Topology of determinantal singularities

Lectures VI and VII of a course in the CIMPA research school 2022 at the ICMC at USP, São Carlos, Brazil

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Outline of the lectures

- Vanishing topology of (isolated) determinantal hypersurfaces $f = \det(A)$
- Rational double points, Tjurina modifications and resolution in family
- Rational triple points and beyond
- Some further explicit case studies
- Computation of Euler characteristics via hyperplane sections
- The graph transformation and Bouquet decomposition for EIDS
- Vanishing cohomology of generic linear EIDS

The geometric setup for determinantal singularities

Schematic picture of a smoothable IDS:

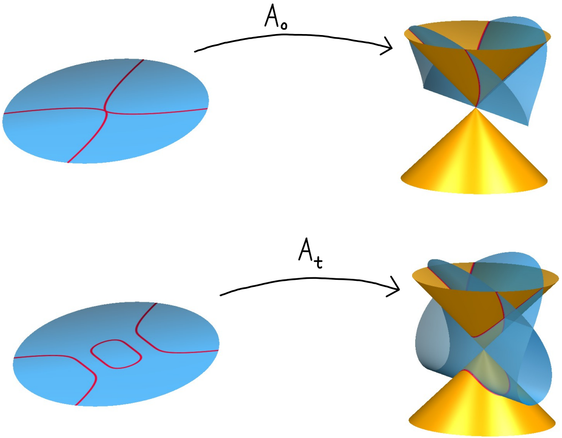


Figure: A schematic picture of a matrix A_0, the intersection of its image with a singular variety $M^s_{m,n}$ in its codomain (pictured as a double cone) and the preimage of that variety $(X^s_A, 0)$. The second row shows a fiber $X^s_{\mathbf{A}}(t) = A_t^{-1}(M^s_{m,n})$ in the deformation of that singularity which is induced from an unfolding $\mathbf{A}(x,t) = A_t(x)$ of A_0.

Schematic picture of the non-smoothable case:

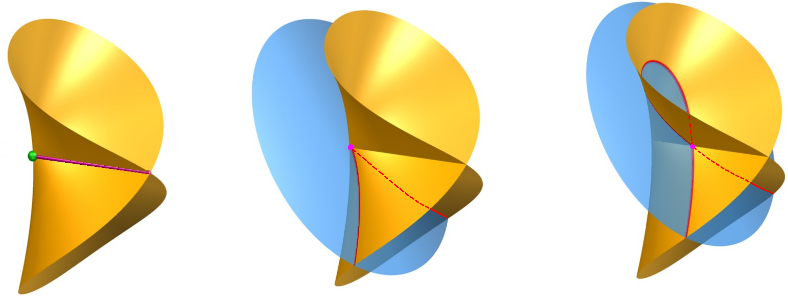


Figure: The so-called "Whitney umbrella" $W \subset \mathbb{R}^3$ with its decomposition into three strata: The origin, the open half of a coordinate axis and the remainder. The other two pictures show the immersion of an affine plane D: In the middle picture, the intersection of D with W is not transverse at the origin. In the picture on the right hand side, D is transverse to W in a stratified sense, despite the fact that the intersection is not a smooth manifold.

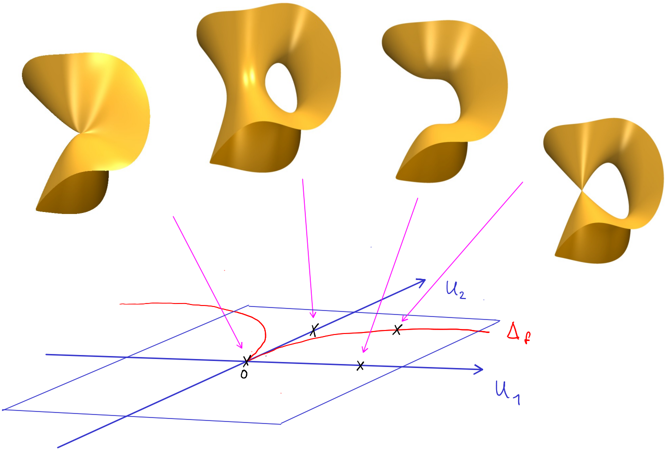
Consider the function $f = x^3 + y \cdot z \in \mathbb{C}[x, y, z]$. We can *choose* a matrix-structure for f, for instance

$$\begin{aligned} f = \det\left(f\right) &= \det\begin{pmatrix} x^3 & y \\ -z & 1 \end{pmatrix} &=& \quad \det\begin{pmatrix} x^2 & y \cdot z \\ -1 & x \end{pmatrix} \\ &= \det\begin{pmatrix} x^2 & y \\ -z & x \end{pmatrix} &=& \quad \det\begin{pmatrix} x^2 & y - x^2 \\ -z & x + z \end{pmatrix} \end{aligned}$$

Question: How many different ways are there to write $f = \det(A)$ for some matrix A? Up to which equivalence?

The semi-universal unfolding (up to right-equivalence) of $f = x^3 + y \cdot z$ is given by

$$\mathbf{F}(x, y, z; u_1, u_2) = x^3 + y \cdot z - u_1 \cdot 1 - u_2 \cdot x \colon (\mathbb{C}^3, 0) \times (\mathbb{C}^2, 0) \to (\mathbb{C}, 0).$$



The semi-universal unfolding of the matrix A (up to GL-equivalence) is given by

$$\mathbf{A}(x, y, z; t_1, t_2) = \begin{pmatrix} x^2 & y \\ -z & x \end{pmatrix} - t_1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - t_2 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

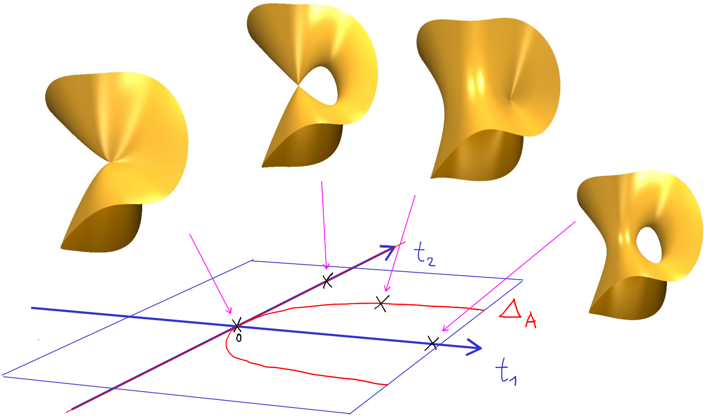


Figure: The deformation of $\{\det(A) = 0\}$ induced by the semi-universal unfolding of A

