

Point-Set Topology

HECHEN HU

December 30, 2017

Contents

1	Basic Set Theory	1
1.1	Sets and Operations on Them	1
1.1.1	Naive Set Theory	1
1.1.2	ZFC: Zermelo-Fraenkel Axioms and Axiom of Choice .	1
1.1.3	The <i>Cardinality</i> of a Set(<i>Cardinal Numbers</i>)	2
1.1.4	Operations on Sets	3
1.2	Countable and Uncountable Sets	3
1.2.1	The Cardinality of the Continuum	5

1

Basic Set Theory

1.1 Classification of Relations

Definition. An *equivalence relation* is a relation that satisfy the following properties:

aRa (Reflexivity);
 $aRb \Rightarrow bRa$ (Symmetry);
 $(aRb) \wedge (bRc) \Rightarrow aRc$ (Transitivity).

An equivalence relation is denoted by the special symbol \sim . $a \sim b$ means a is *equivalent* to b .

Definition. Let $R(\sim)$ be an equivalence relation on A . If $a \in A$, the *equivalence class* of a (denoted \bar{a}) is the class of all those elements of A that are equivalent to a . The class of all equivalence classes in A is denoted A/R and called the *quotient class* of A by R .

Theorem 1.1.1. *Two equivalence classes are either disjoint or equal.*

Definition. A *partial ordering* on a set X^2 is a relation R that have the following properties:

aRa (Reflexivity);
 $(aRb) \wedge (bRc) \Rightarrow aRc$ (Transitivity).
 $(aRb) \wedge (bRa) \Rightarrow (a = b)$ (Anti-symmetry);

We often write $a \preceq b$ and say that b *follows* a . If the condition

$$\forall a \forall b ((aRb) \vee (bRa))$$

holds in addition to transitivity and anti-symmetry defining a partial ordering relation (this means any two elements of X is comparable), the relation R is called an *ordering*, and the set X is said to be *linearly ordered*.

Definition. A relation \prec is called a *strict partial order* if it's nonreflexive and transitive.

Theorem 1.1.2 (The Maximum Principle). *Let A be a set and \prec be a strict partial order on A . Then there exists a maximal simply ordered (linearly ordered) subset of A .*

Lemma (Zorn's Lemma). *Let A be a set that is strictly partially ordered. If every simply ordered subset of A has an upper bound in A , then A has a maximal element.*

Definition. If X is a set and $<$ is an order relation on X , and if $a < b$, the notation (a, b) to denote the set

$$\{x | a < x < b\}$$

it is called an *open interval* in X . If this set is empty, a is called the *immediate predecessor* of b , and b is called the *immediate successor* of a .

Definition. Suppose that A and B are two sets with order relations $<_A$ and $<_B$ respectively. We say that A and B have the same *order type* if there is a bijective correspondence between them that preserves order, that is, if $f : A \rightarrow B$ is a bijection

$$a_1 <_A a_2 \Rightarrow f(a_1) <_B f(a_2)$$

Example 1.1. *The interval $(1, 1)$ of real numbers has the same order type as \mathbb{R} , for the function $f : (-1, 1) \rightarrow \mathbb{R}$ given by*

$$f(x) = \frac{x}{1 - x^2}$$

is an order-preserving bijection.

Definition. Suppose A and B are two sets with order relations $<_A$ and $<_B$ respectively. Define an order relation $<$ on $A \times B$ by defining

$$a_1 \times b_1 < a_2 \times b_2 \tag{1.1}$$

if $a_1 <_A a_2$, or if $a_1 = a_2$ and $b_1 <_B b_2$. It is called the *dictionary order relation* on $A \times B$.

Functions and Their Graphs

Definition. A relation R is said to be functional if

$$(xRy_1) \wedge (xRy_2) \Rightarrow (y_1 = y_2)$$

and it is called a *function*.

$R \subset X \times Y$ is a *mapping from X into Y* , or a *function from X into Y* .

