Lecture 2: Equisingularity and Determinantal Singularities

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1 Equisingularity and Determinantal Varieties: Introduction

In this section we begin the study of the equisingularity of determinantal varieties. You might wonder why, since ICIS are ubiquitous. Indeed, if we look at all polynomial maps $P\colon \mathbb{C}^n \to \mathbb{C}^p$, $n \geq p$ where the degree of each component function is $\leq d$, these maps form a vector space, and the maps which define an ICIS define a Z-open space. Further as $d \to \infty$, the codimension of the complement goes to ∞.

However, there are many situations in which the set studied has additional structure, and we want to study families which preserve this structure. For example, the set may be the image of a finite map, and we may want to preserve this property in deformations. Deciding on the structure we want to preserve in deformations is the first step in deciding on the landscape of the singularity, and the possible deformations depend on this choice. For example, the Whitney umbrella, which is defined by $z^2 - x^2y = 0$, can be thought of as a hypersurface, the image of a map-germ, or as a hypersurface with smooth singular locus. The allowable deformations depend on the choice of landscape; the last two choices have only trivial deformations, while every hypersurface can be deformed to a smooth manifold.

In section two, we study determinantal singularities in the landscape of such singularities. The first step is to understand the allowable deformations.

Any deformation of the defining equations of an ICIS X^d, produces a family of ICIS of the same dimension. This is not true for determinantal singularities as the next example shows.

Example 1.1. *Let X be defined by the maximal minors of*

$$F_X = \begin{bmatrix} x & 0 \\ 0 & y \\ z & z \end{bmatrix}.$$

The maximal minors are xy, yz, xz, so X is the three coordinate axes in $\mathbb{C}^3$. Consider the family defined by $xy - tx, yz - ty, xz - tz$. For $t \neq 0$, $X(t)$ is the origin and the point (t, t, t). So the dimension drops for $t \neq 0$.

The obvious way to get families which are determinantal of the same type is to vary the entries of the matrix, then take minors. This suggests looking at all matrices of a fixed type and viewing our matrix as a map into the space of such matrices. Denote the set of $(n+k) \times n$ matrices with entries in $\mathbb{C}$ by $\mathbb{C}^{n+k\times n}$, and the matrices of rank$< t$ by $M^t(n+k, n)$. In our example, $f\colon \mathbb{C}^3 \to \mathbb{C}^{3\times 2}$, and $X = f^{-1}(M^2(3,2))$ then appears as a non-linear section of $M^2(3,2)$. We write ${}_2X$ to show that X is the pullback of M^2.

This suggests a further generalization, considering varieties that arise as non-linear sections of a "universal variety". In the next diagram V is the universal variety and X is the inverse image of V by f_0.

$$\begin{tikzcd}
k^q,0 \arrow[r, "f_0"] & k^N,0 \\
X,0 \arrow[u, hook] \arrow[r] & V,0 \arrow[u, hook]
\end{tikzcd}$$

This set-up includes determinantal singularities, symmetric singularities, skew-symmetric singularities, and discriminants of $\mathcal{A}$-finitely determined germs. The adaptation of elements of the Mather theory to non-linear sections of spaces has been done by Damon in [8]. In the last section we discuss the extension of the material here to this more general situation.

In the case of determinantal singularities, the universal objects are the $M^t(n+k,n)$. They have a locally holomorphically trivial stratification given by rank.

Recall ${}_tX$ is an essentially isolated determinantal singularity or EIDS if ${}_tX = f^{-1}(M^t(n+k,n))$, and $f\colon \mathbb{C}^q,0 \to \mathbb{C}^{n+k\times n}$, is transverse to the rank stratification of $M^t(n+k,n)$ except perhaps at the origin. Let $\tilde{F} : U \subset \mathbb{C}^q \times \mathbb{C} \to \mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^{n+k})$ be a one parameter deformation of $F : (\mathbb{C}^q,0) \to \mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^{n+k})$, such that $\tilde{F}_s$ is a stabilization of F for all $s \neq 0$ and hence $\widetilde{\mathcal{X}}_s = \widetilde{F}_s^{-1}(\Sigma^t)$ an *essential smoothing* of $X = F^{-1}(\Sigma^t)$ for all $s \neq 0$. We call $\tilde{F}$ a *stabilization family*, $\widetilde{\mathcal{X}}$ an *essential smoothing family.*

In a previous lecture we have seen that if $q < (n+k-i+2)(n-i+2)$, $1 < i \leq n$, then ${}_iX$ has a determinantal smoothing–the generic element of this type is smooth; if $i = 1$, then ${}_1X$ is an ICIS, and always has a smoothing. If $q \geq (n+k-i+2)(n-i+2)$, $1 < i \leq n$, then ${}_iX$ does not have a smoothing, but deforms to a transverse slice of M^i. Knowing the generic element to which a determinatal variety deforms is part of knowing the landscape.

Knowing the possible deformations of X allows us to determine the possible infinitesimal deformations of X.

This lecture builds on the previous lecture on the equisingularity of ICIS. Our goal is to generalize the following two theorems to the EIDS case:

Theorem 1.2. *Let $\mathcal{X}$ be a family of ICIS over Y^k as in the basic setup. Suppose $e(mJM(X(y),0))$ is independent of y. Then $X - Y$ is smooth, and the pair $(X - Y, Y)$ satisfies W.*

Theorem 1.3. *(Necessity) Suppose $\mathcal{X}$ is a family of ICIS, and the pair $(\mathcal{X} - Y, Y)$ satisfies W at the origin. Then, the μ_* sequence of $X(y)$ is independent of y, as is $e(m_yJM(X(y)))$.*

There are several obstacles to overcome to do this.

The Jacobian module of X and its invariants remain a basic building block in the study of equisingularity, because its integral closure is linked to condition W. However, the set of infinitesimal deformations of X is not a free $\mathcal{O}_X$ module. In section two, we will define the determinantal normal module $N(X)$, which will be the necessary infinitesimal deformations in our case. Further, $e(JM(X))$ is not defined in almost all cases considered; we will replace it by the multiplicity of the pair, $e(JM(X),N(X))$, showing in section 2 that this is well-defined for EIDS, and discussing the definition in section 3.

In the ICIS case, we can take a smoothing of X, and a generic linear form l; then the number of critical points of l on the fibers of the smoothing is an invariant, $m_d(X)$. In the ICIS case, this is $e(JM(X))$. This is no longer true in the determinantal case. We introduce the invariant $m_d(X^d)$ in the fourth section, and show its connection to the topology of the stabilization of X and its generic sections.

In order to understand the connection between $e(JM(X),N(X))$ and $m_d(X)$ we will need to understand how $e(JM(X),N(X))$ varies in a family. We do this in the third section using the Multiplicity-

Polar theorem. In section 4, We use the Multiplicity-Polar theorem to relate $e(JM(X), N(X))$ and $m_d(X)$. A new term appears, which measures the curvature of $N(X)$.

This package of ideas is the landscape of the singularity. (For more discussion of the idea of landscape, cf. [23].)

Having surmounted all of our obstacles in Section 5 we prove our generalizations of the ICIS equisingularity theorems. These notes are essentially a report on [30] which is a joint work with Maria Aparecida Ruas.

2 EIDS and the determinantal normal module

In this section we study the determinantal normal module, which is the first order infinitesimal deformations for our choice of landscape.

We describe how to construct $N(X)$,the determinantal normal module of X, where $X = f^{-1}(M^t(n+k, n))$, and $f\colon \mathbb{C}^q, 0 \to \mathbb{C}^{n+k\times n}$. Let $\Delta(l_1, \ldots, l_t, m_1, \ldots, m_t) = \Delta(L, M)$ denote the minor of f obtained using rows $(l_1, \ldots, l_t)$, and columns $m_1, \ldots, m_t$. (We assume $l_i < l_{i+1}, m_i < m_{i+1}$ for $1 \leq i \leq t-1$.) Let $\delta_{i,j}$ be the $(n+k) \times n$ matrix with 1 in the (i, j) entry, other entries 0. Consider the deformation of X given by $f + t\delta_{i,j}$. Each such deformation gives a column in a matrix of generators for $N(X)$ as follows: the $M_{L,M}$ entry of the (i, j) column is gotten by taking the (L, M) minor of $M_X + t\delta_{i,j}$, and taking the linear part of this in t. This entry is the cofactor of $f_{i,j}$ in the expansion of $\Delta(L, M)$. Of course, it is 0 if $i \notin L$ or $j \notin M$.

