Lecture 2: Equisingularity and Determinantal Singularities

by Terence Gaffney

Notation

$$\begin{tikzcd} X^d(0) \subset \arrow[d] & \mathcal{X}^{d+k} \subset \arrow[d, "p_Y"] & Y \times \mathbb{C}^N \arrow[dl, "\pi_Y"] \\ 0 \in & Y = \mathbb{C}^k & \end{tikzcd}$$

- The parameter space is Y, $X^d(0)$ denotes the fiber of the family over $\{0\}$.
- $\mathcal{X}^{d+k}$ denotes the total space of the family which is contained in $Y \times \mathbb{C}^N$. We always assume $\mathcal{X}^{d+k}$ is equidimensional with equidimensional fibers.
- We usually assume $Y \subset \mathcal{X}^{d+k}$,
- $\mathcal{X} = F^{-1}(0)$, $X(y) = {f_y}^{-1}(0)$, where $f_y(z) = F(y, z)$
- $S(X)$ the singular locus of X.

Introduction: Our Goal, Landscapes and EIDS

- In the notes for the ICIS course we showed two theorems.

Theorem

(Sufficiency) Let $\mathcal{X}$ be a family of ICIS over Y^k as in the basic setup. Suppose $e(mJM(X(y),0))$ is independent of y. Then $X - Y$ is smooth, and the pair $(X - Y, Y)$ satisfies W.

Theorem

(Necessity) Suppose $\mathcal{X}$ is a family of ICIS, and the pair $(\mathcal{X} - Y, Y)$ satisfies W at the origin. Then, the μ_* sequence of $X(y)$ is independent of y, as is $e(m_y JM(X(y)))$.

- The invariants depend only on the members of the family $X(y)$, not the family. They also depend only on data at a point of $X(y)$. Can we do this for non-isolated singularities?
- EIDS are an example where we can. The invariants are independent of family, but do depend on the landscape.
- This is a report on a joint work with Maria Aparecida Ruas. (G-Ruas 2021.)

Introduction: Landscapes and Determinantal Singularities

- Why non-isolated singularities?
- There are many situations in which the set studied has additional structure which force the singularities to be non-isolated in general, and we want to study families which preserve this structure.
- For example, the set may be the image of a finite map, and we may want to preserve this property in deformations.
- Deciding on the structure we want to preserve in deformations is the first step in deciding on the landscape of the singularity, and the possible deformations depend on this choice.
- The Whitney umbrella, which is defined by $z^2 - x^2y = 0$, can be thought of as a hypersurface, the image of a map-germ, or as a hypersurface with smooth singular locus. The allowable deformations depend on the choice of landscape; the last two choices have only trivial deformations, while every hypersurface can be deformed to a smooth manifold.

The Landscape of Determinantal Singularities

- We choose to represent a determinantal singularity X, by finding $f\colon \mathbb{C}^q, 0 \to \mathbb{C}^{n+k\times n}$, where $X = f^{-1}(M^t(n+k, n))$ for some t.
- The allowable deformations of X are gotten by deforming f and taking minors of size t.
- The generic objects in our landscape are the transverse sets ie. $X = f^{-1}(M^t(n+k, n))$, where f is transverse to the rank stratification of $M^t(n+k, n)$.
- Let $\tilde{F} : U \subset \mathbb{C}^q \times \mathbb{C} \to \mathbb{C}^{n+k\times n}$ be a one parameter deformation of f such that $\tilde{F}_s$ is transverse to the rank stratification of $M^t(n+k, n)$. $\tilde{F}_s$ is called a stabilization of f for all $s \neq 0$,
- $\widetilde{\mathcal{X}}_s = \widetilde{F}_s^{-1}(M^t)$ an *essential smoothing* of $X = F^{-1}(M^t)$ for all $s \neq 0$. We call $\tilde{F}$ a *stabilization family*, $\widetilde{\mathcal{X}}$ an *essential smoothing family*.

The Landscape of Determinantal Singularities II

- So every determinantal singularity has a deformation to a transverse set, and if we fix a representative of an EIDS $X, 0$, the germ X, x is a generic object.
- Given an allowable deformation of $X, 0$, it is reasonable to hope that data from X at 0 which measures the failure of the transversality of f, and from the universal object M^t will determine if the deformation is W equisingular.
- ICIS case–the universal object was $0 \in \mathbb{C}^p$, and $e(JM(X))$ measured the failure of transversality.

