Applying Lie Group Methods and Characteristic Cohomology for the Topology of Matrix Singularities

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Perspective of Isolated Hypersurface Singularities

Topology of Milnor Fibration, Link, and Complement

$f : \mathbb{C}^{n+1}, 0 \to \mathbb{C}, 0$, $X_0 = f^{-1}(0)$ of dimension n:

Milnor Fibration: $\mathcal{X} = f^{-1}(B^*_\delta) \cap B_\varepsilon \to B^*_\delta$, for $0 < \delta << \varepsilon$
Milnor Fiber: $X_t = f^{-1}(t) \cap B_\varepsilon$

- If f_0 has isolated singularity: use Morse theory to determine topology
- structure of Milnor Fiber: is $(n-1)$–connected and $X_t \simeq_{he} \vee_{i=1}^{\mu} S^n$ bouquet of n–spheres ("compact model");
- algebraic formula for Milnor number: $\mu =$ dimension of Milnor algebra;
- Link $L(X_0) = X_0 \cap S_\varepsilon$ $(n-2)$-connected $(2n-1)$-dim compact manifold $\simeq$ boundary of closed Milnor fiber
- monodromy + Wang sequence to relate link and Milnor fiber

Nonisolated Singularities from Perspective of Isolated Singularities

Theorem (Kato-Matsumoto ('73)): For a hypersurface singularity $\mathcal{V}_0, 0 \subset \mathbb{C}^{n+1}, 0$ defined by $f_0 : \mathbb{C}^{n+1}, 0 \to \mathbb{C}, 0$, if $\dim(\text{sing}(\mathcal{V}_0)) = k$, then Milnor fiber $\mathcal{V}_w$ is $(n-1-k)$–connected. Thus,

$$\widetilde{H}^j(\mathcal{V}_w) = \begin{cases} 0 & \textit{for} \quad 0 \leq j \leq n-k-1 \\ ? & \textit{for} \quad n-k \leq j \leq n \end{cases}$$

Is Milnor fiber $\simeq_{he}$ bouquet of spheres and of which dimensions?

Depends on properties of $\text{sing}(\mathcal{V}_0)$. If $\dim_{\mathbb{C}} \text{sing}(\mathcal{V}_0) \leq 2$, then for special form for $\Sigma = \text{sing}(\mathcal{V}_0)$, e.g. ICIS, and special transverse types of f_0 on Σ, there are a number of results due to (Siersma, Pellikaan, Tibar (a bouquet theorem), Nemethi, Zaharia, Van Straten, etc, see e.g. survey in Siersma ('01)).

Distinguished Classes of Nonisolated Singularities

$$\{\text{nonisolated singularities}\}$$

$$\bigcup$$

$$\{\text{singularities of "Universal Type"}\,\mathcal{V}\}$$

$$\bigcup$$

$$\{\mathcal{V} \text{ are Exceptional Orbit Varieties of Prehomogeneous Spaces}\}$$

$$\bigcup$$

$$\{\text{Matrix Singularities}\}$$

Category of Singularities of Type $\mathcal{V}, 0$

Objects: singularities $\mathcal{V}_0 = f_0^{-1}(\mathcal{V})$ arising as "nonlinear sections" defined by a holomorphic germ (allowing $n \geq N$)

$$\begin{array}{ccccc} & & \mathbb{C}^n, 0 & \xrightarrow{f_0} & \mathbb{C}^N, 0 \\ & & \uparrow & & \uparrow \\ f_0^{-1}(\mathcal{V}) & = & \mathcal{V}_0, 0 & \longrightarrow & \mathcal{V}, 0 \end{array} \qquad (1)$$

Morphisms $\varphi : \mathcal{W}_0, 0 \to \mathcal{V}_0, 0$, for $\mathcal{W}_0 = g_0^{-1}(\mathcal{V})$ defined by $g_0 : \mathbb{C}^p, 0 \to \mathbb{C}^N, 0$, is induced by a germ $\tilde{\varphi} : \mathbb{C}^p, 0 \to \mathbb{C}^n, 0$ so that $g_0 = f_0 \circ \tilde{\varphi}$.

A subcategory are those defined by f_0 which are transverse to $\mathcal{V}$ in an appropriate sense. Then, $\mathrm{codim}_{\mathbb{C}}(sing(\mathcal{V}_0)) = \mathrm{codim}_{\mathbb{C}}(sing(\mathcal{V}))$.

Categories of Singularities of Type $\mathcal{V}$

Examples of "Universal Singularities $\mathcal{V}, 0$", and Singularities of Type $\mathcal{V}$:

- **i)** Discriminants of Stable Germs and discriminants of finitely determined germs;
- **ii)** Bifurcation Sets of $\mathcal{G}$-versal unfoldings, and Bifurcation Sets of $\mathcal{G}$ unfoldings ($\mathcal{G}$ geometric subgroup of $\mathcal{A}$ or $\mathcal{K}$);
- **iii)** Special Central Hyperplane Arrangements, generic hyperplane and hypersurface arrangements of these types;
- **iv)** Exceptional Orbit Hypersurfaces of Prehomogeneous Spaces, e.g. Discriminants for Representation Spaces of Quivers of Finite Type and for Cholesky-type Factorizations of matrices, determinantal Arrangements;
- **v)** Varieties of Singular $m \times m$ Complex Matrices which are General, Symmetric or Skew-symmetric (m even), or $m \times p$ matrices, and matrix singularities of these types.

Example: Isolated Cohen-Macaulay Surface Singularity in $\mathbb{C}^4$

$$f_0(x,y,z,w) = \begin{pmatrix} x & y & z \\ y & z & w \end{pmatrix} \tag{2}$$

$\mathcal{V} = \{$matrices of rank $\leq 1\}$. $\mathcal{V}_0$ is defined by the ideal of 2×2 minors and is Cohen-Macaulay by Hilbert-Burch Theorem

$$(xz - y^2, xw - yz, yw - z^2)$$

Example: Obstructed Isolated Surface Singularity in $\mathbb{C}^5$

$$f_0(x,y,z,u,v) = \begin{pmatrix} x & y & z & u \\ y & z & u & v \end{pmatrix} \tag{3}$$

$\mathcal{V} = \{$matrices of rank $\leq 1\}$. $\mathcal{V}_0$ is defined by the ideal of 2×2 minors

$$(xz - y^2, xu - yz, xv - yu, yu - z^2, yv - zu, zv - u^2)$$

Example (D- '87): Isolated Gorenstein Surface Singularity in $\mathbb{C}^5$

$$f_0(x,y,z,u,v) = \begin{pmatrix} 0 & x & u & v & z \\ -x & 0 & y & z & v \\ -u & -y & 0 & x & 0 \\ -v & -z & -x & 0 & y \\ -z & -v & 0 & -y & 0 \end{pmatrix} \qquad (4)$$

$\mathcal{V}$ consists of the "singular matrices" having rank ≤ 2; and $\mathcal{V}_0$ is defined by the ideal of 4×4 Pfaffians

$$(y^2 + xv, uy + xz, xy + z^2 - v^2, yz - uv, x^2 + yv - uz).$$

By Buchsbaum-Eisenbud Theorem, $\mathcal{V}_0$ is an isolated Gorenstein surface singularity (in fact, unimodal Gorenstein elliptic surface singularity (not ICIS) with topologically trivial versal unfolding).

Example (Bruce ('03): Simple Symmetric Matrix Singularity

$$f_0(x,y) = \begin{pmatrix} x & y^k \\ y^k & xy \end{pmatrix} \tag{5}$$

The matrix is symmetric, $\mathcal{V}$ consists of symmetric matrices of rank ≤ 1, and $\mathcal{V}_0$ is an isolated singularity of type D_{2k+1} defined by $x^2y - y^{2k}$.

This is one of simple symmetric matrix singularities classified by Bruce.

