Point-Set Topology

Hechen Hu

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Basic Set Theory

1.1 Sets and Operations on Them

1.1.1 Naive Set Theory

1^0. A set may consist of any distinguishable objects($x \in A \Rightarrow \exists! x \in A$)
2^0. A set is unambiguously determined by the collection of objects that comprise it.
3^0. Any property defines the set of objects having that property($A = \{x|P(x)\} \Rightarrow P(A)$).

However, this will lead to Russell's Paradox:
Let's have$P(M) \coloneqq M \notin M$
Consider the class $K = \{M|P(M)\}$. If so K is not a set, since whether $P(K)$ is true or false, contradiction arises.

1.1.2 ZFC: Zermelo-Fraenkel Axioms and Axiom of Choice

1^0. **(Axiom of Extensionality)** Sets A and B are equal iff they have the same elements. $(A = B) \Leftrightarrow (\forall x((x \in A) \Leftrightarrow (x \in B)))$
2^0. **(Axiom of Seperation)** To any set A and any property P there corresponds a set B whose elements are those elements of A, and only those, having property P(if A is a set, then $B = \{x \in A|P(x)\}$ is also a set).
3^0. **(Union Axiom)** For any set M whose elements are sets there exists a set $\bigcup M$, called the union of M and consisting of those elements and only those that belong to some element of M $(x \in \bigcup M \Leftrightarrow \exists X((X \in M) \wedge (x \in X)))$
Similarly, the intersection of the set M is defined as:

$$\bigcap M \coloneqq \{x \in \bigcup M | \forall X((X \in M) \Rightarrow (x \in X))\}$$

4^0 **(Pairing Axiom)** For any sets X and Y there exists a set Z such that X and Y are its only elements.
5^0 **(Power Set Axiom)** For any set X there exists a set $P(X)$ having each subset of X as an element, and having no other elements.

Definition. The *successor* X^+ of the set X is $X^+ = X \cup \{X\}$.

Definition. An *inductive* set is a set that $\varnothing$ is one of its elements and the successor of each of its elements aso belongs to it.

6^0 **(Axiom of Infinity)** There exist inductive sets (Example: $\mathbb{N}_0$).
7^0 **(Axiom of Replacement)** Let $F(x, y)$ be a statement(a formula) such that for every $x_0 \in X$ there exists a unique object y_0 such that $F(x_0, y_0)$ is true. Then the objects y for which there exists an element $x \in X$ such that $F(x, y)$ is true form a set.
And finally, an axiom that is independent of ZF.

Definition. A choice function is a function f, defined on a collection X of nonempty sets, such that for every set A in X, $f(A)$ is an element of A.

8^0 **(Axiom of Choice/Zermelo's Axiom)** For any set X of nonempty sets, there exists a choice function f defined on X.$(\forall X[\varnothing \notin X \Rightarrow \exists f : X \mapsto \bigcup X \quad \forall A \in X(f(A) \in A)])$

1.1.3 The *Cardinality* of a Set(*Cardinal Numbers*)

Definition. The set X is said to be *equipollent* to the set Y if there exists a bijective mapping of X onto Y(then $X \sim Y$).

Definition. *Cardinality* is a measure of the number of elements of the set. If $X \sim Y$, we write $\operatorname{card} X = \operatorname{card} Y$.

If X is equipollent to some subset of Y, we say $\operatorname{card} X \leqslant \operatorname{card} Y$, thus

$$(\operatorname{card} X \leqslant \operatorname{card} Y) \coloneqq \exists Z \subset Y(\operatorname{card} X = \operatorname{card} Z)$$

A set is called *finite* if it is not equipollent to any proper subset of itself; otherwise it is called *infinite*.
It has the properties below:

1^0 $(cardX \leqslant \operatorname{card} Y) \wedge (\operatorname{card} Y \leqslant \operatorname{card} Z) \Rightarrow (\operatorname{card} X \leqslant \operatorname{card} Z)$.
2^0 $(\operatorname{card} X \leqslant \operatorname{card} Y) \wedge (\operatorname{card} Y \leqslant \operatorname{card} X) \Rightarrow (\operatorname{card} X = \operatorname{card} Y)$(The

Schröder–Bernstein theorem).
3^0 $\forall X \forall Y (\operatorname{card} X \leqslant \operatorname{card} Y) \vee (\operatorname{card} Y \leqslant \operatorname{card} X)$(Cantor's theorem).