Sectional Singularities

- This suggests a further generalization, considering varieties that arise as non-linear sections of a "universal variety". In the next diagram V is the universal variety and X is the inverse image of V by f_0.
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$$\begin{tikzcd} k^q, 0 \arrow[r, "f_0"] & k^N, 0 \\ X, 0 \arrow[u, hook] \arrow[r] & V, 0 \arrow[u, hook] \end{tikzcd}$$
- These sectional singularities include determinantal singularities, symmetric singularities, skew-symmetric singularities, and discriminants of $\mathcal{A}$-finitely determined germs.
- The adaptation of elements of the Mather theory to non-linear sections of spaces has been done by Damon-87. In the last section we discuss the extension of the material here to this more general situation.

Return to determinantal singularities: Obstacles

- What is the module N of infinitesimal deformations of X? (in the ICIS case it was a free $\mathcal{O}_X$ module.)
- How do we measure the difference between $JM(X)$, the infinitesimally trivial deformations, and N? (They should be the same if X is transverse.)
- If X an ICIS, $\tilde{X}$ a smoothing, l a generic linear form, H defined by l, then the number of critical points of l on a generic fiber of $\tilde{X}$ is an invariant $m_d(X)$, and $m_d(X) = \mu(X) + \mu(X \cap H) = e(JM(X))$. How do we relate m_d to the geometry of X and the modules $JM(X)$ and N?

The Determinantal Normal module

- We construct elements of the determinantal normal module of X, $N(X)$, by first deforming X by deforming f, taking minors of the right size, then taking the first order terms in t.
- The determinantal normal module of $M^t(n+k, n)$ denoted N^t is defined using the identity map on $\mathbb{C}^{n+k\times n}$. It has some nice properties we explore in an example.

Example: N^3 for $M(3,3)$

- $$det(A(t)) = det\left(\begin{bmatrix} a_{1,1}+t & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}\right)$$
- defines a deformation of $M^3(3,3)$.
- Take the derivative with respect to t and set t to 0; we get the cofactor of $a_{1,1}$. This is an element of N^3.
- This is also $\frac{\partial det(A)}{\partial a_{1,1}}$.
- $$det((A\circ f)(t)) = det\left(\begin{bmatrix} a_{1,1}\circ f+t & a_{1,2}\circ f & a_{1,3}\circ f \\ a_{2,1}\circ f & a_{2,2}\circ f & a_{2,3}\circ f \\ a_{3,1}\circ f & a_{3,2}\circ f & a_{3,3}\circ f \end{bmatrix}\right)$$
- defines a deformation of ${}_3X = f^{-1}(M^3)$.
- Take the derivative with respect to t and set t to 0; we get the cofactor of $a_{1,1}\circ f$. This is an element of $N(X)$. So the generators of N^t pullback to generators of $N(X)$.

Universal objects for Determinantal Singularities

- The Universal objects are the $M^t(n+k,n)$.
- $M^t(n+k,n)$ have a locally holomorphically trivial stratification given by rank.
- $N^t = JM(M^t)$. This is because the process of computing the partial derivatives of the defining equations of M^t is the same as computing the columns of N^t. This equality means that every infinitesimal determinantal deformation of M^t is infinitesimally trivial. We say that N^t and M^t are *stable*.
- If ${}_tX$ is defined by $f\colon \mathbb{C}^q, 0 \to \mathbb{C}^{n+k\times n}$, $N(X) = f^*(N^t)$. This follows because f is not involved in the process calculating the elements of N^t and $N(X)$. Because of this property we say N^t is *universal*.
- **Problem** Show $N(\mathcal{X})(y) = N(\mathcal{X}(y))$ using universality.

EIDS and $e(JM(X), N(X))$

Given the germ of a smooth subspace $S, x \subset \mathbb{C}^N$, T is a direct transversal to S, x if it is a transversal to S at x of complementary dimension to S.