1.2 Sets and Operations on Them

1.2.1 Naive Set Theory

- 1⁰. A set may consist of any distinguishable objects ($x \in A \Rightarrow \exists! x \in A$)
- 2⁰. A set is unambiguously determined by the collection of objects that comprise it.
- 3⁰. Any property defines the set of objects having that property ($A = \{x | P(x)\} \Rightarrow P(A)$).

However, this will lead to Russell's Paradox:

Let's have $P(M) := M \notin M$

Consider the class $K = \{M | P(M)\}$. If so K is not a set, since whether $P(K)$ is true or false, contradiction arises.

1.2.2 ZFC: Zermelo-Fraenkel Axioms and Axiom of Choice

- 1⁰. **(Axiom of Extensionality)** Sets A and B are equal iff they have the same elements. ($A = B \Leftrightarrow (\forall x)((x \in A) \Leftrightarrow (x \in B))$)
- 2⁰. **(Axiom of Separation)** To any set A and any property P there corresponds a set B whose elements are those elements of A , and only those, having property P (if A is a set, then $B = \{x \in A | P(x)\}$ is also a set).
- 3⁰. **(Union Axiom)** For any set \mathcal{M} whose elements are sets there exists a set $\bigcup \mathcal{M}$, called the union of \mathcal{M} and consisting of those elements and only those that belong to some element of \mathcal{M} ($x \in \bigcup \mathcal{M} \Leftrightarrow \exists X((X \in \mathcal{M}) \wedge (x \in X))$)

Similarly, the intersection of the set \mathcal{M} is defined as:

$$\bigcap \mathcal{M} := \{x \in \bigcup \mathcal{M} | \forall X((X \in \mathcal{M}) \Rightarrow (x \in X))\}$$

- 4⁰. **(Pairing Axiom)** For any sets X and Y there exists a set Z such that X and Y are its only elements.
- 5⁰. **(Power Set Axiom)** For any set X there exists a set $P(X)$ having each subset of X as an element, and having no other elements.

Definition. The *successor* X^+ of the set X is $X^+ = X \cup \{X\}$.

Definition. An *inductive* set is a set that \emptyset is one of its elements and the successor of each of its elements also belongs to it.

- 6⁰. **(Axiom of Infinity)** There exist inductive sets (Example: \mathbb{N}_0).
- 7⁰. **(Axiom of Replacement)** Let $F(x, y)$ be a statement (a formula) such

that for every $x_0 \in X$ there exists a unique object y_0 such that $F(x_0, y_0)$ is true. Then the objects y for which there exists an element $x \in X$ such that $F(x, y)$ is true form a set.

And finally, an axiom that is independent of ZF.

⁸⁰ **(Axiom of Choice/Zermelo's Axiom)** Given a collection of disjoint nonempty sets, there exists another set consisting of exactly one element from each element of the original set.

Definition. A choice function is a function f , defined on a collection X of nonempty sets, such that for every set A in X , $f(A)$ is an element of A .

Corollary. *There exists a choice function for any collection of nonempty sets.*

1.2.3 The *Cardinality* of a Set (*Cardinal Numbers*)

Definition. The set X is said to be *equipollent* to the set Y if there exists a bijective mapping of X onto Y (then $X \sim Y$).

Definition. *Cardinality* is a measure of the number of elements of the set. If $X \sim Y$, we write $\text{card } X = \text{card } Y$.

If X is equipollent to some subset of Y , we say $\text{card } X \leq \text{card } Y$, thus

$$(\text{card } X \leq \text{card } Y) := \exists Z \subset Y (\text{card } X = \text{card } Z)$$

A set is called *finite* if it is not equipollent to any proper subset of itself; otherwise it is called *infinite*.