There is a body of further work on simple $m \times m$ matrix singularities both symmetric and skew-symmetric (m even) - Bruce, Tari, Haslinger, Goryunov, Zakalyukin

Topology of Nonisolated Singularities of Type $\mathcal{V}, 0$

**Goal: Determine the topology of the Milnor fiber of $\mathcal{V}_0$ for a hypersurface singularity $\mathcal{V}, 0$,
and the topology of the complement, and link of $\mathcal{V}_0, 0$ for general $\mathcal{V}, 0$.**

There are two contributions:

- **a)** The contribution from the topology of the germ f_0 and its geometric interaction with $\mathcal{V}$.
- **b)** The contribution inherited from the topology of $\mathcal{V}$.
- **c)** Determine how these two contributions combine to give the topology of $\mathcal{V}_0$.

Overview of Main Points

- General Category of Singularities $\mathcal{V}_0, 0$ of Type $\mathcal{V}, 0$
- Obtain answers to a) using stabilization of nonlinear sections to obtain "singular Milnor fiber" of f_0.
  - **i)** involves equivalence groups for nonlinear sections of $\mathcal{V}, 0$
  - **ii)** determining the topology of the singular Milnor fibers
  - **iii)** analogues of the Tjurina number τ: codimensions of the extended tangent spaces for the equivalence group orbits
  - **iv)** obtaining "$\mu = \tau$"-type results
- Obtain answers to b) via) Characteristic Cohomology for Singularities of Type $\mathcal{V}, 0$
  - **i)** involves capturing the topology of $\mathcal{V}_0, 0$ inherited from $\mathcal{V}, 0$
  - **ii)** for Milnor fibers, complements, and links
  - **iii)** has invariance and functorial properties
  - **iv)** there is a criterion for detecting nonvanishing subgroups
- Apply Lie Group Methods to the exceptional orbit varieties to obtain their topology
- Apply the preceding to Matrix Singularities

i) Lie Group Methods for $\mathcal{V}$: begin with the complement and deduce the topology of the link and Milnor fiber.

ii) replacing the local Milnor fiber by a global Milnor fiber, which is a smooth affine hypersurface that has a "model complex geometry" resulting from the transitive action of an associated linear algebraic group, yielding as a deformation retract a compact submanifold;

iii) using the relation between the two algebraic group actions and the topology of maximal compact subgroups to deduce the cohomological triviality of an associated fibration of the groups;

iv) tools: Hopf structure theorem, Cartan's results on classical symmetric spaces, Wang sequence, Leray-Hirsch theorem, homotopy long exact sequence of fibration, Bott periodicity theorem;

v) using the preceding to determine the topology: (co)homology and homotopy groups of the Milnor fiber, link, and complement

Gauge-Type $\mathcal{K}$-Equivalence Groups for Singularities of Type $\mathcal{V}$:

i) $\mathcal{K}_{\mathcal{V}}$–**equivalence:** on f_0 as a "nonlinear section of $\mathcal{V}$"(gives equivalence of $\mathcal{V}_0, 0 \subset \mathbb{C}^n, 0$), via actions of pairs (Φ, φ) of (holomorphic) diffeomorphisms, preserving $\mathbb{C}^n \times \mathcal{V}$, acting on *graph*$(f_0)$.

$$\begin{array}{ccccc} \mathbb{C}^n \times \mathbb{C}^N, 0 & \xrightarrow{\Phi} & \mathbb{C}^n \times \mathbb{C}^N, 0 & \xleftarrow{i} & \mathbb{C}^n \times \mathcal{V}, 0 \\ \pi \downarrow & & \pi \downarrow & & \\ \mathbb{C}^n, 0 & \xrightarrow{\varphi} & \mathbb{C}^n, 0 & & \end{array} \tag{6}$$

ii) $\mathcal{K}_H$– **equivalence:** where Φ preserves all of the level sets of H, a "good defining equation" for $\mathcal{V}$.

iii) $\mathcal{K}_G$–**equivalence** ($\mathcal{K}_M$ **for Matrix Singularities):**

$$f_0(x) \mapsto f_1(x) \; = \; B(x) \cdot f_0 \circ \varphi(x)$$

for a representation of G on $\mathbb{C}^N$, $B(x) \in G$ acts pointwise at each x, and $\varphi : \mathbb{C}^n, 0 \to \mathbb{C}^n, 0$ is a germ of a diffeomorphism.

Prehomogeneous Spaces

Let $\rho : G \to GL(V)$ be a complex representation of a (connected) complex linear algebraic group G with an open orbit $\mathcal{U}$. Then, V is called a prehomogeneous (vector) space (due to Sato).
The complement $\mathcal{E} = V \backslash \mathcal{U}$ is the variety of orbits of positive codimension, which we call exceptional orbit variety.

Sato and Kimura: Classification of prehomogeneous spaces arising from irreducible representations of semisimple algebraic groups.

Actions of $GL_m(\mathbb{C})$ on the spaces of $m \times m$ matrices which are general (under left multiplication), or symmetric or skew-symmetric (m even) via $B \cdot A = BAB^T$. The exceptional orbit varieties are determinant varieties: hypersurfaces defined by $\det : M \to \mathbb{C}$:

$\mathcal{D}_m^{sy}$ for $M = Sym_m$; $\mathcal{D}_m$ for $M = M_{m,m}$; and
$\mathcal{D}_m^{sk}$ for $M = Sk_m$, $(m = 2k)$ defined by $Pf : Sk_m \to \mathbb{C}$.

$GL_m(\mathbb{C}) \times GL_p(\mathbb{C})$ on the space of $m \times p$ matrices $M = M_{m,p}$.

Lie Group Representations and Lie Algebra of Representation Vector Fields

Let $\rho : G \to GL(\mathbb{C}^N)$ be a representation of complex Lie group G, with Lie algebra $\mathbf{g}$. There is a natural commutative diagram (functorial under equivariant inclusions $(G, \mathbb{C}^N) \hookrightarrow (H, \mathbb{C}^Q)$).

$$
\begin{array}{ccccc}
\mathbf{g} & \xrightarrow{\tilde{\rho}} & \mathbf{gl}(\mathbb{C}^N) & \xrightarrow{\tilde{i}} & \theta_N \\
{\scriptstyle \exp}\downarrow & & {\scriptstyle \exp}\downarrow & & {\scriptstyle \exp}\downarrow \\
G & \xrightarrow{\rho} & GL(\mathbb{C}^N) & \xrightarrow{i} & \mathcal{D}_N
\end{array}
\qquad (7)
$$

Note: $\tilde{i}$ and $\tilde{\rho}$ are Lie algebra homomorphisms.
For any $u \in \mathbf{g}$ we denote the image by ξ_u and refer to it as the "associated representation vector field".
If G has an open orbit, with exceptional orbit variety $\mathcal{E}$, then the ξ_u are tangent to $\mathcal{E}$.

Extended Tangent Spaces of Equivalence Groups

For a germ $\mathcal{V}, 0 \subset \mathbb{C}^N, 0$, the *module of germs of vector fields tangent to* $\mathcal{V}$ is

$$\mathrm{Derlog}(\mathcal{V}) = \{\zeta \in \theta_N : \zeta(I(\mathcal{V})) \subseteq I(\mathcal{V})\}$$

where θ_N denotes the $\mathcal{O}_N$-module of germs of vector fields on $\mathbb{C}^N, 0$ (alternate notation: $\theta_{\mathcal{V}}$, for hypersurfaces $\mathrm{Derlog}(-\mathcal{V})$).