Lemma

Let X be a stratified subset of $(\mathbb{C}^N, x)$.

i) *Suppose X has a locally holomorphically trivial stratification. Let S_x denote the stratum containing x. Suppose T is a direct transversal to S at x. Then $JM(X) \subset JM(X)_T$.*

ii) *Suppose the stratification is a Whitney stratification. Suppose T is a direct transversal to S, the stratum at x. Then $JM(X) \subset \overline{JM(X)_T}$.*

Lemma

Suppose $F : (\mathbb{C}^q, 0) \to (\mathbb{C}^N, x)$, $x \in X$, X a stratified subset of $(\mathbb{C}^N, x)$.

i) *Suppose X has a Whitney stratification. Let S_x denote the stratum containing x. Suppose F is transverse to S_x at x. Let X_F denote $F^{-1}(X)$. Then $F^*(JM(X)) \subset \overline{JM(X_F)}$.*

ii) *If the stratification is holomorphically trivial, then $F^*(JM(X)) \subset JM(X_F)$.*

EIDS and $e(JM(X), N(X))$ II

- Sketch of proofs
- Let T is a direct transversal to S; move T along S. By intersecting with X this defines a smoothly trivial family.
- Because the family is smoothly trivial, $\frac{\partial G}{\partial y_i} \in JM(G)_T$, y_i any coordinate on S, G defines X.
- Let $F : (\mathbb{C}^q, 0) \to (\mathbb{C}^N, x)$, $x \in X$, X a stratified subset of $(\mathbb{C}^N, x)$, and let F be transverse to S_x at x. Then the image of $DF(0)$ contains a direct transversal T.
- Applying the chain rule to $G \circ F$ we get $JM(X) \supset F^*JM(G)_T$ hence $JM(X) \supset F^*JM(G) = F^*N^t$.

Proposition

If $X, 0$ is an EIDS defined by $F : (\mathbb{C}^q, 0) \to (\mathbb{C}^{(n+k)n})$, then $e(JM(X), N(X))$ is well defined.

- **Proof** By the lemmas, $JM(X, x) = F^*N^t$ except possibly for $x = 0$.

The invariant $m_d(X^d)$

- Let $\tilde{F}$ be a stabilization family of X, $\widetilde{\mathcal{X}}$ an essential smoothing family of X. p a linear function so that H, kernel of p is not a limiting tangent hyperplane to X at the origin, and that $p\mid_{(\widetilde{\mathcal{X}}_s)_{reg}} \to \mathbb{C}$ is a Morse function for $s \neq 0$. Then:

Definition

With the above notation $m_d(X) = n_0$ where n_0 is the number of critical points of $p\mid_{(\widetilde{\mathcal{X}}_s)_{reg}}$.

- In the ICIS case $m_d(X) = \mu(X) + \mu(X \cap H)$.
- For simplicity, let $\chi(X,0)$, $\chi(X \cap H, 0)$ denote the Euler Characteristic of a stabilization of X, $X \cap H$. Let $n_{it} = (-1)^{(k)(t-i)}\binom{n-i}{n-t}$.
- Then, in the determinantal case

$$(-1)^d\chi(X,0) + (-1)^{d-1}\chi(X \cap H, 0) = \sum_{i=1}^{t} n_{it} m_{d_i}({}_iX, 0)$$

The invariant $m_d(X^d)$ II

- In the ICIS case, $m_d(X) = e(JM(X))$
- Then, in the determinantal case

$$m_d({}_iX) = e(JM({}_iX, N({}_iX))) + F(\mathbb{C}^q) \cdot \Gamma_d(M^i)$$

- The term $F(\mathbb{C}^q) \cdot \Gamma_d(M^i)$, the intersection of $F(\mathbb{C}^q)$ with the polar of M^i of complementary dimension is a correction term due to the curvature of $JM(M^t)$.
- The image of $D(F)$ does not lie in a limit tangent hyperplane to M^t at $F(0)$, if and only if $e(JM({}_iX, N({}_iX))) = 0$. So this term measures the failure of transversality.
- To prove this formula, we need to discuss the polar varieties of a space and of a module, and to describe how the multiplicity changes in a family.

