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ISBN 978-1-4614-9595-6 ISBN 978-1-4614-9596-3 (eBook)
DOI 10.1007/978-1-4614-9596-3
Springer New York Heidelberg Dordrecht London

Library of Congress Control Number: 2013956616

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

We first became interested in the interaction between mechanical deformation and magnetic or electric fields just over 10 years ago. This interest was in part motivated by the development of elastomeric materials capable of large deformations that can be generated by the application of an electric or magnetic field. These were beginning to be used in various devices, ranging from activators and sensors to vibration and damping controls. Modelling of the behaviour of such materials requires an adequate theory of nonlinear deformations that can also accommodate electric or magnetic fields in the constitutive description. Although there were nonlinear theories in the literature based on continuum mechanics, we didn't find that they lent themselves easily to applications, in particular to the formulation of boundary-value problems. We therefore decided to seek an approach that could simplify the structure of the constitutive equations and, as a result, the governing equations of equilibrium and motion. We feel that this aim has been achieved in the last few years, and the subject is now at a point where it would be useful to write a connected account of this recent work, and this monograph is the result.

After an introductory chapter, we begin with a chapter (Chap. 2) that summarizes the necessary background from electrostatics, magnetostatics and electrodynamics, aimed primarily at those who have not previously been exposed to much of this theory. This is followed by Chap. 3, which summarizes the essential ingredients of continuum mechanics and nonlinear elasticity theory, partly for the benefit of those who have not attended a course of continuum mechanics. For those with relevant backgrounds the material in these first two chapters will be familiar, but they provide the basic theory and notations that are required in order to merge these distinct subject areas and to derive a coupled nonlinear theory of electroelastic interactions and, separately, of magnetoelastic interactions. In each case the theory is applied to simple representative problems to illustrate the influences of the electric and magnetic fields on the elastic behaviour of materials in the finite deformation regime. We also include a chapter on variational approaches to both electroelasticity and magnetoelasticity. We then provide a discussion of the (linearized) incremental equations superimposed on an underlying configuration consisting of a finite deformation in the presence of either an electric field or a magnetic field. This is used, first in the electroelastic case, to evaluate the stability of the underlying configuration for some simple body geometries, and, for magnetoelastic materials, the magnetoacoustic approximation is adopted in order to study the propagation of magnetoelastic homogeneous plane waves and surface waves.

In the course of writing we have obtained a significant number of new results, which are included here but otherwise unpublished. We have also tried to unify the notation, and therefore much of the notation in the later chapters differs from that in our various papers.

We have been very much helped by the encouragement of colleagues and friends who have been very positive about this project and by those researchers who have taken on board our approach and developed it in different directions.

This monograph is aimed at researchers and graduate students whose interests are at the interface of electromagnetism and continuum mechanics, whether they are mathematicians, engineers or physicists. It requires some familiarity with vector and tensor calculus and some basic knowledge of electromagnetic theory and continuum mechanics.

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Contents

Chapter 1
Introduction

Abstract In this introductory chapter we describe the background that motivates the nonlinear continuum theory of electromechanical and magnetomechanical interactions in elastomeric and polymeric materials with detailed references to the literature. We summarize the content of subsequent chapters in this monograph, beginning with the basics of electromagnetic theory and of the continuum theory of nonlinear elasticity. We outline the content of the remaining chapters that focus on electromechanical and magnetomechanical interactions and their application in the solution of representative boundary value problems. Particular attention is paid on maintaining a consistent notation throughout.

The nonlinear theory describing electromechanical coupling has received considerable attention in the last few years because of the rapid development of elastomeric and polymeric materials which, in response to the application of an electric field, can undergo large deformations (Bar-Cohen 2002a,b; Bossis et al. 2001). Such materials, often referred to as 'smart materials', are being used in a variety of applications, ranging from high-speed actuators and sensors to artificial muscles and other biomedical applications (Broch and Pei 2010; Carpi et al. 2008; Carpi and Smela 2009; Kofod 2008; O'Halloran et al. 2008; Pelrine et al. 2000), and the development of highly deformable and polarizable materials offers exciting possibilities for many new devices. A typical actuator, for example, consists of a thin film of electro-sensitive elastomer on the major surfaces of which are coated two flexible electrodes. Application of a potential difference between the electrodes causes thinning of the film and its lateral expansion (Blok and LeGrand 1969; Pelrine et al. 1998) along with that of the electrodes, thus converting electrical energy into mechanical action and activation. Pelrine et al. (2000), using thin films of silicone and acrylic elastomers, recorded actuation strains well above 100 %. The key point is that the mechanical properties of the materials can be changed rapidly and reversibly by externally applied electric fields, and the coupling between the

L. Dorfmann and R.W. Ogden, *Nonlinear Theory of Electroelastic and Magnetoelastic Interactions*, DOI 10.1007/978-1-4614-9596-3_1,

mechanics and the electric fields is both strong and highly nonlinear. This has prompted much interest in the theory of nonlinear electroelasticity that governs such behaviour.

A parallel phenomenon exists when an elastic material is subject to a magnetic field. Anyone who has ever played with a permanent magnet has been intrigued by how metal objects are attracted by the magnetic force. The force acts not only on the object as a whole but also on each bit of material, inducing a change in shape and/or size of the object commonly known as magnetostriction. For typical metals this change is very small and the associated variations in the magnetic and mechanical properties of the material can be neglected. In such circumstances use of the linear theory of magnetoelasticity is appropriate. Only recently have researchers come to appreciate the profound potential of multifunctional compliant magneto-sensitive materials as new compounds have been synthesized. These mechanically soft materials are capable of large elastic deformations under the influence of an external magnetic field, much larger than in conventional magnetostriction. To put this advance in perspective, the new materials represent a step change by several orders of magnitude compared to conventional magnetic metals in both high magneto-mechanical compliance and large elastic deformability. The new materials are highly deformable and magnetizable polymers composed of a rubberlike base matrix embedded with micron-sized magnetoactive particles. Like a typical rubber, they have low mechanical stiffness and are very compliant, especially in low-dimensional structures such as membranes and rods, while demonstrating good magnetic susceptibility (Barham et al. 2007). The small particle size ensures that the materials are effectively homogeneous, and the material processing has already been advanced to the point where robust material characteristics can be developed. The transformative concepts of nonlinearity in the response and the magneto-mechanical coupling of these materials open the door for many new devices, impacting a range of applications that could not be addressed with previously available materials.

The linear theory of magnetoelasticity, applicable to infinitesimal deformations and weak fields, neglects the magneto-mechanical coupling in the sense that there is no change in mechanical properties due to the applied magnetic field and no change in the magnetic properties due to mechanical deformations. The availability of materials that can operate in a highly nonlinear magneto-mechanical regime offers very exciting possibilities and challenges from the perspectives of device design, materials science, constitutive modelling and magneto-mechanical theory. The nonlinear magneto-mechanical coupling in magneto-sensitive materials has generated much interest recently (see, e.g., Albanese and Cunefare 2003; Dapino 2004; Farshad and Benine 2004; Ginder et al. 1999, 2000, 2001; Kordonsky 1993; Li and Zhang 2008; Varga et al. 2005, 2006; Yalcintas and Dai 2004). The effects of a magnetic field on the mechanical response of magneto-sensitive materials are well illustrated in Rigbi and Jilkén (1983), Jolly et al. (1996), Ginder et al. (2002), Bellan and Bossis (2002), Lokander and Stenberg (2003), Gong et al. (2005) and Boczkowska and Awietjan (2009), for example.

The purpose of this monograph is to present an overview of fundamental concepts of both electromagnetic theory and nonlinear elasticity and the coupling of these theories for the development of a consistent theoretical framework that can be used to describe the nonlinear behaviour of electro-sensitive and magneto-sensitive materials. The nonlinear theory of electroelasticity was originally developed by Toupin (1956) for the static situation and extended to include thermal effects by Tiersten (1971). Relevant background information is provided in the books by Truesdell and Toupin (1960) and Landau and Lifshitz (1960). For nonlinear magnetoelastic interactions, the book by Brown (1966) provided a basis for much subsequent work.

Interest in the theory is evidenced by a range of books dealing with the nonlinear interaction between mechanical and electromagnetic fields, including Hutter and van de Ven (1978), Nelson (1979), Maugin (1988), Eringen and Maugin (1990), Tiersten (1990), Maugin et al. (1992), Kovetz (2000), Yang (2005) and Hutter et al. (2006), and the articles in the volume of lecture notes by Ogden and Steigmann (2011). Recent theoretical and computational developments concerning nonlinear electromechanical interactions are included in articles by, for example, Borcea and Bruno (2001), Dorfmann and Ogden (2003), Brigadnov and Dorfmann (2003), Dorfmann and Brigadnov (2004), Dorfmann and Ogden (2004a), Steigmann (2004), Kankanala and Triantafyllidis (2004), Dorfmann and Ogden (2004b), Dorfmann and Ogden (2005a), McMeeking and Landis (2005), McMeeking et al. (2007), Fosdick and Tang (2007), Vu et al. (2007), Vu and Steinmann (2007), Suo et al. (2008), Bustamante et al. (2008), Bustamante (2009), Bustamante et al. (2009a), Bustamante et al. (2009b), Steigmann (2009), Barham et al. (2010), Dorfmann and Ogden (2010a), Dorfmann and Ogden (2010b), Bustamante (2010), Ponte Castañeda and Galipeau (2011) and Barham et al. (2012).

The book is intended for graduate students, engineers and scientists with an interest in the mathematical theory of electroelastic and magnetoelastic interactions. We have made an effort to present the theory in a self-contained manner using consistent notation throughout. We believe that some knowledge of tensor calculus is beneficial, at least initially. The presentation of the material is such that the reader will be exposed, as the need arises, to more complex vector and tensor operations that are included in some detail. While the theories of electromagnetism and of nonlinear solid mechanics are beautiful in their own right, the coupled theories of electroelasticity and magnetoelasticity provide rather elegant generalizations of the theories.

In this monograph we begin in Chap. 2 with a review of the basic concepts of the electromagnetic theory. We review physical quantities such as point and distributed charges, the Lorentz force, Coulomb's Law, charge conservation, the electrostatic potential, Gauss's Theorem and the force and couple on a dipole in an electric field. Then, we review the ideas of the theory of magnetostatics, specifically the Biot–Savart Law, scalar and vector potentials, Ampère's Circuital Law and the force and couple on a dipole in a magnetic field. Next, for time-dependent fields, we construct Faraday's Law of Induction as a prelude to summarizing the governing equations of electromagnetism, known as Maxwell's equations. For electromagnetic continua, the electric and magnetic properties of a material are defined through the notions of *polarization* and *magnetization*. In the final part of Chap. 2, we derive jump conditions that have to be satisfied by the various electric and magnetic field vectors on surfaces of discontinuity within a material or on the boundary of a material body. An excellent and more detailed treatment of electromagnetism is contained in the classic text by Jackson (1999).

The chapter mentioned above does not account for the deformability of materials. As a prelude to coupling electromagnetic and mechanical effects we next, in Chap. 3, develop sufficient background from nonlinear continuum mechanics to facilitate the subsequent fusing of the distinct theories. An extensive treatment of the mechanics of continua and nonlinear elasticity can be found in the texts by, for example, Truesdell and Noll (1965), Ogden (1997) and Holzapfel (2000). Here we merely provide a summary of the main ideas needed in the following chapters, including kinematics, balance laws, mechanical stress and elastic constitutive laws, and illustrate the theory by its application to some representative boundary-value problems.

The first part of Chap. 4 focuses on some of the different ways in which the equations of *mechanical equilibrium* can be written in the presence of electromechanical interactions. We provide a brief overview of different energy formulations based on Eulerian forms of the electric field, electric displacement and polarization vectors and then introduce the Lagrangian counterparts of the electric and electric displacement fields (but we do not use a Lagrangian version of the polarization) and the associated governing equations, boundary conditions and constitutive equations, with particular attention paid to isotropic electroelastic materials. The resulting Lagrangian formulations are based on two alternative forms of a so-called total energy (density) function and lead to compact systems of equations. The theory is then applied in Chap. 5 to a selection of boundary-value problems that involve the coupling of electric fields with elastic deformations.

Chapter 6 provides the theory of magnetoelastic interactions in an analogous development to that for electroelastic interactions in Chap. 4, and in Chap. 7 the theory is applied to representative boundary-value problems similar to those in Chap. 5.

Based on the theories in Chaps. 4 and 6, we provide in Chap. 8 separate variational treatments of the equations of electroelasticity and magnetoelasticity using both scalar and vector potential functions in each case.

In Chap. 9 we cast the equations of continuum electrodynamics in Lagrangian form and then provide the (linearized) equations associated with ‘small’ incremental motions and electromagnetic fields superimposed on a known finite motion and accompanying electromagnetic field. Thereafter the equations are specialized to the situation in which there is an underlying *static* deformation and electric (or magnetic) field upon which are superimposed *static* incremental deformations and electric (or magnetic) fields. In particular, we derive expressions for the components of the electroelastic and magnetoelastic moduli tensors for isotropic materials.

Chapter 10 illustrates the application of the equations governing the incremental behaviour of electroelastic solids in the analysis of the stability of a homogeneously deformed half-space with an electric field normal to the surface of the half-space and the corresponding problem for a plate of finite thickness. In each case the electric field exterior to the material is accounted for, the analysis is restricted to an underlying plane strain deformation, and for simplicity, the increments are also restricted to the considered plane. The results for the plate problem are compared with those for the problem in which a dielectric material is confined between two flexible electrodes, in which case there is no electric field outside the material. For each problem numerical results are illustrated in respect of a neo-Hookean electroelastic material model.

Finally, in Chap. 11, incremental *motions* are considered superimposed on a static finitely deformed configuration of a magnetoelastic material based on the quasimagnetostatic approximation and applied first to the propagation of homogeneous plane waves for a general magnetoelastic material. Then, for an isotropic magnetoelastic material, the theory is applied to the analysis of surface waves on a homogeneously deformed half-space for various orientations of the underlying magnetic field. For purposes of illustration, two simple prototype models of a magnetoelastic material are adopted, and detailed numerical results are provided.

In an appendix we list some useful formulas from vector and tensor calculus that are used in various places throughout the text.

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Chapter 2
Electromagnetic Theory

Abstract This chapter contains a summary of the main ingredients of the theory of electrostatics and the theory of magnetostatics required for the subsequent formulation of the theory governing electromechanical interactions in electro-sensitive and magneto-sensitive materials. In particular, expressions for the force and couple on an electric dipole in an electric field and on a magnetic dipole in a magnetic field are derived, which are important for examining polarization and magnetization in material media. The full system of four Maxwell equations that govern electromagnetic phenomena in material media is then obtained and the notions of polarization and magnetization that distinguish polarizable and magnetizable materials, respectively, from vacuum or from materials that are not sensitive to electric or magnetic fields are introduced. Maxwell's equations are then cast in their standard form in terms of free charge density and free current density. Based on Maxwell's equations, the final section of the chapter provides a derivation of the conditions that must be satisfied by the electric field, electric displacement, magnetic field and magnetic induction vectors on material boundaries.

2.1 Electrostatics

2.1.1 Preliminary Remarks

The equations that describe the forces generated by charged particles are well established and can be found in any textbook on electrostatics. In this section we begin with a short review of these equations and the associated field quantities and the Lorentz Law which gives the force acting on a charged particle in an electromagnetic field as a prelude to the introduction, in the next three sections, of magnetic fields and the full set of Maxwell's equations that completely describe the interaction between electric and magnetic fields in the nonrelativistic setting. Historically, electromagnetic theory has been developed with reference to observable

L. Dorfmann and R.W. Ogden, *Nonlinear Theory of Electroelastic and Magnetoelastic Interactions*, DOI 10.1007/978-1-4614-9596-3_2,

macroscopic events occurring in vacuum or in condensed matter, but the theory is first based on microscopic quantities such as point charges, dipoles and small current-carrying circuits and their distributions, which together with their forces of interaction are built into a continuum theory that describes experimentally observed phenomena. In this section we are concerned primarily with the equations of electrostatics. Of course, the notions of point charge, dipole, etc., are convenient mathematical idealizations used to build the basic theory and are associated with singularities. For example, the field of a point charge is infinite when evaluated at the point where the charge is located. Continuum theory avoids such singularities.

For detailed background covering the material in this section and Sects. 2.2–2.4, we refer to Becker and Sauter (1964), Landau and Lifshitz (1960), Jackson (1999) and Stratton (2007) and the recent book by Kovetz (2000).

2.1.2 *The Electric Field*

Consider a time-independent spatial distribution of charged particles that interact with one another by generating electrostatic forces. These interacting forces enable the electric field, denoted $\mathbf{E}$, to be defined at an arbitrary location $\mathbf{x}$. Consider the resultant force $\mathbf{f}$ of the considered distribution of particles acting on a test particle with point charge e placed at position $\mathbf{x}$. The point charge must be small enough not to alter the original arrangement of the particles. As the magnitude of e approaches zero, it is clear that the measured force must approach zero as well. However, in the limit the ratio of force $\mathbf{f}$ to the charge e remains finite and identifies the *electric field vector* $\mathbf{E}$ at the point $\mathbf{x}$, i.e.

$$\mathbf{E}(\mathbf{x}) = \lim_{e \to 0} \frac{\mathbf{f}}{e}, \tag{2.1}$$

from which it follows that the electric field has dimensions of force per unit charge.

2.1.3 *The Lorentz Law of Force*

The Lorentz Law of force (Hendrik Antoon Lorentz, 1853–1928) is one of the fundamental elements of the classical theory of electromagnetism and defines the force exerted by an electromagnetic field on a charged particle. We first consider the case of a stationary point charge at rest at location $\mathbf{x}$ subject to an electric field $\mathbf{E}$. This field exerts a force $\mathbf{f}$ on the particle given by

$$\mathbf{f}(\mathbf{x}) = e\mathbf{E}(\mathbf{x}), \tag{2.2}$$

where e is the charge of the particle, which is small enough not to disturb the sources of the electric field and so not alter the electric field. This is a special case of the Lorentz Law in which the particle is stationary. For completeness, we now consider the general case in which the particle is moving, which requires the introduction of a magnetic field. Suppose that, in addition to the electric field $\mathbf{E}$, there is a time-independent magnetic field, described in terms of the magnetic induction vector $\mathbf{B}$, which will be discussed in detail in Sect. 2.2, and consider the particle, instead of being at rest, to be moving with velocity $\mathbf{v}$ and instantaneously located at the point $\mathbf{x}$. The particle experiences an additional force perpendicular to its direction of motion and proportional to the magnitude of $\mathbf{v}$. The additional force is maximal when the motion of the particle is perpendicular to the direction of the magnetic induction $\mathbf{B}$ and vanishes when the orientations of $\mathbf{v}$ and $\mathbf{B}$ coincide, and the total force on the particle, consisting of electric and magnetic components, is then

$$\mathbf{f}(\mathbf{x}) = e[\mathbf{E}(\mathbf{x}) + \mathbf{v} \times \mathbf{B}(\mathbf{x})]. \tag{2.3}$$

This is known as the *Lorentz force*.

2.1.4 Coulomb's Law

The laws of electrostatics have their origin in the experimental work performed by Coulomb (Charles Augustin de Coulomb, 1736–1806), who investigated the forces of interaction generated by a distribution of charged particles at rest. In particular, Coulomb was able to quantify the force of interaction between two charged particles. If the two particles have charges e_1 and e_2 and are placed at locations $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively, the interaction force is given by *Coulomb's Law*

$$\mathbf{f} = k\, e_1 e_2 \frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3}, \tag{2.4}$$

where k is a constant of proportionality that depends on the units used. Coulomb's Law shows that the force depends linearly on the magnitude of each charge, is inversely proportional to the square of the distance between the two particles and is directed along the line connecting the two charges. It is attractive if one charge is positive and one negative, as illustrated in Fig. 2.1, and repulsive if they are both positively charged or both negatively charged.

Coulomb's Law can be generalized to the case of N interacting particles. The resultant net force acting on a test charge e_1 due to all other charged particles is obtained by using the principle of linear superposition and given by

$$\mathbf{f} = k\, e_1 \sum_{i=2}^{N} e_i \frac{\mathbf{x}_1 - \mathbf{x}_i}{|\mathbf{x}_1 - \mathbf{x}_i|^3}. \tag{2.5}$$

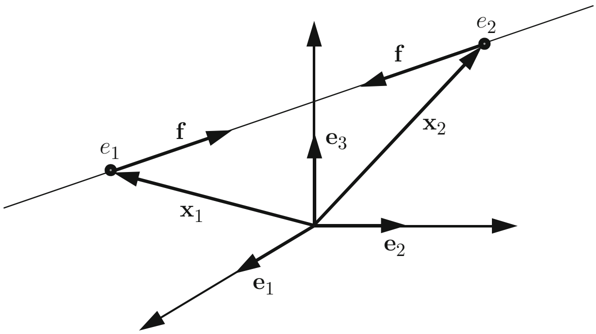

Fig. 2.1 The (attractive) force **f** of interaction between two charged particles e_1 and e_2 for the case in which one charge is positive and one negative. Cartesian basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are also shown together with the position vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ of e_1 and e_2

Coulomb also showed that the electric field **E** generated by an isolated and stationary particle is proportional to its charge e and to the inverse square of the distance from the charge. The field at the point $\mathbf{x}$ due to a point charge e placed at the origin is therefore given by

$$\mathbf{E}(\mathbf{x}) = k\, e\frac{\mathbf{x}}{r^3} = k\, e\frac{\hat{\mathbf{x}}}{r^2}, \tag{2.6}$$

where $r = |\mathbf{x}|$ and $\hat{\mathbf{x}} = \mathbf{x}/r$ is a unit vector. If the charged particle is located at the fixed point $\mathbf{x}'$ instead of the origin, then (2.6) is replaced by

$$\mathbf{E}(\mathbf{x}) = k\, e\frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}. \tag{2.7}$$

This formula for a single particle is easily extended to a distribution of N charged particles located at $\mathbf{x}_i$, $i = 1, \ldots, N$, by the principle of linear superposition to give the electric field at the point $\mathbf{x} \neq \mathbf{x}_i$, $i = 1, \ldots, N$, as

$$\mathbf{E}(\mathbf{x}) = k\sum_{i=1}^{N} e_i\, \frac{\mathbf{x} - \mathbf{x}_i}{|\mathbf{x} - \mathbf{x}_i|^3}. \tag{2.8}$$

2.1.5 Charge Conservation

The definition of the electric field up to this point assumes the existence of a discrete spatial distribution of charged particles. We now generalize this concept within the continuum framework by considering a charge distributed over a small volume in the neighbourhood of a point $\mathbf{x}$. Consider an infinitesimal element of volume $\mathrm{d}V$ and let $\rho_e dV$ be the total charge within this element. Then ρ_e is the *charge density*, which may be positive or negative and depends, in general, on the position $\mathbf{x}$ and time t: $\rho_e = \rho_e(\mathbf{x}, t)$. Thus, the individual charged particles within dV are 'smoothed out' in the continuum theory to form a continuous density function.

The velocities of the individual particles are treated similarly, and we denote by $\mathbf{v}$ the mean velocity of the individual charges in dV. Then, we define the *current density* $\mathbf{J}$ at $\mathbf{x}$ by

$$\mathbf{J} = \rho_e \mathbf{v}, \tag{2.9}$$

from which can be determined the rate at which charges cross a unit surface of any orientation and the current flowing across an arbitrary surface, as discussed below.

The Lorentz force for a point charge subject to electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ is given in (2.3). For a charge distribution with density ρ_e and current density $\mathbf{J}$, the Lorentz force per unit volume is given by

$$\mathbf{f} = \rho_e \mathbf{E} + \mathbf{J} \times \mathbf{B}. \tag{2.10}$$

Consider an arbitrary volume V *fixed* in space and bounded by a closed surface S with unit outward normal $\mathbf{n}$. The charge density per unit volume within V is ρ_e, and the rate at which charge flows out of V across S is given by $\mathbf{J} \cdot \mathbf{n}$ per unit area. The rate of increase of charge within V must arise from the *influx* of charge across S. Thus,

$$\frac{d}{dt} \int_V \rho_e \, dV = -\int_S \mathbf{J} \cdot \mathbf{n} \, dS, \tag{2.11}$$

which says that any change in charge within a confined volume must be balanced by the charge flowing across the bounding surface. By using the divergence theorem to convert the surface integral to an integral over the volume V, we obtain

$$\int_V \left(\frac{\partial \rho_e}{\partial t} + \operatorname{div} \mathbf{J} \right) dV = 0, \tag{2.12}$$

which must hold for arbitrary V. Provided the integrand in (2.12) is continuous, we may deduce the local form of the *charge conservation equation* as

$$\frac{\partial \rho_e}{\partial t} + \operatorname{div} \mathbf{J} = 0. \tag{2.13}$$

In a steady state situation (no time dependence) we have $\partial \rho_e / \partial t = 0$ and (2.13) reduces to

$$\operatorname{div} \mathbf{J} = 0, \tag{2.14}$$

and the corresponding integral form is

$$\int_S \mathbf{J} \cdot \mathbf{n} \, \mathrm{d}S = 0, \tag{2.15}$$

where S is an arbitrary closed surface.

2.1.6 Units

In this book we use the SI system of units in which the basic units are length (metres, m), mass (kilograms, kg), time (seconds, s) and electric charge (Coulomb, C). The electric charge on an electron, for example, is $e = -1.602 \times 10^{-19}\,\mathrm{C}$. From the equations connecting the electromagnetic field variables, it is then possible to derive the dimensions of all other quantities in terms of these four basic units. Force, for example, has dimensions $\mathrm{kg\,m\,s^{-2}}$ and is expressed in Newtons (N): 1 Newton is equal to 1 $\mathrm{kg\,m\,s^{-2}}$. From (2.9) we find the dimensions of the current density $\mathbf{J}$ to be $\mathrm{C\,m^{-2}\,s^{-1}}$ or Ampères per square metre ($\mathrm{A\,m^{-2}}$), where an Ampère (A) has dimensions $\mathrm{C\,s^{-1}}$.

In SI units the constant k in (2.4), which defines the interaction force between two charged particles, has dimensions of $\mathrm{kg\,m^3\,s^{-2}\,C^{-2}}$. From (2.6) we find that the dimension of the electric field $\mathbf{E}$ is $\mathrm{kg\,m\,s^{-2}\,C^{-1}}$, alternatively $\mathrm{N\,C^{-1}}$ or volt per metre ($\mathrm{V\,m^{-1}}$). In SI units the numerical value of the constant of proportionality k in (2.4) is given by $1/4\pi\varepsilon_0$, and Coulomb's Law (2.4) becomes

$$\mathbf{f} = \frac{e_1 e_2}{4\pi\varepsilon_0} \frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3}, \tag{2.16}$$

where $\varepsilon_0 \approx 8.854 \times 10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}}$ is the *permittivity of free space*. Similarly, the electric field at location $\mathbf{x}$ of a point charge e at $\mathbf{x}'$ is given by (2.7), which, using the expression for k, gives

$$\mathbf{E}(\mathbf{x}) = \frac{e}{4\pi\varepsilon_0} \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}. \tag{2.17}$$

2.1.7 The Field of a Static Charge Distribution

If we replace the constant of proportionality k by its value appropriate to the units adopted herein, then (2.6) takes on the form

$$\mathbf{E}(\mathbf{x}) = \frac{e}{4\pi\varepsilon_0} \frac{\hat{\mathbf{x}}}{r^2} = -\frac{e}{4\pi\varepsilon_0} \,\mathrm{grad}\left(\frac{1}{r}\right), \tag{2.18}$$

where again $r = |\mathbf{x}|$, while grad represents the gradient operator with respect to $\mathbf{x}$. When the charged particle is placed at the position $\mathbf{x}'$ instead of at the origin, the electric field is given by (2.7) or, alternatively, by

$$\mathbf{E}(\mathbf{x}) = \frac{e}{4\pi\varepsilon_0}\frac{\mathbf{R}}{R^3} = -\frac{e}{4\pi\varepsilon_0}\,\mathrm{grad}\left(\frac{1}{R}\right), \tag{2.19}$$

where we have introduced the notations $\mathbf{R} = \mathbf{x} - \mathbf{x}'$ and $R = |\mathbf{R}|$. As in (2.18), the gradient operator is with respect to $\mathbf{x}$, i.e. the point at which $\mathbf{E}$ is determined.

