Determinantal Singularities.

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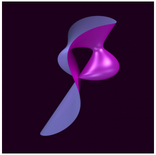

Lecture 1-Part I: Basics on Determinantal Singularities
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ICIS

$$X = V(I) \subset (\mathbb{C}^N, 0),\ I = \langle f_1, \ldots, f_p \rangle,\ p = \mathrm{cod}_{\mathbb{C}^N} X,\ Sing(X) \cap U = \{0\}.$$

$(f_1, \ldots, f_p)$ is a regular sequence and X has the expected dimension

- $\mathscr{K}$-finite germs and their versal deformations

$$f : (\mathbb{C}^N, 0) \to (\mathbb{C}^p, 0),\ \ f = (f_1, \ldots, f_p),\ \ X = f^{-1}(0).$$

- Milnor fibration: the fiber F_θ is smooth and has the *homotopy type of a bouquet of spheres.*

  The Milnor number: $\mu(X) = \#$ spheres of the bouquet .

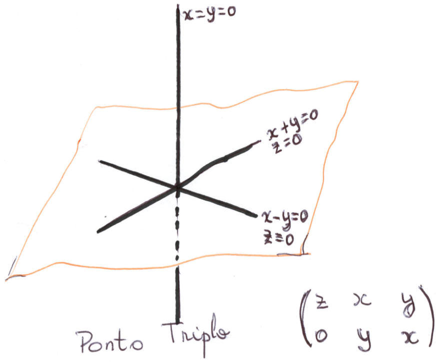
- Equisingularity of families of ICIS: the integral closure of the Jacobian module $JM(F)$ controls W-regularity in families of ICIS (*Whitney equisingularity*).

Beyond ICIS: Determinantal singularities

• Determinantal singularities generalizes ICIS in various aspects. We will explain this and show *new aspects* of this class of singularities.

- A determinantal variety is defined by the zeros of minors of a given size of a matrix, with the condition that it has the expected dimension.
- Why to study them?
  - Algebraic geometry and commutativa algebra (Segre varieties,Veronese varieties, Rational normal scrolls - for example, have a determinantal structure).
  - Singular set and discriminants of smooth mappings are DV.

- Simple analytic sets, for instance, space curves, admit an analytic representation as DV.



$$I_2 = \langle xz, yz, x^2 - y^2 \rangle, \ X = V(I_2)$$

$(xz, yz, x^2 - y^2)$ do not form a regular sequence, as the dimension of X is 1.

Determinantal ideals

Let R be a commutative ring with unity. We write $R^{m\times n}$ for the space of $m \times n$-matrices with entries in R and $R^{m\times m}_{sym}$ and $R^{m\times m}_{sk}$, respectively the spaces of symmetric and skew symmetric $m \times m$ matrices with entries in R.

Definition

Let A be a $m \times n$ be a matrix with entries in R and $I_t \subset R$ the ideal generated by the t-minors of A. We say that I_t is determinantal ideal of type (m, n, t), or determinantal of type t with matrix A.

Note that the same ideal can be determinantal for different matrices.

Example

The principal ideal I in $\mathbb{C}\{x, y, z, w\}$ generated by $f(x, y, z, w) = xz - yw$, is a determinantal ideal of type 1, but it can also be given as a determinantal ideal of type 2 defined by the matrix

$$A = \begin{pmatrix} x & y \\ w & z \end{pmatrix}$$

Basic material from commutative algebra

- The *height* of a prime ideal P of R is the maximum of the length, n, of a chain of strict inclusions

$$P_0 \subset P_1 \subset \ldots \subset P_n = P,$$

  where all P_i are prime ideals of R.

- The *Krull dimension* (or dimension) of R, denoted $\dim R$, is the maximum of the heights of prime ideals in R.

Example

The dimension of $\mathscr{O}_n$ is n. The case $n = 1$ is easy. For $n > 1$, each of the ideals in the chain

$$0 \subset (x_1) \subset (x_1, x_2) \subset \ldots \subset (x_1, ..., x_n)$$

is prime, and there are no other prime between any two of them, since the successive quotients are isomorphic to $\mathscr{O}_1$.

Cohen-Macaulay modules

Let $(R, \mathcal{M})$ be a local ring and M and R-module.

- A sequence $a_1, \ldots, a_r$ of elements in $\mathcal{M}$ is a regular sequence of M if
  - a_1 is not a zero divisor of M and
  - a_i is not a zero-divisor in $M/(a_1, \ldots, a_{i-1})M, \; i = 2, \ldots, r$.
- The depth of M, denoted depth(M) is the maximal lenght of a regular sequence of M.

Over a Noetherian local ring, it is always true that

$$\operatorname{depth} M \leq \dim M.$$

When the equality holds, the module M is Cohen-Macaulay.

