Multivariable Calculus

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Vectors in $\mathbb{R}^n$ Space

1.1 Definition and Properties

Vector is a geometrical object that has both magnitude and direction. Examples include force and velocity.
Properties in in nth dimension vector space for Euclidean vector where e_i is the basis vector for the ith axis(for convenience we will us **i**, **j**, **k** denote the basis vector in a 3-d space):

1.2 Addition/Subtraction

$$\mathbf{a} \pm \mathbf{b} = \sum_{i=1}^{n} (a_i \pm b_i) e_i \tag{1.1}$$

1.3 Scalar Multiplication

$$k\mathbf{a} = \sum_{i=1}^{n} k a_i . e_{\cdot i} \tag{1.2}$$

1.4 Dot Product

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= \sum_{i=1}^{n} a_i * b_i \\ &= \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta \end{aligned} \tag{1.3}$$

(**Result is a scalar**) Dot product has the following properties:

$$
\begin{aligned}
&(a) \quad \mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} \quad (\textit{Commutative})\\
&(b) \quad \mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c} \quad (\textit{Distributive})\\
&(c) \quad k(\mathbf{a}\cdot\mathbf{b}) = (k\mathbf{a})\cdot\mathbf{b} = \mathbf{a}\cdot(k\mathbf{b}) \quad (\textit{Associative})\\
&(d) \quad \mathbf{a}\cdot\mathbf{a} = \|\mathbf{a}\|^2\\
&(e) \quad \mathbf{0}\cdot\mathbf{a} = 0
\end{aligned}
$$

1.5 Direction Angles\Cosines

The directional angles α, β and γ between the vector $\mathbf{v}$ and basis vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in a 3-d space satisfy the following equations:

$$\cos\alpha = \frac{\mathbf{v}\cdot\mathbf{i}}{\|\mathbf{v}\|\,\|\mathbf{i}\|} = \frac{v_1}{\|\mathbf{v}\|} \tag{1.4}$$

$$\cos\beta = \frac{\mathbf{v}\cdot\mathbf{j}}{\|\mathbf{v}\|\,\|\mathbf{j}\|} = \frac{v_2}{\|\mathbf{v}\|} \tag{1.5}$$

$$\cos\gamma = \frac{\mathbf{v}\cdot\mathbf{k}}{\|\mathbf{v}\|\,\|\mathbf{k}\|} = \frac{v_3}{\|\mathbf{v}\|} \tag{1.6}$$

According to their definitions we can also see that:

$$
\begin{aligned}
\cos^2\alpha + \cos^2\beta + \cos^2\gamma &= \frac{{v_1}^2 + {v_1}^2 + {v_1}^2}{\sqrt{{v_1}^2 + {v_1}^2 + {v_1}^2}^2}\\
&= 1
\end{aligned} \tag{1.7}
$$

1.6 Orthogonal Projections

The orthogonal projection of $\mathbf{v}$ on an arbitrary non-zero vector $\mathbf{b}$ can be written as:

$$proj_{\mathbf{b}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b} \tag{1.8}$$

Moreover, we can see that $\mathbf{v} - proj_{\mathbf{b}}\mathbf{v}$ is the vector component of $\mathbf{v}$ orthogonal to $\mathbf{b}$.

1.7 Cross Product

$$
\begin{aligned}
\mathbf{a}\times\mathbf{b} =& det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}\\
&= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}\\
&= \|\mathbf{a}\|\,\|\mathbf{b}\|\sin\theta\mathbf{n}
\end{aligned} \tag{1.9}
$$

($\mathbf{n}$ is the vector that perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ and its direction is decided by the right hand rule in a right-handed coordinate system.)
(Result is a vector that is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$)

At the same time, we can see that

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \quad \mathbf{j} \times \mathbf{i} = -\mathbf{k} \tag{1.10}$$
$$\mathbf{j} \times \mathbf{k} = \mathbf{i} \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \tag{1.11}$$
$$\mathbf{k} \times \mathbf{i} = \mathbf{j} \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \tag{1.12}$$