These results can be generalized to a region in space containing a continuous distribution of charge. In particular, consider a continuous distribution of charge with density ρ_e within a volume V, with the point charge e replaced by the charge $\rho_\mathrm{e}\,\mathrm{d}V$ in the volume element $\mathrm{d}V$. If $\rho_\mathrm{e} = 0$ outside the specified volume V, then the electric field at location $\mathbf{x}$ is the sum of the contributions from all elements of charge $\rho_\mathrm{e}\,\mathrm{d}V$ within V. It is given by

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho_\mathrm{e}(\mathbf{x}')\frac{\mathbf{R}}{R^3}\mathrm{d}V(\mathbf{x}') = -\frac{1}{4\pi\varepsilon_0}\int_V \rho_\mathrm{e}(\mathbf{x}')\,\mathrm{grad}\left(\frac{1}{R}\right)\mathrm{d}V(\mathbf{x}'), \tag{2.20}$$

where the integration is with respect to the $\mathbf{x}'$ variable. The grad operator is again with respect to $\mathbf{x}$ and can therefore be taken outside the integral, leading to an alternative expression for the electric field at point $\mathbf{x}$, specifically

$$\mathbf{E}(\mathbf{x}) = -\frac{1}{4\pi\varepsilon_0}\,\mathrm{grad}\int_V \frac{\rho_\mathrm{e}(\mathbf{x}')}{R}\mathrm{d}V(\mathbf{x}'). \tag{2.21}$$

The gradient operator in the above equation acts on a scalar function, and it is therefore convenient to introduce a notation, namely, φ to represent this function. It is known as the *electrostatic potential* and allows (2.21) to be written compactly as

$$\mathbf{E}(\mathbf{x}) = -\mathrm{grad}\,\varphi(\mathbf{x}), \tag{2.22}$$

where φ is given by

$$\varphi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_\mathrm{e}(\mathbf{x}')}{R}\mathrm{d}V(\mathbf{x}'). \tag{2.23}$$

The gradient operator in (2.22) operates on a scalar quantity, yielding a vector field. The standard vector identity $\mathrm{curl}\,(\mathrm{grad}\,\varphi) \equiv \mathbf{0}$ for any scalar function φ applied in the present context gives the *first equation of electrostatics*

$$\mathrm{curl}\,\mathbf{E} = \mathbf{0}. \tag{2.24}$$

Far from the specified volume V, the electric field is approximately that of a point charge situated at the origin with a charge equal to the total charge within the distribution. In this case we have $1/R \approx 1/r$ and the electrostatic potential (2.23) can be approximated by

$$\varphi(\mathbf{x}) \approx \frac{e}{4\pi\varepsilon_0 r}, \tag{2.25}$$

where

$$e = \int_V \rho_e(\mathbf{x}')\,\mathrm{d}V(\mathbf{x}') \tag{2.26}$$

is the total charge within V.

2.1.8 Gauss's Theorem

Equation (2.24) on its own is not sufficient to determine the electric field. The set of governing equations is completed by means of Gauss's theorem (Carl Friedrich Gauss, 1777–1855), which is now derived.

Consider first a particle carrying charge e placed at a position $\mathbf{x}'$ within a volume V bounded by a closed surface S. Equation (2.19) gives the associated electric field at location $\mathbf{x}$. Specifically, we need to determine the electric field $\mathbf{E}$ at a point $\mathbf{x}$ on the surface S where the unit outward pointing normal vector is $\mathbf{n}$, say. Let $\mathrm{d}\mathbf{S}$ $(= \mathbf{n}\,\mathrm{d}S)$ be an infinitesimal vector area element on the surface S at $\mathbf{x}$, where $\mathrm{d}S > 0$. Then, the flux of $\mathbf{E}$ across $\mathrm{d}\mathbf{S}$ is given by

$$\mathbf{E}\cdot\mathrm{d}\mathbf{S} = \frac{e}{4\pi\varepsilon_0}\frac{\mathbf{R}\cdot\mathrm{d}\mathbf{S}}{R^3}, \tag{2.27}$$

and the total flux of $\mathbf{E}$ across the closed surface S is

$$\int_S \mathbf{E}\cdot\mathrm{d}\mathbf{S} = \frac{e}{4\pi\varepsilon_0}\int_S \frac{\mathbf{R}\cdot\mathrm{d}\mathbf{S}}{R^3}. \tag{2.28}$$

The integrand on the right-hand side defines the *solid angle*, denoted $\mathrm{d}\Omega$, subtended by $\mathrm{d}\mathbf{S}$ at $\mathbf{x}'$, i.e.

$$\mathrm{d}\Omega = \frac{\mathbf{R}\cdot\mathrm{d}\mathbf{S}}{R^3}, \tag{2.29}$$

a purely geometrical quantity, and (2.28) may therefore be written as

$$\int_S \mathbf{E}\cdot\mathrm{d}\mathbf{S} = \frac{e}{4\pi\varepsilon_0}\int_S \mathrm{d}\Omega. \tag{2.30}$$

If $\mathbf{x}'$ lies within the volume V, then the solid angle is equal to 4π. On the other hand, if $\mathbf{x}'$ lies outside the bounding surface, then positive contributions of $\mathbf{R}\cdot\mathrm{d}\mathbf{S}/R^3$ to the integral are balanced by negative contributions and the integral vanishes, as can be shown by a simple application of the divergence theorem, noting that $1/R$ is a fundamental solution of Laplace's equation, i.e. $\nabla^2(1/R) = 0$. Thus,

$$\int_S \mathbf{E}\cdot\mathrm{d}\mathbf{S} = \begin{cases} e/\varepsilon_0 & \text{if } e \text{ is within } V \\ 0 & \text{if } e \text{ is outside } V. \end{cases} \tag{2.31}$$

We now extend this result to the case of a charge distribution within a volume V' which intersects V. The electric field at point $\mathbf{x}$ on the surface S is

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho_\mathrm{e}(\mathbf{x}')\,\mathbf{R}}{R^3}\mathrm{d}V(\mathbf{x}'), \tag{2.32}$$

and the flux of $\mathbf{E}$ across the closed surface S, the boundary of V, is

$$\int_S \mathbf{E}(\mathbf{x})\cdot\mathrm{d}\mathbf{S}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \rho_\mathrm{e}(\mathbf{x}')\,\mathrm{d}V(\mathbf{x}')\int_S \frac{\mathbf{R}\cdot\mathrm{d}\mathbf{S}(\mathbf{x})}{R^3}. \tag{2.33}$$

Using again the properties of solid angle, we have

$$\int_S \frac{\mathbf{R}\cdot\mathrm{d}\mathbf{S}}{R^3} = \begin{cases} 4\pi & \text{if } \mathbf{x}' \text{ is within } V \\ 0 & \text{if } \mathbf{x}' \text{ is outside } V, \end{cases} \tag{2.34}$$

and hence

$$\int_S \mathbf{E}\cdot\mathrm{d}\mathbf{S} = \frac{1}{\varepsilon_0}\int_{V'(V)} \rho_\mathrm{e}(\mathbf{x}')\,\mathrm{d}V(\mathbf{x}'), \tag{2.35}$$

where $V'(V)$ is that part of V' contained within V. If e denotes the total charge inside the volume V, the above equation can be written more compactly as

$$\int_S \mathbf{E}\cdot\mathrm{d}\mathbf{S} = \frac{e}{\varepsilon_0}, \tag{2.36}$$

which is *Gauss's theorem*. It states simply that the resultant flux of the electric field $\mathbf{E}$ across any closed surface S is proportional to the total charge e contained within S.

To derive the associated local equation, we rewrite Gauss's theorem as

$$\int_S \mathbf{E}\cdot\mathrm{d}\mathbf{S} = \frac{1}{\varepsilon_0}\int_V \rho_\mathrm{e}\,\mathrm{d}V, \tag{2.37}$$

where the integration on the right-hand side is restricted to the volume bounded by the surface S. Then, by use of the divergence theorem, (2.37) becomes

$$\int_V \left(\operatorname{div}\mathbf{E} - \frac{\rho_e}{\varepsilon_0}\right) \mathrm{d}V = 0, \tag{2.38}$$

which must hold for arbitrary V. Therefore, provided the integrand in (2.38) is continuous, we deduce that

$$\operatorname{div}\mathbf{E} = \frac{\rho_e}{\varepsilon_0}, \tag{2.39}$$

which is the local form of Gauss's theorem and the *second equation of electrostatics*.

The equations

$$\operatorname{curl}\mathbf{E} = \mathbf{0}, \quad \operatorname{div}\mathbf{E} = \frac{\rho_e}{\varepsilon_0} \tag{2.40}$$

together govern the electrostatic field $\mathbf{E}$. Since $(2.40)_1$ is equivalent to $\mathbf{E} = -\operatorname{grad}\varphi$, we may substitute this into $(2.40)_2$ to obtain *Poisson's equation*

$$\nabla^2\varphi = -\frac{\rho_e}{\varepsilon_0} \tag{2.41}$$

for the scalar potential φ for a given charge distribution with density ρ_e. It is easy to verify that the scalar potential (2.23) satisfies this equation whether $\mathbf{x}$ is within or outside V (see, e.g., Jackson (1999), Sect. 1.7). *A fortiori*, the integrals in (2.20) and (2.23) are finite when $\mathbf{x}$ is within V. For regions where $\rho_e = 0$, (2.41) reduces to *Laplace's equation*

$$\nabla^2\varphi = 0. \tag{2.42}$$

Of course, for any particular boundary-value problem, appropriate boundary conditions need to accompany the equations. These will be given in a general form in Sect. 2.5 for a fixed surface and in Sect. 9.1 for a moving surface.

2.1.9 The Field of a Dipole

The equations of electrostatics discussed up to this point are concerned with the interactions of time-independent charges and fields in free space. When an electric field is applied to a solid medium, the configuration of charges is altered, and this leads, in particular, to the production of dipoles within the medium at the microscopic level, and the material is said to be *polarized*. A dipole can be visualized as two localized concentrations of charge with the same magnitude and opposite

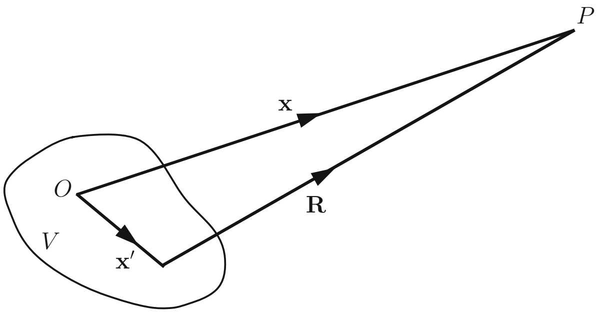

Fig. 2.2 Depiction of a volume V containing a charge distribution with density $\rho_e(\mathbf{x}')$ at point $\mathbf{x}'$ relative to an origin O within V. There are no charges outside V. A field point P exterior to V has position vector $\mathbf{x}$ relative to the origin O and $\mathbf{R} = \mathbf{x} - \mathbf{x}'$

signs separated by a small distance. We shall consider polarization in Sect. 2.4, but here we discuss the field due to an idealized isolated dipole.

Equation (2.20) determines the electric field at location $\mathbf{x}$ for a charge density contained within a volume V, and (2.23) gives the corresponding potential. To determine the field generated by an electric dipole, we again consider a distribution of charge with density $\rho_e(\mathbf{x}')$, but now confined to a *small* volume V, where $\mathbf{x}'$ is the position vector of a typical point in V relative to an origin O located within V and $\rho_e = 0$ outside V. Let $\mathbf{x}$ be the position vector of a point P far from V at which the electrostatic field is to be calculated, as depicted in Fig. 2.2.

Then $|\mathbf{x}'| \ll |\mathbf{x}|$ for all $\mathbf{x}'$ in V, and we may use the Taylor expansion to obtain the approximation

$$\frac{1}{R} \equiv \frac{1}{|\mathbf{x} - \mathbf{x}'|} \approx \frac{1}{r} - \mathbf{x}' \cdot \mathrm{grad}\left(\frac{1}{r}\right), \tag{2.43}$$

recalling that $r = |\mathbf{x}|$. Hence, from (2.23), the electrostatic potential at $\mathbf{x}$ is approximated as

$$\varphi(\mathbf{x}) \approx \frac{e}{4\pi\varepsilon_0 r} - \frac{1}{4\pi\varepsilon_0}\mathbf{p} \cdot \mathrm{grad}\left(\frac{1}{r}\right), \tag{2.44}$$

where e is the total charge in V given by the formula (2.26) and $\mathbf{p}$ is defined by

$$\mathbf{p} = \int_V \rho_e(\mathbf{x}')\mathbf{x}'\mathrm{d}V(\mathbf{x}'). \tag{2.45}$$

If $e \neq 0$, then the origin can be translated to the centre of charge (analogous to the centre of mass in mechanics) so that $\mathbf{p} = \mathbf{0}$, in which case

$$\varphi(\mathbf{x}) \approx \frac{e}{4\pi\varepsilon_0 r}, \tag{2.46}$$

which is the field of a point charge e located at the origin. Then, to a first approximation, the field at a large distance from a charge distribution is indistinguishable from that of a point charge, as already indicated in Sect. 2.1.7. On the other hand, if $e = 0$ and $\mathbf{p} \neq \mathbf{0}$, we have

$$\varphi(\mathbf{x}) \approx -\frac{1}{4\pi\varepsilon_0}\mathbf{p}\cdot\operatorname{grad}\left(\frac{1}{r}\right) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\mathbf{x}}{r^3}. \tag{2.47}$$

This is the potential due to an *electric dipole* of strength $\mathbf{p}$ situated at the origin. This is equivalent to having two charges of magnitude e and of equal and opposite signs very close together, say at distances $\pm\mathbf{d}/2$ from the origin, in which case $\mathbf{p} = e\mathbf{d}$. The above formula becomes exact in the limit in which $\mathbf{d}$ approaches $\mathbf{0}$ as $e \to \infty$, while $\mathbf{p}$ remains finite.

2.1.10 The Force and Couple on a Dipole in an Electric Field

We now calculate the total electric force on a distribution of charge contained within a volume V. Using that part of the Lorentz force density (2.3) due to the electric field $\mathbf{E}$, this is denoted F_e, the subscript e signifying 'electric', and given by

$$\mathsf{F}_\mathrm{e} = \int_V \rho_\mathrm{e}(\mathbf{x}')\mathbf{E}(\mathbf{x}')\,\mathrm{d}V(\mathbf{x}'). \tag{2.48}$$

Now take the origin to be within V and let V be a small volume. Then, by expanding $\mathbf{E}(\mathbf{x}')$ to the first order in $\mathbf{x}'$, we may approximate F_e as

$$\mathsf{F}_\mathrm{e} = \int_V \rho_\mathrm{e}\,\mathrm{d}V(\mathbf{x}')\,\mathbf{E}(\mathbf{0}) + \int_V \rho_\mathrm{e}(\mathbf{x}')\mathbf{x}'\mathrm{d}V(\mathbf{x}')\cdot[\operatorname{grad}\mathbf{E}(\mathbf{0})], \tag{2.49}$$

where $\operatorname{grad}\mathbf{E}(\mathbf{0})$ is $\operatorname{grad}\mathbf{E}$ evaluated at the origin, and we are using the convention that the second-order tensor $\operatorname{grad}\mathbf{E}$ is defined by $(\operatorname{grad}\mathbf{E})\mathbf{a} = (\mathbf{a}\cdot\operatorname{grad})\mathbf{E}$ for an arbitrary vector $\mathbf{a}$ (see (A.22) in Appendix A for the component form of this definition). If the total charge in V vanishes then, by (2.45), this becomes

$$\mathsf{F}_\mathrm{e} = (\mathbf{p}\cdot\operatorname{grad})\mathbf{E}, \tag{2.50}$$

where $\operatorname{grad}\mathbf{E}$ is evaluated at the origin. This is the force acting on a dipole of strength $\mathbf{p}$ located at a point in an electric field, in this case the origin.

The total couple acting on the charge distribution, about the origin, is denoted G_e and given by

$$\mathsf{G}_\mathrm{e} = \int_V \rho_\mathrm{e}(\mathbf{x}')\mathbf{x}'\times\mathbf{E}(\mathbf{x}')\,\mathrm{d}V(\mathbf{x}'), \tag{2.51}$$

and when V is small, this is approximated, to the first order, as

$$\mathbf{G}_e = \mathbf{p} \times \mathbf{E}, \tag{2.52}$$

with $\mathbf{E}$ evaluated at the origin. The formulas (2.50) and (2.52) for $\mathbf{F}_e$ and $\mathbf{G}_e$ are exact in the idealized limit of an isolated dipole.

We note that the equation $\operatorname{curl} \mathbf{E} = \mathbf{0}$ is equivalent to the symmetry $(\operatorname{grad} \mathbf{E})^{\mathrm{T}} = \operatorname{grad} \mathbf{E}$, where $^{\mathrm{T}}$ denotes the transpose of a second-order tensor, and this fact can be used to rewrite the electric Lorentz force density, using (2.39), as

$$\rho_e \mathbf{E} = \varepsilon_0 (\operatorname{div} \mathbf{E})\mathbf{E} = \operatorname{div} \boldsymbol{\tau}_e, \tag{2.53}$$

where $\boldsymbol{\tau}_e$ is the so-called *electrostatic Maxwell stress tensor*, which is defined by

$$\boldsymbol{\tau}_e = \varepsilon_0 [\mathbf{E} \otimes \mathbf{E} - \tfrac{1}{2}(\mathbf{E} \cdot \mathbf{E})\mathbf{I}], \tag{2.54}$$

where $\mathbf{I}$ is the identity tensor and $\otimes$ indicates the tensor product of two vectors, which, in Cartesian component form, is defined by $(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j$. The Maxwell stress tensor plays an important role in subsequent developments.

Finally in this section, it is also convenient as a precursor for later developments to introduce the so-called *electric displacement vector*, denoted $\mathbf{D}$, which for free space is related to $\mathbf{E}$ simply by $\mathbf{D} = \varepsilon_0 \mathbf{E}$, and hence

$$\operatorname{div} \mathbf{D} = \rho_e. \tag{2.55}$$

As we shall see in Sect. 2.4, (2.55) is one of Maxwell's equations, and it applies within material media, with ρ_e then replaced by the free charge density ρ_f (to be defined in Sect. 2.4), as well as in vacuum (where $\rho_e = 0$), and also when there is time dependence.

We also note that the force (2.50) can then be written as

$$\mathbf{F}_e = \varepsilon_0^{-1} (\mathbf{p} \cdot \operatorname{grad}) \mathbf{D}. \tag{2.56}$$

For an isolated dipole, the formulas (2.50) and (2.56) are equivalent, but, as we shall see in Chap. 4, their counterparts are not equivalent in polarizable media.

2.2 Magnetostatics

2.2.1 Preliminary Remarks

In electrostatics the electric field is determined in terms of point charges according to (2.6) for a single point charge placed at the origin or in terms of a distribution of charges according to (2.21), for example. Magnetism is fundamentally different since the analogue of a point charge (a magnetic monopole) does not exist. The basic unit of magnetism is the magnetic dipole, but magnetic fields are generated by moving charges and a magnetic dipole is equivalent to an idealized small current-carrying circuit. That currents generate magnetic fields is demonstrated by placing a small bar magnetic near a fixed wire carrying current. The magnet is deflected by the magnetic force generated by the current. Equally, when a current-carrying wire is placed in the vicinity of a fixed magnet, the wire is deflected. For example, a straight wire carrying current I (measured in Ampéres) produces an azimuthal magnetic flux density (or magnetic induction) field of magnitude $\mu_0 I/2\pi r$ at a perpendicular distance r from the wire, where the constant μ_0 is the *magnetic permeability of free space*, whose value is $4\pi \times 10^{-7}\,\mathrm{NA}^{-2}$. There is an important connection between μ_0 and the permittivity of free space ε_0 introduced in Sect. 2.1.6, namely that $1/\sqrt{\varepsilon_0\mu_0} = c$, the speed of light, which is very slightly less than $3 \times 10^8\,\mathrm{m\,s^{-1}}$.

In this section we are concerned with steady currents and magnetostatic fields, i.e. no time dependence is considered. For a distribution of current, the current density $\mathbf{J}$ then satisfies (2.15), where S is a closed surface and the net flux of current out of the enclosed volume vanishes. Geometrically, we can think of *lines of current flow* within S having tangent in the direction of $\mathbf{J}$ at each point. For example, a so-called *tube of current flow* is defined as the surface formed by all such lines that intersect a given closed curve, analogous to lines and tubes of flow in fluid dynamics. It follows that the flux of $\mathbf{J}$ across a cross section of the tube is the same for all cross sections. *Steady current* therefore consists of closed tubes of current flow. The total current I passing across an *open surface* S is just the flux of $\mathbf{J}$ across S and is given by

$$I = \int_S \mathbf{J} \cdot \mathbf{n}\,\mathrm{d}S. \tag{2.57}$$

In practice, a thin conducting wire is a tube of flow of small cross section $\mathrm{d}\mathbf{S}$ and current $I \approx \mathbf{J} \cdot \mathrm{d}\mathbf{S}$.

Each infinitesimal segment of a wire contributes to the magnetic field produced by the complete circuit. Let $\mathrm{d}\mathbf{x}'$ be such a segment. Then, the (infinitesimal) contribution to the magnetic field induced (the magnetic induction) at the field point $\mathbf{x}$, say $\mathrm{d}\mathbf{B}(\mathbf{x})$, is given by

$$\mathrm{d}\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi}\frac{\mathrm{d}\mathbf{x}' \times \mathbf{R}}{R^3}, \tag{2.58}$$

where again $\mathbf{R} = \mathbf{x} - \mathbf{x}'$ and $R = |\mathbf{R}|$, $\mathbf{x}'$ being the point at which $\mathrm{d}\mathbf{x}'$ is situated. This is the counterpart for magnetostatics of Coulomb's Law in electrostatics and also has an inverse square character. This is the essence of the Biot–Savart Law, deduced on the basis of experiments of Biot and Savart and Ampére and the analysis of Ampére in the nineteenth century (Jean-Baptiste Biot, 1774–1862; Felix Savart, 1791–1841; André-Marie Ampère, 1775–1836).

2.2.2 The Biot–Savart Law and the Vector Potential

To obtain the total magnetic field $\mathbf{B}(\mathbf{x})$ generated by the entire length of the wire, we superpose linearly the contributions provided by all the segments $\mathrm{d}\mathbf{x}'$ of the wire to obtain

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \int_C \frac{\mathrm{d}\mathbf{x}' \times \mathbf{R}}{R^3}, \tag{2.59}$$

where C is the closed circuit of the wire. By replacing $I\,\mathrm{d}\mathbf{x}'$ by $\mathbf{J}(\mathbf{x}')\mathrm{d}V(\mathbf{x}')$, this formula is generalized to that for a current distribution of density $\mathbf{J}(\mathbf{x}')$ within a volume V, giving

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{x}') \times \mathbf{R}}{R^3}\,\mathrm{d}V(\mathbf{x}'). \tag{2.60}$$

This important formula is known as the *Biot–Savart Law* for a volume current distribution. As for the formula (2.20), this applies for $\mathbf{x}$ inside or outside V.

The integrand in the above formula can be written as

$$\frac{\mathbf{J}(\mathbf{x}') \times \mathbf{R}}{R^3} = \operatorname{grad}\left(\frac{1}{R}\right) \times \mathbf{J}(\mathbf{x}') = \operatorname{curl}\left(\frac{\mathbf{J}(\mathbf{x}')}{R}\right), \tag{2.61}$$

where the operators grad and curl are with respect to $\mathbf{x}$, and hence, on taking the curl operation outside the integral, (2.60) can be rewritten as

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \operatorname{curl}\left[\int_V \frac{\mathbf{J}(\mathbf{x}')}{R}\,\mathrm{d}V(\mathbf{x}')\right]. \tag{2.62}$$

This prompts the introduction of a vector function $\mathbf{A}$ defined by

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{x}')}{R}\mathrm{d}V(\mathbf{x}'), \tag{2.63}$$

which is known as the *magnetostatic vector potential*. Equation (2.62) can now be written in the more concise form

$$\mathbf{B} = \operatorname{curl}\mathbf{A}, \tag{2.64}$$

from which it follows that $\mathbf{B}$ satisfies the equation

$$\operatorname{div}\mathbf{B} = 0. \tag{2.65}$$

This is a fundamental equation of magnetostatics. Integration of this equation over a volume V and then use of the divergence theorem shows that the magnetic flux through any closed surface within a magnetic field is always zero. It expresses the fact that magnetic poles cannot be isolated, i.e. there is no counterpart in magnetostatics of the electrostatic point charge. In fact, (2.65) is general and holds even when there is time dependence and electromagnetic coupling, both in free space and in material media.

2.2.3 Scalar Magnetic Potential

Consider again (2.59), which gives the magnetic field due to a thin closed current-carrying circuit C. We now write this in the alternative forms

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi}\int_C \operatorname{grad}\left(\frac{1}{R}\right)\times \mathrm{d}\mathbf{x}' = \frac{\mu_0 I}{4\pi}\operatorname{curl}\int_C \frac{\mathrm{d}\mathbf{x}'}{R}, \tag{2.66}$$

where again we recall that the derivatives are with respect to $\mathbf{x}$, not $\mathbf{x}'$.

For points $\mathbf{x}$ distant from C for which $|\mathbf{x}'| \ll |\mathbf{x}|$ for all $\mathbf{x}'$ on C, we may use the Taylor expansion (2.43), i.e.

$$\frac{1}{R} \equiv \frac{1}{|\mathbf{x}-\mathbf{x}'|} \approx \frac{1}{r} - \mathbf{x}'\cdot\operatorname{grad}\left(\frac{1}{r}\right),$$

and since C is a closed curve, the first term in the integral vanishes and we obtain

$$\mathbf{B}(\mathbf{x}) = -\frac{\mu_0}{4\pi}\operatorname{curl}\left[\boldsymbol{\mathcal{M}}\operatorname{grad}\left(\frac{1}{r}\right)\right], \tag{2.67}$$

where the second-order tensor $\boldsymbol{\mathcal{M}}$ is defined by

$$\boldsymbol{\mathcal{M}} = I\int_C \mathrm{d}\mathbf{x}'\otimes\mathbf{x}' \tag{2.68}$$

and $\otimes$ signifies the tensor product of two vectors. For any two vectors $\mathbf{a}$ and $\mathbf{b}$, for example, this product is defined, in Cartesian components, by $(\mathbf{a}\otimes\mathbf{b})_{ij} = a_i b_j$, as noted in Sect. 2.1.10, while $\boldsymbol{\mathcal{M}}$ has components $\mathcal{M}_{ij}$ and $(\boldsymbol{\mathcal{M}}\mathbf{a})_i = \mathcal{M}_{ij}a_j$, $i, j \in \{1, 2, 3\}$, with summation over j from 1 to 3 (here we are using the summation convention for repeated indices). Moreover, $\boldsymbol{\mathcal{M}}$ is a skew-symmetric tensor, since, because C is a closed circuit,

$$\boldsymbol{\mathcal{M}} + \boldsymbol{\mathcal{M}}^{\mathrm{T}} = I\int_C \mathrm{d}(\mathbf{x}'\otimes\mathbf{x}') = \mathbf{O}, \tag{2.69}$$

the zero tensor, where $^{\mathrm{T}}$ signifies the transpose of a second-order tensor.

The tensor $\boldsymbol{\mathcal{M}}$ is referred to as the *magnetic moment tensor*. Let $\mathbf{m}$ denote the associated axial vector, defined by $\mathbf{m} = -\frac{1}{2}\boldsymbol{\epsilon}\boldsymbol{\mathcal{M}}$, where $\boldsymbol{\epsilon}$ is the alternating tensor (see Appendix A.1). Expressed in components, this is written as $m_i = -\frac{1}{2}\epsilon_{ijk}\mathcal{M}_{jk}$, with summation over indices j and k from 1 to 3. Then, for any vector $\mathbf{a}$, $\boldsymbol{\mathcal{M}}\mathbf{a} = \mathbf{m} \times \mathbf{a}$, and, since $\mathbf{m}$ is independent of $\mathbf{x}$, (2.67) becomes

$$\mathbf{B}(\mathbf{x}) = -\frac{\mu_0}{4\pi}\operatorname{curl}\left[\mathbf{m} \times \operatorname{grad}\left(\frac{1}{r}\right)\right] = \frac{\mu_0}{4\pi}\operatorname{curl}\operatorname{curl}\left(\frac{\mathbf{m}}{r}\right). \tag{2.70}$$

Since $1/r$ satisfies Laplace's equation (provided $r \neq 0$), we may use the standard identity $\operatorname{curl}\operatorname{curl} = \operatorname{grad}\operatorname{div} - \nabla^2$ to rewrite the above as

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\operatorname{grad}\operatorname{div}\left(\frac{\mathbf{m}}{r}\right) = -\frac{\mu_0}{4\pi}\operatorname{grad}\left(\frac{\mathbf{m}\cdot\mathbf{x}}{r^3}\right). \tag{2.71}$$

Thus, we may introduce a scalar potential function $\psi(\mathbf{x})$ defined by

$$\psi(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\cdot\mathbf{x}}{r^3} \tag{2.72}$$

such that

$$\mathbf{B}(\mathbf{x}) = -\operatorname{grad}\psi(\mathbf{x}). \tag{2.73}$$

The potential (2.72) has the same structure as the potential associated with an electric dipole given in (2.47). Thus (2.72) is interpreted as the magnetostatic potential of a *magnetic dipole* of strength $\mathbf{m}$ situated at the origin. Moreover, since

$$\mathbf{m} = -\frac{1}{2}\boldsymbol{\epsilon}\boldsymbol{\mathcal{M}} = \frac{1}{2}I\int_C \mathbf{x}' \times \mathrm{d}\mathbf{x}', \tag{2.74}$$

the *potential due to a magnetic dipole is equivalent to that due to a small current loop*. More particularly, if C is a planar loop, then

$$\mathbf{m} = I\,\mathrm{d}\mathbf{S} = I\,\mathbf{n}\,\mathrm{d}S, \tag{2.75}$$

where $\mathrm{d}S$ is the plane area enclosed by the loop and $\mathbf{n}$ is the unit normal to the plane of the loop, directed in the positive sense.