The ring R is called Cohen-Macaulay, if R is a Cohen-Macaulay R-module

Theorem

(Eagon-Hochster) Let R be a Noetherian ring of dimension d. Let A be a $p \times q$ matrix with entries in R, and I_r the ideal generated by the r minors of A. Then the following hold:

- $\dim(R/I_r) \geq d - (p-r+1)(q-r+1)$
- If R is Cohen-Macaulay and $\dim(R/I_r) = d - (p-r+1)(q-r+1)$, then the ring R/I_r is Cohen-Macaulay.

In analogy to the complete intersection case, we will refer to the bound $(p-r+1)(q-r+1)$ as the expected codimension of the determinantal ideal I_r.

- For determinantal ideals of symmetric matrices, the expected codimension for the ideal of r-minors of an $n \times n$ matrix is $\frac{1}{2}(n-r+2)(n-r+1)$.
- For skew-symmetric matrices $A \in R_{sk}^{m \times m}$ the expected codimension of the Pfaffian ideal is $\frac{1}{2}(m-2r+2)(m-2r+1)$.

$\mathcal{O}_N := \{f : (\mathbb{C}^N, 0) \to \mathbb{C}\}$ is the local ring of germs of holomorphic functions at 0.

Definition

A germ of complex space $(X, 0) \subset (\mathbb{C}^N, 0)$ is a determinantal variety of type (m, n, t), $1 \leq t \leq min\{m, n\}$, if there is a $m \times n$ matrix A with entries in $\mathcal{O}_N$, such that the local ring $\mathcal{O}_{(X,0)}$ is given by $\mathcal{O}_N / I_t$ where I_t is the ideal generated by the $t \times t$ minors of A, and moreover codim(X)=(m-t+1)(n-t+1).

The variety $(X, 0)$ may have different representations as a determinantal variety.

Example

(Pinkham's example, 1974) $(X^2, 0) \subset (\mathbb{C}^5, 0)$,
$I_2 = \langle x_1x_3 - x_2^2, x_1x_4 - x_2x_3, x_1x_5 - x_2x_4, x_2x_4 - x_3^2, x_2x_5 - x_3x_4, x_3x_5 - x_4^2 \rangle$

$$A = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_5 \end{pmatrix}, \quad B = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \\ x_3 & x_4 & x_5 \end{pmatrix}$$

$\mathcal{O}_{(X,0)} \backsimeq \mathcal{O}_5 / I_2$, $\text{cod}X^2 = 3 = (4 - 2 + 1)(2 - 2 + 1)$.

Deforming the matrix A or B will lead to topologically distinct deformations of X.

Geometrical/infinitesimal approach to determinantal singularities

• From the geometric viewpoint of this course, we choose from the beginning a given representation of a determinantal singularity. The allowable deformations of the matrix are the deformations in the space of the chosen representation matrix. Following Terry Gaffney, we say the chosen representation is the landscape of the singularity.

• In a more general context, we want to understand determinantal singularities as objects defined as (nonlinear) sections of certain universal (or model) singularities. This more general context will be explained by Jim Damon in his lectures.

$$\{\text{Determinantal Singularities}\, \mathcal{V}_0\} \xrightarrow{F} \{\text{Singularities of Universal type} \mathcal{V}\}$$

$$\mathcal{V}_0 = F^{-1}(\mathcal{V}),$$

with some transversality assumption on F.

Generic determinantal varieties.

Universal determinantal varieties in $\mathbb{C}^{m\times n}$

Our universal (model) varieties will be the generic determinantal varieties we now introduce.

Definition

• $\mathbb{C}^{m\times n}$: *space of* $m \times n$ *matrices with complex entries.*
For all $1 \leq t \leq \min\{m, n\}$ *let*

$$M^t_{m,n} = \{A \in \mathbb{C}^{m\times n} | rank(A) < t\}.$$

(Vanishing locus of $t \times t$ minors)

This set is called generic determinantal variety.

Convention: $\min\{m, n\} = n$.

Basic properties of $M^t_{m,n}$

1. $M^t_{m,n}$ are irreducible algebraic varieties, $1 \leq t \leq n$.
2. $\operatorname{cod} M^t_{m,n} = (n-t+1)(m-t+1)$ in $\mathbb{C}^{m\times n}$.
3. The singular set of $M^t_{m,n}$ is $M^{t-1}_{m,n}$.
4. The decomposition $M^t_{m,n} = \cup_{i=1}^t \{M^i_{m,n} \setminus M^{i-1}_{m,n}\}$ is a Whitney stratification.
5. $M^t_{m,n}$ is Cohen-Macaulay.