We say $\operatorname{card} X < \operatorname{card} Y$ if $(\operatorname{card} X \leqslant \operatorname{card} Y) \wedge (\operatorname{card} X \neq \operatorname{card} Y)$.
let $\varnothing$ be the empty set and $P(X)$ the set of all subsets(thus, the power set) of the set X. Then:

Theorem 1.1.1. $\operatorname{card} X < \operatorname{card} P(X)$

Proof. The assertion is obvious for the empty set, and we shall assume that $X \neq \varnothing$.
Since $P(X)$ contains all the one-element subsets of X, $cardX \leqslant cardP(X)$. Suppose, contrary to the assertion, that there exists a bijective mapping $f : X \to P(X)$. Let set $A = \{x \in X : x \notin f(x)\}$ consisting of the elements $x \in X$ that do not belong to the set $f(x) \in P(X)$ assigned to them by the bijection. Because $A \in P(X)$, there exists $a \in X$ such that $f(a) = A$. For the element a the relation $a \in A$ or $a \notin A$ is impossible by the definition of A(Similar to Russell's Paradox). □

1.1.4 Operations on Sets

Notation	Meaning	Definition
$A \subset B$	A is a subset of B	$\forall x((x \in A) \Rightarrow (x \in B))$
$A = B$	A equals to B	$(A \subset B) \wedge (B \subset A)$
$\varnothing$	Empty Set	$\{x \mid x \neq x\}$
$A \cup B$ $A \cap B$	The union of A and B The intersection of A and B	$\{x \mid x \in A \vee x \in B\}$ $\{x \mid x \in A \wedge x \in B\}$
$A \setminus B$	The difference between A and B	$\{x \mid x \in A \wedge x \notin B\}$
$C_M A$	The complement of A in M	$\{x \mid x \in M \wedge x \notin A\} where A \subset M$
$A \times B$	The Cartesian Product of A and B	$\{(x, y) \mid x \in A \wedge y \in B\}$
A^2	$A \times A$	

In the ordered pair$z = (x_1, x_2)$ where$Z = X_1 \times X_2, z \in Z, x_1 \in X_1, x_2 \in X_2$, x_1 is called the *first projection* of the pair z and denoted $\mathrm{pr}_1 z$ while x_2 is called the *second projection* of the pair z and denoted $\mathrm{pr}_2 z$.

1.2 Countable and Uncountable Sets

Definition. A set X is *countable* if it is equipollent with the set $\mathbb{N}$ of natural numbers, that is, $\operatorname{card} X = \operatorname{card} \mathbb{N}$.

Proposition. An infinite subset of a countable set is countable.

Proof. Let's consider a countable set E. There is a minimal element of $E_1 \coloneqq E$, which we assign to $1 \in \mathbb{N}$ and denote $e_1 \in E$. E is infinite, so $E_2 \coloneqq E \setminus e_1$ is not empty. Following the principle of induction, we can construct a injective mapping from $\{1, 2...\}$ to $\{e_1, e_2, ...\}$.

Now we have to prove that this mapping is also surjective. Suppose the contrary, that an element $e \in E$ does not have a natural number assigned to it. The set $K = \{n \in E | n \leqslant e\}$ is finite, since it's a subset of $\mathbb{N}$ bounded both from below and above. According to our previous construction, we assign 1 to $\min K$, denoted as e_1, and we can acquire a sequence $e_1, e_2, ...e_{k=\operatorname{card} K}$. But $e_{k=\operatorname{card} K}$ is $\max K$, and because $e \in K \wedge (\forall n \in K(n \leqslant e))$, $e = \max K$. Therefore $e = e_k$, or otherwise it will contradict the uniqueness of maximal element. □

Proposition. The Union of the sets of a finite or countable system of countable sets is also a countable set.