For a dipole situated at the point $\mathbf{x}'$, the potential in (2.72) is replaced by

$$\psi(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\cdot\mathbf{R}}{R^3}. \tag{2.76}$$

Now consider a circuit C of finite dimensions carrying current I, as depicted in Fig. 2.3. Let S be any regular surface that is bounded by C. Imagine that a fine network of curves is constructed on S such that each mesh is infinitesimal,

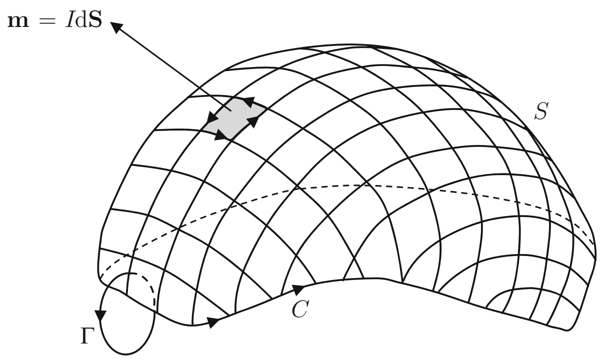

Fig. 2.3 An open surface S bounded by a closed circuit C carrying current I. On S is shown a network of curves made up of small current loops with current I corresponding to magnetic dipoles with magnetic moment $\mathbf{m} = I\mathrm{d}\mathbf{S}$, where $\mathrm{d}\mathbf{S}$ is the directed area element on S related to the direction of the current by the right-hand screw rule. The closed curve Γ encircles C once and hence cuts S

effectively plane and with vector area element $\mathrm{d}\mathbf{S}$. We may regard the current I as flowing in each curve of the mesh because it cancels out on adjoining meshes. In effect, we have a surface S consisting of a distribution of magnetic dipoles $I\,\mathrm{d}\mathbf{S}$. The potential at $\mathbf{x}$ is due to contributions from all such dipoles. Inserting $\mathbf{m} = I\,\mathrm{d}\mathbf{S}$ into (2.76) and integrating over S, we obtain the potential

$$\psi(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \int_S \frac{\mathbf{R} \cdot \mathrm{d}\mathbf{S}}{R^3}, \tag{2.77}$$

and with reference to Sect. 2.1.8, we see that

$$\int_S \frac{\mathbf{R} \cdot \mathrm{d}\mathbf{S}}{R^3} = \Omega(\mathbf{x}) \tag{2.78}$$

is the solid angle subtended by S at $\mathbf{x}$. Thus,

$$\psi(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \Omega(\mathbf{x}). \tag{2.79}$$

The solid angle $\Omega(\mathbf{x})$ has the property that its value changes by 4π as the point $\mathbf{x}$ crosses the surface S. This means the potential function ψ is multi-valued and changes in value by $\mu_0 I$ each time $\mathbf{x}$ traverses a curve which cuts S once. Otherwise ψ is continuous.

2.2.4 Ampère's Circuital Law

Consider again an open surface S bounded by the circuit C carrying current I. Let the closed curve Γ encircle the circuit C just once, therefore cutting S, with the direction around Γ related to the direction of the current in C by the right-hand screw rule, as shown in Fig. 2.3. At any point of Γ, the magnetic induction is given by (2.73) with (2.79). The line integral of $\mathbf{B}$ around Γ is

$$\int_\Gamma \mathbf{B}\cdot \mathrm{d}\mathbf{x} = -\int_\Gamma \operatorname{grad}\psi \cdot \mathrm{d}\mathbf{x} = -[\,\psi\,]_\Gamma, \tag{2.80}$$

where $[\,\psi\,]_\Gamma$ is the change in ψ as Γ is traversed once. This is non-zero because ψ is multi-valued, and since Γ cuts S just once in the sense described above, Ω increases by -4π, and hence ψ by $-\mu_0 I$, for a single traversal of Γ. Therefore,

$$\int_\Gamma \mathbf{B}\cdot \mathrm{d}\mathbf{x} = \mu_0 I, \tag{2.81}$$

where we note that the right-hand side is independent of the curve Γ. Now let Σ be an open surface bounded by Γ, and let I be the total current flowing through Σ. Then, the above argument can be applied to a current distribution $\mathbf{J}$ such that $\int_\Sigma \mathbf{J}\cdot \mathrm{d}\mathbf{S} = I$, leading to the formula

$$\int_\Gamma \mathbf{B}\cdot \mathrm{d}\mathbf{x} = \mu_0 \int_\Sigma \mathbf{J}\cdot \mathrm{d}\mathbf{S}. \tag{2.82}$$

This is a mathematical statement of *Ampère's Circuital Law*. By applying Stokes' theorem to (2.82), we obtain

$$\int_\Sigma (\operatorname{curl}\mathbf{B} - \mu_0 \mathbf{J})\cdot \mathrm{d}\mathbf{S} = 0, \tag{2.83}$$

which holds for any open surface Σ associated with a Γ with the considered properties. Provided the integrand in (2.83) is continuous, we obtain the local form of one of the fundamental equations of magnetostatics, specifically

$$\operatorname{curl}\mathbf{B} = \mu_0 \mathbf{J}. \tag{2.84}$$

We recall that in deriving this equation, it has been assumed that $\mathbf{J}$ is time independent.

Returning to (2.64) we note that it is not affected by the addition of the gradient of an arbitrary scalar function (say φ) to the magnetic vector potential, i.e.

$$\mathbf{A} \to \mathbf{A} + \operatorname{grad}\varphi, \tag{2.85}$$

which is known as a *gauge transformation*. This flexibility enables a restriction to be imposed on $\mathbf{A}$, a *gauge condition*, which is usually taken in the form

$$\operatorname{div}\mathbf{A} = 0. \tag{2.86}$$

Using (2.84) and (2.64), we have

$$\operatorname{curl}(\operatorname{curl}\mathbf{A}) = \mu_0\mathbf{J}, \tag{2.87}$$

and, by using a standard vector identity, (2.87) can be written in the equivalent form

$$\operatorname{grad}(\operatorname{div}\mathbf{A}) - \nabla^2\mathbf{A} = \mu_0\mathbf{J}. \tag{2.88}$$

Equation (2.86) is then used to reduce (2.88) to

$$\nabla^2\mathbf{A} = -\mu_0\mathbf{J}, \tag{2.89}$$

which, for given $\mathbf{J}$, is Poisson's equation for the magnetostatic vector potential. It can be verified that the expression for $\mathbf{A}$ given in (2.63) is a solution of (2.89) whether $\mathbf{x}$ is inside or outside V.

2.2.5 Force and Couple on a Dipole in a Magnetic Field

We now derive expressions for the (mechanical) force and couple on a magnetic dipole placed in a magnetic field. For this purpose we recall from Sect. 2.1.3 that the Lorentz force acting on a charged particle e moving with velocity $\mathbf{v}$ in an electromagnetic field with electric field $\mathbf{E}$ and magnetic induction $\mathbf{B}$ is $e\mathbf{E} + e\mathbf{v}\times\mathbf{B}$. In the case of a continuous distribution of charge with density ρ_e and current with density $\mathbf{J}$, the Lorentz force per unit volume is given by (2.10) as $\rho_e\mathbf{E} + \mathbf{J}\times\mathbf{B}$. We now focus on the magnetic contribution $\mathbf{J}\times\mathbf{B}$ to the Lorentz force.

Consider a material volume V in which there is a current distribution with density $\mathbf{J}$, and let $\mathbf{B}$ be the magnetic induction field permeating the material. Then, the magnetic contribution to the Lorentz force acting on V, which we denote by F_m, is

$$\mathsf{F}_m = \int_V \mathbf{J}\times\mathbf{B}\,dV, \tag{2.90}$$

where the subscript m signifies 'magnetic'. Now suppose that V consists simply of a single current loop C carrying current I. Then, we may replace the volume integral by a line integral around C, and (2.90) becomes

$$\mathsf{F}_m = I\int_C d\mathbf{x}\times\mathbf{B} = I\int_S (d\mathbf{S}\times\operatorname{grad})\times\mathbf{B}, \tag{2.91}$$

where S is a regular open surface bounded by C and the latter integral has been obtained by an application of Stokes' theorem.

Next, we take C and S to be infinitesimal so that the derivatives of $\mathbf{B}$ are approximately uniform over S. Then (2.91) is approximated as $\mathbf{F}_\mathrm{m} \approx I(\mathrm{d}\mathbf{S} \times \mathrm{grad}) \times \mathbf{B}$, and by setting $I\,\mathrm{d}\mathbf{S} = \mathbf{m}$ to be the equivalent magnetic dipole and taking the limit $I \to \infty$ as $\mathrm{d}S \to 0$ while keeping $\mathbf{m}$ finite, we obtain the exact result $\mathbf{F}_\mathrm{m} = (\mathbf{m} \times \mathrm{grad}) \times \mathbf{B}$, which is evaluated at the location of the dipole. By standard vector identities and the fact that $\mathrm{div}\,\mathbf{B} = 0$, this *force on a dipole* $\mathbf{m}$ *in a magnetic induction field* $\mathbf{B}$ may be written as

$$\mathbf{F}_\mathrm{m} = (\mathrm{grad}\,\mathbf{B})^\mathrm{T}\mathbf{m}. \tag{2.92}$$

In (2.92) and henceforth, similarly to Sect. 2.1.10, we adopt the following conventions: for two vector fields $\mathbf{u}$ and $\mathbf{v}$, we define the products $(\mathrm{grad}\,\mathbf{u})^\mathrm{T}\mathbf{v}$ and $(\mathrm{grad}\,\mathbf{u})\mathbf{v} \equiv (\mathbf{v}\cdot\mathrm{grad})\mathbf{u}$ via their index notation representations $u_{j,i}v_j$ and $u_{i,j}v_j$, respectively, where $_{,j} = \partial/\partial x_j$ and $(\mathrm{grad}\,\mathbf{u})_{ij} = u_{i,j}$.

The (magnetic) couple on V, denoted $\mathbf{G}_\mathrm{m}$, about a fixed origin due to the magnetic Lorentz force is given by

$$\mathbf{G}_\mathrm{m} = \int_V \mathbf{x} \times (\mathbf{J} \times \mathbf{B})\,\mathrm{d}V, \tag{2.93}$$

where $\mathbf{x}$ is the position vector relative to the chosen origin. When V consists of just a current loop C, this becomes

$$\mathbf{G}_\mathrm{m} = I\int_C \mathbf{x} \times (\mathrm{d}\mathbf{x} \times \mathbf{B}) = I\int_C (\mathrm{d}\mathbf{x} \otimes \mathbf{x})\mathbf{B} - I\int_C (\mathbf{x}\cdot\mathrm{d}\mathbf{x})\mathbf{B}. \tag{2.94}$$

Once more we take C to be infinitesimal, but now it suffices, as a first approximation, to take $\mathbf{B}$ to be uniform over C so that it can be taken outside the integrals. Then, since C is a closed circuit, the final integral in (2.94) vanishes, and on use of (2.68) $\mathbf{G}_\mathrm{m}$ can be written compactly as

$$\mathbf{G}_\mathrm{m} = \boldsymbol{\mathcal{M}}\mathbf{B} = \mathbf{m} \times \mathbf{B}, \tag{2.95}$$

again with $\mathbf{B}$ evaluated at the location of the dipole, and this is exact in the limit described above. This is the *couple on a dipole* $\mathbf{m}$ *in a magnetic induction field* $\mathbf{B}$.

Thus far the development has been based entirely on the use of the *magnetic induction* vector $\mathbf{B}$, but at this point it is instructive to introduce the so-called *magnetic field* vector, which is denoted by $\mathbf{H}$. For the field due to an isolated dipole placed in a vacuum, for example, $\mathbf{B}$ and $\mathbf{H}$ are simply related by $\mathbf{B} = \mu_0\mathbf{H}$, where the constant μ_0 is again the permeability of free space. This relationship applies at any point in free space or in non-magnetizable materials, whatever the source of the magnetic field, in which case $\mathbf{B}$ and $\mathbf{H}$ satisfy the same equations. In particular, $\mathrm{curl}\,\mathbf{H} = \mathbf{0}$, or equivalently $(\mathrm{grad}\,\mathbf{H})^\mathrm{T} = \mathrm{grad}\,\mathbf{H}$, and (2.92) and (2.95) can be written in the alternative forms

$$\mathbf{F}_\mathrm{m} = \mu_0(\mathbf{m}\cdot\mathrm{grad})\mathbf{H}, \quad \mathbf{G}_\mathrm{m} = \mu_0\mathbf{m} \times \mathbf{H}. \tag{2.96}$$

We emphasize that while the two expressions for $\mathbf{F}_{\mathrm{m}}$ are equivalent in the present context, their counterparts are not equivalent in magnetizable media, and the distinction will be recognized as important, in particular when dealing with deformable media.

Similarly to the electric Maxwell stress tensor introduced in Sect. 2.1.10, we derive a magnetic Maxwell stress tensor. The magnetic contribution to the Lorentz force density may be rewritten, using (2.84), as

$$\mathbf{J} \times \mathbf{B} = \mu_0^{-1}(\operatorname{curl}\mathbf{B}) \times \mathbf{B} = \mu_0^{-1}[(\operatorname{grad}\mathbf{B})\mathbf{B} - (\operatorname{grad}\mathbf{B})^{\mathrm{T}}\mathbf{B}] = \operatorname{div}\boldsymbol{\tau}_{\mathrm{m}}, \tag{2.97}$$

where $\boldsymbol{\tau}_{\mathrm{m}}$ is the *magnetic Maxwell stress tensor*, defined by

$$\boldsymbol{\tau}_{\mathrm{m}} = \mu_0^{-1}[\mathbf{B} \otimes \mathbf{B} - \tfrac{1}{2}(\mathbf{B} \cdot \mathbf{B})\mathbf{I}]. \tag{2.98}$$

Here, the subscript m indicates ‘magnetic’, not ‘Maxwell’. As with its electric counterpart, this tensor has an important role to play subsequently.

2.3 Faraday’s Law of Induction

2.3.1 Preliminary Remarks

Having summarized the basic equations for both electrostatics and magnetostatics, we now consider time-dependent fields, in which case there is in general a strong coupling between the electric and magnetic fields. In experiments in 1819, Oersted (Hans Christian Ørsted, 1777–1851) showed that a steady current produces a steady magnetic field and established a connection between the electric current and the magnetic field. Faraday (Michael Faraday, 1791–1867) in his initial experiments investigated the possibility of a steady magnetic field producing a steady electric current, which, as we now know, is not possible. In the process, however, Faraday made the transformational discovery that a *time-varying* magnetic field will induce the flow of an electric current in a closed circuit and therefore an electric field. This phenomenon is known as *electromagnetic induction* and requires the additional information that is embodied in *Faraday’s Law of Induction*, which we examine in detail in this section.

Before discussing Faraday’s Law, we note that a time-varying electric field always generates a magnetic field. Equation (2.39) shows that a changing electric field is necessarily associated with a charge density ρ_{e} that depends on time. The equation of charge conservation (2.13) connects a time-varying ρ_{e} to the current density $\mathbf{J}$, which then generates a magnetic field, as quantified by the Biot–Savart Law (2.60).

2.3.2 Electromotive Force

Consider the uniform flow of an electric current in a closed circuit, which is equivalent to the average motion of charges (conduction electrons) along the wire. This motion suffers resistance analogous to dynamic friction, and therefore some force is required to drive the electrons along the wire and to maintain the current flow. This driving force is known as the *electromotive force*, abbreviated as emf. A uniform current in a closed loop of wire can therefore only be achieved if the component of the driving force tangential to the wire does *net work* in driving the electrons once round the loop in the direction of the current.

Let $\mathbf{f}$ denote the driving force per unit charge. Then the net work done around the closed circuit C is

$$\int_C \mathbf{f} \cdot \mathrm{d}\mathbf{x} \neq 0, \tag{2.99}$$

which implies that $\mathbf{f}$ is non-conservative. In the case of a battery, for example, this is the voltage (which drives the current). The use of an *electrostatic* field as the driving force can be excluded because the field $\mathbf{E}$ is the gradient of the electrostatic potential, so that $\mathbf{f} = \mathbf{E} = -\mathrm{grad}\varphi$ and $\mathbf{E}$ is conservative provided φ is single valued.

2.3.3 Flux of a Magnetic Field Through a Moving Circuit

For the development that follows, it is important to remember that Faraday has shown experimentally that the same electromotive force can be induced in a closed circuit whether the closed circuit is moved while keeping the applied magnetic field stationary or whether the closed circuit is fixed and the applied magnetic field varies in time. Faraday also observed that the emf generated is proportional to the change of the magnetic field per unit time, faster change inducing a larger emf. He also noticed that the induced emf is proportional to the area bounded by the closed circuit.

The objective now is to determine an expression for the electromotive force in a small closed circuit C that is moving with velocity $\mathbf{v}$ without rotation in a *magnetostatic field* $\mathbf{B}$. Let $\mathbf{u}$ denote the velocity of the charges relative to the wire so that $\mathbf{v} + \mathbf{u}$ is their resultant velocity. The magnetic force per unit charge is then given by

$$\mathbf{f} = (\mathbf{v} + \mathbf{u}) \times \mathbf{B}, \tag{2.100}$$

and the electromotive force is the integral of this around the circuit C:

$$\int_C \mathbf{f} \cdot \mathrm{d}\mathbf{x} = \int_C [(\mathbf{v} + \mathbf{u}) \times \mathbf{B}] \cdot \mathrm{d}\mathbf{x} = -\int_C \mathbf{B} \cdot [(\mathbf{v} + \mathbf{u}) \times \mathrm{d}\mathbf{x}]. \tag{2.101}$$

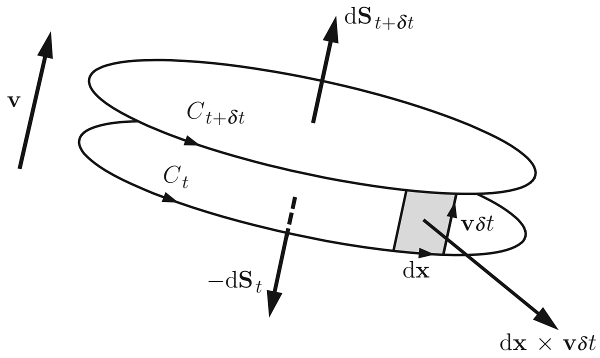

Fig. 2.4 A small circuit moves with uniform velocity $\mathbf{v}$ without rotation. At time t it is at C_t and the approximately plane surface bounded by C_t is denoted by S_t, with vector area element $\mathrm{d}\mathbf{S}_t$ related to the direction of C_t by the right-hand screw rule. At an infinitesimal time δt later, the curve is at $C_{t+\delta t}$ and S_t has moved to $S_{t+\delta t}$ with vector area element $\mathrm{d}\mathbf{S}_{t+\delta t}$. The normal vector to the ribbon-like surface which completes the boundary of the volume formed by the motion of the curve is also shown, pointing out of the volume

The magnetic force generated by the motion of the charges along the wire is $\mathbf{u} \times \mathbf{B}$, which is perpendicular to the current flow and therefore does no work around the circuit. The expression for the electromotive force therefore reduces to

$$\int_C \mathbf{f} \cdot \mathrm{d}\mathbf{x} = -\int_C \mathbf{B} \cdot (\mathbf{v} \times \mathrm{d}\mathbf{x}). \tag{2.102}$$

In the time interval from time t to $t + \delta t$, the closed circuit C moves from the location C_t to location $C_{t+\delta t}$, where δt is infinitesimal. Let S_t denote an approximately plane surface bounded by C_t and $S_{t+\delta t}$ the corresponding surface bounded by $C_{t+\delta t}$ with surface normal vectors related to the direction of C by the right-hand screw rule. The surfaces consisting of the ribbon-like surface swept out by the motion of C and the surfaces S_t and $S_{t+\delta t}$ form the boundary of a closed volume. The outward normals to the three surfaces are in the directions $\mathrm{d}\mathbf{x} \times \mathbf{v}$, $-\mathrm{d}\mathbf{S}_t$ and $\mathrm{d}\mathbf{S}_{t+\delta t}$, respectively, as illustrated in Fig. 2.4. Clearly,

$$\int_C \mathbf{B} \cdot (\mathrm{d}\mathbf{x} \times \mathbf{v})\delta t, \quad -\int_{S_t} \mathbf{B} \cdot \mathrm{d}\mathbf{S}, \quad \int_{S_{t+\delta t}} \mathbf{B} \cdot \mathrm{d}\mathbf{S} \tag{2.103}$$

are the fluxes of $\mathbf{B}$ across the respective surfaces *out* of the enclosed volume.

Since $\operatorname{div}\mathbf{B} = 0$, it follows from the divergence theorem that

$$\int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0 \tag{2.104}$$

for any closed surface S. Hence, the sum of the fluxes in (2.103) must vanish:

$$\int_C \mathbf{B} \cdot (\mathrm{d}\mathbf{x} \times \mathbf{v})\delta t + \int_{S_{t+\delta t}} \mathbf{B} \cdot \mathrm{d}\mathbf{S} - \int_{S_t} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0. \tag{2.105}$$

Thus, from (2.102) and (2.105), in the limit as $\delta t \to 0$, we obtain

$$\int_C \mathbf{f} \cdot \mathrm{d}\mathbf{x} = \lim_{\delta t \to 0} \left\{ -\frac{1}{\delta t} \left[\int_{S_{t+\delta t}} \mathbf{B} \cdot \mathrm{d}\mathbf{S} - \int_{S_t} \mathbf{B} \cdot \mathrm{d}\mathbf{S} \right] \right\} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S}. \quad (2.106)$$

This shows the important property that the electromotive force is equal to the change of the magnetic flux across any surface S bounded by C. The minus sign on the right-hand side of (2.106) indicates that the induced electric current will produce a magnetic field that always opposes the change of the magnetic flux. The latter connection is known as Lenz's Law (Heinrich Friedrich Emil Lenz, 1804–1865).

If the circuit C is moved through a magnetic field in such a way that the flux through C changes, then an electromotive force is induced and a current will flow. For a rigid closed circuit moving with uniform translation (no rotation), a current will flow provided **B** changes with position. If **B** is uniform, then the flux will not change unless C rotates.

Since the circuit C is moving, we may write, with reference to the integral on the right-hand side of (2.106),

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \int_S \mathbf{B}_{,t} \cdot \mathrm{d}\mathbf{S}, \quad (2.107)$$

where $\mathbf{B}_{,t}$ denotes the *material time derivative* (or total time derivative), which accounts for the motion of C (and hence of S) and is given by

$$\mathbf{B}_{,t} = \frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v} \cdot \mathrm{grad})\mathbf{B}, \quad (2.108)$$

where $\partial \mathbf{B}/\partial t$ is the time derivative at fixed $\mathbf{x}$. For a definition of the material time derivative for a moving and deforming material, we refer to Sect. 3.1.4.1. Since $\mathbf{v}$ is independent of position on S, it is easy to show from the identity (A.19) in Appendix A.2 that $(\mathbf{v} \cdot \mathrm{grad})\mathbf{B} = -\mathrm{curl}\,(\mathbf{v} \times \mathbf{B})$. Using this in the above equation together with (2.107) and then applying Stokes' theorem, we obtain from (2.106) the formula

$$\int_C \mathbf{f} \cdot \mathrm{d}\mathbf{x} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S} + \int_C (\mathbf{v} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{x}. \quad (2.109)$$

Equation (2.109) gives an expression for the electromotive force alternative to (and equivalent to) that in (2.106). The right-hand side in each case is the rate of change of the flux of **B** through S. For an observer in a frame of reference moving with C, S appears fixed and (2.109) admits the possibility that **B** can vary with t as well as **x**. We recall that in deriving (2.106), we assumed that **B** was time independent. The derivation carries over to the case when **B** depends on time. Thus, the flux through the circuit C may be changed by either a time-dependent magnetic field or by the motion of C, or both simultaneously.

2.3.4 Faraday's Law

In the previous subsection we quantified the electromotive force generated by moving a closed circuit in a magnetostatic field. We have shown that a magnetic force is the driving agency that generates the flow of charges. Here, on the other hand, we consider an observer rigidly attached to the moving circuit, i.e. the circuit is held fixed in the moving frame of reference. Let $\mathbf{E}'$ and $\mathbf{B}'$ be the electric and magnetic induction fields as measured in this frame of reference, which we take to have *constant* velocity $\mathbf{v}$. We now relate these to the corresponding fields in the fixed frame.

Consider the force on a unit point charge moving with velocity $\mathbf{u}$ in an electromagnetic field. In a fixed frame of reference the magnetic force (2.1.3) is

$$\mathbf{f} = \mathbf{E} + (\mathbf{u} + \mathbf{v}) \times \mathbf{B}. \tag{2.110}$$

The force measured in the moving frame is

$$\mathbf{f}' = \mathbf{E}' + \mathbf{u} \times \mathbf{B}' \tag{2.111}$$

since the point charge has velocity $\mathbf{u}$ relative to this frame of reference. According to Newton's Second Law, the force must be the same in both frames. Therefore,

$$\mathbf{E} + (\mathbf{u} + \mathbf{v}) \times \mathbf{B} = \mathbf{E}' + \mathbf{u} \times \mathbf{B}', \tag{2.112}$$

which must hold for arbitrary $\mathbf{u}$. By taking $\mathbf{u} = \mathbf{0}$, it follows that

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \tag{2.113}$$

and hence on substituting back in (2.112), we obtain

$$\mathbf{B}' = \mathbf{B}. \tag{2.114}$$

As already noted, the magnetic force $\mathbf{u} \times \mathbf{B}$ in (2.100) does not contribute to the electromotive force given by (2.101) and (2.102). According to the moving observer the driving force cannot be magnetic since the stationary circuit does not experience magnetic forces in a magnetostatic field. However, the moving observer perceives a non-conservative electric field so that electric force becomes the driving agency that moves the charges. The electric force per unit charge measured in the moving frame is $\mathbf{f} = \mathbf{E}'$, and the electromotive force on the complete circuit C is then given by

$$\int_C \mathbf{E}' \cdot \mathrm{d}\mathbf{x} = -\int_S \mathbf{B}'_{,t} \cdot \mathrm{d}\mathbf{S}, \tag{2.115}$$

which shows that, contrary to the electrostatic case, the electric field inside the conducting circuit is non-conservative when there is a change in the magnetic flux. Note that not only is $\mathbf{B}' = \mathbf{B}$ but also $\mathbf{B}'_{,t} = \mathbf{B}_{,t}$ since the latter is the time derivative of $\mathbf{B}$ following the motion of the circuit. By Stokes' theorem, (2.115) can be written as

$$\int_S (\operatorname{curl} \mathbf{E}' + \mathbf{B}'_{,t}) \cdot \mathrm{d}\mathbf{S} = 0, \tag{2.116}$$

and since S is arbitrary, it follows that

$$\operatorname{curl} \mathbf{E}' + \mathbf{B}'_{,t} = \mathbf{0}, \tag{2.117}$$

as measured in the frame of reference of the moving circuit.

Substitution of (2.113) into (2.117) gives

$$\operatorname{curl}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \mathbf{B}_{,t} = \mathbf{0}, \tag{2.118}$$

and hence, using (2.108) and the formula $\operatorname{curl}(\mathbf{v} \times \mathbf{B}) = -(\mathbf{v} \cdot \operatorname{grad})\mathbf{B}$, we obtain the important equation

$$\operatorname{curl} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}, \tag{2.119}$$

connecting the fields $\mathbf{E}$ and $\mathbf{B}$ measured in a fixed frame of reference. This shows that the structure of (2.117) is invariant under changes of uniformly moving frames of reference.

Equation (2.106) is known as *Faraday's Law of Induction*, which we repeat here compactly as

$$\int_C \mathbf{f} \cdot \mathrm{d}\mathbf{x} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S}. \tag{2.120}$$

Faraday's Law is very general and allows for both time and space variation of $\mathbf{E}$ and $\mathbf{B}$ and is independent of observer. Different observers measure different voltages (emfs), the same flux, but different rates of change of that flux. Faraday's Law is based on experiments in which the magnetic flux through a thin wire circuit is made to vary in a variety of ways. It is the basis of the dynamo and the electric motor (which involve rotations of wire loops in magnetic fields).

