Point-Set Topology

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January 2, 2018

Contents

1

Basic Set Theory

1.1 Classification of Relations

Definition. An *equivalence relation* is a relation that satisfy the following properties:

aRa (Reflexivity);
$aRb \Rightarrow bRa$ (Symmetry);
$(aRb) \wedge (bRc) \Rightarrow aRc$ (Transitivity).

An equivalence relation is denoted by the special symbol $\sim$. $a \sim b$ means a is *equivalent* to b.

Definition. Let $R(\sim)$ be an equivalence relation on A. If $a \in A$, the *equivalence class* of a (denoted $\bar{a}$) is the class of all those elements of A that are equivalent to a. The class of all equivalence classes in A is denoted A/R and called the *quotient class* of A by R.

Theorem 1.1.1. *Two equivalence classes are either disjoint or equal.*

Definition. A *partial ordering* on a set X^2 is a relation R that have the following properties:

aRa (Reflexivity);
$(aRb) \wedge (bRc) \Rightarrow aRc$ (Transitivity).
$(aRb) \wedge (bRa) \Rightarrow (a = b)$ (Anti-symmetry);

We often write $a \preceq b$ and say that b *follows* a. If the condition

$$\forall a \forall b ((aRb) \vee (bRa)$$

holds in addition to transitivity and anti-symmetry defining a partial ordering relation(this means any two elements of X is comparable), the relation R is called an *ordering*, and the set X is said to be *linearly ordered*.

Definition. A relation $\prec$ is called a *strict partial order* if it's nonreflexive and transitive.

Theorem 1.1.2 (The Maximum Principle)**.** *Let A be a set and $\prec$ be a strict partial order on A. Then there exists a maximal simply ordered(linearly ordered) subset of A.*

Lemma (Zorn's Lemma)**.** *Let A be a set that is strictly partially ordered. If every simply ordered subset of A has an upper bound in A, then A has a maximal element.*

Definition. If X is a set and $<$ is an order relation on X, and if $a < b$, the notation (a, b) to denote the set

$$\{x|a < x < b\}$$

it is called an *open interval* in X. If this set is empty, a is called the *immediate predecessor* of b, and b is called the *immediate successor* of a.

Definition. Suppose that A and B are two sets with order relations $<_A$ and $<_B$ respectively. We say that A and B have the same *order type* if there is a bijective correspondence between them that preserves order, that is, if $f : A \to B$ is a bijection

$$a_1 <_A a_2 \Rightarrow f(a_1) <_B f(a_2)$$

***Example* 1.1.** *The interval $(1, 1)$ of real numbers has the same order type as $\mathbb{R}$, for the function $f : (-1, 1) \to \mathbb{R}$ given by*

$$f(x) = \frac{x}{1 - x^2}$$

is an order-preserving bijection.

Definition. Suppose A and B are two sets with order relations $<_A$ and $<_B$ respectively. Define an order relation $<$ on $A \times B$ by defining

$$a_1 \times b_1 < a_2 \times b_2 \tag{1.1}$$

if $a_1 <_A a_2$, or if $a_1 = a_2$ and $b_1 <_B b_2$. It is called the *dictionary order relation* on $A \times B$.

Functions and Their Graphs

Definition. A relation R is said to be functional if

$$(xRy_1) \wedge (xRy_2) \Rightarrow (y_1 = y_2)$$

and it is called a *function.*
$R \subset X \times Y$ is a *mapping from X into Y*, or a *function from X into Y*.

1.2 Sets and Operations on Them

1.2.1 Naive Set Theory

1^0. A set may consist of any distinguishable objects($x \in A \Rightarrow \exists! x \in A$)
2^0. A set is unambiguously determined by the collection of objects that comprise it.
3^0. Any property defines the set of objects having that property($A = \{x|P(x)\} \Rightarrow P(A)$).

However, this will lead to Russell's Paradox:
Let's have$P(M) \coloneqq M \notin M$
Consider the class $K = \{M|P(M)\}$. If so K is not a set, since whether $P(K)$ is true or false, contradiction arises.

1.2.2 ZFC: Zermelo-Fraenkel Axioms and Axiom of Choice

1^0. **(Axiom of Extensionality)** Sets A and B are equal iff they have the same elements. $(A = B) \Leftrightarrow (\forall x((x \in A) \Leftrightarrow (x \in B)))$
2^0. **(Axiom of Seperation)** To any set A and any property P there corresponds a set B whose elements are those elements of A, and only those, having property P(if A is a set, then $B = \{x \in A|P(x)\}$ is also a set).
3^0. **(Union Axiom)** For any set $\mathcal{M}$ whose elements are sets there exists a set $\bigcup M$, called the union of M and consisting of those elements and only those that belong to some element of $\mathcal{M}$ ($x \in \bigcup \mathcal{M} \Leftrightarrow \exists X((X \in \mathcal{M}) \wedge (x \in X))$)
Similarly, the intersection of the set $\mathcal{M}$ is defined as:

$$\bigcap \mathcal{M} \coloneqq \{x \in \bigcup \mathcal{M}|\forall X((X \in \mathcal{M}) \Rightarrow (x \in X))\}$$

4^0 **(Pairing Axiom)** For any sets X and Y there exists a set Z such that X and Y are its only elements.
5^0 **(Power Set Axiom)** For any set X there exists a set $P(X)$ having each subset of X as an element, and having no other elements.

Definition. The *successor* X^+ of the set X is $X^+ = X \cup \{X\}$.

Definition. An *inductive* set is a set that $\varnothing$ is one of its elements and the successor of each of its elements aso belongs to it.

6^0 **(Axiom of Infinity)** There exist inductive sets (Example: $\mathbb{N}_0$).
7^0 **(Axiom of Replacement)** Let $F(x, y)$ be a statement(a formula) such

that for every $x_0 \in X$ there exists a unique object y_0 such that $F(x_0, y_0)$ is true. Then the objects y for which there exists an element $x \in X$ such that $F(x, y)$ is true form a set.
And finally, an axiom that is independent of ZF.

