Algorithmic Methods for Singularities

Talk 3: Elimination and Normalization as computational tasks

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Finding equations for parametrized varieties

Equations for (the closure of) the Im(f)?

$$\begin{array}{rcl} f : \mathbb{C}^m & \longrightarrow & \mathbb{C}^n \\ (t_1, \ldots, t_m) & \longmapsto & (f_1(\underline{t}), \ldots, f_n(\underline{t})) \end{array}$$

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on the algebraic side:

$$\begin{array}{rcl} f^\# : \mathbb{C}[x_1, \ldots, x_n] & \longrightarrow & \mathbb{C}[t_1, \ldots, t_m] \\ x_i & \longmapsto & f_i(\underline{t}) \end{array}$$

Equivalent task: Compute $\ker(f^\#)$

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Equivalent task: Compute $\ker(f^\#)$

Solution: Elimination of variables!

Elimination = Intersection with subring

Fact:

G SB of $I \subset \mathbb{C}[\underline{t}, \underline{x}]$ w.r.t elimination ordering for $\underline{t}$

$$\Longrightarrow \{g \in G \mid LT(g) \in \mathbb{C}[\underline{x}]\} \text{ standard basis of } I \cap \mathbb{C}[\underline{x}]$$

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key point: elimination ordering for $\underline{t}$ on $\mathbb{C}[\underline{t}, \underline{x}]$

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conditions on ordering:

- first comparison only on the $\underline{t}$
- $1 < t_i \ \forall i$

Computing a kernel by elimination

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Back to our map:

$$\begin{aligned} f : \mathbb{C}[x_1, \ldots, x_n] &\longrightarrow \mathbb{C}[t_1, \ldots, t_m] \\ x_i &\longmapsto f_i(\underline{t}) \end{aligned}$$

Denote by $\langle g_1, \ldots, g_n \rangle$ the ideal of the graph of f.

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Denote by $\langle g_1, \ldots, g_n \rangle$ the ideal of the graph of f.

More explicitly: $g_i = x_i - f_i(\underline{t})$

Then:

$$\ker(f) = \langle g_1, \ldots, g_n \rangle \cap \mathbb{C}[\underline{x}]$$

Example 1

Find the ideal of the curve C with parametrization $f(t) = (t^4, t^5, t^6)$!

Step 1: Choose elimination ordering
e.g. lex on t followed by deglex on x,y,z

Step 2: Setup ideal of the graph
$I = \langle x - t^4, y - t^5, z - t^6 \rangle$

Step 3: Compute standard basis G of I

Step 4: Drop all elements of G involving some t

Result: $I_C = \langle x^3 - z^2, xz - y^2 \rangle$

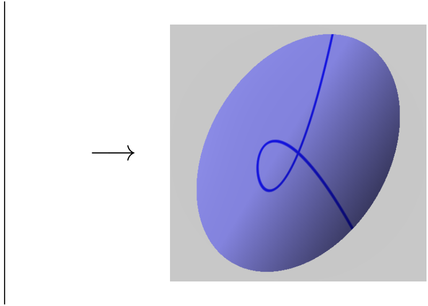
another example: $(s, t) \longmapsto (s^3, s^2t, st^2, t^3)$
(see exercises on Wednesday)

Worksheet 3, Block 1

Finding a Parametrization of a Curve?

The opposite question: Finding a parametrization

Possible for curves via normalization!



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Integral closure and normal rings

Let $A \subset B$ be rings.

Recall:
$b \in B$ integral over A, if

$$\exists p \in A[t] \text{ monic } : p(b) = 0$$

integral closure of A in B

$$C(A, B) = \{b \in B \mid b \text{ integral over } A\}$$

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A integrally closed or normal, if

$$A = C(A, \mathrm{Quot}(A))$$

$\overline{A} = C(A, \mathrm{Quot}(A))$ is called the normalization of A.

Idea of the algorithm

setting:
A noetherian reduced ring,
$J \subset A$ ideal, $x \in J$ non-zerodivisor for A

inclusion of rings:

$$A \subseteq \underbrace{Hom_A(J,J) \cong \frac{1}{x}(xJ : J)}_{\phi \longmapsto \frac{\phi(x)}{x}} \subseteq \overline{A}$$

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Idea of algorithm:

- $Hom_A(J,J)$ computable
- $A \subsetneq Hom_A(J,J)$, if $A \subsetneq \overline{A}$
- use a radical ideal J and iterate process

A normality criterion

Non-normal locus:

$$N(A) = \{P \in \mathsf{Spec}(A) \mid A_P \text{ not normal } \}$$

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Normality Criterion:
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- $\exists x \in J$ non-zerodivisor for A
- $J = \sqrt{J}$
- $N(A) \subset V(J)$

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Then

$$A \text{ normal } \Longleftrightarrow A = Hom_A(J, J) \Longleftrightarrow \langle x \rangle = (xJ : J)$$