Thus, the electric and magnetic fields are in general intimately connected when there is time variation or motion.

Equation (2.119) replaces the electrostatic equation $\operatorname{curl} \mathbf{E} = \mathbf{0}$ for time-varying situations. By taking the divergence of (2.119) we obtain

$$\operatorname{div}\left(\frac{\partial \mathbf{B}}{\partial t}\right) = \frac{\partial}{\partial t}(\operatorname{div} \mathbf{B}) = 0. \tag{2.121}$$

If $\operatorname{div}\mathbf{B} = 0$ at some initial time, then it follows from (2.121) that $\operatorname{div}\mathbf{B} = 0$ for all time. Thus, the magnetostatic equation $\operatorname{div}\mathbf{B} = 0$ still holds in the time-varying situation.

2.4 Maxwell's Equations

2.4.1 The Full Set of Maxwell's Equations

We begin this section by first recalling the fundamental equations governing time-independent electric and magnetic fields that were discussed, respectively, in Sects. 2.1 and 2.2. The equations of electrostatics were derived in Sects. 2.1.7 and 2.1.8 and are

$$\operatorname{curl}\mathbf{E} = \mathbf{0}, \quad \operatorname{div}\mathbf{E} = \frac{\rho_e}{\varepsilon_0}. \tag{2.122}$$

Similarly, for the magnetostatic field we have the two equations

$$\operatorname{curl}\mathbf{B} = \mu_0\mathbf{J}, \quad \operatorname{div}\mathbf{B} = 0, \tag{2.123}$$

from Sects. 2.2.3 and 2.2.2, respectively, where we emphasize that $\mathbf{J}$ is a *steady* current density. Applying the divergence operator to both sides of $(2.123)_1$ shows that

$$\operatorname{div}\mathbf{J} = 0. \tag{2.124}$$

For time-dependent fields, the equation of charge conservation (2.124) is no longer valid and must be replaced by (2.13), which we write here as

$$\operatorname{div}\mathbf{J} + \frac{\partial\rho_e}{\partial t} = 0. \tag{2.125}$$

In Sect. 2.3 we have seen that a steady current produces a magnetic field and that a time-varying magnetic field will induce a flow of electric charges and therefore produce an electric field. Clearly, the fundamental equations describing static fields need to be modified to reflect these experimental facts. Equations $(2.122)_2$ and $(2.123)_2$ remain unchanged, while $(2.122)_1$ is replaced by (2.119), i.e.

$$\operatorname{curl}\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = \mathbf{0}, \tag{2.126}$$

which is the local form of Faraday's Law.

This leaves $(2.123)_1$, which no longer holds since it implies (2.124) and not (2.125). To compensate for this difference we write, instead of $(2.123)_1$,

$$\operatorname{curl}\mathbf{B} = \mu_0\mathbf{J} + \mathbf{G}, \tag{2.127}$$

where $\mathbf{G}$ is an unknown vector function that has to be determined. On taking the divergence of this equation and using $(2.122)_2$ and (2.125) we obtain

$$\operatorname{div}\mathbf{G} = -\mu_0\operatorname{div}\mathbf{J} = \mu_0\frac{\partial\rho_e}{\partial t} = \mu_0\varepsilon_0\operatorname{div}\left(\frac{\partial\mathbf{E}}{\partial t}\right). \tag{2.128}$$

The equations are now self-consistent if we set

$$\mathbf{G} = \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}, \tag{2.129}$$

so that

$$\mu_0^{-1}\operatorname{curl}\mathbf{B} = \mathbf{J} + \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}. \tag{2.130}$$

A divergence-free vector can be added to $\mathbf{G}$, but this is inessential.

We now collect together the four fundamental differential equations for time-dependent fields as

$$\operatorname{div}\mathbf{E} = \frac{\rho_e}{\varepsilon_0}, \qquad \operatorname{div}\mathbf{B} = 0, \tag{2.131}$$

$$\operatorname{curl}\mathbf{B} = \mu_0\mathbf{J} + \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}, \quad \operatorname{curl}\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}. \tag{2.132}$$

These are the four *Maxwell equations* (James Clerk Maxwell, 1831–1879) that govern the fields $\mathbf{E}$ and $\mathbf{B}$ everywhere when the charge density ρ_e and current density $\mathbf{J}$ are known. When coupled with the Lorentz Law of force, they constitute an exact and complete description of classical (non-relativistic) electromagnetic phenomena.

On taking the curl of $(2.132)_2$ and making use of $(2.132)_1$, we obtain

$$\operatorname{curl}(\operatorname{curl}\mathbf{E}) = -\frac{\partial}{\partial t}(\operatorname{curl}\mathbf{B}) = -\frac{\partial}{\partial t}\left(\mu_0\mathbf{J} + \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right). \tag{2.133}$$

Combining this with the identity $\operatorname{curl}(\operatorname{curl}\mathbf{E}) = \operatorname{grad}(\operatorname{div}\mathbf{E}) - \nabla^2\mathbf{E}$ and $(2.131)_1$, we arrive at the equation

$$\nabla^2\mathbf{E} - \mu_0\varepsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2} = \operatorname{grad}\left(\frac{\rho_e}{\varepsilon_0}\right) + \mu_0\frac{\partial\mathbf{J}}{\partial t}. \tag{2.134}$$

This is the *inhomogeneous wave equation* for $\mathbf{E}$, where $c = (\mu_0\varepsilon_0)^{-1/2}$ is the speed of light anticipated at the beginning of Sect. 2.2 (i.e. the speed of electromagnetic effects in free space). The right-hand side of (2.134) is the *source* term. Similarly, taking the curl of $(2.132)_1$ and using $(2.131)_2$ and $(2.132)_2$, the corresponding wave equation for the magnetic induction $\mathbf{B}$ is obtained as

$$\nabla^2\mathbf{B} - \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} = -\mu_0 \operatorname{curl}\mathbf{J}. \tag{2.135}$$

In free space, where $\rho_e = 0$ and $\mathbf{J} = \mathbf{0}$, we obtain the *homogeneous wave equations*

$$\nabla^2\mathbf{E} = \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}, \quad \nabla^2\mathbf{B} = \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}. \tag{2.136}$$

Now, from $(2.131)_2$, we may write $\mathbf{B} = \operatorname{curl}\mathbf{A}$, $\mathbf{A}$ being a time-dependent vector potential. Substitution of this into $(2.132)_2$ yields

$$\operatorname{curl}\left(\mathbf{E} + \frac{\partial\mathbf{A}}{\partial t}\right) = \mathbf{0}, \tag{2.137}$$

and hence we may introduce a scalar field φ such that

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} - \operatorname{grad}\varphi. \tag{2.138}$$

On substituting this into $(2.131)_1$ we may rearrange it as a wave equation for φ, specifically

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho_e}{\varepsilon_0} - \frac{\partial}{\partial t}\left(\operatorname{div}\mathbf{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t}\right). \tag{2.139}$$

Similarly, substitution into $(2.132)_1$ leads to a wave equation for $\mathbf{A}$, i.e.

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J} - \operatorname{grad}\left(\operatorname{div}\mathbf{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t}\right). \tag{2.140}$$

Since there is flexibility in the definition of $\mathbf{A}$ (as noted in Sect. 2.2.4, the gradient of an arbitrary scalar function may be added to $\mathbf{A}$), these equations suggest that the additional condition

$$\operatorname{div}\mathbf{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0 \tag{2.141}$$

should be adopted. This is a gauge condition, extending that in (2.86) to the time-varying situation and known as the *Lorenz condition* (Ludwig Valentin Lorenz, 1829–1891). Note that Lorentz and Lorenz are different. The wave equations then become

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho_e}{\varepsilon_0}, \quad \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}. \tag{2.142}$$

These have solutions analogous to those in the static case given by (2.23) and (2.63), respectively, namely

$$\varphi(\mathbf{x}, t) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_e(\mathbf{x}', t')}{R}\mathrm{d}V(\mathbf{x}'), \quad \mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{x}', t')}{R}\mathrm{d}V(\mathbf{x}'), \tag{2.143}$$

where t' is the retarded time $t - R/c$. Thus, the structure of the potentials carries over to the dynamic situation.

2.4.2 Polarization and Magnetization in Materials

In Sects. 2.1.10 and 2.2.5, respectively, we introduced the *electric displacement vector* $\mathbf{D}$ and the *magnetic field vector* $\mathbf{H}$ in free space. These are simply related to $\mathbf{E}$ and $\mathbf{B}$, respectively, by a constant factor in each case. Thus,

$$\mathbf{D} = \varepsilon_0\mathbf{E}, \quad \mathbf{B} = \mu_0\mathbf{H}, \tag{2.144}$$

where ε_0 is the electric permittivity and μ_0 the magnetic permeability of free space introduced earlier. In material media these relations do not hold in general, and to aid the description of the electric and magnetic properties of materials, we introduce two additional vectors, defined by

$$\mathbf{P} = \mathbf{D} - \varepsilon_0\mathbf{E}, \quad \mathbf{M} = \mu_0^{-1}\mathbf{B} - \mathbf{H}. \tag{2.145}$$

The vector $\mathbf{P}$ is called the *polarization density* and $\mathbf{M}$ the *magnetization density*. We now provide physical interpretations for these quantities.

We recall from (2.44) that the electrostatic potential at $\mathbf{x}$ due to a point charge e and dipole $\mathbf{p}$ at the origin is given by

$$\varphi(\mathbf{x}) = \frac{e}{4\pi\varepsilon_0 r} - \frac{1}{4\pi\varepsilon_0}\mathbf{p}\cdot\mathrm{grad}\left(\frac{1}{r}\right), \tag{2.146}$$

where we have replaced the approximation by an equality by neglecting higher-order terms. Thus, we are considering an isolated point charge and an isolated dipole situated at the origin. There is no net charge on a dipole since it consists of equal amounts of positive and negative charge (these are said to be *bound charges*), whereas e is regarded as a *free charge*. We now generalize these notions and consider a continuous distribution of free charges and dipoles in a volume V with densities $\rho_f(\mathbf{x}')$ and $\mathbf{P}(\mathbf{x}')$ at the point $\mathbf{x}'$, where $\rho_f(\mathbf{x}')$ represents the *free charge density*. Then, the potential at $\mathbf{x}$ due to this distribution is

$$\varphi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \left[\frac{\rho_\mathrm{f}(\mathbf{x}')}{R} + \mathbf{P}(\mathbf{x}')\cdot \mathrm{grad}'\left(\frac{1}{R}\right)\right]\mathrm{d}V(\mathbf{x}'), \tag{2.147}$$

where $R = |\mathbf{x}-\mathbf{x}'|$ and grad' is the gradient with respect to $\mathbf{x}'$. We also note that

$$\mathrm{grad}'\left(\frac{1}{R}\right) = -\mathrm{grad}\left(\frac{1}{R}\right). \tag{2.148}$$

By the divergence theorem, (2.147) can be rewritten as

$$\varphi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_\mathrm{f}(\mathbf{x}') - \mathrm{div}'\mathbf{P}(\mathbf{x}')}{R}\mathrm{d}V(\mathbf{x}') + \int_S \frac{\mathbf{P}(\mathbf{x}')\cdot\mathrm{d}\mathbf{S}(\mathbf{x}')}{R}, \tag{2.149}$$

where div' is the divergence operator with respect to $\mathbf{x}'$ and S is the bounding surface of V.

From the first integral in (2.149) it can be seen that the term $-\mathrm{div}'\mathbf{P}(\mathbf{x}')$ acts like an additional charge density. It is referred to as the *bound charge density* and denoted ρ_b, i.e.

$$\rho_\mathrm{b}(\mathbf{x}) = -\mathrm{div}\,\mathbf{P}(\mathbf{x}) \tag{2.150}$$

at any point $\mathbf{x}$ in V. Thus, the total charge density consists of free charge and bound charge, and we write, for any point $\mathbf{x}$ in V,

$$\rho_\mathrm{e}(\mathbf{x}) = \rho_\mathrm{f}(\mathbf{x}) + \rho_\mathrm{b}(\mathbf{x}). \tag{2.151}$$

This provides the interpretation of $\mathbf{P}$. The first term in the formula (2.149) can then be recognized as the same as that in (2.23), which did not account for the surface term included here. The complete expression (2.149) is a solution of Poisson's equation (2.41).

It follows from $(2.131)_1$ and $(2.145)_1$ that

$$\mathrm{div}\,\mathbf{D} = \rho_\mathrm{f}. \tag{2.152}$$

This is the equation that replaces $(2.131)_1$ in the case of polarizable materials, and it applies for both static and time-dependent fields.

Turning now to $(2.132)_1$, we may use (2.145), (2.150) and (2.151) to rewrite it as

$$\mathrm{curl}\,\mathbf{H} = \frac{\partial\mathbf{D}}{\partial t} + \mathbf{J} - \mathrm{curl}\,\mathbf{M} - \frac{\partial\mathbf{P}}{\partial t}. \tag{2.153}$$

We now interpret the different terms on the right-hand side of this equation. The application of a magnetic field to a material generates a flow of electrons and an alignment of intrinsic magnetic dipoles known as magnetization and quantified by the magnetization density (or magnetic moment per unit volume) introduced in (2.145)$_2$. The effect of the magnetization is to induce a *bound current density* resulting from the motion of bound charges in atoms. We denote this by $\mathbf{J}_\mathrm{b}$. Moreover, when the polarization changes in time, it generates an additional current, characterized by the *polarization current density*, which we denote by $\mathbf{J}_\mathrm{p}$, and the difference $\mathbf{J} - \mathbf{J}_\mathrm{b} - \mathbf{J}_\mathrm{p}$ is the *free current density*, which we denote by $\mathbf{J}_\mathrm{f}$. Thus,

$$\mathbf{J} = \mathbf{J}_\mathrm{f} + \mathbf{J}_\mathrm{b} + \mathbf{J}_\mathrm{p}. \tag{2.154}$$

The connections

$$\mathbf{J}_\mathrm{b} = \mathrm{curl}\,\mathbf{M}, \quad \mathbf{J}_\mathrm{p} = \frac{\partial \mathbf{P}}{\partial t} \tag{2.155}$$

then follow.

To see why curl $\mathbf{M}$ can be interpreted as a current density, consider the following. From Sect. 2.2.3 the vector potential at the point $\mathbf{x}$ associated with an isolated magnetic dipole situated at the origin is

$$\mathbf{A}(\mathbf{x}) = -\frac{\mu_0}{4\pi}\boldsymbol{\mathcal{M}}\,\mathrm{grad}\left(\frac{1}{r}\right), \tag{2.156}$$

where $\boldsymbol{\mathcal{M}}$ is the magnetization tensor given by (2.68), which is skew-symmetric and is related to the magnetic moment vector $\mathbf{m}$ (the axial vector of $\boldsymbol{\mathcal{M}}$) by $\mathbf{m} = -\frac{1}{2}\boldsymbol{\epsilon}\boldsymbol{\mathcal{M}}$. Conversely, $\boldsymbol{\mathcal{M}}$ is given in terms of $\mathbf{m}$ by $\boldsymbol{\mathcal{M}} = -\boldsymbol{\epsilon}\mathbf{m}$, or, in components, $\mathcal{M}_{ij} = -\epsilon_{ijk}m_k$.

Suppose now there is a distribution of dipoles with density $\mathbf{M}(\mathbf{x}')$ and tensor density $\boldsymbol{\mathcal{M}}(\mathbf{x}')$ within a volume V, vanishing outside V. Then, the vector potential at $\mathbf{x}$ is given by the integral

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_V \boldsymbol{\mathcal{M}}(\mathbf{x}')\,\mathrm{grad}'\left(\frac{1}{R}\right)\mathrm{d}V(\mathbf{x}'), \tag{2.157}$$

where (2.148) has again been used. By applying the divergence theorem, we obtain

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathrm{div}'\boldsymbol{\mathcal{M}}(\mathbf{x}')}{R}\mathrm{d}V(\mathbf{x}') + \frac{\mu_0}{4\pi}\int_S \frac{\boldsymbol{\mathcal{M}}(\mathbf{x}')\mathrm{d}\mathbf{S}(\mathbf{x}')}{R}, \tag{2.158}$$

where the skew-symmetry of $\boldsymbol{\mathcal{M}}$ has been used. But it is easy to show that $\mathrm{div}'\boldsymbol{\mathcal{M}}(\mathbf{x}') = \mathrm{curl}'\mathbf{M}(\mathbf{x}')$ and hence

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathrm{curl}'\mathbf{M}(\mathbf{x}')}{R}\mathrm{d}V(\mathbf{x}') + \frac{\mu_0}{4\pi}\int_S \frac{\boldsymbol{\mathcal{M}}(\mathbf{x}')\mathrm{d}\mathbf{S}(\mathbf{x}')}{R}. \tag{2.159}$$

Comparison of the first term in the above with (2.63) shows that, for a distribution of dipoles, curl $\mathbf{M}$ behaves like a current density and therefore has the interpretation indicated above. The expression (2.159) is a solution of Poisson's equation (2.89) for $\mathbf{J} = \mathbf{J}_\mathrm{b}$.

Clearly $\operatorname{div}\mathbf{J}_\mathrm{b} = 0$ and, by definition, $\mathbf{J}_\mathrm{p}$ satisfies the charge conservation equation

$$\operatorname{div}\mathbf{J}_\mathrm{p} = \frac{\partial}{\partial t}(\operatorname{div}\mathbf{P}) = -\frac{\partial \rho_\mathrm{b}}{\partial t}, \tag{2.160}$$

from which it may be deduced that the free charge satisfies separately the charge conservation equation

$$\operatorname{div}\mathbf{J}_\mathrm{f} = -\frac{\partial \rho_\mathrm{f}}{\partial t}. \tag{2.161}$$

It follows from $(2.132)_1$ and $(2.145)_2$ that

$$\operatorname{curl}\mathbf{H} = \mathbf{J}_\mathrm{f} + \frac{\partial \mathbf{D}}{\partial t}, \tag{2.162}$$

which shows that only the free electric current density $\mathbf{J}_\mathrm{f}$ remains in Maxwell's equation. The term $\partial\mathbf{D}/\partial t$, the time derivative of the electric displacement, plays a role similar to a current density and is known as the *displacement current*.

To summarize, the four Maxwell equations in material matter may be written as

$$\operatorname{div}\mathbf{D} = \rho_\mathrm{f}, \quad \operatorname{div}\mathbf{B} = 0, \tag{2.163}$$

$$\operatorname{curl}\mathbf{H} = \mathbf{J}_\mathrm{f} + \frac{\partial \mathbf{D}}{\partial t}, \quad \operatorname{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.164}$$

which are equivalent to (2.131) and (2.132). For a detailed treatment of Maxwell's equations, see, for example, the classic texts by Jackson (1999), Landau and Lifshitz (1960) and Stratton (2007), and, for an interesting historical overview, we refer to the book by Maugin (1988).

In (2.145) there are three vector fields associated with electric effects and three vector fields associated with magnetic effects. In each case there is one connection between the three vectors. These apply to all polarizable or magnetizable materials. To distinguish between different materials an additional connection is needed in each case. Such a connection is known as a *constitutive equation*. For polarizable materials this may take the form of an explicit expression for $\mathbf{P}$ in terms of either $\mathbf{D}$ or $\mathbf{E}$, with either $\mathbf{E}$ or $\mathbf{D}$ as the independent electric variable, or of either $\mathbf{E}$ or $\mathbf{D}$ in terms of one of the other variables. Similarly, for magnetizable materials any one of $\mathbf{H}$, $\mathbf{B}$ or $\mathbf{M}$ may be adopted as the independent variable and the constitutive law specified accordingly.

Basic examples of constitutive laws include those for *linear isotropic media*, for which the equations in (2.144) are replaced by

$$\mathbf{D} = \varepsilon_\mathrm{r}\varepsilon_0\mathbf{E}, \quad \mathbf{B} = \mu_\mathrm{r}\mu_0\mathbf{H}, \tag{2.165}$$

where ε_r and μ_r are the *relative dielectric permittivity* and *relative magnetic permeability*, respectively. From (2.145) and (2.165), the polarization and magnetization are given by

$$\mathbf{P} = \frac{\varepsilon_\mathrm{r} - 1}{\varepsilon_\mathrm{r}}\mathbf{D}, \quad \mathbf{M} = \frac{\mu_\mathrm{r} - 1}{\mu_0 \mu_\mathrm{r}}\mathbf{B}, \tag{2.166}$$

so that $\mathbf{P}$ and $\mathbf{M}$, respectively, are parallel to the electric displacement $\mathbf{D}$ and the magnetic induction $\mathbf{B}$. Also, the units of the electric polarization $\mathbf{P}$ and the electric displacement $\mathbf{D}$ as well as the units of the magnetization vector $\mathbf{M}$ and the magnetic field vector $\mathbf{H}$ coincide; see (2.165) and (2.166). In vacuo or in non-polarizable material, $\varepsilon_\mathrm{r} = 1$, while in vacuo or in non-magnetizable media, $\mu_\mathrm{r} = 1$. In polarizable materials $\varepsilon_\mathrm{r} > 1$ and $\mathbf{P}$ is in the same direction as $\mathbf{D}$. For most materials $\mu_\mathrm{r} > 1$; however, there are some magnetizable materials for which $\mu_\mathrm{r} < 1$ and $\mathbf{M}$ is therefore opposite in direction to $\mathbf{B}$. For details of the permittivity and permeability constants of dielectric and magnetic materials, a convenient source of information is Wikipedia (2013), which contains references to multiple sources.

2.5 Boundary Conditions

Maxwell's equations (2.163) and (2.164) are valid for any material medium provided $\mathbf{D}$ and $\mathbf{H}$ are given by appropriate constitutive laws. To these equations we need to append boundary conditions in order to formulate and solve boundary-value problems. In general the field vectors $\mathbf{E}, \mathbf{D}, \mathbf{B}$ and $\mathbf{H}$ are discontinuous across surfaces between different media or across a surface bounding the material. In this section we derive, using (2.163) and (2.164) in integral form together with the divergence and Stokes' theorems, as appropriate, the equations satisfied by the discontinuities. We consider only stationary surfaces. The results will be generalized to moving surfaces in Chap. 9.

2.5.1 Boundary Conditions for E *and* D

Let S be a *stationary* surface which carries free surface charge σ_f per unit area. The two sides of S are distinguished as side 1 and side 2, and field vectors on the two sides of S are identified with subscripts 1 and 2. Let $\mathbf{n}$ be the unit normal to S pointing from side 1 to side 2. The 'jump' in a vector on S is the difference between its values on side 2 and side 1, evaluated on S. Thus $\mathbf{E}$, for example, has jump $\mathbf{E}_2 - \mathbf{E}_1$, which is denoted $[\![\mathbf{E}]\!]$, and similarly for the other vectors. The jump conditions satisfied by $\mathbf{E}$ and $\mathbf{D}$ are summarized as

$$\mathbf{n} \times [\![\mathbf{E}]\!] = \mathbf{0}, \quad \mathbf{n} \cdot [\![\mathbf{D}]\!] = \sigma_\mathrm{f}. \tag{2.167}$$

We now establish these results.

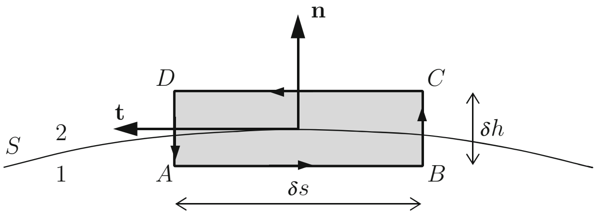

Fig. 2.5 A small plane area intersecting the surface S with corners A, B, C, D in the plane of the unit normal $\mathbf{n}$ to the surface and a unit tangent vector $\mathbf{t}$. The unit normal points from side 1 to side 2 of the surface. The bounding curve is traversed in the direction of the arrows along the path $ABCDA$

Consider the Maxwell equation $(2.164)_2$ integrated over an open surface Σ with bounding curve Γ. After application of Stokes' theorem, it becomes

$$\int_{\Gamma} \mathbf{E} \cdot \mathrm{d}\mathbf{x} = -\int_{\Sigma} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S}. \tag{2.168}$$

Let Σ be an infinitesimal plane rectangular surface with Γ identified by its corner points $ABCD$ lying in the plane of the unit normal $\mathbf{n}$ to a surface S and a unit tangent vector $\mathbf{t}$ to the surface and intersecting S, as shown in Fig. 2.5. The sides AB and CD of Γ are parallel to $\mathbf{t}$ and have lengths δs. The sides BC and DA are parallel to $\mathbf{n}$ and have lengths δh. Then, application of (2.168) to Σ and Γ yields the approximate result

$$-\int_{AB} \mathbf{E}\cdot\mathbf{t}\mathrm{d}s + \int_{BC} \mathbf{E}\cdot\mathbf{n}\mathrm{d}h + \int_{CD} \mathbf{E}\cdot\mathbf{t}\mathrm{d}s - \int_{DA} \mathbf{E}\cdot\mathbf{n}\mathrm{d}h \approx -\frac{\partial \mathbf{B}}{\partial t}\cdot(\mathbf{n}\times\mathbf{t})\delta h \delta s. \tag{2.169}$$

Taking the limit as $\delta h \to 0$ and then dividing by δs and letting $\delta s \to 0$, we obtain $\mathbf{E}_2 \cdot \mathbf{t} - \mathbf{E}_1 \cdot \mathbf{t} = 0$, i.e. $\mathbf{t} \cdot [\![\mathbf{E}]\!] = 0$. This holds for an arbitrary $\mathbf{t}$ normal to $\mathbf{n}$, and hence the result $(2.167)_1$ follows.

Now consider a cylinder (or 'pill box') of infinitesimal height δh and cross-sectional area $\delta \mathbf{S} = \mathbf{n}\delta S$ straddling the surface S, as depicted in Fig. 2.6. Equation $(2.163)_1$ is integrated over the volume V of the cylinder and the divergence theorem then applied to give

$$\int_{\Sigma} \mathbf{D} \cdot \mathrm{d}\mathbf{S} = \int_{V} \rho_{\mathrm{f}}\, \mathrm{d}V, \tag{2.170}$$

where Σ is the bounding surface of the cylinder.

Since δh is infinitesimal and the flux of $\mathbf{D}$ across the lateral surface of the cylinder becomes negligible as $\delta h \to 0$, the only contributions to the surface integral come from the top and bottom surfaces of the cylinder. The right-hand side of (2.170) is the total free charge in V, which consists of the surface charge $\sigma_{\mathrm{f}}\delta S$.

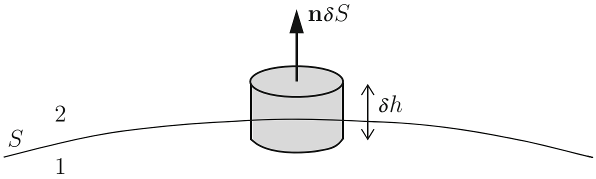

Fig. 2.6 A 'pill-box' of height δh and cross-sectional area δS straddling the surface S with unit normal $\mathbf{n}$ pointing from side 1 to side 2 of S

Equation (2.170) is therefore approximated simply as $\mathbf{D}_2 \cdot \mathbf{n}\,\delta S - \mathbf{D}_1 \cdot \mathbf{n}\,\delta S \approx \sigma_{\mathrm{f}}\delta S$, which, after dividing by δS and taking the limit $\delta S \to 0$, yields $\mathbf{n} \cdot [\![\mathbf{D}]\!] = \sigma_{\mathrm{f}}$, and hence $(2.167)_2$ is established. Clearly, if the surface S is free of distributed charge σ_{f}, then the normal component of $\mathbf{D}$ is continuous.

If the material medium is surrounded by a non-polarizable medium or a vacuum (where $\mathbf{P} = \mathbf{0}$), the boundary conditions (2.167) can be written in the alternative forms

$$[\![\mathbf{D}]\!] = \sigma_{\mathrm{f}}\mathbf{n} + (\mathbf{n} \cdot \mathbf{P})\,\mathbf{n} - \mathbf{P}, \quad \varepsilon_0[\![\mathbf{E}]\!] = \sigma_{\mathrm{f}}\mathbf{n} + (\mathbf{n} \cdot \mathbf{P})\mathbf{n}, \tag{2.171}$$

where use has been made of the connection $(2.145)_1$.

2.5.2 *Boundary Conditions for* B *and* H

The counterparts of the boundary conditions (2.167) for the magnetic vectors are

$$\mathbf{n} \times [\![\mathbf{H}]\!] = \mathbf{K}_{\mathrm{f}}, \quad \mathbf{n} \cdot [\![\mathbf{B}]\!] = 0, \tag{2.172}$$

where $\mathbf{K}_{\mathrm{f}}$ is the free current surface density on the surface S per unit area. The proof of (2.172) follows the same pattern as for (2.167) and is given below.