What's more, the area A of the parallelogram that has $\mathbf{a}$ and $\mathbf{b}$ as adjacent sides is:

$$\boldsymbol{A} = \|\mathbf{a} \times \mathbf{b}\| \tag{1.13}$$

Thus, $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ if $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors. More useful properties of cross product:

$$\begin{aligned}
&(\boldsymbol{a}) \quad \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \quad (\boldsymbol{Anti - Commutative})\\
&(\boldsymbol{b}) \quad \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \quad (\boldsymbol{Distributive})\\
&(\boldsymbol{c}) \quad (\mathbf{b} + \mathbf{c}) \times \mathbf{a} = \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a}\\
&(\boldsymbol{d}) \quad \boldsymbol{k}(\mathbf{a} \times \mathbf{b}) = (\boldsymbol{k}\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\boldsymbol{k}\mathbf{b}) \quad (\boldsymbol{Associative})\\
&(\boldsymbol{e}) \quad \mathbf{a} \times \mathbf{0} = \mathbf{0} \times \mathbf{a} = \mathbf{0}\\
&(\boldsymbol{f}) \quad \mathbf{a} \cdot \mathbf{a} = \mathbf{0}
\end{aligned}$$

1.8 Scalar triple product

$$(\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}) = \boldsymbol{det} \begin{bmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} \\ \boldsymbol{b_1} & \boldsymbol{b_2} & \boldsymbol{b_3} \\ \boldsymbol{c_1} & \boldsymbol{c_2} & \boldsymbol{c_3} \end{bmatrix} \tag{1.14}$$

If we switch two rows of this matrix, the product will be multiplied by -1.
The absolute value of scalar triple product will give us the volume of the parallelepiped that has $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ as adjacent edges. Therefore, $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$ iff they lie on the same plane.

2

Lines and Planes

2.1 Equations of Lines

The line in 3-d space that passes through the point $\boldsymbol{P_0(x_0, y_0, z_0)}$ and is parallel to the non-zero vector $\mathbf{v} =< \boldsymbol{a, b, c} >= \boldsymbol{a}\mathbf{i}+\boldsymbol{b}\mathbf{j}+\boldsymbol{c}\mathbf{k}$ has equations:

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct \quad (Parametric) \tag{2.1}$$

$$l =< x_0, y_0, z_0 > +t < a, b, c > \quad (Vector) \tag{2.2}$$

If two lines doesn't intercept or parallel to each other in a 3-d space, they are skew.

2.2 Equations of Planes

Definition: A vector perpendicular to a plane is called a **normal** to that plane.
A plane which passing through $\boldsymbol{P_0(x_0, y_0, z_0)}$ and having $\mathbf{n} =< \boldsymbol{a, b, c} >$as its normal has equations:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad (Point - Normalform) \tag{2.3}$$

$$ax + by + cz + d = 0 \quad (d = -ax_0 - by_0 - cz_0)(Generalform) \tag{2.4}$$

2.3 Angle between Planes

For two planes that have $\mathbf{n_1}$ and $\mathbf{n_2}$ as its normal, the acute angle between them $\boldsymbol{\theta}$ can be obtained from the following equation:

$$\cos\theta = \frac{|\mathbf{n_1} \cdot \mathbf{n_2}|}{\|\mathbf{n_1}\| \, \|\mathbf{n_2}\|} \tag{2.5}$$

2.4 Distance

The distance $\boldsymbol{D}$ between a point $\boldsymbol{P_0(x_0, y_0, z_0)}$ and the plane $\boldsymbol{ax + by + cz + d = 0}$ is:

$$\boldsymbol{D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}} \tag{2.6}$$