Recall: Basic Constructions for ideals and modules

- Given a submodule M of a free $\mathcal{O}_{X^d}$ module F of rank p, we can associate a subalgebra $\mathcal{R}(M)$ of the symmetric $\mathcal{O}_{X^d}$ algebra on p generators called the Rees algebra of M.
- If $(m_1, \ldots, m_p)$ is an element of M then $\sum m_i T_i$ is the corresponding element of $\mathcal{R}(M)$.
- $\mathrm{Projan}(\mathcal{R}(M))$, the projective analytic spectrum of $\mathcal{R}(M)$ is the closure of the projectivised row spaces of M at points where the rank of a matrix of generators of M is maximal.
- $\mathrm{Projan}(\mathcal{R}(JM(X)))$ is the conormal space of X. It consists of the tangent hyperplanes to X_0 and the closure of this space in $X \times \mathbb{P}^{n-1}$, $X \subset \mathbb{C}^n$.
- Denote the projection to X^d by c, or by c_M where there is ambiguity.
- If $M \subset N$ or $h \in N$, then h and M generate ideals on $\mathrm{Projan}\mathcal{R}(N)$; denote them by $\rho(h)$ and $\mathcal{M}$.
- If we can express h in terms of a set of generators $\{n_i\}$ of N as $\sum g_i n_i$, then in the chart in which $T_1 \neq 0$, we can express a generator of $\rho(h)$ by $\sum g_i T_i / T_1$.

Multiplicity of Pairs of Modules: Intersection Theoretic Definition

- The next diagram shows the spaces that come into the definition of $e(M, N)$.

$$\begin{tikzcd} B_{\mathcal{M}}(\mathrm{Projan}\,\mathcal{R}(N)) \arrow[r, "\pi_N"] \arrow[d, "\pi_M"] & \mathrm{Projan}\,\mathcal{R}(N) \arrow[d, "\pi_{X\,N}"] \\ \mathrm{Projan}\,\mathcal{R}(M) \arrow[r, "\pi_{X\,M}"] & X \end{tikzcd}$$

- $$e(M, N) = \sum_{j=0}^{d+g-2} \int D_{M,N} \cdot c_M^{d+g-2-j} \cdot c_N^j.$$
- $D_{M,N}$ the exceptional divisor of $B_{\mathcal{M}}(\mathrm{Projan}\mathcal{R}(N))$, g is the generic rank of N and M. $D_{M,N}$ is compact, since $\overline{M} = \overline{N}$, except at 0.
- On the blow up $B_{\mathcal{M}}(\mathrm{Projan}\mathcal{R}(N))$ we have two tautological bundles which are pullbacks of the bundles on $\mathrm{Projan}\mathcal{R}(N)$and $\mathrm{Projan}\mathcal{R}(M)$.
- Denote the corresponding Chern classes by c_M and c_N.

Understanding the Intersection Theoretic Definition

- We have $\mathrm{Projan}\mathcal{R}(N) \subset X^d \times \mathbb{P}^{g(N)-1}$, so $B_{\mathcal{M}}(\mathrm{Projan}\mathcal{R}(N)) \subset X^d \times \mathbb{P}^{g(N)-1} \times \mathbb{P}^{g(M)-1}$ where $g(N)$ is the number of generators of N.
- To calculate $\int D_{M,N} \cdot c_M^{d+g-2-j} \cdot c_N^j$, intersect $B_{\mathcal{M}}(\mathrm{Projan}\mathcal{R}(N))$ with j hyperplanes from $\mathbb{P}^{g(N)-1}$ and $d+g-2-j$ hyperplanes from $\mathbb{P}^{g(M)-1}$. This will define a curve. Then the order of a generic element of $\mathcal{M}$ on this curve will be the intersection number.
- **Problem** If $M = \begin{bmatrix} x & y & 0 \\ 0 & x & y \end{bmatrix}$, $N = \mathcal{O}_2^2$, show $e(M) = 3$ using the intersection theoretic definition.

Polar Varieties of a Module

- Teissier defined the polar varieties $\Gamma_l(X^d)$, $0 \le l < d$, of an analytic germ $(X^d, x) \subset \mathbb{C}^n$ of codimension l as follows: take a generic projection π of $X^d \to \mathbb{C}^{d-l+1}$, and take the closure of the points where the rank of $\pi|X_0$ is less than $d - l + 1$.
- Let X be defined by $F(x, y, z) = z^2 - y^2 - x^2 = 0$. Use the projection π to the $y - z$ plane to define the polar curve, $\Gamma_1(X)$, of $X, 0$. Let $G = (\pi, F)$. $\Gamma_1(X)$ is defined by $F = 0$ and $det(D(G)) = F_x = -2x = 0$, so $\Gamma_1(X)$ is two lines with equations $x = 0, z^2 - y^2 = 0$.
- In the example, varying the projection a little does not change the multiplicity of the polar curve. Teissier-'81 showed that for generic projections, the multiplicity of the polar variety is independent of projection.
- $\Gamma_0(X^d) = X^d$ because the rank of $\pi < d + 1$ at all points.