Consider again the cylinder of infinitesimal height δh and cross-sectional area $\delta\mathbf{S} = \mathbf{n}\delta S$ straddling the surface S in Fig. 2.6. Equation $(2.163)_2$, when integrated over the volume V of the cylinder followed by an application of the divergence theorem, yields

$$\int_{\Sigma} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0. \tag{2.173}$$

Again, since δh is infinitesimal and the flux of $\mathbf{B}$ across the lateral surface of the cylinder becomes negligible as $\delta h \to 0$ and only the integrals over the top and bottom surfaces of the cylinder contribute non-negligible values. Equation (2.173) is therefore approximated simply as $\mathbf{B}_2 \cdot \mathbf{n}\,\delta S - \mathbf{B}_1 \cdot \mathbf{n}\,\delta S \approx 0$, which, after dividing by δS and taking the limit $\delta S \to 0$, yields $\mathbf{n}\cdot[\![\mathbf{B}]\!] = 0$, and hence $(2.172)_2$ is established.

Next, consider (2.164)$_2$ integrated over the open surface Σ with bounding curve Γ shown in Fig. 2.5. On application of Stokes' theorem, it becomes

$$\int_{\Gamma} \mathbf{H} \cdot \mathrm{d}\mathbf{x} = \int_{\Sigma} \left(\mathbf{J}_{\mathrm{f}} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot \mathrm{d}\mathbf{S}, \tag{2.174}$$

which yields the approximate result

$$-\int_{AB} \mathbf{H} \cdot \mathbf{t}\, \mathrm{d}s + \int_{BC} \mathbf{H} \cdot \mathbf{n}\, \mathrm{d}h + \int_{CD} \mathbf{H} \cdot \mathbf{t}\, \mathrm{d}s - \int_{DA} \mathbf{H} \cdot \mathbf{n}\, \mathrm{d}h$$
$$\approx \left[\left(\int_{BC} \mathbf{J}_{\mathrm{f}}\, \mathrm{d}h + \frac{\partial \mathbf{D}}{\partial t}\, \mathrm{d}h \right) \times \mathbf{n} \right] \cdot \mathbf{t}\, \delta s. \tag{2.175}$$

In the limit as $\delta h \to 0$ the term in $\partial \mathbf{D}/\partial t$ in the integral on the right-hand side becomes negligible, as do the integrals along BC and DA on the left-hand side, while the term in $\mathbf{J}_{\mathrm{f}}$ becomes the surface current density $\mathbf{K}_{\mathrm{f}}$ with $\mathbf{n} \cdot \mathbf{K}_{\mathrm{f}} = 0$. Then dividing by δs and letting $\delta s \to 0$, we obtain $\mathbf{H}_2 \cdot \mathbf{t} - \mathbf{H}_1 \cdot \mathbf{t} = (\mathbf{K}_{\mathrm{f}} \times \mathbf{n}) \cdot \mathbf{t}$. Setting $\mathbf{n} \times \mathbf{t} = \mathbf{k}$ and noting that $\mathbf{k} \times \mathbf{n} = \mathbf{t}$, it follows that $\{\mathbf{n} \times [\![\mathbf{H}]\!]\} \cdot \mathbf{k} = \mathbf{K}_{\mathrm{f}} \cdot \mathbf{k}$. Since $\mathbf{t}$ is an arbitrary tangent, then so is $\mathbf{k}$. This holds for arbitrary $\mathbf{k}$ normal to $\mathbf{n}$, and hence the result (2.172)$_2$ follows.

Note that if outside the material is a vacuum or a non-magnetizable material $\mathbf{M} = \mathbf{0}$ outside the material, in which case, by combining the two boundary conditions (2.172) and using the connection (2.145)$_2$, we obtain

$$[\![\mathbf{H}]\!] = (\mathbf{n} \cdot \mathbf{M})\mathbf{n} - \mathbf{n} \times \mathbf{K}_{\mathrm{f}}, \quad [\![\mathbf{B}]\!] = \mu_0 \mathbf{n} \times (\mathbf{n} \times \mathbf{M}) - \mu_0 \mathbf{n} \times \mathbf{K}_{\mathrm{f}}. \tag{2.176}$$

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Chapter 3
Nonlinear Elasticity Background

Abstract In this chapter we summarize the main ingredients of continuum mechanics and their application to nonlinear elasticity with a view to the subsequent development of the nonlinear theories of electroelastic and magnetoelastic interactions. The setting is purely mechanical without reference to any electromagnetic effects. In particular, we review the necessary kinematics of deformation and motion; some required integral theorems involving tensors; the balance equations of mass, linear and angular momentum; and the associated stress tensors. Balance of energy is then used to introduce the strain-energy function of a hyperelastic material, following which we discuss the notions of objectivity and material symmetry as applied to the constitutive equations, and we provide some simple examples of constitutive equations for isotropic and transversely isotropic materials. We then give a general formulation of boundary-value problems in nonlinear elasticity, which is applied to some representative problems involving non-homogeneous deformations that will also feature in later chapters dealing with electroelastic and magnetoelastic boundary-value problems.

3.1 Continuum Kinematics

3.1.1 Deformation of a Solid Continuum

For more detailed background on nonlinear elasticity, we refer to the books by Ogden (1997), Ciarlet (1988), Holzapfel (2000) and Truesdell and Noll (1965) and the collection of articles in Fu and Ogden (2001).

A deformable continuous *body* is a contiguous collection of *particles*, also referred to as *material points*, that reside in three-dimensional Euclidean point space. The region occupied by the body is referred to as a *configuration*. Under the action of applied forces the configuration will in general change as the body deforms. We may identify an arbitrarily chosen configuration as a *reference*

L. Dorfmann and R.W. Ogden, *Nonlinear Theory of Electroelastic and Magnetoelastic Interactions*, DOI 10.1007/978-1-4614-9596-3_3,

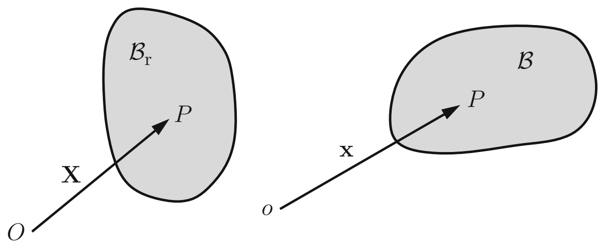

Fig. 3.1 Depiction of a material body in its reference configuration $\mathcal{B}_r$ and deformed configuration $\mathcal{B}$ with position vectors $\mathbf{X}$ and $\mathbf{x}$, respectively, of a material point P relative to origins O and o

configuration from which the deformation is measured. Typically this is taken to be a configuration in which there are no applied forces. We denote this reference configuration by $\mathcal{B}_r$, where the subscript r signifies 'reference'. Let $\mathcal{B}$ denote any other configuration into which the body is deformed relative to $\mathcal{B}_r$. Then, any particle, P say, of the body may be identified by its position vector $\mathbf{X}$ in $\mathcal{B}_r$ relative to some origin O. Let $\mathbf{x}$ be the position vector of P in the configuration $\mathcal{B}$ relative to an origin o (which may, but need not in general, coincide with O), as depicted in Fig. 3.1. We refer to $\mathcal{B}$ as the *deformed configuration*, which is arrived at by a continuous transformation from $\mathcal{B}_r$. For the moment we do not consider time dependence so that the configurations are purely statically related.

We define a bijection mapping $\boldsymbol{\chi} : \mathcal{B}_r \to \mathcal{B}$ such that

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) \quad \text{for all } \mathbf{X} \in \mathcal{B}_r, \quad \mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathcal{B}. \tag{3.1}$$

The mapping $\boldsymbol{\chi}$ is called the *deformation* of the body *from* $\mathcal{B}_r$ *to* $\mathcal{B}$. It is usual to assume that $\boldsymbol{\chi}(\mathbf{X})$ is twice continuously differentiable, although there are situations where this requirement needs to be relaxed. An example of this is across a phase boundary where one or more of the first or second derivatives of $\boldsymbol{\chi}$ is discontinuous.

For the most part we shall refer vectors and tensors to rectangular Cartesian coordinate systems. For this purpose we define basis vectors $\{\mathbf{E}_\alpha\}$ and $\{\mathbf{e}_i\}$ for $\mathcal{B}_r$ and $\mathcal{B}$, respectively, with corresponding coordinates X_α, $\alpha = 1, 2, 3$, for $\mathbf{X}$, and x_i, $i = 1, 2, 3$, for $\mathbf{x}$. We shall use Greek indices for vectors and tensors associated with $\mathcal{B}_r$ and Roman indices for those associated with $\mathcal{B}$. The X_α are called *referential*, *material* or *Lagrangian* coordinates, while the x_i are referred to as *spatial* or *Eulerian* coordinates.

Thus, relative to the origins O and o, respectively, we have

$$\mathbf{X} = X_\alpha \mathbf{E}_\alpha, \quad \mathbf{x} = x_i \mathbf{e}_i. \tag{3.2}$$

In (3.2) the summation convention over repeated indices applies. It will also apply henceforth except where stated otherwise. In general, $\mathbf{E}_\alpha$ and $\mathbf{e}_i$ may be chosen to have different orientations, but it is often convenient to let them coincide.

In the development of the basic principles of continuum mechanics a body is endowed with various physical properties which are represented by scalar, vector and tensor fields, such as mass density, displacement or strain, defined on *either* $\mathcal{B}_\mathrm{r}$ *or* $\mathcal{B}$. For $\mathcal{B}_\mathrm{r}$ the position vector $\mathbf{X}$ serves as the independent variable, and the fields are then said to be defined in terms of the *referential*, *material* or *Lagrangian* description. Alternatively, in the case of $\mathcal{B}$, $\mathbf{x}$ is used as the independent variable and the description is said to be *spatial* or *Eulerian*.

3.1.2 Spatial Derivatives of Field Variables

In the following we shall make use of the vector differential operators Grad, Div and Curl, which are the counterparts of grad, div and curl when the independent variable is $\mathbf{X}$ rather than $\mathbf{x}$. For this purpose we need to identify the appropriate conventions for the ordering of indices when these operate on vectors and tensors, and we restrict attention to rectangular Cartesian coordinates.

We also make use of the tensor product operator $\otimes$, defined, for a pair of vectors $\mathbf{a}$ and $\mathbf{b}$, by $(\mathbf{a}\otimes\mathbf{b})\mathbf{c} = (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$ for an arbitrary vector $\mathbf{c}$, the component form of which was introduced in Sect. 2.1.10. Similarly, for a second-order tensor $\mathbf{T}$ and a vector $\mathbf{a}$, the tensor product is defined by $(\mathbf{T}\otimes\mathbf{a})\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{T}$, which is different from the product $(\mathbf{a}\otimes\mathbf{T})\mathbf{c} = \mathbf{a}\otimes\mathbf{T}\mathbf{c}$.

Let Φ, $\mathbf{A}$ and $\mathbf{T}$ be, respectively, scalar, vector and second-order tensor functions of position $\mathbf{X}$. The operations of Grad on these functions with respect to the basis $\{\mathbf{E}_\alpha\}$ are defined by

$$\mathrm{Grad}\,\Phi = \frac{\partial\Phi}{\partial X_\alpha}\mathbf{E}_\alpha, \tag{3.3}$$

$$\mathrm{Grad}\,\mathbf{A} = \frac{\partial\mathbf{A}}{\partial X_\beta}\otimes\mathbf{E}_\beta = \frac{\partial}{\partial X_\beta}(A_\alpha\mathbf{E}_\alpha)\otimes\mathbf{E}_\beta = \frac{\partial A_\alpha}{\partial X_\beta}\mathbf{E}_\alpha\otimes\mathbf{E}_\beta, \tag{3.4}$$

$$\mathrm{Grad}\,\mathbf{T} = \frac{\partial\mathbf{T}}{\partial X_\alpha}\otimes\mathbf{E}_\alpha = \frac{\partial}{\partial X_\alpha}(T_{\beta\gamma}\mathbf{E}_\beta\otimes\mathbf{E}_\gamma)\otimes\mathbf{E}_\alpha = \frac{\partial T_{\beta\gamma}}{\partial X_\alpha}\mathbf{E}_\beta\otimes\mathbf{E}_\gamma\otimes\mathbf{E}_\alpha. \tag{3.5}$$

In the present context we do not need to distinguish between tensor products such as $(\mathbf{E}_\alpha\otimes\mathbf{E}_\beta)\otimes\mathbf{E}_\gamma$, $\mathbf{E}_\alpha\otimes(\mathbf{E}_\beta\otimes\mathbf{E}_\gamma)$ and $\mathbf{E}_\alpha\otimes\mathbf{E}_\beta\otimes\mathbf{E}_\gamma$.

The divergence operations on the vector and tensor fields are defined as

$$\mathrm{Div}\,\mathbf{A} = \frac{\partial A_\alpha}{\partial X_\alpha}, \quad \mathrm{Div}\,\mathbf{T} \equiv \frac{\partial T_{\beta\gamma}}{\partial X_\alpha}\mathbf{E}_\gamma(\mathbf{E}_\beta\cdot\mathbf{E}_\alpha), \tag{3.6}$$

the latter corresponding to contraction of $\mathrm{Grad}\,\mathbf{T}$ on the first and third basis vectors, and since $\mathbf{E}_\beta\cdot\mathbf{E}_\alpha = \delta_{\alpha\beta}$, where $\delta_{\alpha\beta}$ is the Kronecker delta, we obtain

$$\mathrm{Div}\,\mathbf{T} = \frac{\partial T_{\beta\gamma}}{\partial X_\beta}\mathbf{E}_\gamma. \tag{3.7}$$

The Curl operations on $\mathbf{A}$ and $\mathbf{T}$ are defined using the alternating symbol $\epsilon_{\alpha\beta\gamma}$ by

$$\operatorname{Curl}\mathbf{A} = \epsilon_{\alpha\beta\gamma}\frac{\partial A_\gamma}{\partial X_\beta}\mathbf{E}_\alpha, \quad \operatorname{Curl}\mathbf{T} = \epsilon_{\alpha\beta\gamma}\frac{\partial T_{\gamma\delta}}{\partial X_\beta}\mathbf{E}_\alpha \otimes \mathbf{E}_\delta. \tag{3.8}$$

Note, in particular, that in the divergence of a second-order tensor the derivative operates on the first index. In the literature it is sometimes defined with respect to the second index. Curl is also defined with respect to its operation on the first index. Corresponding conventions apply to the Eulerian forms of the differential operators.

3.1.3 The Deformation Gradient

We define the *deformation gradient tensor*, which is denoted $\mathbf{F}$, by

$$\mathbf{F}(\mathbf{X}) = \operatorname{Grad}\mathbf{x} \equiv \operatorname{Grad}\boldsymbol{\chi}(\mathbf{X}). \tag{3.9}$$

In components,

$$\mathbf{F} = F_{i\alpha}\mathbf{e}_i \otimes \mathbf{E}_\alpha, \quad F_{i\alpha} = \frac{\partial x_i}{\partial X_\alpha}, \tag{3.10}$$

with respect to the chosen basis vectors, where $x_i = \chi_i(\mathbf{X})$. Take particular note of the convention for the ordering of indices here.

We assume that $\det\mathbf{F} \neq 0$ (to be justified shortly) so that $\mathbf{F}$ has an inverse $\mathbf{F}^{-1}$, given by

$$\mathbf{F}^{-1} = \operatorname{grad}\mathbf{X}, \quad (\mathbf{F}^{-1})_{\alpha j} = \frac{\partial X_\alpha}{\partial x_j}, \tag{3.11}$$

where $X_\alpha = (\boldsymbol{\chi}^{-1})_\alpha(\mathbf{x})$.

It follows that

$$\mathrm{d}\mathbf{x} = \mathbf{F}\mathrm{d}\mathbf{X}, \quad \mathrm{d}\mathbf{X} = \mathbf{F}^{-1}\mathrm{d}\mathbf{x}. \tag{3.12}$$

Equation $(3.12)_1$ describes how small *line elements* $\mathrm{d}\mathbf{X}$ of material at $\mathbf{X}$ transform under the deformation into $\mathrm{d}\mathbf{x}$ (which consists of the same material as $\mathrm{d}\mathbf{X}$) at $\mathbf{x}$. It shows that *line elements* transform *linearly* since $\mathbf{F}$ depends on $\mathbf{X}$ (and not on $\mathrm{d}\mathbf{X}$). Thus, at each $\mathbf{X}$, $\mathbf{F}$ is a *linear mapping*, i.e. a second-order tensor, in fact a two-point tensor since it relates two different configurations. We justify taking $\mathbf{F}$ to be *non-singular* ($\det\mathbf{F} \neq 0$) by noting that $\mathbf{F}\mathrm{d}\mathbf{X} \neq \mathbf{0}$ if $\mathrm{d}\mathbf{X} \neq \mathbf{0}$, i.e. a line element cannot be annihilated by the deformation process. Similarly, the inverse $\mathbf{F}^{-1}$ is non-singular.

We shall use the standard notation

$$J = \det\mathbf{F}. \tag{3.13}$$

Now let $\phi(\mathbf{x})$ and $\mathbf{a}(\mathbf{x})$, respectively, be scalar and vector fields associated with the deformed configuration $\mathcal{B}$, i.e. they are Eulerian fields. They may be expressed in the Lagrangian description by means of the transformation $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$, and for this purpose we use the notations

$$\Phi(\mathbf{X}) \equiv \phi(\mathbf{x}) = \phi(\boldsymbol{\chi}(\mathbf{X})), \quad \mathbf{A}(\mathbf{X}) \equiv \mathbf{a}(\mathbf{x}) = \mathbf{a}(\boldsymbol{\chi}(\mathbf{X})). \tag{3.14}$$

We may then establish the very useful formulas

$$\mathrm{Grad}\,\Phi = \mathbf{F}^{\mathrm{T}}\mathrm{grad}\,\phi, \quad \mathrm{Grad}\,\mathbf{A} = (\mathrm{grad}\,\mathbf{a})\mathbf{F}, \tag{3.15}$$

$$\mathrm{Div}\,\mathbf{A} = J\,\mathrm{div}\,(J^{-1}\mathbf{F}\mathbf{A}), \quad \mathrm{Curl}\,\mathbf{A} = J\mathbf{F}^{-1}\mathrm{curl}\,(\mathbf{F}^{-\mathrm{T}}\mathbf{A}), \tag{3.16}$$

where $^{\mathrm{T}}$ denotes the transpose of a second-order tensor. Corresponding formulas may be written down for the gradient and divergence of a tensor field, following the conventions outlined in Sect. 3.1.2. We also note the kinematical identities

$$\mathrm{Div}\,(J\mathbf{F}^{-1}) = \mathbf{0}, \quad \mathrm{div}\,(J^{-1}\mathbf{F}) = \mathbf{0}, \quad \mathrm{Curl}\,(\mathbf{F}^{\mathrm{T}}) = \mathbf{0}, \quad \mathrm{curl}\,(\mathbf{F}^{-\mathrm{T}}) = \mathbf{0}. \tag{3.17}$$

The proofs of these formulas are straightforward and can be found in standard continuum mechanics texts. Equations (3.15)–(3.17) are valuable for converting formulas between Eulerian and Lagrangian descriptions.

3.1.3.1 Deformation of Surface and Volume Elements

Let $\mathrm{d}S$ and $\mathrm{d}s$ be surface area elements based at points $\mathbf{X}$ and $\mathbf{x}$ on some surface $\mathcal{S}_\mathrm{r}$ in $\mathcal{B}_\mathrm{r}$ and $\mathcal{S}$ in $\mathcal{B}$, respectively, and let $\mathbf{N}$ and $\mathbf{n}$ be unit normals to $\mathcal{S}_\mathrm{r}$ and $\mathcal{S}$ at points $\mathbf{X}$ and $\mathbf{x}$, respectively, as depicted in Fig. 3.2. Then the vector area element $\mathbf{N}\mathrm{d}S$ transforms into $\mathbf{n}\mathrm{d}s$ according to *Nanson's formula*

$$\mathrm{d}\mathbf{s} \equiv \mathbf{n}\mathrm{d}s = J\mathbf{F}^{-\mathrm{T}}\mathbf{N}\mathrm{d}S \equiv J\mathbf{F}^{-\mathrm{T}}\mathrm{d}\mathbf{S}. \tag{3.18}$$

For a volume element $\mathrm{d}V$ in $\mathcal{B}_\mathrm{r}$ the corresponding volume element $\mathrm{d}v$ in $\mathcal{B}$ is given by

$$\mathrm{d}v = J\mathrm{d}V. \tag{3.19}$$

By convention we take volume elements to be positive, and hence we deduce that $J > 0$.

From (3.19) we see that J is a local measure of the change in volume under the deformation. If the deformation is such that there is no change in volume, the deformation is said to be *isochoric*, and then

$$J \equiv \det\mathbf{F} = 1. \tag{3.20}$$

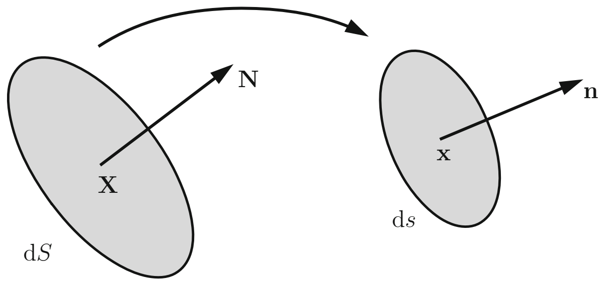

Fig. 3.2 Transformation of the area element dS at $\mathbf{X}$ with unit normal $\mathbf{N}$ in the reference configuration $\mathcal{B}_\mathrm{r}$ into the area element ds at $\mathbf{x}$ with unit normal $\mathbf{n}$ in the deformed configuration $\mathcal{B}$

For some materials many deformations are such that (3.20) holds to a good approximation, and (3.20) is adopted as a constraint, which is, of course, an *idealization*. A material for which (3.20) holds for *all* deformations is called an *incompressible material*.

3.1.3.2 Measures of Deformation and Strain

For the deformation gradient tensor, or for any second-order tensor $\mathbf{F}$ such that $\det \mathbf{F} > 0$, there exist unique, positive definite, symmetric tensors, $\mathbf{U}$ and $\mathbf{V}$, and a unique proper orthogonal tensor $\mathbf{R}$ such that

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}. \tag{3.21}$$

This is known as the *polar decomposition theorem*.

Let λ_i, $i = 1, 2, 3$, be the eigenvalues of $\mathbf{U}$ and $\mathbf{u}^{(i)}$, $i = 1, 2, 3$, be the corresponding unit eigenvectors. Then, since $\mathbf{U}$ is positive definite, $\lambda_i > 0$, $i = 1, 2, 3$. Moreover, the λ_i are also the eigenvalues of $\mathbf{V}$ with unit eigenvectors $\mathbf{v}^{(i)} = \mathbf{Ru}^{(i)}$, as can be seen from the calculation

$$\mathbf{V}(\mathbf{Ru}^{(i)}) = \mathbf{VRu}^{(i)} = \mathbf{RUu}^{(i)} = \mathbf{R}(\lambda_i \mathbf{u}^{(i)}) = \lambda_i(\mathbf{Ru}^{(i)}). \tag{3.22}$$

The eigenvalues λ_i, $i = 1, 2, 3$, are called the *principal stretches* of the deformation, and $\mathbf{U}$ and $\mathbf{V}$ are, respectively, the *right* and *left stretch tensors*. Since they are symmetric they enjoy spectral decompositions, which we write using their eigenvalues and eigenvectors as

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}, \quad \mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}. \tag{3.23}$$

It follows from the above that $\mathbf{F}$ can be decomposed in the form

$$\mathbf{F} = \sum_{i=1}^{3} \lambda_i \mathbf{v}^{(i)} \otimes \mathbf{u}^{(i)} \tag{3.24}$$

and that

$$J \equiv \det \mathbf{F} = \det \mathbf{U} = \det \mathbf{V} = \lambda_1 \lambda_2 \lambda_3. \tag{3.25}$$

Stretch is an important notion and can be defined in a general way as follows. Let $\mathbf{M}$ be a unit vector in the direction of an arbitrary line element $\mathrm{d}\mathbf{X}$ in $\mathcal{B}_\mathrm{r}$ so that $\mathrm{d}\mathbf{X} = \mathbf{M}|\mathrm{d}\mathbf{X}|$. Then, since $\mathrm{d}\mathbf{x} = \mathbf{F}\mathrm{d}\mathbf{X}$, we obtain $|\mathrm{d}\mathbf{x}| = |\mathbf{FM}||\mathrm{d}\mathbf{X}|$, and this prompts the definition of the *stretch in the direction* $\mathbf{M}$ at $\mathbf{X}$ as the ratio of the length of a line element in the deformed configuration to that in the undeformed configuration, i.e. $|\mathbf{FM}|$, which we denote by $\lambda(\mathbf{M})$. This is given by

$$\lambda(\mathbf{M}) = [\mathbf{M} \cdot (\mathbf{F}^\mathrm{T}\mathbf{F}\mathbf{M})]^{1/2}. \tag{3.26}$$

Since both $\mathbf{F}$ and its inverse are non-singular, we have $0 < \lambda(\mathbf{M}) < \infty$ for all unit vectors $\mathbf{M}$. Note that, in accordance with the definition (3.26), $\lambda_i = \lambda(\mathbf{u}^{(i)})$—hence the terminology *principal stretch*.

Using the polar decomposition (3.21), we may also form the tensor measures of deformation

$$\mathbf{c} = \mathbf{F}^\mathrm{T}\mathbf{F} = \mathbf{U}^2, \quad \mathbf{b} = \mathbf{F}\mathbf{F}^\mathrm{T} = \mathbf{V}^2, \tag{3.27}$$

which are called the *right* and *left Cauchy–Green deformation tensors*, respectively. Note that we are using lower case letters for the tensors $\mathbf{c}$ and $\mathbf{b}$ here, rather than the usual upper case $\mathbf{C}$ and $\mathbf{B}$ in order to avoid a conflict with the notation $\mathbf{B}$ used for the magnetic induction vector.

Since $|\mathrm{d}\mathbf{x}|^2 = \mathrm{d}\mathbf{X} \cdot (\mathbf{F}^\mathrm{T}\mathbf{F}\mathrm{d}\mathbf{X})$, we see that $\mathrm{d}\mathbf{X} \cdot (\mathbf{F}^\mathrm{T}\mathbf{F} - \mathbf{I})\mathrm{d}\mathbf{X}$ is a measure of the change in the squared length of a line element. The material is said to be *unstrained* at $\mathbf{X}$ if no line element changes length, which is the case if $\mathbf{F}^\mathrm{T}\mathbf{F} - \mathbf{I} = \mathbf{O}$, the zero tensor, or equivalently $\lambda(\mathbf{M}) = 1$ for all *unit* vectors $\mathbf{M}$ at $\mathbf{X}$. Thus, the tensor $\mathbf{F}^\mathrm{T}\mathbf{F} - \mathbf{I}$ is a measure of strain, from which is formed the so-called *Green* or *Green–Lagrange strain tensor*, denoted here with the lower case letter $\mathbf{e}$ instead of the usual $\mathbf{E}$, which conflicts with the notation for the electric field. It is defined by

$$\mathbf{e} = \frac{1}{2}(\mathbf{F}^\mathrm{T}\mathbf{F} - \mathbf{I}) = \frac{1}{2}(\mathbf{c} - \mathbf{I}) = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I}). \tag{3.28}$$

This is just one of the many possible choices of strain tensor, but we shall not make use of the other ones in this text, but refer to Ogden (1997), for example, for discussion of other strain tensors.

Note that if $\mathbf{F}$ is just a rotation $\mathbf{R}$, then, since $\mathbf{R}$ is orthogonal, $\mathbf{R}^\mathrm{T}\mathbf{R} = \mathbf{I}$, i.e. there is no strain associated with a pure rotation.

Finally in this section it is useful to note that the *displacement* $\mathbf{u}$ of a particle is defined as

$$\mathbf{u} = \mathbf{x} - \mathbf{X}, \tag{3.29}$$

so that

$$\mathbf{x} = \mathbf{X} + \mathbf{u} \tag{3.30}$$

and

$$\mathbf{F} = \mathrm{Grad}\,\mathbf{x} = \mathbf{I} + \mathrm{Grad}\,\mathbf{u}, \tag{3.31}$$

where $\mathrm{Grad}\,\mathbf{u}$ is the *displacement gradient*. Here we have used the result $\mathrm{Grad}\,\mathbf{X} = \mathbf{I}$, the identity tensor (in components $\partial X_\alpha / \partial X_\beta = \delta_{\alpha\beta}$).

