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Solving the Non-Commutativity Problem in Matrix Multiplication

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Resumen

En este artículo se utiliza un algoritmo que permite resolver el problema de la no-conmutatividad de la multiplicación de matrices. La solución a este problema no es directa, sin embargo en este trabajo se presenta un algoritmo que permite una solución directa al problema mencionado. La solución conlleva un paso intermedio incrementando el orden de las matrices, un cálculo directo y el paso a las variables originales. La solución se trata primero intuitivamente y luego se formaliza utilizando el producto tensorial de Kronecker. El método se torna útil y con más fuerza dada la actual tecnología.

Palabras Clave: Cálculo de matrices, Conmutatividad en la multiplicación de matrices, producto tensorial, algoritmo aplicado, problemas iterativos..

Abstract

An algorithm that allows solving the problem of non-commutativity in matrix multiplication has been utilized in this paper. The solutions to this problem are not direct, however, this work presents a direct solving algorithm for the above-mentioned problem. This solution implies an increase in the matrix order, a direct calculation and then a return to the original variables. Solution is first sought intuitively and then the result is formalized by using the Kronecker's tensor product properties. Technology development increasingly reinforces this method's power and use.

Keywords: Computations on matrices, Commutativity of matrix multiplication, tensor product, applied algorithm, iterative problems.

1. Introducción

The problem of matrix commutativity has classically been studied and the solution is not easy. This paper presents a solution to the above-mentioned problem through a simple algorithm. This procedure could be derived from Lancaster&Tismenetsky (1985). Given current technology, this method has shown to be a good option in computer simulations and/or in problems solved by applying numerical methods. In this paper, ' means prime; the transpose is designated by ^{T}, and ⊗ is Kronecker 's tensor product.

matrix of
of order n.

2. Génesis and form of the problem

A matrix equation has been obtained by simulating the heat transfer phenomenon from the inside the outside medium in the wall of a Teniente Type Copper Converter and using finite differences. This equation results from the discretization of convection equations and from fixed temperatures (imposed by the liquid) at the inside wall, from convection equations at the outside border and from conduction equations in the inner side of the wall. Special care was applied when approaching the equations in finite differences, both regarding space and time, with the purpose of achieving a stable and conditioned solution. Börger (1996) shows the details to develop (1) where this matrix equation has the following form:

$$\mathbf{A}\cdot\mathbf{X}+\mathbf{X}\cdot\mathbf{B}+\mathbf{C}\cdot\mathbf{X}\cdot\mathbf{D}=\mathbf{E} \tag{1}$$

Where $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$ are parameter matrixes, $\mathbf{X}$ is the variable matrix (temperature) and $\mathbf{E}$ is a matrix known from the previous iteration. This paper solves this problem using a transformation, herein called the B transformation.

3. The B Transformation

Let B be a transformation from the matrixes space $n\times n$ to the matrixes space $n^2\times 1$, initially over the complex field:

$$\mathrm{B}: M_{n\times n}(C)\rightarrow M_{n^2\times 1}(C) \tag{2}$$

This transformation takes matrix A and converts it into $\mathrm{B}(\mathbf{A})=\mathbf{A}^{*}$ (3)

$$\mathbf{A}\rightarrow \mathrm{B}(\mathbf{A})=\mathbf{A}^{*} \tag{4}$$

Where A is a nxn matrix, then

$$\mathbf{A}=\left(a_{ij}\right)_{n\times n} \tag{5}$$

and the transformed matrix is a column order $n^2\times 1$ composed by n column matrixes

$$\mathbf{A}^{*}=\begin{bmatrix}\mathbf{A}_1^C\\ \mathbf{A}_2^C\\ \cdot\\ \mathbf{A}_n^C\end{bmatrix} \tag{6}$$

$$\mathbf{A}_h^C=\begin{bmatrix}a_{1h}\\ a_{2h}\\ \cdot\\ a_{nh}\end{bmatrix} \tag{7}$$

It is clear that this transformation is linear, K. Hoffman, R.Kunze (1973), since the void nxn matrix is transformed into a void $n^2\times 1$ matrix, $\mathbf{O}_{n\times n}\longrightarrow \mathbf{O}^{*}_{n^2\times 1}$ and it is easy to demonstrate – by multiplying by a scalar - that:

$$\mathrm{B}(\alpha\cdot\mathbf{A})=\alpha\cdot\mathrm{B}(\mathbf{A})=\alpha\cdot\mathbf{A}^{*} \tag{8}$$

$$\mathrm{B}(\alpha\cdot\mathbf{A})\rightarrow \alpha\cdot\mathrm{B}(\mathbf{A}) \tag{9}$$

with α scalar

Proving that the transformation is an isomorphism I.N. Herstein (1977) is easy too, because it is injective and Superjective.