An Alternate construction of $\Gamma_l(X)$

- $\pi\colon X^d \subset \mathbb{C}^n \to \mathbb{C}^{d-l+1}$, so dim ker$\pi = n-d+l-1$.
- Apply DF to a basis of $ker\pi$. This gives $n-d+l-1$ generic linear combinations of the generators of $JM(X)$. Put them into a matrix A
- Take the points on X_0 where the $rkA < g$, g the generic rank of DF; $g = n-d$, and take the closure. The expected codimension of the set is $(n-d+l-1)-(g-1) = l$.
- Use a similar construction for $\Gamma_l(M)$, M, a module with generic rank g. Take $g+l-1$ generic generators of M, put them into a matrix A. Take the points on X, where $rk(M) = g$, $rk(A) < g$, and take the closure.
- **Example:** Let $M = \begin{bmatrix} x & y & 0 \\ 0 & x & y \end{bmatrix}$, $X = \mathbb{C}^2$; for $\Gamma_1(M)$ we need $2+1-1 = 2$ generic generators.
- Take $A = \begin{bmatrix} y & x \\ x+ay & by \end{bmatrix}$ for our generic generators, since M has rank 2 except at $(0,0)$ for generic a, b, $\Gamma_1(M)$ defined by $det(A) = 0$ is a pair of lines intersecting transversely.

An Alternate construction of $\Gamma_l(M)$

- Claim: $\Gamma_l(M)$, is constructed by intersecting $\mathrm{Projan}\mathcal{R}(M)$ with $X \times H_{g+l-1}$ where H_{g+l-1} is a general plane of codimension $g+l-1$, then projecting to X.
- Picking equations of H is the same thing as picking linear combinations of the generators of M, and asking that they are linearly dependent.
- **Example** Suppose H is defined by $T_1 = 0, \ldots, T_{g+l-1} = 0$. $X \times H \cap \mathrm{Projan}\mathcal{R}(M)$ consists of points where some linear combination of the rows of $[M]$ is zero in the first $g+l-1$ places.
- If $\Gamma_l(M,0)$ is empty, then the fiber of $Projan\mathcal{R}(M)$ over 0 has dimension less than $g+l-1$.
- Polar varieties of M control the size of the fibers of $Projan\mathcal{R}(M)$.
- Suppose $\mathcal{X}$ is a stabilization family for X^d. Then the degree over Y of $\Gamma_d(JM_z(\mathcal{X}) = m_d(X)$.

Multiplicity Polar Theorem

- $M \subset N \subset F$, a free $\mathcal{O}_X$ module, X equidimensional, a family of sets over Y,with equidimensional fibers, Y smooth.
- $\overline{M} = \overline{N}$ off a set C of dimension k which is finite over Y.
- $\Delta(e(M,N)) =$ $e(M(0), N(0), \mathcal{O}_{X(0)}, 0) - \sum_{x \in p_y^{-1}(y)} e(M(y), N(y), \mathcal{O}_{X(y)}, (y,x))$ is the change in the multiplicity of the pair (M,N) as the parameter changes from 0 to y.
- Formula (MPT, Arxiv '07):

$$\Delta(e(M,N)) = mult_y \Gamma_d(M) - mult_y \Gamma_d(N)$$

Application to equisingularity

- A family of determinantal singularities is *good* if there exists a neighborhood U of Y such that F_y is transverse to the rank stratification off the origin for all $y \in U$.
- The good condition ensures that we only have to control the pairs of strata (V, Y), V some stratum in the canonical stratification of $\mathcal{X} - Y$.
- Good implies all strata (except perhaps Y) have the expected dimension, and the types of dimension zero are controlled by the colength of the defining ideals.
- $N(X) = f^*(JM(M^t))$, and $\Gamma_l(N(X)) = f^{-1}\Gamma_l(M^t)$.