How are the unfoldings of f and A related?
Expanding the determinant of $\mathbf{A}$ gives an unfolding of f:

$$\det \mathbf{A}(x,y,z;t_1,t_2) = \det \begin{pmatrix} x^2 - t_1 & y \\ -z & x - t_2 \end{pmatrix} = f - t_1 \cdot x - t_2 \cdot x^2 + t_1 \cdot t_2.$$

Since F is semi-universal, it must be possible to find maps

$$\begin{aligned} \theta \colon (\mathbb{C}^2,0) &\to (\mathbb{C}^2,0), & (t_1,t_2) &\mapsto (u_1,u_2) = \theta(t_1,t_2) \\ \Phi \colon (\mathbb{C}^3,0) \times (\mathbb{C}^2,0) &\to (\mathbb{C}^3,0) & (x,y,z;t_1,t_2) &\mapsto \Phi_t(x,y,z) \end{aligned}$$

with Φ_0 the identity such that

$$F(x,y,z;\theta(t_1,t_2)) = \det \mathbf{A}(\Phi_t(x,y,z);t_1,t_2).$$

It turns out that the *Tschirnhaus transformation* $x \mapsto x + \frac{1}{3}t_2$ serves as Φ_t; then

$$\begin{aligned} \det \mathbf{A}(\Phi_t(x,y,z);t) &= \det \begin{pmatrix} x^2 + \frac{2}{3}t_2 \cdot x + \frac{1}{9}t_2^2 - t_1 & y \\ -z & x - \frac{2}{3}t_2 \end{pmatrix} \\ &= f - \underbrace{\left(t_1 + \frac{1}{3}t_2^2\right)}_{u_2} \cdot x - \underbrace{\frac{2}{3}t_2\left(\frac{1}{9}t_2^2 - t_1\right)}_{u_1} \end{aligned}$$

We can draw the following conclusions:

- The map

$$\theta\colon (t_1, t_2) \mapsto (u_1, u_2) = \left(t_1 + \frac{1}{3} t_2^2, \frac{2}{3} t_2 \left(\frac{1}{9} t_2^2 - t_1 \right) \right)$$

  is neither an embedding, nor a submersion, but a *branched covering* of degree 3.

- The discriminant Δ_A of A decomposes into parts

$$\Delta_A = \Delta_A^{(0)} \cup \Delta_A^{(1)} = \{t_2^2 - t_1 = 0\} \cup \{t_1 = 0\}$$

  which are characterized by $\mathbf{A}_t$ being non-transverse to the stratum $V_{2,2}^r$ at a *generic* point of $\Delta_A^{(r)}$.

- Since the Milnor fiber M_f pulls back to the *determinantal Milnor fiber* M_A^2, we necessarily have $\theta^{-1}(\Delta_f) = \Delta_A$.
  Indeed, substituting θ in the defining equation of Δ_f yields

$$\theta^*(4u_2^3 - 27u_1^2) = -4t_1 \cdot (t_2^2 - t_1)^2.$$

Question: Is this a coincidence? Which other parts of singularity theory does this relate to?

Goryunov and Zakalyukin [8] have found an interesting relationship between the universal unfoldings of the simple symmetric matrices in two variables from the classification by Bruce and Tari [4] and those of the associated hypersurface singularities.

Normal form	associated hypersurface	GL-codim.
$\begin{pmatrix} y^k & x \\ x & y^l \end{pmatrix}$, $k \geq 1, l \geq 2$	A_{k+l-1}	$k+l-1$
$\begin{pmatrix} x & 0 \\ 0 & y^2+x^k \end{pmatrix}$, $k \geq 2$	D_{k+2}	$k+2$
$\begin{pmatrix} x & 0 \\ 0 & xy+y^k \end{pmatrix}$, $k \geq 2$	D_{2k}	$2k$
$\begin{pmatrix} x & y^k \\ y^k & xy \end{pmatrix}$, $k \geq 2$	D_{2k+1}	$2k+1$
$\begin{pmatrix} x & y^2 \\ y^2 & x^2 \end{pmatrix}$	E_6	6
$\begin{pmatrix} x & 0 \\ 0 & x^2+y^3 \end{pmatrix}$	E_7	7

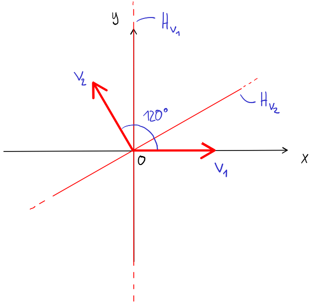
Simple singularities of symmetric matrices of size $m = 2$ in two variables, [3, Theorem 1.1]

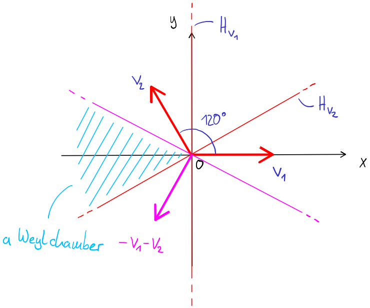
For every A-D-E singularity f, there is a corresponding Lie group G which comes with a Lie algebra $\mathfrak{g}$ and a **root system** Φ – a finite set of non-zero vectors $v_1, \ldots, v_N$ in Euclidean space $\mathbb{R}^\mu$ satisfying certain properties and symmetries.
In fact, the **Dynkin diagrams** encode these root systems with the vertices corresponding to a subset of *simple positive roots* in Φ.
For any root $v \in \mathbb{R}^\mu$ we have an orthogonal hyperplane H_v and an associated reflection

$$\mathbb{R}^\mu \to \mathbb{R}^\mu, \quad w \mapsto w - \frac{2\langle v, w\rangle}{\langle v, v\rangle} \cdot v.$$

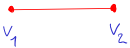
Now consider the complexification of this, i.e. the induced reflections in $\mathbb{R}^\mu \otimes_\mathbb{R} \mathbb{C} \cong \mathbb{C}^\mu$: The subgroup $W \subset \mathrm{Aut}(\mathbb{C}^\mu)$ generated by these reflections at the **mirrors** $H_v \otimes_\mathbb{R} \mathbb{C} \subset \mathbb{C}^\mu$ is called the **Weyl group** of G. It is finite and the quotient $\mathbb{C}^\mu / W \cong \mathbb{C}^\mu$ turns out to again be smooth.

