I. Linear Algebra

I.4. Inner Product Spaces

Lecture based on

https://github.com/gwthomas/math4ml (Garrett Thomas, 2018)

https://mml-book.github.io/ (Deisenroth et al., 2020, Mathematics for Machine Learning)

Prof. Dr. Christoph Lippert

Digital Health & Machine Learning

An **inner product** on a real vector space V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ satisfying

(i) $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, with equality if and only if $\mathbf{x} = \mathbf{0}$

(ii) $\langle \alpha \mathbf{x} + \beta \mathbf{y}, \mathbf{z} \rangle = \alpha \langle \mathbf{x}, \mathbf{z} \rangle + \beta \langle \mathbf{y}, \mathbf{z} \rangle$

(iii) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$

for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and all $\alpha, \beta \in \mathbb{R}$.

A vector space endowed with an inner product is called an **inner product space**.

Note that any inner product on V induces a norm on V:

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$$

- One can verify that the axioms for norms are satisfied under this definition and follow (almost) directly from the axioms for inner products.
- Therefore any inner product space is also a normed space (and hence also a metric space).[1]

[1] If an inner product space is complete with respect to the distance metric induced by its inner product, we say that it is a **Hilbert space**.

Two vectors $\mathbf{x}$ and $\mathbf{y}$ are said to be **orthogonal** if

$$\langle \mathbf{x}, \mathbf{y} \rangle = 0;$$

we write $\mathbf{x} \perp \mathbf{y}$ for shorthand.

- Orthogonality generalizes the notion of perpendicularity from Euclidean space.
- If two orthogonal vectors $\mathbf{x}$ and $\mathbf{y}$ additionally have unit length (i.e. $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$), then they are described as **orthonormal**.

The standard inner product on $\mathbb{R}^n$ is given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i = \mathbf{x}^\top \mathbf{y}$$

- The inner product on $\mathbb{R}^n$ is also often written $\mathbf{x} \cdot \mathbf{y}$ (hence the alternate name **dot product**).
- $\| \cdot \|_2$ on $\mathbb{R}^n$ is induced by this inner product.

$$\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{\sum_{i=1}^{n} x_i^2} = \|\mathbf{x}\|_2$$

The well-known **Pythagorean theorem** generalizes naturally to arbitrary inner product spaces.

Theorem

If $\mathbf{x} \perp \mathbf{y}$, *then*

$$\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$

Proof.

Suppose $\mathbf{x} \perp \mathbf{y}$, i.e. $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.
Then

$$\|\mathbf{x}+\mathbf{y}\|^2 = \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$

as claimed. $\square$

The **Cauchy-Schwarz inequality** is sometimes useful in proving bounds:

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|$$

for all $\mathbf{x}, \mathbf{y} \in V$.

Equality holds exactly when $\mathbf{x}$ and $\mathbf{y}$ are scalar multiples of each other (or equivalently, when they are linearly dependent).

Summary

- Inner product spaces generalize normed spaces.
- The well-known **Pythagorean theorem** and the **Cauchy-Schwarz inequality** generalize naturally to arbitrary inner product spaces.