# Multivariable Calculus

Hechen Hu

November 25, 2017

ii

# Contents

# 1

# Vectors in $\mathbb{R}^n$ Space

## 1.1  Definition and Properties

Vector is a geometrical object that has both magnitude and direction. Examples include force and velocity.

Properties in in $n$th dimension vector space for Euclidean vector where $e_i$ is the basis vector for the $i$th axis(for convenience we will us $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ denote the basis vector in a 3-d space):

## 1.2  Addition/Subtraction

$$\mathbf{a} \pm \mathbf{b} = \sum_{i=1}^{n}(a_i \pm b_i)e_i \tag{1.1}$$

## 1.3  Scalar Multiplication

$$k\mathbf{a} = \sum_{i=1}^{n} ka_i.e_{\cdot i} \tag{1.2}$$

## 1.4  Dot Product

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i * b_i$$
$$= \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta \tag{1.3}$$

(**Result is a scalar**) Dot product has the following properties:

$(a)$   $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$   (*Commutative*)

$(b)$   $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$   (*Distributive*)

$(c)$   $k(\mathbf{a} \cdot \mathbf{b}) = (k\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (k\mathbf{b})$   (*Associative*)

$(d)$   $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$

$(e)$   $\mathbf{0} \cdot \mathbf{a} = 0$

## 1.5   Direction Angles\Cosines

The directional angles $\alpha$, $\beta$ and $\gamma$ between the vector $\mathbf{v}$ and basis vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in a 3-d space satisfy the following equations:

$$\cos\alpha = \frac{\mathbf{v} \cdot \mathbf{i}}{\|\mathbf{v}\| \, \|\mathbf{i}\|} = \frac{v_1}{\|\mathbf{v}\|} \qquad (1.4)$$

$$\cos\beta = \frac{\mathbf{v} \cdot \mathbf{j}}{\|\mathbf{v}\| \, \|\mathbf{j}\|} = \frac{v_2}{\|\mathbf{v}\|} \qquad (1.5)$$

$$\cos\gamma = \frac{\mathbf{v} \cdot \mathbf{k}}{\|\mathbf{v}\| \, \|\mathbf{k}\|} = \frac{v_3}{\|\mathbf{v}\|} \qquad (1.6)$$

According to their definitions we can also see that:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{v_1{}^2 + v_1{}^2 + v_1{}^2}{\sqrt{v_1{}^2 + v_1{}^2 + v_1{}^2}^2}$$

$$= 1 \qquad (1.7)$$

## 1.6   Orthogonal Projections

The orthogonal projection of $\mathbf{v}$ on an arbitrary non-zero vector $\mathbf{b}$ can be written as:

$$proj_{\mathbf{b}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b} \qquad (1.8)$$

Moreover, we can see that $\mathbf{v} - proj_{\mathbf{b}}\mathbf{v}$ is the vector component of $\mathbf{v}$ orthogonal to $\mathbf{b}$.

## 1.7   Cross Product

$$\mathbf{a} \times \mathbf{b} = det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

$$= (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$$

$$= \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta\mathbf{n} \qquad (1.9)$$

($\mathbf{n}$ is the vector that perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ and its direction is decided by the right hand rule in a right-handed coordinate system.)
(Result is a vector that is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$)

At the same time, we can see that

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \quad \mathbf{j} \times \mathbf{i} = -\mathbf{k} \tag{1.10}$$

$$\mathbf{j} \times \mathbf{k} = \mathbf{i} \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \tag{1.11}$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{j} \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \tag{1.12}$$

What's more, the area A of the parallelogram that has $\mathbf{a}$ and $\mathbf{b}$ as adjacent sides is:

$$A = \|\mathbf{a} \times \mathbf{b}\| \tag{1.13}$$

Thus, $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ if $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors. More useful properties of cross product:

$(a)$  $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$  $(\boldsymbol{Anti - Commutative})$

$(b)$  $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$  $(\boldsymbol{Distributive})$

