Algorithmic Methods for Singularities

Talk 2: Invariants from the computational perspective

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Dimension from definition to computation

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Efficiently computable: the last two items!

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Let $P(t) = \sum_{\nu=0}^{d} a_i t^d$, then

- $\dim(A) = d$
- $\mathrm{mult}(A) = d! \cdot a_d$

A monomial example

Consider the 3 coordinate axes in $(\mathbb{C}^3, 0)$:

$$I = \langle xy, xz, yz \rangle \subseteq \mathbb{C}[x,y,z]_{\langle x,y,z \rangle}$$
$$A = \mathbb{C}[x,y,z]_{\langle x,y,z \rangle}/I$$

k	new monomials	total number
1	1	1
2	x, y, z	4
3	x^2, y^2, z^2	7
4	x^3, y^3, z^3	10
$\vdots$		

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Obviously, $P(t) = 3 \cdot t - 2$

Hence: $\dim(X) = 1$ and $\mathrm{mult}(X) = 3$ as expected

From ideal to monomial ideal

Key to computing previous example: Monomial ideal

And in the non-monomial case?

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And in the non-monomial case? **standard bases**

- K a (exact computable) field
- $>$ suitable total ordering on $\mathrm{Mon}(\underline{x})$
- $LT(f)$ largest monomial in $f \in K[\underline{x}]$ w.r.t. $>$
- $L(I) := \langle LT(f) \mid f \in I \rangle$

$$G = \{g_1, \ldots, g_s\} \text{ standard basis of } I$$

$$\Longleftrightarrow$$

$$L(G) = L(I) \text{ and } \langle G \rangle = I$$

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Facts:

- $\dim(K[\underline{x}]_{\langle \underline{x} \rangle}/I) = \dim(K[\underline{x}]_{\langle \underline{x} \rangle}/L(I))$
- $\mathrm{mult}(K[\underline{x}]_{\langle \underline{x} \rangle}/I) = \mathrm{mult}(K[\underline{x}]_{\langle \underline{x} \rangle}/L(I))$

Example

Consider

- $I = \langle x^2 + y^2, xy \rangle \subseteq \mathbb{Q}[x, y]_{\langle x,y \rangle}$
- $>$ negative degree reverse lexicographical ordering
- $L(x^2 + y^2) = x^2$, $L(xy) = xy$

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Actually, standard basis of I:

$$G = \{x^2 + y^2, xy, y^3\}$$

A multiplicity 4 space curve example

Consider

- $I = \langle xy, x^2 + y^2 + z^2 \rangle \subseteq \mathbb{Q}[x, y, z]_{\langle x,y,z \rangle}$
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$P(t) = 4 \cdot t - 4$, dim $= 1$, mult $= 4$

Jupyter Notebook 2, Block 1

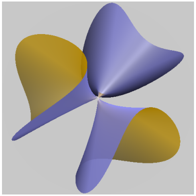
Milnor number of a Hypersurface

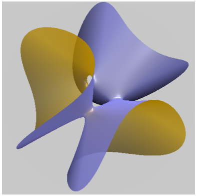
Let $(X, 0) = (V(f), 0) \subseteq \mathbb{C}\{\underline{x}\}$.

Milnor number:

$$\mu := \dim_{\mathbb{C}} \mathbb{C}\{\underline{x}\}/\langle \frac{\partial f}{\partial x_1}, \ldots \frac{\partial f}{\partial x_n} \rangle$$

geometric interpretation of the Milnor number:





Milnor number: Example

Consider $f = x^3 + y^5$

$$\mu = \dim_{\mathbb{C}} \mathbb{C}\{x, y\} / \langle x^2, y^4 \rangle$$

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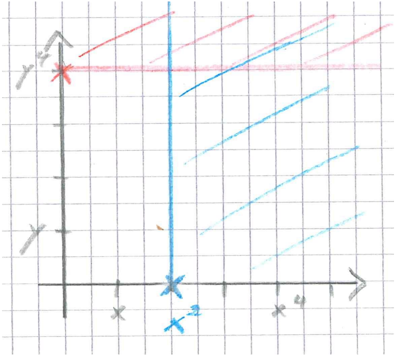

Milnor number: non-monomial setting

Consider $f = x \cdot y \cdot (x + y - 1)$

Jacobian matrix of f: $(2xy + y^2 - y \quad x^2 + 2xy - x)$

L(I) w.r.t. local degree odering: $\langle x, y \rangle \Longrightarrow \mu = 1$

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L(I) w.r.t. local degree odering: $\langle x, y\rangle \Longrightarrow \mu = 1$

L(I) w.r.t. global degree ordering: $\langle x^2, xy, y^3\rangle \Longrightarrow \mu = 4$

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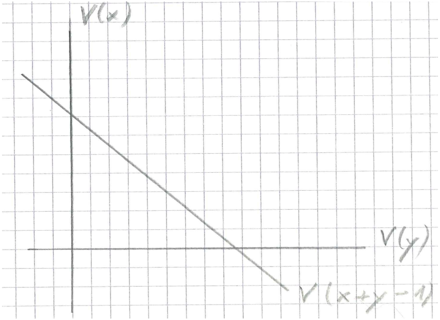