8^0 **(Axiom of Choice/Zermelo's Axiom)** Given a collection of disjoint nonempty sets, there exists another set consisting of exactly one element from each element of the original set.

Definition. A choice function is a function f, defined on a collection X of nonempty sets, such that for every set A in X, $f(A)$ is an element of A.

Corollary. *There exists a choice function for any collection of nonempty sets.*

1.2.3 The *Cardinality* of a Set(*Cardinal Numbers*)

Definition. The set X is said to be *equipollent* to the set Y if there exists a bijective mapping of X onto Y(then $X \sim Y$).

Definition. *Cardinality* is a measure of the number of elements of the set. If $X \sim Y$, we write $\operatorname{card} X = \operatorname{card} Y$.

If X is equipollent to some subset of Y, we say $\operatorname{card} X \leqslant \operatorname{card} Y$, thus

$$(\operatorname{card} X \leqslant \operatorname{card} Y) \coloneqq \exists Z \subset Y(\operatorname{card} X = \operatorname{card} Z)$$

A set is called *finite* if it is not equipollent to any proper subset of itself; otherwise it is called *infinite*.
It has the properties below:

1^0 $(cardX \leqslant \operatorname{card} Y) \wedge (\operatorname{card} Y \leqslant \operatorname{card} Z) \Rightarrow (\operatorname{card} X \leqslant \operatorname{card} Z)$.
2^0 $(\operatorname{card} X \leqslant \operatorname{card} Y) \wedge (\operatorname{card} Y \leqslant \operatorname{card} X) \Rightarrow (\operatorname{card} X = \operatorname{card} Y)$(The Schröder–Bernstein theorem).
3^0 $\forall X \forall Y (\operatorname{card} X \leqslant \operatorname{card} Y) \vee (\operatorname{card} Y \leqslant \operatorname{card} X)$(Cantor's theorem).

We say $\operatorname{card} X < \operatorname{card} Y$ if $(\operatorname{card} X \leqslant \operatorname{card} Y) \wedge (\operatorname{card} X \neq \operatorname{card} Y)$.
let $\varnothing$ be the empty set and $P(X)$ the set of all subsets(thus, the power set) of the set X. Then:

Theorem 1.2.1. $\operatorname{card} X < \operatorname{card} P(X)$

Proof. The assertion is obvious for the empty set, and we shall assume that $X \neq \varnothing$.
Since $P(X)$ contains all the one-element subsets of X, $\operatorname{card} X \leqslant \operatorname{card} P(X)$. Suppose, contrary to the assertion, that there exists a bijective mapping $f : X \to P(X)$. Let set $A = \{x \in X : x \notin f(x)\}$ consisting of the elements

$x \in X$ that do not belong to the set $f(x) \in P(X)$ assigned to them by the bijection. Because $A \in P(X)$, there exists $a \in X$ such that $f(a) = A$. For the element a the relation $a \in A$ or $a \notin A$ is impossible by the definition of A(Similar to Russell's Paradox). □

1.2.4 Operations on Sets

Notation	Meaning	Definition
$A \subset B$	A is a subset of B	$\forall x((x \in A) \Rightarrow (x \in B))$
$A \subsetneq B$	A is a proper subset of B	$A \neq B \wedge A \subset B$
$A = B$	A equals to B	$(A \subset B) \wedge (B \subset A)$
$\varnothing$	Empty Set	$\{x \| x \neq x\}$
$A \cup B$ $A \cap B$	The union of A and B The intersection of A and B	$\{x \| x \in A \vee x \in B\}$ $\{x \| x \in A \wedge x \in B\}$
$A \setminus B$	The difference between A and B	$\{x \| x \in A \wedge x \notin B\}$
$C_M A$	The complement of A in M	$\{x \| x \in M \wedge x \notin A\} where A \subset M$
$A \times B$	The Cartesian Product of A and B	$\{(x, y) \| x \in A \wedge y \in B\}$
A^2	$A \times A$	

In the ordered pair$z = (x_1, x_2)$ where$Z = X_1 \times X_2, z \in Z, x_1 \in X_1, x_2 \in X_2$, x_1 is called the *first projection* of the pair z and denoted $\operatorname{proj}_1 z$ while x_2 is called the *second projection* of the pair z and denoted $\operatorname{proj}_2 z$.

1.3 Countable and Uncountable Sets

Definition. A set X is *countable* if it is equipollent with the set $\mathbb{N}$ of natural numbers, that is, $\operatorname{card} X = \operatorname{card} \mathbb{N}$.

Proposition. An infinite subset of a countable set is countable.

Proof. Let's consider a countable set E. There is a minimal element of $E_1 := E$, which we assign to $1 \in \mathbb{N}$ and denote $e_1 \in E$. E is infinite, so $E_2 := E \setminus e_1$ is not empty. Following the principle of induction, we can construct a injective mapping from $\{1, 2...\}$ to $\{e_1, e_2, ...\}$.

Now we have to prove that this mapping is also surjective. Suppose the contrary, that an element $e \in E$ does not have a natural number assigned to it. The set $K = \{n \in E | n \leqslant e\}$ is finite, since it's a subset of $\mathbb{N}$ bounded both from below and above. According to our previous construction, we assign 1 to $\min K$, denoted as e_1, and we can acquire a sequence $e_1, e_2, ...e_{k=\operatorname{card} K}$. But $e_{k=\operatorname{card} K}$ is $\max K$, and because $e \in K \wedge (\forall n \in K(n \leqslant e))$, $e = \max K$. Therefore $e = e_k$, or otherwise it will contradict the uniqueness of maximal element. □

Proposition. The Union of the sets of a finite or countable system of countable sets is also a countable set.