$(c)$  $(\mathbf{b} + \mathbf{c}) \times \mathbf{a} = \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a}$

$(d)$  $k(\mathbf{a} \times \mathbf{b}) = (k\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (k\mathbf{b})$  $(\boldsymbol{Associative})$

$(e)$  $\mathbf{a} \times \mathbf{0} = \mathbf{0} \times \mathbf{a} = \mathbf{0}$

$(f)$  $\mathbf{a} \cdot \mathbf{a} = \mathbf{0}$

## 1.8  Scalar triple product

$$(\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}) = det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \tag{1.14}$$

**If we switch two rows of this matrix, the product will be multiplied by $-1$.**
The absolute value of scalar triple product will give us the volume of the parallelepiped that has $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ as adjacent edges. Therefore, $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$ iff they lie on the same plane.

# 2

# Lines and Planes

## 2.1 Equations of Lines

The line in 3-d space that passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to the non-zero vector $\mathbf{v} = <a, b, c> = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ has equations:

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct \quad (Parametric) \quad (2.1)$$

$$l = <x_0, y_0, z_0> + t <a, b, c> \quad (Vector) \quad (2.2)$$

If two lines doesn't intercept or parallel to each other in a 3-d space, they are skew.

## 2.2 Equations of Planes

Definition: A vector perpendicular to a plane is called a **normal** to that plane.

A plane which passing through $P_0(x_0, y_0, z_0)$ and having $\mathbf{n} = <a, b, c>$ as its normal has equations:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad (Point - Normal form) \quad (2.3)$$

$$ax + by + cz + d = 0 \quad (d = -ax_0 - by_0 - cz_0)(General form) \quad (2.4)$$

## 2.3 Angle between Planes

For two planes that have $\mathbf{n_1}$ and $\mathbf{n_2}$ as its normal, the acute angle between them $\theta$ can be obtained from the following equation:

$$\cos \theta = \frac{|\mathbf{n_1} \cdot \mathbf{n_2}|}{\|\mathbf{n_1}\| \, \|\mathbf{n_2}\|} \quad (2.5)$$

## 2.4   Distance

The distance $D$ between a point $P_0(x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is:

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \qquad (2.6)$$

**Example: Distance between two skew lines:**

**Solution** Find two parallel planes passing through the two skew lines by taking the cross product of their direction vectors, and find the distance between these two planes.

**Example: Find the equation of the plane that passes through the line of intersection of the planes $P_1 : x - z = 1$ and $P_2 : y + 2z = 3$ and is perpendicular to the plane $P_3 : x + y - 2z = 1$.**

**Solutions:** Find the direction vector for the line of intersection of $P_1$ and $P_2$ by taking the cross product of their normal vectors. Then take the cross product of the direction vector of the line and the normal vector of $P_3$ to find the normal vector of the desired plane. Then find a point on the line of intersection and solve for the plane.

## 2.5   Cylindrical and Spherical Coordinates

**Definition.** The three coordinates $(r, \theta, z)$ of a point $P$ are defined as:
$r$ is the Euclidean distance from the $z$-axis to the point $P$.
$\theta$ is the angle between $x$-axis and the line from the origin to the projection of $P$ on $xy$-plane.
The height $z$ is the signed distance from the $xy$-plane to the point $P$.

**Conversion between Rectangular and Cylindrical Coordinates:**

Suppose the rectangular and cylindrical coordinate of the same point is

respectively $(x, y, z)$ and $(r, \theta, z)$.

<div align="center">

Rectangular to Cylindrical

</div>

$$r = \sqrt{x^2 + y^2}$$
$$\tan\theta = \frac{y}{x}$$
$$z = z$$

<div align="center">

Cylindrical to Rectangular

</div>

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Note that $\theta$ is not always $\tan^{-1}\frac{y}{x}$. To find the correct $\theta$, we must know which quadrant of $xy$-plane the point lies over.