The Weyl group for the A_2-singularity

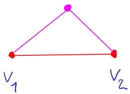




Dynkin Diagram:

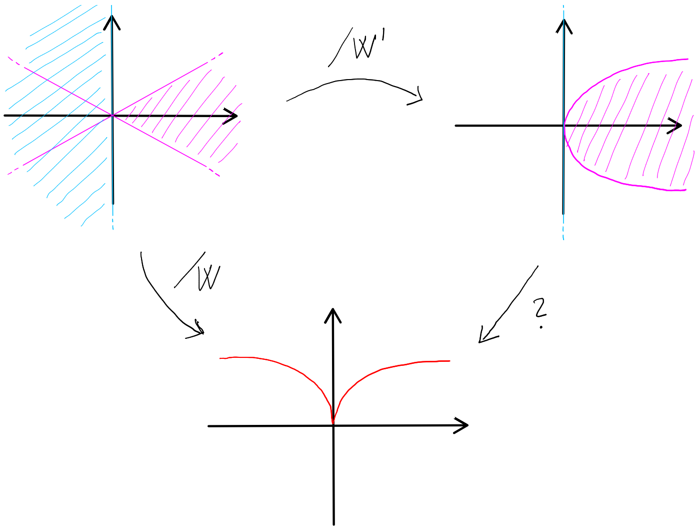


affine Dynkin Diagram:



The Weyl group for the A_2-singularity

Let W be the Weyl group generated by two mirrors and $W' \subset W$ the subgroup generated by only one mirror.



Excursion continued: Unfoldings of A-D-E-singularities

Let us summarize some well known facts about A-D-E singularities; the following appears, for instance, in [2], but also in [8]:

Proposition 1

Let $f \in \mathbb{C}\{x, y\}$ be a simple hypersurface singularity. Then the Milnor number μ is equal to the dimension of the vector space for the associated root system and the quotient $\mathbb{C}^\mu/W$ by the Weyl group can be identified with the base of the semi-universal unfolding of f such that the following holds.

Let $\mathcal{A} \subset \mathbb{C}^\mu$ be the reflection hyperplanes of the roots. Then the quotient map

$$q : \mathbb{C}^\mu \xrightarrow{/W} \mathbb{C}^\mu/W \cong \mathbb{C}^\mu$$

takes the mirrors to the discriminant of f with $\mathcal{A} = q^{-1}(\Delta_f)$.

Goryunov and Zakalyukin have observed the following [8]:

There is a $1:1$-correspondence between pairs of Weyl groups $(W; W')$ with W of type A, D, or E and $W' \subset W$ the subgroup obtained by either

- deleting two vertices of degree one from the affine Dynkin diagram $\tilde{W}$ of W and the simple symmetric 2×2-matrices, or
- deleting one vertex of degree two from the affine Dynkin diagram and the simple symmetric 3×3-matrices

such that the following theorem (below) holds.

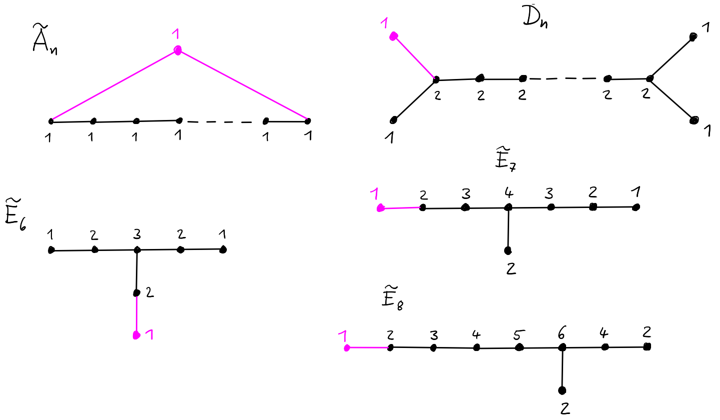


Figure: The affine Dynkin diagrams of the simple singularities

Theorem 1 ([8])

Let $f = \det(A)$ be as in the above correspondence; let $\mathcal{A}_W \subset \mathbb{C}^\mu$ be the mirror arrangement for W, $\mathcal{A}_{W'} \subset \mathcal{A}_W$ the one for W'.
Then

$$(\mathbb{C}^\mu, \mathcal{A}_W)/W' \cong (\mathbb{C}^\mu, \Delta_A).$$

Moreover, this biholomorphism provides isomorphisms

$$\mathcal{A}_{W'}/W \cong \Delta_A^{(1)} \quad \text{and} \quad \mathcal{A}_{W\setminus W'}/W' \cong \Delta_A^{(0)}.$$

and the quotient maps can be completed to a commutative diagram

$$\begin{tikzcd}
 & (\mathbb{C}^\mu, \mathcal{A}_W) \arrow[dl, "/W'"'] \arrow[dr, "/W"] & \\
(\mathbb{C}^\mu, \Delta_A) \arrow[rr, "\theta"] & & (\mathbb{C}^\mu, \Delta_f)
\end{tikzcd}$$

with the induced map θ in the base spaces of the semi-universal unfoldings of A and f, respectively.

Simple symmetric matrices in two variables

Normal form	associated hypersurface	GL-codim.	Pair of Weyl groups
$\begin{pmatrix} y^k & x \\ x & y^l \end{pmatrix}$, $k \geq 1, l \geq 2$	A_{k+l-1}	$k+l-1$	$(A_{k+l-1}; A_{k-1} \oplus A_{l-1})$
$\begin{pmatrix} x & 0 \\ 0 & y^2+x^k \end{pmatrix}$, $k \geq 2$	D_{k+2}	$k+2$	$(D_{k-2}; D_{k-3})$
$\begin{pmatrix} x & 0 \\ 0 & xy+y^k \end{pmatrix}$, $k \geq 2$	D_{2k}	$2k$	$(D_{2k}; A_{2k-1})$
$\begin{pmatrix} x & y^k \\ y^k & xy \end{pmatrix}$, $k \geq 2$	D_{2k+1}	$2k+1$	$(D_{2k+1}; A_{2k})$
$\begin{pmatrix} x & y^2 \\ y^2 & x^2 \end{pmatrix}$	E_6	6	$(E_6; D_5)$
$\begin{pmatrix} x & 0 \\ 0 & x^2+y^3 \end{pmatrix}$	E_7	7	$(E_7; E_6)$

Simple singularities of symmetric matrices of size $m = 2$ in two variables, [3, Theorem 1.1], extended according to [8, Table 1]

The previous theorem was found by considering "1-parameter suspension" of the hypersurface singularities f (i.e. replacing $f(x, y)$ by $f(x, y) + z^2$). A closer inspection of the tables in [3] and [4] yields that we have a correspondence

$$
\begin{tikzcd}
\{\text{simple symmetric } 2 \times 2\text{-matrices in 2 variables}\} \arrow[d, bend left, "1:1"] \\
\{\text{simple square } 2 \times 2\text{-matrices in 3 variables}\} \arrow[u, bend left]
\end{tikzcd}
\tag{1}
$$

which is given by

$$A \in \mathbb{C}\{x, y\}^{2\times 2}_{\mathrm{sym}} \qquad \leftrightarrow \qquad A + \begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix} \in \mathbb{C}\{x, y, z\}^{2\times 2}$$

up to GL-equivalence. In particular, this induces the 1-parameter suspension of the associated hypersurface singularities in every case and it identifies the semi-universal unfoldings on both sides.