Proposition

Suppose $\mathcal{X}$ is a one parameter stabilization of an EIDS ${}_tX^d$. Then $f(\mathbb{C}^q) \cdot \Gamma_d(M^t) = m_{\mathbb{C}}(\Gamma_d(N(\mathcal{X})))$.

Proof

- Let F be the map from $\mathbb{C} \times \mathbb{C}^q \to \mathbb{C}^{(n+k)n}$ which defines $\mathcal{X}$.
- Arrange for f_y and $\Gamma_d(M^t)$ to be transverse, $y \neq 0$.
- $f(\mathbb{C}^q) \cdot \Gamma_d(M^t)$ is the number of points in which f_y intersects $\Gamma_d(M^t)$.
- This is the multiplicity over $\mathbb{C}$ of $\Gamma_d(N(\mathcal{X}))$ at the origin, as $F^{-1}(\Gamma_d(M^t)) = \Gamma_d(N(\mathcal{X}))$

$m_d(X)$ and $e(JM(X), N(X))$

Proposition

Suppose $\mathcal{X}$ is a one parameter stabilization of an EIDS ${}_tX^d$ then

$$e(JM(X), N(X)) + f(\mathbb{C}^q) \cdot \Gamma_d(M^t) = m_d(X).$$

- The proof is an application of the Multiplicity-Polar Theorem.
- Let F be the map from $\mathbb{C} \times \mathbb{C}^q \to \mathbb{C}^{(n+k)n}$ which defines $\mathcal{X}$, f_y and $\Gamma_d(M^t)$ transverse, $y \neq 0$.
- For $y \neq 0$, $\overline{JM(\tilde{X}(y))} = \overline{f_y^*(JM(M^t))}$. So $e(JM(\tilde{X}(y)), N(\tilde{X}(y))) = 0, y \neq 0$. Then,
- $e(JM(X), N(X)) + mult_C \Gamma_d(N(\mathcal{X})) = mult_C \Gamma_d(JM_z(\mathcal{X})) = m_d(X)$.
- $e(JM(X), N(X)) + f(\mathbb{C}^q) \cdot \Gamma_d(M^t) = m_d(X)$.

A family of examples

- The singularities X_l are a space curves defined by the minors of

$$F_{X_l} = \begin{bmatrix} z & x \\ y & z \\ x^l & y \end{bmatrix}$$

  We assume $l - 1$ is not divisible by 3.

- Then
  $m_1(X_l) = e(JM(X_l), N(X_l)) + f(\mathbb{C}^3) \cdot \Gamma_1(M^2) = 2l - 2 + 4 = 2l + 2.$

W equisingularity of a good family of EIDS

Theorem

Suppose $\mathcal{X}^{d+k}$ is a good k-dimensional family of EIDS. Then the family is Whitney equisingular if and only if the invariants

$$e(m_Y JM({}_i\mathcal{X}(y)), N({}_i\mathcal{X}(y))) + \tilde{F}(y)(\mathbb{C}^q) \cdot \Gamma_{d_i}(M^i)$$

are independent of y.

- Proof: Since the family is good we only need to control the pairs of strata (V, Y), V some stratum in the canonical stratification of $\mathcal{X} - Y$.
- Condition W holds for each open stratum if and only if the fiber of the blow-up of $Projan\mathcal{R}(JM_z({}_i\mathcal{X}))$ by m_Y is equidimensional over Y. (Teissier -1981, G-Rangachev -2016).
- The bound on the fiber of Y holds iff $\Gamma_{d_i}(m_Y JM_z({}_i\mathcal{X}))$ is empty, iff all $m_{d_i}(mJM({}_i\mathcal{X}(y)))$ are independent of y.

The expansion formula and a corollary

Corollary

Suppose $\mathcal{X}^{d+k}$ is a good k-dimensional family of EIDS. Then the family is Whitney equisingular if and only if the polar multiplicities at the origin of ${}_i\mathcal{X}(y)$ and the invariants

$$e(JM({}_i\mathcal{X}(y)), N({}_i\mathcal{X}(y))) + \tilde{F}(y)(\mathbb{C}^q) \cdot \Gamma_{d(i)}(M^i)$$

are independent of y.