Local and global orderings

Examples of global monomial orderings ($1 < x_i \ \forall i$):

lex $\underline{x}^\alpha > \underline{x}^\beta \iff$
$\exists 1 \leq i \leq n$: $(\alpha_j = \beta_j \ \forall j < i)$ and $(\alpha_i > \beta_i)$

deglex $\underline{x}^\alpha > \underline{x}^\beta \iff$
$(\deg(\underline{x}^\alpha) > \deg(\underline{x}^\beta))$ or
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$\tau_{(0,0)} = 1$, $\tau_{(1,0)} = 1$, $\tau_{(0,1)} = 1$

$\tau_{global} = \tau_{(0,0)} + \tau_{(1,0)} + \tau_{(0,1)} = 3$

μ versus τ: inhomogeneous example

Fact for hypersurface germs: $\tau \leq \mu$

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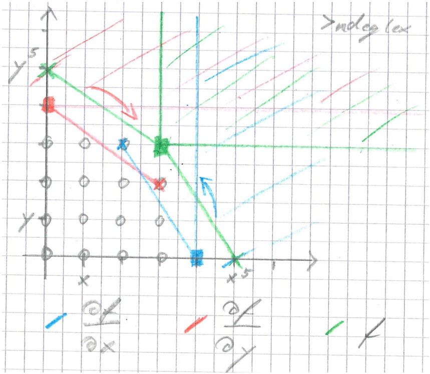
Consider μ and τ for $(V(x^5 + y^5 - t \cdot x^3 y^3), 0)$:



Worksheet 2, Block 2

Sideremark: Newton Polyhedron

A different perspective:

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$K(f) = Conv(\Delta(f) \cup \{0\})$

$K_0(f) =$ closure of $(K(f) \setminus \Delta(f)) \cup \{0\}$

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Fact:
f Newton non-degenerate
$\Longrightarrow \mu(f) =$ Minkowski mixed volume of $K_0(f)$

Newton polyhedron in a picture

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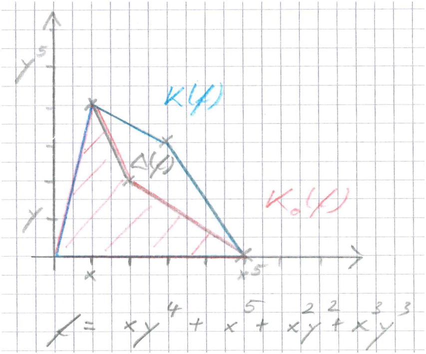

μ for ICIS: Lê-Greuel formula

The setting:

- $q \geq 2$
- $X = (V(f_1, \ldots, f_q), 0) \subseteq (\mathbb{C}^n, 0)$ ICIS germ
- $J_k = \langle m \in \mathbb{C}\{\underline{x}\} \mid m\ k-\mathrm{minor}\ of \left(\frac{\partial f_i}{\partial x_j}\right)_{\substack{1 \leq i \leq k \\ 1 \leq j \leq n}} \rangle$
- $I_k = \langle f_1, \ldots, f_{k-1} \rangle + J_k$
- all I_k have finite colength in $\mathbb{C}\{\underline{x}\}$

Then:

$$\mu(X, 0) = \sum_{\nu=1}^{q} (-1)^{(q-\nu)} \dim_{\mathbb{C}} \left(\mathbb{C}\{\underline{x}\}/I_\nu\right)$$

The colength condition on I_k

The condition: $\dim_{\mathbb{C}}(\mathbb{C}\{\underline{x}\}/I_k) < \infty$

where $I_k = \langle f_1, \ldots, f_{k-1} \rangle +$
$$\langle m \in \mathbb{C}_{\underline{x}} \mid m \; k-\text{minor of} \left(\frac{\partial f_i}{\partial x_j} \right)_{\substack{1 \leq i \leq k \\ 1 \leq j \leq n}} \rangle$$

Warning example: S_5-Singularity $V(xy, x^2 + y^2 + z^2)$

$f_1 = xy, f_2 = x^2 + y^2 + z^2$
$$\Longrightarrow I_2 = \langle xy, xz, yx, x^2 + y^2 \rangle$$
Problem: $z^i \notin I_k \; \forall i \in \mathbb{N}$

$f_1 = x^2 + y^2 + z^2, f_2 = xy$
$$\Longrightarrow I_2 = \langle x^2 + y^2 + z^2, xz, xy, x^2 + y^2 \rangle$$
no problem: finite colength

Tjurina number and T^1 for ICIS

$X = (V(\underbrace{f_1, \ldots, f_q}_{=:I}), 0) \subseteq (\mathbb{C}^n, 0)$ ICIS germ, $q \geq 1$

Then

$$T^1_{(X,0)} = (\mathbb{C}\{\underline{x}\})^q / (I \cdot (\mathbb{C}\{\underline{x}\})^q + J)$$

where $J = \langle \frac{\partial \underline{f}}{\partial x_1}, \ldots, \frac{\partial \underline{f}}{\partial x_n} \rangle \subset (\mathbb{C}\{\underline{x}\})^q$.

Note: no condition on the generators
(unlike Milnor number!)

See Jupyter Notebook 2, Block 3