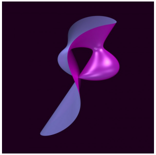
Determinantal Singularities.

J. Damon, T. Gaffney, M. A. S. Ruas, M. Zach
CIMPA School on Singularities and Applications
ICMC-USP



Lecture 1-Part II: Matrices Singularity Theory
M.A.S.Ruas

Determinantal singularities.

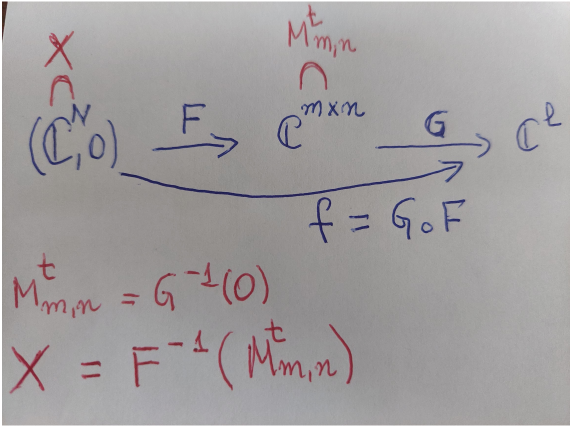


Figura: $X = V(I),\ I = \langle f_1, \ldots, f_\ell \rangle$

EIDS

Definition EIDS:

$(X,0) \subset (\mathbb{C}^N,0)$ has an essentially isolated determinantal singularity at the origin (EIDS) if any sufficiently small representative $F : U \to \mathbb{C}^{m\times n}$ is transverse to all strata $M^i_{m,n} \setminus M^{i-1}_{m,n}$ of the rank stratification of $M^t_{m,n}$ in a punctured neighbourhood of the origin.

If $x \in U$, $x \neq 0$, $rank\, F(x) = i-1$, then $F \pitchfork M^i_{m,n} \setminus M^{i-1}_{m,n}$ at x.

Deformations of EIDS

Deformations (in particular, smoothings) of determinantal singularities are themselves determinantal ones.

Definition

(Ebeling and Gusein Zade (2009)) *An essential smoothing* $\tilde{X}$ *of the EIDS* $(X, 0)$ *is a subvariety lying in a neighbourhood* U *of the origin in* $\mathbb{C}^N$ *and defined by a perturbation* $\tilde{F} : U \to \mathbb{C}^{m\times n}$ *of the germ* F *such that* $\tilde{F}$ *is transversal to all the strata* $M^i_{m,n} \setminus M^{i-1}_{m,n}$*, with* $i \leq t$.

Cohen-Macauly codimension 2 singularities

It follows from Hilbert-Burch Theorem that the germ of a Cohen-Macaulay variety of codimension 2 can be expressed as the maximal minors of $(s+1)\times s$ matrices and vice-versa. In the same way, flat deformations of these varieties can be represented by perturbations of the matrix $F(x) = (f_{i,j}(x))$ and any perturbation of the matrix gives rise to a flat deformation of the variety.

The following result follows as a corollary of Hilbert-Burch Theorem:

Theorem

Suppose $I \subset \mathbb{C}\{x_1, \ldots, x_p\}$ is an ideal with $\mathbb{C}\{x_1, \ldots, x_p\}/I$ Cohen-Macaulay of dimension $p-2$. Then I has a resolution of the form

$$0 \longrightarrow \mathbb{C}^n\{x\} \xrightarrow{A} \mathbb{C}^{n+1}\{x\} \xrightarrow{f} \mathbb{C}\{x\} \longrightarrow \mathbb{C}\{x\}/I \longrightarrow 0,$$

for some matrix $(n+1)\times n$ matrix A with entries in $\mathbb{C}\{x\}$, and $I = I_n$ is the ideal generated by the maximal minors of A.