The determinantal normal module of $M^t(n+k, n)$ denoted N^t is defined using the identity map on $\mathbb{C}^{n+k\times n}$. The modules $N(X)$ and N^t have many nice properties which we collect here.

Proposition 2.1. *Let $X = f^{-1}(M^t(n+k, n))$, $\mathcal{X}$ a deformation of X and $f\colon \mathbb{C}^q, 0 \to \mathbb{C}^{n+k\times n}$ then*

1. $N^t = JM(M^t)$
2. $N(X) = f^*(N^t)$
3. $N(\mathcal{X})(y) = N(\mathcal{X}(y))$.

Proof. 1) $N^t = JM(M^t)$. M^t is defined by G^t the tuple of $t \times t$ minors of identity map. Then 1) follows because the construction of the (L, M)-th element of the (i, j) column of the generating matrix of N^t gives the same result as the definition of the partial derivative with respect to $a_{i,j}$ of the (L, M) minor.

2) $N(X) = f^*(N^t)$. Composition with f takes the generators of N^t to the generators of $N(X)$; since $f^*(N^t)$ is the $\mathcal{O}_X$ module generated by these pullbacks, the result follows.

3) $N(\mathcal{X})(y) = N(\mathcal{X}(y))$. Exercise. $\square$

If a singularity has the property that its Jacobian module is the same as its module of allowable infinitesimal deformations, we say the singularity is *stable*. If the module of allowable infinitesimal deformations of any non-linear section of a space X is the pullback of $N(X)$ we say the module $N(X)$ is *universal*. By 1) and 2) of the last proposition we see that N^t is universal, and M^t is stable. (Sometimes we say N^t is stable as well.)

We have that $JM(X)$ is generated by the columns of $D(G^t \circ f)$, while the proposition shows $N(X)$ is generated by the columns of $(DG) \circ f$. This is a useful observation, because it shows us exactly how $JM(X)$ sits inside $N(X)$. The chain rule will allow us to relate Df and $JM(X)$ and $N(X)$.

When we studied ICIS singularities, we always considered the relation between $JM(X)$ and $\mathcal{O}^p_X$. Indeed, X^d is an ICIS if and only if $JM(X)$ has finite colength inside $\mathcal{O}^{n-d}_X$. This length condition is equivalent to $JM(X,x) = \mathcal{O}_{X,x}^{n-d}$ for $x \neq 0$. In the ICIS case the module of allowable infinitesimal deformations is $\mathcal{O}_X^{n-d}$. We will show that X is an EIDS if and only if $\mathcal{O}_X^p$ has finite colength inside $N(X)$.

This will imply that the multiplicity of the pair $e(JM(X), N(X))$ is defined. (The multiplicity of a pair of modules will be studied in the next section.) We will do this by showing that the transversality of f to the strata of M^t off the origin implies that off the origin the two modules agree. To do this we use the holomorphic triviality of the stratification of M^t. To show that our multiplicity is well defined, it is actually enough to have a Whitney stratification, so we treat this case as well in the next lemmas.

We first work on the target, then on the source. Recall that if (S,x) is the germ of a submanifold at x, then a direct transversal to S at x is a plane T of dimension complementary to S which is transverse to S. Given such a plane T, we may make a change of coordinates so that the ambient space is holomorphic to $T \times S$. In this coordinate system we denote by $JM(X)_T$, the submodule of $JM(X)$ generated by the partial derivatives of the defining functions with respect to a basis of T.

Lemma 2.2. *Let X be a stratified subset of $(\mathbb{C}^N, x)$.*

i) *Suppose X has a locally holomorphically trivial stratification. Let S_x denote the stratum containing x. Suppose T is a direct transversal to S at x. Then $JM(X) \subset JM(X)_T$.*

ii) *Suppose the stratification is a Whitney stratification, and let S_x denote the stratum containing x. Suppose T is a direct transversal to S at x. Then $JM(X) \subset \overline{JM(X)_T}$.*

Proof. i) Choose local coordinates so that the ambient space is holomorphic to $S_x \times T$, and $X = f^{-1}(0)$. Since S_x is a stratum in a holomorphically trivial stratification, with trivialization of form $(s, r(s,t))$, $r(s,0) = (0)$, for all i, $\frac{\partial}{\partial s_i}$ lifts to a holomorphic field ξ tangent to every stratum of X, $\xi = \frac{\partial}{\partial s_i} + \sum h_j(s,t)\frac{\partial}{\partial t_j}$, $h_j(s,t) \in m_S$. This implies that $Df(\xi) = 0$, hence, $\frac{\partial f}{\partial s_i} = -\sum h_j(s,t)\frac{\partial F}{\partial t_j}$, hence $JM(X)_S \subset m_S JM(X)_T$.

ii) Since S_x is a stratum in a Whitney stratification, it is known ([14], Theorem 2.5) that $JM(X)_S \subset \overline{m_S JM(X)_T}$, from which the result follows.

□

Lemma 2.3. *Suppose $F : (\mathbb{C}^q, 0) \to (\mathbb{C}^N, x)$, $x \in X$, X a stratified subset of $(\mathbb{C}^N, x)$.*

i) *Suppose X has a Whitney stratification. Let S_x denote the stratum containing x. Suppose F is transverse to S_x at x. Let X_F denote $F^{-1}(X)$. Then $F^*(JM(X)) \subset \overline{JM(X_F)}$.*

ii) *Suppose X has a locally holomorphically trivial stratification. Let S_x denote the stratum containing x. Suppose F is transverse to S_x at x. Let X_F denote $F^{-1}(X)$. Then $F^*(JM(X)) \subset JM(X_F)$.*

Proof. i) Since F is transverse to S_x at x, there is a linear space T_F of $\mathbb{C}^q$, whose image under DF is a direct transversal to S_x at x, $DF(0)$ injective on T_F. We assume coordinates on the target chosen to fit with the previous lemma and coordinates on the source so that $\mathbb{C}^q$ is the product of T_F and a direct transversal A, the $DF(0)$ takes T_F to T in these coordinates, z coordinates on T_F, y' coordinates on A. Let G be the map defining X, so that $G \circ F$ defines X_F. Let $\phi : \mathbb{C}, 0 \to X_F, x$ be a curve. By the chain rule $JM(X_F)$ is a submodule of $F^*JM(X)$. The generators of $JM(X_F)$, when composed with ϕ, by the chain rule and choice of coordinates are

$$\{\frac{\partial(G \circ F)}{\partial z_i} \circ \phi = \frac{\partial G}{\partial z_i} \circ F \circ \phi\} \mod m_1 \phi^* F^* JM(X)$$

as the other partial derivatives of $G \circ F$ are $0 \mod m_1 \phi^* F^* JM(X)$. Since $\frac{\partial G}{\partial z_i}$ pull back to generators of $\phi^* F^*(JM(X))$ on the curve $F \circ \phi$ by the previous lemma, by Nakayama's lemma the $\{\frac{\partial(G \circ F)}{\partial z_i} \circ \phi\}$ generate $\phi^* F^*(JM(X))$ as well. Hence, $F^*(JM(X)) \subset \overline{JM(X_F)}$ by the curve criterion.

The proof of ii) is even easier. We again use the chain rule and local coordinates, this time applying Nakayama's lemma to $\{\frac{\partial(G\circ F)}{\partial z_i}\}$ and $F^*JM(X)$.

□

Proposition 2.4. *Let $F\colon \mathbb{C}^q \to \mathbb{C}^{mn}$, $X = F^{-1}(M^t)$, F transverse to the rank stratification of M^t at the origin. Then $JM(X) = F^*(N(M^t)) = N(X)$.*

Proof. Since the stratification of M^t by rank is holomorphically trivial, by the last proposition we know that $JM(X) = F^*(JM(M^t))$. Since N is stable and universal, $F^*(N(M^t)) = N(X)$,which implies the result.

□

In a later lecture, the notion of *holonomic strata* will be defined. The strata in the rank stratification of M^t are all holonomic; hence f transversal to the rank stratification implies f is *log transversal.*

3 Multiplicity of pairs of modules

Now we want to give an intersection theoretic definition of the multiplicity of a module. It agrees with the length theoretic definition of Buchsbaum-Rim when the latter is well defined ([38]). This definition extends the notion of multiplicity to pairs of modules as well.

We recall briefly some ideas from our ICIS lecture needed to understand the definition.