Let $\mathcal{D}_n$ denote the group of germs of diffeomorphisms; it has extended tangent space θ_n. If $\mathrm{Derlog}(\mathcal{V}) = \mathcal{O}_N\{\eta_1, \dots, \eta_\ell\}$, then

$$\begin{aligned} T\mathcal{K}_{\mathcal{V},e} f_0 &= df_0(\theta_n) + \mathcal{O}_n\{f_0^*(\mathrm{Derlog}(\mathcal{V})\}) \\ &= \mathcal{O}_n\{\frac{\partial f_0}{\partial x_i}, i = 1 \dots n, \eta_j \circ f_0, j = 1 \dots \ell\} \end{aligned}$$

For $\mathcal{V}$ a hypersurface, with "good defining equation" H, $T\mathcal{K}_{H,e} f_0$ is obtained by replacing $\mathrm{Derlog}(\mathcal{V})$ by

$$\mathrm{Derlog}(H) = \{\eta \in \mathrm{Derlog}(\mathcal{V}) : \eta(H) = 0\}$$

"*logarithmic tangent space*" $T_{log}\mathcal{V}_y = \{\eta(y) : \eta \in \mathrm{Derlog}(\mathcal{V})\}$ is a subspace of the tangent space T_yS for S the stratum of the canonical Whitney stratification of $\mathcal{V}$ containing y.
Likewise, $T_{log}H_y = \{\eta(y) : \eta \in \mathrm{Derlog}(H)\}$.
Then,

$$T_{log}H_y \subseteq T_{log}\mathcal{V}_y \subseteq T_yS$$

We say (after Saito) *$\mathcal{V}$ is holonomic* if the second inclusion is an equality for y in a neighborhood of 0. If moreover, both inclusions are equalities for y in a neighborhood of 0, we say *$\mathcal{V}$ is H-holonomic*.
Then, we say that f_0 is *algebraically transverse to $\mathcal{V}$* if transversality holds using $T_{log}\mathcal{V}_y$ in place of T_yS for all $y \in \mathcal{V}$.
If $\mathcal{V}$ is holonomic, then algebraic transversality to $\mathcal{V}$ is equivalent to transversality to $\mathcal{V}$ (i.e. to the canonical Whitney stratification of $\mathcal{V}$).

Relevance of Thom-Mather Approach for Matrix Singularities:

a) All three groups are "geometric subgroups of $\mathcal{A}$ or $\mathcal{K}$"so basic theorems of singularity theory apply : finite determinacy, versal unfoldings, infinitesimal stability implies def-stability, etc

b) finite $\mathcal{K}_{\mathcal{V}}$-determinacy of f_0 is equivalent to "algebraic transversality of f_0 to $\mathcal{V}$"off 0.

c) Lie group classification methods apply, including those of Bruce, Du Plessis, Wall, Kirk, etc.

d) If f_0 and the hypersurface $\mathcal{V}$ are weighted homogeneous for the same set of weights, then the extended tangent spaces of f_0 for $\mathcal{K}_{\mathcal{V}}$ and $\mathcal{K}_H$ are the same. Hence,

$$\mathcal{K}_{\mathcal{V},e} - \text{codim}(f_0) = \mathcal{K}_{H,e} - \text{codim}(f_0)$$

e) For $m \times m$ matrices which are general, symmetric, or skew-symmetric (m even), $\mathcal{K}_M$ and $\mathcal{K}_{\mathcal{V}}$ have the same tangent spaces (by Józefiak, , Pragacz, Gulliksen, and Negård); hence they give the same equivalence.

Free Divisors

Definition: A hypersurface germ $\mathcal{V}, 0 \subset \mathbb{C}^p$ is a *free divisor* if Derlog$(\mathcal{V})$ is a free $\mathcal{O}_p$-module (necessarily) of rank p.

Saito's Criterion ('80): If there are p vector fields $\zeta_i = \sum_i b_{i,j} \dfrac{\partial}{\partial z_j}$ in Derlog$(\mathcal{V})$ so that the "coefficient determinant" $\det(b_{i,j})$ defines $\mathcal{V}$ with reduced structure then $\mathcal{V}$ is a free divisor and Derlog$(\mathcal{V})$ is a free $\mathcal{O}_p$-module generated by the $\{\zeta_i\}$.

Basic Examples : Discriminants (Saito, Looijenga), Bifurcation sets (Terao, Bruce), Hyperplane arrangements (Terao), see e.g. survey D- ('01)

General Theorem (D- '98, '03, '06): For a geometric subgroup $\mathcal{G}$ of $\mathcal{A}$ or $\mathcal{K}$:

If $\mathcal{G}$ is "*Cohen-Macaulay and generically has Morse-Type singularities*", then $\mathcal{G}$-discriminants for $\mathcal{G}$-versal unfoldings are free divisors.

Equidimensional Representations: $\dim G = \dim V$.
Then, $\mathcal{E}$ is a hypersurface.
Examples:

i) Reductive groups: include quivers of finite representation type. The exceptional orbit variety is called the the "discriminant". These are linear free divisors (Buchweitz and Mond)

ii) Solvable linear algebraic groups: *block representations* criteria for exceptional orbit varieties being free or free* divisors. Examples: (modified) Cholesky factorizations (D'and B. Pike)

iii) General linear algebraic groups formed as extensions of reductive groups by solvable linear algebraic groups: block representations yielding exceptional orbit varieties free or free* divisors.

Overview of the Strategy for Nonisolated Matrix Singularities

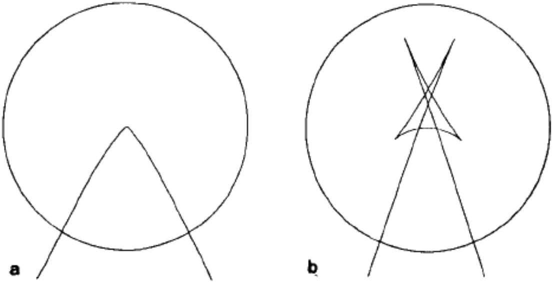
1) Suppose $\mathcal{V}, 0$ is a holonomic hypersurface singularity and f_0 has finite $\mathcal{K}_{\mathcal{V}}$-codimension. A stabilization $f_t : B_\varepsilon \to \mathbb{C}^N$ of f_0 is a perturbation of f_0 so for $0 < t < \delta$, f_t is tranverse to $\mathcal{V}$.

2) $\mathcal{V}_t = f_t^{-1}(\mathcal{V})$ is the "singular Milnor fiber" for a stabilization f_t. By results of D- and Mond ('91) (using a result of Lê ('86)), $\mathcal{V}_t$ is homotopy equivalent to a bouquet of spheres of real dimension $n-1$, whose number we denote by $\mu_{\mathcal{V}}(f_0)$.

3) If moreover $\mathcal{V}$ is an H-holonomic free divisor, then the results in D- and Mond ('91) and D- ('96) (using a result of Siersma ('91)), give a formula for the singular Milnor number

$$\mu_{\mathcal{V}}(f_0) \quad = \quad \mathcal{K}_{H,e} - \mathrm{codim}(f_0)$$

4) If $n < \mathrm{codim}(\mathrm{sing}(\mathcal{V}))$, then the perturbation f_t misses $\mathrm{sing}(\mathcal{V})$, so $f_t^{-1}(\mathcal{V})$ is smooth and gives a Milnor fiber of $H \circ f_0$.



Discriminant of $f_{\mathbf{R},0}(x, y) = (x, xy + y^6)$; **b** Discriminant of the stable perturbation $f_{\mathbf{R},t}$ of $f_{\mathbf{R},0}$

a) Discriminant of a germ of $\mathcal{A}_e$-codimension 6, which is a section i of the discriminant Δ of stable unfolding of the germ. b) Resulting "Discriminant Milnor Fiber" is the "Singular Milnor Fiber" arising from the stabilization of the section i, which (even in the real case) is homotopy equivalent to a bouquet of 6 S^1's.

$$6 = \mathcal{A}_e - \text{codim}(f_0) = \mathcal{K}_{\Delta,e} - \text{codim}(i) = \mathcal{K}_{H,e} - \text{codim}(i)$$

Bruce ('03) observed a "$\mu = \tau$"-type result:
Simple symmetric matrix singularities f_0, $n = 2 < \text{codim}(\text{sing}(\mathcal{V}))$ for $\mathcal{V} = \mathcal{D}_2^{sy}$, (are weighted homogeneous, $H \circ f_0 (= det(f_0)$ define simple hypersurface singularities)

$$(\mu_{\mathcal{V}}(f_0) =)\, \mu(H \circ f_0) \;=\; \mathcal{K}_{M,e} - \text{codim}(f_0) (= \mathcal{K}_{H,e} - \text{codim}(f_0))$$

The Variety of singular $m \times m$ matrices $\mathcal{D}_m^*$ is essentially never a free divisor. In general, corrections are thus needed for the formula $\mu_{\mathcal{V}}(f_0)$. There are two different approaches for corrections:

- Goryunov and Mond ('05) for small n and arbitrary size matrices; or
- D- and Pike ('14) for arbitrary n but small size (including 2×3) matrices.

