Algorithmic Methods for Singularities

Talk 2: Invariants from the computational perspective

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Dimension from definition to computation

Let $X = V(I) \subseteq \mathbb{A}^n_{\mathbb{C}}$ be irreducible.

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Efficiently computable: the last two items!

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Let $P(t) = \sum_{\nu=0}^{d} a_i t^d$, then

- $\dim(A) = d$
- $\mathrm{mult}(A) = d! \cdot a_d$

A monomial example

Consider the 3 coordinate axes in $(\mathbb{C}^3, 0)$:

$$I = \langle xy, xz, yz \rangle \subseteq \mathbb{C}[x, y, z]_{\langle x,y,z \rangle}$$

$$A = \mathbb{C}[x, y, z]_{\langle x,y,z \rangle}/I$$

k	new monomials	total number
1	1	1
2	x, y, z	4
3	x^2, y^2, z^2	7
4	x^3, y^3, z^3	10
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Obviously, $P(t) = 3 \cdot t - 2$

Hence: $\dim(X) = 1$ and $\mathrm{mult}(X) = 3$ as expected

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Key to computing previous example: Monomial ideal

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And in the non-monomial case? **standard bases**

- K a (exact computable) field
- $>$ suitable total ordering on $\mathrm{Mon}(\underline{x})$
- $LT(f)$ largest monomial in $f \in K[\underline{x}]$ w.r.t. $>$
- $L(I) := \langle LT(f) \mid f \in I \rangle$

$$G = \{g_1, \ldots, g_s\} \text{ standard basis of } I$$

$$\Longleftrightarrow$$

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Facts:

- $\dim(K[\underline{x}]_{\langle \underline{x} \rangle}/I) = \dim(K[\underline{x}]_{\langle \underline{x} \rangle}/L(I))$
- $\mathrm{mult}(K[\underline{x}]_{\langle \underline{x} \rangle}/I) = \mathrm{mult}(K[\underline{x}]_{\langle \underline{x} \rangle}/L(I))$

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Actually, standard basis of I:

$$G = \{x^2 + y^2, xy, y^3\}$$

A multiplicity 4 space curve example

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$P(t) = 4 \cdot t - 4$, dim $= 1$, mult $= 4$

Jupyter Notebook 2, Block 1