It has the properties below:

$$1^0 \quad (\text{card } X \leq \text{card } Y) \wedge (\text{card } Y \leq \text{card } Z) \Rightarrow (\text{card } X \leq \text{card } Z).$$

$$2^0 \quad (\text{card } X \leq \text{card } Y) \wedge (\text{card } Y \leq \text{card } X) \Rightarrow (\text{card } X = \text{card } Y) \text{ (The Schröder–Bernstein theorem).}$$

$$3^0 \quad \forall X \forall Y (\text{card } X \leq \text{card } Y) \vee (\text{card } Y \leq \text{card } X) \text{ (Cantor's theorem).}$$

We say $\text{card } X < \text{card } Y$ if $(\text{card } X \leq \text{card } Y) \wedge (\text{card } X \neq \text{card } Y)$.

let \emptyset be the empty set and $P(X)$ the set of all subsets (thus, the power set) of the set X . Then:

Theorem 1.2.1. $\text{card } X < \text{card } P(X)$

Proof. The assertion is obvious for the empty set, and we shall assume that $X \neq \emptyset$.

Since $P(X)$ contains all the one-element subsets of X , $\text{card } X \leq \text{card } P(X)$. Suppose, contrary to the assertion, that there exists a bijective mapping $f : X \rightarrow P(X)$. Let set $A = \{x \in X : x \notin f(x)\}$ consisting of the elements

$x \in X$ that do not belong to the set $f(x) \in P(X)$ assigned to them by the bijection. Because $A \in P(X)$, there exists $a \in X$ such that $f(a) = A$. For the element a the relation $a \in A$ or $a \notin A$ is impossible by the definition of A (Similar to Russell's Paradox). \square

1.2.4 Operations on Sets

Notation	Meaning	Definition
$A \subset B$	A is a subset of B	$\forall x((x \in A) \Rightarrow (x \in B))$
$A \subsetneq B$	A is a proper subset of B	$A \neq B \wedge A \subset B$
$A = B$	A equals to B	$(A \subset B) \wedge (B \subset A)$
\emptyset	Empty Set	$\{x x \neq x\}$
$A \cup B$	The union of A and B	$\{x x \in A \vee x \in B\}$
$A \cap B$	The intersection of A and B	$\{x x \in A \wedge x \in B\}$
$A \setminus B$	The difference between A and B	$\{x x \in A \wedge x \notin B\}$
$C_M A$	The complement of A in M	$\{x x \in M \wedge x \notin A\}$ where $A \subset M$
$A \times B$	The Cartesian Product of A and B	$\{(x, y) x \in A \wedge y \in B\}$
A^2	$A \times A$	

In the ordered pair $z = (x_1, x_2)$ where $Z = X_1 \times X_2, z \in Z, x_1 \in X_1, x_2 \in X_2$, x_1 is called the *first projection* of the pair z and denoted $\text{pr}_1 z$ while x_2 is called the *second projection* of the pair z and denoted $\text{pr}_2 z$.

1.3 Countable and Uncountable Sets

Definition. A set X is *countable* if it is equipollent with the set \mathbb{N} of natural numbers, that is, $\text{card } X = \text{card } \mathbb{N}$.

Proposition. An infinite subset of a countable set is countable.

Proof. Let's consider a countable set E . There is a minimal element of $E_1 := E$, which we assign to $1 \in \mathbb{N}$ and denote $e_1 \in E$. E is infinite, so $E_2 := E \setminus e_1$ is not empty. Following the principle of induction, we can construct a injective mapping from $\{1, 2, \dots\}$ to $\{e_1, e_2, \dots\}$.

Now we have to prove that this mapping is also surjective. Suppose the contrary, that an element $e \in E$ does not have a natural number assigned to it. The set $K = \{n \in \mathbb{N} | n \leq e\}$ is finite, since it's a subset of \mathbb{N} bounded both from below and above. According to our previous construction, we assign 1 to $\min K$, denoted as e_1 , and we can acquire a sequence $e_1, e_2, \dots, e_{k=\text{card } K}$. But $e_{k=\text{card } K}$ is $\max K$, and because $e \in K \wedge (\forall n \in K (n \leq e))$, $e = \max K$. Therefore $e = e_k$, or otherwise it will contradict the uniqueness of maximal element. \square

Proposition. The Union of the sets of a finite or countable system of countable sets is also a countable set.