Proof. Let $X_1, X_2..., X_n, ...$ is a countable system of sets and each set $X_m = \{x_m^1, ..., x_m^n, ...\}$ is itself countable. Since $\forall m \in \mathbb{N}(\operatorname{card}(X = \bigcup_{n\in\mathbb{N}} X_n) \geqslant X_m)$, X is an infinite set. The ordered pair (m, n) identifies the element $x_m^n \in X_m$. We can construct a mapping, like $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \coloneqq (m, n) \to \frac{(m+n-2)(m+n-1)}{2} + m$, such that it is bijective. Thus X is countable. Then because $\operatorname{card} X \leqslant \operatorname{card} \mathbb{N}$ and the fact that X is infinite, we conclude that $\operatorname{card} X = \operatorname{card} \mathbb{N}$. □

If it is known that a set is either finite or countable, we say it is *at most countable*($\operatorname{card} X \leqslant \mathbb{N}$).

Corollary. $\operatorname{card} \mathbb{Z} = \operatorname{card} \mathbb{N}$

Corollary. $\operatorname{card} \mathbb{N}^2 = \operatorname{card} \mathbb{N}$*(The direct product of countable sets is countable).*

Corollary. $\operatorname{card} \mathbb{Q} = \operatorname{card} \mathbb{N}$*, that is, the set of rational numbers is countable.*

Proof. Let (m, n) denote a rational number $\frac{m}{n}$. It is known that the pair (m, n) and (m', n') define the same number iff they are proportional. Thus $\mathbb{Q}$ is equipollent to some infinite subset of the set $\mathbb{Z}\times\mathbb{Z}$. Since $\operatorname{card} \mathbb{Z}^2 = \operatorname{card} \mathbb{N}$, we can conclude that $\operatorname{card} \mathbb{Q} = \operatorname{card} \mathbb{N}$. □

Corollary. *The set of algebraic numbers is countable.*

Proof. It can be observed that $\operatorname{card} \mathbb{Q} \times \mathbb{Q} = \operatorname{card} \mathbb{N}$. By the principle of induction, $\forall k \in \mathbb{N}(\operatorname{card} \mathbb{Q}^k = \operatorname{card} \mathbb{N})$. Let $r \in \mathbb{Q}^k$ be an ordered set $(r_1, r_2, ..., r_k)$ consists of k rational numbers.

An algebraic equation of degree k with rational coefficient can be writtne in the reduced form $x^k + r_1 x^{k-1} + \cdots + r_k = 0$. Thus there are as many different algebraic equations of degree k as there are different ordered sets

$(r_1, ..., r_k)$ of rational numbers, that is, a countable set.

The algebraic equation with rational coefficients (of arbitrary degree) is the union of sets consisting of algebraic equation (of a fixed degree) which is countable, and this union is countable. Each such equation has only a finite number of roots. Hence the set of algebraic numbers is at most countable. But it is infinite, and therefore countable. □

1.2.1 The Cardinality of the Continuum

Definition. The set $\mathbb{R}$ of real numbers is also called the *number continuum*(from Latin *continuum*, meaning continuous, or solid), and its cardinality the *cardinality of the continuum.*

Theorem 1.2.1 (Cantor)**.** $\operatorname{card}\mathbb{N} < \operatorname{card}\mathbb{R}$

Proof by Nested Interval Lemma. It is sufficient to show that even $[0, 1]$ in an uncountable set.

