I. Linear Algebra

I.9. Fundamental Theorem of Linear Algebra

Lecture based on

https://github.com/gwthomas/math4ml (Garrett Thomas, 2018)

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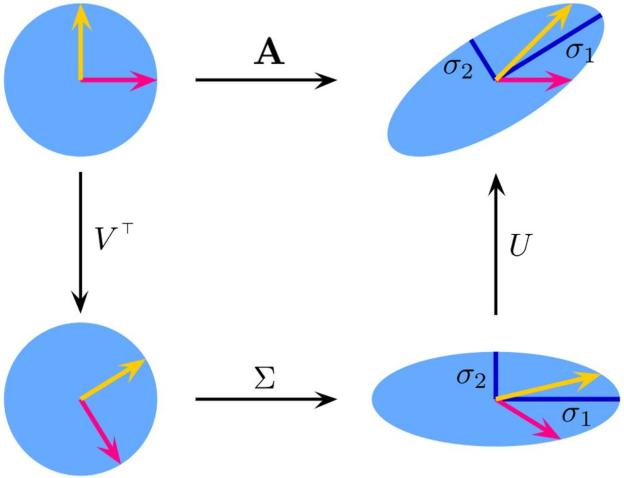
$$\underbrace{\mathbf{x} \rightarrow \mathbf{A}\mathbf{x}}_{\text{linear map } \mathbb{R}^n \rightarrow \mathbb{R}^m}$$

$$\mathbf{A}\mathbf{x} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^\top\mathbf{x}$$



(modified from https://en.wikipedia.org/wiki/Singular_value_decomposition)

Theorem (Fundamental Theorem of Linear Algebra)

If $\mathbf{A} \in \mathbb{R}^{m \times n}$, *then*

(i) $\mathrm{null}(\mathbf{A}) = \mathrm{range}(\mathbf{A}^\top)^\perp$

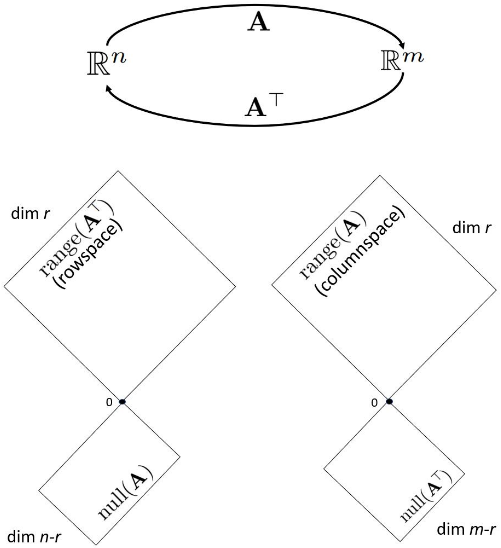
(ii) $\mathrm{null}(\mathbf{A}) \oplus \mathrm{range}(\mathbf{A}^\top) = \mathbb{R}^n$

(iii) $\underbrace{\dim \mathrm{range}(\mathbf{A})}_{\mathrm{rank}(\mathbf{A})} + \dim \mathrm{null}(\mathbf{A}) = n$ ***(rank-nullity)***

(iv) *If* $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$ *is the SVD of* $\mathbf{A}$, *then the columns of* $\mathbf{U}$ *and* $\mathbf{V}$ *form the* ***fundamental subspaces*** *of* $\mathbf{A}$:

Subspace	*Columns*
$\mathrm{range}(\mathbf{A}^\top)$	*The first* r *columns of* $\mathbf{U}$
$\mathrm{range}(\mathbf{A})$	*The first* r *columns of* $\mathbf{V}$
$\mathrm{null}(\mathbf{A})$	*The last* $m - r$ *columns of* $\mathbf{U}$
$\mathrm{null}(\mathbf{A}^\top)$	*The last* $n - r$ *columns of* $\mathbf{U}$

where $r = \mathrm{rank}(\mathbf{A})$



$$\mathrm{null}(\mathbf{A}) = \mathrm{range}(\mathbf{A}^\top)^\perp$$