Theorem (Goryunov-Mond ('05))

For any of the three types of $m \times m$ matrices (with m even in skew-symmetric case) provided $n \leq \text{codim}(\text{sing}(\mathcal{V}))$ ($= 3$ symmetric case, 6 skew-symmetric case, or 4 for general case). Then, the Milnor number of the isolated singularity $H \circ f_0$ is given by

$$\mu(H \circ f_0) \;=\; \tau + (\beta_0 - \beta_1)$$

where $\tau = \mathcal{K}_{H,e} - \text{codim}(f_0)$ and $\beta_0 - \beta_1$ is a two term Euler characteristic with $\beta_i = \text{rank}(\text{Tor}_i^{\mathcal{O}_N}(\mathcal{O}_N/J(H), \mathcal{O}_n))$, where $N = \dim M$.

In addition, in the case $n = \text{codim}(\text{sing}(\mathcal{V})) - 1$, the correction term vanishes (implying the observed results of Bruce in the symmetric case).

Theorem (D - Pike ('14))

For any of the three types of $m \times m$ matrices (with m even in skew-symmetric case) for small m, but for any n in each case, the singular Milnor number $\mu_{\mathcal{V}}(f_0)$ for $\mathcal{V} = \mathcal{D}_m^*$ is given as a linear combination with integer coefficients

$$\mu_{\mathcal{V}}(f_0) \;=\; \sum_i a_i \, \mu_{\mathcal{W}_i}(f_0) \tag{8}$$

where the $\mathcal{W}_i$ are associated free divisors on linear subspaces of M. Each is computed as a length of a determinantal module.

This extends to a formula for the vanishing Euler characteristic for the singular Milnor fiber of f_0 for $\mathcal{V} = \mathcal{D}_{2,3} \subset M_{2,3}$.

Example: For germs $f_0 = (a_{i,j}) \in Sym_2$ transverse off 0 to the associated free divisor $\mathcal{E}_2^{sy}$ given by $a_{1,1} \cdot \det(a_{i,j}) = 0$,

$$\mu_{\mathcal{D}_2^{sy}}(f_0) \;=\; \mu_{\mathcal{E}_2^{sy}}(f_0) \,-\, (\mu(a_{1,1}) \,+\, \mu(a_{1,1}, a_{1,2}))$$

$\mu_{\mathcal{E}_2^{sy}}(f_0) = \operatorname{codim}(T\mathcal{K}_{B_2,e} f_0)$ and $(\mu(a_{1,1}) \,+\, \mu(a_{1,1}, a_{1,2}))$ is the length of a determinantal module by the Lê-Greuel formula.

Characteristic Cohomology for Singularities of Type $\mathcal{V}$

Given a singularity $\mathcal{V}_0$ of type $\mathcal{V}$ defined by $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$.

Milnor Fibers (for the hypersurface $\mathcal{V}$):
with $H : \mathbb{C}^N, 0 \to \mathbb{C}, 0$ the defining equation, and for $0 < \eta << \delta, \varepsilon$ sufficiently small so $f_0(B_\varepsilon) \subset B_\delta$,
$H^{-1}(B^*_\eta) \cap B_\delta \to B^*_\eta$ is a Milnor fibration for H and
$(H \circ f_0)^{-1}(B^*_\eta) \cap B_\varepsilon \to B^*_\eta$ is the Milnor fibration for $H \circ f_0$. Also, we may arrange for $w \in B^*_\eta$, both $F_w = H^{-1}(w) \cap B_\delta$ is a Milnor fiber for $\mathcal{V}$; $\mathcal{V}_w = f_0^{-1}(F_w) \cap B_\varepsilon$ is the Milnor fiber of $\mathcal{V}_0$, and $\widetilde{f}_0 = f_0 | \mathcal{V}_w : \mathcal{V}_w \to F_w$.

$$\mathcal{A}_{\mathcal{V}}(f_0, R) \overset{def}{=} \widetilde{f}_0^*(H^*(F_w; R))$$

is the characteristic subalgebra of $H^*(\mathcal{V}_w; R)$. This is independent (up to Milnor fiber cohomology algebra isomorphism) of choices for sufficiently small $0 < \eta << \delta, \varepsilon$ and $w \in B^*_\eta$.

Characteristic Cohomology for Complements and Links

Complements and Links (for general $\mathcal{V}$) :
$S^{2N-1}_{\delta'}$ is transverse to $\mathcal{V}$ for $0 < \delta' < \delta$, so $(B_\delta, B_\delta \cap \mathcal{V})$ is homeomorphic to the cone on $(S^{2N-1}_{\delta}, S^{2N-1}_{\delta} \cap \mathcal{V})$.
Similarly, for $0, \varepsilon' < \varepsilon$, $S^{2n-1}_{\varepsilon'}$ is transverse to $\mathcal{V}_0$, and $(B_\varepsilon, B_\varepsilon \cap \mathcal{V}_0)$ is homeomorphic to the cone on $(S^{2n-1}_{\varepsilon}, S^{2n-1}_{\varepsilon} \cap \mathcal{V}_0)$.

Then, for each δ' and ε' with $f_0(\overline{B}_{\varepsilon'}) \subset B_{\delta'}$, we have an induced map on cohomology

$$f_0^* : H^*(\overline{B}_{\delta'} \backslash \mathcal{V}; R) \to H^*(\overline{B}_{\varepsilon'} \backslash \mathcal{V}_0; R)$$

Then,

$$\mathcal{C}_{\mathcal{V}}(f_0, R) \overset{def}{=} \varinjlim f_0^*(H^*(B_{\delta'} \backslash \mathcal{V}); R)$$

is a characteristic subalgebra of $H^*(\mathbb{C}^n \backslash \mathcal{V}_0; R)$ (note this is in local cohomology as $\mathcal{V}_0$ is a germ).

Characteristic Cohomology for Links

By Alexander Duality for Taut subspaces of spheres (for $\mathbf{k}$ a field of characteristic 0) for $(S^{2n-1}_{\varepsilon'}, S^{2n-1}_{\varepsilon'} \cap \mathcal{V}_0)$

$$\alpha : \widetilde{H}^q(S^{2n-1}_{\varepsilon'} \cap \mathcal{V}_0; \mathbf{k}) \simeq \widetilde{H}_{2n-2-q}(S^{2n-1}_{\varepsilon'} \backslash \mathcal{V}_0; \mathbf{k}) \simeq \widetilde{H}^{2n-2-q}(S^{2n-1}_{\varepsilon'} \backslash \mathcal{V}_0; \mathbf{k})$$

The characteristic cohomology of the Link in $H^*(L(\mathcal{V}_0); \mathbf{k})$

$$\mathcal{B}_{\mathcal{V}}(f_0, \mathbf{k}) \overset{def}{=} \varinjlim \alpha^{-1} \circ j^*_{\varepsilon'}(f^*_0(\widetilde{H}^*(B_{\delta'} \backslash \mathcal{V}); \mathbf{k})))$$

for the homotopy equivalence $j_{\varepsilon'} : S^{2n-1}_{\varepsilon'} \backslash \mathcal{V}_0 \subset \overline{B}_{\varepsilon'} \backslash \mathcal{V}_0$.
This is independent of $\varepsilon' < \varepsilon$

Unlike $\mathcal{A}_{\mathcal{V}}(f_0, R)$ and $\mathcal{C}_{\mathcal{V}}(f_0, R)$, which are subalgebras, $\mathcal{B}_{\mathcal{V}}(f_0, \mathbf{k})$ is only a graded $\mathbf{k}$-vector subspace.

