

# Multivariable Calculus

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# 1

## Vectors in $\mathbb{R}^n$ Space

### 1.1 Definition and Properties

Vector is a geometrical object that has both magnitude and direction. Examples include force and velocity.

Properties in  $n$ th dimension vector space for Euclidean vector where  $e_i$  is the basis vector for the  $i$ th axis (for convenience we will use  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  denote the basis vector in a 3-d space):

### 1.2 Addition/Subtraction

$$\mathbf{a} \pm \mathbf{b} = \sum_{i=1}^n (a_i \pm b_i) e_i \quad (1.1)$$

### 1.3 Scalar Multiplication

$$k\mathbf{a} = \sum_{i=1}^n k a_i \cdot e_i \quad (1.2)$$

### 1.4 Dot Product

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= \sum_{i=1}^n a_i * b_i \\ &= \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta \end{aligned} \quad (1.3)$$

(Result is a scalar) Dot product has the following properties:

- (a)  $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$  (*Commutative*)
- (b)  $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$  (*Distributive*)
- (c)  $k(\mathbf{a} \cdot \mathbf{b}) = (k\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (k\mathbf{b})$  (*Associative*)
- (d)  $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$
- (e)  $\mathbf{0} \cdot \mathbf{a} = 0$

## 1.5 Direction Angles\Cosines

The directional angles  $\alpha$ ,  $\beta$  and  $\gamma$  between the vector  $\mathbf{v}$  and basis vectors  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  in a 3-d space satisfy the following equations:

$$\cos \alpha = \frac{\mathbf{v} \cdot \mathbf{i}}{\|\mathbf{v}\| \|\mathbf{i}\|} = \frac{v_1}{\|\mathbf{v}\|} \quad (1.4)$$

$$\cos \beta = \frac{\mathbf{v} \cdot \mathbf{j}}{\|\mathbf{v}\| \|\mathbf{j}\|} = \frac{v_2}{\|\mathbf{v}\|} \quad (1.5)$$

$$\cos \gamma = \frac{\mathbf{v} \cdot \mathbf{k}}{\|\mathbf{v}\| \|\mathbf{k}\|} = \frac{v_3}{\|\mathbf{v}\|} \quad (1.6)$$

According to their definitions we can also see that:

$$\begin{aligned} \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma &= \frac{v_1^2 + v_2^2 + v_3^2}{\sqrt{v_1^2 + v_2^2 + v_3^2}^2} \\ &= 1 \end{aligned} \quad (1.7)$$

## 1.6 Orthogonal Projections

The orthogonal projection of  $\mathbf{v}$  on an arbitrary non-zero vector  $\mathbf{b}$  can be written as:

$$proj_{\mathbf{b}} \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{b}}{\|\mathbf{b}\|^2} \mathbf{b} \quad (1.8)$$

Moreover, we can see that  $\mathbf{v} - proj_{\mathbf{b}} \mathbf{v}$  is the vector component of  $\mathbf{v}$  orthogonal to  $\mathbf{b}$ .

## 1.7 Cross Product

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \\ &= (a_2 b_3 - a_3 b_2) \mathbf{i} - (a_1 b_3 - a_3 b_1) \mathbf{j} + (a_1 b_2 - a_2 b_1) \mathbf{k} \\ &= \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta \mathbf{n} \end{aligned} \quad (1.9)$$

( $\mathbf{n}$  is the vector that perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$  and its direction is decided by the right hand rule in a right-handed coordinate system.)

(Result is a vector that is orthogonal to both  $\mathbf{a}$  and  $\mathbf{b}$ )

At the same time, we can see that

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \quad \mathbf{j} \times \mathbf{i} = -\mathbf{k} \quad (1.10)$$

$$\mathbf{j} \times \mathbf{k} = \mathbf{i} \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \quad (1.11)$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{j} \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \quad (1.12)$$

What's more, the area  $A$  of the parallelogram that has  $\mathbf{a}$  and  $\mathbf{b}$  as adjacent sides is:

$$A = \|\mathbf{a} \times \mathbf{b}\| \quad (1.13)$$

Thus,  $\mathbf{a} \times \mathbf{b} = \mathbf{0}$  if  $\mathbf{a}$  and  $\mathbf{b}$  are parallel vectors. More useful properties of cross product:

- (a)  $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$  (*Anti - Commutative*)
- (b)  $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$  (*Distributive*)
- (c)  $(\mathbf{b} + \mathbf{c}) \times \mathbf{a} = \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a}$
- (d)  $k(\mathbf{a} \times \mathbf{b}) = (k\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (k\mathbf{b})$  (*Associative*)
- (e)  $\mathbf{a} \times \mathbf{0} = \mathbf{0} \times \mathbf{a} = \mathbf{0}$
- (f)  $\mathbf{a} \cdot \mathbf{a} = 0$

## 1.8 Scalar triple product

$$(\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}) = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad (1.14)$$

**If we switch two rows of this matrix, the product will be multiplied by  $-1$ .**

The absolute value of scalar triple product will give us the volume of the parallelepiped that has  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  as adjacent edges. Therefore,  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$  iff they lie on the same plane.





## 2

# Lines and Planes

### 2.1 Equations of Lines

The line in 3-d space that passes through the point  $P_0(x_0, y_0, z_0)$  and is parallel to the non-zero vector  $\mathbf{v} = \langle a, b, c \rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$  has equations:

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct \quad (\textit{Parametric}) \quad (2.1)$$

$$l = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle \quad (\textit{Vector}) \quad (2.2)$$

If two lines doesn't intercept or parallel to each other in a 3-d space, they are skew.

### 2.2 Equations of Planes

Definition: A vector perpendicular to a plane is called a **normal** to that plane.

A plane which passing through  $P_0(x_0, y_0, z_0)$  and having  $\mathbf{n} = \langle a, b, c \rangle$  as its normal has equations:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad (\textit{Point - Normal form}) \quad (2.3)$$

$$ax + by + cz + d = 0 \quad (d = -ax_0 - by_0 - cz_0) (\textit{General form}) \quad (2.4)$$

### 2.3 Angle between Planes

For two planes that have  $\mathbf{n}_1$  and  $\mathbf{n}_2$  as its normal, the acute angle between them  $\theta$  can be obtained from the following equation:

$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (2.5)$$

## 2.4 Distance

The distance  $D$  between a point  $P_0(x_0, y_0, z_0)$  and the plane  $ax + by + cz + d = 0$  is:

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \quad (2.6)$$

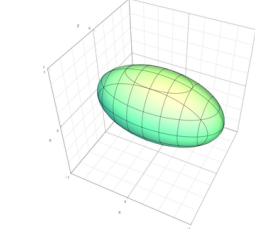
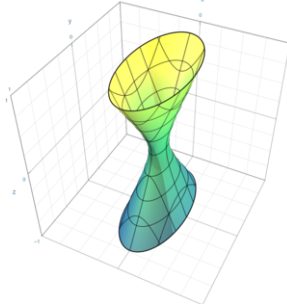
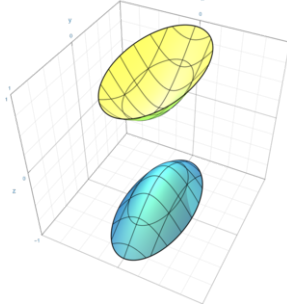
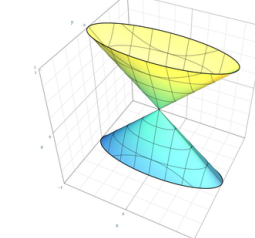
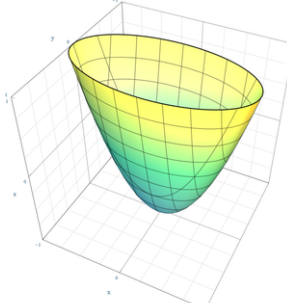
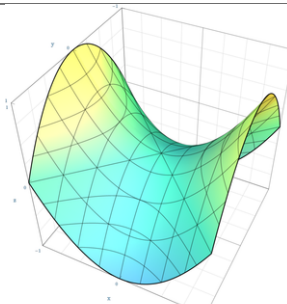
# 3

## Quadric Surfaces

### 3.1 Traces

To help graphing a complex surface in a 3-d space, we obtain traces, or the curves(mesh lines) formed by cutting this surface with well-chosen planes. Usually, surfaces are built up from traces in planes that are parallel to the coordinate planes.