3.1.3.3 Homogeneous Deformations

There is an important class of deformations, known as *homogeneous deformations*, for which $\mathbf{F}$ is independent of $\mathbf{X}$ and hence $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) = \mathbf{F}\mathbf{X} + \mathbf{d}$, where $\mathbf{d}$ is a constant vector. Some special cases of this are listed in the following, and we shall make use of these later in the text.

Pure homogeneous strain For this deformation $\mathbf{F} = \mathbf{U} = \mathbf{V}$, and there is no change in the orientation of the principal axes of $\mathbf{U}$ during the deformation. Thus, the deformation gradient has the form

$$\mathbf{F} = \mathbf{U} = \lambda_1 \mathbf{u}^{(1)} \otimes \mathbf{u}^{(1)} + \lambda_2 \mathbf{u}^{(2)} \otimes \mathbf{u}^{(2)} + \lambda_3 \mathbf{u}^{(3)} \otimes \mathbf{u}^{(3)}, \tag{3.32}$$

where the principal stretches are constants.

Simple elongation This is a special case of pure homogeneous strain. Consider, for example, the uniform axial extension of a solid right circular cylinder with accompanying lateral contraction. Let the principal axis $\mathbf{u}^{(1)}$ lie along the cylinder axis and let λ_1 be the corresponding principal stretch. Then, assuming there is symmetry perpendicular to the axis (which would be the case if the material properties are isotropic in the transverse plane), $\lambda_2 = \lambda_3$ and hence the deformation gradient may be written as

$$\mathbf{F} = \mathbf{U} = \lambda_1 \mathbf{u}^{(1)} \otimes \mathbf{u}^{(1)} + \lambda_2 (\mathbf{u}^{(2)} \otimes \mathbf{u}^{(2)} + \mathbf{u}^{(3)} \otimes \mathbf{u}^{(3)}). \tag{3.33}$$

Pure dilatation This is defined by $\lambda_1 = \lambda_2 = \lambda_3 = J^{1/3}$, so that $\mathbf{F} = J^{1/3}\mathbf{I}$.

Simple shear Simple shear is defined by the equation

$$\mathbf{x} = \mathbf{X} + \gamma(\mathbf{N} \cdot \mathbf{X})\mathbf{M}, \tag{3.34}$$

where γ is a constant called the *amount of shear* and $\mathbf{M}$ and $\mathbf{N}$ are orthogonal unit vectors defined in the reference configuration. Then

$$\mathbf{F} = \mathbf{I} + \gamma \mathbf{M} \otimes \mathbf{N} \tag{3.35}$$

and $\tan^{-1} \gamma$ is the angle of shear of the directions $\mathbf{M}$ and $\mathbf{N}$, i.e. the change in the angle between $\mathbf{M}$ and $\mathbf{N}$ due to the deformation.

If we take $\mathbf{M} = \mathbf{E}_1$ and $\mathbf{N} = \mathbf{E}_2$, then deformation gradient has matrix of components, which we denote by F, given by

$$\mathsf{F} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.36}$$

3.1.4 Motion of a Continuum

We now consider a material body that is moving and deforming so that its motion may be parametrized with time. We then denote the configuration it occupies at time $t \in I \subset \mathbb{R}$ by $\mathcal{B}_t$ instead of $\mathcal{B}$, where I is an appropriate interval of time. The family of configurations $\{\mathcal{B}_t : t \in I\}$ is called a *motion* of the body, and $\mathcal{B}_t$, which is called the *current configuration*, changes continuously with t. The deformation $\boldsymbol{\chi}(\mathbf{X})$ becomes time dependent and is written as

$$\mathbf{x} = \boldsymbol{\chi}_t(\mathbf{X}), \tag{3.37}$$

with inverse $\mathbf{X} = \boldsymbol{\chi}_t^{-1}(\mathbf{x})$ for each $t \in I$, or as

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) \quad \text{for all } \mathbf{X} \in \mathcal{B}_\mathrm{r},\ t \in I. \tag{3.38}$$

For each particle P this describes the motion of P with t as parameter and hence the motion of $\mathcal{B}_t$.

The *velocity* $\mathbf{v}$ of the particle P is defined as

$$\mathbf{v} = \frac{\partial}{\partial t}\boldsymbol{\chi}(\mathbf{X}, t), \tag{3.39}$$

i.e. the rate of change of position of P (or $\partial/\partial t$ at fixed $\mathbf{X}$). The *acceleration* $\mathbf{a}$ of P is

$$\mathbf{a} = \frac{\partial^2}{\partial t^2}\boldsymbol{\chi}(\mathbf{X}, t). \tag{3.40}$$

3.1.4.1 The Material Time Derivative

Let ϕ be a scalar field defined on $\mathcal{B}_t$, i.e. $\phi(\mathbf{x}, t)$. Since $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$, we may write

$$\phi(\mathbf{x}, t) = \phi[\boldsymbol{\chi}(\mathbf{X}, t), t] \equiv \Phi(\mathbf{X}, t), \tag{3.41}$$

which defines the notation Φ. Thus, any field defined on $\mathcal{B}_t$ (respectively $\mathcal{B}_\mathrm{r}$) can, through (3.38) (respectively its inverse), equally be defined on $\mathcal{B}_\mathrm{r}$ (respectively $\mathcal{B}_t$).

The *material time derivative* of ϕ is the rate of change of ϕ at fixed *material point* P, i.e. at fixed $\mathbf{X}$. By definition and by use of the chain rule for partial derivatives, we obtain

$$\frac{\partial}{\partial t}\Phi(\mathbf{X}, t) = \frac{\partial}{\partial t}\phi + \mathbf{v} \cdot \operatorname{grad}\phi. \tag{3.42}$$

The left-hand expression in (3.42) represents the material time derivative in the material description, while the right-hand side is its equivalent in the spatial description. We will also write the material time derivative as $\Phi_{,t}$ when it is understood that the Lagrangian field Φ has $\mathbf{X}$ and t as independent variables. Moreover, we will write the material time derivative as $\phi_{,t}$ for the Eulerian field ϕ, although this means the right-hand side of (3.42) and not just the first term on the right-hand side. More generally, a subscript t following a comma will indicate the material time derivative when attached to any scalar, vector or tensor field. For example, the velocity and acceleration may be written $\mathbf{v} = \mathbf{x}_{,t}$ and $\mathbf{a} = \mathbf{v}_{,t} = \mathbf{x}_{,tt}$.

Similarly to (3.42), for a vector field

$$\mathbf{a}(\mathbf{x}, t) = \mathbf{a}[\boldsymbol{\chi}(\mathbf{X}, t), t] \equiv \mathbf{A}(\mathbf{X}, t), \tag{3.43}$$

wherein $\mathbf{A}$ is defined, we obtain

$$\mathbf{A}_{,t} \equiv \frac{\partial}{\partial t}\mathbf{A}(\mathbf{X}, t) \equiv \mathbf{a}_{,t} = \frac{\partial \mathbf{a}}{\partial t} + (\mathbf{v} \cdot \operatorname{grad})\mathbf{a}. \tag{3.44}$$

In particular, the acceleration is given by

$$\mathbf{a} = \mathbf{v}_{,t} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \operatorname{grad})\mathbf{v}. \tag{3.45}$$

3.1.4.2 The Velocity Gradient and Related Quantities

The *velocity gradient tensor*, denoted $\boldsymbol{\Gamma}$, is an Eulerian tensor defined by

$$\boldsymbol{\Gamma} = \operatorname{grad}\mathbf{v}, \tag{3.46}$$

which has components

$$\Gamma_{ij} = \frac{\partial v_i}{\partial x_j} \tag{3.47}$$

with respect to the basis $\{\mathbf{e}_i\}$.

Using the second identity in (3.15) and the fact that $\mathbf{X}$ and t are independent variables in the Lagrangian description of the motion, we obtain the important connection

$$\mathbf{F}_{,t} = \mathbf{\Gamma}\mathbf{F}. \tag{3.48}$$

By using the fact that $\mathbf{F}\mathbf{F}^{-1} = \mathbf{I}$, the useful formula

$$(\mathbf{F}^{-1})_{,t} = -\mathbf{F}^{-1}\mathbf{\Gamma} \tag{3.49}$$

can then be derived.

Next, we give an expression for the material time derivative of J. This requires use of the expression in component form for the determinant of $\mathbf{F}$, specifically

$$J = \frac{1}{6}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}F_{i\alpha}F_{j\beta}F_{k\gamma}. \tag{3.50}$$

Differentiating this and then pre-multiplying by $\mathbf{F}$ and making use of the formula $\epsilon_{ijk}F_{i\alpha}F_{j\beta}F_{k\gamma} = J\epsilon_{\alpha\beta\gamma}$, we obtain

$$\mathbf{F}\frac{\partial J}{\partial \mathbf{F}} = J\mathbf{I}, \quad \frac{\partial J}{\partial \mathbf{F}} = J\mathbf{F}^{-1}. \tag{3.51}$$

Using this result we obtain

$$J_{,t} = \operatorname{tr}\left(\frac{\partial J}{\partial \mathbf{F}}\mathbf{F}_{,t}\right) = J\operatorname{tr}(\mathbf{F}^{-1}\mathbf{F}_{,t}). \tag{3.52}$$

From (3.48) we then deduce that

$$J_{,t} = J\operatorname{tr}\mathbf{\Gamma} \equiv J\operatorname{div}\mathbf{v}, \tag{3.53}$$

since $\operatorname{tr}\mathbf{\Gamma} = \Gamma_{ii} = \partial v_i/\partial x_i = \operatorname{div}\mathbf{v}$.

Thus, $\operatorname{div}\mathbf{v}$ measures the rate at which volume changes during the motion. For an *isochoric* motion $J \equiv 1$, $J_{,t} = 0$ and hence

$$\operatorname{div}\mathbf{v} = 0. \tag{3.54}$$

While the deformation gradient $\mathbf{F}$ describes how material line elements change their length and orientation during deformation, the velocity gradient $\mathbf{\Gamma}$ describes the rate of these changes. Note that $\mathbf{F}$ relates $\mathcal{B}_t$ to $\mathcal{B}_\mathrm{r}$, but that $\mathbf{\Gamma}$ is independent of $\mathcal{B}_\mathrm{r}$.

The deformation gradient can be decomposed multiplicatively via the polar decomposition theorem, but the corresponding decomposition of $\boldsymbol{\Gamma}$ is additive. Specifically, we may decompose $\boldsymbol{\Gamma}$ into the sum of a symmetric and an antisymmetric part as

$$\boldsymbol{\Gamma} = \boldsymbol{\Sigma} + \boldsymbol{\Omega}, \tag{3.55}$$

where

$$\boldsymbol{\Sigma} = \frac{1}{2}(\boldsymbol{\Gamma} + \boldsymbol{\Gamma}^{\mathrm{T}}), \quad \boldsymbol{\Omega} = \frac{1}{2}(\boldsymbol{\Gamma} - \boldsymbol{\Gamma}^{\mathrm{T}}), \tag{3.56}$$

which are, respectively, symmetric and antisymmetric.

Some useful connections between the material time derivatives of Eulerian and Lagrangian vector fields can be obtained by considering the pull back operations of $J\mathbf{F}^{-1}$ and $\mathbf{F}^{\mathrm{T}}$ on an Eulerian vector field $\mathbf{a}$. First, let $\mathbf{A} = J\mathbf{F}^{-1}\mathbf{a}$. Then, by taking the material time derivative of both sides of this equation, using the results above and then rearranging, we obtain

$$J^{-1}\mathbf{F}\mathbf{A}_{,t} = \frac{\partial \mathbf{a}}{\partial t} + (\mathbf{v} \cdot \operatorname{grad})\mathbf{a} + (\operatorname{div}\mathbf{v})\mathbf{a} - \boldsymbol{\Gamma}\mathbf{a}, \tag{3.57}$$

or equivalently

$$J^{-1}\mathbf{F}\mathbf{A}_{,t} = \frac{\partial \mathbf{a}}{\partial t} - \operatorname{curl}(\mathbf{v} \times \mathbf{a}) + (\operatorname{div}\mathbf{a})\mathbf{v}. \tag{3.58}$$

Next, by setting $\mathbf{A} = \mathbf{F}^{\mathrm{T}}\mathbf{a}$ instead and again performing the material time derivative and rearranging, we obtain

$$\mathbf{F}^{-\mathrm{T}}\mathbf{A}_{,t} = \frac{\partial \mathbf{a}}{\partial t} + (\mathbf{v} \cdot \operatorname{grad})\mathbf{a} + \boldsymbol{\Gamma}^{\mathrm{T}}\mathbf{a}. \tag{3.59}$$

3.1.5 Integral Theorems

Here we record the standard forms of the divergence and Stokes' theorems for a continuum, but in terms of Lagrangian variables. Their Eulerian counterparts are listed in Appendix A.3. Consider some volume $\mathcal{D}_{\mathrm{r}}$ within the reference configuration $\mathcal{B}_{\mathrm{r}}$, and let $\partial\mathcal{D}_{\mathrm{r}}$ denote its boundary. Also, let $\mathcal{S}_{\mathrm{r}}$ denote an open surface within $\mathcal{B}_{\mathrm{r}}$ and $\partial\mathcal{S}_{\mathrm{r}}$ its bounding closed curve, all with appropriate regularity for the validity of the integral theorems. Let $\mathbf{A}(\mathbf{X})$ be a Lagrangian vector field. Then the standard forms of the divergence and Stokes' theorems are stated as

$$\int_{\mathcal{D}_{\mathrm{r}}} \operatorname{Div}\mathbf{A}\,\mathrm{d}V = \int_{\partial\mathcal{D}_{\mathrm{r}}} \mathbf{A}\cdot\mathbf{N}\,\mathrm{d}S, \tag{3.60}$$

and

$$\int_{\mathcal{S}_r} \mathrm{Curl}\,\mathbf{A} \cdot \mathbf{N}\,\mathrm{d}S = \int_{\partial \mathcal{S}_r} \mathbf{A} \cdot \mathrm{d}\mathbf{X}, \tag{3.61}$$

respectively. These can be written equivalently in tensor form as

$$\int_{\mathcal{D}_r} \mathrm{Grad}\,\mathbf{A}\,\mathrm{d}V = \int_{\partial \mathcal{D}_r} \mathbf{A} \otimes \mathbf{N}\,\mathrm{d}S, \tag{3.62}$$

and

$$\int_{\mathcal{S}_r} (\mathbf{N} \times \mathrm{Grad})\mathbf{A}\,\mathrm{d}S = \int_{\partial \mathcal{S}_r} \mathrm{d}\mathbf{X} \otimes \mathbf{A}, \tag{3.63}$$

respectively.

Equally, $\mathbf{A}$ may be replaced by a tensor $\mathbf{T}$ to give

$$\int_{\mathcal{D}_r} \mathrm{Grad}\,\mathbf{T}\,\mathrm{d}V = \int_{\partial \mathcal{D}_r} \mathbf{T} \otimes \mathbf{N}\,\mathrm{d}S, \tag{3.64}$$

and

$$\int_{\mathcal{S}_r} (\mathbf{N} \times \mathrm{Grad})\mathbf{T}\,\mathrm{d}S = \int_{\partial \mathcal{S}_r} \mathrm{d}\mathbf{X} \otimes \mathbf{T}. \tag{3.65}$$

If, in particular, $\mathbf{T}$ is a second-order tensor, then contraction of (3.64) yields the standard divergence theorem for a second-order tensor, namely

$$\int_{\mathcal{D}_r} \mathrm{Div}\,\mathbf{T}\,\mathrm{d}V = \int_{\partial \mathcal{D}_r} \mathbf{T}^{\mathrm{T}}\mathbf{N}\,\mathrm{d}S, \tag{3.66}$$

and we recall the convention for the divergence of a second-order tensor given in (3.7). This is an important formula and will be used frequently in this text, either in Lagrangian form, as here, or in its counterpart Eulerian form.

A similar contraction of (3.65) yields

$$\int_{\mathcal{S}_r} \mathbf{N} \cdot (\mathrm{Curl}\,\mathbf{T})\,\mathrm{d}S = \int_{\partial \mathcal{S}_r} \mathbf{T}^{\mathrm{T}}\mathrm{d}\mathbf{X}, \tag{3.67}$$

where $\mathrm{Curl}\,\mathbf{T}$ is defined in (3.8).

3.1.6 Transport Formulas

Let us now consider a tensor field $\mathbf{T}$ that depends on time as well as position. Suppose it is an Eulerian field so that $\mathbf{T} = \mathbf{T}(\mathbf{x}, t)$. In many situations we are faced with evaluating line, surface and volume integrals that involve a tensor $\mathbf{T}$ and determining how those integrals depend on time. Let $\mathcal{C}_t$, $\mathcal{S}_t$ and $\mathcal{D}_t$ be, respectively, a curve, surface and volume at time t which come from $\mathcal{C}_\mathrm{r}$, $\mathcal{S}_\mathrm{r}$ and $\mathcal{D}_\mathrm{r}$ in the reference configuration and consist of the same material curve, surface and volume, respectively. Then, bearing in mind the integrals considered in the previous subsection, we wish to evaluate the time derivatives of

$$\int_{\mathcal{C}_t} \mathrm{d}\mathbf{x} \otimes \mathbf{T}, \quad \int_{\mathcal{S}_t} \mathbf{T} \otimes \mathbf{n}\,\mathrm{d}s, \quad \int_{\mathcal{D}_t} \mathbf{T}\,\mathrm{d}v. \tag{3.68}$$

In order to carry out the time differentiation we first of all convert the integrals to integrals over the appropriate region in $\mathcal{B}_\mathrm{r}$ by using the transformations

$$\mathrm{d}\mathbf{x} = \mathbf{F}\mathrm{d}\mathbf{X}, \quad \mathbf{n}\mathrm{d}s = J\mathbf{F}^{-\mathrm{T}}\mathbf{N}\mathrm{d}S, \quad \mathrm{d}v = J\,\mathrm{d}V, \tag{3.69}$$

to obtain

$$\int_{\mathcal{C}_\mathrm{r}} \mathbf{F}\mathrm{d}\mathbf{X} \otimes \mathbf{T}, \quad \int_{\mathcal{S}_\mathrm{r}} \mathbf{T} \otimes J\mathbf{F}^{-\mathrm{T}}\mathbf{N}\,\mathrm{d}S, \quad \int_{\mathcal{D}_\mathrm{r}} \mathbf{T}J\,\mathrm{d}V, \tag{3.70}$$

with $\mathbf{T}$ now treated as a function of $\mathbf{X}$ and t via the motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ (in this case we retain the notation $\mathbf{T}$), and then using the formulas

$$\mathbf{F}_{,t} = \boldsymbol{\Gamma}\mathbf{F}, \quad (J\mathbf{F}^{-\mathrm{T}})_{,t} = J[(\mathrm{tr}\,\boldsymbol{\Gamma})\mathbf{I} - \boldsymbol{\Gamma}^\mathrm{T}]\mathbf{F}^{-\mathrm{T}}, \quad J_{,t} = J\,\mathrm{tr}\,\boldsymbol{\Gamma} \tag{3.71}$$

to obtain, after converting back to the current configuration,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{C}_t} \mathrm{d}\mathbf{x} \otimes \mathbf{T} = \int_{\mathcal{C}_t} (\mathrm{d}\mathbf{x} \otimes \mathbf{T}_{,t} + \boldsymbol{\Gamma}\mathrm{d}\mathbf{x} \otimes \mathbf{T}), \tag{3.72}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{S}_t} \mathbf{T} \otimes \mathbf{n}\,\mathrm{d}s = \int_{\mathcal{S}_t} [\mathbf{T}_{,t} \otimes \mathbf{n} + (\mathrm{tr}\,\boldsymbol{\Gamma})\mathbf{T} \otimes \mathbf{n} - \mathbf{T} \otimes \boldsymbol{\Gamma}^\mathrm{T}\mathbf{n}]\,\mathrm{d}s, \tag{3.73}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{D}_t} \mathbf{T}\,\mathrm{d}v = \int_{\mathcal{D}_t} [\mathbf{T}_{,t} + (\mathrm{tr}\,\boldsymbol{\Gamma})\mathbf{T}]\,\mathrm{d}v, \tag{3.74}$$

within which $\mathbf{T}$ may be replaced by a vector or scalar with appropriate use of the operator $\otimes$ as necessary.

3.2 Mechanical Balance Equations

The equations that govern the motion of a continuum are based on the balance laws of mass, linear momentum, angular momentum and energy. We now discuss these in turn.

3.2.1 *Mass Conservation*

We denote by $\rho = \rho(\mathbf{x}, t)$ the mass density in the arbitrary region $\mathcal{D}_t$, i.e. the mass per unit volume in $\mathcal{D}_t$ (which should be distinguished from the charge density ρ_e introduced in Chap. 2). The corresponding reference density, defined per unit volume in $\mathcal{D}_\mathrm{r}$, is denoted $\rho_\mathrm{r} = \rho_\mathrm{r}(\mathbf{X}, t)$. If we consider an infinitesimal volume $\mathrm{d}V$ in $\mathcal{D}_\mathrm{r}$ that transforms into $\mathrm{d}v$ in $\mathcal{D}_t$, the mass of the element is unchanged if $\rho \mathrm{d}v = \rho_\mathrm{r} \mathrm{d}V$. Then, by (3.19), we obtain

$$\rho J = \rho_\mathrm{r}, \tag{3.75}$$

which is one form of the *mass conservation equation*. Another (equivalent) form is obtained by taking the material time derivative of (3.75) and then using (3.53), which yields

$$\rho_{,t} + \rho \operatorname{div} \mathbf{v} = 0. \tag{3.76}$$

In the context of fluid dynamics this is known as the *continuity equation*.

3.2.2 *Forces and Momenta*

The forces that act on a solid continuum are of two types, body forces and surface forces. Let the body force density, i.e. the body force per unit mass, be denoted by $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$ at point $\mathbf{x}$ within an arbitrary region $\mathcal{D}_t$, in general also dependent on time t, and let $\mathbf{t} = \mathbf{t}(\mathbf{x}, t; \mathbf{n})$ denote the surface force per unit area at a point $\mathbf{x}$ on the surface $\partial\mathcal{D}_t$ due to the surrounding material and/or applied loads, where $\mathbf{n}$ is the unit outward normal to $\partial\mathcal{D}_t$ and, in general, itself depends on $\mathbf{x}$ and t. The vector $\mathbf{t}$ is called the *stress vector*, and, in accordance with Cauchy's (Augustin-Louis Cauchy, 1789–1857) stress principle (see, e.g., Ogden 1997), it is assumed to depend continuously on $\mathbf{n}$.

The total mechanical force acting on $\mathcal{D}_t$ and its boundary is then written as

$$\mathsf{F}(\mathcal{D}_t) = \int_{\mathcal{D}_t} \rho \mathbf{f}\, \mathrm{d}v + \int_{\partial\mathcal{D}_t} \mathbf{t}\, \mathrm{d}s. \tag{3.77}$$

The corresponding total moment of the forces about an arbitrary origin o is written as

$$\mathbf{G}(\mathcal{D}_t; o) = \int_{\mathcal{D}_t} \rho \mathbf{x} \times \mathbf{f} \, \mathrm{d}v + \int_{\partial \mathcal{D}_t} \mathbf{x} \times \mathbf{t} \, \mathrm{d}s, \tag{3.78}$$

and it is assumed that there are no intrinsic mechanical surface or body couples. When we come to deal with interactions between the mechanics and electric or magnetic fields, then in general a body couple (independent of o) needs to be included in the volume integral in (3.78). In this monograph we shall not consider surface couples other than those generated by surface tractions, but we will consider electric and magnetic body couples in subsequent chapters.

3.2.3 Euler's Laws of Motion

Euler's laws of motion, which together are the counterpart of Newton's Second Law for particles and rigid bodies, are now written as

$$\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} = \mathbf{F}, \quad \frac{\mathrm{d}\mathbf{H}}{\mathrm{d}t} = \mathbf{G}, \tag{3.79}$$

where $\mathbf{M}$ and $\mathbf{H}$ are, respectively, the linear momentum and angular momentum (about o) of the material in $\mathcal{D}_t$, which are defined by

$$\mathbf{M}(\mathcal{D}_t) = \int_{\mathcal{D}_t} \rho \mathbf{v} \, \mathrm{d}v, \tag{3.80}$$

and

$$\mathbf{H}(\mathcal{D}_t; o) = \int_{\mathcal{D}_t} \rho \mathbf{x} \times \mathbf{v} \, \mathrm{d}v. \tag{3.81}$$

Note that the two equations in (3.79) are independent, in contrast to the situation in classical particle and rigid-body mechanics, where $(3.79)_2$ is a consequence of $(3.79)_1$. These are independent of the choice of origin o, although $\mathbf{G}$ and $\mathbf{H}$ do depend on such a choice. The equations in (3.79) are the integral forms of the equations of *linear* and *angular momentum balance*.

By converting the integrals on the left-hand sides of (3.79) to integrals over $\mathcal{D}_\mathrm{r}$ carrying through the differentiation under the integral, using the definitions of velocity and acceleration from (3.39) and (3.40) and then converting back to integrals over $\mathcal{D}_t$, or by using the transport formula (3.74) along with (3.76), we may express the balance laws as

$$\int_{\mathcal{D}_t} \rho(\mathbf{a}-\mathbf{f})\,\mathrm{d}v = \int_{\partial\mathcal{D}_t} \mathbf{t}\,\mathrm{d}s, \tag{3.82}$$

$$\int_{\mathcal{D}_t} \rho\mathbf{x}\times(\mathbf{a}-\mathbf{f})\,\mathrm{d}v = \int_{\partial\mathcal{D}_t} \mathbf{x}\times\mathbf{t}\,\mathrm{d}s, \tag{3.83}$$

where $\mathbf{a}$ is the acceleration.

3.2.3.1 The Cauchy Stress Tensor

The next step is to convert (3.82) and (3.83) into local form. For this purpose we need to make use of *Cauchy's theorem*, which enables $\mathbf{t}$ to be expressed as a linear function of the unit normal $\mathbf{n}$ on any suitably regular surface (for a proof, e.g., see Ogden 1997). We write this in the form

$$\mathbf{t} = \boldsymbol{\sigma}^{\mathrm{T}}\mathbf{n}, \tag{3.84}$$

where $\boldsymbol{\sigma}$ is a second-order tensor *independent of* $\mathbf{n}$, which is called the *Cauchy stress tensor*. It is convenient here to write this with a transpose.

Application of the divergence theorem to (3.82) enables the latter to be written as

$$\int_{\mathcal{D}_t} (\operatorname{div}\boldsymbol{\sigma} + \rho\mathbf{f} - \rho\mathbf{a})\,\mathrm{d}v = \mathbf{0}, \tag{3.85}$$

and since this holds for arbitrary $\mathcal{D}_t$, we deduce, provided the integrand is continuous (which we assume to be the case), that

$$\operatorname{div}\boldsymbol{\sigma} + \rho\mathbf{f} = \rho\mathbf{a}. \tag{3.86}$$

This is the local form of the equation of linear momentum balance, otherwise known as the *equation of motion* for the considered continuum.

Use of (3.86) in (3.83) yields

$$\int_{\mathcal{D}_t} \mathbf{x}\times\operatorname{div}\boldsymbol{\sigma}\,\mathrm{d}v = \int_{\partial\mathcal{D}_t} \mathbf{x}\times(\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{n})\,\mathrm{d}s, \tag{3.87}$$

which may also be written as

$$\int_{\mathcal{D}_t} \boldsymbol{\epsilon}(\mathbf{x}\otimes\operatorname{div}\boldsymbol{\sigma})\,\mathrm{d}v = \int_{\partial\mathcal{D}_t} \boldsymbol{\epsilon}(\mathbf{x}\otimes\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{n})\,\mathrm{d}s, \tag{3.88}$$

where $\boldsymbol{\epsilon}$ is the alternating tensor. Recall that for a pair of vectors $\mathbf{u}$ and $\mathbf{v}$, $\boldsymbol{\epsilon}(\mathbf{u}\otimes\mathbf{v}) = \mathbf{u}\times\mathbf{v}$. For a general second-order tensor $\mathbf{T}$, we define the product

$(\boldsymbol{\epsilon}\mathbf{T})_i = \epsilon_{ijk}T_{jk}$. Application of the divergence theorem to the right-hand side of (3.88) then reduces (3.88) to

$$\int_{\mathcal{D}_t} \boldsymbol{\epsilon}\boldsymbol{\sigma}\,\mathrm{d}v = \mathbf{0}, \tag{3.89}$$

and since this holds for arbitrary $\mathcal{D}_t$, we obtain $\boldsymbol{\epsilon}\boldsymbol{\sigma} = \mathbf{0}$, or equivalently

$$\boldsymbol{\sigma}^{\mathrm{T}} = \boldsymbol{\sigma}, \tag{3.90}$$

i.e. $\boldsymbol{\sigma}$ is symmetric. We emphasize that couple stresses are not included here.