**Definition.** The spherical coordinates of a point $P$ are then defined as follows:
The radius $\rho$ is the Euclidean distance from the origin $O$ to $P$.
The azimuth $\theta$ is the signed angle measured from the azimuth reference direction to the orthogonal projection of the line segment $OP$ on the $xy$-plane.
The inclination $\phi$ is the angle between the $z$-axis and the line segment $OP$.

**Conversion Between Spherical and Other Two kinds of Coordinates:**

Suppose the rectangular, cylindrical, and spherical coordinate of the same point is respectively $(x, y, z)$, $(r, \theta, z)$, and $(\rho, \theta, \phi)$. $\theta$ in cylindrical coordinate is the same as $\theta$ in spherical coordinate.

<div align="center">

Spherical to Rectangular or Cylindrical

</div>

$$r = \rho\sin\phi$$
$$x = r\cos\theta = \rho\sin\phi\cos\theta$$
$$y = r\sin\theta = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

<div align="center">

Rectangular to Spherical

</div>

$$\rho = \sqrt{r^2 + z^2} = \sqrt{x^2 + y^2 + z^2}$$
$$\tan\theta = \frac{y}{x}$$
$$\cos\phi = \frac{z}{\rho} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

# 3

# Quadric Surfaces
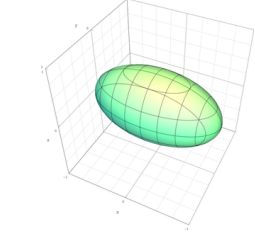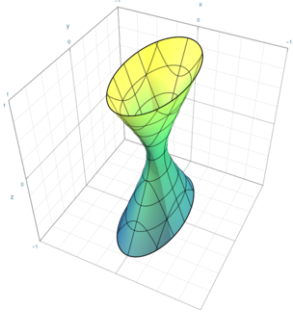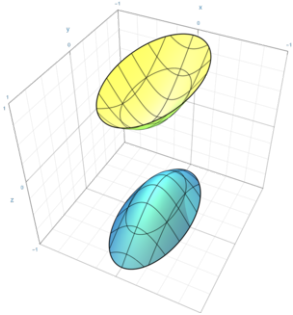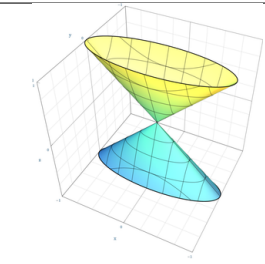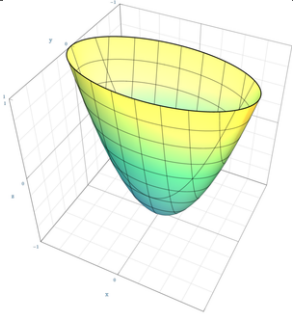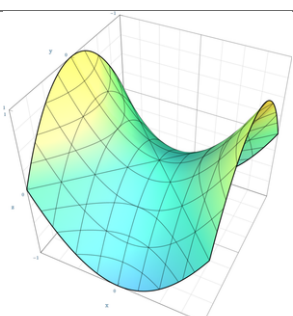
## 3.1 Traces

To help graphing a complex surface in a 3-d space, we obtain traces, or the curves(mesh lines) formed by cutting this surface with well-chosen planes. Usually, surfaces are built up from traces in planes that are parallel to the coordinate planes.

## 3.2 Type of Quadric Surfaces

| Name | Equation | Figure |
|------|----------|--------|
| Ellipsoid | $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ |  |
| Hyperboloid of one sheet | $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ |  |
| Hyperboloid of two sheets | $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ |  |
| Elliptic cone | $z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ |  |
| Elliptic paraboloid | $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ |  |
| Hyperbolic paraboloid | $z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$ |  |

# 4

# Calculus of Vector-Valued Functions

## 4.1 Orientation/Direction of its graph

The direction a graph of a vector-valued function goes when its parameter, $t$, increases is called the *orientation* or *direction of increasing parameter.*

## 4.2 Domain and Natural Domain

The domain of a vector-valued function is the set of all allowable values of $t$. The natural domain of a vector-valued function is the intersection of its component functions' domain.