Normal form	alt. normal form	GL-codimension
$\begin{pmatrix} x & z^k \\ \pm z^l & y \end{pmatrix}$, $1 \leq k \leq l$	$\begin{pmatrix} x^k & y+z \\ y-z & x^l \end{pmatrix}$, $1 \leq k \leq l$	$k+l-1$
$\begin{pmatrix} x & -y \\ y+z^k & x \end{pmatrix}$	$\begin{pmatrix} y & x^k+z \\ x^k-z & y \end{pmatrix}$	$2k-1$
$\begin{pmatrix} x & y \\ z^2 \pm y^k & x \end{pmatrix}$, $2 \leq k$	$\begin{pmatrix} y^2+x^k & z \\ -z & x \end{pmatrix}$, $2 \leq k$	$k+2$
$\begin{pmatrix} x & y \\ y^2 & x+z^2 \end{pmatrix}$	$\begin{pmatrix} y & x^2+z \\ x^2-z & y^2 \end{pmatrix}$	6
$\begin{pmatrix} x & y \\ y^2+z^3 & x \end{pmatrix}$	$\begin{pmatrix} y^2+x^3 & z \\ -z & y \end{pmatrix}$	7
$\begin{pmatrix} x & y \\ yz & x+z^k \end{pmatrix}$, $2 \leq k$	$\begin{pmatrix} y & x^k+z \\ x^k-z & yx \end{pmatrix}$, $2 \leq k$	$2k+1$
$\begin{pmatrix} x & y \\ yz+z^k & x \end{pmatrix}$, $3 \leq k$	$\begin{pmatrix} yx+x^k & z \\ -z & y \end{pmatrix}$, $3 \leq k$	$2k$

Simple singularities of square matrices in three variables of size $m=2$ from [4] with alternate forms up to GL-equivalence

Exercise: A possible connection to McKay-correspondence?

Recall the McKay-correspondence for A-D-E singularities $(X,0)=\{f=0\}$ and their minimal resolutions $\pi\colon(\tilde{X},E)\to(X,0)$:

$$
\begin{tikzcd}
\{\text{vertices in the Dynkin diagram}\} \arrow[d, bend left, "\text{dual resolution graph}"] \\
\{\text{components } E_i \text{ of the exceptional divisor}\} \arrow[u, bend left] \arrow[d, bend left, "c_1(\pi^*M/\mathrm{tor}(\pi^*(M))) \text{ is non-trivial on } E_i"] \\
\{\text{maximal Cohen Macaulay modules } M \text{ on } (X,0)\} \arrow[u, bend left] \arrow[d, bend left, "\text{presentation of } M \text{ as a } \mathbb{C}\{x,y,z\}\text{-module}"] \\
\left\{\text{matrices } A\in\mathbb{C}\{x,y,z\}^{r\times r} \text{ with } \det(A)=f^k\right\} \arrow[u, bend left]
\end{tikzcd}
\tag{2}
$$

see e.g. [5].

Exercise: For the admissible choices of vertices for the 2×2-matrices from the Theorem by Goryunov and Zakalyukin: Do we get precisely those matrices A from the McKay correspondence which are of size 2 and satisfy $\det(A)=f$?

Definition 2

Let $(X_0, 0) \hookrightarrow (X, 0) \xrightarrow{\varphi} (B, 0)$ be a deformation of a singularity $(X, 0)$ over some parameter space $(B, 0)$. A (partial) *resolution in family* for φ is a commutative diagram

$$\begin{tikzcd}
(Y_0, E) \arrow[r, hook] \arrow[d, "\pi_0"] & (Y, E) \arrow[d, "\pi"] \\
(X_0, E) \arrow[r, hook] \arrow[d] & (X, 0) \arrow[d, "\varphi"] \\
\{0\} \arrow[r, hook] & (B, 0)
\end{tikzcd}$$

of flat families over $(B, 0)$ such that for every parameter $t \in B$, the map on the fibers $Y_t \to X_t$ is a (partial) resolution of singularities.

Theorem 3 (Tjurina [15], Brieskorn [2])

The rational double points admit a resolution in family after a Galois base change.

At times, we can construct a resolution in family by choosing matrix structures and applying the so-called *Tjurina modification* (see, e.g. [11] [14], [7], ...)

$$
\begin{tikzcd}
(Y_0, E) \arrow[r, hook] \arrow[d, "\pi_0"] & (Y, E) \arrow[r, "\hat{\mathbf{A}}"] \arrow[d, "\pi"] & \hat{M}^s_{m,n} \arrow[d, "\hat{\nu}"] \arrow[dr, "\hat{\Phi}"] & \\
(X_0, 0) \arrow[r, hook] \arrow[d] & (X, 0) \arrow[r, "\mathbf{A}"] \arrow[d] & M^s_{m,n} \arrow[r, dashed, "\Phi"] & \mathrm{Grass}(s-1, m) \\
\{0\} \arrow[r, hook] & (\mathbb{C}^k, 0) & &
\end{tikzcd}
$$

$$\Phi\colon\ V^{s-1}_{m,n} \to \mathrm{Grass}(s-1, m), \quad \varphi \mapsto \mathrm{Im}(\varphi)$$

We pick up on the previous example for $f = \det \begin{pmatrix} x^2 & y \\ -z & x \end{pmatrix}$ and its semi-universal unfoldings.

The equations for the Tjurina transform in family are given by

$$0 = \mathbf{A}(x, y, z; t_1, t_2) \cdot \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} a_0(x^2 - t_1) + a_1 y \\ -a_0 z + a_1(x - t_2) \end{pmatrix}$$

where $(a_0 : a_1)$ are the homogeneous coordinates of $\mathbb{P}^1$.

The chart $\{a_1 \neq 0\}$ is always smooth.

In the other one, we find (the deformation of) one A_1-singularity (Y_0, p) at the origin:

$$x^2 + \frac{a_1}{a_0} \cdot y = t_1.$$

Setting $t_1 = 0$, we can again write this as a matrix singularity for the matrix

$$A' = \begin{pmatrix} x & \frac{a_1}{a_0} \\ -y & x \end{pmatrix}.$$

But to also capture the deformation, we need to make another base change!