3.2.3.2 The Nominal Stress Tensor

Using Nanson's formula (3.18) the traction on an area element $\mathbf{n}\mathrm{d}s$ in the current configuration can be written as

$$\mathbf{t}\mathrm{d}s = \boldsymbol{\sigma}\mathbf{n}\mathrm{d}s = J\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}\mathbf{N}\mathrm{d}S \equiv \mathbf{S}^{\mathrm{T}}\mathbf{N}\mathrm{d}S,$$

wherein the *nominal stress tensor* $\mathbf{S}$ is defined as

$$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}. \tag{3.91}$$

Its transpose $\mathbf{S}^{\mathrm{T}}$ is the so-called *first Piola–Kirchhoff stress tensor* (Gabrio Piola, 1794–1850; Gustav Robert Kirchhoff, 1824–1887). The nominal stress is a measure of the force *per unit reference area*, while $\boldsymbol{\sigma}$ measures the force *per unit deformed area*. In general, $\mathbf{S}$ is not symmetric but satisfies, by the symmetry of $\boldsymbol{\sigma}$,

$$\mathbf{F}\mathbf{S} = \mathbf{S}^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}. \tag{3.92}$$

The equation of motion (3.86) can also be written in terms of $\mathbf{S}$ by making use of the identity (3.16) with $\mathbf{A} = \mathbf{S}$ and (3.91) to give $J\,\mathrm{div}\,\boldsymbol{\sigma} = \mathrm{Div}\,\mathbf{S}$, which, with $J = \rho_{\mathrm{r}}/\rho$ from (3.75), gives

$$\mathrm{Div}\,\mathbf{S} + \rho_{\mathrm{r}}\mathbf{f} = \rho_{\mathrm{r}}\mathbf{a}. \tag{3.93}$$

3.2.3.3 Examples of Boundary Tractions

Consider a region $\mathcal{D}_t$ with boundary $\partial\mathcal{D}_t$, and let $\mathbf{n}$ be the unit normal to $\partial\mathcal{D}_t$ at point $\mathbf{x}$. Let $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$ be the load per unit area at $\mathbf{x}$ due to the surrounding material, or to an applied load if $\partial\mathcal{D}_t$ is a boundary of a material body. The *normal traction*, i.e. the normal component of the stress vector, is $\mathbf{n}\cdot\mathbf{t} = \mathbf{n}\cdot(\boldsymbol{\sigma}\mathbf{n})$. We may also refer to this as the *normal stress* on the surface $\partial\mathcal{D}_t$. It is tensile (compressive) when positive (negative).

Let $\mathbf{m}$ be a unit tangent vector to the surface at $\mathbf{x}$, so that $\mathbf{m} \cdot \mathbf{n} = 0$. Then the *shear stress* in the direction $\mathbf{m}$ is defined as $\mathbf{m} \cdot \mathbf{t} = \mathbf{m} \cdot (\boldsymbol{\sigma}\mathbf{n})$.

We now take particular note of two types of boundary traction, namely a *pressure load* and a *dead load*. If $P > 0$ denotes the pressure (per unit area of $\partial\mathcal{D}_t$ and independent of $\mathbf{n}$), then $\mathbf{t} = -P\mathbf{n}$ and $\boldsymbol{\sigma} = -P\mathbf{I}$, and by Nanson's formula the corresponding nominal traction (per unit area of $\partial\mathcal{D}_\mathrm{r}$) is given by

$$\mathbf{S}^\mathrm{T}\mathbf{N} = -JP\mathbf{F}^{-\mathrm{T}}\mathbf{N}. \tag{3.94}$$

Clearly, changing geometry affects the pressure load. By contrast, for a dead load the nominal traction $\mathbf{S}^\mathrm{T}\mathbf{N}$ is independent of the deformation and is a function of $\mathbf{X}$.

3.2.4 Energy Balance

The *power* or *rate of working* of the forces acting on the region $\mathcal{D}_t$ and its boundary during its motion is denoted $\mathcal{P}(\mathcal{D}_t)$ and defined by

$$\mathcal{P}(\mathcal{D}_t) = \int_{\mathcal{D}_t} \rho\mathbf{f} \cdot \mathbf{v}\,\mathrm{d}v + \int_{\partial\mathcal{D}_t} \mathbf{t} \cdot \mathbf{v}\,\mathrm{d}s. \tag{3.95}$$

This is partially converted into *kinetic energy*, which is denoted here by $\mathcal{K}(\mathcal{D}_t)$ and defined by

$$\mathcal{K}(\mathcal{D}_t) = \frac{1}{2}\int_{\mathcal{D}_t} \rho\mathbf{v} \cdot \mathbf{v}\,\mathrm{d}v = \frac{1}{2}\int_{\mathcal{D}_\mathrm{r}} \rho_\mathrm{r}\mathbf{v} \cdot \mathbf{v}\,\mathrm{d}V, \tag{3.96}$$

where the latter is obtained by using the connection $\mathrm{d}v = J\mathrm{d}V = \rho_\mathrm{r}\mathrm{d}V/\rho$. By taking the derivative of the final term in (3.96) with respect to t and then converting back to an integral over $\mathcal{D}_t$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}(\mathcal{D}_t) = \int_{\mathcal{D}_t} \rho\mathbf{a} \cdot \mathbf{v}\,\mathrm{d}v, \tag{3.97}$$

where $\mathbf{a} = \mathbf{v}_{,t}$ is the acceleration.

On the use of the equation of motion (3.86), the connection $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$ and the symmetry of $\boldsymbol{\sigma}$, we obtain

$$\mathcal{P}(\mathcal{D}_t) - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}(\mathcal{D}_t) = -\int_{\mathcal{D}_t} \mathbf{v} \cdot (\operatorname{div}\boldsymbol{\sigma})\,\mathrm{d}v + \int_{\partial\mathcal{D}_t} (\boldsymbol{\sigma}\mathbf{v}) \cdot \mathbf{n}\,\mathrm{d}s, \tag{3.98}$$

and then by use of the divergence theorem, we arrive at

$$\mathcal{P}(\mathcal{D}_t) - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}(\mathcal{D}_t) = \int_{\mathcal{D}_t} \mathrm{tr}\,(\boldsymbol{\sigma}\boldsymbol{\Gamma})\,\mathrm{d}v. \tag{3.99}$$

This is the *energy balance equation*, which we now write as

$$\mathcal{P}(\mathcal{D}_t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}(\mathcal{D}_t) + \mathcal{R}(\mathcal{D}_t), \tag{3.100}$$

where

$$\mathcal{R}(\mathcal{D}_t) = \int_{\mathcal{D}_t} \mathrm{tr}\,(\boldsymbol{\sigma}\boldsymbol{\Gamma})\,\mathrm{d}v. \tag{3.101}$$

This states that the work being done by the body and surface forces is converted into changing kinetic energy and $\mathcal{R}(\mathcal{D}_t)$. The latter represents the internal rate of working of the stresses and may be associated with stored (or potential) energy, with energy dissipated in the form of heat or a combination of the two. Note that since $\boldsymbol{\sigma}$ is symmetric, the integrand $\mathrm{tr}\,(\boldsymbol{\sigma}\boldsymbol{\Gamma})$ may also be written as $\mathrm{tr}\,(\boldsymbol{\sigma}\boldsymbol{\Sigma})$, where $\boldsymbol{\Sigma}$ is the symmetric part of $\boldsymbol{\Gamma}$ given by (3.56).

It is instructive to convert (3.101) to an integral over the reference region $\mathcal{D}_\mathrm{r}$. Thus,

$$\mathcal{R}(\mathcal{D}_t) = \int_{\mathcal{D}_\mathrm{r}} J\,\mathrm{tr}\,(\boldsymbol{\sigma}\boldsymbol{\Gamma})\,\mathrm{d}V. \tag{3.102}$$

The integrand in (3.102) is the rate of working of the stresses per unit reference volume. Using (3.48) and (3.91), we obtain

$$J\,\mathrm{tr}\,(\boldsymbol{\sigma}\boldsymbol{\Gamma}) = \mathrm{tr}\,(\mathbf{FS}\boldsymbol{\Gamma}) = \mathrm{tr}\,(\mathbf{S}\boldsymbol{\Gamma}\mathbf{F}) = \mathrm{tr}\,(\mathbf{SF}_{,t}), \tag{3.103}$$

which shows that the density (per unit reference volume) of the stress rate of working is simply $\mathrm{tr}\,(\mathbf{SF}_{,t})$ and the variables $\mathbf{S}$ and $\mathbf{F}$ are said to be *work conjugate*.

Many other pairs of work conjugate variables can be constructed (see, e.g., Ogden 1997). Here we consider one such pair based on the Green strain tensor $\mathbf{e}$ defined in (3.28), from which we obtain $\mathbf{e}_{,t} = (\mathbf{F}^\mathrm{T}\mathbf{F}_{,t} + \mathbf{F}_{,t}{}^\mathrm{T}\mathbf{F})/2$. Then, noting that $\mathbf{SF}^{-\mathrm{T}} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}$ is symmetric, we obtain

$$\mathrm{tr}\,(\mathbf{SF}_{,t}) = \mathrm{tr}\,(\mathbf{SF}^{-\mathrm{T}}\mathbf{F}^\mathrm{T}\mathbf{F}_{,t}) = \mathrm{tr}\,(\mathbf{Pe}_{,t}), \tag{3.104}$$

where we have introduced the notation $\mathbf{P}$ for the so-called *second Piola–Kirchhoff stress tensor*, which is given in terms of the other stress tensors by

$$\mathbf{P} = \mathbf{SF}^{-\mathrm{T}} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}. \tag{3.105}$$

The variables $\mathbf{P}$ and $\mathbf{e}$ constitute a pair of work conjugate stress and strain tensors. Note that the tensor $\mathbf{P}$ is only being used temporarily here and should not be confused with the polarization vector introduced in Sect. 2.4.2 which will be used in later chapters.

3.3 Constitutive Equations for Hyperelastic Solids

3.3.1 Hyperelasticity

A hyperelastic material is one for which the stress rate of working is associated entirely with stored elastic energy, which is described in terms of an *elastic stored energy* density, here denoted $W(\mathbf{F})$, per unit volume in $\mathcal{D}_\mathrm{r}$ such that its material time derivative is

$$W(\mathbf{F})_{,t} = \mathrm{tr}\,(\mathbf{S}\mathbf{F}_{,t}). \tag{3.106}$$

Note that $W(\mathbf{F})$ is also commonly referred to as the *strain energy* or *potential energy* (per unit volume in $\mathcal{D}_\mathrm{r}$) and a hyperelastic material is also referred to as *Green elastic* (George Green, 1793–1841). Then, we have

$$\int_{\mathcal{D}_t} \mathrm{tr}\,(\boldsymbol{\sigma}\boldsymbol{\Gamma})\,\mathrm{d}v = \int_{\mathcal{D}_\mathrm{r}} W(\mathbf{F})_{,t}\,\mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{D}_\mathrm{r}} W(\mathbf{F})\,\mathrm{d}V. \tag{3.107}$$

The energy balance (3.100) can now be written as

$$\mathcal{P}(\mathcal{D}_t) = \frac{\mathrm{d}}{\mathrm{d}t}[\mathcal{K}(\mathcal{D}_t) + \mathcal{W}(\mathcal{D}_t)], \tag{3.108}$$

where

$$\mathcal{W}(\mathcal{D}_t) = \int_{\mathcal{D}_t} J^{-1} W(\mathbf{F})\,\mathrm{d}v = \int_{\mathcal{D}_\mathrm{r}} W(\mathbf{F})\,\mathrm{d}V \tag{3.109}$$

is the total elastic strain energy in the region $\mathcal{D}_t$. We remark that $W(\mathbf{F})$ represents the work done (per unit volume at $\mathbf{X}$) by the stress in deforming the material from $\mathcal{B}_\mathrm{r}$ to $\mathcal{B}_t$ (and the deformation gradient from $\mathbf{I}$ to $\mathbf{F}$) and is independent of the path taken in deformation space.

Since W depends only on $\mathbf{F}$, we have

$$W(\mathbf{F})_{,t} = \mathrm{tr}\left(\frac{\partial W}{\partial \mathbf{F}}\mathbf{F}_{,t}\right), \tag{3.110}$$

and comparison with (3.106) shows that the nominal stress is given by

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}. \tag{3.111}$$

In component form

$$S_{\alpha i} = \left(\frac{\partial W}{\partial \mathbf{F}}\right)_{\alpha i} = \frac{\partial W}{\partial F_{i\alpha}}, \tag{3.112}$$

which identifies our convention for the order of the indices in differentiation with respect to a second-order tensor.

The Cauchy stress tensor is then obtained from (3.91) as

$$\boldsymbol{\sigma} = J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{F}}. \tag{3.113}$$

Thus, for a hyperelastic material both $\mathbf{S}$ and $\boldsymbol{\sigma}$ are functions of $\mathbf{F}$, and it is convenient to write them in the form

$$\mathbf{S} = \mathbf{h}(\mathbf{F}), \quad \boldsymbol{\sigma} = \mathbf{g}(\mathbf{F}), \tag{3.114}$$

where $\mathbf{h}$ and $\mathbf{g}$ are the *response functions* of the material associated with $\mathbf{S}$ and $\boldsymbol{\sigma}$, respectively, *relative to* $\mathcal{B}_\mathrm{r}$, and

$$\mathbf{g}(\mathbf{F}) = J^{-1}\mathbf{F}\mathbf{h}(\mathbf{F}). \tag{3.115}$$

3.3.2 Objectivity

The dependence of W on $\mathbf{F}$ is restricted by both physical and mathematical considerations. An important first consideration is the notion of *objectivity*, also known as the *principle of material frame indifference* (or indifference to observer transformations). For a hyperelastic material a simple way to interpret objectivity is that if the material is subject to an arbitrary rigid motion *after* deformation then its stored energy is unchanged.

A general rigid-body motion superimposed on the motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ is described by

$$\mathbf{x}^* = \mathbf{Q}(t)\mathbf{x} + \mathbf{c}(t), \tag{3.116}$$

where $\mathbf{Q}(t)$ is a proper orthogonal tensor representing a rotation and the vector $\mathbf{c}(t)$ represents a rigid translation. The resulting deformation gradient, $\mathbf{F}^*$ say, is then given by

$$\mathbf{F}^* = \mathbf{Q}\mathbf{F}. \tag{3.117}$$

Objectivity requires that, for any deformation gradient $\mathbf{F}$, W be unaffected by this superimposed motion, i.e.

$$W(\mathbf{F}^*) \equiv W(\mathbf{QF}) = W(\mathbf{F}) \quad \text{for all rotations } \mathbf{Q}. \tag{3.118}$$

It is easy to show that the consequences of objectivity for the response functions $\mathbf{h}$ and $\mathbf{g}$ are

$$\mathbf{S}^* \equiv \mathbf{h}(\mathbf{QF}) = \mathbf{h}(\mathbf{F})\mathbf{Q}^{\mathrm{T}} \equiv \mathbf{SQ}^{\mathrm{T}}, \tag{3.119}$$

and

$$\boldsymbol{\sigma}^* \equiv \mathbf{g}(\mathbf{QF}) = \mathbf{Qg}(\mathbf{F})\mathbf{Q}^{\mathrm{T}} \equiv \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}}, \tag{3.120}$$

for all proper orthogonal $\mathbf{Q}$.

3.3.2.1 Objective Scalar, Vector and Tensor Functions

Consider the Eulerian scalar, vector and (second-order) tensor fields $\phi, \mathbf{u}, \mathbf{T}$ defined on $\mathcal{B}_t$ as functions of $\mathbf{x}$ and t, and suppose $\phi^*, \mathbf{u}^*, \mathbf{T}^*$ are the corresponding fields defined on $\mathcal{B}_t^*$, where $\mathcal{B}_t^*$ is obtained from $\mathcal{B}_t$ by the rigid motion (3.116). Bearing in mind the above discussion of W, $\mathbf{h}$ and $\mathbf{g}$, these fields are said to be *objective* if, for all such motions,

$$\phi^* = \phi, \quad \mathbf{u}^* = \mathbf{Qu} \equiv \mathbf{uQ}^{\mathrm{T}}, \quad \mathbf{T}^* = \mathbf{QTQ}^{\mathrm{T}}. \tag{3.121}$$

Examples of objective fields are the density ρ, the unit normal $\mathbf{n}$ and the Cauchy stress tensor $\boldsymbol{\sigma}$. Examples of non-objective functions are the velocity vector $\mathbf{v}$ and the material time derivative $\boldsymbol{\sigma}_{,t}$ of the Cauchy stress tensor. It is also straightforward to show that for objective fields $\phi, \mathbf{u}, \mathbf{T}$, their derivatives $\operatorname{grad}\phi$, $\operatorname{grad}\mathbf{u}$, $\operatorname{grad}\mathbf{T}$, $\operatorname{div}\mathbf{u}$, $\operatorname{div}\mathbf{T}$, $\operatorname{curl}\mathbf{u}$ and $\operatorname{curl}\mathbf{T}$ are objective.

There is an equivalent definition of objectivity in terms of Lagrangian fields. An objective Lagrangian field is one that is unaffected by the superimposed rotation (3.116). For example, if $\mathbf{u}$ is an objective Eulerian vector field, then the pull back vector fields $\mathbf{F}^{\mathrm{T}}\mathbf{u}$ and $J\mathbf{F}^{-1}\mathbf{u}$ are objective Lagrangian vectors since $(\mathbf{F}^{\mathrm{T}}\mathbf{u})^* = \mathbf{F}^{*\mathrm{T}}\mathbf{u}^* = \mathbf{F}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}\mathbf{Qu} = \mathbf{F}^{\mathrm{T}}\mathbf{u}$, and similarly $(J\mathbf{F}^{-1}\mathbf{u})^* = J\mathbf{F}^{-1}\mathbf{u}$.

In considering the material time derivative, we note that for the objective vector field $\mathbf{u}$, $\mathbf{u}_{,t}$ is not objective, but its Lagrangian counterpart $(\mathbf{F}^{\mathrm{T}}\mathbf{u})_{,t}$ clearly is objective, and hence $\mathbf{u}_{,t} + \boldsymbol{\Gamma}^{\mathrm{T}}\mathbf{u}$ is an objective Eulerian tensor. Similarly, $(\mathbf{F}^{\mathrm{T}}\boldsymbol{\sigma}\mathbf{F})_{,t}$ is objective and hence $\boldsymbol{\sigma}_{,t} + \boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{\sigma} + \boldsymbol{\sigma}\boldsymbol{\Gamma}$ is an objective Eulerian stress rate.

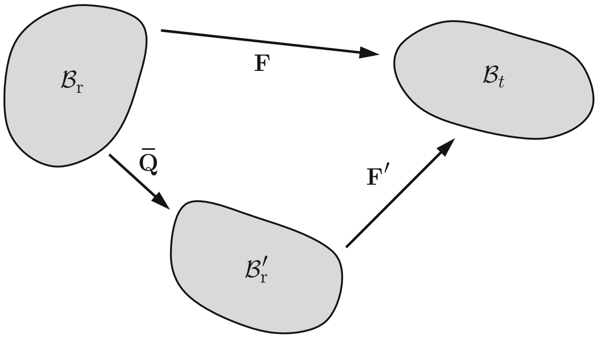

Fig. 3.3 Depiction of the current (deformed) configuration $\mathcal{B}_t$ with deformation gradients $\mathbf{F}$ and $\mathbf{F}'$ relative to reference configurations $\mathcal{B}_r$ and $\mathcal{B}'_r$, respectively, which are themselves connected by the deformation gradient $\bar{\mathbf{Q}}$

3.3.3 *Material Symmetry*

Some materials exhibit an intrinsic symmetry in their undeformed configuration which can be characterized by considering the effect of a change of reference configuration. Let us consider a reference configuration $\mathcal{B}_r$ and a second reference configuration $\mathcal{B}'_r$, so that the position vectors of a material particle in these configurations are $\mathbf{X}$ and $\mathbf{X}'$, respectively. Let $\bar{\mathbf{Q}} = \mathrm{Grad}\,\mathbf{X}'$ be the deformation gradient relating $\mathcal{B}'_r$ to $\mathcal{B}_r$. Consider also a deformed (or current) configuration $\mathcal{B}_t$ such that the deformation gradients relative to $\mathcal{B}_r$ and $\mathcal{B}'_r$ are $\mathbf{F}$ and $\mathbf{F}'$, respectively, i.e. $\mathbf{F} = \mathrm{Grad}\,\mathbf{x}$ and $\mathbf{F}' = \mathrm{Grad}'\mathbf{x}$, where Grad' is the gradient with respect to $\mathbf{X}'$ (see Fig. 3.3).

Then it is easy to see that

$$\mathbf{F} = \mathbf{F}'\bar{\mathbf{Q}}. \tag{3.122}$$

In general, the strain-energy function relative to $\mathcal{B}'_r$, W' say, will be different from that relative to $\mathcal{B}_r$, i.e. W. For certain changes of reference configuration, however, i.e. for certain $\bar{\mathbf{Q}}$, we may have $W' = W$, in which case

$$W(\mathbf{F}') = W(\mathbf{F}) \tag{3.123}$$

for all deformation gradients $\mathbf{F}$ and for all such $\bar{\mathbf{Q}}$ such that (3.123) holds with $\mathbf{F} = \mathbf{F}'\bar{\mathbf{Q}}$.

The set of tensors $\bar{\mathbf{Q}}$ for which (3.123) holds with (3.122) defines the *symmetry of the material relative to* $\mathcal{B}_r$, and the larger the set is, the more symmetry the material possesses. The set of $\bar{\mathbf{Q}}$ forms a multiplicative group, which is referred to as the *symmetry group of the material relative to* $\mathcal{B}_r$.

Here we shall only consider those $\bar{\mathbf{Q}}$ that correspond to rotations. Then, the material symmetry relative to $\mathcal{B}_\mathrm{r}$ corresponds to the set of rotations $\bar{\mathbf{Q}}$ such that, for any deformation gradient $\mathbf{F}$,

$$W(\mathbf{F}\bar{\mathbf{Q}}^\mathrm{T}) = W(\mathbf{F}). \tag{3.124}$$

Consequences of this symmetry for the response functions $\mathbf{h}$ and $\mathbf{g}$ are that

$$\mathbf{h}(\mathbf{F}\bar{\mathbf{Q}}^\mathrm{T}) = \bar{\mathbf{Q}}\mathbf{h}(\mathbf{F}), \quad \mathbf{g}(\mathbf{F}\bar{\mathbf{Q}}^\mathrm{T}) = \mathbf{g}(\mathbf{F}) \tag{3.125}$$

for all rotations $\bar{\mathbf{Q}}$ in the symmetry group. In particular, the Cauchy stress $\boldsymbol{\sigma} = \mathbf{g}(\mathbf{F})$ is unaffected by a change of reference configuration corresponding to a member of the symmetry group.

3.3.3.1 Example: Isotropy

An important example of symmetry is that of *isotropy*, for which (3.124) holds for *arbitrary* rotations $\bar{\mathbf{Q}}$. In this case the material is said to be *isotropic relative to* $\mathcal{B}_\mathrm{r}$.

We now combine the requirements of objectivity and isotropy, i.e.

$$W(\mathbf{Q}\mathbf{F}) = W(\mathbf{F}), \quad W(\mathbf{F}\bar{\mathbf{Q}}^\mathrm{T}) = W(\mathbf{F}), \tag{3.126}$$

for all rotations $\bar{\mathbf{Q}}$ and $\mathbf{Q}$. We emphasize that $\mathbf{Q}$ is a rotation in $\mathcal{B}_t$, while $\bar{\mathbf{Q}}$ is a rotation in $\mathcal{B}_\mathrm{r}$. Using the polar decomposition $\mathbf{F} = \mathbf{V}\mathbf{R}$ with $\bar{\mathbf{Q}} = \mathbf{R}$ we obtain $W(\mathbf{F}) = W(\mathbf{V})$, and then we consider $W(\mathbf{Q}\mathbf{F}\bar{\mathbf{Q}}^\mathrm{T}) = W(\mathbf{F}\bar{\mathbf{Q}}^\mathrm{T}) = W(\mathbf{F})$ by objectivity and isotropy in turn. By choosing $\bar{\mathbf{Q}} = \mathbf{Q}\mathbf{R}$, it then follows that

$$W(\mathbf{Q}\mathbf{V}\mathbf{Q}^\mathrm{T}) = W(\mathbf{V}) \quad \text{for all rotations } \mathbf{Q}. \tag{3.127}$$

This is a statement that W is an *isotropic function* of $\mathbf{V}$. It may be expressed equivalently using the polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U}$ as

$$W(\bar{\mathbf{Q}}\mathbf{U}\bar{\mathbf{Q}}^\mathrm{T}) = W(\mathbf{U}) \quad \text{for all rotations } \bar{\mathbf{Q}}. \tag{3.128}$$

One consequence of isotropy is that W depends on $\mathbf{V}$ only through its invariants. In particular, we may consider the principal invariants of $\mathbf{V}$, which are defined by

$$i_1 = \mathrm{tr}\,\mathbf{V}, \quad i_2 = \frac{1}{2}[i_1^2 - \mathrm{tr}\,(\mathbf{V}^2)], \quad i_3 = \det\mathbf{V} = J. \tag{3.129}$$

These are the coefficients in the Cayley–Hamilton theorem for $\mathbf{V}$, which states that

$$\mathbf{V}^3 - i_1\mathbf{V}^2 + i_2\mathbf{V} - i_3\mathbf{I} = \mathbf{O}, \tag{3.130}$$

where again $\mathbf{I}$ is the second-order identity tensor and $\mathbf{O}$ is the zero tensor. They may be expressed in terms of the principal stretches as

$$i_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad i_2 = \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2, \quad i_3 = \lambda_1\lambda_2\lambda_3. \tag{3.131}$$

For a general discussion of invariants, see Spencer (1971).

Thus, W depends on only three scalar measures of deformation. We can use the principal invariants i_1, i_2, i_3 as its arguments or any other set of three independent invariants, or, equivalently, as a symmetric function of the principal stretches $\lambda_1, \lambda_2, \lambda_3$.

The principal invariants of $\mathbf{b} = \mathbf{V}^2$, equivalently of $\mathbf{c} = \mathbf{U}^2$, are denoted I_1, I_2, I_3 and are defined by

$$I_1 = \mathrm{tr}\,\mathbf{c}, \quad I_2 = \frac{1}{2}[I_1^2 - \mathrm{tr}\,(\mathbf{c}^2)], \quad I_3 = \det\mathbf{c} = J^2. \tag{3.132}$$

In terms of the principal stretches we have

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2, \quad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{3.133}$$

Let us consider W to be a function of I_1, I_2, I_3, which we write as $\bar{W}(I_1, I_2, I_3)$. Then we can express the nominal stress tensor in the form

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}} = \sum_{i=1}^{3} \frac{\partial \bar{W}}{\partial I_i}\frac{\partial I_i}{\partial \mathbf{F}}. \tag{3.134}$$

From the definitions (3.132) we calculate

$$\frac{\partial I_1}{\partial \mathbf{F}} = 2\mathbf{F}^{\mathrm{T}}, \quad \frac{\partial I_2}{\partial \mathbf{F}} = 2(I_1\mathbf{F}^{\mathrm{T}} - \mathbf{F}^{\mathrm{T}}\mathbf{b}), \quad \frac{\partial I_3}{\partial \mathbf{F}} = 2I_3\mathbf{F}^{-1}, \tag{3.135}$$

and substitution into (3.134) yields

$$\mathbf{S} = 2\bar{W}_1\mathbf{F}^{\mathrm{T}} + 2\bar{W}_2(I_1\mathbf{F}^{\mathrm{T}} - \mathbf{F}^{\mathrm{T}}\mathbf{b}) + 2\bar{W}_3 I_3\mathbf{F}^{-1}, \tag{3.136}$$

where $\bar{W}_i = \partial\bar{W}/\partial I_i$, $i = 1, 2, 3$.

The corresponding Cauchy stress is then obtained as

$$\boldsymbol{\sigma} = 2J^{-1}[\bar{W}_1\mathbf{b} + \bar{W}_2(I_1\mathbf{b} - \mathbf{b}^2) + I_3\bar{W}_3\mathbf{I}]. \tag{3.137}$$

Note, in particular, that $\boldsymbol{\sigma}$ is coaxial with $\mathbf{b}$, which is another consequence of isotropy.

It follows that $\boldsymbol{\sigma} = \mathbf{g}(\mathbf{V})$ and $\mathbf{V}^{-1}\boldsymbol{\sigma} = \boldsymbol{\sigma}\mathbf{V}^{-1}$, and since $\mathbf{V}^{-1}\mathbf{g}(\mathbf{V}) = J\mathbf{h}(\mathbf{V})$, we deduce that $\mathbf{h}(\mathbf{V})$ is symmetric. Similarly, $\mathbf{h}(\mathbf{U})$ is symmetric and it follows that

$$\mathbf{h}(\mathbf{F}) = \mathbf{R}^{\mathrm{T}}\mathbf{h}(\mathbf{V}) = \mathbf{h}(\mathbf{U})\mathbf{R}^{\mathrm{T}}. \tag{3.138}$$

This symmetry is a consequence of isotropy, and if the material is not isotropic, then neither $\mathbf{h}(\mathbf{U})$ nor $\mathbf{h}(\mathbf{V})$ is in general symmetric.