- There is an expansion formula for $e(m_Y JM({}_i\mathcal{X}(y)), N({}_i\mathcal{X}(y))$ as a sum of terms with the polar multiplicities of ${}_i\mathcal{X}(y)$ with combinatorial coefficients, and the term $e(JM({}_i\mathcal{X}(y)), N({}_i\mathcal{X}(y)))$.
- So W implies the polar multiplicities are constant, and $e(m_Y JM({}_i\mathcal{X}(y)), N({}_i\mathcal{X}(y)) + \tilde{F}(y)(\mathbb{C}^q) \cdot \Gamma_{d_i}(\Sigma^i)$ is constant.
- So the terms $e(JM({}_i\mathcal{X}(y)), N({}_i\mathcal{X}(y)) + \tilde{F}(y)(\mathbb{C}^q) \cdot \Gamma_{d_i}(\Sigma^i)$ are independent of y as well.

Two examples

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_1 + x_5^2 + t_{10}x_3 \end{pmatrix}. \tag{1}$$

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \\ x_3 & x_4 & x_1 + x_5^2 + t_{10}x_3 \end{pmatrix}. \tag{2}$$

pause

- Both families give deformations of the same determinantal surface singularity, X.
- Both are W equisingular,
- value of $m_2(X)$ depends on the landscape. (8 in the rectangular case, 6 in the symmetric case.)

Epilogue: ESSI

- Given the diagram:

$$
\begin{tikzcd}
k^q, 0 \arrow[r, "f_0"] & k^N, 0 \\
X, 0 \arrow[u, hook] \arrow[r] & V, 0 \arrow[u, hook]
\end{tikzcd}
$$

- Suppose V is stable and $N(V)$ is universal. V has a minimal Whitney stratification by Teissier 1981. If f_0 is transverse to this minimal Whitney stratification of V, except perhaps at the origin, we say that X is an *Essentially Isolated Sectional Singularity* or *EISS*.
- Stabilizations exist.
- Since f_0 is transverse to the Whitney stratification of V , it follows that $\overline{JM(X)} = \overline{f_0^*(JM(V))}$ except at the origin, hence $e(JM(X), f_0^*(JM(V)), 0)$ is well defined.

Definition

Suppose X^d is an EISS, and F is a stabilization family with base $\mathbb{C}$. Define $m_d(X)$ as the degree over $\mathbb{C}$ of the polar curve of $JM_z(F)$.

Epilogue II

Proposition

The invariant $m_d(X)$ is independent of the choice of F.

- **Proof:** Apply the multiplicity polar theorem to $e(JM(X), f_0^*(JM(V)), 0)$, and $\Gamma^1(JM_z(F))$, and $\Gamma^1(F^*(JM(V)))$. We get as before:
- $$m_d(X) = e(JM(X), f_0^*(JM(V)), 0) + m_{\mathbb{C}}(\Gamma^1(F^*(JM(V)))$$ $$= e(JM(X), f_0^*(JM(V)), 0) + (f_0(\mathbb{C}^q) \cdot \Gamma_q(V).$$
- In the right-hand side of the last equality, the terms are independent of F.
- Define a *good* family of EISS as before. Define a filtration of V by V_i, the union of all strata of V of dimension $\leq i$. Each V_i is equidimensional. Let i_X denote $f^{-1}(V_i)$.

W Equisingularity of ESSI

Theorem

Suppose $\mathcal{X}^{d+k}$ is a good k-dimensional family of EISS. Then the following are equivalent:

1. *The family is Whitney equisingular.*
2. *The invariants*

$$e(m_Y JM({}_i\mathcal{X}(y)), N({}_i\mathcal{X}(y))) + \tilde{F}(y)(\mathbb{C}^q) \cdot \Gamma_{d_i}(V_i)$$

   are independent of y.

3. *The polar multiplicities at the origin of ${}_i\mathcal{X}(y)$ and the invariants*

$$e(JM({}_i\mathcal{X}(y)), N({}_i\mathcal{X}(y))) + \tilde{F}(y)(\mathbb{C}^q) \cdot \Gamma_{d(i)}(V_i)$$

   are independent of y.

- Since all of the actors are on stage, the proof is a reprise of the ideas of the lecture.
- **Challenge Problem** Relate $m_{d_i i}(X)$ to the topology of V.