Proof. Let $X_1, X_2..., X_n, ...$ is a countable system of sets and each set $X_m = \{x_m^1, ..., x_m^n, ...\}$ is itself countable. Since $\forall m \in \mathbb{N}(\operatorname{card}(X = \bigcup_{n\in\mathbb{N}} X_n) \geqslant X_m)$, X is an infinite set. The ordered pair (m, n) identifies the element $x_m^n \in X_m$. We can construct a mapping, like $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N} := (m, n) \to \frac{(m+n-2)(m+n-1)}{2} + m$, such that it is bijective. Thus X is countable. Then because $\operatorname{card} X \leqslant \operatorname{card} \mathbb{N}$ and the fact that X is infinite, we conclude that $\operatorname{card} X = \operatorname{card} \mathbb{N}$. □

If it is known that a set is either finite or countable, we say it is *at most countable*($\operatorname{card} X \leqslant \mathbb{N}$).

Corollary. $\operatorname{card} \mathbb{Z} = \operatorname{card} \mathbb{N}$

Corollary. $\operatorname{card} \mathbb{N}^2 = \operatorname{card} \mathbb{N}$*(The direct product of countable sets is countable).*

Corollary. $\operatorname{card} \mathbb{Q} = \operatorname{card} \mathbb{N}$*, that is, the set of rational numbers is countable.*

Proof. Let (m, n) denote a rational number $\frac{m}{n}$. It is known that the pair (m, n) and (m', n') define the same number iff they are proportional. Thus $\mathbb{Q}$ is equipollent to some infinite subset of the set $\mathbb{Z}\times\mathbb{Z}$. Since $\operatorname{card} \mathbb{Z}^2 = \operatorname{card} \mathbb{N}$, we can conclude that $\operatorname{card} \mathbb{Q} = \operatorname{card} \mathbb{N}$. □

Corollary. *The set of algebraic numbers is countable.*

Proof. It can be observed that $\operatorname{card} \mathbb{Q} \times \mathbb{Q} = \operatorname{card} \mathbb{N}$. By the principle of induction, $\forall k \in \mathbb{N}(\operatorname{card} \mathbb{Q}^k = \operatorname{card} \mathbb{N})$. Let $r \in \mathbb{Q}^k$ be an ordered set $(r_1, r_2, ..., r_k)$ consists of k rational numbers.

An algebraic equation of degree k with rational coefficient can be writtne in the reduced form $x^k + r_1 x^{k-1} + \cdots + r_k = 0$. Thus there are as many different algebraic equations of degree k as there are different ordered sets $(r_1, ..., r_k)$ of rational numbers, that is, a countable set.

The algebraic equation with rational coefficients (of arbitrary degree) is the union of sets consisting of algebraic equation (of a fixed degree) which is countable, and this union is countable. Each such equation has only a finite number of roots. Hence the set of algebraic numbers is at most countable. But it is infinite, and therefore countable. □

1.3.1 The Cardinality of the Continuum

Definition. The set $\mathbb{R}$ of real numbers is also called the *number continuum*(from Latin *continuum*, meaning continuous, or solid), and its cardinality the *cardinality of the continuum.*

Theorem 1.3.1 (Cantor)**.** $\operatorname{card}\mathbb{N} < \operatorname{card}\mathbb{R}$

Proof by Nested Interval Lemma. It is sufficient to show that even $[0, 1]$ in an uncountable set.

Assume it is countable, that is, can be written as a sequence $x_1, x_2, ..., x_n, ...$. Take x_1 on $I_0 = [0, 1]$, and find I_1 such that $x_1 \notin I_1$. Then construct the nested interval I_n such that $x_{n+1} \notin I_{n+1}$ and $|I_n| > 0$. It follows the nested interval lemma that there exist a point $c \in [0, 1]$ belonging to all I_n. But by our construction, $c \in \mathbb{R}$ and c cannot be any point of the sequence $x_1, x_2, ..., x_n, ...$. □

Proof by Cantor's Diagonal Argument. Let's first consider an the set L and write out the infinite sequence of distinct binary numbers in it which has the form:

$$s1 = (0, 0, 0, 0, 0, 0, 0, ...) \tag{1.2}$$
$$s2 = (1, 1, 1, 1, 1, 1, 1, ...) \tag{1.3}$$
$$s3 = (0, 1, 0, 1, 0, 1, 0, ...) \tag{1.4}$$
$$s4 = (1, 0, 1, 0, 1, 0, 1, ...) \tag{1.5}$$
$$s5 = (1, 1, 0, 1, 0, 1, 1, ...) \tag{1.6}$$
$$s6 = (0, 0, 1, 1, 0, 1, 1, ...) \tag{1.7}$$
$$s7 = (1, 0, 0, 0, 1, 0, 0, ...) \tag{1.8}$$
$$... \tag{1.9}$$
$$\tag{1.10}$$

We then constrcut a number s such that its first digit is the complementary (swapping 0s for 1s and vice versa) of the first digit of s_1 and etc.

$$s1 = (\mathbf{0}, 0, 0, 0, 0, 0, 0, ...) \tag{1.11}$$
$$s2 = (1, \mathbf{1}, 1, 1, 1, 1, 1, ...) \tag{1.12}$$
$$s3 = (0, 1, \mathbf{0}, 1, 0, 1, 0, ...) \tag{1.13}$$
$$s4 = (1, 0, 1, \mathbf{0}, 1, 0, 1, ...) \tag{1.14}$$
$$s5 = (1, 1, 0, 1, \mathbf{0}, 1, 1, ...) \tag{1.15}$$
$$s6 = (0, 0, 1, 1, 0, \mathbf{1}, 1, ...) \tag{1.16}$$
$$s7 = (1, 0, 0, 0, 1, 0, \mathbf{0}, ...) \tag{1.17}$$
$$... \tag{1.18}$$
$$s = (\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{1}, ..) \tag{1.19}$$

By construction s differs from s_n at the nth digit, so s is not in this sequence, and thus L is uncountable.

We can now define a mapping $f : L \to \mathbb{R}. f(s_n) = r_n \in \mathbb{R}$ means that s_n and r_n have the same digit while r_n is under base 10 and s_n is under base 2. For $s_n \neq s_m \Rightarrow (r_n = f(s_n)) \neq (r_m = f(s_m))$, f is injective, and with the fact that all s_n corresponds to a r_n together give us $\operatorname{card} f(L) = \operatorname{card} L$. Since $f(L)$ is a subset of $\mathbb{R}$, we can see that $\mathbb{R}$ is also uncountable. $\square$

The proof above illustrates the theorem below.