### 3.2 Type of Quadric Surfaces

Name	Equation	Figure
Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	
Hyperboloid of one sheet	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	
Hyperboloid of two sheets	$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	
Elliptic cone	$z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$	
Elliptic paraboloid	$z = \frac{y^2}{a^2} + \frac{x^2}{b^2}$	
Hyperbolic paraboloid	$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$	

## 4

# Calculus of Vector-Valued Functions

### 4.1 Orientation/Direction of its graph

The direction a graph of a vector-valued function goes when its parameter,  $t$ , increases is called the *orientation* or *direction of increasing parameter*.

### 4.2 Domain and Natural Domain

The domain of a vector-valued function is the set of all allowable values of  $t$ . The natural domain of a vector-valued function is the intersection of its component functions' domain.

### 4.3 Radius Vector/Position Vector

If a function can be expressed as  $\mathbf{F}(t) = \langle f(t), g(t), h(t) \rangle$ , then the position vector of it at  $t = k$  is  $\langle f(k), g(k), h(k) \rangle$ .

### 4.4 Vector Form of A Line Segment

For two vectors  $\mathbf{r}_0$  and  $\mathbf{r}_1$  that has its initial point at origin, the line passes through the terminal points of them can be written as:

$$\mathbf{r} = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0) \quad (4.1)$$

And this is called the *two-point vector form o a line*.

## 4.5 Calculus of Vector-Valued Functions

The Calculus of vector-valued functions in 2-d and 3-d space is similar to "normal" functions: **just apply each operator to its component functions and "sum" them up.** The definition of integrable, differentiable and continuous is also similar: **each property requires its component functions have the corresponding property.**

The tangent line of the graph at point  $\mathbf{r}(t_0)$ :

$$\mathbf{r} = \mathbf{t}_0 + t\mathbf{r}'(t_0) \quad (4.2)$$

For the dot product and cross product, which are unique to vector-valued functions, the derivative is defined as following:

$$\frac{d}{dt}[\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)] = \mathbf{r}_1(t) \cdot \frac{d\mathbf{r}_2}{dt} + \frac{d\mathbf{r}_1}{dt} \cdot \mathbf{r}_2(t) \quad (4.3)$$

$$\frac{d}{dt}[\mathbf{r}_1(t) \times \mathbf{r}_2(t)] = \mathbf{r}_1(t) \times \frac{d\mathbf{r}_2}{dt} + \frac{d\mathbf{r}_1}{dt} \times \mathbf{r}_2(t) \quad (4.4)$$

In 2-d space, the tangent line to a circle is perpendicular to the radius at the point of tangency. Similarly, in for a vector-valued function, if  $\|\mathbf{r}(t)\|$  is constant for all  $t$ , then:

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \quad (4.5)$$

that is,  $\mathbf{r}(t)$  and  $\mathbf{r}'(t)$  are orthogonal for all  $t$ .

## 4.6 Arc Length

In a 2-d space, the arc length  $L$  of a parametric curve  $\mathbf{x} = \mathbf{x}(t), \mathbf{y} = \mathbf{y}(t), (a \leq t \leq b)$  can be given as:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad (4.6)$$

**Lemma 1.** *In a 2-d space, the arc length  $L$  of a function  $\mathbf{y} = \mathbf{f}(\mathbf{x})$  that itself and its derivative is continuous on  $[a, b]$  is:*