Theorem

(Gaffney-Grulha-R.) The local Euler obstruction of $M^t_{m,n}$, at the origin is

$$e(t,n) = \binom{n}{t-1}, \ 1 \leq t \leq n.$$

• The smooth varieties $M^r_{m,n} \setminus M^{r-1}_{m,n}$, for any $1 \leq r \leq \min\{m,n\}$ are invariant under the action of the Lie group $GL(m) \times GL(n)$

$$\begin{aligned} GL(m) \times GL(n) \times \mathbb{C}^{m\times n} &\to \mathbb{C}^{m\times n} \\ ((H,K),A) &\mapsto H \circ A \circ K^{-1} \end{aligned}$$

So the decomposition $M^r_{m,n} = \cup_{i=1}^r \{M^i_{m,n} \setminus M^{i-1}_{m,n}\}$ is a Whitney stratification .

Exercise

Let

$$F = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

be a $m \times n$ matrix with $A \in \mathbb{C}^{k\times k}$, $B \in \mathbb{C}^{k\times(n-k)}$, $C \in \mathbb{C}^{(m-k)\times k}$, $D \in \mathbb{C}^{(m-k)\times(n-k)}$. Suppose that F has rank k. Then

1. $\operatorname{rank}(A) = k$ if and only if $D - CA^{-1}B = 0$.
2. The set $S^{r-1} = M^r_{m,n} \setminus M^{r-1}_{m,n}$ of $m \times n$ matrices of rank equal to $r-1$ is a smooth manifold of codimension $(m-r+1)(n-r+1)$.

Determinantal variety in $(\mathbb{C}^N, 0)$, again.

The representation matrix of a determinantal variety determines a map:

$$F : U \subset \mathbb{C}^N \to \mathbb{C}^{m \times n},$$

$F(x) = (f_{ij}(x))$, where f_{ij} are complex analytic functions on U.

The pre-image of $M^t_{m,n}$ by F,

$$X = F^{-1}(M^t_{m,n})$$

is a determinantal variety of type (m, n, t) on $U \subset \mathbb{C}^N$ if
$\mathrm{cod}(X) = \mathrm{cod}(M^t_{m,n}) = (m - t + 1)(n - t + 1)$ in $\mathbb{C}^N$.

Essentially Isolated Determinantal Singularities

EIDS: defined by Ebeling and Gusein-Zade in [Proc. Steklov Inst. Math. (2009)].

$$F : (\mathbb{C}^N, 0) \to \mathbb{C}^{m\times n}, \quad X = F^{-1}(M^t_{m,n}),$$

a determinantal variety of type (m, n, t).

Definition EIDS:

$(X, 0) \subset (\mathbb{C}^N, 0)$ has an essentially isolated determinantal singularity at the origin (EIDS) if any sufficiently small representative $F : U \to \mathbb{C}^{m\times n}$ is transverse to all strata $M^i_{m,n} \setminus M^{i-1}_{m,n}$ of the stratification of $M^t_{m,n}$ in a punctured neighbourhood of the origin.

If $x \in U$, $x \neq 0$, $rank\, F(x) = i - 1$, then $F \pitchfork M^i_{m,n} \setminus M^{i-1}_{m,n}$ at x.

We assume $F(0) = 0$.

Consequences of the transversality assumption

Let $X = F^{-1}(M^t_{m,n})$ be an EIDS of type (m, n, t) in $\mathbb{C}^N$.

- $cod_{\mathbb{C}^N} X = (n - t + 1)(m - t + 1)$.
- The singular set of $X = F^{-1}(M^t_{m,n})$ is the EIDS $F^{-1}(M^{t-1}_{m,n})$, (or empty).
- X is Cohen Macaulay.
- X has an isolated singularity at the origin if and only if $N \leq (m - t + 2)(n - t + 2)$.

  When $N < (m - t + 2)(n - t + 2)$, Isolated Determinantal Singularity- IDS.

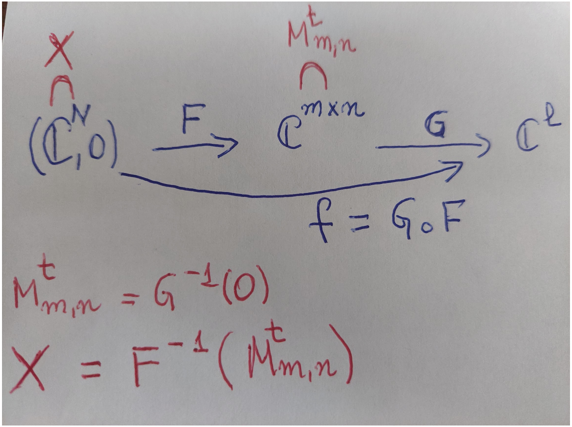


Figure: $X = V(I), \; I = \langle f_1, \ldots, f_\ell \rangle$

Essential smoothings of an EIDS

Deformations (in particular smoothings) of an EIDS of type (m, n, t) are EIDS of the same type.