Given a submodule M of a free $\mathcal{O}_{X^d}$ module F of rank p, we can associate a subalgebra $\mathcal{R}(M)$ of the symmetric $\mathcal{O}_{X^d}$ algebra on p generators. This is known as the *Rees algebra* of M. If $(m_1,\dots,m_p)$ is an element of M then $\sum m_iT_i$ is the corresponding element of $\mathcal{R}(M)$. Then Projan$(\mathcal{R}(M))$, the projective analytic spectrum of $\mathcal{R}(M)$ is the closure of the projectivised row spaces of M at points where the rank of a matrix of generators of M is maximal. Denote the projection to X^d by c, or by c_M where there is ambiguity.

If M is a submodule of N or h is a section of N, then h and M generate ideals on Projan $\mathcal{R}(N)$; denote them by $\rho(h)$ and $\mathcal{M}$. If we can express h in terms of a set of generators $\{n_i\}$ of N as $\sum g_in_i$, then in the chart in which $T_1 \neq 0$, we can express a generator of $\rho(h)$ by $\sum g_iT_i/T_1$.

The next diagram shows the spaces that come into the definition of $e(M,N)$.

$$\begin{array}{ccc} B_{\mathcal{M}}(\text{Projan}\,\mathcal{R}(N)) & \xrightarrow{\pi_N} & \text{Projan}\,\mathcal{R}(N) \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \pi_{XN}} \\ \text{Projan}\,\mathcal{R}(M) & \xrightarrow{\pi_{XM}} & X \end{array}$$

On the blow up $B_{\mathcal{M}}(\text{Projan}\,\mathcal{R}(N))$ we have two tautological bundles. One is the pullback of the bundle on Projan $\mathcal{R}(N)$. The other comes from Projan $\mathcal{R}(M)$. Denote the corresponding Chern classes by c_M and c_N, and denote the exceptional divisor by $D_{M,N}$. Suppose the generic rank of N (and hence of M) is g.

Then the multiplicity of a pair of modules M,N is:

$$e(M,N) = \sum_{j=0}^{d+g-2} \int D_{M,N} \cdot c_M^{d+g-2-j} \cdot c_N^j.$$

Kleiman and Thorup show that this multiplicity is well defined at $x \in X$ as long as $\overline{M} = \overline{N}$ on a deleted neighborhood of x. This condition implies that $D_{M,N}$ lies in the fiber over x, hence is compact.

Notice that when $N = F$ and M has finite colength in F then $e(M,N)$ is the Buchsbaum-Rim multiplicity $e(M, \mathcal{O}^p_{X,x})$.

Kleiman and Thorup also showed that $e(M,N)$ vanishes if and only if M and N have the same integral closure, provided the support of N is equidimensional. ([38], (6.3)(ii).)

Remark 3.1. *We have seen that there is a map from* $\operatorname{Projan}\mathcal{R}(N) \setminus V(\mathcal{F}) \to \operatorname{Projan}\mathcal{R}(M)$. *The diagram used in the definition of* $e(M,N)$ *can be used to make this more precise. Namely, the complement of* $\pi_M D_{M,N}$ *is the largest open subset* V *of* $\operatorname{Projan}\mathcal{R}(M)$ *such that the map* $\pi_M^{-1}V \setminus D_{M,N} \to V$ *is finite. Plainly,* π_N *is an isomorphism over the complement* U *of* $V(\mathcal{F})$*, and* $\pi_N^{-1}U$ *contains* $\pi_M^{-1}V$.

Let's re-calculate two examples using this definition.

Example 3.2. *Let* $M = I = (x^2, xy, y^2) \subset \mathcal{O}_2$. *Then* $e(M) = 4$.

We can simplify our work by replacing M by a reduction (x^2, y^2), which we also denote by M. Then $d = 2$, $p = g = 1$, $\operatorname{Projan}\mathcal{R}(N) = \mathbb{C}^2$, hence $c_N^j = 0, j \neq 0, 1$ if $j = 0$, $\operatorname{Projan}(\mathcal{M}) = B_I(\mathbb{C}^2) = B_{\mathcal{M}}(\operatorname{Projan}\mathcal{R}(N))$, and $\operatorname{Projan}(\mathcal{M}) \subset \mathbb{C}^2 \times \mathbb{P}^1$. So the only term we need to calculate is $\int D_{M,N} \cdot c_M$. We can calculate this term as follows: Intersect $B_I(\mathbb{C}^2)$ with $\mathbb{C}^2 \times H$, H a generic hyperplane in $\mathbb{P}^1$, which represents $c(M)$. Project this curve to $\mathbb{C}^2$, and calculate the order of I on the curve. Projecting the curve to $\mathbb{C}^2$ amounts to setting a generic combination of the generators to zero, and looking at the curve obtained, removing any components in $V(I)$. In this case a generic curve is $x^2 - ay^2 = 0$, $a \neq 0$. This consists of two branches ($x - y = 0$ and $x + y = 0$ if $a = 1$) and the colength of the ideal on each branch is 2 so the multiplicity is $2 + 2 = 4$.

Example 3.3. *Let* $[M] = \begin{bmatrix} x & y & 0 \\ 0 & x & y \end{bmatrix}$. *Show* $e(M) = 3$.

Here $d = 2$, $p = g = 2$, $N = \mathcal{O}_2^2$, $\operatorname{Projan}\mathcal{R}(N) = \mathbb{C}^2 \times \mathbb{P}^1$, $\operatorname{Projan}\mathcal{R}(M) \subset \mathbb{C}^2 \times \mathbb{P}^2$, dimension of $B_{\mathcal{M}}(\operatorname{Projan}\mathcal{R}(N))$ is 3. So we need to calculate $\int D_{M,N} \cdot c_M^2$, $\int D_{M,N} \cdot c_M \cdot c_N$ (Notice that $c_N^2 = 0$, since we are working on $\operatorname{Projan}\mathcal{R}(N) = \mathbb{C}^2 \times \mathbb{P}^1$.) Now we have two choices: as before we intersect a representative of each class with the blow-up then push down to X, then see what the multiplicity of M is on each curve. Or, we can push down to $\operatorname{Projan}\mathcal{R}(N)$ and evaluate $\mathcal{M}$ on each curve. (For details of how this approach works, the reader should consult [16] Theorem 3.1 and the two examples which follow.)

Taking the second route, projecting the intersection of the blow-up with a hyperplane from $\mathbb{C}^2 \times \mathbb{P}^1$ and a hyperplane from $\mathbb{C}^2 \times \mathbb{P}^2$, is a curve on $\mathbb{C}^2 \times \mathbb{P}^1$, defined by a linear relation $T_1 = aT_2$, and by setting one of the elements of $\mathcal{M}$ restricted to this set to zero. The restriction of $\mathcal{M}$ to the locus $T_1 = aT_2$ is the ideal generated by the entries of the linear combination of the first row and a times the second row from the original matrix. A generic curve is given by setting $x + ay = 0$, and the multiplicity of $\mathcal{M}$ on this curve is 1. So, $\int D_{M,N} \cdot c_M \cdot c_N = 1$.

Projecting the intersection of the blow-up with two hyperplanes from $\mathbb{C}^2 \times \mathbb{P}^2$, amounts to setting two generic elements of $\mathcal{M}$ to zero and removing any components of $V(\mathcal{M})$. Setting $xT_1 + yT_2$ and $yT_1 + xT_2 = 0$ gives two curves. One curve is $x = y$, $T_1 = 1 = -T_2$ and the other curve is $x = -y$, $T_1 = 1 = T_2$.

The restriction of $\mathcal{M}$ to the first curve is x so the multiplicity is 1; as it is on the second curve as well, for a total of 3.

Notice that in the last example $3 = e(M) \neq e(J(M)) = 4$. (J(M) is the ideal of maximal order non-vanishing minors, and is (x^2, xy, y^2) in this case.) But,

Problem 3.4. *Suppose* $M \subset N \subset F$ *are* m *primary* $\mathcal{O}_{X,x}$ *modules,* X, x *equidimensional. Show that* $e(M) = e(N)$ *if and only if* $e(J(M)) = e(J(N))$.

There are examples though, where there is a family of ICIS singularities where $e(JM(X_y))$ is independent of y, but $e(J(JM(X_y)))$ is not. In the example due to Henry and Merle, the embedding dimension of the singularity changes at $y = 0$–the singularity goes from being codimension 2 to being codimension 1, because one of the defining equations is no longer singular off the origin. Is this the only way for the connection between the two invariants to break?