## 4.3 Radius Vector/Position Vector

If a function can be expressed as $F(t) =< f(t), g(t), h(t) >$, then the position vector of it at $t = k$ is $< f(k), g(k), h(k) >$.

## 4.4 Vector Form of A Line Segment

For two vectors $\mathbf{r_0}$ and $\mathbf{r_1}$ that has its initial point at origin, the line passes through the terminal points of them can be written as:

$$\mathbf{r} = \mathbf{r_0} + t(\mathbf{r_1} - \mathbf{r_0}) \tag{4.1}$$

And this is called the ***two-point vector form o a line***.

## 4.5    Calculus of Vector-Valued Functions

The Calculus of vector-valued functions in 2-d and 3-d space is similar to
"normal" functions:**just apply each operator to its component func-
tions and "sum" them up.** The definition of integrable, differentiable
and continuous is also similar:**each property requires its component
functions have the corresponding property.**
The tangent line of the graph at point $\mathbf{r}(t_0)$:

$$\mathbf{r} = \mathbf{t_0} + t\mathbf{r}'(t_0) \tag{4.2}$$

For the dot product and cross product, which are unique to vector-valued
functions, the derivative is defined as following:

$$\frac{d}{dt}[\mathbf{r_1}(t) \cdot \mathbf{r_2}(t)] = \mathbf{r_1}(t) \cdot \frac{d\mathbf{r_2}}{dt} + \frac{d\mathbf{r_1}}{dt} \cdot \mathbf{r_2}(t) \tag{4.3}$$

$$\frac{d}{dt}[\mathbf{r_1}(t) \times \mathbf{r_2}(t)] = \mathbf{r_1}(t) \times \frac{d\mathbf{r_2}}{dt} + \frac{d\mathbf{r_1}}{dt} \times \mathbf{r_2}(t) \tag{4.4}$$

In 2-d space, the tangent line to a circle is perpendicular to the radius
at the point of tangency. Similarly, in for a vector-valued function, if $\|\mathbf{r}(t)\|$
is constant for all $t$, then:

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \tag{4.5}$$

that is, $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal for all $t$.

## 4.6    Arc Length

In a 2-d space, the arc length $L$ of a parametric curve $x = x(t), y = y(t), (a \le t \le b)$ can be given as:

$$L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt \tag{4.6}$$

**Lemma.** *In a 2-d space, the arc length $L$ of a function $y = f(x)$ that itself
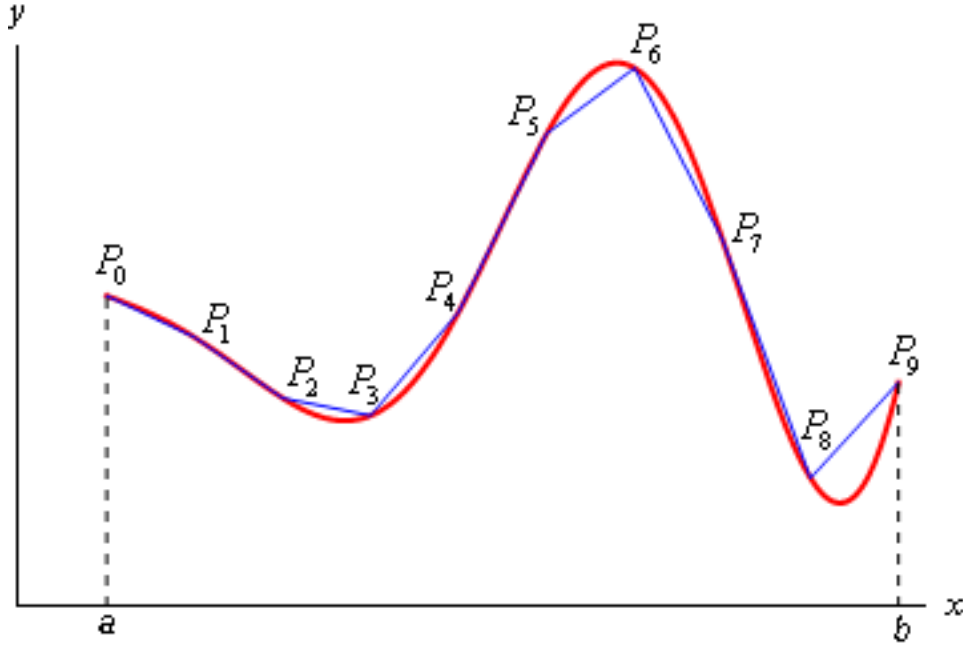and its derivative is continuous on $[a, b]$ is:*

