Problem Set

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1 Mathematical Analysis

Problem 1. These questions deal with measure.

1. Show that a set of measure zero has no interior points.
2. Show that not having interior points by no means guarantees that a set is of measure zero.
3. Construct a set having measure zero whose closure is the entire space $\mathbb{R}^n$.

Problem 2. These questions deal with Riemann integrals.

1. Construct the analogue of the Dirichlet function in $\mathbb{R}^n$ and show that a bounded function $f : I \to \mathbb{R}$ equal to zero at almost every point of the interval I may still fail to belong to $\mathcal{R}(I)$.
2. Show that if $f \in \mathcal{R}(I)$ and $f(x) = 0$ at almost all points of the interval I, then $\int_I f(x)dx = 0$.

Problem 3. (The Brunn-Minkowski inequality) Given two nonempty sets $A, B \subset \mathbb{R}^n$, we form their vector sum in the sense of Minkowski $A + B := \{a + b | a \in A, b \in B\}$. Let $V(E)$ denote the content of a set $V(E) \subset \mathbb{R}^n$.

1. Verify that if A and B are standard n-dimensional intervals(parallelepipeds), then

$$V^{1/n}(A + B) \geqslant V^{1/n}(A) + V^{1/n}(B)$$

2. Now prove the preceding inequality for arbitrary measurable compact sets A and B.

Problem 4. These questions deal with functions and sets.

1. Construct a subset of the square $I \subset \mathbb{R}^2$ such that on the one hand its intersection with any vertical line and any horizontal line consists of at most one point, while on the other hand its closure equals I.
2. Construct a function $f : I \to \mathbb{R}$ for which both of the iterated integrals that occur in Fubini's theorem exist and equal, yet $f \notin \mathcal{R}(I)$.

Problem 5. Consider the sequence of integrals

$$F_0(x) = \int_0^x f(y)dy \qquad F_n(x) = \int_0^x \frac{(x-y)^n}{n!} f(y)dy \quad n \in \mathbb{N}$$

where $f \in C(\mathbb{R}, \mathbb{R})$.

1. Verify that $F_n'(x) = F_{n-1}(x)$, $F_n^{(k)}(0) = 0$ if $k \leqslant n$, and $F_n^{n+1}(x) = f(x)$.
2. Show that

$$\int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} f(x_n)dx_n = \frac{1}{n!} \int_0^x (x-y)^n f(y)dy$$

2 Topology

Problem 1. Let X be a set.

1. If $\{\mathcal{T}_\alpha\}$ is a family of topologies on X, show that $\bigcap \mathcal{T}_\alpha$ is a topology on X. Is $\bigcup \mathcal{T}_\alpha$ a topology on X?
2. Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections $\mathcal{T}_\alpha$, and a unique largest topology contained in all $\mathcal{T}_\alpha$.
3. If $X = \{a, b, c\}$, let

$$\mathcal{T}_1 = \{\varnothing, X, \{a\}, \{a, b\}\} \quad \text{and} \quad \mathcal{T}_1 = \{\varnothing, X, \{a\}, \{b, c\}\}$$

   Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$, and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.

Problem 2. Show that X is Hausdorff iff the diagonal $\Delta = \{x \times x | x \in X\}$ is closed in $X \times X$.

Problem 3. (Kuratowski) Consider the collection of all subsets A of the topological space X. The operations of closure $A \to \bar{A}$ and complementation $A \to X \setminus A$ are functions from this collection to itself.

1. Show that starting with a given A, one can form no more than 14 distinct sets by applying these two operations successively.
2. Find a subset A of $\mathbb{R}$ (in its usual topology) for which the maximum of 14 is obtained.

Problem 4. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point.

Problem 5. Let X be the subset of $\mathbb{R}^\omega$ consisting of all sequences $\mathbf{x}$ such that $\sum x_i^2$ converges. Then the formula

$$d(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{\infty} (x_i - y_i)^2\right]^{1/2}$$

defines a metric on X. On X we have the three topologies inherits from the box, uniform, and product topologies on $\mathbb{R}^\omega$. We have also the topology given by the metric d, which we call the ℓ^2-topology.

1. Show that on X, we have the inclusions

$$\text{box topology} \supset \ell^2\text{-topology} \supset \text{uniform topology}$$

2. The set $\mathbb{R}^\infty$ of all sequences that are eventually zero is contained in X. Show that the four topologies that $\mathbb{R}^\infty$ inherits as a subspace of X are all distinct.
3. The set

$$H = \prod_{n \in \mathbb{Z}_+} [0, 1/n]$$

   is contained in X; it is called the *Hilbert cube*. Compare the four topologies that H inherits as a subspace of X.