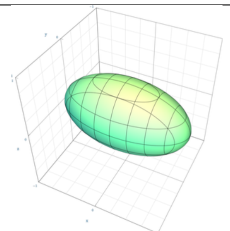
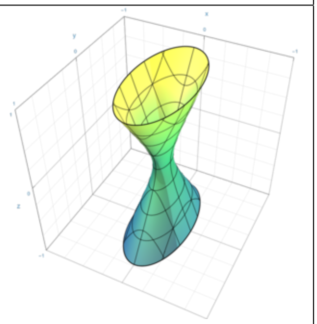
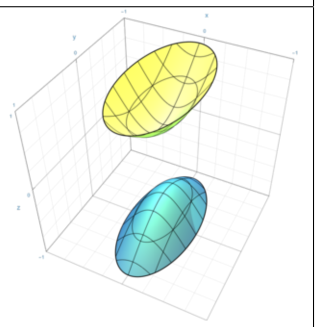
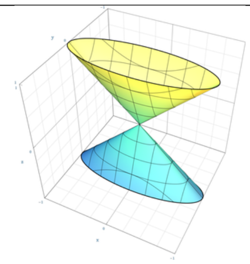
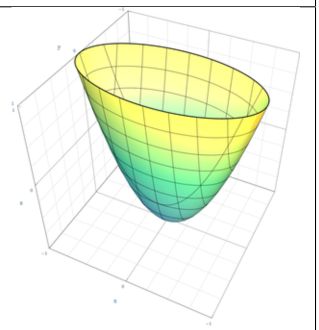
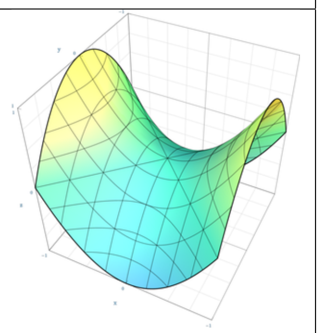
3

Quadric Surfaces

3.1 Traces

To help graphing a complex surface in a 3-d space, we obtain traces, or the curves(mesh lines) formed by cutting this surface with well-chosen planes. Usually, surfaces are built up from traces in planes that are parallel to the coordinate planes.

3.2 Type of Quadric Surfaces

Name	Equation	Figure
Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	
Hyperboloid of one sheet	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	
Hyperboloid of two sheets	$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	
Elliptic cone	$z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$	
Elliptic paraboloid	$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$	
Hyperbolic paraboloid	$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$	

4

Calculus of Vector-Valued Functions

4.1 Orientation/Direction of its graph

The direction a graph of a vector-valued function goes when its parameter, $\boldsymbol{t}$, increases is called the *orientation* or *direction of increasing parameter*.

4.2 Domain and Natural Domain

The domain of a vector-valued function is the set of all allowable values of $\boldsymbol{t}$. The natural domain of a vector-valued function is the intersection of its component functions' domain.

4.3 Radius Vector/Position Vector

If a function can be expressed as $\boldsymbol{F(t)} =< \boldsymbol{f(t), g(t), h(t)} >$, then the position vector of it at $\boldsymbol{t = k}$ is $< \boldsymbol{f(k), g(k), h(k)} >$.

4.4 Vector Form of A Line Segment

For two vectors$\mathbf{r_0}$ and $\mathbf{r_1}$ that has its initial point at origin, the line passes through the terminal points of them can be written as:

$$\mathrm{r} = \mathrm{r_0} + t(\mathrm{r_1} - \mathrm{r_0}) \tag{4.1}$$

And this is called the ***two-point vector form o a line***.

4.5 Calculus of Vector-Valued Functions

The Calculus of vector-valued functions in 2-d and 3-d space is similar to "normal" functions:**just apply each operator to its component functions and "sum" them up.** The definition of integrable, differentiable and continuous is also similar:**each property requires its component functions have the corresponding property.**
The tangent line of the graph at point $\mathbf{r}(t_0)$:

$$\mathbf{r} = \mathbf{t_0} + t\mathbf{r}'(t_0) \tag{4.2}$$

For the dot product and cross product, which are unique to vector-valued functions, the derivative is defined as following:

$$\frac{d}{dt}[\mathbf{r_1}(t) \cdot \mathbf{r_2}(t)] = \mathbf{r_1}(t) \cdot \frac{d\mathbf{r_2}}{dt} + \frac{d\mathbf{r_1}}{dt} \cdot \mathbf{r_2}(t) \tag{4.3}$$

$$\frac{d}{dt}[\mathbf{r_1}(t) \times \mathbf{r_2}(t)] = \mathbf{r_1}(t) \times \frac{d\mathbf{r_2}}{dt} + \frac{d\mathbf{r_1}}{dt} \times \mathbf{r_2}(t) \tag{4.4}$$

In 2-d space, the tangent line to a circle is perpendicular to the radius at the point of tangency. Similarly, in for a vector-valued function, if $||\mathbf{r}(t)||$ is constant for all t, then:

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \tag{4.5}$$

that is, $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal for all t.