Tjurina transformation (in family)

The Tjurina transformation (Z, E') of the A_1-singularity (Y_0, p) is smooth:

$$A' \cdot \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = \begin{pmatrix} x & a_1 \\ -y & x \end{pmatrix} \cdot \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = 0$$

so after the base change

$$\theta' : (t_1', t_2') \mapsto (t_1, t_2) = (t_1'^2, t_2')$$

the Tjurina transformation in family for the matrix

$$\mathbf{A}' = \begin{pmatrix} x - t_1' & a_1 \\ -y & x + t_1' \end{pmatrix}$$

is *topologically trivial*.

We arrive at the following big diagram:

$$\begin{tikzcd}
Z_0 \arrow[r, hook] \arrow[d] & Z \arrow[d] & & \\
Y_0 \arrow[r, hook] \arrow[d] & Y' \arrow[r] \arrow[d] & Y \arrow[d] & \\
X_0 \arrow[r, hook] \arrow[d] & X^2_{\mathbf{A}'} \arrow[r] \arrow[d] & X^2_{\mathbf{A}} \arrow[r] \arrow[d] & X_F \arrow[d] \\
\{0\} \arrow[r, hook] & \mathbb{C}^2 \arrow[r, "\theta'"] & \mathbb{C}^2 \arrow[r, "\theta"] & \mathbb{C}^2
\end{tikzcd}$$

where

- X_F is the semi-universal unfolding of the hypersurface f,
- $Y \to X^2_{\mathbf{A}}$ is the Tjurina transformation in family for the unfolding $\mathbf{A}$,
- Y' is the fiber product for the base change θ' and the unfolding $\mathbf{A}'$ of the matrix structure of the A_1-singularity in Y.
- $Z \to Y'$ is the Tjurina transformation in family for $\mathbf{A}'$.

The Tjurina transformation had already been used *previously* in [14] to construct resolutions in family for the *rational triple points* $(X_0, 0) \subset (\mathbb{C}^4, 0)$.

These are given as $(X_0, 0) = (X_A^2, 0)$ for some matrix $A \in \mathbb{C}\{x, y, z, w\}^{2\times 3}$.

They coincide precisely with the *simple, isolated Cohen-Macaulay codimension 2 singularities* in $(\mathbb{C}^4, 0)$ from the classification by Frühbis-Krüger and Neumer [6].

Example 4

Consider the $A_{0,l-1,k-1}$-singularity from [14]:

$$A = \begin{pmatrix} w^l & y & x \\ z & w & y^k \end{pmatrix}, \quad l \geq k \geq 2.$$

The Tjurina transform is smooth away from the points $0, \infty \in \mathbb{P}^1$ where we find an A_{l-1}- and an A_{k-1}-singularity, respectively.

Tjurina argues that the remaining singularities in the Tjurina transform for the defining matrix can only be rational double points. For these, the construction of Tjurina modification in family is already known and, hence, a full resolution in family can be constructed as before.

More generally, for arbitrary rational surfaces $(X_0, 0) \subset (\mathbb{C}^p, 0)$ we have the following theory:

- Wahl [16]: If a rational surface singularity has embedding dimension $p \geq 4$, then, if it is determinantal, it must be determinantal of type $(2, p-2, 2)$.
- Wahl [16], Röhr [12], de Jong [9]: A rational surface singularity of embedding dimension p is determinantal if and only if the dual graph of a minimal resolution of singularities consists of one $-(p-1)$-curve and possibly some -2-curves.
- Building on Brieskorn's work on resolutions, Artin has shown in [1] that for an *arbitrary* rational surface singularity with semi-universal deformation $(X_0, 0) \hookrightarrow (X, 0) \xrightarrow{\varphi} (S, 0)$ there is a *smooth* space $(R, 0)$ parametrizing those deformations of a minimal resolution which *blow down* to deformations of $(X_0, 0)$. Moreover, the comparison map $\Phi\colon (R, 0) \to (S, 0)$ is *finite* and takes $(R, 0)$ surjectively onto one component of $(S, 0)$, the so-called *Artin component*.
- Wahl [17]: For a *determinantal* rational surface singularity of embedding dimension $p \geq 4$, the Artin component consists precisely of the determinantal deformations.

In particular, the smoothing over the Artin component of a determinantal rational surface singularity is always homotopy equivalent to a bouquet of 2-dimensional spheres.

We consider determinantal singularities at the origin given by *linear* maps

$$A\colon \mathbb{C}^p \to \mathbb{C}^{m\times n}.$$

Recall that $(X_A^s, 0) = (A^{-1}(M_{m,n}^s), 0)$ has *isolated singularity* at the origin, iff

$$\operatorname{codim} \mathrm{Sing}(M_{m,n}^s) = \operatorname{codim}(M_{m,n}^{s-1}) = (m-s+2)(n-s+2) \leq p$$

and that $(X_A^s, 0)$ admits a determinantal smoothing iff this inequality is strict.

Schematic picture of a smoothable IDS:

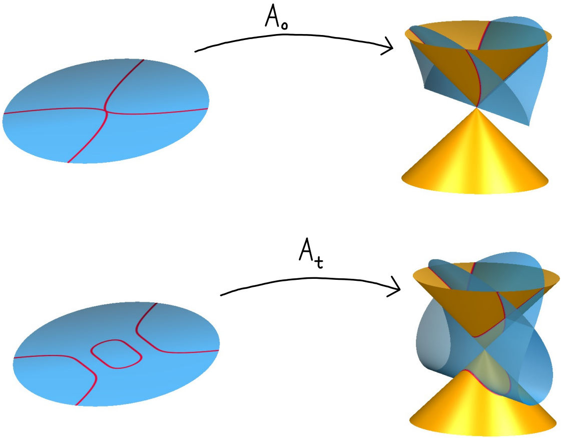


Figure: A schematic picture of a matrix A_0, the intersection of its image with a singular variety $M^s_{m,n}$ in its codomain (pictured as a double cone) and the preimage of that variety $(X^s_A, 0)$. The second row shows a fiber $X^s_{\mathbf{A}}(t) = A_t^{-1}(M^s_{m,n})$ in the deformation of that singularity which is induced from an unfolding $\mathbf{A}(x,t) = A_t(x)$ of A_0.