$$L = \int ds \tag{4.7}$$

*where*

$$ds = \sqrt{1 + (\frac{dy}{dx})^2} dx \quad if \quad y = f(x), a \le x \le b$$

$$ds = \sqrt{1 + (\frac{dx}{dy})^2} dy \quad if \quad x = g(y), c \le y \le d$$

*Proof.* As we can see in the figure below, the arc length is the sum of distance between $n$ consecutive points when $n \to \infty$



Arc Length $\boldsymbol{L}$ can be written as:

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}$$

Additionally, we can see that

$$\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2} = \sqrt{\Delta x^2 + \Delta y_i^2}$$

According to Mean Value Theorem, there exists an $\bar{x}$ such that

$$\Delta y_i = f'(\bar{x}_i) \Delta x$$

Thus

$$\sqrt{\Delta x^2 + \Delta y_i^2} = \sqrt{\Delta x^2 + \Delta y_i^2}$$
$$= \sqrt{\Delta x^2 + (f'(\bar{x}_i) \Delta x)^2}$$
$$= \sqrt{1 + [f'(\bar{x}_i)]^2} \Delta x$$

The exact length of the given curve is

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + [f'(\bar{x}_i)]^2} \Delta x$$

$$= \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

$$= \int_a^b \sqrt{1 + (\frac{dy}{dx})^2} dx$$

$\square$

Now we can prove Theorem (4.6):

*Proof.* Recall that $x = x(t), y = y(t)$, therefore

$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2} dx$$

$$= \int_a^b \sqrt{1 + \frac{(\frac{dy}{dt})^2}{(\frac{dx}{dt})^2}} \frac{dx}{dt} dt$$

$$= \int_a^b \sqrt{1 + \frac{(\frac{dy}{dt})^2}{(\frac{dx}{dt})^2}} \frac{dx}{dt} dt$$

$$= \int_a^b \frac{1}{|\frac{dx}{dt}|} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \frac{dx}{dt} dt$$

If we assume that $\frac{dx}{dt} \geq 0$, then

$$L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$$

$\square$

Analogously, the arc length $L$ of a smoothly parametrized function(**have a continuously turning tangent vector**) in 3-d space is

$$L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2} dt$$

$$= \int_a^b \left\| \frac{d\mathbf{r}}{dt} \right\| dt \qquad\qquad (4.8)$$

## 4.7   Arc Length as A Parameter

Sometime it would be more convenient to replace $t$ with $s$, which is the length of arc measured along the curve from some fixed reference point. There are three steps:

**Step 1.** Select an reference point.

**Step 2.** Choose one direction from the reference point as the positive direction.

**Step 3.** Change the length $s$ to a "signed" length, which means $s$ is positive if $s$ "moves along the curve" to its positive direction.

Note that there are infinitely many different arc length parameterizations.