Challenge Problem 3.5. *Give a geometric characterization of when $e(JM(X_y))$ is independent of y, but $e(J(JM(X_y)))$ is not.*

This problem is connected with the difference between using the conormal modification to study equisingularity conditions and using the Nash modification, which is why it is interesting. In the ICIS case a difference in the value of the multiplicity between the generic point y and the origin implies there is a jump in the dimension of the fiber of the exceptional divisor over the origin. So if the value of $e(JM(X_y))$ is independent of y, but $e(J(JM(X_y)))$ is not, then the set of limiting tangent planes has a jump in dimension at the origin, but the set of limiting tangent hyperplanes does not.

Reading In section 3 of [16] these ideas are developed further. It also contains the example due to Henry and Merle mentioned above.

There is an important case where it is easy to calculate the multiplicity of the pair. Suppose we are given $\mathcal{O}_X$ modules $M \subset N \subset F$, where F is free, X has dimension 1, and $e(M, N)$ is defined. We want a procedure to calculate $e(M, N)$. The first step is to find a normalization $\tilde{X}, n$ of the curve. Then we can use the following proposition.

Proposition 3.6. *Suppose X is a curve singularity, then $e(M, N) = e(n^*(M), n^*(N))$.*

Proof. This is a corollary of theorem 5.1 of [36]. □

We'll illustrate the rest of the procedure with an example taken from [28]. The procedure is also described in [36].

The curves we consider are the X_l, defined by the minors of

$$F_l = \begin{bmatrix} z & x \\ y & z \\ x^l & y \end{bmatrix}.$$

We assume $l - 1$ is not divisible by 3. With this assumption we have a normalization given by $(\mathbb{C}, n_l)$ where $n_l(t) = (t^3, t^{2l+1}, t^{l+2})$. The assumption on l means that the exponents on the first and last terms in the formula for n are relatively prime. The form of n is a reflection of the fact that X_l is weighted homogeneous with weights $(3, 2l + 1, l + 2)$.

In this example the module N is $F_l^*(JM(\Sigma^2))$ where Σ^2 is the linear maps of rank< 2, and we view F_l as map from $\mathbb{C}^3 \to Hom(\mathbb{C}^2, \mathbb{C}^3)$. Then $M = JM(X_l)$.

The next step is to find a minimal set of generators for $n_l^*(N)$ and $n_l^*(M)$. Pulling back the generators of $JM(\Sigma^2)$ using $F_l \circ n_l$, we get:

$$n_l^*(N) = \begin{bmatrix} t^{l+2} & -t^3 & 0 & -t^{2l+1} & t^{l+2} & 0 \\ 0 & t^{2l+1} & -t^{l+2} & 0 & -t^{3l} & t^{2l+1} \\ t^{2l+1} & 0 & -t^3 & -t^{3l} & 0 & t^{l+2} \end{bmatrix}.$$

As this matrix has generic rank 2, $n_l^*(N)$ can be generated freely by 2 generators since we are working over $\mathcal{O}_1$, so a matrix of generators R_N of $n_l^*(N)$ with a minimal number of columns is

$$R_N = \begin{bmatrix} -t^3 & 0 \\ t^{2l+1} & -t^{l+2} \\ 0 & -t^3 \end{bmatrix}.$$

A calculation shows that $n_l^*(JM(X))$ is generated by the columns of:

$$R_{JM} = \begin{bmatrix} -t^3 & 2t^{l+2} \\ 2t^{2l+1} & -t^{3l} \\ t^{l+2} & t^{2l+1} \end{bmatrix}.$$

Note that

$$R_{JM} = R_N \begin{bmatrix} 1 & -2t^{l-1} \\ -t^{l-1} & -t^{2l-2} \end{bmatrix}.$$

Denote the submodule of $\mathcal{O}_1^2$ whose matrix of generators is the 2×2 matrix in the last line by K. Since $n_l^*(N)$ is freely generated, it is isomorphic to $\mathcal{O}_1^2$. The isomorphism carries the pair $(n_l^*(JM(X)), n_l^*(N))$ to $(K, \mathcal{O}_1^2)$. Then $e(n_l^*(JM(X)), n_l^*(N)) = e(K, \mathcal{O}_1^2)$. Since $\mathcal{O}_1$ is Cohen-Macaulay, the multiplicity of the second pair is the colength of the determinant of the matrix of generators of K, which is $2l - 2$.

Polar Varieties of a Module

Intuitively, the polar varieties of a module measure the "curvature" of $\operatorname{Projan} \mathcal{R}(M)$, and we have encountered them in the examples of the previous paragraph. As we shall see, the projection of $B_{\mathcal{M}}(\operatorname{Projan} \mathcal{R}(N)) \cdot c_M^2$ to $\mathbb{C}^2$, studied in Example 3.3 is the polar curve of M.

The *polar variety of codimension* l of M in X, denoted $\Gamma_l(M)$, is constructed by intersecting $\operatorname{Projan} \mathcal{R}(M)$ with $X \times H_{g+l-1}$ where H_{g+l-1} is a general plane of codimension $g+l-1$, then projecting to X.

So, in the setting of Example 3.3, $g = 2$, and $g + l - 1 = 2 + 1 - 1 = 2$, and the projection of $B_{\mathcal{M}}(\operatorname{Projan} \mathcal{R}(N)) \cdot c_M^2$ to $\operatorname{Projan} \mathcal{R}(M)$ is the intersection of $\mathbb{C}^2 \times H_2$ with $\operatorname{Projan} \mathcal{R}(M)$. Thus the projection of $B_{\mathcal{M}}(\operatorname{Projan} \mathcal{R}(N)) \cdot c_M^2$ to $\mathbb{C}^2$ is $\Gamma_1(M)$.

The polar varieties of M can be constructed by working only on X. The plane H_{g+l-1} consists of all hyperplanes containing a fixed plane H_K of dimension $g + l - 1$. By multiplying the matrix of generators of M by a basis of H_K we obtain a submodule of M denoted M_H.

Proposition 3.7. *In this set-up the polar variety of codimension l consists of the closure in X of the set of points where the rank of M_H is less than g, and the rank of M is g.*

Proof. Since H_{g+l-1} is generic, the general point of $\operatorname{Projan} \mathcal{R}(M) \cap X \times H_{g+l-1}$ lies over points where the rank of M is g. Choose coordinates so that a basis for H_K consists of the last $g + l - 1$ elements of the standard basis of $\mathbb{C}^j$, j the number of generators of M. We can find v such that $v[M_H] = 0$ but $v[M] \neq 0$ if and only if we are at a point where the rank of $M_H < g$. The existence of v is equivalent to being able to find a combination of the rows of $[M]$, such that the last $g + l - 1$ entries are 0. This row is a hyperplane which lies in H_{g+l-1}. $\square$

Teissier ([51], [52]) defined the polar varieties of an analytic germ $(X^d, x) \subset \mathbb{C}^n$ of codimension l as follows: take a generic projection π of $X^d \to \mathbb{C}^{d-l+1}$, and take the closure of the critical points of the restriction of the projection to the smooth points of X. Using the last proposition, it is easy to see that these polar varieties are the polar varieties of the Jacobian module of X.

For, given $(X^d, x) \subset \mathbb{C}^n$, the generic rank g of the Jacobian module of X is $n-d$. The kernel of a generic projection to $\mathbb{C}^{d-l+1}$ has dimension $n-d+l-1 = g+l-1$. Let the fixed plane H_K in the previous proposition be the kernel of π. Then the rank of M_H is less than maximal at a smooth point of X if and only if the tangent space of X has larger than expected intersection with the kernel of π. Thus, a tangent hyperplane of X contains H_K at a smooth point of X if and only if x is a critical point for the restriction of the projection to X at x. Thus the two notions of polar variety coincide.

If M is an ideal and we are working on X, then M_H is a sheaf of ideals and the polar varieties are the closure of the set defined by this sheaf on the complement of $V(M)$.

Problem 3.8. *Given $M \subset N \subset \mathcal{O}^p_{X,x}$, M and N both $\mathcal{O}_X$ modules, M induces an ideal sheaf on* $\operatorname{Projan}\mathcal{R}(N)$*, and we can define the polar varieties of this ideal sheaf. (To do this we must work on the fiber of* $\operatorname{Projan}\mathcal{R}(N)$ *over x.) Show that the projection of the polar of dimension d defined in this way to X is $\Gamma^d(M)$.*

Thus, there are 4 different settings for studying the polar varieties. It is often useful in proofs to move between them.