Schematic picture of the non-smoothable case:

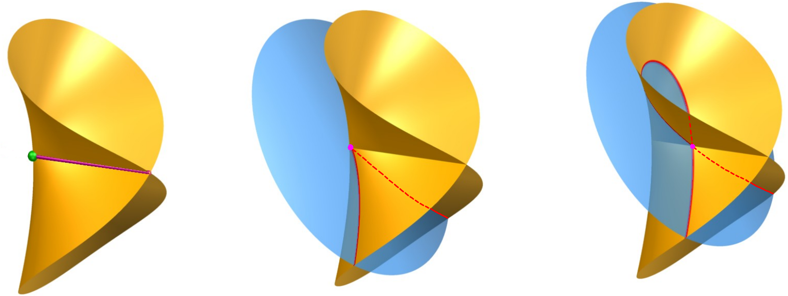


Figure: The so-called "Whitney umbrella" $W \subset \mathbb{R}^3$ with its decomposition into three strata: The origin, the open half of a coordinate axis and the remainder. The other two pictures show the immersion of an affine plane D: In the middle picture, the intersection of D with W is not transverse at the origin. In the picture on the right hand side, D is transverse to W in a stratified sense, despite the fact that the intersection is not a smooth manifold.

Let $(X_A^s, 0) \subset (\mathbb{C}^p, 0)$ be an EIDS of type (m, n, s).

Observe:

- The Tjurina transformation is equidimensional iff $\dim \mathrm{Grass}(s-1, m) \leq \dim(X_A^s, 0) = p - (m-s+1)(n-s+1)$.
- In general, the strict Tjurina transformation will meet the exceptional set in a set of positive dimension, so the Tjurina transform will have *non-isolated* singularities!
- The Tjurina transformation is a local complete intersection if and only if $p - m(m-s+1) > \dim X_A^1$ ([11, Proposition 5.1]).

Some further examples: Linear EIDS

The following table lists the Tjurina transforms and Milnor fibers of generic linear EIDS:

m	n	s	p	Tj. Transf.	Milnor fiber
2	n	2	n-1	-	n points
2	n	2	n	$n \times A_1$	$\bigvee_{i=1}^{n-1} S^1$
2	n	2	$n < p < 2n$	$\left\lvert \mathcal{O}(-(2n-p+1)) \oplus \mathcal{O}(-1)^{p-n-1} \right\rvert$	S^2
2	n	2	$p \geq 2n$	$\mathbb{C}^{p-2n} \times \lvert \mathcal{O}(-1)^n \rvert$	$\{\text{pt.}\}$
3	4	3	5	$10 \times A_1$	$\bigvee_{i=1}^{9} S^3$
3	4	3	4	non-isol. singularity	?

Lemma 5 (cf. e.g. [10])

Let $(\mathbb{C}^p, 0) \supset (X_0, 0) \hookrightarrow (X, 0) \xrightarrow{\varphi} (\mathbb{C}, 0)$ *be a smoothing of an equidimensional singularity. Then there exists a dense open subset* $\Omega \subset \mathrm{Hom}(\mathbb{C}^p, \mathbb{C})$ *of linear forms such that for* $l \in \Omega$ *and* $D_\eta = l^{-1}(\{\eta\})$ *we have*

1. *the singularities of* $(X_0 \cap D_0, 0)$ *are "not worse" than those of* $(X, 0)$*;*
2. *for every* $1 \gg \varepsilon \gg |t| > 0$ *the function*

$$l \colon M_t = B_\varepsilon \cap X_t \to \mathbb{C}$$

has only Morse critical points on the interior of the Milnor fiber.

Remark 6

The number of Morse points in the above lemma is Gaffney's multiplicity m_d.

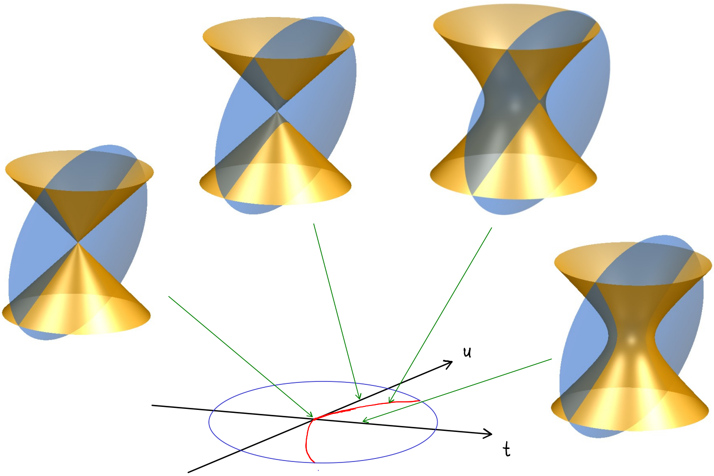


Figure: A smoothing of the double cone $X_t \subset \mathbb{C}^3$ given by $x^2 + y^2 - z^2 = t$ and its hyperplane section by $l = x - 0.2z = u$

Induction on hyperplane sections

Theorem 7

In the previous setup, up to homotopy, the Milnor fiber M_t is obtained from its hyperplane section $M_t^{\pitchfork} = M_t \cap D_\eta$, $1 \gg |t| \gg |\eta| \geq 0$, by attaching cells of dimension $d = \dim M_t$.

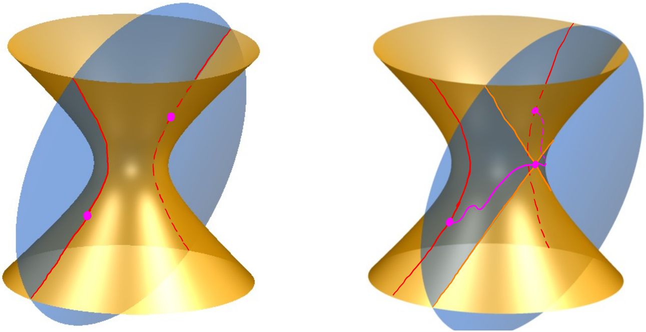


Figure: Attachement of a cell for a Morse critical point of the linear form l on the smoothing X_t of the singularity.

The problem is: It is in general very difficult to determine *where* these attachements take place! But we get the Euler characteristic.

Induction on hyperplane sections

We may take successive hyperplane sections of the smoothing $(X_0, 0) \hookrightarrow (X, 0) \xrightarrow{\varphi} (\mathbb{C}, 0)$ ending with a deformation of fat points.

Corollary 8

The topological Euler characteristic of the Milnor fiber is

$$\chi(M_t) = \sum_{i=0}^{\dim(X_0,0)} (-1)^i \cdot m_i(\varphi)$$

with $m_i(\varphi)$ the i-th polar multiplicity of the smoothing.

Tibăr's bouquet decomposition

Let $(X,0) \subset (\mathbb{C}^p,0)$ be a reduced, equidimensional germ of dimension d, endowed with a Whitney stratification $\mathcal{S} = \{V_i\}_{i \in I}$. Suppose

$$f\colon (X,0) \to (\mathbb{C},0)$$

is a function with an isolated singularity in the stratified sense.