A deformation F_s of F,

$$F_s : U \subset \mathbb{C}^N \to \mathbb{C}^{m \times n}$$

is called a stabilization of F if it is transverse to the rank stratification.

A stabilization F_s determines the EIDS $X_s = F_s^{-1}(M^t_{m,n})$, called essential smoothing of X (Ebeling-Gusein Zade). X_s is not smooth in general

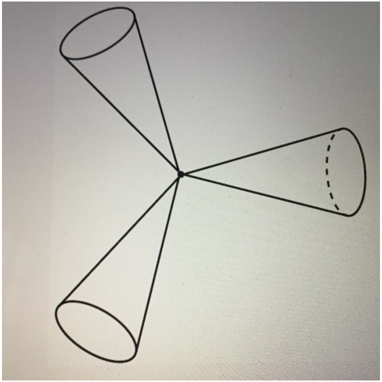
We can define similar notions for 1-parameter families $\widetilde{F}(x, s)$, $\widetilde{F}(x, 0) = F(x)$ and call them, respectively, *stabilization family*, and *essential smoothing family*.

The topology of the essential smoothings is uniquely given for a fixed choice of representation matrix of a given type (m, n, t); they are also called singular Milnor fibers of the EIDS X.

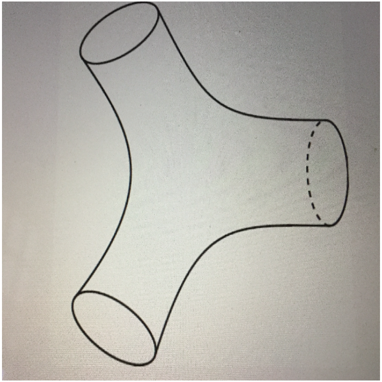
Example

For generic values of a, b, c, $\widetilde{F}_{(a,b,c)}$ gives a smoothing of the determinantal curve in $\mathbb{C}^3$, defined by $F_{(0,0,0)} : (\mathbb{C}^3, 0) \to \mathbb{C}^{2\times 3}$, where:

$$\tilde{F}_{a,b,c}(x, y, z) = \begin{pmatrix} z & y & x \\ c & x + a & y + b \end{pmatrix}$$



X triple point in $\mathbb{C}^3$



$X_{a,b,c}$ smoothing of X.

Cohen-Macauly codimension 2 singularities

(Cohen-Macaulay codimension 2 singularities). It follows from Hilbert-Burch Theorem that the germ of a Cohen-Macaulay variety of codimension 2 can be expressed as the maximal minors of $s+1 \times s$ matrices and vice-versa. In the same way, flat deformations of these varieties can be represented by perturbations of the matrix $F(x) = (f_{i,j}(x)$ and any perturbation of the matrix gives rise to a flat deformation of the variety.

The following result follows as a corollary of Hilbert-Burch Theorem:

Theorem

Suppose $I \subset \mathbb{C}\{x_1, \ldots, x_p\}$ is an ideal with $\mathbb{C}\{x_1, \ldots, x_p\}/I$ Cohen-Macaulay of dimensions $p-2$. Then I has a resolution of the form

$$0 \longrightarrow \mathbb{C}^n\{x\} \xrightarrow{A} \mathbb{C}^{n+1}\{x\} \xrightarrow{f} \mathbb{C}\{x\} \longrightarrow \mathbb{C}\{x\}/I \longrightarrow,$$

for some matrix $(n+1) \times n$ matrix A with entries in $\mathbb{C}\{x\}$, and $I = I_n$ is the ideal generated by the maximal minors of A.

- Enrico Arbarello, Maurizio Cornalba, and Pillip A. Griffiths. Geometry of algebraic curves. Volume II. volume 268 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, pages xxx+963. Springer, Heidelberg, 2011. With a contribution by Joseph Daniel Harris.
- Winfried Bruns and Udo Vetter. *Determinantal rings*, volume 1327 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988.
- J. A. Eagon and M. Hochster. *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci.* 93: 1020-1058, 1971.
- A. Frühbis-Krüger and M. Zach. *Determinantal singularities*, Handbook of Geometry and Topology of Singularities III, Springer-Verlag, 2022.
- S. M. Guseĭn-Zade and V. Èbeling. On the indices of 1-forms on determinantal singularities. *Tr. Mat. Inst. Steklova*, 267(Osobennosti i Prilozheniya):119–131, 2009.
- F. S. Macaulay. *The algebraic theory of modular systems*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1994. Revised reprint of the 1916 original, With an introduction by Paul Roberts.
- H. C. Pinkham. Deformations of cones with negative grading. *J. Algebra*, 30:92–102, 1974.