Functoriality and Invariance for Characteristic Cohomology (D - '21)

Functoriality : Both $\mathcal{A}_{\mathcal{V}}(f_0, R)$ and $\mathcal{C}_{\mathcal{V}}(f_0, R)$ are functorial on the category of singularities of type $\mathcal{V}$.
$\mathcal{B}_{\mathcal{V}}(f_0, \mathbf{k})$ is not functorial (instead Relative Gysin homomorphism).

Invariance : On the space of nonlinear sections $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$, $\mathcal{K}_{\mathcal{V}}$-equivalence induces an ambient equivalence between the singularities $\mathcal{V}_0 = f_0^{-1}(\mathcal{V})$. The subgroup $\mathcal{K}_H$ further preserves the Milnor fibers for $\mathcal{V}$ a hypersurface.

Theorem :

- $\mathcal{A}_{\mathcal{V}}(f_0, R)$ is, up to algebra isomorphism of the cohomology of Milnor fibers, preserved under $\mathcal{K}_H$-equivalence.
- $\mathcal{C}_{\mathcal{V}}(f_0, R)$ is, up to algebra isomorphism of the cohomology of the complement, preserved under $\mathcal{K}_{\mathcal{V}}$-equivalence.
- $\mathcal{B}_{\mathcal{V}}(f_0, \mathbf{k})$ is, up to group isomorphism of the cohomology of the link, preserved under $\mathcal{K}_{\mathcal{V}}$-equivalence.

Analogy with Characteristic Cohomology for Vector Bundles

For a vector bundle $\xi = (E, Y, G)$ with $p : E \to Y$ a vector bundle with structure group G (e.g. O_n, U_n, Sp_n, SO_n, etc.):

a) E is classified via a map $f : Y \to BG$ by a pullback $E = f^*(\gamma_G)$ for γ_G the canonical vector bundle over BG, the classifying space for G;

b) every f' homotopic to f defines, up to isomorphism, the same vector bundle;

c) for an appropriate coefficient ring R, the cohomology $H^*(BG; R) \simeq R[b_1, \ldots, b_k]$, where in most cases there are no relations between the generators;

d) the $b_i(E) = f^*(b_i)$ give the appropriate characteristic classes of E, which are functorial under pullback by $g : X \to Y$.

e) the $b_i(E)$ generate a "characteristic subalgbra" of $H^*(Y; R)$

f) in the case Y is a compact manifold and $E = TY$ the tangent bundle, the $b_i(E)$ capture properties of Y.

Characteristic Cohomology for Matrix Singularities

Let M denote the space of complex $m \times m$ matrices which are either general, symmetric or skew-symmetric with $\mathcal{D}_m^{(*)}, 0 \subset M, 0$ the variety of singular matrices (general (), symmetric (sy), skew-symmetric (sk) (m even)) with reduced defining equations $H = \det$ or Pf.

Given a matrix singularity $\mathcal{V}_0, 0$ defined by $f_0; \mathbb{C}^n, 0 \to M, 0$ for $\mathcal{V} = \mathcal{D}_m^{(*)}$, we denote the characteristic cohomology by

$$\begin{aligned}\mathcal{A}^{(*)}(f_0, R) &= \mathcal{A}_{\mathcal{D}_m^{(*)}}(f_0, R)\\ \mathcal{C}^{(*)}(f_0, R) &= \mathcal{C}_{\mathcal{D}_m^{(*)}}(f_0, R)\\ \mathcal{B}^{(*)}(f_0, \mathbf{k}) &= \mathcal{B}_{\mathcal{D}_m^{(*)}}(f_0, \mathbf{k})\end{aligned}$$

General or Skew-symmetric cases: $R = \mathbb{Z}$ (and hence for any R); Symmetric case: either $R = \mathbf{k}$ a field of characteristic 0, or for the case of Milnor fiber also $R = \mathbb{Z}/2\mathbb{Z}$.

Topology of Exceptional Orbit Varieties

General Approach for Exceptional Orbit Hypersurface $\mathcal{E}$:

i) Local Milnor fiber $\overset{diff}{\simeq}$ Global Milnor fiber (via Morse theory).

ii) Global Milnor fiber $\overset{holo\ diff}{\simeq}$ homogeneous space G'/H' of linear algebraic groups.

iii) $G'/H' \simeq_{h.e.} K'/L'$, maximal compact subgroups yielding a compact model for Milnor fiber.

iv) More generally for Exceptional Orbit Varieties, obtain a similar decomposition $\mathbb{C}^N \backslash \mathcal{E} \overset{holo\ diff}{\simeq} G/H$, and obtain a compact model for the complement.

v) Using iii) compute (co)homology and homotopy groups of Milnor fiber, and with iv), for the complement and link and deduce (rational) cohomological triviality of monodromy.

Milnor Fibers of the Determinant Varieties

Theorem (D- '16): The Milnor fibers of the determinant hypersurfaces are homogeneous spaces homotopy equivalent to classical symmetric spaces (**k** a field of characteristic 0).

Determinant Variety	Milnor Fiber F	Symmetric Space	$H^*(F, \mathbf{k})$
$\mathcal{D}_m^{sy}$ (m = 2k+1)	$SL_m(\mathbb{C})/SO_m(\mathbb{C})$	SU_m/SO_m	$\Lambda^*\mathbf{k}\{e_5, e_9, \dots, e_{4k+1}\}$
$\mathcal{D}_m^{sy}$ (m = 2k)	$SL_m(\mathbb{C})/SO_m(\mathbb{C})$	SU_m/SO_m	$\Lambda^*\mathbf{k}\{e_5, e_9, \dots, e_{4k-3}\}\cdot \{1, e_{2k}\}$
$\mathcal{D}_m$	$SL_m(\mathbb{C})$	SU_m	$\Lambda^*\mathbf{k}\{e_3, e_5, \dots, e_{2m-1}\}$
$\mathcal{D}_m^{sk}, m = 2k$	$SL_{2k}(\mathbb{C})/Sp_k(\mathbb{C})$	SU_{2k}/Sp_k	$\Lambda^*\mathbf{k}\{e_5, e_9, \dots, e_{4k-3}\}$

In general, skew-symmetric cases, replace **k** by $\mathbb{Z}$; symmetric case

$$H^*(F_m^{(sy)}; \mathbb{Z}/2\mathbb{Z}) \simeq \Lambda^*\mathbb{Z}/2\mathbb{Z}\{e_2, e_3, \dots, e_m\} \qquad \text{where } e_i = w_i(\tilde{E}_m)$$

Theorem (D- '16): The cohomology of the complements and links of determinant hypersurfaces are given by the following table, where for the link the cohomology $H^*(K/L, \mathbf{k})$ is upper truncated and shifted.

Determinant Variety	Complement $M \backslash \mathcal{D}$	$H^*(M \backslash \mathcal{D}, \mathbf{k}) \simeq H^*(K/L, \mathbf{k})$	Shift
$\mathcal{D}_m^{sy}$ (m = 2k+1)	$GL_m(\mathbb{C})/O_m(\mathbb{C}) \sim U_m/O_m(\mathbb{R})$	$\Lambda^*\mathbf{k}\langle e_1, e_5, \ldots, e_{2m-1}\rangle$	$\binom{m+1}{2} - 2$
$\mathcal{D}_m^{sy}$ (m = 2k)	$GL_m(\mathbb{C})/O_m(\mathbb{C}) \sim U_m/O_m(\mathbb{R})$	$\Lambda^*\mathbf{k}\langle e_1, e_5, \ldots, e_{2m-3}\rangle$	$\binom{m+1}{2}+$ m - 2
$\mathcal{D}_m$	$GL_m(\mathbb{C}) \sim U_m$	$\Lambda^*\mathbf{k}\langle e_1, e_3, \ldots, e_{2m-1}\rangle$	$m^2 - 2$
$\mathcal{D}_m^{sk}$ (m = 2k)	$GL_{2k}(\mathbb{C})/Sp_k(\mathbb{C}) \sim U_{2k}/Sp_k$	$\Lambda^*\mathbf{k}\langle e_1, e_5, \ldots, e_{2m-3}\rangle$	$\binom{m}{2} - 2$

Numerical Criterion for cohomological triviality of monodromy is satisfied for all cases except $\mathcal{D}_m^{sy}$ for m even.