In section 3, the following expressions will be proven

$$\mathrm{B}(\mathbf{A}\cdot\mathbf{X})=\mathbf{A}'\cdot\mathrm{B}(\mathbf{X})=\mathbf{A}'\cdot\mathbf{X}^{*} \tag{10}$$

with A, X being nxn matrices

$$\mathrm{B}(\mathbf{X}\cdot\mathbf{B})=\mathbf{X}'\cdot\mathrm{B}(\mathbf{B})=\mathbf{X}'\cdot\mathbf{B}^{*}=\mathbf{B}''\cdot\mathbf{X}^{*} \tag{11}$$

$$\mathrm{B}(\mathbf{A}\cdot\mathbf{X}\cdot\mathbf{B})=\mathbf{A}'\cdot\mathrm{B}(\mathbf{X}\cdot\mathbf{B})=\mathbf{A}'\cdot\mathbf{B}''\cdot\mathbf{X}^{*} \tag{12}$$

where A, X, B are nxn matrixes and symbols �` and ‘’ don’t mean transposition, but a reordering of the relevant matrix to be seen in section 3.
So the expression AX+XB=C may be treated as follows:

$$\mathcal{B}\left(\mathbf{A}\cdot\mathbf{X}+\mathbf{X}\cdot\mathbf{B}-\mathbf{C}\right)=\mathcal{B}\left(\mathbf{0}\right) \tag{13}$$

Where A,X,B,C are nxn matrixes and 0 is the nxn void matrix.

$$\mathcal{B}\left(\mathbf{A}\cdot\mathbf{X}+\mathbf{X}\cdot\mathbf{B}-\mathbf{C}\right)=0^{*} \tag{14}$$

$$\mathcal{B}\left(\mathbf{A}\cdot\mathbf{X}\right)+\mathcal{B}\left(\mathbf{X}\cdot\mathbf{B}\right)-\mathcal{B}\left(\mathbf{C}\right)=0^{*} \tag{15}$$

using (3), (10) and (11) this results in

$$\mathbf{A}'\cdot\mathbf{X}^{*}+\mathbf{B}''\cdot\mathbf{X}^{*}-\mathbf{C}^{*}=0^{*} \tag{16}$$

Then, if determinant $\left|\mathbf{A}'+\mathbf{B}''\right|\neq 0$, it is true that

$$\mathbf{X}^{*}=\left(\mathbf{A}'+\mathbf{B}''\right)^{-1}\cdot\mathbf{C}^{*} \tag{17}$$

and X is obtained through the inverse transformation

$$\mathbf{X}=\mathcal{B}^{-1}\left(\mathbf{X}^{*}\right) \tag{18}$$

The expression AX+XB+CXD=E can also be solved,

$$\mathcal{B}\left(\mathbf{A}\cdot\mathbf{X}+\mathbf{X}\cdot\mathbf{B}+\mathbf{C}\cdot\mathbf{X}\cdot\mathbf{D}-\mathbf{E}\right)=\mathcal{B}\left(\mathbf{0}\right) \tag{19}$$

Where A,X,B,C,D,E are nxn matrixes and 0 is the void nxn matrix.

$$\mathcal{B}\left(\mathbf{A}\cdot\mathbf{X}+\mathbf{X}\cdot\mathbf{B}+\mathbf{C}\cdot\mathbf{X}\cdot\mathbf{D}-\mathbf{E}\right)=0^{*} \tag{20}$$

$$\mathcal{B}\left(\mathbf{A}\cdot\mathbf{X}\right)+\mathcal{B}\left(\mathbf{X}\cdot\mathbf{B}\right)+\mathcal{B}\left(\mathbf{C}\cdot\mathbf{X}\cdot\mathbf{D}\right)-\mathcal{B}\left(\mathbf{E}\right)=0^{*} \tag{21}$$

using (3), (10), (11) and (12) this results in

$$\mathbf{A}'\cdot\mathbf{X}^{*}+\mathbf{B}''\cdot\mathbf{X}^{*}+\mathbf{C}'\cdot\mathbf{D}''\cdot\mathbf{X}^{*}-\mathbf{E}^{*}=0^{*} \tag{22}$$