Example

Determinantal surface in $\mathbb{C}^4$ Let F be the following map:

$$\begin{array}{cccc} F & : & \mathbb{C}^4 & \rightarrow & \mathbb{C}^{2\times 3} \\ & & (x,y,z,w) & \mapsto & \left(\begin{array}{ccc} x & y & z \\ w & x & y \end{array}\right) \end{array}$$

Then $X = F^{-1}(M^2_{2,3}) = V(x^2 - wy, xy - wz, y^2 - xz)$, X is a surface in $\mathbb{C}^4$ with isolated singularity at the origin.

Matrices Singularity Theory

The methods of singularity theory apply both to real and complex matrices.

- Arnol'd (1971), square matrices.
- Bruce (2003), simple singularities of symmetric matrices.
- Bruce and Tari (2004), simple singularities of square matrices.
- Haslinger (2001), simple skew-symmetric.
- Frühbis-Krüger (2000) and Frühbis-Krüger and Neumer (2010), Cohen-Macaulay codimension 2 simple singularities.
- Goryunov and Mond (2005), Tjurina and Milnor numbers of square matrices.
- M. Silva Pereira (2010), singularity theory of general $n \times m$ matrices.

The group $\mathcal{G}$ acting on $\mathcal{O}_N^{m\times n} = \{F : (\mathbb{K}^N, 0) \to \mathbb{K}^{m\times n})\}$ $\mathbb{K} = \mathbb{R}, \mathbb{C}.$

$\mathcal{R} = \{\phi : (\mathbb{K}^N, 0) \to (\mathbb{K}^N, 0),$ germs of analytic diffeomorphisms$\}$

$\mathcal{H} = GL_m(\mathcal{O}_N) \times GL_n(\mathcal{O}_N)$ and $\mathcal{G} = \mathcal{R} \times \mathcal{H}$ (semidirect product)

Definition

Given matrices $F_1(x) = (f_{ij}^1(x))_{m\times n}$ and $F_2(x) = (f_{ij}^2(x))_{m\times n}$, we say

$$F_1 \sim_{\mathcal{G}} F_2 \text{ if } \exists\ (\phi, R, L) \in \mathcal{G} \text{ such that } F_1 = L^{-1}(\phi^* F_2)R.$$

Proposition

If $F_1 \sim_{\mathcal{G}} F_2$ then the corresponding determinantal varieties $X_1^t = F_1^{-1}(M_{m,n}^t)$ and $X_2^t = F_2^{-1}(M_{m,n}^t)$, $1 \leq t \leq n$ are isomorphic.

Other groups acting on $\mathcal{O}_N^{m\times n}$

Let $\mathcal{V} = M_{m,n}^t$.

- $\mathcal{K}_{\mathcal{V}}$ equivalence acts on the graph of F by diffeomorphisms preserving $\mathcal{V}$.
- $\mathcal{K}_H$, where $H = 0$ is a good defining equation for $\mathcal{V}$ and the group acts on the level sets $H = c$.

Tangent Space to the $\mathcal{G}$-orbit of F.

We denote by $\Theta(F)$ the free $\mathcal{O}_N$ module of rank nm consisting of all deformations of $F : (\mathbb{K}^N, 0) \to \mathbb{K}^{m\times n}$.

The tangent space to the $\mathcal{G}$-orbit of F, $T\mathcal{G}(F) = T\mathcal{R}(F) + T\mathcal{H}(F)$.

$$T\mathcal{G}(F) = \mathcal{M}_N \left\{ \frac{\partial F}{\partial x_i} \right\} + \mathcal{O}_N \left\{ R_{lk}, C_{ij} \right\}$$

$$T\mathcal{G}_e(F) = \mathcal{O}_N \left\{ \frac{\partial F}{\partial x_i} \right\} + \mathcal{O}_N \left\{ R_{lk}, C_{ij} \right\}$$

where $C_{ij}(M)$ (respectively $R_{lk}(M)$) is the matrix which has the i-column (respectively l-row) equal to the j-column of M (respectively k-row) with zeros in other places.

