Algorithmic Methods for Singularities

Talk 1: First steps in OSCAR

Anne Frühbis-Krüger and Matthias Zach

Institut für Mathematik
Universität Oldenburg

Sao Carlos, July 12th 2022

See Jupyter Notebook 1 – first block

Jacobian Criterion: equidim. case

Let $X = V(f_1, \ldots, f_s) \subset \mathbb{A}^n_{\mathbb{C}}$ be **equidimensional**

$$X \text{ is singular at point } P$$

$$\Longleftrightarrow$$

$$rk\left(\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j}(P)\right) < n - dim(X)$$

singular locus: $V(minors(\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j}, n - dim(X)))$

Computational tasks:

- derivatives ✓ (Jacobian matrix)
- minors ✓ (minors)
- dimension of X ✓ (details see second talk)

Back to the examples from the notebook

Example 1: the polynomial

$$\begin{aligned} X &= V(x \cdot y \cdot (x+y-1)) \\ &= V(x) \cup V(y) \cup V(x+y-1) \subseteq \mathbb{A}^2_{\mathbb{C}} \end{aligned}$$

Three lines meeting pairwise in the points:
$(0,0)$, $(0,1)$ and $(1,0)$

Back to the examples from the notebook

Example 1: the polynomial

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Example 2: the ideal

$$V(zx^2 - y^2, yz - x^5, x^3y - z^2) \subseteq \mathbb{A}^3_{\mathbb{C}}$$

This is an ICMC2 singularity corresponding to the parametrized curve (t^3, t^7, t^8) with singular locus $\{\underline{0}\}$.

Jupyter notebook block 2

Primary Decomposition

Let A be a noetherian ring, $I \subset A$ an ideal.

$$I = \bigcap_{i=1}^{s} Q_i$$

is an irredundant primary decomposition, if

- Q_i primary $\forall 1 \leq i \leq s$
- no Q_i can be omitted
- $\sqrt{Q_i} \neq \sqrt{Q_j} \quad \forall 1 \leq i < j \leq s$

associated prime ideals $\sqrt{Q_i}$ are unique

Primary decomposition and radicals of ideals are computable in $\mathbb{K}[\underline{x}]$,

but may be **expensive**!

Jacobian Criterion: general case

Let $X = \bigcup_{i=1}^{m} X_i \subset \mathbb{A}^n$ be an equidimensional decompositon of X.

Then the singular locus consists of:

- singular loci of the X_i $\forall 1 \leq i \leq m$
- pairwise intersections $X_i \cap X_j$ $\forall 1 \leq i < j \leq m$

Yet another example

Yet another example in $\mathbb{A}^3_{\mathbb{C}}$:

$$V(x^2 - y^3) \cup V(x-1, y^2 - z^2)$$

Union of a surface with 1-dimensional singular locus and a singular curve.

Back to the notebook

Computing in local rings

Up to now:

- computations in $\mathbb{Q}[\underline{x}]$
- affine schemes in $\mathbb{A}^n$

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Theoretically necessary for handling singularities:

- power series ring, e.g. $\mathbb{C}\{\underline{x}\}$
- space germs

For a field K allowing exact computations (e.g. $\mathbb{Q}$, $\mathbb{F}_q$):

$$\underbrace{K[\underline{x}] \subseteq K[\underline{x}]_{\langle \underline{x} \rangle}}_{\text{exact computations}} \subseteq K\langle \underline{x} \rangle \subseteq K[[\underline{x}]]$$

$\Longrightarrow$ reasoning in $\mathbb{C}\{\underline{x}\}$, computations in $K[\underline{x}]_{\langle \underline{x} \rangle}$

See Jupyter Notebook 1 – last block