Vanishing Compact Models for Milnor Fibers and Complements

Def: Vanishing Compact Model for Milnor fibration for $\mathcal{V}$
There is a compact space $Q_{\mathcal{V}}$, smooth curves $\gamma(t) : [0, \eta) \to B_\eta$ satisfying $|\gamma(t)| = t$, $\beta(t) : [0, \eta) \to [0, \delta_0)$, and an embedding

$$\Phi : Q_{\mathcal{V}} \times (0, \delta) \hookrightarrow H^{-1}(B^*_\eta) \cap B_{\delta_0}$$

- **i)** each $H : H^{-1}(\overline{B}^*_{|\gamma(t)|}) \cap B_{\beta(t)} \to \overline{B}^*_{|\gamma(t)|}$ is again a Milnor fibration for H
- **ii)** each $\Phi(Q_{\mathcal{V}} \times \{t\}) \subset F_w$ is a homotopy equivalence, for F_w the Milnor fiber of i) over $w = \gamma(t)$.

Analogous Vanishing Compact Model for Complement of $\mathcal{V}$
There is a compact space $P_{\mathcal{V}}$, a smooth curve
$\gamma(t) : [0, \delta) \to [0, \delta_0)$, and an embedding into the complement of $\mathcal{V}$,
$\Phi : P_{\mathcal{V}} \times (0, \delta) \hookrightarrow B_{\delta_0} \backslash \mathcal{V}$ satisfying analogue of ii) but for the case of the complements $B_{\gamma(t)} \backslash \mathcal{V}$.

Detection Lemmas for Non-Vanishing of Characteristic Cohomology

We say a graded subgroup $E \subset H^*(Q_{\mathcal{V}}; R)$ is detected by a compact subspace $\lambda_E : Q_E \subset Q_{\mathcal{V}}$ if $\lambda_E^* : H^*(Q_{\mathcal{V}}; R) \to H^*(Q_E; R)$ induces an isomorphism from E to $H^*(Q_E; R)$.

Germ of an embedding $i_E : \mathbb{C}^s, 0 \to \mathbb{C}^N, 0$ detects E if for sufficiently small $0 < \eta << \varepsilon < \delta$ there is a vanishing compact model $\Psi : Q_E \times (0, \delta) \hookrightarrow (H \circ i_E)^{-1}(B_\eta^*) \cap B_\varepsilon$ for the Milnor fibration of $H \circ i_E$ so that $i_E \circ \Psi = \Phi \circ (\lambda_E \times id)$.

Detection Lemma for Milnor Fibers: Let $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$ define $\mathcal{V}_0, 0$ of type $\mathcal{V}$. If $g : \mathbb{C}^s, 0 \to \mathbb{C}^n, 0$, satisfies $f_0 \circ g$ is $\mathcal{K}_H$-equivalent to a germ detecting E, then, $\mathcal{A}_{\mathcal{V}}(f_0, R)$ contains a graded subgroup which is isomorphic to E via $f_0^* : H^*(F_w; R) \to H^*(\mathcal{V}_w; R)$.

There is an analogous detection lemma for complements and links.

Cartan Models for Symmetric Spaces

Cartan Model for a symmetric space represented by $Y = G/G^{\sigma}$ for a compact Lie group G with σ an involution of G: submanifold $\mathcal{C} \subset G$ defined by the embedding $G/G^{\sigma} \to \mathcal{C} \overset{def}{=} \{g \cdot \sigma(g^{-1}) : g \in G\}$ sending $g \mapsto g \cdot \sigma(g^{-1})$.

- For $F_m^c = SU_m$, $G = SU_m$ is its own Cartan model
- For $F_m^{(sy),c} = SU_m/SO_m$, $\sigma(A) = \overline{A}$ and the Cartan model is $\mathcal{C}_m^{(sy)} = SU_m \cap Sym_m(\mathbb{C})$;
- For $F_m^{(sk),c} = SU_m/Sp_n$, $(m = 2n)$, $\sigma(A) = J_n \cdot \overline{A} \cdot J_n^{-1}$ and the Cartan model is $\mathcal{C}_m^{(sy)} = (SU_m \cap Sk_m(\mathbb{C})) \cdot J_n^{-1}$.

Theorem (D- '18): Via the Cartan models there are Schubert decompositions for the compact models and extensions to the Milnor fibers. These are directly identified as Kronecker duals to the cohomology generators.

Compact Models for Milnor Fibers of Determinant Hypersurfaces

Global Milnor fiber; its representation as a homogenenous space; compact model as a symmetric space, compact model as subspace and Cartan model.

Milnor Fiber	Quotient Space	Symmetric Space	Compact Model $F_m^{*,c}$	Cartan Model
F_m	$SL_m(\mathbb{C})$	SU_m	SU_m	F_m^c
F_m^{sy}	$SL_m(\mathbb{C})/SO_m(\mathbb{C})$	SU_m/SO_m	$SU_m \cap Sym_m(\mathbb{C})$	$F_m^{sy,c}$
F_m^{sk} $m=2n$	$SL_{2n}(\mathbb{C})/Sp_n(\mathbb{C})$	SU_{2n}/Sp_n	$SU_m \cap Sk_m(\mathbb{C})$	$F_m^{sk,c} \cdot J_n^{-1}$

For matrix singularities, $Q_{\mathcal{V}} = F_m^{(*),c}$, with vanishing compact models $\Phi(A, t) \mapsto t \cdot A$, and $\ell < m$, $\lambda_E : F_\ell^{(*),c} \to F_m^{(*),c}$ for $E = H^*(F_\ell^{(*),c}; R)^{!!}$

Flags and Kite Maps for General and Symmetric Cases

We define the detection maps for matrix singularities using the standard flag $0 \subset \mathbb{C} \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^m$.

Form of elements of a kite space $\mathbf{K}_m^{(*)}(\ell)$ of size ℓ in either the space of general matrices () or symmetric matrices (sy). For general matrices the upper left matrix of size $\ell \times \ell$ is a general matrix, while for symmetric matrices it is symmetric.

$$\begin{pmatrix} * & \cdots & * & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & 0 & \cdots & 0 \\ * & \cdots & * & 0 & \cdots & 0 \\ 0 & \cdots & 0 & * & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & * \end{pmatrix}$$

A kite map of size ℓ is given by inclusion $i_m^{(*)}(\ell) : \mathbf{K}_m^{(*)}(\ell) \hookrightarrow M_m^{(*)}$.

An unfurled kite map of size ℓ is a germ in the $\mathcal{K}_{HM}$-orbit of $i_m^{(*)}(\ell)$.

Kite Maps for Skew-symmetric Case

Form of elements of a linear skew-symmetric kite space of size ℓ (with ℓ even) in the space of skew-symmetric matrices. The upper left $\ell \times \ell$ matrix is a skew-symmetric matrix

$$\begin{pmatrix} * & \cdots & * & 0 & \cdots & 0 \\ * & \cdots & * & 0 & \cdots & 0 \\ * & \cdots & * & 0 & \cdots & 0 \\ 0 & \cdots & 0 & J_1(*) & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & J_1(*) \end{pmatrix}$$

where $J_1(*) = \begin{pmatrix} 0 & * \\ -* & 0 \end{pmatrix}$

There are corresponding skew-symmetric kite maps of size ℓ given by the inclusion $i_m^{(sk)}(\ell) : \mathbf{K}_m^{(sk)}(\ell) \hookrightarrow Sk_m(\mathbb{C})$, and unfurled skew-symmetric kite maps of size ℓ.