$\mathcal{G}_e$-codimension of F

The group $\mathcal{G}$ is a geometric subgroup of the contact group $\mathcal{K}$. Hence the infinitesimal methods of singularity theory applies ([Miriam Pereira's thesis, 2010]).

- F is $\mathcal{G}$-stable if $T\mathcal{G}_e(F) = \Theta(F)$
- $F = (f_{ij})$ is $\mathcal{G}$- finitely determined if there exist k such that for all $G = (g_{ij})$, if $j^k f_{ij} = j^k g_{ij}$ then $F \sim_{\mathcal{G}} G$.
- $\mathcal{G}_e - cod(F) = \dim_{\mathbb{K}} \frac{\Theta_F}{T\mathcal{G}_e(F)}$

We define the determinantal Tjurina number as

$$\tau_d(F) = \mathcal{G}_e - cod(F)$$

For CMC2 singularities $\tau_d(F) = \tau(F)$, the Tjurina number of F.

Theorem

(M.S. Pereira, PhD thesis)

Let $F : U \to \mathbb{K}^{m\times n}$ The following conditions are equivalent

- $\tau_d(F)$ *is finite.*
- F *is* $\mathcal{G}$*-finitely determined.*
- F *has a versal unfolding with* $\mathcal{G}_e - cod(F)$*-parameters.*

$\mathbb{K} = \mathbb{C}$: Geometric criterion

F is $\mathcal{G}$-finitely determined $\iff X = F^{-1}(M^t_{m,n})$ is an EIDS for all $t \leq n$.

Example

$$A_k = \begin{pmatrix} x & y & z \\ w & z^k & x \end{pmatrix}, \forall\, k \geq 1.$$

Normal form of family of simple Cohen-Macaulay codimension 2 singularity (A. Frühbis-Krüger and A. Neumer (2010)).

The surface $X_k \subset \mathbb{C}^4$ is defined by $\langle xz^k - yw, x^2 - zw, xy - z^{k+1} \rangle$.

Versal deformation of F_k :

$$\widetilde{F}_k(x, y, z, w, u_0, u_1, \ldots, u_k) = \begin{pmatrix} x & y & z \\ w & z^k + \Sigma_0^{k-1} u_i z^i & x + u_k \end{pmatrix},$$

Singular fibration of an EIDS

$$F : (\mathbb{C}^N, 0) \to \mathbb{C}^{m\times n},\ (X, 0) = F^{-1}(M^t_{m,n})$$

$$\widetilde{F} : V \times W \subset \mathbb{C}^N \times \mathbb{C}^s \to \mathbb{C}^{m\times n}, \widetilde{F}(x, 0) = F(x),$$

$$\widetilde{F} \pitchfork \{M^i_{m,n} \setminus M^{i-1}_{m,n}\},\ \mathfrak{X} = \widetilde{F}^{-1}(\Sigma^t)$$

$$\begin{array}{ccc} \mathfrak{X} & \subset & V \times W \subset \mathbb{C}^N \times \mathbb{C}^s \\ & & \downarrow \pi \\ Bif(F) & \subset & W \subset \mathbb{C}^s \end{array},$$

where $Bif(F)$ is the bifurcation set.

$Bif(F)$ is a proper analytic subset of W.

For $u \in W \setminus Bif(F)$, $\widetilde{F}_u$ define $\widetilde{X}_u$, an essential smoothing of X.

$W \setminus Bif(F)$ is a connected set.

The generic fiber $\widetilde{X}_u$ is well defined: determinantal Milnor fiber.

(Singular fibration of the EIDS (Matthias Zach's PhD thesis).)

The topology of the determinantal Milnor fiber determines invariants of the EIDS X.