Proof. Let $X_1, X_2, \dots, X_n, \dots$ is a countable system of sets and each set $X_m = \{x_m^1, \dots, x_m^n, \dots\}$ is itself countable. Since $\forall m \in \mathbb{N} (\text{card}(X = \bigcup_{n \in \mathbb{N}} X_n) \geq X_m)$, X is an infinite set. The ordered pair (m, n) identifies the element $x_m^n \in X_m$. We can construct a mapping, like $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} := (m, n) \rightarrow \frac{(m+n-2)(m+n-1)}{2} + m$, such that it is bijective. Thus X is countable. Then because $\text{card } X \leq \text{card } \mathbb{N}$ and the fact that X is infinite, we conclude that $\text{card } X = \text{card } \mathbb{N}$. \square

If it is known that a set is either finite or countable, we say it is *at most countable* ($\text{card } X \leq \aleph_0$).

Corollary. $\text{card } \mathbb{Z} = \text{card } \mathbb{N}$

Corollary. $\text{card } \mathbb{N}^2 = \text{card } \mathbb{N}$ (*The direct product of countable sets is countable*).

Corollary. $\text{card } \mathbb{Q} = \text{card } \mathbb{N}$, *that is, the set of rational numbers is countable*.

Proof. Let (m, n) denote a rational number $\frac{m}{n}$. It is known that the pair (m, n) and (m', n') define the same number iff they are proportional. Thus \mathbb{Q} is equipollent to some infinite subset of the set $\mathbb{Z} \times \mathbb{Z}$. Since $\text{card } \mathbb{Z}^2 = \text{card } \mathbb{N}$, we can conclude that $\text{card } \mathbb{Q} = \text{card } \mathbb{N}$. \square

Corollary. *The set of algebraic numbers is countable.*

Proof. It can be observed that $\text{card } \mathbb{Q} \times \mathbb{Q} = \text{card } \mathbb{N}$. By the principle of induction, $\forall k \in \mathbb{N} (\text{card } \mathbb{Q}^k = \text{card } \mathbb{N})$. Let $r \in \mathbb{Q}^k$ be an ordered set (r_1, r_2, \dots, r_k) consists of k rational numbers.

An algebraic equation of degree k with rational coefficient can be written in the reduced form $x^k + r_1 x^{k-1} + \dots + r_k = 0$. Thus there are as many different algebraic equations of degree k as there are different ordered sets (r_1, \dots, r_k) of rational numbers, that is, a countable set.

The algebraic equation with rational coefficients (of arbitrary degree) is the union of sets consisting of algebraic equation (of a fixed degree) which is countable, and this union is countable. Each such equation has only a finite number of roots. Hence the set of algebraic numbers is at most countable. But it is infinite, and therefore countable. \square

1.3.1 The Cardinality of the Continuum

Definition. The set \mathbb{R} of real numbers is also called the *number continuum* (from Latin *continuum*, meaning continuous, or solid), and its cardinality the *cardinality of the continuum*.

Theorem 1.3.1 (Cantor). $\text{card } \mathbb{N} < \text{card } \mathbb{R}$

Proof by Nested Interval Lemma. It is sufficient to show that even $[0, 1]$ is an uncountable set.