There is a special case which will be important to us. The diagram below represents the smoothing of an isolated singularity.

$$\begin{array}{ccc} X^d(0) \subset & \mathcal{X}^{d+1} \subset Y \times \mathbb{C}^N & \supset \mathcal{X}(y) \\ \downarrow & \downarrow{\scriptstyle p_Y} & \downarrow{\scriptstyle \pi_Y} \\ 0 \in & Y = \mathbb{C} & \supset y \neq 0 \end{array}$$

Let $M = JM_z(\mathcal{X})$, Then $\Gamma_d(\mathcal{X})$ by the previous proposition is defined by selecting $N-1$ generic generators of $JM_z(\mathcal{X})$, and looking to see where they have less than maximal rank. Assume coordinates chosen so that the first $N-1$ columns of $[JM(\mathcal{X})]$ are generic. Then the points where the polar intersects $\mathcal{X}(y)$ are the critical points of z_N restricted to $\mathcal{X}(y)$. The number of such points is the number of sheets of $\Gamma_d(\mathcal{X})$ over Y is the multiplicity of $\Gamma_d(\mathcal{X})$ over Y at the origin. If the smoothing is unique up to diffeomorphism, then the invariant is denoted $m_d(X)$. It is clear that the number of critical points of a generic linear form on a smoothing of X is important to the topology of $\mathcal{X}(y)$, so this number is an important invariant of X. We have already met $m_d(X^d)$ if X is an ICIS in the lecture on the equisingularity of ICIS. In the next section we see that it is well defined if X is an EIDS.

By construction, the existence of a polar variety of M at $x \in X$ is tied to the dimension of the fiber of $\operatorname{Projan}\mathcal{R}(M)$ over x.

Problem 3.9. *Suppose X^d, x equidimensional and M has the same generic rank g on each component of X at x. Show that $\Gamma_l(M, x)$ is non-empty if and only if the dimension of the fiber of* $\operatorname{Projan}\mathcal{R}(M)$ *over x is greater than or equal to $l+g-1$.*

Since the Whitney conditions are controlled by the dimension of the fiber of the exceptional divisor of $B_{m_Y}(C(\mathcal{X}))$, and the dimension of the fibers are detected by the presence of the polar varieties of the relative Jacobian module, it is reasonable to look for a connection between invariants associated with integral closure and those associated with polar varieties.

An approach for linking the behavior of the multiplicity of an ideal in a family to the degree of the exceptional divisor is given by Teissier in [51, p. 345]. We include an excerpt from this reference where this idea is mentioned.

Soient $f: (X,0) \to (\mathbb{D},0)$ un morphisme d'espaces réduits, I un idéal de $\mathcal{O}_X$ définissant un sous-espace $Y \subset X$ tel que $f|Y : Y \to \mathbb{D}$ soit fini, $p: X' \to X$ l'éclatement de Y, $D_{vert.}$ la réunion des composantes du diviseur exceptionnel D (non nécessairement réduites) dont l'image ensembliste par p est 0, $\deg D_{vert.} = \deg(\mathcal{O}_{D_{vert.}}(1))$. Pour tout représentant suffisamment petit du germe de f en 0, on a l'égalité

$$\deg D_{vert.} = e(I \cdot \mathcal{O}_{X(0)}) - e(I \cdot \mathcal{O}_{X(s)}) \quad (\text{pour } s \neq 0) \quad .$$

En particulier, on a "$e(I.\mathcal{O}_{X(s)})$ est indépendant de $s \in \mathbb{D}$" si et seulement si $\dim p^{-1}(0) = \dim X - 2$.

Here is how we can understand Teissier's formula. The fiber of the exceptional divisor over $0 \in X^{d+1}$ is a projective variety so it has a degree. When we intersect this variety with a linear space of complementary dimension, on the one hand, the number of points we get is the degree of the variety, on the other, because intersecting $B_I(X)$ with this linear space defines the polar curve of I, it is the number of points in the polar curve over a generic t value. Call this number $m_d(I, X)$. Now one way to define the polar curve is to pick d generic elements of I, chosen so that they define a reduction of $I(0)$ and are a reduction of I on the total space over $\mathbb{D} - 0$, and see where they are zero. Call this ideal J. By construction the points where they are zero outside of $V(I)$, will be a Z-open and dense set of the polar curve, and at points of $V(I)$, $\overline{I(y)} = \overline{J(y)}$ and so $e(I(y)) = e(J(y))$ at such points. Since J is generated by d elements, a lemma shows that $e(J(y))$ is independent of y. So

$$degD_{vert} = m_d(I, X) = e(J \cdot \mathcal{O}_{X(0)}) - e(J \cdot \mathcal{O}_{X(y)}) = e(I \cdot \mathcal{O}_{X(0)}) - e(I \cdot \mathcal{O}_{X(y)}).$$

If we extend this approach to pairs of modules we find that the polar variety of N enters as well as the polar variety for M.

Set-up: $M \subset N \subset F$, a free $\mathcal{O}_X$ module, X equidimensional, a family of sets over Y, with equidimensional fibers, Y smooth, $\overline{M} = \overline{N}$ off a set C of dimension k which is finite over Y.

Let $\Delta(e(M, N)) = e(M(0), N(0), \mathcal{O}_{X(0)}, 0) - e(M(y), N(y), \mathcal{O}_{X(y)}, (y, x))$ be the change in the multiplicity of the pair (M, N) as the parameter changes from y to 0.

Theorem 3.10. *(Multiplicity Polar Theorem, [15], [16]):*

$$\Delta(e(M, N)) = mult_y \Gamma_d(M) - mult_y \Gamma_d(N)$$

Many applications of this theorem can be found in: [16], [18], [19], [28], [26].

Here are some problems to help you get familiar with the Multiplicity Polar Theorem. The second one may be a little hard.

Problem 3.11. *Let $I(t) = (x^2 - t, y^2 - t)$, $J(t) = (x^2 - t, y^2 - t, xy)$, working on $\mathbb{C} \times \mathbb{C}^2$. Show that the Multiplicity Polar Theorem is true in this case by calculating all of the terms.*

Problem 3.12. *If $I(t)$ is a family of ideals, show that*

$$\Delta(e(I, \mathcal{O}_X)) = mult_y \Gamma_d(I)$$

Problem 3.13. *The multiplicity of ideals is known to be additive–that is if $I \subset J$, then $e(I) = e(J) + e(I, J)$, if $e(I)$ and $e(J)$ are defined. Use this and the last problem to prove the multiplicity polar theorem for pairs of families of ideals $I(t)$ and $J(t)$ where $e(I(t))$ and $e(J(t))$ are defined for all t.*

Notice that the Multiplicity Polar Theorem shows why $e(M, N$ is not upper semi-continuous. If the polar variety of N has a large value, then $mult_y\Gamma_d(M) - mult_y\Gamma_d(N)$ may be negative.

In [24] the following theorem is proved:

Theorem 3.14. *Suppose $(X^{d+k}, 0) \subset (\mathbb{C}^{n+k}, 0)$ is a complete intersection, $X = F^{-1}(0)$, $F : \mathbb{C}^{n+k} \to \mathbb{C}^p$, Y a smooth subset of X, coordinates chosen so that $\mathbb{C}^k \times 0 = Y$. Then the following are equivalent:*

i) the pair $(X - Y, Y)$ satisfies W at 0;

ii)The sets $X(y)$ are complete intersections with isolated singularities and $e(m_yJM(X_y))$ is independent of y for all $y \in Y$ near 0.

To show the power of the Multiplicity Polar theorem we give a simple proof of Theorem 3.14 which links $e(JM(X_y))$ and the Whitney conditions. The proof that that i) implies ii) avoids the use of topology.

Proof. of 3.14: i) implies ii) The Whitney conditions imply that the fiber of $D \subset B_{m_Y}(C(X))$, the exceptional divisor is equidimensional over Y. Because the dimension of the fiber is small, there is no polar variety of codimension d for $m_YJM(X)$. Since $\mathcal{O}^p_X$ has no polar varieties, the Multiplicity Polar Theorem implies that $e(mJM(X_y))$ is independent of y.

ii) implies i) The independence of $e(mJM(X_y))$ from y implies that there is no polar variety of codimension d for $m_YJM(X)$, and hence the fiber of D over Y^k is equidimensional. At this point we apply the theorem of Kleiman-Thorup ([38], [24]). We know that $JM_Y(X)$, the submodule generated by the partial derivatives taken with respect to coordinates on Y, is in the integral closure of $m_YJM(X)$ at points in a Z-open subset of Y. Since the dimension of the set of points of Projan$(m_YJM(X))$ over the set of points where the integral closure condition does not hold is at most $(k-1)+(d+g-2) < (d+k)+(g-1)-1$, it follows that $JM_Y(X)$ is in the integral closure of $m_YJM(X)$ at all points of Y. ☐

4 Invariants of EIDS

Our first invariant is the multiplicity of the pair $(JM(X), N^t(X))$.