Then, if determinant $\left|\mathbf{A}'+\mathbf{B}''+\mathbf{C}'\cdot\mathbf{D}''\right|\neq 0$, it is true that

$$\mathbf{X}^{*}=\left(\mathbf{A}'+\mathbf{B}''+\mathbf{C}'\cdot\mathbf{D}''\right)^{-1}\cdot\mathbf{E}^{*} \tag{23}$$

and X is obtained through the inverse transformation

$$\mathbf{X}=\mathcal{B}^{-1}\left(\mathbf{X}^{*}\right)$$

4. The matrix form AX+XB+CXD=E

3x3 order matrixes will be studied first to show the genesis of the algorithm and then a generalization will be inferred.
Given the following matrixes:

$$\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \mathbf{B}=\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$\mathbf{C}=\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \qquad \mathbf{D}=\begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}$$

$$\mathbf{E}=\begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} \qquad \mathbf{X}=\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

The method basis consists in first changing the form of the matrix as a tool to solve the reordered variables secondly. Therefore, the n by n elements of matrix A will be redistributed in a structure of $n^2 \times n^2$ elements, and $\mathbf{X}^{*}$ of order $n^2 \times 1$ like in (6) that fix A’. Thus, the AX form can be extended to the following form of (10):

$$\mathbf{A}'\mathbf{X}^{*}=\begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{12} \\ x_{22} \\ x_{32} \\ x_{13} \\ x_{23} \\ x_{33} \end{pmatrix}$$

At this point, it must be noted that the previous A’ matrix is a block diagonal matrix. And that in the above mentioned structure the A` matrix remains as a constant (parameter) matrix of order $n^2 \times n^2$ and $\mathbf{X}^{*}$ is a variable matrix of order $n^2 \times 1$.
A similar method is applied to the XB form. First, the matrix is converted into the $\mathbf{X}'\cdot\mathbf{B}^{*}$ form and then to the $\mathbf{B}''\cdot\mathbf{X}^{*}$ form of (11), where the matrix B" is a constant (parameter) matrix of order $n^2 \times n^2$ and $\mathbf{X}^{*}$ is a variable

matrix of order $n^2 \times 1$. Therefore the matrix is reordered as follows:

$$\mathbf{X'}\cdot\mathbf{B}^* = \begin{pmatrix} x_{11} & x_{12} & x_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ x_{21} & x_{22} & x_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ x_{31} & x_{32} & x_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{11} & x_{12} & x_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{21} & x_{22} & x_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{31} & x_{32} & x_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_{11} & x_{12} & x_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & x_{21} & x_{22} & x_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & x_{31} & x_{32} & x_{33} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \\ b_{12} \\ b_{22} \\ b_{32} \\ b_{13} \\ b_{23} \\ b_{33} \end{pmatrix}$$

And when matrices are reordered again, the following is obtained:

$$\mathbf{B''}\mathbf{X}^* = \begin{pmatrix} b_{11} & 0 & 0 & b_{21} & 0 & 0 & b_{31} & 0 & 0 \\ 0 & b_{11} & 0 & 0 & b_{21} & 0 & 0 & b_{31} & 0 \\ 0 & 0 & b_{11} & 0 & 0 & b_{21} & 0 & 0 & b_{31} \\ b_{12} & 0 & 0 & b_{22} & 0 & 0 & b_{32} & 0 & 0 \\ 0 & b_{12} & 0 & 0 & b_{22} & 0 & 0 & b_{32} & 0 \\ 0 & 0 & b_{12} & 0 & 0 & b_{22} & 0 & 0 & b_{32} \\ b_{13} & 0 & 0 & b_{23} & 0 & 0 & b_{33} & 0 & 0 \\ 0 & b_{13} & 0 & 0 & b_{23} & 0 & 0 & b_{33} & 0 \\ 0 & 0 & b_{13} & 0 & 0 & b_{23} & 0 & 0 & b_{33} \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{12} \\ x_{22} \\ x_{32} \\ x_{13} \\ x_{23} \\ x_{33} \end{pmatrix}$$

