Algorithmic Methods for Singularities

Talk 4: Deformations

Anne Frühbis-Krüger and Matthias Zach

Institut für Mathematik
Universität Oldenburg

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Unfoldings and Deformations

Recall:

$$F \in \mathbb{C}\{\underline{x}, \underline{t}\} \text{ unfolding of } f \in \mathbb{C}\{\underline{x}\}$$
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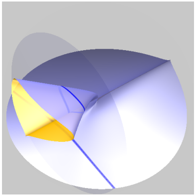
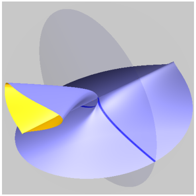
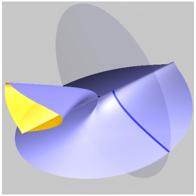
This induces a deformation of $(X, 0) = (V(f), 0) \subset (\mathbb{C}^n, 0)$:

$$\begin{array}{ccc} (X,0) & \hookrightarrow & (\mathcal{X},0) \\ \downarrow & & \downarrow \mathit{flat} \\ pt & \hookrightarrow & (S,0) \end{array}$$

where $(\mathcal{X}, 0) = (V(F), 0) \subset (\mathbb{C}^n \times \mathbb{C}^k, 0)$.

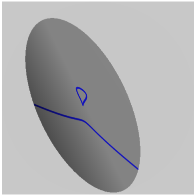
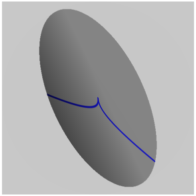
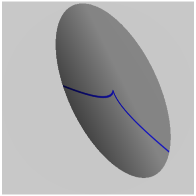
An example in pictures

$F = y^2 - x^5 - t \cdot x^3 \in \mathbb{C}\{x, y, t\}$:

fibres marked in the picture:
$t = -1\ (A_2)$, $t = 0\ (A_4)$, $t = 1\ (A_2)$

Now focus on the fibres:

Questions for deformations of IHS

For a given family:

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For a given central fibre:

- What singularities can appear in other fibres?
- How many different types appear?
- Is there a family inducing any possible deformation?

Equivalences and Determinacy Bounds I

Recall for $f, g \in \mathbb{C}\{\underline{x}\}$:

$$f \sim_R g \Longleftrightarrow \exists \phi \in \mathrm{Aut}(\mathbb{C}\{\underline{x}\}) : f = g \circ \phi$$

$$f \sim_C g \Longleftrightarrow (V(f), 0) \cong (V(g), 0)$$

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f finitely determined, if

$$\exists k \in \mathbb{N} : \quad (f \equiv g \bmod \mathfrak{m}^{k+1} \Longrightarrow f \sim g)$$

Easily computable bounds for determinacy: $\mu + 1$, $\tau + 1$
(usually significantly too high)

Determinacy Bounds II

Better bounds, requiring Ideal Membership Test:

f right-k-determined, if

$$\mathfrak{m}^{k+1} \subset \mathfrak{m}^2 \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle$$

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Exercise: Write a function which takes f and k as input and checks the above condition for right-determinacy, returning true or false.

Versal families

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A deformation

$$(X,x) \hookrightarrow (\mathcal{X},x) \longrightarrow (S,s)$$

is called **versal**, if any other deformation

$$(X,x) \hookrightarrow (\mathcal{Y},x) \longrightarrow (T,t)$$

arises from it by a base change $\phi : (T,t) \longrightarrow (S,s)$.

Task: Find a versal family with given special fibre!

Versal families

Task: Find a versal family with given, finitely determined special fibre $V(f)$!

Construction:

- Compute a K-basis $\{g_1, \ldots, g_\tau\}$ of T^1
- Set
$$F = f + \sum_{i=1}^{\tau} t_i g_i \in K\{\underline{x}, \underline{t}\}$$
- $(\mathcal{X}, 0) = (V(F), 0)$ versal family

Discriminant

Task: What fibres of a family are singular?
Compute the relative T^1:

$$T^1_{rel}(\mathcal{X},0) = \mathbb{C}\{\underline{x},\underline{t}\}/\left(\langle F\rangle + \left\langle \frac{\partial F}{\partial x_1},\ldots,\frac{\partial F}{\partial x_n}\right\rangle\right)$$

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project the support of T^1 to the base:

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Caution: Elimination is expensive!

Worksheet 4, Block 1

The ICIS case and beyond

Let $(X,0) = (V(\underbrace{\langle f_1,\ldots,f_s\rangle}_{:=I}),0)$ be an isolated singularity.

Consider as base space: $(\mathbb{T},0) = \mathbb{C}[\varepsilon]/\langle\varepsilon^2\rangle$

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$(f_1+\varepsilon g_1,\dots,f_s+\varepsilon g_s)$ defines a (flat) deformation
of $(X,0)$ over $(\mathbb{T},0)$

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$$\Longleftrightarrow$$

$\varphi : I \longrightarrow \mathbb{C}\{\underline{x}\}/I$ given by $\varphi(f_i) = g_i$ provides a well-defined element in

$$N_{X,0} = Hom_{\mathbb{C}\{\underline{x}\}}(I, \mathbb{C}\{\underline{x}\}/I)$$

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For ICIS: $N_{X,0} \cong (\mathbb{C}\{\underline{x}\})^s / I \cdot (\mathbb{C}\{\underline{x}\})^s$

Flatness

Example:
$\langle xy - t, xz, yz \rangle\rangle$ does not provide a flat family!

Dimension of fibre for $t = 0$ is 1, but 0 otherwise!

What happened?

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Flatness ensures that all relations lift!

T^1 for ICIS and beyond

$\Theta_n = \langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \rangle_{\mathbb{C}\{\underline{x}\}-\text{module}}$

Consider

$$\begin{array}{rcl} \alpha : \Theta_n & \longrightarrow & N_{(X,0)} \\ \vartheta & \longmapsto & (f \longmapsto \vartheta f) \end{array}$$

Then

$$T^1(X,0) = coker(\alpha)$$

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For ICIS:

$$T^1 = (\mathbb{C}\{\underline{x}\})^s / (I \cdot (\mathbb{C}\{\underline{x}\})^s + \langle \frac{\partial \underline{f}}{\partial x_1}, \ldots, \frac{\partial \underline{f}}{\partial x_n} \rangle_{\mathbb{C}\{\underline{x}\}-\text{module}})$$

Obstructions

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$T^2(X, 0)$ encodes obstructions for lifting a deformation over a fat point to one over an infinitessimally bigger fat point.

T^2

Consider a free presenation of I:

$$\mathcal{O}_n^l \xrightarrow{\psi} \mathcal{O}_n^s \xrightarrow{\varphi} I \longrightarrow 0$$

where $\phi(e_i) = f_i$

We consider

- $I_R = \ker(\varphi)$
- $I_K = \langle f_i e_j - f_j e_i \mid 1 \leq i < j \leq s \rangle$
- $\mathcal{O}_{(X,x)}$-linear map

$$I_R / I_K \longrightarrow \mathcal{O}_n^s / I \mathcal{O}_n^s = \mathcal{O}_{(X,0)}^s$$

Dualize the last map to obtain:

$$\mathrm{Hom}_{\mathcal{O}_{(X,0)}}(\mathcal{O}_{(X,0)}^s, \mathcal{O}_{(X,0)}) \xrightarrow{\phi} \mathrm{Hom}_{\mathcal{O}_{(X,0)}}(I_R / I_K, \mathcal{O}_{(X,0)})$$

$$T^2(X,0) = \mathrm{coker}(\phi)$$

Worksheet 4, Block 2