We first prove a result which will give a geometric characterization of when $e(JM(X), N^t(X)) = 0$.

Proposition 4.1. *Suppose $X = f^{-1}(\Sigma)$, X equidimensional. Then $e(JM(X, x), f^*(JM(\Sigma)), x) = 0$ if and only if the image of $D(f)(x)$ does not lie in a limiting tangent hyperplane to Σ at $f(x)$.*

Proof. Suppose Σ defined by $G = 0$, so $G \circ f$ defines X. Then $DG(F)$ is the matrix of generators of $f^*(JM(\Sigma)), x)$. If we write the generators of $JM(X)$ in terms of the generators of $f^*(JM(\Sigma)), x)$, the coefficients come from Df, by the chain rule. Consider the ideal sheaf induced on Projan $\mathcal{R}(f^*(JM(\Sigma)))$ by the module with matrix Df. Denote it by $\mathcal{F}$. Then for X equidimensional

$$e(JM(X, x), f^*(JM(\Sigma)), x) = 0$$

if and only if

$$\overline{JM(X, x)} = \overline{f^*(JM(\Sigma)), x)}$$

if an only if $V(\mathcal{F})$ is empty. Now the points of Projan $\mathcal{R}(f^*(JM(\Sigma)))$ over x are the limit tangent hyperplanes of Σ at $f(x)$; for $V(\mathcal{F})$ to vanish at a point over x, the defining form of some limiting tangent hyperplane H to Σ at $f(x)$, when applied to the columns of $Df(x)$ must be zero. This is equivalent im$Df(x) \subset H$.

☐

Corollary 4.2. *Suppose $e(JM(X,x), f^*(JM(\Sigma)), x) = 0$; then at all nearby smooth points of Σ, f is transverse to Σ.*

Proof. $e(JM(X,x), f^*(JM(\Sigma)), x) = 0$ implies $\overline{JM(X,x)} = \overline{f^*(JM(\Sigma)), x)}$; this must hold on a neighborhood U of x. Consequently if $f(x')$ is a smooth point of $\Sigma, x' \in U$, then if f is not transverse to Σ at $f(x')$, there is a tangent hyperplane which contains the image of $Df(x')$. $\square$

We next characterize EIDS using the multiplicity. We need some more notation. Given $f\colon \mathbb{C}^q \to \mathbb{C}^{m\times n}$ we denote $f^{-1}(M^t(m,n))$ by $M^t(f)$.

Proposition 4.3. *Suppose $(X,x) = M^{t_0}(f)$ is a determinantal singularity. Then (X,x) is an EIDS if and only if $e(JM(M^t(f)), N^t, x)$ is defined for all $t \leq t_0$.*

Proof. (X,x) is an EIDS if and only if f is transverse to the rank stratification of M^{t_0} except possibly at x. By 2.4, at points x' where f is transverse to the rank stratification of M^{t_0}, and hence of $M^t, t < t_0$, $JM(M^t(f), x') = N^t, x'$. So if (X,x) is an EIDS, then $e(JM(M^t(f)), N^t, x')$ is zero for all $t \leq t_0$ for all $x' close to x$. It follows that $e(JM(M^t(f)), N^t, x)$ is defined for all $t \leq t_0$. Now if $JM(M^t(f), x') = N^t, x'$ for all x' close to x, and $t \leq t_0$, this holds if $x' \in M^t(f) \setminus M^{t-1}$, and $f(x')$ is a smooth point of M^t. Then the last corollary implies that f is transverse to M^t at $f(x')$. $\square$

The invariant $m_d(X^d)$ was introduced in [13] for the study of ICIS singularities, and for isolated singularities whose versal deformations have a smooth base in [16]. It was recently used in [28], [2] and [44] to study a smoothable IDS. In the IDS context, it is the number of critical points that a generic linear form has when restricted to the generic fiber in a smoothing $\mathcal{X}$ of X. It is also the multiplicity of the polar curve of the relative Jacobean module $JM_z(\mathcal{X})$ over the parameter space at the origin.

In the EIDS context, instead of a smoothing, we have a stabilization–a determinantal deformation of X to the generic fiber (the essential smoothing of X.) The stabilization is part of the package of ideas which make up the landscape of X.

Let $\tilde{F}$ be a stabilization family of X, $\widetilde{\mathcal{X}}$ an essential smoothing family of X.

We write

$$\pi : \widetilde{\mathcal{X}} \subset \mathbb{C}^q \times \mathbb{C} \to \mathbb{C},$$

$\pi^{-1}(s) = \widetilde{\mathcal{X}}_s$, $\widetilde{\mathcal{X}}_0 = X$. Let $S(\widetilde{\mathcal{X}})$ be the singular set of $\widetilde{\mathcal{X}}$, so that $(\widetilde{\mathcal{X}})_{reg} = \widetilde{\mathcal{X}} - S(\widetilde{\mathcal{X}})$. Choose a linear form $p\colon \mathbb{C}^q \to \mathbb{C}$, consider the critical set of the map $(\pi, p) \mid (\widetilde{\mathcal{X}} - S(\widetilde{\mathcal{X}}))$, and take its closure in $\widetilde{\mathcal{X}}$. This is the relative polar curve of $\widetilde{\mathcal{X}}$ with respect to p. If the multiplicity of the polar curve at the origin over $\mathbb{C}$ has the generic value among all choices of p, then we call the polar curve with respect to p, the relative polar curve of $\widetilde{\mathcal{X}}$. Of course, this means that the relative polar curve of $\widetilde{\mathcal{X}}$ has many representatives. The relative polar curve of $\widetilde{\mathcal{X}}$ is also the polar curve of $JM_z(\widetilde{\mathcal{X}})$.

Definition 4.4. *Let $P_d(\widetilde{\mathcal{X}})$ be the relative polar curve of an essential smoothing family of X, then $m_d(X) := m_{\mathbb{C}}(P_d(\widetilde{\mathcal{X}}))$, where $m_{\mathbb{C}}$ is the multiplicity of the set $P_d(\widetilde{\mathcal{X}})$ over $\mathbb{C}$ at the origin.*

From the construction $m_d(X)$ is independent of p, and independent of the stabilization, once we fix the landscape.

To understand the invariant better, we can ask that H, kernel of p is not a limiting tangent hyperplane to X at the origin, and that $p\mid_{(\widetilde{\mathcal{X}}_s)_{reg}} \to \mathbb{C}$ is a Morse function for $s \neq 0$. Then:

Proposition 4.5. *With the above notation*

$$m_d(X) = n_0$$

where n_0 is the number of critical points of $p\mid_{(\widetilde{\mathcal{X}}_s)_{reg}}$.

Proof. Let $\widetilde{\mathcal{X}}$ be a family of essential smoothings for X. Then $C(JM_z(\widetilde{\mathcal{X}}))$ has dimension $(d+1)+(q-d-1)=q$, as $d+1$ is the dimension of $\widetilde{\mathcal{X}}$ and $(q-d-1)$ is the fiber dimension of $C(JM_z(\widetilde{\mathcal{X}}))$ over a generic point. It follows that the dimension of $\pi_c^{-1}(S(\widetilde{\mathcal{X}}))$ must be at most $q-1$, so we can choose H so that $\pi_{\mathbb{P}}^{-1}(H)$ intersects $\pi_c^{-1}(S(\widetilde{\mathcal{X}}))$ only at points over the origin, and intersects $C(JM_z(\widetilde{\mathcal{X}}))$ transversely off $p_c^{-1}(0)$. This implies that the fiber of $P_d(\widetilde{\mathcal{X}})$ over a generic point s of $\mathbb{C}$ lies in $(\widetilde{\mathcal{X}}_s)_{reg}$. We can assume H is not a critical value of the projection of $C(JM_z(\widetilde{\mathcal{X}}))\mid_{(\widetilde{\mathcal{X}})_{reg}}$ to $\mathbb{P}^{q-1}$. This implies that $\mathbb{C}^q\times H$ and $C(JM_z(\widetilde{\mathcal{X}}))\mid_{(\widetilde{\mathcal{X}}_s)_{reg}}$ meet transversely for generic s. In turn, this means that the points of intersection project to Morse points of $p\mid_{(\widetilde{\mathcal{X}}_s)_{reg}}$. Since the number of these points is $m_d(X)$, the proposition is proved. □

The invariants $m_d(X)$ are related to the topology of X.