Definition. Let X denote the two element set $\{0, 1\}$. Then X^ω is uncountable.

The cardinality of $\mathbb{R}$ is often denotes as $\mathfrak{c}$.

Corollary. $\mathbb{Q} \neq \mathbb{R}$*, and so irrational numbers exist.*

Corollary. *There exist transcendental numbers, since the set of algebraic numbers is countable.*

***Example* 1.2.** *The cardinality of* $P(X)$*, which is the power set of* X*, satisfy that if* $\operatorname{card} X = n$*,* $\operatorname{card} P(X) = 2^n$*.*

Proof. We can use the principle of induction to complete the proof. If $n = 1$, $X = \{x\}$, then $P(X) = \{\varnothing, X\}$, then $\operatorname{card} P(X) = 2^1$.

Now if $n \in \mathbb{N} \Rightarrow \operatorname{card} P(X) = 2^n$, let X be a set that has x as one of its elements and has the cardinality of $n + 1$. Therefore $Y = X \setminus \{x\}$ has n elements. We can divide $P(X)$ into two parts: the ones containing x and the ones don't. If $x \in A \subset P(X)$, then $A \setminus \{x\} \subset P(Y)$ and vice versa. Thus we can set up a bijection between $P(Y)$ and the elements in $P(X)$ that contains x. Similarly, we can clearly see that a bijection between the subsets of $P(X)$ that does not contains x and $P(Y)$. Thus $\operatorname{card} P(X) = 2^n + 2^n = 2^{n+1}$, and we complete the proof. $\square$

We'll use a script letter to denote the collection of sets, for example, $\mathcal{A}$ for collection of sets and A for individual sets in it.

Definition. A *partition* of a set A, besides the definition we have when studying Riemann Sum, can be defined as a collection of disjoint nonempty subsets of A whose union is all of A.

Theorem 1.3.2. *Given any partition* $\mathcal{D}$ *of* A*, there is exactly one equivalence relation on* A *from which it is derived.*

***Example* 1.3.** *Defined two points in the plane to be equivalent if they lie at the same distance from the origin. The collection of equivalence classes consists of all circles centered at the origin, along with the set consisting of the origin alone.*

Definition. Any set together with a order relation $<$ that satisfy both the following properties

1. $<$ has the least upper bound property.
2. if $x < y$, then there exists an element z such that $x < z < y$.

is called a *linear continuum.*

Definition. Let $\mathcal{A}$ be a nonempty collection of sets. An *indexing function* for $\mathcal{A}$ is a surjective function f from some set J, called the *index set*, to $\mathcal{A}$. The collection $\mathcal{A}$, together with the indexing function is called an *indexed family of sets.* Given $\alpha \in J$, the set $f(\alpha)$ is denoted A_α. The indexed family itself is denoted by

$$\{A_\alpha\}_{\alpha \in J}$$

Theorem 1.3.3 (Principle of Recursive Definition)**.** *Let A be a set. Given a formula that defined $h(1)$ as a unique element of A, and for $i > 1$ defines $h(i)$ uniquely as an element of A in terms of the values of h for positive integers less than i, this formula determines a unique function $h : \mathbb{N} \to A$.*

Definition. A set A with an order relation $<$ is said to be *well-ordered* if every nonempty subset of it has a smallest element.

Theorem 1.3.4. *Any subset of a well-ordered set is well-ordered. The cartesian product of two well-ordered sets is well-ordered.*

Theorem 1.3.5. *Every nonempty finite ordered set has the order type of a section of $\mathbb{N}$, so it's well-ordered.*

Theorem 1.3.6 (Well-ordering theorem, proved by Zermelo)**.** *If A is a set, there exists an order relation on A that is a well-ordering.*

Corollary. *There exists an uncountable well-ordered set.*

Definition. Let X be a well-ordered set. Given $\alpha \in X$, let S_α denote the set

$$S_\alpha = \{x | x \in X \wedge x < \alpha\}$$

It is called the *section* of X by α.

Lemma. *There exists a well-ordered set A having a largest element Ω, such that the section S_Ω of A by Ω is uncountable but every other section of A is countable.*

Proof. We begin with an uncountable well-ordered set B. Let C be the well-ordered set $\{1, 2\} \times B$ in the dictionary order, then some section of C is uncountable. Let Ω be the smallest element of C for which the section of C by Ω is uncountable, then let A consist of this section along with Ω. $\square$

The set S_Ω is called a *minimal uncountable well-ordered set*, and the well-ordered set $A = S_\Omega \cup \{\Omega\}$ by $\bar{S}_\Omega$.

Theorem 1.3.7. *If A is a countable subset of S_Ω, then A has an upper bound in S_Ω.*

Proof. Let A be a countable subset of S_Ω. For each $a \in A$, the section S_a is countable. Therefore, the union $B = \bigcup_{\alpha \in A} S_a$ is also countable. Since $S_\Omega \neq B$, let x be a point of S_Ω that is not in B, and then x is an upper bound for A. $\square$

2

Topological Spaces and Continuous Functions

2.1 Definition for Topological Spaces

Definition. A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. $\varnothing$ and X are in $\mathcal{T}$.
2. The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
3. The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is called a *topological space*. A subset U of X is an *open set* of X if U belongs to the collection $\mathcal{T}$. Then a topological space is a set X with a collection of subsets of X, called open sets, such that X and $\varnothing$ are both open and arbitrary unions and finite intersections of open sets are open.

Definition. If X is any set, the collection of all subsets of X is a topology on X and called the *discrete topology*. The collection consisting of X and $\varnothing$ is also a topology on X and is called the *indiscrete topology* or the *trivial topology*.

Definition. Let X be a set; let $\mathcal{T}_f$ be the collection of all subsets U of X such that $X \setminus U$ either is finite or is all of X. Then $\mathcal{T}_f$ is a topology on X and called the *finite complement topology*.