**Theorem 4.7.1.** ***Chain Rule***    *Let* $\mathbf{r}(t)$ *be a vector-valued function in 2-d/3-d space that is differentiable with respect to* $t$. *If* $t = g(\tau)$ *is a change of parameter in which* $g$ *is differentiable with respect to* $\tau$, *then* $\mathbf{r}(g(\tau))$ *is differentiable with respect to* $\tau$ *and*

$$\frac{d\mathbf{r}}{d\tau} = \frac{d\mathbf{r}}{dt}\frac{dt}{d\tau} \tag{4.9}$$

A change in parameter is smooth if $\mathbf{r}(g(\tau))$ is smooth and $\mathbf{r}(t)$ is smooth. For all $\tau$, $\frac{dt}{d\tau} > 0$ is called a positive change of parameter while $\frac{dt}{d\tau} < 0$ is called a negative change of parameter.

**Theorem 4.7.2.** *Let* $C$ *be the graph of a smooth vector-valued function* $\mathbf{r}(t)$ *in 2-d or 3-d space, and let* $\mathbf{r}(t_0)$ *be any point on* $C$. *Then the following formula defines a positive change of parameter from* $t$ *to* $s$, *where* $s$ *is an arc length parameter having* $\mathbf{r}(t_0)$ *as its reference point:*

$$s = \int_{t_0}^{t} \left\|\frac{d\mathbf{r}}{du}\right\| du \tag{4.10}$$

**Theorem 4.7.3.** *If* $C$ *is the graph of a smooth vector-valued function* $\mathbf{r}(t)$ *in 2-d or 3-d space, where* $t$ *is a general parameter, and if* $s$ *is the arc length parameter for* $C$ *defined by **Theorem 2**, then for every value of* $t$ *the tangent vector has length*

$$\left\|\frac{dr}{dt}\right\| = \frac{ds}{dt} \tag{4.11}$$

*Proof.* This can be derived from applying the Fundamental Theorem of Calculus to Theorem 2. □

**Theorem 4.7.4.** *If $C$ is the graph of a smooth vector-valued function $\mathbf{r}(t)$ in 2-d or 3-d space, where $s$ is the arc length parameter, then for every value of $s$ the tangent vector to $C$ has length*

$$\left\| \frac{d\mathbf{r}}{ds} \right\| = 1 \qquad (4.12)$$

*Proof.* Let $t = s$ in **Theorem 3**. □

**Theorem 4.7.5.** *If $C$ is the graph of a smooth vector-valued function $\mathbf{r}(t)$ in 2-d or 3-d space, and if*

$$\left\| \frac{d\mathbf{r}}{dt} \right\| = 1 \qquad (4.13)$$

*for every value of $t$, then $t$ is an arc length parameter that has its reference point at the point on $C$ where $t = 0$.*

*Proof.* The formula

$$s = \int_0^t \left\| \frac{d\mathbf{r}}{du} \right\| du$$

defines an arc length parameter for $C$ with reference point $\mathbf{r}(0)$. Note that

$$\left\| \frac{d\mathbf{r}}{dt} \right\| = 1$$

by hypothesis. Thus the formula can be rewrite as

$$s = \int_0^t du = t - 0 = t$$

□

## 4.8   Unit Tangent, Normal, and Binormal Vectors

**Definition.** The unit tangent of a smooth vector-valued function $\mathbf{r}(t)$ in 2-d space or 3-d space that points in the direction of increasing parameter can be expressed as:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

and it's called the *unit tangent vector* to $C$ at $t$.

Note that for all smooth parameterization which induce the same direction have the same unit tangent vector.

Recall that if a vector-valued function $\mathbf{r}(t)$ has constant norm, then $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal. Because $\mathbf{T}(t)$ has constant norm $1$, so $\mathbf{T}(t)$ and

$\mathbf{T}'(t)$ are orthogonal. This implies that $\mathbf{T}'(t)$ is perpendicular to the tangent line to $C$ at $t$, so we say that $\mathbf{T}'(t)$ is *normal* to $C$ at $t$. If $\mathbf{T}'(t) \neq \mathbf{0}$, then

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

is the *principle unit normal vector*, or simply *unit normal vector* to $C$ at $t$ and points in the same direction as $\mathbf{T}'(t)$.