Theorem 9 ([13])

The Milnor fiber M_f is obtained from the complex link $\mathcal{L}(X,0)$ to which one attaches cones ("thimbles") over local Milnor fibers of stratified Morse singularities. The image of each such attaching map retracts within $\mathcal{L}(X,0)$ to a point.

This can be generalized to (stratified) isolated complete intersections $f\colon (X,0) \to (\mathbb{C}^k,0)$ and we obtain a formula

$$M_f \cong_{\mathrm{ht}} \mathcal{L}^{k-1}(X,0) \vee \bigvee_{i \in I} \bigvee_{j=1}^{\mu_f(i)} S^{\dim(V_i)}(\mathcal{L}(X,V_i)) \tag{3}$$

where

$$\mathcal{L}^{k-1}(X,0) = X \cap B_\varepsilon \cap l^{-1}(\{\delta\}), \quad 1 \gg \varepsilon \gg |\delta| > 0$$

is the *complex link of codimension* $k-1$ for some generic linear form $l\colon (X,0) \to (\mathbb{C}^k,0)$.

Bouquet decomposition

An EIDS can be transformed into a stratified ICIS. Let

$$A\colon (\mathbb{C}^p, 0) \to (\mathbb{C}^{m\times n}, 0)$$

be the defining matrix and

$$\Gamma_A = \{(x, \varphi) \in \mathbb{C}^p \times \mathbb{C}^{m\times n} : A(x) = \varphi\}$$

its graph defined by the $m \cdot n$ equations $h_{ij}(x, y) = y_{ij} - a_{ij}(x) = 0$. Then

$$X_A^s \cong \Gamma_A \cap \mathbb{C}^p \times M_{m,n}^s$$

and the h_{ij} form a *regular sequence* on $\mathbb{C}^p \times M_{m,n}^s$.

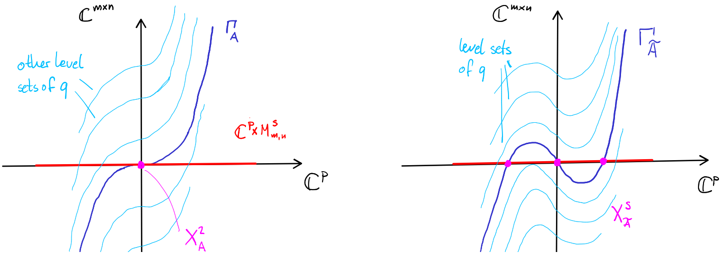


Figure: Schematic picture of the graph transformation

Lemma 10 ([18])

The projection

$$q \colon \mathbb{C}^p \times M^s_{m,n} \to \mathbb{C}^{m\times n}, \quad (x,\varphi) \mapsto \varphi - A(x)$$

is a stratified ICIS with central fiber $(X^s_A, 0)$. Moreover, any unfolding of A induces an unfolding of q.

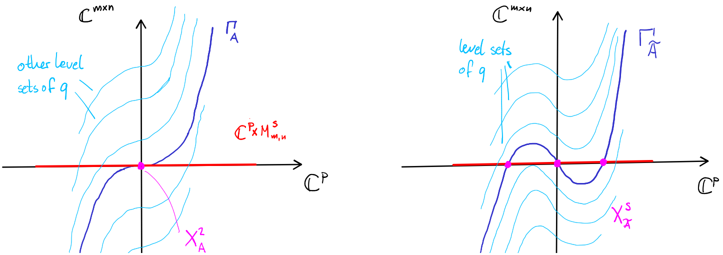


Figure: Schematic picture of the graph transformation

Theorem 11 ([18])

Let $(X_A^s, 0) = (A^{-1}(M_{m,n}^s), 0) \subset (\mathbb{C}^p, 0)$ be an EIDS of type (m, n, t) given by a holomorphic map germ

$$A\colon (\mathbb{C}^p, 0) \to (\mathbb{C}^{m\times n}, 0),$$

$\tilde{A}$ a stabilization of A, and $M_A^s = \tilde{A}^{-1}(M_{m,n}^s)$ the determinantal Milnor fiber. Define

$$r_0 := \min\{r \in \mathbb{N} : (m-r)(n-r) \leq p\}.$$

Then M_A^s has a bouquet decomposition as

$$M_A^s \cong_{ht} L_{m,n}^{s,m\cdot n-p} \vee \bigvee_{r_0 \leq r < s} \bigvee_{i=1}^{\mu_A(r)} S^{p-(m-r)(n-r)+1}(L_{m-r,n-r}^{s-r-1,1}) \tag{4}$$

where $L_{m,n}^{s,k}$ is the intersection $L_{m,n}^{s,k} := H_k \cap M_{m,n}^s$ of a codimension k plane H_k in general position off the origin with the generic determinantal variety and every number $\mu_A(r)$ is a sum of certain polar multiplicities.

Bouquet decomposition: the smoothable case

For a *smoothable* IDS, the bouquet decomposition takes a simpler form. In this case, $s_0 = s - 1$ and there is only *one* stratum contributing to the factors on the right hand side:

$$M_A^s \cong_{ht} L_{m,n}^{s,m\cdot n-p} \vee \bigvee_{i=1}^{\mu_A} S^{\dim(X_A^s,0)} \tag{5}$$

Thus, the vanishing topology splits into a bouquet of spheres of the expected dimension and a mysterious part $L_{m,n}^{s,m\cdot n-p}$ that we still know very little about in general.

However, we now have a pretty good understanding of the cohomology:

Theorem 12 ([19])

Let $0 < s \leq m \leq n$, $k > (m-s+1)(n-s+1)$, and $M_A^s = \mathcal{L}^{k-1}(M_{m,n}^s, 0)$ the complex link of codimension $k-1$ of the generic determinantal variety. Then there is a natural short exact sequence

$$0 \longrightarrow H^{\leq d}(\mathrm{Grass}(s-1,m)) \longrightarrow H^\bullet(M_A^s) \longrightarrow Q \longrightarrow 0$$

where $H^{\leq d}$ denotes the truncated cohomology up to degree $d = \dim M_A^s$ and Q is concentrated in cohomological degree d.

Another “characteristic cohomology”?

Recall the concept of “characteristic cohomology” from Damon's lecture.

$$\begin{tikzcd}
X_A^m \arrow[r, "A"] \arrow[d, hook] & M_{m,m}^m \arrow[r] \arrow[d, hook] & \{0\} \arrow[d, hook] \\
\mathbb{C}^p \arrow[r, "A"] & \mathbb{C}^{m\times m} \arrow[r, "\det"] & \mathbb{C} \\
M_{\det\circ A} \arrow[r, "A"'] \arrow[u, hook] & M_{\det} \arrow[r] \arrow[u, hook] & \{\delta\} \arrow[u, hook]
\end{tikzcd}$$

Note that $M_{\det\circ A} = A^{-1}(M_{\det})$ and $M_{\det} \cong \mathrm{SL}(m;\mathbb{C})$.

