Abstract Algebra

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Groups

1.1 Semigroups, Monoids and Groups

Definition. A *semigroup* is a nonempty set G together with a binary operation on G which is associative.

Definition. A *monoid* is a semigroup G which contains a (two-sided) identity element $e \in G$ such that $ae = ea = a$ for all $a \in G$.

Definition. A *group* is a monoid G such that there exists a (two-sided) inverse element and the operation between the inverse element and the original element yields the identity element regardless of order of operation.

Definition. A semigroup G is said to be *abelian* or *commutative* if its binary operation is commutative.

Definition. The *order* of a group G is the cardinal number $|G|$. G is said to be finite(resp. infinite) if $|G|$ is finite(resp. infinite).

Theorem 1.1.1. *If G is a monoid, then the identity element e is unique. If G is a group, then*

- $c \in G$ *and* $(cc = c) \Rightarrow (c = e)$*;*
- *for all* $a, b, c \in G$ *we have* $(ab = ac) \Rightarrow (b = c)$ *and* $(ba = ca) \Rightarrow (b = c)$ *(left and right cancellation);*
- *for each element in G its inverse element is unique;*
- *for each element in G the inverse of its inverse is itself;*
- *for* $a, b \in G$ *we have* $(ab)^{-1} = b^{-1}a^{-1}$*;*
- *for* $a, b \in G$ *the equation* $ax = b$ *and* $ya = b$ *have unique solutions in* $G : x = a^{-1}b$ *and* $y = ba^{-1}$*.*

Proposition. Let G be a semigroup. G is a group iff the following conditions hold:

- there exists an element $e \in G$ such that $ea = a$ for all $a \in G$ (left identity element);
- for each $a \in G$, there exists an element $a^{-1} \in G$ such that $a^{-1}a = e$ (left inverse).

and an analogous result holds for "right inverses" and a "right identity".

Proposition. Let G be a semigroup. G is a group iff for all $a, b \in G$ the equations $ax = b$ and $ya = b$ have solutions in G.

Proof. Left for Exercise □

***Example* 1.1.** *Let S be a nonempty set and $A(S)$ the set of all bijections $S \to S$. Under the operation of composition of functions, $\circ$, $A(S)$ is a group. The elements of $A(S)$ are called permutations and $A(S)$ is called the group of permutations on the set S. If $S = \{1, 2, 3, \cdots, n\}$, then $A(S)$ is called the symmetric group on n letters and denoted S_n. $|S_n| = n!$.*

Definition. The *direct product* of two groups G and H with identities e_G and e_H is the group whose underlying set is $G \times H$ and whose binary operation is given by:

$$(a, b)(a', b') = (aa', bb'), \qquad \text{where } a, a' \in G; b, b' \in H$$

$G \times H$ is abelian if both G and H are; (e_G, e_H) is the identity and (a^{-1}, b^{-1}) is the inverse of (a, b). Clearly $|G \times H| = |G||H|$.

Theorem 1.1.2. *Let $R(\sim)$ be an equivalence relation on a monoid G such that a_1 a_2 and b_1 b_2 imply a_1b_1 a_2b_2 for all $a_i, b_i \in G$. Then the set G/R of all equivalence classes of G under R is a monoid under the binary operation defined by $(\bar{a})(\bar{b}) = \overline{ab}$, where $\bar{x}$ denoted the equivalence class of $x \in G$. If G is an [abelian] group, then so is G/R.*

*An equivalence relation on a monoid G that satisfies these hypothesis is called a **congruence relation** on G.*

***Example* 1.2.** *The following relation on the additive froup $\mathbb{Q}$ is a congruence relation:*

$$a \sim b \Leftrightarrow a - b \in \mathbb{Z}$$

The set of equivalence classes (denoted $\mathbb{Q}/\mathbb{Z}$) is an infinite abelian group, with addition given by $\bar{a} + \bar{b} = \overline{a + b}$, and called the group of rationals modulo one.

Definition. The *meaningful product* on any sequence of elements of a semigroup G, $\{a_1, a_2, \cdots\}$, $a_1, \cdots, a_n$(in this order), is defined inductively as below: If $n = 1$, the only meaningful product is a_1. If $n > 1$, then a meaningful product is defined to be any product of the form $(a_1 \cdots a_m)(a_{m+1} \cdots a_n)$ where $m < n$ and $(a_1 \cdots a_m)$ and $(a_{m+1} \cdots a_n)$ are meaningful products of m and $n - m$ elements respectively.