Definition. Suppose $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \supset \mathcal{T}$, we say that $\mathcal{T}'$ is *finer* than $\mathcal{T}$; If $\mathcal{T}'$ properly contains $\mathcal{T}$, we say that $\mathcal{T}'$ is *strictly finer* then $\mathcal{T}$. We also say that $\mathcal{T}$ is *coarser* than $\mathcal{T}'$, or *strictly coarser*, in these two respective situations. We say $\mathcal{T}$ is *comparable* with $\mathcal{T}'$ if either $\mathcal{T}' \supset \mathcal{T}$ or $\mathcal{T} \supset \mathcal{T}'$

2.2 Basis for Topology

Definition. If X is a set, a *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X(called *basis elements*) such that

1. For each $x \in X$, there is at least one basis element B containing x.
2. If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the *topology* $\mathcal{T}$ *generated by* $\mathcal{B}$ as follows: A subset U of X is said to be open in X if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ and $x \in B$ and $B \subset U$.

***Example* 2.1.** *If X is any set, the collection of all one-point subsets of X is a basis for the discrete topology on X.*

Lemma. *Let X be a set; Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals toe collection of all unions of elements of $\mathcal{B}$.*

Proof. Given a collection of elements of $\mathcal{B}$, they are also elements of $\mathcal{T}$. Because $\mathcal{T}$ is a topology, their union is in $\mathcal{T}$. Conversely, given $U \in \mathcal{T}$, choose for each $x \in U$ an element B_x of $\mathcal{B}$ such that $x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$, so U equals a union of elements of $\mathcal{B}$. □

Lemma. *Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.*

Proof. First we show that $\mathcal{C}$ is a basis. Given $x \in X$, since X is open, there is by hypothesis an element C of $\mathcal{C}$ such that $x \in C \subset X$. Now let $x \in C_1 \cap C_2$, where C_1 and C_2 are elements of $\mathcal{C}$. The intersection of them is open, and there exists by hypothesis an element C_3 in $\mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$.

Let $\mathcal{T}$ be the collection of open sets of X; we will show that the topology $\mathcal{T}'$ generated by $\mathcal{C}$ equals the topology. First, note that if U belongs to $\mathcal{T}$ and if $x \in U$, then there is by hypothesis an element C of $\mathcal{C}$ such that $x \in C \subset U$. It follows that U belongs to the topology $\mathcal{T}'$ by definition. Conversely, if W belongs to the topology $\mathcal{T}$, then W equals a union of elements of $\mathcal{C}$ by the preceding lemma. Since each element of $\mathcal{C}$ belongs to $\mathcal{T}$ and $\mathcal{T}$ is a topology, W also belongs to $\mathcal{T}$. □

Lemma. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then the following are equivalent:*

1. *$\mathcal{T}'$ is finer than $\mathcal{T}$.*

2. *For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.*

Proof. First, we prove that the second condition implies the first one. Given an element U of $\mathcal{T}$, we wish to show that $U \in \mathcal{T}'$. Let $x \in U$. Since $\mathcal{B}$ generates $\mathcal{T}$, there is a element $B \in \mathcal{B}$ such that $x \in B \subset U$. Condition (2) tells us there exists an element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$. Then $x \in B' \subset U$, so $U \in \mathcal{T}'$ by definition.

Then we prove that the first condition implies the second. We are given $x \in X$ and $B \in \mathcal{B}$. Now B belongs to $\mathcal{T}$ by definition and $\mathcal{T} \subset \mathcal{T}'$ by condition (1); therefore, $B \in \mathcal{T}'$. Since $\mathcal{T}'$ is generated by $\mathcal{B}'$, there is an element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$. □

Definition. If $\mathcal{B}$ is the collection of all intervals in the real line, the topology generated by $\mathcal{B}$ is called the *standard topology* on the real line. If $\mathcal{B}'$ is the collection of all half-open intervals $[a, b)$ where $a < b$, the topology generated by $\mathcal{B}'$ is called the *lower limit topology* on $\mathbb{R}$. When $\mathbb{R}$ is given the lower limit topology, it's denoted $\mathbb{R}_l$. Let K denote the set of all numbers of the form $1/n$ for $n \in \mathbb{N}$, and let $\mathcal{B}''$ be the collection of all open intervals (a, b), along with all sets of the form $(a, b) \setminus K$. The topology generated by $\mathcal{B}''$ will be called the *K-topology* on $\mathbb{R}$. When $\mathbb{R}$ is given this topology, it's denoted by $\mathbb{R}_K$.

Lemma. *The topologies of $\mathbb{R}_l$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$, but are not comparable with one another.*

Proof. Let $\mathcal{T}$, $\mathcal{T}'$, and $\mathcal{T}''$ be the topologies of $\mathbb{R}$, $\mathbb{R}_l$, and $\mathbb{R}_K$ respectively. Given a basis element (a, b) for $\mathcal{T}$ and a point x of (a, b), the basis element $[x, b)$ for $\mathcal{T}'$ contains x and lies in (a, b). On the other hand, given the basis element $[x, d)$ for $\mathcal{T}'$, there is no open interval (a, b) that contains x and lies in $[x, d)$, and thus $\mathcal{T}'$ is strictly finer than $\mathcal{T}$.

A similar argument applies to $\mathbb{R}_K$. Given a basis element (a, b) for $\mathcal{T}$ and a point $x \in (a, b)$, this same interval is a basis for $\mathcal{T}''$ that contains x. On the other hand, given the basis element $B = (-1, 1) \setminus K$ for $\mathcal{T}''$ and the point 0 of B, there is no open interval that contains 0 and lies in B.

