties:

- i  $\mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{A}$
- ii  $\mathbf{A}^\dagger\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger$
- iii It is unique.



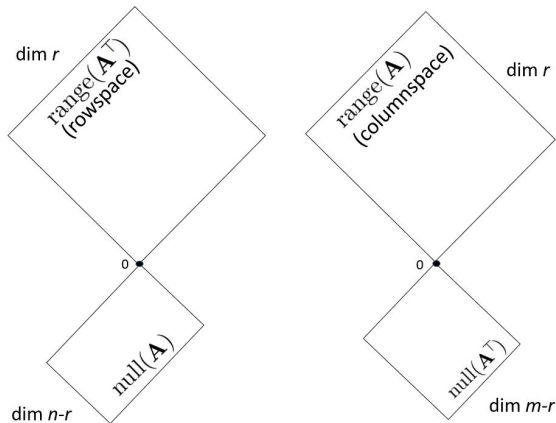
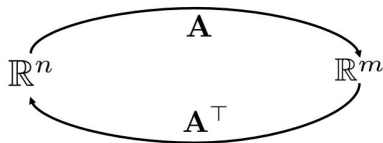
$$\mathbf{A}\mathbf{V} = \mathbf{U}\mathbf{\Sigma}$$

$$\mathbf{A}\mathbf{v}_1 = \sigma_1 \mathbf{u}_1 \quad \dots \quad \mathbf{A}\mathbf{v}_r = \sigma_r \mathbf{u}_r$$

$$\mathbf{A}\mathbf{v}_{r+1} = \mathbf{0} \quad \dots \quad \mathbf{A}\mathbf{v}_n = \mathbf{0}$$

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^\top + \dots + \sigma_r \mathbf{u}_r \mathbf{v}_r^\top$$

$$\mathbf{A}\mathbf{V}_r = \mathbf{U}_r \mathbf{\Sigma}_r$$



## Reduced Form of the SVD

$$\mathbf{A} = \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_r & \dots & \mathbf{u}_n \end{bmatrix} \left[ \begin{array}{ccc|ccc} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ \hline & & & \mathbf{0} & & \\ & & & & \mathbf{0} & \end{array} \right] \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_r & \dots & \mathbf{v}_n \end{bmatrix}^\top$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_r \end{bmatrix}^\top$$

$$\mathbf{A} = \mathbf{U}_r \mathbf{\Sigma}_r \mathbf{V}_r^\top$$