Assume it is countable, that is, can be written as a sequence $x_1, x_2, ..., x_n, ...$. Take x_1 on $I_0 = [0, 1]$, and find I_1 such that $x_1 \notin I_1$. Then construct the nested interval I_n such that $x_{n+1} \notin I_{n+1}$ and $|I_n| > 0$. It follows the nested interval lemma that there exist a point $c \in [0, 1]$ belonging to all I_n. But by our construction, $c \in \mathbb{R}$ and c cannot be any point of the sequence $x_1, x_2, ..., x_n, ...$. □

Proof by Cantor's Diagonal Argument. Let's first consider an the set L and write out the infinite sequence of distinct binary numbers in it which has the form:

$$s1 = (0, 0, 0, 0, 0, 0, 0, ...) \tag{1.1}$$
$$s2 = (1, 1, 1, 1, 1, 1, 1, ...) \tag{1.2}$$
$$s3 = (0, 1, 0, 1, 0, 1, 0, ...) \tag{1.3}$$
$$s4 = (1, 0, 1, 0, 1, 0, 1, ...) \tag{1.4}$$
$$s5 = (1, 1, 0, 1, 0, 1, 1, ...) \tag{1.5}$$
$$s6 = (0, 0, 1, 1, 0, 1, 1, ...) \tag{1.6}$$
$$s7 = (1, 0, 0, 0, 1, 0, 0, ...) \tag{1.7}$$
$$... \tag{1.8}$$
$$\tag{1.9}$$

We then constrcut a number s such that its first digit is the complementary (swapping 0s for 1s and vice versa) of the first digit of s_1 and etc.

$$s1 = (\mathbf{0}, 0, 0, 0, 0, 0, 0, ...) \tag{1.10}$$
$$s2 = (1, \mathbf{1}, 1, 1, 1, 1, 1, ...) \tag{1.11}$$
$$s3 = (0, 1, \mathbf{0}, 1, 0, 1, 0, ...) \tag{1.12}$$
$$s4 = (1, 0, 1, \mathbf{0}, 1, 0, 1, ...) \tag{1.13}$$
$$s5 = (1, 1, 0, 1, \mathbf{0}, 1, 1, ...) \tag{1.14}$$
$$s6 = (0, 0, 1, 1, 0, \mathbf{1}, 1, ...) \tag{1.15}$$
$$s7 = (1, 0, 0, 0, 1, 0, \mathbf{0}, ...) \tag{1.16}$$
$$... \tag{1.17}$$
$$s = (\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{1}, ..) \tag{1.18}$$

By construction s differs from s_n at the nth digit, so s is not in this sequence, and thus L is uncountable.

We can now define a mapping $f : L \to \mathbb{R}. f(s_n) = r_n \in \mathbb{R}$ means that s_n and r_n have the same digit while r_n is under base 10 and s_n is under base 2. For $s_n \neq s_m \Rightarrow (r_n = f(s_n)) \neq (r_m = f(s_m))$, f is injective, and with the fact that all s_n corresponds to a r_n together give us $\operatorname{card} f(L) = \operatorname{card} L$. Since $f(L)$ is a subset of $\mathbb{R}$, we can see that $\mathbb{R}$ is also uncountable. □

The cardinality of $\mathbb{R}$ is often denotes as $\mathfrak{c}$.

Corollary. *$\mathbb{Q} \neq \mathbb{R}$, and so irrational numbers exist.*

Corollary. *There exist transcendental numbers, since the set of algebraic numbers is countable.*

***Example* 1.1.** *The cardinality of $P(X)$, which is the power set of X, satisfy that if $\operatorname{card} X = n$, $\operatorname{card} P(X) = 2^n$.*

Proof. We can use the principle of induction to complete the proof. If $n = 1$, $X = \{x\}$, then $P(X) = \{\varnothing, X\}$, then $\operatorname{card} P(X) = 2^1$.

Now if $n \in \mathbb{N} \Rightarrow \operatorname{card} P(X) = 2^n$, let X be a set that has x as one of its elements and has the cardinality of $n + 1$. Therefore $Y = X \setminus \{x\}$ has n elements. We can divide $P(X)$ into two parts: the ones containing x and the ones don't. If $x \in A \subset P(X)$, then $A \setminus \{x\} \subset P(Y)$ and vice versa. Thus we can set up a bijection between $P(Y)$ and the elements in $P(X)$ that contains x. Similarly, we can clearly see that a bijection between the subsets of $P(X)$ that does not contains x and $P(Y)$. Thus $\operatorname{card} P(X) = 2^n + 2^n = 2^{n+1}$, and we complete the proof. □