Now we show that $\mathbb{R}_l$ and $\mathbb{R}_K$ are not comparable. For any basis element in $\mathbb{R}_l$ that has 0 as its lower limit, it always contains number of the form $1/n$, thus not any subset of sets of the form $(a, b) \setminus K$. The rest of this argument is trivial. □

Definition. A *subbasis* $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The *topology generated by the subbasis* $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

For the purpose of checking whether $\mathcal{T}$ is a topology, it's sufficient to show that the collection $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ is a basis, for then the collection $\mathcal{T}$ of all unions of elements of $\mathcal{B}$ is a topology. Given $x \in X$, it belongs to an element of $\mathcal{S}$ and hence to an element of $\mathcal{B}$; to check the second condition, let

$$B_1 = S_1 \cap \cdots \cap S_m \quad \text{and} \quad B_2 = S_1' \cap \cdots \cap S_n'$$

to be two elements of $\mathcal{B}$. Their intersection

$$B_1 \cap B_2 = (S_1 \cap \cdots \cap S_m) \cap (S_1' \cap \cdots \cap S_n')$$

is also a finite intersection of elements of $\mathcal{S}$, so it belongs to $\mathcal{B}$.

2.3 The Order Topology

Definition. Let X be a set with a linear order relation; assume X has, more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

1. All open intervals (a, b) in X.
2. All interval of the form $[a_0, b)$, where a_0 is the smallest element of X.
3. All intervals of the form $(a, b_0]$, where b_0 is the largest element of X.

The collection $\mathcal{B}$ is a basis for a topology on X, which is called the *order topology*.

Definition. If X is an ordered set, and a is an element of X, there are four subsets of X that are called the rays determined by a. The *open rays* are $(a, +\infty)$ and $(-\infty, a)$, the *closed rays* are $[a, +\infty)$ and $(-\infty, a]$.

2.4 The Product Topology on $X \times Y$

Definition. Let X and Y be topological spaces. The *product topology* on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where U and V are open sets of X and Y respectively.

Theorem 2.4.1. *If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y, then the collection*

$$\mathcal{D} = \{B \times C | B \in \mathcal{B} \wedge C \in \mathcal{C}\}$$

is a basis for the topology of $X \times Y$.

Proof. Given an open set W of $X \times Y$ and a point $x \times y$ of W, by definition of the product topology there is a basis element $U \times V$ such that $x \times y \in U \times V \subset W$. Then $\mathcal{D}$ is a basis for $X \times Y$. $\square$

Theorem 2.4.2. *The collection*

$$\mathcal{S} = \{\text{proj}_1^{-1} U | U \text{ open in } X\} \cup \{\text{proj}_2^{-1} V | V \text{ open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

Proof. Let $\mathcal{T}$ denote the product topology on $X \times Y$; Let $\mathcal{T}'$ be the topology generated by $\mathcal{S}$. Because every element of $\mathcal{S}$ belongs to $\mathcal{T}$, so do arbitrary unions of finite intersections of elements of $\mathcal{S}$. Thus $\mathcal{T}' \subset \mathcal{T}$. On the other hand, every basis element $U \times V$ for the topology $\mathcal{T}$ is a finite intersection of elements in $\mathcal{S}$, since

$$U \times V = \text{proj}_1^{-1}(U) \cap \text{proj}_2^{-1}(V)$$

Therefore, $U \times V$ belongs to $\mathcal{T}'$, so that $\mathcal{T} \subset \mathcal{T}'$. $\square$

2.5 The Subspace Topology

Definition. Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection

$$\mathcal{T}_Y = \{Y \cap U | U \in \mathcal{T}\}$$

is a topology on Y, called the *subspace topology*. With this topology, Y is called a *subspace* of X; its open sets consist of all intersections of open sets of X with Y.

Lemma. *If $\mathcal{B}$ is a basis for the topology of X then the collection*

$$\mathcal{B}_Y = \{B \cap Y | B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

Proof. Given U open in X and given $y \in U \subset Y$, we can choose an element B of $\mathcal{B}$ such that $y \in B \subset U$. Then $y \in B \cap Y \subset U \cap Y$. It follows that $\mathcal{B}$ is a basis for the subspace topology on Y. $\square$

Definition. If Y is a subspace of X, a set U is *open in Y* (or *open relative to Y*) if it belongs to the topology of Y; this implies in particular that it is a subset of Y. We say that U is *open in X* if it belongs to the topology of X.

Lemma. *Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.*

Proof. Since U is open in Y, $U = Y \cap V$ for some set V open in X. Since Y and V are both open in X, so is $Y \cap V$. □

Theorem 2.5.1. *If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.*

Proof. The set $U \times V$ is the general basis element for $X \times Y$, where U is open in X and V is open in Y. Therefore, $(U \times V) \cap (A \times B)$ is the general basis element for the subspace topology on $A \times B$. Now

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$$

The set $(U \cap A) \times (V \cap B)$ is the general basis element for the product topology on $A \times B$. □

If X is an ordered set and Y is a subset of it. The order relation on X, when restricted to Y, makes Y into an ordered set. However, **the resulting order topology on Y need not be the same as the topology inherits as a subspace of X.** We'll give two examples below.

***Example* 2.2.** *Let Y be the subset $[0, 1) \cup \{2\}$ of $\mathbb{R}$. In the subspace topology on Y the one-point set $\{2\}$ is open, because it is the intersection of the open set $(\frac{3}{2}, \frac{5}{2})$ with Y. But in the order topology on Y, $\{2\}$ is not open. Any basis element for the order topology on Y that contains 2 is of the form*

$$\{x | x \in Y \wedge a < x \leqslant 2\}$$

for some $a \in Y$; such a set necessarily contains points of Y less than 2.

***Example* 2.3.** *Let $I = [0, 1]$. The dictionary order on $I \times I$ is just the restriction to $I \times I$ of the dictionary order on the plane $\mathbb{R} \times \mathbb{R}$. However, the dictionary order topology on $I \times I$ is not the same as the subspace topology on $I \times I$ obtained from the dictionary topology on $\mathbb{R} \times \mathbb{R}$. For example, the set $\{1/2\} \times (1/2, 1]$ is open in $I \times I$ in the subspace topology, but not in the order topology.*

The set $I \times I$ in the dictionary order topology will be called the *ordered square* and denoted by I_o^2.

Definition. Given an ordered set X, a subset Y is *convex* in X if for each pair of points $a < b$ of Y, the entire interval (a, b) of points of X lies in Y. Intervals and rays in X are convex in X.