4.6 Arc Length

In a 2-d space, the arc length L of a parametric curve $x = x(t), y = y(t), (a \leq t \leq b)$ can be given as:

$$L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt \tag{4.6}$$

Lemma. *In a 2-d space, the arc length L of a function $y = f(x)$ that itself and its derivative is continuous on $[a, b]$ is:*

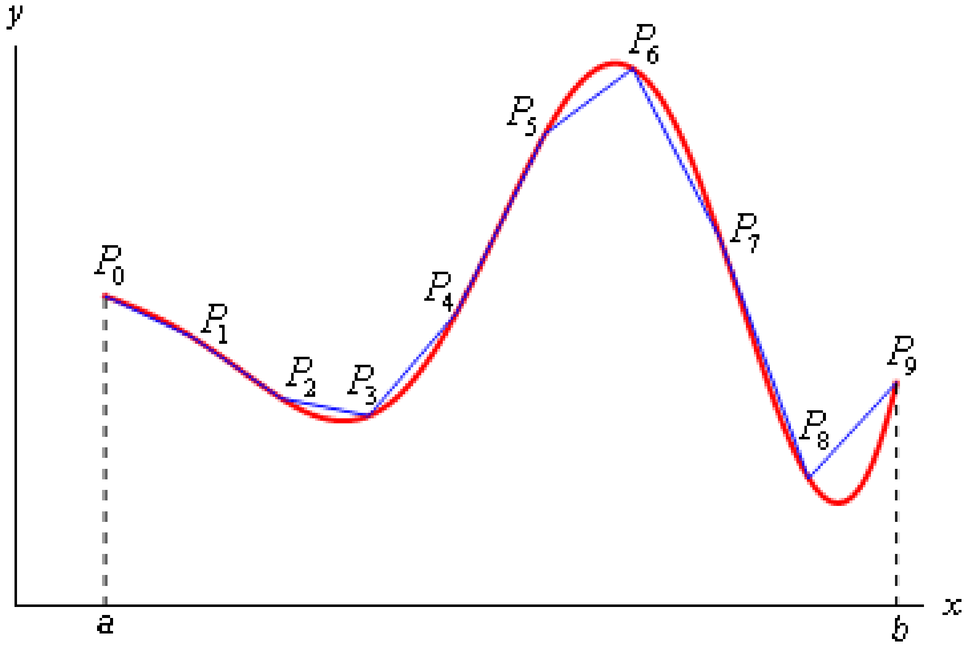
$$L = \int ds \tag{4.7}$$

where

$$ds = \sqrt{1 + (\frac{dy}{dx})^2} dx \quad if \quad y = f(x), a \leq x \leq b$$

$$ds = \sqrt{1 + (\frac{dx}{dy})^2} dy \quad if \quad x = g(y), c \leq y \leq d$$

Proof. As we can see in the figure below, the arc length is the sum of distance between n consecutive points when $n \to \infty$



Arc Length L can be written as:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}$$

Additionally, we can see that

$$\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2} = \sqrt{\Delta x^2 + \Delta y_i^2}$$

According to Mean Value Theorem, there exists an $\bar{x}$ such that

$$\Delta y_i = f'(\bar{x}_i)\Delta x$$

Thus

$$\begin{aligned}
\sqrt{\Delta x^2 + \Delta y_i^2} &= \sqrt{\Delta x^2 + \Delta y_i^2} \\
&= \sqrt{\Delta x^2 + (f'(\bar{x}_i)\Delta x)^2} \\
&= \sqrt{1 + [f'(\bar{x}_i)]^2}\Delta x
\end{aligned}$$