Examples of Kite Maps

Linear Kite Map of size 4 into 5×5 general matrices

$$\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & 0 \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & 0 \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & 0 \\ 0 & 0 & 0 & 0 & x_{5,5} \end{pmatrix}$$

Unfurled kite map of size 3 into 4×4 symmetric matrices

$$\begin{pmatrix} x_{1,1} + 2x_{4,4}x_{1,3} & x_{1,2} + 2x_{4,4}x_{2,3} & x_{1,3} + 2x_{4,4}x_{3,3} & 0 \\ x_{1,2} + 2x_{4,4}x_{2,3} & x_{2,2} - 35x_{1,2}x_{4,4} & x_{2,3} & (7x_{1,2} - 5)x_{4,4} \\ x_{1,3} & x_{2,3} & x_{3,3} & 0 \\ 0 & (7x_{1,2} - 5)x_{4,4} & 0 & x_{4,4} \end{pmatrix}$$

Detecting Nonvanishing Characteristic Cohomology using Kite Maps

Theorem (D- '21): Let $f_0 : \mathbb{C}^n, 0 \to M, 0$ define an $m \times m$ matrix singularity of one of the three types containing an unfurled kite map of size ℓ of the corresponding type (i.e. there exists $g : \mathbf{K}_m^{(*)}(\ell), 0 \to \mathbb{C}^n, 0$ so that $f_0 \circ g$ is $\mathcal{K}_{HM}$-equivalent to $i_m^{(*)}(\ell)$).

a) For general matrices,

$$\mathcal{A}(f_0, R) \; \supseteq \; \Lambda^* R\langle e_3, e_5, \dots, e_{2\ell-1}\rangle \, .$$

b) For skew-symmetric matrices (with m, ℓ both even),

$$\mathcal{A}^{(sk)}(f_0, R) \; \supseteq \; \Lambda^* R\langle e_5, e_9, \dots, e_{2\ell-3}\rangle \, .$$

c) For symmetric matrices,

$$\mathcal{A}^{(sy)}(f_0, \mathbf{k}) \; \supseteq \Lambda^* \mathbf{k}\langle e_3, e_5, \dots, e_{2\ell-1}\rangle \qquad char(\mathbf{k}) = 0,$$
$$\mathcal{A}^{(sy)}(f_0, \mathbb{Z}/2\mathbb{Z}) \; \supseteq \Lambda^* \mathbb{Z}/2\mathbb{Z}\langle e_2, e_3, \dots, e_\ell\rangle \qquad (e_j = w_j(f_0^*(\tilde{E}_m))$$

Nonvanishing Characteristic Cohomology for Complements and Links

Theorem (D- '21): Let $f_0 : \mathbb{C}^n, 0 \to M, 0$ define a matrix singularity $\mathcal{V}_0$ of any of the three types. If f_0 contains a kite map of size ℓ, then, $\mathcal{C}^{(*)}(f_0, \mathbf{k})$ contains an exterior algebra given by the Table. Also, $\mathcal{B}^{(*)}(f_0, \mathbf{k})$, as a graded vector space contains the graded subspace given by the Table by truncating the subalgebra in the top degree and shifting by the amount listed in the last column.

Determinantal Hypersurface Type	$\mathcal{C}^{(*)}(f_0, \mathbf{k})$ contains subalgebra	Shift for Link
$\mathcal{D}_m^{sy}$ ℓ odd	$\Lambda^*\mathbf{k}\langle e_1, e_5, \dots, e_{2\ell-1}\rangle$	$2n - \binom{\ell+1}{2} - 2$
$\mathcal{D}_m^{sy}$ ℓ even	$\Lambda^*\mathbf{k}\langle e_1, e_5, \dots, e_{2\ell-3}\rangle$	$2n - \binom{\ell}{2} - 2$
$\mathcal{D}_m$	$\Lambda^*\mathbf{k}\langle e_1, e_3, \dots, e_{2\ell-1}\rangle$	$2n - \ell^2$ - 2
$\mathcal{D}_m^{sk}$ (m = 2k) ℓ even	$\Lambda^*\mathbf{k}\langle e_1, e_5, \dots, e_{2\ell-3}\rangle$	$2n - \binom{\ell}{2} - 2$

Example 1

$f_0(\mathbf{x},\mathbf{y}); \mathbb{C}^{21}, 0 \to M_5(\mathbb{C}), 0$ where $\mathbf{x} = (x_{1,1}, \ldots, x_{4,4}, x_{5,5})$ and $\mathbf{y} = (y_1, y_2, y_3, y_4)$ and $g_i(\mathbf{x}, 0) = 0$

$$\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & g_1(\mathbf{x},\mathbf{y}) \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & g_2(\mathbf{x},\mathbf{y}) \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & g_3(\mathbf{x},\mathbf{y}) \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & g_4(\mathbf{x},\mathbf{y}) \\ y_1 & y_2 & y_3 & y_4 & x_{5,5} \end{pmatrix}$$

Then, f_0 contains a kite map $i_5(4)$ and by the Theorems.

$$\begin{aligned} \mathcal{A}(f_0, R) &\supseteq \Lambda^* R\langle e_3, e_5, e_7 \rangle \\ \mathcal{C}(f_0, R) &\supseteq \Lambda^* R\langle e_1, e_3, e_5, e_7 \rangle \\ \mathcal{B}(f_0, \mathbf{k}) &\supseteq \widetilde{\Lambda^* \mathbf{k}}\langle e_1, e_3, e_5, e_7 \rangle[24] \end{aligned}$$

Example 2

$f_0(\mathbf{x}, \mathbf{y}); \mathbb{C}^9, 0 \to Sym_4(\mathbb{C}), 0$ where
$\mathbf{x} = (x_{1,1}, x_{1,2}, x_{1,3}, x_{2,2}, x_{2,3}, x_{3,3}, x_{4,4})$ and $\mathbf{y} = (y_1, y_2)$.

$$\begin{pmatrix} x_{1,1} + 2x_{4,4}x_{1,3} & x_{1,2} + 2x_{4,4}x_{2,3} & x_{1,3} + 2x_{4,4}x_{3,3} & y_1 \\ x_{1,2} + 2x_{4,4}x_{2,3} & x_{2,2} - 35x_{1,2}x_{4,4} & x_{2,3} + y_1 x_{1,1}^2 & (7x_{1,2} - 5)x_{4,4} \\ x_{1,3} + 2x_{4,4}x_{3,3} & x_{2,3} + y_1 x_{1,1}^2 & x_{3,3} + y_2 x_{2,2}^2 & y_2 \\ y_1 & (7x_{1,2} - 5)x_{4,4} & y_2 & x_{4,4} \end{pmatrix}$$

Then, f_0 contains an unfurled kite map $i_4^{(sy)}(3)$ when $\mathbf{y} = (0, 0)$ so by the Theorems.

$$\begin{aligned} \mathcal{A}^{(sy)}(f_0, \mathbb{Z}/2\mathbb{Z}) &\supseteq \Lambda^*\mathbb{Z}/2\mathbb{Z}\langle e_2, e_3\rangle \\ \mathcal{A}^{(sy)}(f_0, \mathbf{k}) &\supseteq \Lambda^*\mathbf{k}\langle e_5\rangle \\ \mathcal{C}^{(sy)}(f_0, \mathbf{k}) &\supseteq \Lambda^*\mathbf{k}\langle e_1, e_5\rangle \\ \mathcal{B}^{(sy)}(f_0, \mathbf{k}) &\supseteq \widetilde{\Lambda^*\mathbf{k}}\langle e_1, e_5\rangle[10] \end{aligned}$$

Open Questions Regarding Topology of Singularities of Type $\mathcal{V}$

If R is a field of characteristic 0, then for a general hypersurface singularity we write the cohomology of the singular Milnor fiber of f_0 as a direct sum.

$$H^*(\mathcal{V}_w; R) \simeq \mathcal{A}_{\mathcal{V}}(f_0, R) \oplus \mathcal{W}_{\mathcal{V}}(f_0, R) \tag{9}$$

We then may ask several questions about the properties of the summand $\mathcal{W}_{\mathcal{V}}(f_0, R)$.

i) Does $R^{\mu}[n-1]$ for $\mu = \mu_{\mathcal{V}}(f_0)$ occur as a summand?

ii) Does $\mathcal{W}_{\mathcal{V}}(f_0, R)$ vanish below degree $n-1$?

iii) If i) holds, is there an additional contribution in degree $n-1$ to $\mathcal{W}_{\mathcal{V}}(f_0, R)$?

iv) If ii) does not hold, can $\mathcal{W}_{\mathcal{V}}(f_0, R)$ be chosen to be an $\mathcal{A}_{\mathcal{V}}(f_0, R)$-submodule?