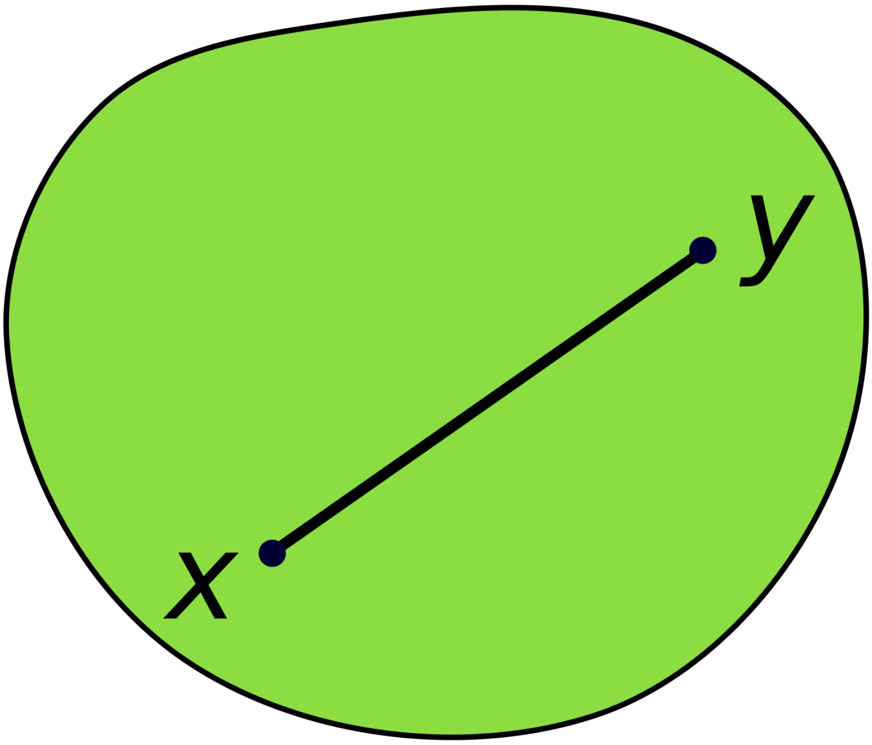

Figure 2.1: A convex set

Theorem 2.5.2. *Let X be an ordered set in the order topology; Let Y be a subset of X that is convex in X. Then the order topology on Y is the same as the topology Y inherits as a subspace of X.*

Proof. Consider the ray $(a, +\infty)$ in X. If $a \in Y$, its intersection with Y is

$$(a, +\infty) \cap Y = \{x | x \in Y \wedge x > a\}$$

this is an open ray of the ordered set Y. If $a \notin Y$, then a is either a lower bound on Y or an upper bound on Y since Y is convex. In the former case, the set $(a, +\infty) \cap Y$ equals all of Y; in the latter case, it is empty.

A similar argument holds for the intersection of $(-\infty, a)$ and Y. Since the sets $(a, +\infty) \cap Y$ and $(-\infty, a) \cap Y$ form a subbasis for the subspace topology on Y, and since each is open in the order topology, the order topology contains(or is coarser than) the subspace topology.

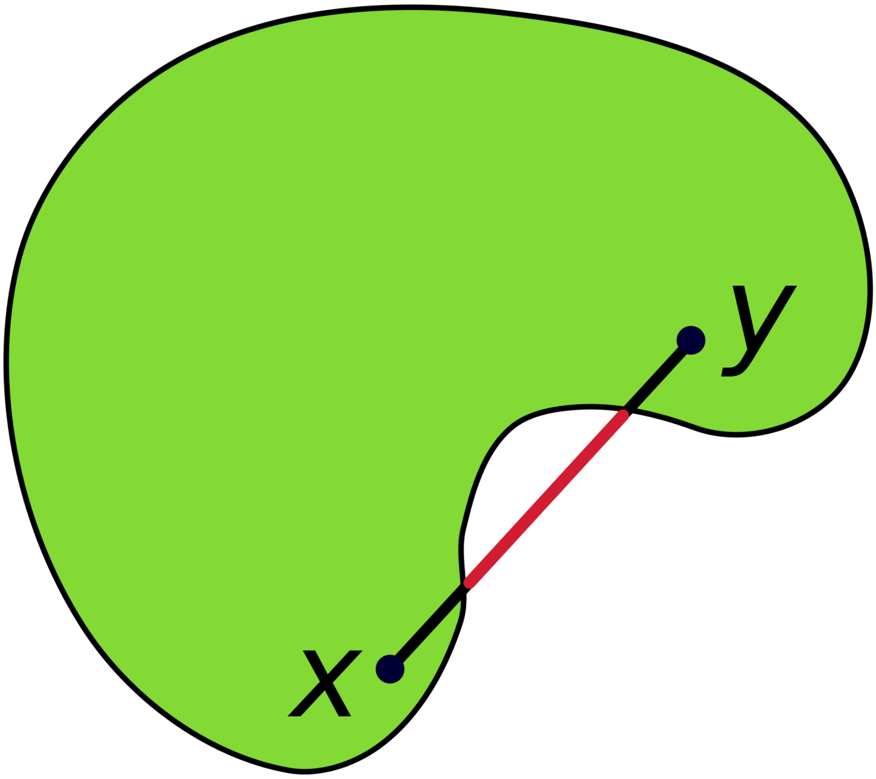

Figure 2.2: A non-convex set

Note that any open ray of Y equals the intersection of an open ray of X with Y, so it is open in the subspace topology on Y. Since the open rays of Y are a subbasis for the order topology on Y, this topology is contained in(or finer than) the subspace topology. $\square$

To avoid ambiguity, if X is an ordered set in the order topology and Y is a subset of X, we shall assume that Y is given the subspace topology unless specified.