The unit normal vector always points toward the concave side of $C$ in 2-d space.

According to **Theorem 4.7.4**, $\|\mathbf{r}'(t)\| = 1$. Thus

$$\mathbf{T}(s) = \mathbf{r}'(s)$$

and consequently

$$\mathbf{N}(s) = \frac{\mathbf{r}''(s)}{\|\mathbf{r}''(s)\|}$$

**Definition.** The binormal vector to $C$ at $t$ can be defined as

$$\begin{aligned}
\mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t) \\
&= \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}
\end{aligned}$$

that is, the cross product of its unit tangent vector and unit normal vector and the direction of binormal vector is determined by the right-hand rule. $\|\mathbf{B}(t)\| = 1$ since $\|\mathbf{T}(t) \times \mathbf{N}(t)\| = \|\mathbf{T}(t)\| \, \|\mathbf{N}(t)\| \sin(\pi/2) = 1$.

In terms of arc length parameterion, it can be expressed as

$$\mathbf{B}(s) = \frac{\mathbf{r}'(s) \times \mathbf{r}''(s)}{\|\mathbf{r}''(s)\|}$$

Together with unit tangent vector and unit normal vector, the binormal vector define three mutually perpendicular planes that point through that point – the **TB**-plane (called the *rectifying plane*), the **TN**-plane (called the *osculating plane*), and the **NB**-plane (called the *normal plane*). The coordinate system(right-hand) system determined by these three vectors is called the **TNB**-frame.

## 4.9 Curvature

**Definition.** If $C$ is a smooth curve in 2-d space or 3-d space that is parametrized by arc length, then the *curvature* of $C$, denoted by $\kappa = \kappa(s)$, is defined by

$$\kappa(s) = \left\| \frac{d\mathbf{T}}{ds} \right\| = \|\mathbf{r}''(s)\|$$

**Theorem 4.9.1.** *If* $\mathbf{r}(t)$ *is a smooth vector-valued function in 2-d space or 3-d space, then for each value of* $t$ *at which* $\mathbf{T}(t)$ *and* $\mathbf{r}''(t)$ *exist, the curvature* $\kappa$ *can be expressed as*

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$
$$= \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

**Definition.** If at $P$ $\kappa \neq 0$, then there exists a unique circle $\Omega$ passing through $P$ on the concave side of $C$ such that the curvature of $\Omega$ is $\kappa$ and it shares a common tangent vector with $C$. Then $\Omega$ is called a *osculating circle* or *circle of curvature* at $P$, $1/\kappa$ is called the radius of curvature at $P$, and the center of $\Omega$ is called the center of curvature at $P$.

**Definition.** Let $\phi(s)$ be the angle measured counterclockwise from the direction of the positive $x$-axis to the unit tangent vector $T$ **in terms of arc length**. Then

$$\kappa(s) = |\frac{d\phi}{ds}|$$

## 4.10   Motion Along A Curve

**Definition.** If a particle moves along a smooth vector-valued function $\mathbf{r}(t)$, then the velocity of this particle is

$$\mathbf{v}(t) = \mathbf{r}'(t) = \frac{ds}{dt}\mathbf{T}(t)$$

where $\mathbf{v}(t)$ is the tangent vector at $t$. The direction of $\mathbf{v}(t)$ gives the instantaneous direction of motion at $t$. The magnitude of $\mathbf{v}(t)$ is the instantaneous rate of change of arc length as a function of time, or just simply speed.

**Definition.** The acceleration of this particle with velocity of $\mathbf{v}(t)$ is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$. What's more

$$\mathbf{a}(t) = \frac{d^2 s}{dt^2}\mathbf{T}(t) + \kappa(t)(\frac{ds}{dt})^2\mathbf{N}(t)$$

If the particle is travelling on a circle with radius $r$, the speed $v_0$ is constant, and $\|\mathbf{a}(t)\| = \frac{v_0^2}{r}$.