Matrix B" was reordered to obtain our previous matrix $\mathbf{X}^*$ of equation (6).
When the form CXD is converted into the (12) form:

$$\mathbf{C'}\cdot\mathbf{D''}\cdot\mathbf{X}^* = \begin{pmatrix} (d_{11}\cdot\mathbf{C}) & (d_{12}\cdot\mathbf{C}) & (d_{13}\cdot\mathbf{C}) \\ (d_{21}\cdot\mathbf{C}) & (d_{22}\cdot\mathbf{C}) & (d_{23}\cdot\mathbf{C}) \\ (d_{31}\cdot\mathbf{C}) & (d_{32}\cdot\mathbf{C}) & (d_{33}\cdot\mathbf{C}) \end{pmatrix} \cdot \begin{pmatrix} \mathbf{X}_1^{\mathbf{C}} \\ \mathbf{X}_2^{\mathbf{C}} \\ \mathbf{X}_3^{\mathbf{C}} \end{pmatrix}$$

where d_{ij} are elements of matrix D, $\mathbf{X}_{\mathbf{h}}^{\mathbf{C}}$ defined column matrix (7), and C the nxn matrix.
The following structures are obtained from the above equations:

$$\mathbf{E}^* = \begin{pmatrix} e_{11} \\ e_{21} \\ e_{31} \\ e_{12} \\ e_{22} \\ e_{32} \\ e_{13} \\ e_{23} \\ e_{33} \end{pmatrix} \qquad \mathbf{X}^* = \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{12} \\ x_{22} \\ x_{32} \\ x_{13} \\ x_{23} \\ x_{33} \end{pmatrix}$$

The solution to form (1) is direct since it was converted into (22)
As seen in (23), matrix $\mathbf{X}^*$ is solved from (22). X is obtained from X* by simple inspection or by the following algorithm,

$$\begin{aligned} \mathbf{X} = {} & (\mathbf{I}_{3x3}\ \mathbf{0}_{3x3}\ \mathbf{0}_{3x3})\cdot(\mathbf{X}^*)_{9x1}\cdot(1\,0\,0)_{1x3} + \\ & (\mathbf{0}_{3x3}\ \mathbf{I}_{3x3}\ \mathbf{0}_{3x3})\cdot(\mathbf{X}^*)_{9x1}\cdot(0\,1\,0)_{1x3} + \\ & (\mathbf{0}_{3x3}\ \mathbf{0}_{3x3}\ \mathbf{I}_{3x3})\cdot(\mathbf{X}^*)_{9x1}\cdot(0\,0\,1)_{1x3} \end{aligned} \qquad (24)$$

This method allows solving the commutativity problem, basically by inverting a matrix of higher order. This is not difficult given today’s technology. Also, given the forms obtained in the new matrices (one block diagonal and another five-diagonal –in the 3x3 order case), there are methods to solve this kind of matrices in problems of higher order, which backs this paper. The above is also applicable to solve specific problems in computationally developed algorithms, as is the case of resolutions in finite differences or finite elements. In iterative solutions, the problem of solution conditioning and stability arises additionally. This method allows a stability analysis on the (22) and (23) transformed matrix equations, based on the parameter matrix to be inverted. Conditioning based on the residual vectors method can also been studied, for example.
The final structures are:

$$\mathbf{C'}\cdot\mathbf{D''} = \begin{pmatrix} (d_{11}\cdot\mathbf{C}) & (d_{12}\cdot\mathbf{C}) & (d_{13}\cdot\mathbf{C}) & \dots & (d_{1n}\cdot\mathbf{C}) \\ (d_{21}\cdot\mathbf{C}) & (d_{22}\cdot\mathbf{C}) & (d_{23}\cdot\mathbf{C}) & \dots & (d_{2n}\cdot\mathbf{C}) \\ (d_{31}\cdot\mathbf{C}) & (d_{32}\cdot\mathbf{C}) & (d_{33}\cdot\mathbf{C}) & \dots & (d_{3n}\cdot\mathbf{C}) \\ . & . & . & \dots & . \\ & & & \dots & \\ (d_{n1}\cdot\mathbf{C}) & (d_{n2}\cdot\mathbf{C}) & (d_{n3}\cdot\mathbf{C}) & \dots & (d_{nn}\cdot\mathbf{C}) \end{pmatrix}$$

being d_{ij} the elements of matrix D, and C the nxn matrix.