Invariants of an EIDS

Definition

(Damon and Pike [Geom. Topol., **18**(2) (2014)], Ebeling and Gusein-Zade (2009)) *The* *singular vanishing Euler characteristic of X*, *is*

$$\bar{\chi}(X) = \bar{\chi}(\widetilde{X}_u) = \chi(\widetilde{X}_u) - 1.$$

(Nuño-Ballesteros, Oréfice-Okamoto and Tomazella [Israel J. Math. **197** (2013), 475-495.]) *When $\widetilde{X}_u$ is smooth,*

$$\nu(X) = (-1)^{dim(X)}(\chi(\widetilde{X}_u) - 1).$$

The polar multiplicity $m_d(X)$, dim X=d

Introduced by T. Gaffney for ICIS, [Topology, 1993]

$$\pi : \widetilde{\mathcal{X}} \subset \mathbb{C}^N \times \mathbb{C} \to \mathbb{C},$$

a essential smoothing family of X,
$\pi^{-1}(0) = X$, $\pi^{-1}(s) = \widetilde{\mathcal{X}}_s$.

$S(\widetilde{\mathcal{X}})$: singular set of $\widetilde{\mathcal{X}}$, $\widetilde{\mathcal{X}}_{reg} = \widetilde{\mathcal{X}} - S(\widetilde{\mathcal{X}})$.

Choose $p : \mathbb{C}^N \to \mathbb{C}$ linear form , and let

$$(\pi, p)|_{\widetilde{\mathcal{X}}_{reg}} : \widetilde{\mathcal{X}}_{reg} \to \mathbb{C} \times \mathbb{C}$$

The relative polar curve of $\widetilde{\mathcal{X}}$ with respect to p:

$$P_d(\widetilde{\mathcal{X}}, p) = \overline{\Sigma(\pi, p)|_{\widetilde{(\mathcal{X})}_{reg}}}$$

If the *multiplicity* of the polar curve at the origin has the generic (minimum) value among all choices of p, then we call the polar curve with respect to p, the the relative polar curve of $\widetilde{\mathcal{X}}$, denoted $P_d(\widetilde{\mathcal{X}})$.

$$m_d(X) = m_0(P_d(\widetilde{\mathcal{X}})).$$

Proposition

[Gaffney-R] *With the above notation*

$$m_d(X) = n_0$$

where n_0 is the number of non degenerate critical points of $p|_{\widetilde{(\mathcal{X}_s)}_{reg}}$.

IDS admitting smoothing: N < (m-t+2)(n-t+2)

Theorem (Nuno-Ballesteros, Oréfice-Okamoto and Tomazella, Israel J. 2013)

Let $p : \mathbb{C}^N \to \mathbb{C}$ be a generic linear projection. Then

$$\nu(X,0) + \nu(X \cap p^{-1}(0), 0) = \#\Sigma(p|_{\tilde{X}}),$$

where $\#\Sigma(p|_{\tilde{X}})$ denotes the number of critical points of $p|_{\tilde{X}}$.

When $\dim X = 2$, $\nu(X) = \mu(X)$ (also in Pereira and Ruas [Math. Scand., 2014], Damon and Pike [Geom. Topol. 2014].)

$\#\Sigma(p|_{\tilde{X}}) = m_d(X).$

Analogous result for any EIDS were given in [Gaffney and Ruas,] based on a formula due to [Ebeling and Gusein-Zade, 2009] for the Poincaré-Hopf-Nash index of a holomorphic form with isolated zero defined on an EIDS:

Proposition (Prop. 1. Ebeling-Gusein Zade, 2009)

The PHN-index $ind_{PHN}\omega$ of the 1-form ω with isolated zero on the EIDS (X,0) is given by

$$ind_{PHN}\omega = n_0,$$

where n_0 is the number of non degenerate singular points of a generic deformation $\widetilde{\omega}$ of ω on $\widetilde{(\mathcal{X}_s)}_{reg}$.

Taking $\omega = dp$, p a generic linear form, $ind_{PHN}\omega = m_d(X)$.

Nash transform $\hat{\Sigma}^t$ of Σ^t is smooth.