***Example* 4.1.** *Over a time interval* $[t_1, t_2]$, *the distance traveled by a particle is*

$$s = \int_{t_1}^{t_2} \|\mathbf{v}(t)\| \, dt$$

**Definition.** If $a_T = \frac{d^2 s}{dt^2}$ and $a_N = \kappa(t)(\frac{ds}{dt})^2$, then

$$\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$$

and $\boldsymbol{a_T}$ is called the *tangential component of acceleration* and $\boldsymbol{a_N}$ is called *normal component of acceleration* . $\boldsymbol{a_T}\mathbf{T}$ is the *tangential vector component of acceleration* and $\boldsymbol{a_N}\mathbf{N}$ is called the *normal vector component of acceleration*

***Example* 4.2.**

$$a_T = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|}$$

$$a_N = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|}$$

$$\kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}$$

In 2-d space, the cross product can be computed by viewing $\mathbf{r}(t) =< x(t), y(t) >$ as $\mathbf{r}(t) =< x(t), y(t), \mathbf{0} >$ in 3-d space.

***Example* 4.3.**

$$\|\mathbf{a}\|^2 = \mathbf{a}_T^2 + \mathbf{a}_N^2$$

*for all smooth position functions.*

## 4.11 Projectile Motion

**Theorem 4.11.1.** *For a projectile motion*

$$\mathbf{r}(t) = -\frac{gt^2}{2}\mathbf{j} + t\mathbf{v_0} + s_0\mathbf{j}$$

$$= (-\frac{gt^2}{2} + s_0)\mathbf{j} + t\mathbf{v_0}$$

*where $\boldsymbol{g}$ is the gravitational acceleration, $\boldsymbol{v_0}$ is the initial velocity, and $\boldsymbol{s_0}$ is the initial height.*

# 5

# Multivariable Calculus

## 5.1 Definition of Multivariable functions and their graph

**Definition.** A *real-valued function $f$ of $n$ real variables* is a mapping that assign an ordered pair of $n$ real numbers to a real number in $D \subset \mathbb{R}^n$.

**Definition.** If $f$ is a function of two variables and $k$ is a real number, then the *level curve of $f$ of height $k$* is $\{(x, y) | f(x, y) = k\}$. A collection of many level curves for a function $f$ all drawn in the same $xy$-plane is called a *contour plot/map* of $f$.

## 5.2 Properties of Sets in 2-d and 3-d space

**Definition.** For $r > 0$ and a point $P$, the *open ball of radius $r$ centered at $P$* means the set of all points whose distance to $P$ is less than $r$ and denotes $B_r(P)$.

Suppose $D \subset \mathbb{R}^2$ or $D \subset \mathbb{R}^3$.

**Definition.** A point $P$ is said to be an *interior point* of $D$ if $\exists r > 0(B_r(P) \subset D)$.

**Definition.** A point $P$ is said to be an *boundary point* of $D$ if $\forall r > 0((B_r(P) \cap D \neq \emptyset) \wedge (B_r(P) \setminus D \neq \emptyset))$

**Definition.** For a set $D$ in 2-d and 3-d space, the set of all its interior points is called the *interior* of $D$. The set of all its boundary points is called the *boundary* of $D$.

**Definition.** A set is to be said *open* if it contains none of its boundary points, and when it contains all of its boundary points, it's called a *closed* set.

However, some sets are **neither** open or close.

***Example*** **5.1.** *A set $D$ in $n$-space is both open and closed iff $D = \mathbb{R}^n$ or $D = \emptyset$.*

**Definition.** A subset $D$ of $\mathbb{R}^n$ is said to be *bounded* if $\exists r > 0 \exists P(D \subset B_r(P))$. That is, the set is contained in some set with finite radius. It's *unbounded* if it's not bounded.

**Definition.** A point $P$ is said to be the *accumulation point* of some $D \subset \mathbb{R}^n$ if $B_r(P) \subset D \wedge \forall r > 0(B_r(P) \setminus D \setminus \{P\} \neq \emptyset)$.