Assume it is countable, that is, can be written as a sequence $x_1, x_2, \dots, x_n, \dots$. Take x_1 on $I_0 = [0, 1]$, and find I_1 such that $x_1 \notin I_1$. Then construct the nested interval I_n such that $x_{n+1} \notin I_{n+1}$ and $|I_n| > 0$. It follows the nested interval lemma that there exist a point $c \in [0, 1]$ belonging to all I_n . But by our construction, $c \in \mathbb{R}$ and c cannot be any point of the sequence $x_1, x_2, \dots, x_n, \dots$. \square

Proof by Cantor's Diagonal Argument. Let's first consider the set L and write out the infinite sequence of distinct binary numbers in it which has the form:

$$s_1 = (0, 0, 0, 0, 0, 0, 0, \dots) \quad (1.2)$$

$$s_2 = (1, 1, 1, 1, 1, 1, 1, \dots) \quad (1.3)$$

$$s_3 = (0, 1, 0, 1, 0, 1, 0, \dots) \quad (1.4)$$

$$s_4 = (1, 0, 1, 0, 1, 0, 1, \dots) \quad (1.5)$$

$$s_5 = (1, 1, 0, 1, 0, 1, 1, \dots) \quad (1.6)$$

$$s_6 = (0, 0, 1, 1, 0, 1, 1, \dots) \quad (1.7)$$

$$s_7 = (1, 0, 0, 0, 1, 0, 0, \dots) \quad (1.8)$$

$$\dots \quad (1.9)$$

$$(1.10)$$

We then construct a number s such that its first digit is the complementary (swapping 0s for 1s and vice versa) of the first digit of s_1 and etc.

$$s_1 = (\mathbf{0}, 0, 0, 0, 0, 0, 0, \dots) \quad (1.11)$$

$$s_2 = (1, \mathbf{1}, 1, 1, 1, 1, 1, \dots) \quad (1.12)$$

$$s_3 = (0, 1, \mathbf{0}, 1, 0, 1, 0, \dots) \quad (1.13)$$

$$s_4 = (1, 0, 1, \mathbf{0}, 1, 0, 1, \dots) \quad (1.14)$$

$$s_5 = (1, 1, 0, 1, \mathbf{0}, 1, 1, \dots) \quad (1.15)$$

$$s_6 = (0, 0, 1, 1, 0, \mathbf{1}, 1, \dots) \quad (1.16)$$

$$s_7 = (1, 0, 0, 0, 1, 0, \mathbf{0}, \dots) \quad (1.17)$$

$$\dots \quad (1.18)$$

$$s = (\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{1}, \dots) \quad (1.19)$$

By construction s differs from s_n at the n th digit, so s is not in this sequence, and thus L is uncountable.

We can now define a mapping $f : L \rightarrow \mathbb{R}$. $f(s_n) = r_n \in \mathbb{R}$ means that s_n and r_n have the same digit while r_n is under base 10 and s_n is under base 2. For $s_n \neq s_m \Rightarrow (r_n = f(s_n)) \neq (r_m = f(s_m))$, f is injective, and with the fact that all s_n corresponds to a r_n together give us $\text{card } f(L) = \text{card } L$. Since $f(L)$ is a subset of \mathbb{R} , we can see that \mathbb{R} is also uncountable. \square

The proof above illustrates the theorem below.

Definition. Let X denote the two element set $\{0, 1\}$. Then X^ω is uncountable.

The cardinality of \mathbb{R} is often denotes as \mathfrak{c} .

Corollary. $\mathbb{Q} \neq \mathbb{R}$, and so irrational numbers exist.

Corollary. There exist transcendental numbers, since the set of algebraic numbers is countable.

Example 1.2. The cardinality of $P(X)$, which is the power set of X , satisfy that if $\text{card } X = n$, $\text{card } P(X) = 2^n$.

Proof. We can use the principle of induction to complete the proof. If $n = 1$, $X = \{x\}$, then $P(X) = \{\emptyset, X\}$, then $\text{card } P(X) = 2^1$.

