Problem Set

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1 Mathematical Analysis

Problem 1. Let $f \in C^{(\infty)}(\mathbb{R})$. Show that for $x \neq 0$

$$\frac{1}{x^{n+1}} f^{(n)}(\frac{1}{x}) = (-1)^n \frac{d^n}{dx^n}(x^{n-1} f(\frac{1}{x}))$$

Problem 2. Show that the function

$$f(x) = \begin{cases} \exp(-\frac{1}{(1+x)^2} - \frac{1}{(1-x)^2}) \text{ for } -1 < x < 1, \\ 0 \quad \text{for } 1 \leqslant |x| \end{cases}$$

is infinitely differentiable on $\mathbb{R}$.

Problem 3. Find $\lim\limits_{x \to \infty} x[\frac{1}{e} - (\frac{x}{x+1})^x]$.

Problem 4. Show that if a function f is defined and differentiable on an open interval I and $[a,b] \subset I$, then
a) the function $f'(x)$ (even if it is not continuous) assumes on $[a,b]$ all the values between $f'(a)$ and $f'(b)$ (*the theorem of Darboux*).
b) if f'' also exists in (a,b), then there is a point $\xi \in (a,b)$ such that $f'(b) - f'(a) = f''(\xi)(b-a)$.

Problem 5. Let $x = (x_1, \cdots, x_n)$ and $\alpha = (\alpha_1, \cdots, \alpha_n)$, where $x_i > 0, \alpha_i > 0$ and $\sum_{i=1}^n \alpha_i = 1$. For any number $t \neq 0$ we consider the *mean of order* t *of the numbers* $x_1, \cdots, x_n$ *with weight* α_i:

$$M_t(x, \alpha) = (\sum_{i=1}^n \alpha_i x_i^t)^{1/t}$$

In particular, when $\alpha_1 = \cdots = \alpha_n = \frac{1}{n}$, we obtain the harmonic, arithmetic, and quadratic means for $t = -1, 1, 2$ respectively.

Show that
a) $\lim\limits_{t \to 0} M_t(x, \alpha) = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, that is, in the limit one can obtain the geometric mean;
b) $\lim\limits_{t \to +\infty} M_t(x, \alpha) = \max_{1 \leqslant i \leqslant n} x_i$;
c) $\lim\limits_{t \to -\infty} M_t(x, \alpha) = \min_{1 \leqslant i \leqslant n} x_i$;
d) $M_t(x, \alpha)$ is a nondecreasing function of t on $\mathbb{R}$ and is strictly increasing if $n > 1$ and the numbers x_i are all nonzero.

2 Abstract Algebra

Problem 1. Prove the following proposition:

Proposition. Let G be a semigroup. G is a group iff for all $a, b \in G$ the equations $ax = b$ and $ya = b$ have solutions in G.

Problem 2. Prove the following theorem:

Theorem 2.1 (Generalized Commutative Law)**.** *If G is a commutative semigroup and $a_1, \cdots, a_n \in G$, then for any permutation $i_1, \cdots, i_n$ of $1, 2, \cdots, n$, $a_1 a_2 \cdots a_n = a_{i_1} a_{i_2} \cdots a_{i_n}$.*