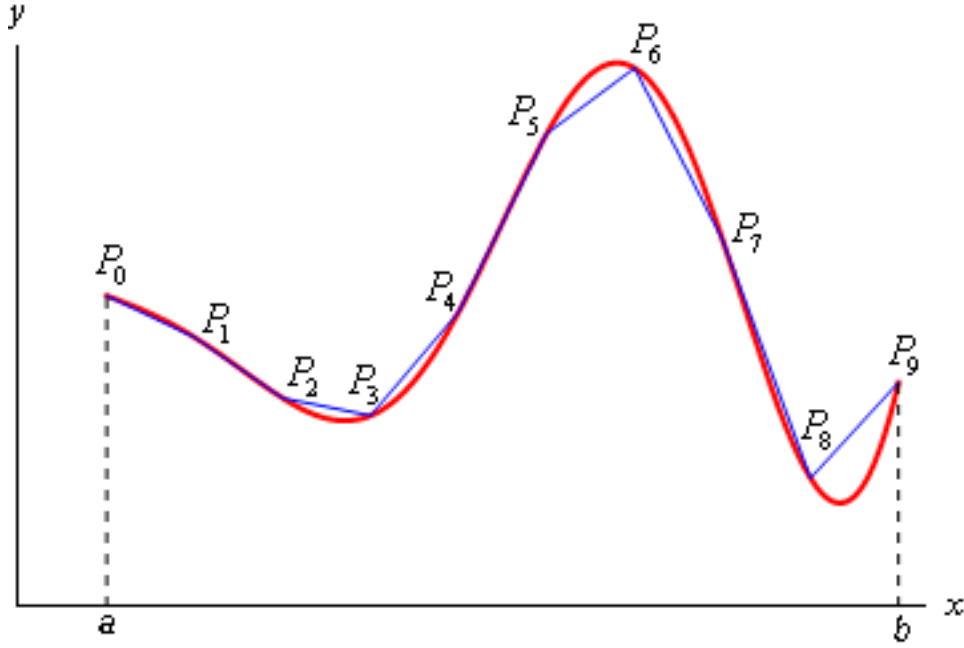
$$L = \int ds \quad (4.7)$$

where

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \text{if } y = f(x), a \leq x \leq b$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \quad \text{if } x = g(y), c \leq y \leq d$$

*Proof.* As we can see in the figure below, the arc length is the sum of distance between  $n$  consecutive points when  $n \rightarrow \infty$



Arc Length  $L$  can be written as:

$$L = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}$$

Additionally, we can see that

$$\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2} = \sqrt{\Delta x^2 + \Delta y_i^2}$$

According to Mean Value Theorem, there exists an  $\bar{x}$  such that

$$\Delta y_i = f'(\bar{x}_i) \Delta x$$

Thus

$$\begin{aligned} \sqrt{\Delta x^2 + \Delta y_i^2} &= \sqrt{\Delta x^2 + \Delta y_i^2} \\ &= \sqrt{\Delta x^2 + (f'(\bar{x}_i) \Delta x)^2} \\ &= \sqrt{1 + [f'(\bar{x}_i)]^2} \Delta x \end{aligned}$$

The exact length of the given curve is

$$\begin{aligned}
 L &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2} \\
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{1 + [f'(\bar{x}_i)]^2} \Delta x \\
 &= \int_a^b \sqrt{1 + [f'(x)]^2} dx \\
 &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx
 \end{aligned}$$

□

Now we can prove Theorem (4.6):

*Proof.* Recall that  $x = x(t)$ ,  $y = y(t)$ , therefore

$$\begin{aligned}
 L &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\
 &= \int_a^b \sqrt{1 + \frac{\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2} \frac{dx}{dt}} dt \\
 &= \int_a^b \sqrt{1 + \frac{\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2} \frac{dx}{dt}} dt \\
 &= \int_a^b \frac{1}{\left|\frac{dx}{dt}\right|} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \frac{dx}{dt} dt
 \end{aligned}$$

If we assume that  $\frac{dx}{dt} \geq 0$ , then

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

□

Analogously, the arc length  $L$  of a smoothly parametrized function (**have a continuously turning tangent vector**) in 3-d space is

$$\begin{aligned}
 L &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \\
 &= \int_a^b \left\| \frac{d\mathbf{r}}{dt} \right\| dt
 \end{aligned} \tag{4.8}$$



## 4.7 Arc Length as A Parameter

Sometime it would be more convenient to replace  $\mathbf{t}$  with  $\mathbf{s}$ , which is the length of arc measured along the curve from some fixed reference point. There are three steps:

**Step 1.** Select an reference point.

**Step 2.** Choose one direction from the reference point as the positive direction.

**Step 3.** Change the length  $\mathbf{s}$  to a "signed" length, which means  $\mathbf{s}$  is positive if  $\mathbf{s}$  "moves along the curve" to its positive direction.

Note that there are infinitely many different arc length parameterizations.