Open Questions Regarding Topology of Matrix Singularities

1) For the monomial classes outside of the detected nonvanishing exterior algebras, determine which ones map to nonzero elements in $H^*(\mathcal{V}_w; R)$, resp. $H^*(\mathbb{C}^n \backslash \mathcal{V}_0; R)$.
2) Determine the nonvanishing cohomology of the link outside of $\mathcal{B}^{(*)}(f_0; \mathbf{k})$ (Role of Lê-Hamm Local Lefschetz theorem ?)
3) For the symmetric case with $\mathbb{Z}/2\mathbb{Z}$-coefficients, determine the remaining nonvanishing Stiefel-Whitney classes of the vector bundle $f_0^*(\tilde{E}_m)$.
4) Determine the answers to i)- iv) for matrix singularities

Bibiography I: Topology of Milnor Fibers of Nonisolated Singularities

This bibliography is to supplement those provided by Terry Gaffney, Maria Ruas, and Matthias Zach. In addition, there is a further bibliography in the preprint about Thom-Mather theory (below).

Kato, M. and Matsumoto, Y. *On the Connectivity of the Milnor Fiber of a Holomorphic Function at a Critical Point*, Manifolds-Tokyo 1973 (Proc. Int'l. Conf., Tokyo, 1973), Univ. Tokyo Press (1975) 131–136.

Lê Dũng Tráng *Le concept de singularité isolée de fonction analytique* Adv. studies in Pure Math. **8** (1986), 215-227.

Siersma, D. *The Vanishing Topology of Non-isolated Singularities* New Developments in Singularity Theory, Eds. D. Siersma, C. T. C. Wall, V. Zakalyukin, Nato Science Series **21**, Kluwer Acad. Publ. (2001), 447-472.

Bibiography II: Equivalence Groups

Damon, J. *Deformations of Sections of Singularities and Gorenstein Surface Singularities* Amer. J. Math. **109** (1987) 695-722.

Damon, J. *The unfolding and determinacy theorems for subgroups of $\mathcal{A}$ and $\mathcal{K}$*, Mem. Amer. Math. Soc. **50 no. 306** (1984)

Damon, J. *Thom-Mather Theory at 50 years of Age*, preprint available at jndamon.sites.oasis.unc.edu and CIMPA school webpage.

Gervais, J. J. *Germes de G-détermination fini*, C. R. Acad. Sci. Paris Ser. A-B **284A** (1977) 291–293

Gervais, J. J. *Critères de G-Stabilité en terms de Transversalité*, Canad. J. Math. **31** (1979) 264–273

Bibiography III: Free Divisors

Buchweitz, R. O. and Mond, D. *Linear Free Divisors and Quiver Representations* in Singularities and Computer Algebra London math. Soc. Lect. Ser. **vol 324** Cambridge Univ. Press, 2006, 41–77

Damon, J. *On the Legacy of Free Divisors: Discriminants and Morse Type Singularities*, Amer. Jour. Math. **120**, (1998), 453-492; *On the Legacy of Free Divisors III: Functions and Divisors on Complete Intersections*, Oxford Quart. Jour. Math. **57** (2006) 49-79

Damon, J. and Pike, B. *Solvable Groups, Free Divisors and Nonisolated Matrix Singularities I: Towers of Free Divisors*, Annales de l'Inst. Fourier **65 no. 3** (2015) 1251-1300.

Granger, M., Mond, D., Nieto-Reyes, A., and Schulze,M. *Linear Free Divisors and the Global Logarithmic Comparison Theorem*, Annales Inst. Fourier **59** 811–850

Bibiography IV: Topology of Singular Milnor Fibers of Singularities of Type $\mathcal{V}$

Damon, J. *Higher Multiplicities and Almost Free Divisors and Complete Intersections*, Memoirs Amer. Math. Soc. **123 no 589** (1996).

Damon, J. *Nonlinear Sections of Non-isolated Complete Intersections*, New Developments in Singularity Theory, Eds. D. Siersma, C. T. C. Wall, V. Zakalyukin, Nato Science Series **21**, Kluwer Acad. Publ. (2001), 405-445

Damon, J. and Mond, D. *$\mathcal{A}$–Codimension and the Vanishing Topology of Discriminants*, Invent. Math. **106** (1991), 217–242.

Siersma, D. *Vanishing cycles and special fibres* Singularity Theory and its Applications: Warwick , Part I, Springer Lecture Notes **1462** (1991), 292-301.

Bibiography V: Topology of Exceptional Orbit Varieties of Prehomogeneous Spaces

Damon, J. *Topology of Exceptional Orbit Hypersurfaces of Prehomogeneous Spaces*, Journal of Topology **9 no. 3** (2016) 797-825.

Damon, J. *Schubert Decomposition for Milnor Fibers of the Varieties of Singular Matrices*, Special Issue in Honor of E. Brieskorn, Journal of Singularities **vol 18** (2018) 358–396.

Sato, M. *Theory of Prehomogeneous Vector Spaces (algebraic part)*, English translation of Sato's lectures from notes by T. Shintani, Nagoya Math. Jour. **120** (1990), 1-34

Sato, M. and Kimura, T. *A Classification of Irreducible Prehomogeneous Vector Spaces and Their Relative Invariants*, Nagoya Math. Jour. **65**, (1977) 1–155

Bibiography VI: Topology of (Singular) Milnor Fibers of Matrix Singularities

Bruce, J.W. *Families of symmetric matrices* Moscow Math. Jour **3 no. 2**, Special Issue in honor of V. I. Arnol'd, (2003) 335-360

Damon, J. and Pike, B. *Solvable Groups, Free Divisors and Nonisolated Matrix Singularities II: Vanishing Topology*, Geom. and Top. **18 no. 2** (2014) 911-962.

Damon, J. and Pike, B. *Solvable Group Representations and Free Divisors whose Complements are $K(\pi, 1)$'s*, Proc. Conference on Singularities and Generic Geometry, in Top. and its Appl. **159** (2012) 437-449.

Goryunov, V. and Mond, D. *Tjurina and Milnor Numbers of Matrix Singularities*, J. London Math. Soc. **72 (2)** (2005), 205-224.

Goryunov, V. *Vanishing cycles of Matrix Singularities*, preprint

Bibiography VII: Characteristic Cohomology and Topology of Matrix Singularities

Damon, J. *Characteristic Cohomology I : Singularities of Given Type*, Oxford Quarterly Jour. of Math. **71** (2021) 1055-1076

Damon, J. *Characteristic Cohomology II : Matrix Singularities*, submitted for publication, see arXiv:1911.02102.

Hamm, H. and Lê, D. T. *Une Théorème de Zariski du Type de Lefschetz*, Ann. Sci. École Norm. Sup. **6 no. 4** (1973) 317-355