Now if $n \in \mathbb{N} \Rightarrow \text{card } P(X) = 2^n$, let X be a set that has x as one of its elements and has the cardinality of $n + 1$. Therefore $Y = X \setminus \{x\}$ has n elements. We can divide $P(X)$ into two parts: the ones containing x and the ones don't. If $x \in A \subset P(X)$, then $A \setminus \{x\} \subset P(Y)$ and vice versa. Thus we can set up a bijection between $P(Y)$ and the elements in $P(X)$ that contains x . Similarly, we can clearly see that a bijection between the subsets of $P(X)$ that does not contains x and $P(Y)$. Thus $\text{card } P(X) = 2^n + 2^n = 2^{n+1}$, and we complete the proof. \square

We'll use a script letter to denote the collection of sets, for example, \mathcal{A} for collection of sets and A for individual sets in it.

Definition. A *partition* of a set A , besides the definition we have when studying Riemann Sum, can be defined as a collection of disjoint nonempty subsets of A whose union is all of A .

Theorem 1.3.2. Given any partition \mathcal{D} of A , there is exactly one equivalence relation on A from which it is derived.

Example 1.3. Defined two points in the plane to be equivalent if they lie at the same distance from the origin. The collection of equivalence classes consists of all circles centered at the origin, along with the set consisting of the origin alone.

Definition. Any set together with an order relation $<$ that satisfy both the following properties

1. $<$ has the least upper bound property.
2. if $x < y$, then there exists an element z such that $x < z < y$.

is called a *linear continuum*.

Definition. Let \mathcal{A} be a nonempty collection of sets. An *indexing function* for \mathcal{A} is a surjective function f from some set J , called the *index set*, to \mathcal{A} . The collection \mathcal{A} , together with the indexing function is called an *indexed family of sets*. Given $\alpha \in J$, the set $f(\alpha)$ is denoted A_α . The indexed family itself is denoted by

$$\{A_\alpha\}_{\alpha \in J}$$

Theorem 1.3.3 (Principle of Recursive Definition). *Let A be a set. Given a formula that defines $h(1)$ as a unique element of A , and for $i > 1$ defines $h(i)$ uniquely as an element of A in terms of the values of h for positive integers less than i , this formula determines a unique function $h : \mathbb{N} \rightarrow A$.*

Definition. A set A with an order relation $<$ is said to be *well-ordered* if every nonempty subset of it has a smallest element.

Theorem 1.3.4. *Any subset of a well-ordered set is well-ordered. The cartesian product of two well-ordered sets is well-ordered.*

Theorem 1.3.5. *Every nonempty finite ordered set has the order type of a section of \mathbb{N} , so it's well-ordered.*

Theorem 1.3.6 (Well-ordering theorem, proved by Zermelo). *If A is a set, there exists an order relation on A that is a well-ordering.*

Corollary. *There exists an uncountable well-ordered set.*

Definition. Let X be a well-ordered set. Given $\alpha \in X$, let S_α denote the set

$$S_\alpha = \{x \mid x \in X \wedge x < \alpha\}$$

It is called the *section* of X by α .

Lemma. *There exists a well-ordered set A having a largest element Ω , such that the section S_Ω of A by Ω is uncountable but every other section of A is countable.*

Proof. We begin with an uncountable well-ordered set B . Let C be the well-ordered set $\{1, 2\} \times B$ in the dictionary order, then some section of C is uncountable. Let Ω be the smallest element of C for which the section of C by Ω is uncountable, then let A consist of this section along with Ω . \square

The set S_Ω is called a *minimal uncountable well-ordered set*, and the well-ordered set $A = S_\Omega \cup \{\Omega\}$ by \bar{S}_Ω .

Theorem 1.3.7. *If A is a countable subset of S_Ω , then A has an upper bound in S_Ω .*

Proof. Let A be a countable subset of S_Ω . For each $a \in A$, the section S_a is countable. Therefore, the union $B = \bigcup_{a \in A} S_a$ is also countable. Since $S_\Omega \neq B$, let x be a point of S_Ω that is not in B , and then x is an upper bound for A . \square

2

Topological Spaces and Continuous Functions