$$
\mathbf{A'}=\begin{pmatrix}
a_{11} & a_{12} & . & a_{1n} & 0 & 0 & . & 0 & \cdots & 0 & 0 & . & 0 & 0 & 0 & . & 0 \\
a_{21} & . & . & a_{2n} & 0 & . & . & 0 & \cdots & 0 & . & . & 0 & 0 & . & . & 0 \\
. & . & . & . & . & . & . & . & \cdots & . & . & . & . & . & . & . & . \\
a_{n1} & . & . & a_{nn} & 0 & . & . & 0 & & 0 & . & . & 0 & 0 & . & . & 0 \\
0 & 0 & . & 0 & a_{11} & a_{12} & . & a_{1n} & & 0 & 0 & . & 0 & 0 & 0 & . & 0 \\
0 & . & . & 0 & a_{21} & . & . & a_{2n} & & 0 & . & . & 0 & 0 & . & . & 0 \\
. & . & . & . & . & . & . & . & & . & . & . & . & . & . & . & . \\
0 & . & . & 0 & a_{n1} & . & . & a_{nn} & \cdots & 0 & . & . & 0 & 0 & . & . & 0 \\
0 & 0 & . & 0 & 0 & 0 & . & 0 & \cdots & | & | & | & | & | & | & | & | \\
0 & . & . & 0 & 0 & . & . & 0 & & | & | & | & | & | & | & | & | \\
. & . & . & . & . & . & . & . & & | & | & | & | & | & | & | & | \\
0 & . & . & 0 & 0 & . & . & 0 & & | & | & | & | & | & | & | & | \\
| & | & | & | & | & | & | & | & & a_{11} & a_{12} & . & a_{1n} & 0 & 0 & . & 0 \\
| & | & | & | & | & | & | & | & & a_{21} & . & . & . & 0 & . & . & 0 \\
| & | & | & | & | & | & | & | & & . & . & . & . & . & . & . & . \\
| & | & | & | & | & | & | & | & & a_{n1} & . & . & a_{nn} & 0 & . & . & 0 \\
0 & 0 & . & 0 & 0 & 0 & . & 0 & \cdots & 0 & 0 & . & 0 & a_{11} & a_{12} & . & a_{1n} \\
0 & . & . & 0 & 0 & . & . & 0 & & 0 & . & . & 0 & a_{21} & . & . & a_{2n} \\
. & . & . & . & . & . & . & . & & . & . & . & . & . & . & . & . \\
0 & . & . & 0 & 0 & . & . & 0 & & 0 & . & . & 0 & a_{n1} & . & . & a_{nn}
\end{pmatrix}
$$

These matrixes are general and allow a general solution for our main issue. It is formalized by using the tensor product

$$
\mathbf{B}''=\begin{pmatrix}
b_{11} & 0 & . & 0 & b_{21} & 0 & . & 0 & \cdots & b_{n-1,1} & 0 & . & 0 & b_{n,1} & 0 & . & 0 \\
0 & b_{11} & 0 & . & 0 & b_{21} & 0 & . & & 0 & b_{n-1,1} & 0 & . & 0 & b_{n,1} & 0 & . \\
. & . & . & 0 & . & . & . & 0 & \cdots & . & . & . & 0 & . & . & . & 0 \\
0 & . & 0 & b_{11} & 0 & . & 0 & b_{21} & & 0 & . & 0 & b_{n-1,1} & 0 & . & 0 & b_{n,1} \\
b_{12} & 0 & . & 0 & b_{22} & 0 & . & 0 & \cdots & b_{n-1,2} & 0 & . & 0 & b_{n,2} & 0 & . & 0 \\
0 & b_{12} & 0 & . & 0 & b_{22} & 0 & . & & 0 & b_{n-1,2} & 0 & . & 0 & b_{n,2} & 0 & . \\
. & . & . & 0 & . & . & . & 0 & & . & . & . & 0 & . & . & . & 0 \\
0 & . & 0 & b_{12} & 0 & . & 0 & b_{22} & \cdots & 0 & . & 0 & b_{n-1,2} & 0 & . & . & b_{n,2} \\
0 & 0 & . & 0 & b_{23} & 0 & . & 0 & & | & | & | & | & | & | & | & | \\
0 & . & . & 0 & 0 & b_{23} & 0 & . & & | & | & | & | & | & | & | & | \\
. & . & . & . & . & . & . & 0 & & | & | & | & | & | & | & | & | \\
0 & . & . & 0 & 0 & . & 0 & b_{23} & & | & | & | & | & | & | & | & | \\
| & | & | & | & | & | & | & | & \cdots & b_{n-1,n-1} & 0 & . & 0 & 0 & 0 & . & 0 \\
| & | & | & | & | & | & | & | & & 0 & b_{n-1,n-1} & 0 & . & 0 & . & . & 0 \\
| & | & | & | & | & | & | & | & \cdots & . & . & . & 0 & . & . & . & . \\
| & | & | & | & | & | & | & | & & 0 & . & 0 & b_{n-1,n-1} & 0 & . & . & 0 \\
b_{1,n} & 0 & . & 0 & b_{2,n} & 0 & . & 0 & & b_{n-1,n} & 0 & . & 0 & b_{n,n} & 0 & . & 0 \\
0 & b_{1,n} & 0 & . & 0 & b_{2,n} & 0 & . & & 0 & b_{n-1,n} & 0 & . & 0 & b_{n,n} & 0 & . \\
. & . & . & 0 & . & . & . & 0 & \cdots & . & . & . & 0 & . & . & . & 0 \\
0 & . & . & b_{1,n} & 0 & . & 0 & b_{2,n} & & 0 & . & 0 & b_{n-1,n} & 0 & . & 0 & b_{n,n}
\end{pmatrix}
$$