If $X=\cup_i X$, for $1\le i\le t$, then it makes sense to consider $m_{d_i}({}_iX)$ where d_i is the dimension of ${}_iX$, and the expected dimension of ${}_iX$ is greater than 0. If the expected dimension of ${}_iX$ is zero, we define $m_0({}_iX)$ to be the colength of the ideal defining ${}_iX$. As the next proposition shows, $m_0({}_iX)$ is the number of points of type ${}_iX$ on a generic fiber of the stabilization.

Proposition 4.6. *Suppose X is a determinantal variety, and ${}_iX$ has expected dimension 0. Then $m_0({}_iX)$ is the number of points of type ${}_iX$ on a generic fiber of the stabilization of X.*

Proof. In a stabilization $\mathcal{X}$ of X, ${}_i\mathcal{X}$ is a determinantal variety, hence Cohen-Macaulay. Consider the projection π of ${}_i\mathcal{X}$ to the base Y^1, where y is the coordinate on the base. Then the degree of π restricted to ${}_i\mathcal{X}$ is the multiplicity of $(y\circ\pi)$ in the local ring of ${}_i\mathcal{X}$ at the origin. Since ${}_i\mathcal{X}$ is Cohen-Macaulay, this multiplicity is just the dimension of the ring mod $(y\circ\pi)$, in turn this is just the colength of the ideal defining ${}_iX$. □

The invariant $m_d(X^d)$ for an EIDS as we have defined it, appears in Ebeling and Gusein-Zade's paper [10] as the specialization of Poincaré-Hopf-Nash index of a differential 1-form ω denoted $\operatorname{ind}_{\mathrm{PHN}}\omega$. The invariant $m_d(X^d)$ is $\operatorname{ind}_{\mathrm{PHN}} dl$, l a generic linear form. This follows from Proposition 1, [10] or Proposition 2.3, [46] which shows that the value of this invariant for a generic linear form is the number of Morse points of the restriction of the linear form to the generic fiber of a stabilization, and 4.5 shows the same is true for m_d. We will use formulas developed for $\operatorname{ind}_{\mathrm{PHN}}\omega$ to connect the values of $m_d({}_iX,0)$ with the topology of an essential smoothing of ${}_iX$.

Proposition 4.7. *In a family of EIDS the $m_{d_i}({}_iX,0)$, $d_i=\dim_i X$, are constant for all i if and only if*

$$(-1)^{d_i}\chi({}_iX,0)+(-1)^{d_i-1}\chi({}_iX\cap H,0)$$

are constant for all i.

Proof. The following formula relating the radial index of a differential form ω and the $\operatorname{ind}_{\mathrm{PHN}}\omega$ is due to Ebeling and Gusein-Zade ([10] proposition 4).

$$\operatorname{ind}_{\mathrm{rad}}(\omega,X,0)=\sum\nolimits_{i=1}^{t} n_{it}\operatorname{ind}_{\mathrm{PHN}}(\omega,{}_iX,0)+(-1)^{d-1}\overline{\chi}(X,0), \tag{1}$$

where $d=\dim X$, $\overline{\chi}(X,0)$ is the reduced Euler characteristic of the intersection of an essential smoothing of X with a small ball centered at the origin, and the integers n_{it} are given by the formulas $n_{it}=(-1)^{(k)(t-i)}\binom{n-i}{n-t}$.

Assuming $X={}_tX$, we can re-write Ebeling-Gusein Zade's formula in (1) using dl for ω and $m_{d_i}({}_iX)$ for $\operatorname{ind}_{\mathrm{PHN}}(dl,{}_iX,0)$, $d_i=\dim_i X_,$. Further, if $\omega=dl$, l is a generic linear form, then $\operatorname{ind}_{\mathrm{rad}}(dl,X,0)=(-1)^{d-1}\overline{\chi}(\widetilde{X\cap H})$, $H=l^{-1}(\epsilon)$ for ϵ sufficiently small ([9], Theorem 3). This invariant does not depend on the generic hyperplane H or on the stabilization of the determinantal section $X \cap H$, so we simply write $\mathrm{ind}_{\mathrm{rad}}(dl, X, 0) = (-1)^{d-1}\overline{\chi}(X \cap H)$. We apply the formula with these substitutions to each stratum in turn getting:

$$(-1)^d \chi(X, 0) + (-1)^{d-1}\chi(X \cap H, 0) = \sum_{i=1}^{t} n_{it} m_{d_i}({}_iX, 0)$$

$$(-1)^{d_{t-1}} \chi({}_{t-1}X, 0) + (-1)^{d_{t-1}-1}\chi({}_{t-1}X \cap H, 0) = \sum_{i=1}^{t-1} n_{i(t-1)} m_{d_i}({}_iX, 0)$$

$$\cdots$$

$$(-1)^{d_1} \chi({}_1X, 0) + (-1)^{d_1-1}\chi({}_1X \cap H, 0) = m_{d_1}({}_1X).$$

It is understood that if the last equation corresponds to a stratum of dimension zero, then both sides of the last equation have only 1 term, since $\chi({}_1X \cap H, 0) = 0$ and the right hand side is the colength of an ideal as explained above.

□

Problem 4.8. *This problem is designed to give you practice with the formula and to review some of the ideas from the ICIS course. Show that if $X^d, 0$, $d > 0$ is an ICIS, that the formula*

$$(-1)^d \chi(X, 0) + (-1)^{d-1}\chi(X \cap H, 0) = \sum_{i=1}^{t} n_{it} m_{d_i}({}_iX, 0)$$

of the last result becomes:

$$\mu(X) + \mu(X \cap H) = e(JM(X))$$

We would like to get a formula for determinantal singularities relating $m_{d_i}({}_iX, 0)$ to the multiplicity of a module. Naively, we might hope that $m_d(X^d) = e(JM(X), N(X))$. But $N(X)$ is not free and has its own links to topology. The multiplicity polar theorem suggests that the multiplicity of the polar curve of $N(\mathcal{X})$ at the origin should be part of the formula where $\mathcal{X}$ is a one parameter stabilization of X. However, we would like a formula which does not make explicit reference to a stabilization. Because $N(X) = f^*(JM(M^t))$, we can do this using intersection theory. Consider the graph of f in $\mathbb{C}^q \times \mathbb{C}^{n+k \times n}$, then $f(\mathbb{C}^q) \cdot \Gamma_d(M^t)$ is the intersection of the graph with $\mathbb{C}^q \times \Gamma_d(M^t)$, where $\Gamma_d(M^t)$ has codimension d in M^t. (Of course, this implies $\Gamma_d(M^t)$ has codimension q in $\mathbb{C}^{n+k \times n}$.)

Notice that if $q \geq n(n+k)$ then $\Gamma_d(\Sigma^i)$ must be empty, so the intersection number is 0. If $n = 2, k = 1$, then this is true for all ${}_2X$, for $q \geq 6$. (In fact, it is true in this case for $q \geq 5$.)

Now we connect our infinitesimal invariants with the polar invariants.

Proposition 4.9. *Suppose $\mathcal{X}$ is a one parameter stabilization of an EIDS ${}_iX^d$ then*

$$e(JM(X), N(X)) + F(\mathbb{C}^q) \cdot \Gamma_d(\Sigma^i) = m_d(X).$$

Proof. Let $\tilde{F}$ be the map from $\mathbb{C} \times \mathbb{C}^q \to \mathbb{C}^{(n+k)n}$ which defines $\mathcal{X}$. We can arrange for $\tilde{F}$ and $\Gamma_d(\Sigma^i)$ to be transverse. Then $F(\mathbb{C}^q) \cdot \Gamma_d(\Sigma^i)$ is the number of points in which $\tilde{F}(y)$ intersects $\Gamma_d(\Sigma^i)$. In turn, this is the multiplicity over $\mathbb{C}$ of $\Gamma_d(N(\mathcal{X}))$ at the origin, as $\tilde{F}^{-1}(\Gamma_d(\Sigma^i)) = \Gamma_d(N(\mathcal{X}))$. Now we apply the multiplicity polar theorem to $JM_z(\mathcal{X})$ and $N(\mathcal{X})$ we get:

$$e(JM(X), N(X)) + mult_C\Gamma_d(N(\mathcal{X})) = mult_C\Gamma_d(JM_z(\mathcal{X})) = m_d(X).$$

which gives the result. Notice that we used the universality of N to identify $N(X)$ with $N(\mathcal{X})(0)$. □

We can now complete the link between our infinitesimal invariants and our topological invariants. In the following corollary, we adopt the convention that if the expected dimension of a singularity type is 0 for X, we use the colength of the ideal defining the singularity type.