The non-normal locus

Setting:

- $I \subset K[\underline{x}]$ prime ideal
- J radical of ideal of singular locus of $V(I)$
- $x \in J$ (obviously non-zerodivisor for $K[\underline{x}]/I$)

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- $x \in J$ (obviously non-zerodivisor for $K[\underline{x}]/I$)

Then:

$$N(K[\underline{x}]/I) = V(\langle x \rangle : (xJ : J))$$

Example:

$I = \langle z^2 - y^3 \rangle \subset K[y, z]$, $J = \langle z, y \rangle$, $x = z$

$$\begin{aligned} N(K[y,z]/I) &= (\langle z \rangle : (zJ : J)) \\ &= (\langle z \rangle : (z, y^2)) \\ &= \langle z, y \rangle \end{aligned}$$

Iteration step: ugly technical details

$\{u_0 = x, u_1, \dots, u_s\}$ generators for $(xJ : J)$ as A-module

Then

$$Hom_A(J, J) \cong A[t_1, \dots, t_s]/(I_{prod} + I_{syz})$$

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where

- ring structure of $Hom_A(J, J)$ implies: $u_i u_j \in (xJ : J)$
  $$\Longrightarrow \exists\, \xi_k^{ij} \in A : u_i u_j = \sum_{k=0}^{s} x \xi_k^{ij} u_k$$
  due to A-module structure

  $I_{prod} = \langle t_i t_j - \sum_{k=0}^{s} \xi_k^{ij} t_k \mid 1 \leq i \leq j \leq s \rangle$
  where $t_0 = 1$

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- relations among the u_i exist, hence
  $I_{syz} = \langle \sum_{k=0}^{s} \eta_k t_k \mid \sum_{k=0}^{s} \eta_k u_k = 0 \rangle$
  (suffices to consider generators for $syz(u_0, \ldots, u_s)$)

in our example

$u_0 = z, \ u_1 = y^2$

$u_1^2 = y^4 \equiv z^2 y = \underbrace{y}_{\xi_0^{11}} \cdot z \cdot u_0$

$$\Longrightarrow \quad I_{prod} = \langle t_1^2 - y \rangle$$

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A-module of syzygies generated by $\langle zu_1 - y^2 u_0, yu_1 - zu_0 \rangle$

$$\implies \quad I_{syz} = \langle zt_1 - y^2, yt_1 - z \rangle$$

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Hence

$$A_{new} := Hom_A(J, J) \cong K[t_1, y, z] / \langle t_1^2 - y, z t_1 - y^2, y t_1 - z \rangle$$

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Hence

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In general: continue with a new J, x and A_{new} to determine $Hom_{A_{new}}(J_{new}, J_{new})$...

Integral Closure of an Ideal

$A \subset B$ Ring extension, $I \subseteq A$ ideal

Recall:

$b \in B$ integral over I, if

$$\exists\ p = \sum_{i=0}^{n} a_{n-i} t^i \in A[t] : p(b) = 0$$

such that $a_0 = 1$, $a_1, \ldots, a_n \in I$

$$C(I, B) := \{b \in B \mid b \text{ integral over } I\}$$

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such that $a_0 = 1,\ a_1, \ldots, a_n \in I$

$$C(I, B) := \{b \in B \mid b \text{ integral over } I\}$$

Fact:

$$IC(A, B) \subseteq C(I, B) = \sqrt{IC(A, B)}$$

radical computable, $C(A, \mathrm{Quot(A)})$ computable
$\implies C(I, Quot(A)) \subseteq C(A, Quot(A))$ computable

Worksheet 3, Block 1

Conductor and δ-invariant

For $A = K[\underline{x}]/P$, P prime ideal:

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and $N(A) = V(\mathcal{C})$.

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For Curves:

$$\delta = \dim_K(\overline{A}/A)$$

Example: A space curve

Consider space curve $X = V(y^3 - x^2z, xy^2 - z^2, x^3 - yz)$

$$\mathbb{C}[x,y,z]/I_X \cong \mathbb{C}[t^4, t^5, t^7] \hookrightarrow \mathbb{C}[t]$$

Delta_abzhlen.png

$$\delta = 4 \quad C = \langle t^7 \rangle$$

The Serre criterion

Recall: $\mathrm{ht}(P) = \dim(A_P)$ for prime ideals $P \subset A$

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Serre's Criterion:

reduced noetherian ring A normal

$$\Longleftrightarrow$$

(R1) A_P regular local ring $\forall\, P \subset A$ prime of height 1

(S2) $\mathrm{minAss}(\langle f \rangle) = \mathrm{Ass}(\langle f \rangle)$ for every non-zerodivisor $f \in A$

Warning: normal varieties may be singular!!

Example: $V(xy - z^2) \subset \mathbb{A}^3_C$ is normal

Worksheet 3, last Block