$$
\mathbf{E}^*=\begin{pmatrix} e_{11} \\ e_{21} \\ . \\ \underline{e_{n1}} \\ e_{12} \\ e_{22} \\ . \\ \underline{e_{n2}} \\ e_{13} \\ e_{23} \\ . \\ \underline{e_{n3}} \\ . \\ . \\ \underline{.} \\ e_{1n} \\ e_{2n} \\ . \\ e_{nn} \end{pmatrix}
\qquad
\mathbf{X}^*=\begin{pmatrix} x_{11} \\ x_{21} \\ . \\ \underline{x_{n1}} \\ x_{12} \\ x_{22} \\ . \\ \underline{x_{n2}} \\ x_{13} \\ x_{23} \\ . \\ \underline{x_{n3}} \\ . \\ . \\ \underline{.} \\ x_{1n} \\ x_{2n} \\ . \\ x_{nn} \end{pmatrix}
$$

5. In the form of kronecker's tensor product .

Based on the Tensor Product or Direct Product(⊗)A. Graham (1981) and also based on the Stack Operator, the transformed matrix expression (22) has the following matrixes A', B'' and C'D'' depending on A, B, C and D:

$$\mathbf{A'}=\left(\mathbf{I}\otimes\mathbf{A}\right) \tag{25}$$

$$\mathbf{B''}=\left(\mathbf{B^T}\otimes\mathbf{I}\right) \tag{26}$$

$$\mathbf{C'}\cdot\mathbf{D''}=\left(\mathbf{D^T}\otimes\mathbf{C}\right) \tag{27}$$

A column matrix as $\mathbf{Col_h}=[\mathbf{0\,..\,0\,1\,0\,..\,1}]^{\mathbf{T}}$ must first be defined, with n rows and a 1 element in row h. Therefore, matrixes are expressed as follows:

$$\mathbf{X}^* = \mathbf{X^S} = \sum_{\mathbf{h=1}}^{\mathbf{n}} \mathbf{Col_h}\otimes\mathbf{X}\cdot\mathbf{Col_h} \tag{28}$$

For (16):

$$\mathbf{E}^* = \mathbf{E^S} = \sum_{\mathbf{h=1}}^{\mathbf{n}} \mathbf{Col_h}\otimes\mathbf{E}\cdot\mathbf{Col_h} \tag{29}$$

Where $\mathbf{X^S}$ and $\mathbf{E^S}$ are the stack operators of matrixes X and E respectively.

With these results, we can transform the original equation (1) to solve it in $\mathbf{X}^*$; then, with (23), the expression to return to the original form X is as follows:

$$\mathbf{X} = \left(\sum_{h=1}^{n} \left[\mathbf{I_n} \otimes \left(\mathbf{Col_h^T} \right) \right] \cdot \left(\mathbf{X}^* \cdot \mathbf{Col_h^T} \right) \right)^{\mathbf{T}} \tag{30}$$

$$\mathbf{X} = \sum_{h=1}^{n} \left(\left(\mathbf{I_n} \otimes \left(\mathbf{Col_h^T} \right) \right) \cdot \left(\mathbf{X}^* \cdot \mathbf{Col_h^T} \right) \right)^{\mathbf{T}} \tag{31}$$

$$\mathbf{X} = \sum_{h=1}^{n} \left(\left(\mathbf{Col_h} \cdot \mathbf{X}^{*\mathbf{T}} \right) \cdot \left(\mathbf{I_n} \otimes \mathbf{Col_h} \right) \right) \tag{32}$$