2.6 Closed Sets and Limit Points

2.6.1 Closed Sets

Theorem 2.6.1. *Let X be a topological space. Then the following conditions hold:*

1. *$\varnothing$ and X are closed.*

2. *Arbitrary intersections of closed sets are closed.*

3. *Finite unions of closed sets are closed.*

Proof. (1) is obvious. Given a collection of closed sets $\{A_\alpha\}\alpha \in J$, we have $X - \bigcap_{\alpha\in J} = \bigcup_{\alpha\in J}(X - A_\alpha)$. Since the set on the right hand side is open, $\bigcap A_\alpha$ is closed. The third condition can be verified similarly. $\square$

If Y is a subspace of X, we say that a set A is *closed in* Y if A is a subset of Y and if A is closed in the subspace topology of Y(that is, if $Y \setminus A$ is open in Y).

Theorem 2.6.2. *Let Y be a subspace of X. Then a set A is closed in Y iff it equals the intersection of a closed set of X with Y.*

Proof. Assume that $A = C \cap Y$, where C is closed in X. Then $X \setminus C$ is open in X, so that $(X \setminus C) \cap Y$ is open in Y, but then $(X \setminus C) \cap Y = Y \setminus A$, and hence A is closed in Y. Conversely, assume that A is closed in Y. Then $Y \setminus A$ is open in Y, and by definition it equals the intersection of an open set U of X with Y. The set $X \setminus U$ is closed in X, so $A = Y \cap (X \setminus U)$, as desired. $\square$

Theorem 2.6.3. *Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.*

2.6.2 Closure and Interior of a Set

Definition. Given a subset A of a topological space X, the *interior* of A is defined as the union of all open sets contained in A, and the *closure* of A is defined as the intersection of all closed sets containing A. The interior is denoted by $\operatorname{Int} A$ and the closure of A is denoted by $\operatorname{Cl} A$ or $\bar{A}$. Obviously $\operatorname{Int} A$ is open and $\bar{A}$ is closed. Furthermore

$$\operatorname{Int} A \subset A \subset \bar{A}$$

If A is open, then $A = \operatorname{Int} A$; if A is closed, $A = \bar{A}$.

Theorem 2.6.4. *Let Y be a subspace of X; let A be a subset of Y; let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equals $\bar{A} \cap Y$.*

Proof. Let B denote the closure of A in Y. The set $\bar{A}$ is closed in X, so $\bar{A} \cap Y$ is closed. Since $\bar{A}$ contains A, we have that $B \subset (\bar{A} \cap Y)$.

On the other hand, B is closed in Y. Hence $B = C \cap Y$ for some C closed in X. Then C is a closed set of X containing A; because $\bar{A}$ is the intersection of all such closed sets, we have $\bar{A} \subset C$. Then $(\bar{A} \cap Y) \subset (C \cap Y) = B$. $\square$

Theorem 2.6.5. *Let A be a subset of the topological space X.*

1. *Then $x \in \bar{A}$ iff every open set U containing x intersects A.*
2. *Supposing the topology of X is given by a basis, then $x \in \bar{A}$ iff every basis element B containing x intersects A.*

Proof. If $x \notin \bar{A}$, the set $U = X \setminus \bar{A}$ is an open set containing x that does not intersect A. Conversely, if there exists an open set U containing x which does not intersect A, then $X \setminus U$ is a closed set containing A. By the definition of the closure, the set $X \setminus U$ must contain $\bar{A}$; therefore, $x \notin \bar{A}$. The second statement follows from our proof, since any basis element is open. □

2.6.3 Limit Points

Definition. If A is a subset of the topological space X and if x is a point of X, then x is a *limit point*(or *cluster point* or *point of accumulation*) of A if every neighborhood of x intersects A in some point other than x itself. Equivalently, x is a limit point of A if it belongs to the closure of $A \setminus x$.

Theorem 2.6.6. *Let A be a subset of the topological space X; let A' be the set of all limit points of A. Then*

$$\bar{A} = A \cup A'$$

Proof. If x is in A', every neighborhood of x intersects A. Therefore, x belongs to $\bar{A}$. Hence $A' \subset \bar{A}$. By definition $A \subset \bar{A}$, and it follows that $A \cup A' \subset \bar{A}$.

Now let x be a point of $\bar{A}$. If x lies in A, the relation $x \in A \cup A'$ is trivial; suppose that x is not in A. Since $x \in \bar{A}$, we know that every neighborhood U of x intersects A. Then $x \in A'$, so that $x \in A \cup A'$. □

Corollary. *A subset of a topological space is closed iff it contains all its limit points.*

2.7 Hausdorff Spaces

Definition. A topological space is called a *Hausdorff space* if each pair of points has disjoint neighborhoods.

Theorem 2.7.1. *Every finite point set in a Hausdorff space is closed.*

Proof. It's sufficient to prove that the closure of any one-point set is itself, and this is true since for any point of the complement of this one-point set there exists a neighborhood that does not intersect with the original set. □

Definition. The condition that finite point sets be closed is called the T_1 *axiom.*

Theorem 2.7.2. *Let X be a space satisfying the T_1 axiom; let A be a subset of X. Then the point x is a limit point of A iff every neighborhood of x contains infinitely many points of A.*

Proof. If every neighborhood of a limit point x only intersects A at finitely many number of points, then by the T_1 axiom, any neighborhood of x is closed, and the complement of any deleted neighborhood of x is open. Then the intersection of the original neighborhood of x with this complement is a neighborhood of x, but it does not intersect with A at all, and this contradicts to the fact that x is a limit point. □

Theorem 2.7.3. *If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X.*

Proof. Suppose that x_n is a sequence of points of X that converges to x. If $y \neq x$, let U and V be disjoint neighborhoods of x and y, respectively. Since U contains x_n for all but a finitely many values of n, the set V cannot. Then x_n cannot converge to y. □

Theorem 2.7.4. *Every simply ordered set is a Hausdorff space in the order topology. The product of two Hausdorff spaces is a Hausdorff space. A subspace of a Hausdorff space is a Hausdorff space.*

Proof. The first two statements can be verified easily. For any pair of points of this subspace, one can choose two disjoint neighborhoods, and thus their intersection with this subspace are disjoint neighborhoods of this two points respectively in the subspace, but since the subspace topology is just the intersection of open sets in the original topological space with the subspace, we had done selecting two disjoint neighborhoods of two points in the subspace topology. □