The characteristic cohomology of the Milnor fiber $M_{\det\circ A}$ is defined as the image of the pullback $A^*\colon H^\bullet(M_{\det}) \to H^\bullet(M_{\det\circ A})$.

Theorem 13 (Hopf)

We have an isomorphism of graded vector spaces

$$H^\bullet(\mathrm{SL}(m,\mathbb{C})) \cong \bigwedge^\bullet(\mathbb{Z}e_3 \oplus \mathbb{Z}e_5 \oplus \cdots \oplus \mathbb{Z}e_{2m-1})$$

with e_i of cohomological degree i.

Another "characteristic cohomology"?

For a *smoothable* IDS $(X_A^s, 0)$ given by $A\colon (\mathbb{C}^p, 0) \to (\mathbb{C}^{m\times n})$ we have that

$$M_A^s = \tilde{A}^{-1}(V_{m,n}^{s-1})$$

is the preimage of a *single, smooth stratum* $V_{m,n}^{s-1}$ under a transverse map.

Idea: Replace the "universal Milnor fiber" $M_{\det} \cong \mathrm{SL}(m; \mathbb{C})$ by the stratum $V_{m,n}^{s-1}$!

Every such stratum has the structure of an orbit under the action of the Lie group

$$G := \mathrm{GL}(m; \mathbb{C}) \times \mathrm{GL}(n; \mathbb{C}) \curvearrowright \mathbb{C}^{m\times n}, \quad (P, Q) * \varphi = P \cdot \varphi \cdot Q^{-1}.$$

Then $V_{m,n}^r = G * \mathbf{1}_{m,n}^r$ where

$$\mathbf{1}_{m,n}^r = \begin{pmatrix} \mathbf{1}^r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \in \mathbb{C}^{m\times n}$$

is the block matrix with an $r \times r$ unit matrix in the upper left corner.

There are formulas to compute the cohomology of such orbits!

Another "characteristic cohomology"?

Proposition 2 (–, '21)

Let $0 < r \leq m \leq n$ *be integers. The cohomology of the stratum* $V^r_{m,n}$ *is graded isomorphic (as a* $\mathbb{Z}$*-module) to the Koszul algebra*

$$H^\bullet\left(V^r_{m,n}\right) \cong \bigwedge\left(R^r_m \cdot \eta_1 \oplus \cdots \oplus R^r_m \cdot \eta_r\right)$$

over the ring $R^r_m = H^\bullet(\mathrm{Grass}(r,m))$ *with each* η_j *a free generator of degree* $2n - 2j + 1$*.*

Moreover, the commutative subring R^r_m is generated by the Chern classes of the tautological quotient bundle modulo the image of the matrices in the stratum.

For an arbitrary EIDS $(X^s_A, 0)$ and its essential smoothing M^s_A we can define the *characteristic cohomology of the strata* to be the image of the pullbacks

$$\tilde{A}^* : H^\bullet(V^r_{m,n}) \to H^\bullet(M^{r+1}_A \setminus M^r_A).$$

Note:

- $(X^s_A, 0)$ is (determinantally) smoothable iff $M^{s-1}_A = \emptyset$.
- somehow magically, in this case, all η_j pull back to vanishing degrees above the middle dimension of M^s_A.

Remaining question: How do the classes in R^{s-1}_m pull back in degrees $\leq \dim_{\mathbb{C}} M^s_A$?

Another "characteristic cohomology"?

Let $D_i \subset \mathbb{C}^{m\times n}$ be a plane of codimension i in general position and

$$\mathcal{K}^i = M^s_{m,n} \cap D_i \cap \partial B_\varepsilon, \quad 1 \gg \varepsilon > 0$$

the real and

$$\mathcal{L}^i = M^s_{m,n} \cap D_i \cap B_\varepsilon \cap l^{-1}(\{\delta\}), \quad 1 \gg \varepsilon \gg |\delta| > 0$$

the complex link of codimension i of $(M^s_{m,n}, 0)$. They will be smooth within the range

$$(m+n)(r-1) - (r-1)^2 \leq i < (m+n)r - r^2.$$

Theorem 14

For i in the above range we have

$$\begin{aligned} H^k(\mathrm{Grass}(r,m)) &\cong H^k(\mathcal{K}^i(M^{r+1}_{m,n}, 0)) \quad \text{for} \quad k < d(i), && (6)\\ H^k(\mathrm{Grass}(r,m)) &\cong H^k(\mathcal{L}^i(M^{r+1}_{m,n}, 0)) \quad \text{for} \quad k < d(i), && (7)\\ H^{d(i)}(\mathrm{Grass}(r,m)) &\subset H^{d(i)}(\mathcal{K}^i(M^{r+1}_{m,n}, 0)) \subset H^{d(i)}(\mathcal{L}^i(M^{r+1}_{m,n}, 0)) && (8) \end{aligned}$$

where $d(i) = \dim_{\mathbb{C}} \mathcal{L}^i(M^{r+1}_{m,n}, 0) = (m+n)r - r^2 - i - 1$. The left hand sides all agree with the cohomology of $V^r_{m,n}$ in this range for k and the above maps are given by the pullback in cohomology for the natural inclusions of the real and complex links into that stratum.

Theorem 15 ([19])

Let $0 < s \leq m \leq n$, $k > (m-s+1)(n-s+1)$, and $M_A^s = \mathcal{L}^{k-1}(M_{m,n}^s, 0)$ the complex link of codimension $k-1$ of the generic determinantal variety. Then there is a natural short exact sequence

$$0 \longrightarrow H^{\leq d}(\mathrm{Grass}(s-1,m)) \longrightarrow H^{\bullet}(M_A^s) \longrightarrow Q \longrightarrow 0$$

where $H^{\leq d}$ denotes the truncated cohomology up to degree $d = \dim M_A^s$ and Q is concentrated in cohomological degree d.

Corollary 16

Let $(X,0) \subset (\mathbb{C}^5, 0)$ be a Cohen-Macaulay germ of codimension 2 with an isolated singularity at the origin and M its smoothing. Then $H^2(M) \cong \mathbb{Z}$ is free of rank one and generated by the Chern class of the canonical bundle on M.

Corollary 17

For $1 < s \leq m < n$, i.e. for non-square matrices, the constant sheaf $\mathbb{Z}_{M_{m,n}^s}[\dim M_{m,n}^s]$ on the generic determinantal variety, shifted by its dimension, is never a perverse sheaf for the middle perversity.

The End

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