In (30), $\mathbf{Col_h^T}$ is the transpose matrix of previous matrix $\mathbf{Col_h}$, meaning a row matrix with a 1 element in column h and of order n.
Using Kronecker's product can solve the problem

$$\mathbf{A} \cdot \mathbf{X} = \mathbf{F} \Rightarrow \left(\mathbf{I} \otimes \mathbf{A} \right) \cdot \mathbf{X}^* = \mathbf{F}^*$$

a non-zero determinant of A is sufficient.
To solve the problem using Kronecker

$$\mathbf{X} \cdot \mathbf{B} = \mathbf{F} \Rightarrow \left(\mathbf{B^T} \otimes \mathbf{I} \right) \cdot \mathbf{X}^* = \mathbf{F}^*$$

a non-zero determinant of B is sufficient.
To solve the problem using Kronecker

$$\mathbf{C} \cdot \mathbf{X} \cdot \mathbf{D} = \mathbf{F} \Rightarrow \left(\mathbf{D^T} \otimes \mathbf{C} \right) \cdot \mathbf{X}^* = \mathbf{F}^*$$

a non-zero determinant of C and a non-zero determinant of D are sufficient.
To solve the problem using Kronecker

$$\begin{gathered} \mathbf{A} \cdot \mathbf{X} + \mathbf{X} \cdot \mathbf{B} + \mathbf{C} \cdot \mathbf{X} \cdot \mathbf{D} = \mathbf{E} \Rightarrow \\ \left(\mathbf{I} \otimes \mathbf{A} + \mathbf{B^T} \otimes \mathbf{I} + \mathbf{D^T} \otimes \mathbf{C} \right) \cdot \mathbf{X}^* = \mathbf{E}^* \end{gathered} \tag{33}$$

a non-zero determinant of $\left(\mathbf{I} \otimes \mathbf{A} + \mathbf{B^T} \otimes \mathbf{I} + \mathbf{D^T} \otimes \mathbf{C} \right)$ is sufficient.

6. Example in a programming language.

An example to solve (1) in MATLAB programming language is shown below:

```
% A,B,C,D,E Matrixes of order 3 by 3

A=[1 1 1;1 0 1;-1 2 3];
B=[-1 1 -1;1 -1 3;4 7 9];
C=[1 1 3;-3 3 10;3 1 8];
D=[0 2 0;-1 -5 -2;1 5 -9]
E=[1 8 8;1 7 1;1 5 5]

%Identity matrix of order 3 by 3
Id=eye(3)

%Computation of  A' y B''
Aprima=kron(Id,A);
Bprimaprima=Kron(B',Id);  (In this row ‛ denotes transposition)
CpimaDprimaprima=kron(D',C); (In this row ‛ denotes transposition)

%Column matrices

F=[1 0 0]';
G=[0 1 0]';
H=[0 0 1]';

%Calculation of E*

Easter=kron(F,E*F)+kron(G,E*G)+kron(H,E*H);

% Calculation of the  (A'+B''+C'D'') Inverse
Inversaprimas=inv(Aprima+Bprimaprima+CpimaDprimaprima);

% Calculation of X*

Xaster=Inversaprimas*Easter;

%Calculation of X from X*

X=F*Xaster'*kron(Id,F)+G*Xaster'*kron(Id,G)+H*Xaster'* kron(Id,H) (In this row ‛ denotes transposition)
```

7. Conclusión

This paper tried to develop a computational method aimed at solving the matrix commutativity problem, as a support to many repeated problems in different areas. It is useful not only in case of single calculations, but also in iterative calculations, to analyze calculation stability in (33), by calculating the auto-values of parameter matrix $\left(\mathbf{I} \otimes \mathbf{A} + \mathbf{B^T} \otimes \mathbf{I} + \mathbf{D^T} \otimes \mathbf{C} \right)$ (a non zero determinant suffices) and, conditioning the solution, for example, by using the residual vectors method, which are inherent when solving iterative equations. This algorithm is very easy to apply and understand, which represents additional advantages. For instance, with an extended matrix we can affect only properties of one discretization element, since one variable of the variable matrix matches one element or parameter from the parameter matrix, i.e. in a space discretization. And from time to time it can be affected by time discretization.

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