The exact length of the given curve is

$$\begin{aligned}
L &= \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{(x_{i+1}-x_i)^2+(y_{i+1}-y_i)^2} \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{1+[f'(\bar{x}_i)]^2}\Delta x \\
&= \int_a^b \sqrt{1+[f'(x)]^2}dx \\
&= \int_a^b \sqrt{1+(\frac{dy}{dx})^2}dx
\end{aligned}$$

□

Now we can prove Theorem (4.6):

Proof. Recall that $x = x(t), y = y(t)$, therefore

$$\begin{aligned}
L &= \int_a^b \sqrt{1+(\frac{dy}{dx})^2}dx \\
&= \int_a^b \sqrt{1+\frac{(\frac{dy}{dt})^2}{(\frac{dx}{dt})^2}}\frac{dx}{dt}dt \\
&= \int_a^b \sqrt{1+\frac{(\frac{dy}{dt})^2}{(\frac{dx}{dt})^2}}\frac{dx}{dt}dt \\
&= \int_a^b \frac{1}{|\frac{dx}{dt}|}\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\frac{dx}{dt}dt
\end{aligned}$$

If we assume that $\frac{dx}{dt} \geq 0$,then

$$L = \int_a^b \sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}dt$$

□

Analogously, the arc length L of a smoothly parametrized function(**have a continuously turning tangent vector**) in 3-d space is

$$\begin{aligned}
L &= \int_a^b \sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2+(\frac{dz}{dt})^2}dt \\
&= \int_a^b \left\|\frac{dr}{dt}\right\| dt \qquad (4.8)
\end{aligned}$$

4.7 Arc Length as A Parameter

Sometime it would be more convenient to replace t with s, which is the length of arc measured along the curve from some fixed reference point. There are three steps:
Step 1. Select an reference point.
Step 2. Choose one direction from the reference point as the positive direction.
Step 3. Change the length s to a "signed" length, which means s is positive if s "moves along the curve" to its positive direction.
Note that there are infinitely many different arc length parameterizations.

Theorem 4.7.1. ***Chain Rule*** *Let* $\mathbf{r}(t)$ *be a vector-valued function in 2-d/3-d space that is differentiable with respect to* t. *If* $t = g(\tau)$ *is a change of parameter in which* g *is differentiable with respect to* τ, *then* $\mathbf{r}(g(\tau))$ *is differentiable with respect to* τ *and*

$$\frac{d\mathbf{r}}{d\tau} = \frac{d\mathbf{r}}{dt}\frac{dt}{d\tau} \tag{4.9}$$

A change in parameter is smooth if $\mathbf{r}(g(\tau))$ is smooth and $\mathbf{r}(t)$ is smooth. For all τ, $\frac{dt}{d\tau} > 0$ is called a positive change of parameter while $\frac{dt}{d\tau} < 0$ is called a negative change of parameter.

Theorem 4.7.2. *Let* C *be the graph of a smooth vector-valued function* $\mathbf{r}(t)$ *in 2-d or 3-d space, and let* $\mathbf{r}(t_0)$ *be any point on* C. *Then the following formula defines a positive change of parameter from* t *to* s, *where* s *is an arc length parameter having* $\mathbf{r}(t_0)$ *as its reference point:*

$$s = \int_{t_0}^{t} \left\| \frac{d\mathbf{r}}{du} \right\| du \tag{4.10}$$

Theorem 4.7.3. *If* C *is the graph of a smooth vector-valued function* $\mathbf{r}(t)$ *in 2-d or 3-d space, where* t *is a general parameter, and if* s *is the arc length parameter for* C *defined by* ***Theorem 2***, *then for every value of* t *the tangent vector has length*

$$\left\| \frac{d\mathbf{r}}{dt} \right\| = \frac{ds}{dt} \tag{4.11}$$

Proof. This can be derived from applying the Fundamental Theorem of Calculus to Theorem 2. □

Theorem 4.7.4. *If C is the graph of a smooth vector-valued function $\mathbf{r}(t)$ in 2-d or 3-d space, where s is the arc length parameter, then for every value of s the tangent vector to C has length*

$$\left\|\frac{dr}{ds}\right\| = 1 \tag{4.12}$$

Proof. Let $t = s$ in **Theorem 3**. □

Theorem 4.7.5. *If C is the graph of a smooth vector-valued function $\mathbf{r}(t)$ in 2-d or 3-d space, and if*

$$\left\|\frac{dr}{dt}\right\| = 1 \tag{4.13}$$

for every value of t, then t is an arc length parameter that has its reference point at the point on C where $t = 0$.