Corollary 4.10. *For an EIDS X,*

$$(-1)^d\chi(\tilde{X},0)+(-1)^{d-1}\chi(\widetilde{X\cap H},0)=$$
$$\sum_{i=1}^{t} n_{it}(e(JM(_iX^{d_i}),N(_iX^{d_i})))+F(\mathbb{C}^q)\cdot\Gamma_{d_i}(\Sigma^i)).$$

Further, in a family of EIDS $\mathcal{X}$, the invariants

$$e(JM(_i\mathcal{X}(y)^{d_i}),N(_i\mathcal{X}(y)^{d_i}))+\tilde{F}(y)(\mathbb{C}^q)\cdot\Gamma_{d_i}(\Sigma^i)$$

are independent of y iff the invariants

$$(-1)^{\dim_i\mathcal{X}(y)}\chi(_i\mathcal{X}(y),0)+(-1)^{dim_i\mathcal{X}(y)-1}\chi(_i\widetilde{X\cap H}(y),0)$$

are independent of y for all i.

The intersection numbers $F(\mathbb{C}^q)\cdot\Gamma_d(\Sigma^i)$ depend only on the presentation matrix F. In the case of Σ^n, this number has been computed in terms of the entries of F in [28].

5 Equisingularity of EIDS

Now we connect our results with Whitney equisingularity. Recall that in the complex analytic case, this is the same as Verdier equisingularity. We first state a condition on our families.

Definition 5.1. *A s-parameter family $\mathcal{X}$ of EIDS is good if there exists a neighborhood U of Y such that $F_{\mathcal{X}(y)}$ is transverse to the rank stratification off the origin for all $y\in U$.*

Theorem 5.2. *Suppose $\mathcal{X}^{d+k}$ is a good k-dimensional family of EIDS. Then the family is Whitney equisingular if and only if the invariants*

$$e(m_YJM(_i\mathcal{X}(y)),N(_i\mathcal{X}(y)))+\tilde{F}(y)(\mathbb{C}^q)\cdot\Gamma_{d_i}(\Sigma^i)$$

are independent of y.

Proof. Since the family is good we only need to control the pairs of strata (V,Y), V some stratum in the canonical stratification of $\mathcal{X}-Y$. From the good hypothesis we know all strata (except perhaps Y) have the expected dimension, and the types of dimension zero are controlled by the colength of the defining ideals. To show the positive dimensional strata are Whitney over Y, we apply the multiplicity polar theorem ([15]). The Whitney conditions hold for the open stratum of our singularity, provided the fiber of the blow-up of the conormal modification by m_Y is equidimensional over Y. This occurs if and only the polar variety of codimension d of $JM_z(\mathcal{X})$ is empty. The independence of our invariants from y hold if and only if this polar is empty.

□

Corollary 5.3. *Suppose $\mathcal{X}^{d+k}$ is a good k-dimensional family of EIDS. Then the family is Whitney equisingular if and only if the polar multiplicities at the origin of $_i\mathcal{X}(y)$ and the invariants*

$$e(JM(_i\mathcal{X}(y)),N(_i\mathcal{X}(y)))+\tilde{F}(y)(\mathbb{C}^q)\cdot\Gamma_{d(i)}(\Sigma^i)$$

are independent of y.

Proof. There is an expansion formula for $e(m_Y JM({}_i\mathcal{X}(y)), N({}_i\mathcal{X}(y))$ as a sum of terms with the polar multiplicities of ${}_i\mathcal{X}(y)$ with combinatorial coefficients, and the term $e(JM({}_i\mathcal{X}(y)), N({}_i\mathcal{X}(y)))$. So Whitney implies the polar multiplicities are constant, and $e(m_Y JM({}_i\mathcal{X}(y)), N({}_i\mathcal{X}(y)) + \tilde{F}(y)(\mathbb{C}^q) \cdot \Gamma_{d_i}(\Sigma^i)$ are constant. So the terms $e(JM({}_i\mathcal{X}(y)), N({}_i\mathcal{X}(y)) + \tilde{F}(y)(\mathbb{C}^q) \cdot \Gamma_{d_i}(\Sigma^i)$ are independent of y as well. The other direction is clear. □

For good families of IDS, the following result from [3] follows as a corollary of the above result and Proposition 4.9.

Proposition 5.4. (Theorem 5.3, [3]) *Let $\mathcal{X}^{d+k}$ be a good k-dimensional family of IDS. Then the family is Whitney equisingular if and only if all polar multiplicities $m_i(X,0)$, $i = 0,\dots,d$ are constant.*

6 Extending results to EISS

We return now to the study of non-linear sections of universal spaces. Recall the commutative diagram:

$$\begin{tikzcd} \mathbb{C}^q, 0 \arrow[r, "f_0"] & \mathbb{C}^N, 0 \\ X, 0 \arrow[u, hook] \arrow[r] & V, 0 \arrow[u, hook] \end{tikzcd}$$

In order to extend 5.2 and 5.3 to this setting, we will want V to be stable and $N(V)$ to be universal. V has a minimal Whitney stratification by [51]. If f_0 is transverse to this minimal Whitney stratification of V, except perhaps at the origin, we say that X is an *Essentially Isolated Sectional Singularity* or *EISS*. Then we can deform f_0 so that f_t is transverse to the Whitney stratification of V and f_t is transverse to the polar variety of V of codimension q in $\mathbb{C}^N$. Then again $f_t^{-1}(V)$ is called a stabilization of X. Since f_0 is transverse to the Whitney stratification of V by 2.3, it follows that $\overline{JM(X)} = \overline{f_0^*(JM(V))}$ except at the origin, hence $e(JM(X), f_0^*(JM(V)), 0)$ is well defined.

Definition 6.1. *Suppose X^d is an EISS, and F is a stabilization family with base $\mathbb{C}$. Define $m_d(X)$ as the degree over $\mathbb{C}$ of the polar curve of $JM_z(F)$.*

Proposition 6.2. *The invariant $m_d(X)$ is independent of the choice of F.*

Proof. We apply the multiplicity polar theorem to $e(JM(X), f_0^*(JM(V)), 0)$, and $\Gamma^1(JM_z(F))$, and $\Gamma^1(F^*(JM(V)))$. We get

$$m_d(X) = e(JM(X), f_0^*(JM(V)), 0) + m_{\mathbb{C}}(\Gamma^1(F^*(JM(V)))$$

$$= e(JM(X), f_0^*(JM(V)), 0) + (f_0(\mathbb{C}^q) \cdot \Gamma_q(V).$$

Now, in the right-hand side of the last equality, the terms are independent of F. □

We can again define a *good* family of EISS. Define a filtration of V by V_i, the union of all strata of V of dimension $\leq i$. Each V_i is equidimensional. Let i_X denote $f^{-1}(V_i)$.

Theorem 6.3. *Suppose $\mathcal{X}^{d+k}$ is a good k-dimensional family of EISS. Then the following are equivalent:*

1. *The family is Whitney equisingular.*

2. The invariants

$$e(m_Y JM({}_i\mathcal{X}(y)), N({}_i\mathcal{X}(y))) + \tilde{F}(y)(\mathbb{C}^q) \cdot \Gamma_{d_i}(V_i)$$

are independent of y.

3. The polar multiplicities at the origin of ${}_i\mathcal{X}(y)$ and the invariants

$$e(JM({}_i\mathcal{X}(y)), N({}_i\mathcal{X}(y))) + \tilde{F}(y)(\mathbb{C}^q) \cdot \Gamma_{d(i)}(V_i)$$

are independent of y.

Proof. 1. if and only if 2. follows exactly the proof of 5.2.

2. if and only if 3.

The expansion formula for $e(m_Y JM({}_i\mathcal{X}(y)), N({}_i\mathcal{X}(y)))$ as a sum of terms with the polar multiplicities of ${}_i\mathcal{X}(y)$ with combinatorial coefficients, and the term $e(JM({}_i\mathcal{X}(y)), N({}_i\mathcal{X}(y)))$ still holds. So 3. implies 2. Assume 2. Since 2. implies 1., all of the polar multiplicities of ${}_i\mathcal{X}(y)$ are independent of y, and so 2. implies that $e(JM({}_i\mathcal{X}(y)), N({}_i\mathcal{X}(y))) + \tilde{F}(y)(\mathbb{C}^q) \cdot \Gamma_{d(i)}(V_i)$ are independent of y as well. $\square$

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