Definition. The *standard n product* $\prod_{i=1}^{n} a_i$ is defined as follows:

$$\prod_{i=1}^{1} a_i = a_i; \quad \text{for } n > 1, \prod_{i=1}^{n} a_i = (\prod_{i=1}^{n-1} a_i)a_n$$

Theorem 1.1.3 (Generalized Associative Law)**.** *If G is a semigroup and $a_1, \cdots, a_n \in G$, then any two meaningful products of $a_1, \cdots, a_n$ in this order are equal.*

Theorem 1.1.4 (Generalized Commutative Law)**.** *If G is a commutative semigroup and $a_1, \cdots, a_n \in G$, then for any permutation $i_1, \cdots, i_n$ of $1, 2, \cdots, n$, $a_1 a_2 \cdots a_n = a_{i_1} a_{i_2} \cdots a_{i_n}$.*

Definition. Let G be a semigroup, $a \in G$ and $n \in \mathbb{N}$. The element $a^n \in G$ is defined to be the standard n product $\prod_{i=1}^{n} a_i$ with $a_i = a$ for $1 \leqslant i \leqslant n$. If G is a monoid, a^0 is defined to be the identity element e. If G is a group, then for each $n \in \mathbb{N}$, a^{-n} is defined to be $(a^{-1})^n \in G$.

Theorem 1.1.5. *If G is a group(resp. semigroup, monoid) and $a \in G$, then for all $m, n \in \mathbb{Z}$ (resp. $\mathbb{N}$ and $\mathbb{N} \cup \{0\}$) :*

- $a^m a^n = a^{m+n}$
- $(a^m)^n = a^{mn}$

1.2 Homomorphisms and Subgroups

Definition. Let G and H be semigroups. A function $f : G \to H$ is a *homomorphism* provided

$$f(ab) = f(a)f(b) \text{ for all } a, b \in G$$

If f is injective as a map of sets, f is said to be a *monomorphism.* If f is surjective, f is called an *epimorphism.* If f is bijective, f is called an *isomorphism.* In this case G and H are said to be *isomorphic* (written $G \cong H$). A homomorphism $f : G \to G$ is called an *endomorphism* of G and an isomorphism $f : G \to G$ is called an *automorphism* of G.

Definition. Let $f : G \to H$ be a homomorphism of groups. The *kernel* of f(denoted Ker f) is $\{a \in G | f(a) = e \in H\}$. If A is a subset of G, then $f(A) = \{b \in H | b = f(a) \text{ for some } a \in A\}$ is the *image of* A. $f(G)$ is called the *image of* f and denoted Im f. If B is a subset of H, $f^{-1}(B) = \{a \in G | f(a) \in B\}$ is the *inverse image* of B.

Theorem 1.2.1. *Let $f : G \to H$ be a homomorphism of groups. Then*

- *f is a monomorphism iff Ker $f = \{e\}$.*
- *f is an isomorphism iff there is a homomorphism $f^{-1} : H \to G$ such that $ff^{-1} = 1_H$ and $f^{-1}f = 1_G$.*

Definition. Let G be a semigroup and H a nonempty subset of it. If for every $a, b \in H$ we have $ab \in H$, we say that H is *closed* under the product in G. This is the same as saying that the binary operation on G, when restricted to H, is a binary operation on H.

Definition. Let G be a group and H a nonempty subset that is closed under the product in G. If H is itself a group under the product in G, then H is said to be a *subgroup* of G, denoted $H < G$.

Definition. If a subgroup H is not G itself or the *trivial subgroup*, which consists only of the identity element, is called a *proper subgroup.*

Theorem 1.2.2. *Let H be a nonempty subset of a group G. Then H is a subgroup of G iff $ab^{-1} \in H$ for all $a, b \in H$.*

Corollary. *If G is a group and $\{H_i | i \in I\}$ is a nonempty family of subgroups, then $\bigcap\limits_{i \in I} H_i$ is a subgroup of G.*

Proof. Left for Exercise □

Definition. Let G be a group and X a subset of G. Let $\{H_i | i \in I\}$ be the family of all subgroups of G which contain X. Then $\bigcap\limits_{i \in I} H_i$ is called the *subgroup of G generated by the set X* and denoted $\langle X \rangle$. The elements of X are the *generators* of $\langle X \rangle$. If $G = \langle a_1, \cdots, a_n \rangle, (a_i \in G)$, G is said to be finitely generated. If $a \in G$, the subgroup $\langle a \rangle$ is called the *cyclic (sub)group* generated by a.

Theorem 1.2.3. *If G is a group and X a nonempty subset of G, then the subgroup $\langle X \rangle$ generated by X consists of all finite products $a_1^{n_1} a_2^{n_2} \cdots a_t^{n_t} (a_i \in X; n_i \in \mathbb{Z})$. In particular for every $a \in G$, $\langle a \rangle = \{a^n | n \in \mathbb{Z}\}$.*

Proof. Left for Exercise □

Definition. The subgroup $\langle \bigcap\limits_{i \in I} H_i \rangle$ generated by the set $\bigcap\limits_{i \in I} H_i$ is called the *subgroup generated by the groups* $\{H_i | i \in I\}$. If H and K are subgroups, the subgroup $\langle H \cup K \rangle$ generated by H and K is called the *join* of H and K and is denoted $H \vee K$.

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