**Theorem 1. Chain Rule** *Let  $\mathbf{r}(\mathbf{t})$  be a vector-valued function in 2-d/3-d space that is differentiable with respect to  $\mathbf{t}$ . If  $\mathbf{t} = \mathbf{g}(\boldsymbol{\tau})$  is a change of parameter in which  $\mathbf{g}$  is differentiable with respect to  $\boldsymbol{\tau}$ , then  $\mathbf{r}(\mathbf{g}(\boldsymbol{\tau}))$  is differentiable with respect to  $\boldsymbol{\tau}$  and*

$$\frac{d\mathbf{r}}{d\boldsymbol{\tau}} = \frac{d\mathbf{r}}{d\mathbf{t}} \frac{d\mathbf{t}}{d\boldsymbol{\tau}} \quad (4.9)$$

A change in parameter is smooth if  $\mathbf{r}(\mathbf{g}(\boldsymbol{\tau}))$  is smooth and  $\mathbf{r}(\mathbf{t})$  is smooth. For all  $\boldsymbol{\tau}$ ,  $\frac{d\mathbf{t}}{d\boldsymbol{\tau}} > \mathbf{0}$  is called a positive change of parameter while  $\frac{d\mathbf{t}}{d\boldsymbol{\tau}} < \mathbf{0}$  is called a negative change of parameter.

**Theorem 2.** *Let  $\mathbf{C}$  be the graph of a smooth vector-valued function  $\mathbf{r}(\mathbf{t})$  in 2-d or 3-d space, and let  $\mathbf{r}(\mathbf{t}_0)$  be any point on  $\mathbf{C}$ . Then the following formula defines a positive change of parameter from  $\mathbf{t}$  to  $\mathbf{s}$ , where  $\mathbf{s}$  is an arc length parameter having  $\mathbf{r}(\mathbf{t}_0)$  as its reference point:*

$$\mathbf{s} = \int_{\mathbf{t}_0}^{\mathbf{t}} \left\| \frac{d\mathbf{r}}{d\mathbf{u}} \right\| d\mathbf{u} \quad (4.10)$$

**Theorem 3.** *If  $\mathbf{C}$  is the graph of a smooth vector-valued function  $\mathbf{r}(\mathbf{t})$  in 2-d or 3-d space, where  $\mathbf{t}$  is a general parameter, and if  $\mathbf{s}$  is the arc length parameter for  $\mathbf{C}$  defined by **Theorem 2**, then for every value of  $\mathbf{t}$  the tangent vector has length*

$$\left\| \frac{d\mathbf{r}}{d\mathbf{t}} \right\| = \frac{d\mathbf{s}}{d\mathbf{t}} \quad (4.11)$$

*Proof.* This can be derived from applying the Fundamental Theorem of Calculus to Theorem 2.  $\square$

**Theorem 4.** *If  $\mathbf{C}$  is the graph of a smooth vector-valued function  $\mathbf{r}(\mathbf{t})$  in 2-d or 3-d space, where  $s$  is the arc length parameter, then for every value of  $s$  the tangent vector to  $\mathbf{C}$  has length*

$$\left\| \frac{d\mathbf{r}}{ds} \right\| = 1 \quad (4.12)$$

*Proof.* Let  $\mathbf{t} = s$  in **Theorem 3**. □

**Theorem 5.** *If  $\mathbf{C}$  is the graph of a smooth vector-valued function  $\mathbf{r}(\mathbf{t})$  in 2-d or 3-d space, and if*

$$\left\| \frac{d\mathbf{r}}{dt} \right\| = 1 \quad (4.13)$$

*for every value of  $\mathbf{t}$ , then  $\mathbf{t}$  is an arc length parameter that has its reference point at the point on  $\mathbf{C}$  where  $\mathbf{t} = \mathbf{0}$ .*

*Proof.* The formula

$$s = \int_0^t \left\| \frac{d\mathbf{r}}{du} \right\| du$$

defines an arc length parameter for  $\mathbf{C}$  with reference point  $\mathbf{r}(\mathbf{0})$ . Note that

$$\left\| \frac{d\mathbf{r}}{dt} \right\| = 1$$

by hypothesis. Thus the formula can be rewrite as

$$s = \int_0^t du = t - 0 = t$$

□