Proof. The formula

$$s = \int_0^t \left\|\frac{d\mathbf{r}}{du}\right\| du$$

defines an arc length parameter for C with reference point $\mathbf{r}(0)$. Note that

$$\left\|\frac{dr}{dt}\right\| = 1$$

by hypothesis. Thus the formula can be rewrite as

$$s = \int_0^t du = t - 0 = t$$

□

4.8 Unit Tangent, Normal, and Binormal Vectors

Definition. The unit tangent of a smooth vector-valued function $\mathbf{r}(t)$ in 2-d space or 3-d space that points in the direction of increasing parameter can be expressed as:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

and it's called the *unit tangent vector* to C at t.

Recall that if a vector-valued function $\mathbf{r}(t)$ has constant norm, then $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal. Because $\mathbf{T}(t)$ has constant norm 1, so $\mathbf{T}(t)$ and $\mathbf{T}'(t)$ are orthogonal. This implies that $\mathbf{T}'(t)$ is perpendicular to the tangent line to C at t, so we say that $\mathbf{T}'(t)$ is *normal* to C at t. If $\mathbf{T}'(t) \neq 0$, then

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

is the *principle unit normal vector*, or simply *unit normal vector* to $\boldsymbol{C}$ at $\boldsymbol{t}$ and points in the same direction as $\mathbf{T}'(\boldsymbol{t})$.

The unit normal vector always points toward the concave side of $\boldsymbol{C}$ in 2-d space.

According to **Theorem 4.7.4**, $\|\mathbf{r}'(\boldsymbol{t})\| = \mathbf{1}$. Thus

$$\mathbf{T}(s) = \mathbf{r}'(s)$$

and consequently

$$\mathbf{N}(s) = \frac{\mathbf{r}''(s)}{\|\mathbf{r}''(s)\|}$$

Definition. The binormal vector to $\boldsymbol{C}$ at $\boldsymbol{t}$ can be defined as

$$\begin{aligned}\mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t) \\ &= \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}\end{aligned}$$

that is, the cross product of its unit tangent vector and unit normal vector and the direction of binormal vector is determined by the right-hand rule. $\|\mathbf{B}(t)\| = \mathbf{1}$ since $\|\mathbf{T}(t) \times \mathbf{N}(t)\| = \|\mathbf{T}(t)\| \, \|\mathbf{N}(t)\| \sin(\pi/2) = \mathbf{1}$.

In terms of arc length parameteriation, it can be expressed as

$$\mathbf{B}(s) = \frac{\mathbf{r}'(s) \times \mathbf{r}''(s)}{\|\mathbf{r}''(s)\|}$$

Together with unit tangent vector and unit normal vector, the binormal vector define three mutually perpendicular planes that point through that point – the **TB**-plane (called the *rectifying plane*), the **TN**-plane (called the *osculating plane*), and the **NB**-plane (called the *normal plane*). The coordinate system(right-hand) system determined by these three vectors is called the **TNB**-frame.

4.9 Curvature

Definition. If $\boldsymbol{C}$ is a smooth curve in 2-d space or 3-d space that is parametrized by arc length, then the *curvature* of $\boldsymbol{C}$, denoted by $\boldsymbol{\kappa} = \boldsymbol{\kappa}(s)$, is defined by

$$\kappa(s) = \left\| \frac{d\mathbf{T}}{ds} \right\| = \|\mathbf{r}''(s)\|$$

Theorem 4.9.1. *If* $\mathbf{r}(t)$ *is a smooth vector-valued function in 2-d space or 3-d space, then for each value of* t *at which* $\mathbf{T}(t)$ *and* $\mathbf{r}''(t)$ *exist, the curvature* κ *can be expressed as*

$$\begin{aligned}\kappa(t) &= \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} \\ &= \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}\end{aligned}$$