If we consider W as a function of $\lambda_1, \lambda_2, \lambda_3$, which we write as $\tilde{W}(\lambda_1, \lambda_2, \lambda_3)$, and form the derivatives $\lambda_i \partial\tilde{W}/\partial\lambda_i$, $i = 1, 2, 3$, then we find that

$$\lambda_i \frac{\partial \tilde{W}}{\partial \lambda_i} = 2\bar{W}_1\lambda_i^2 + 2\bar{W}_2(I_1\lambda_i^2 - \lambda_i^4) + 2I_3\bar{W}_3, \tag{3.139}$$

which are precisely the principal components of $J\boldsymbol{\sigma}$ in (3.137). We denote the principal Cauchy stresses by σ_i, $i = 1, 2, 3$, and then we have the simple relations

$$\sigma_i = J^{-1}\lambda_i \frac{\partial \tilde{W}}{\partial \lambda_i}, \quad i = 1, 2, 3, \quad (\text{no summation over } i), \tag{3.140}$$

where now

$$J = \lambda_1\lambda_2\lambda_3. \tag{3.141}$$

The Cauchy stress may be expressed in the spectral form

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}, \tag{3.142}$$

and the nominal stress in the form

$$\mathbf{S} = \sum_{i=1}^{3} s_i \mathbf{u}^{(i)} \otimes \mathbf{v}^{(i)}, \tag{3.143}$$

where we recall that $\mathbf{u}^{(i)}$ and $\mathbf{v}^{(i)}$ are the unit eigenvectors of $\mathbf{U}$ and $\mathbf{V}$, respectively, and we note that this introduces the notation s_i, which is defined by

$$s_i = \frac{\partial \tilde{W}}{\partial \lambda_i}, \quad i = 1, 2, 3. \tag{3.144}$$

It is usual to set $W(\mathbf{I}) = 0$, which corresponds to $\bar{W}(3, 3, 1) = 0$ when W is considered as a function of I_1, I_2, I_3 or $\tilde{W}(1, 1, 1) = 0$ when a function of $\lambda_1, \lambda_2, \lambda_3$. If the reference configuration is stress-free, then we also have

$$\frac{\partial W}{\partial \mathbf{F}}(\mathbf{I}) = \mathbf{O}, \quad \bar{W}_1(3, 3, 1) + 2\bar{W}_2(3, 3, 1) + \bar{W}_3(3, 3, 1) = 0, \tag{3.145}$$

and

$$\tilde{W}_i(1, 1, 1) = 0, \quad i = 1, 2, 3, \tag{3.146}$$

for these three representations, respectively, where $\tilde{W}_i = \partial\tilde{W}/\partial\lambda_i$.

For the classical linear theory of isotropic elasticity, $\tilde{W}$ takes on the special form

$$\tilde{W}(\lambda_1, \lambda_2, \lambda_3) = \mu(e_1^2 + e_2^2 + e_3^2) + \frac{1}{2}\lambda(e_1 + e_2 + e_3)^2, \tag{3.147}$$

where e_1, e_2, e_3 are the principal components of the infinitesimal strain tensor and may be written as $e_i = \lambda_i - 1$, $i = 1, 2, 3$, and the constants λ and μ are the Lamé moduli. From this it is easy to see that

$$\tilde{W}_{ii} = \lambda + 2\mu, \quad \tilde{W}_{ij} = \lambda, \ i \neq j, \ i = 1, 2, 3, \tag{3.148}$$

where $\tilde{W}_{ij} = \partial^2\tilde{W}/\partial\lambda_i\partial\lambda_j$. These conditions must be satisfied by any isotropic strain-energy function $\tilde{W}(\lambda_1, \lambda_2, \lambda_3)$ when $\lambda_1 = \lambda_2 = \lambda_3 = 1$ for consistency with the classical theory.

3.3.3.2 Example: Transverse Isotropy

Another important example is that associated with fibre-reinforced materials, where the fibre direction is specified locally by a unit vector in the reference configuration. We denote this vector by $\mathbf{M}$, which is also referred to as a *preferred direction* in the reference configuration and should not be confused with the magnetization vector $\mathbf{M}$ introduced in Sect. 2.4.2, which is not used in this chapter. Without the preferred direction, the material would be isotropic relative to $\mathcal{B}_{\mathrm{r}}$. In general $\mathbf{M}$ varies with position $\mathbf{X}$ and is a vector field which, when the strain-energy function is endowed with suitable properties, can be regarded as modelling the fibres as a continuous distribution. The material response is therefore indifferent to arbitrary rotations about the direction $\mathbf{M}$. Also, it is usual to make no physical distinction between the directions $\mathbf{M}$ and $-\mathbf{M}$. Thus, the response must also be unaffected by interchange of $\mathbf{M}$ and $-\mathbf{M}$ and can be represented in terms of $\mathbf{M} \otimes \mathbf{M}$.

The strain energy $W(\mathbf{F})$ must satisfy $W(\mathbf{FQ}) = W(\mathbf{F})$ for all rotations $\mathbf{Q}$ (in the reference configuration) such that $\mathbf{QM} = \pm\mathbf{M}$. Equivalently, such a material can be characterized by a strain energy that is an isotropic function of $\mathbf{F}$ and the tensor $\mathbf{M} \otimes \mathbf{M}$ jointly. Since, by objectivity, W depends on $\mathbf{F}$ only through the right stretch tensor $\mathbf{U}$ (or, equivalently, $\mathbf{c} = \mathbf{U}^2$), this means that, on writing the dependence as $W(\mathbf{c}, \mathbf{M} \otimes \mathbf{M})$, we must have

$$W(\mathbf{QcQ}^{\mathrm{T}}, \mathbf{QM} \otimes \mathbf{QM}) = W(\mathbf{c}, \mathbf{M} \otimes \mathbf{M}) \quad \text{for all proper orthogonal } \mathbf{Q}. \tag{3.149}$$

The requirement (3.149) implies that W depends on five invariants, namely the principal invariants I_1, I_2, I_3 of $\mathbf{c}$, defined by (3.132), together with two independent invariants that depend on $\mathbf{M}$. It is usual to take these as

$$I_4 = \mathbf{M} \cdot (\mathbf{cM}), \quad I_5 = \mathbf{M} \cdot (\mathbf{c}^2\mathbf{M}). \tag{3.150}$$

Note that I_4 has a direct kinematical interpretation since, in accordance with (3.26), $\sqrt{I_4}$ represents the stretch in the direction $\mathbf{M}$. In general, however, there is no immediate simple interpretation for I_5, but see Merodio and Ogden (2002) and Holzapfel and Ogden (2010) for an alternative choice of invariant that does have such an interpretation based on Nanson's formula (3.18). We use the notation

$$\bar{W}(I_1, I_2, I_3, I_4, I_5) \tag{3.151}$$

to represent the strain energy when treated as a function of the invariants noted above.

In order to calculate the stresses we require the derivatives

$$\frac{\partial I_4}{\partial \mathbf{F}} = 2\mathbf{M} \otimes \mathbf{FM}, \quad \frac{\partial I_5}{\partial \mathbf{F}} = 2(\mathbf{M} \otimes \mathbf{FcM} + \mathbf{cM} \otimes \mathbf{FM}), \tag{3.152}$$

together with the derivatives of I_1, I_2, I_3 given by (3.135). The resulting nominal stress tensor is given by

$$\begin{aligned}\mathbf{S} = {} & 2\bar{W}_1\mathbf{F}^{\mathrm{T}} + 2\bar{W}_2(I_1\mathbf{I} - \mathbf{c})\mathbf{F}^{\mathrm{T}} + 2I_3\bar{W}_3\mathbf{F}^{-1} + 2\bar{W}_4\mathbf{M} \otimes \mathbf{FM} \\ & + 2\bar{W}_5(\mathbf{M} \otimes \mathbf{FcM} + \mathbf{cM} \otimes \mathbf{FM}),\end{aligned} \tag{3.153}$$

where $\bar{W}_i = \partial\bar{W}/\partial I_i, i = 1, \ldots, 5$. The result for an isotropic material is recovered by omitting the terms in $\bar{W}_4$ and $\bar{W}_5$. Equation (3.153) gives the stress in a fibre-reinforced material for which the fibre direction corresponds to $\mathbf{M}$ locally in the reference configuration. The Cauchy stress can be calculated from (3.153) using the general formula $J\boldsymbol{\sigma} = \mathbf{FS}$. This gives

$$\boldsymbol{\sigma} = 2J^{-1}[\bar{W}_1\mathbf{b} + \bar{W}_2(I_1\mathbf{b} - \mathbf{b}^2) + I_3\bar{W}_3\mathbf{I} + \bar{W}_4\mathbf{m} \otimes \mathbf{m} + \bar{W}_5(\mathbf{m} \otimes \mathbf{bm} + \mathbf{bm} \otimes \mathbf{m})], \tag{3.154}$$

where $\mathbf{m} = \mathbf{FM}$.

In the reference configuration we take the energy to vanish, so that $\bar{W}(3, 3, 1, 1, 1) = 0$, and in a stress-free reference configuration

$$(\bar{W}_1 + 2\bar{W}_2 + \bar{W}_3)\mathbf{I} + (\bar{W}_4 + 2\bar{W}_5)\mathbf{M} \otimes \mathbf{M} = \mathbf{O}, \tag{3.155}$$

from which it follows that

$$\bar{W}_1 + 2\bar{W}_2 + \bar{W}_3 = 0, \quad \bar{W}_4 + 2\bar{W}_5 = 0 \tag{3.156}$$

evaluated for $I_1 = I_2 = 3$ and $I_3 = I_4 = I_5 = 1$.

These expressions need to be modified when there is a residual stress in the reference configuration. For details of the relevant conditions, we refer to Shams et al. (2011) and Ogden and Singh (2011) and for earlier analysis of residual stress to the papers by Hoger (1985, 1986, 1993), for example.

3.3.4 Incompressible Materials

Some important classes of materials can be considered to be incompressible to a very good approximation. These include rubberlike materials and electroactive and magnetoactive elastomers. In this section we therefore consider the modifications in the constitutive law required to accommodate incompressibility.

For an *incompressible* material the deformation gradient must satisfy the *internal constraint*

$$J \equiv \det \mathbf{F} = 1 \tag{3.157}$$

at each point of the material. From (3.52) it follows that

$$\mathrm{tr}\,(\mathbf{F}^{-1}\mathbf{F}_{,t}) = 0. \tag{3.158}$$

Now consider the relation (3.106), which we write as $W_{,t} = \mathrm{tr}\,(\mathbf{S}\mathbf{F}_{,t})$, and from which the nominal stress was obtained as (3.111), i.e. $\mathbf{S} = \partial W/\partial \mathbf{F}$. When the constraint holds the components of $\mathbf{F}$ are no longer independent and the latter expression for $\mathbf{S}$ requires modification. For this purpose we note that in view of (3.158) we may now write

$$\mathrm{tr}\,(\mathbf{S}\mathbf{F}_{,t}) = W_{,t} = W_{,t} - p\,\mathrm{tr}\,(\mathbf{F}^{-1}\mathbf{F}_{,t}), \tag{3.159}$$

where p is an arbitrary scalar (in general a function of $\mathbf{x}$). This is a Lagrange multiplier introduced to take care of the constraint so that the derivatives of W with respect to the components of $\mathbf{F}$ can be formed independently of the constraint. With this additional variable the components of $\mathbf{F}_{,t}$ are also considered independent and hence

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}} - p\,\mathbf{F}^{-1}, \tag{3.160}$$

which is the required modified expression for $\mathbf{S}$ for an incompressible material. Then, from (3.113) with $J = 1$, the Cauchy stress $\boldsymbol{\sigma}$ is given by

$$\boldsymbol{\sigma} = \mathbf{F}\frac{\partial W}{\partial \mathbf{F}} - p\,\mathbf{I}, \tag{3.161}$$

which shows that p may be interpreted as a hydrostatic pressure (recall Sect. 3.2.3.3).

The above relations apply independently of material symmetry. For the special case of isotropy they can be specialized in terms of the principal stretches, as follows. From (3.24) and (3.143) we have

$$\mathbf{F} = \sum_{i=1}^{3} \lambda_i \mathbf{v}^{(i)} \otimes \mathbf{u}^{(i)}, \quad \mathbf{S} = \sum_{i=1}^{3} s_i \mathbf{u}^{(i)} \otimes \mathbf{v}^{(i)}, \tag{3.162}$$

respectively, and it can then be shown that $W_{,t} = \mathrm{tr}\,(\mathbf{SF}_{,t}) = \sum_{i=1}^{3} s_i\lambda_{i,t}$. Noting that in terms of the stretches, the incompressibility constraint can be written as $\lambda_1\lambda_2\lambda_3 = 1$, it follows that

$$\sum_{i=1}^{3} s_i\lambda_{i,t} = \tilde{W}_{,t} - p\sum_{i=1}^{3}\lambda_i^{-1}\lambda_{i,t}, \tag{3.163}$$

and hence

$$s_i = \tilde{W}_i - p\lambda_i^{-1}, \quad i = 1, 2, 3, \tag{3.164}$$

which is the specialization of (3.160) to principal axes. The corresponding specialization of (3.161) is obtained similarly as

$$\sigma_i = \lambda_i s_i = \lambda_i \tilde{W}_i - p, \quad i = 1, 2, 3;\ \text{no summation.} \tag{3.165}$$

Since $I_3 = 1$ for an incompressible material, in terms of the invariants I_1, I_2 we have $\bar{W}(I_1, I_2)$ and

$$\boldsymbol{\sigma} = 2\bar{W}_1\mathbf{b} + 2\bar{W}_2(I_1\mathbf{b} - \mathbf{b}^2) - p\mathbf{I}. \tag{3.166}$$

The counterpart of this for a transversely isotropic material, for which we write $\bar{W}(I_1, I_2, I_4, I_5)$, is

$$\boldsymbol{\sigma} = 2\bar{W}_1\mathbf{b}+2\bar{W}_2(I_1\mathbf{b}-\mathbf{b}^2)+2\bar{W}_4\mathbf{m}\otimes\mathbf{m}+2\bar{W}_5(\mathbf{m}\otimes\mathbf{bm}+\mathbf{bm}\otimes\mathbf{m})-p\mathbf{I}. \tag{3.167}$$

If the reference configuration is stress-free, then instead of (3.156), we have

$$2\bar{W}_1 + 4\bar{W}_2 - p_{\mathrm{r}} = 0, \quad \bar{W}_4 + 2\bar{W}_5 = 0 \tag{3.168}$$

evaluated for $I_1 = I_2 = 3$ and $I_4 = I_5 = 1$, where p_{r} is the value of p in the reference configuration. For an isotropic material only the first of these is relevant.

Note that by means of the Cayley–Hamilton theorem, the term $I_1\mathbf{b} - \mathbf{b}^2$ may also be written as $I_2\mathbf{I} - \mathbf{b}^{-1}$ and the term in $\mathbf{I}$ absorbed into p.

3.3.4.1 Examples of Strain-Energy Functions

Many different strain-energy functions are available in the literature to model the behaviour of rubberlike solids and other materials. Here we provide a limited number of examples for incompressible elastic materials.

A basic isotropic strain-energy function, known as the *neo-Hookean* material, has the form

$$\bar{W} = \frac{1}{2}\mu(I_1 - 3), \tag{3.169}$$

where μ (> 0) is a material constant referred to as the *shear modulus* of the material in the reference configuration. This is a prototype model for rubber elasticity. The associated Cauchy and nominal stresses are given by

$$\boldsymbol{\sigma} = \mu \mathbf{b} - p\mathbf{I}, \quad \mathbf{S} = \mu \mathbf{F}^{\mathrm{T}} - p\mathbf{F}^{-1}, \tag{3.170}$$

respectively. A slightly more general model is the *Mooney–Rivlin* material, defined by

$$\bar{W} = \frac{1}{2}\mu_1(I_1 - 3) - \frac{1}{2}\mu_2(I_2 - 3), \tag{3.171}$$

where μ_1 (≥ 0) and μ_2 (≤ 0) are constants such that $\mu_1 - \mu_2 = \mu$ (> 0). The Cauchy stress can be calculated from (3.166) as

$$\boldsymbol{\sigma} = \mu_1 \mathbf{b} - \mu_2(I_1\mathbf{b} - \mathbf{b}^2) - p\mathbf{I}. \tag{3.172}$$

A simple model for transverse isotropy, known as the *standard reinforcing model* (see, e.g., Qiu and Pence 1997), is

$$\bar{W} = \frac{1}{2}\mu(I_1 - 3) + \frac{1}{2}\mu\gamma(I_4 - 1)^2, \tag{3.173}$$

where μ is again the shear modulus in the reference configuration (in the plane normal to the fibre direction) and $\gamma > 0$ is a parameter that reflects the stiffness of the fibre reinforcement. Note that $2\bar{W}_1 = \mu$ and, consistently with $(3.168)_2$, $\bar{W}_4 = 0$ in the reference configuration.

The Cauchy stress is given by

$$\boldsymbol{\sigma} = \mu \mathbf{b} + \mu\gamma(I_4 - 1)\mathbf{m} \otimes \mathbf{m} - p\mathbf{I}. \tag{3.174}$$

3.3.4.2 Some Homogeneous Deformations of Incompressible Materials

We recall from Sect. 3.1.3.3 that for a homogeneous deformation, the deformation gradient $\mathbf{F}$ is independent of position. An important example of this is the so-called *pure homogeneous strain* defined in (3.32) which we now write as

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3, \tag{3.175}$$

where the principal stretches $\lambda_1, \lambda_2, \lambda_3$ are constants and (X_1, X_2, X_3) and (x_1, x_2, x_3) are Cartesian coordinates, each associated with the same (fixed) principal axes $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \mathbf{u}^{(3)}$.

For an isotropic elastic material $W = \tilde{W}(\lambda_1, \lambda_2, \lambda_3)$, with $\lambda_1\lambda_2\lambda_3 = 1$ for an incompressible material, on which we now focus. The principal Cauchy stresses are, from (3.165), $\sigma_i = \lambda_i \tilde{W}_i - p$, $i = 1, 2, 3$. Since only two of the stretches are independent, it is useful in some applications to consider $\tilde{W}$ as a function of just two stretches. We therefore define

$$\hat{W}(\lambda_1, \lambda_2) = \tilde{W}(\lambda_1, \lambda_2, \lambda_1^{-1}\lambda_2^{-1}). \tag{3.176}$$

Then

$$\sigma_1 - \sigma_3 = \lambda_1 \frac{\partial \hat{W}}{\partial \lambda_1}, \quad \sigma_2 - \sigma_3 = \lambda_2 \frac{\partial \hat{W}}{\partial \lambda_2}, \tag{3.177}$$

and we note that p no longer appears explicitly. The conditions that $\hat{W}$ has to satisfy in an unstressed reference configuration for consistency with the classical linear theory of incompressible isotropic elasticity are

$$\hat{W}(1,1) = 0, \quad \hat{W}_1 = \hat{W}_2 = 0, \quad \hat{W}_{12} = 2\mu, \quad \hat{W}_{11} = \hat{W}_{22} = 4\mu, \tag{3.178}$$

where again μ is the shear modulus.

We next consider the pure homogeneous strain of an incompressible transversely isotropic elastic material. We take the direction $\mathbf{M}$ of transverse isotropy in the reference configuration to lie in the (X_1, X_2) principal plane with components $(\cos\varphi, \sin\varphi, 0)$. Then, the invariants I_4 and I_5 defined by (3.150) are given by

$$I_4 = \lambda_1^2 \cos^2\varphi + \lambda_2^2 \sin^2\varphi, \quad I_5 = \lambda_1^4 \cos^2\varphi + \lambda_2^4 \sin^2\varphi, \tag{3.179}$$

and, in terms of the (independent) stretches λ_1 and λ_2, I_1 and I_2 in (3.133) become

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2}, \quad I_2 = \lambda_1^2\lambda_2^2 + \lambda_1^{-2} + \lambda_2^{-2}. \tag{3.180}$$

From (3.167) the components of $\boldsymbol{\sigma}$ are calculated as

$$\begin{aligned}\sigma_{11} = &-p + 2\bar{W}_1\lambda_1^2 + 2\bar{W}_2\lambda_1^2(\lambda_2^2 + \lambda_3^2)\\ &+ 2\bar{W}_4\lambda_1^2\cos^2\varphi + 4\bar{W}_5\lambda_1^4\cos^2\varphi,\end{aligned} \tag{3.181}$$

$$\begin{aligned}\sigma_{22} = &-p + 2\bar{W}_1\lambda_2^2 + 2\bar{W}_2\lambda_2^2(\lambda_1^2 + \lambda_3^2)\\ &+ 2\bar{W}_4\lambda_2^2\sin^2\varphi + 4\bar{W}_5\lambda_2^4\sin^2\varphi,\end{aligned} \tag{3.182}$$

$$\sigma_{12} = 2[\bar{W}_4 + \bar{W}_5(\lambda_1^2 + \lambda_2^2)]\lambda_1\lambda_2 \sin\varphi\cos\varphi, \tag{3.183}$$

$$\sigma_{33} = -p + 2\bar{W}_1\lambda_3^2 + 2\bar{W}_2\lambda_3^2(\lambda_1^2 + \lambda_2^2), \quad \sigma_{13} = \sigma_{23} = 0. \tag{3.184}$$

Note that σ_{12} does not in general vanish, unlike the situation for an isotropic material. This means that (as a result of lack of symmetry) shear stress is required to maintain the pure homogeneous strain in this case, and it vanishes only if the preferred direction is along one of the coordinate axes. This illustrates the fact that the principal axes of $\boldsymbol{\sigma}$ do not in general coincide with the Eulerian principal axes (which, here, are the coordinate axes).

From (3.181), (3.182) and (3.184), we obtain

$$\sigma_{11} - \sigma_{33} = 2\lambda_1^{-2}\lambda_2^{-2}(\lambda_1^4\lambda_2^2 - 1)(\bar{W}_1 + \lambda_2^2\bar{W}_2) + 2\bar{W}_4\lambda_1^2\cos^2\varphi + 4\bar{W}_5\lambda_1^4\cos^2\varphi, \tag{3.185}$$

$$\sigma_{22} - \sigma_{33} = 2\lambda_1^{-2}\lambda_2^{-2}(\lambda_1^2\lambda_2^4 - 1)(\bar{W}_1 + \lambda_1^2\bar{W}_2) + 2\bar{W}_4\lambda_2^2\sin^2\varphi + 4\bar{W}_5\lambda_2^4\sin^2\varphi. \tag{3.186}$$

Equations (3.179) and (3.180) show that I_1, I_2, I_4, I_5, and hence the strain energy, depend only on λ_1, λ_2 and the angle φ. We express this dependence by extending the notation $\hat{W}$ defined in (3.176) to the present situation. Thus, we define

$$\hat{W}(\lambda_1, \lambda_2, \varphi) = \bar{W}(I_1, I_2, I_4, I_5), \tag{3.187}$$

where φ, a material parameter, is included in the argument to emphasize the dependence on the preferred direction. It is important to note that, in general, in contrast to the isotropic situation, $\hat{W}(\lambda_1, \lambda_2, \varphi)$ is *not symmetric* in λ_1 and λ_2. It is, however, easy to show that (3.185) and (3.186) may be written in the simple forms as

$$\sigma_{11} - \sigma_{33} = \lambda_1 \frac{\partial \hat{W}}{\partial \lambda_1}, \quad \sigma_{22} - \sigma_{33} = \lambda_2 \frac{\partial \hat{W}}{\partial \lambda_2}. \tag{3.188}$$

The formulas in (3.188) are identical in form to the corresponding formulas (3.177) in the isotropic case, except that here σ_{11} and σ_{22} are *not* in general principal stresses since the shear stress σ_{12} does not in general vanish.

Plane Strain It is interesting to examine the simplifications that arise in the case of a plane deformation. We consider a plane deformation in which $\lambda_3 = 1$. It then follows that $\lambda_1\lambda_2 = 1$ and from (3.179) and (3.180) that

$$I_2 = I_1, \quad I_5 = (I_1 - 1)I_4 - 1. \tag{3.189}$$

Thus, we may regard the energy as a function of just two independent invariants, such as I_1 and I_4, and we write

$$\bar{\bar{W}}(I_1, I_4) = \bar{W}\left(I_1, I_1, I_4, I_1 I_4 - I_4 - 1\right). \tag{3.190}$$

It can then be shown that the in-plane Cauchy stress is given simply by

$$\boldsymbol{\sigma} = 2\bar{\bar{W}}_1\mathbf{b} + 2\bar{\bar{W}}_4\mathbf{FM} \otimes \mathbf{FM} - p\mathbf{I}, \tag{3.191}$$

wherein all vectors and tensors are restricted to the $(1,2)$ plane. This should be compared with (3.167) but note that p is in general different from the p in the latter equation.

For pure homogeneous strain, the in-plane components of $\boldsymbol{\sigma}$ are obtained from (3.191) as

$$\sigma_{11} = 2\bar{\bar{W}}_1\lambda_1^2 + 2\bar{\bar{W}}_4\lambda_1^2\cos^2\varphi - p, \tag{3.192}$$

$$\sigma_{22} = 2\bar{\bar{W}}_1\lambda_2^2 + 2\bar{\bar{W}}_4\lambda_2^2\sin^2\varphi - p, \tag{3.193}$$

$$\sigma_{12} = 2\bar{\bar{W}}_4\sin\varphi\cos\varphi, \tag{3.194}$$

with $\lambda_1\lambda_2 = 1$.

For the simple shear deformation discussed in Sect. 3.1.3.3, we obtain, from (3.191),

$$\sigma_{11} = 2\bar{\bar{W}}_1(1+\gamma^2) + 2\bar{\bar{W}}_4(\cos\varphi + \gamma\sin\varphi)^2 - p, \tag{3.195}$$

$$\sigma_{22} = 2\bar{\bar{W}}_1 + 2\bar{\bar{W}}_4\sin^2\varphi - p, \tag{3.196}$$

$$\sigma_{12} = 2\gamma\bar{\bar{W}}_1 + 2\bar{\bar{W}}_4\sin\varphi(\cos\varphi + \gamma\sin\varphi), \tag{3.197}$$

with

$$I_1 = 3 + \gamma^2, \quad I_4 = 1 + \gamma\sin 2\varphi + \gamma^2\sin^2\varphi. \tag{3.198}$$

For an isotropic material these reduce to

$$\sigma_{11} = 2\bar{\bar{W}}_1(1+\gamma^2) - p, \quad \sigma_{22} = 2\bar{\bar{W}}_1 - p, \quad \sigma_{12} = 2\gamma\bar{\bar{W}}_1, \tag{3.199}$$

from which we obtain the *universal relation*

$$\sigma_{11} - \sigma_{22} - \gamma\sigma_{12} = 0 \tag{3.200}$$

between the components of the Cauchy stress, i.e. a relation independent of the form of the (incompressible, isotropic) strain-energy function. There is no counterpart of this relation in general for a transversely isotropic material. For a review of universal relations in finite elasticity we refer to Saccomandi (2001).

3.4 Boundary-Value Problems

We now consider the formulation of static (equilibrium) boundary-value problems. Specifically, we consider the equilibrium equation in the absence of body forces. The appropriate specialization of the equation of motion (3.86) is then

$$\operatorname{div}\boldsymbol{\sigma} = \mathbf{0}, \tag{3.201}$$

or, in terms of nominal stress,

$$\operatorname{Div}\mathbf{S} = \mathbf{0}. \tag{3.202}$$

Equations (3.201) and (3.202) have to be taken in conjunction with the stress-deformation relations

$$\boldsymbol{\sigma} = J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{F}}, \quad \mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}, \tag{3.203}$$

respectively, in the case of an unconstrained material, with the deformation gradient $\mathbf{F}$ given by $\mathbf{F} = \operatorname{Grad}\mathbf{x}$ with $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$. For an incompressible material the stress-deformation relations (3.203) are replaced by

$$\boldsymbol{\sigma} = \mathbf{F}\frac{\partial W}{\partial \mathbf{F}} - p\mathbf{I}, \quad \mathbf{S} = \frac{\partial W}{\partial \mathbf{F}} - p\mathbf{F}^{-1}, \quad \det\mathbf{F} \equiv 1. \tag{3.204}$$

Appropriate boundary conditions are required in order to formulate a boundary-value problem. Typical boundary conditions arising in problems of nonlinear elasticity are those in which $\mathbf{x}$ is specified on part of the boundary, $\partial\mathcal{B}_{\mathrm{r}}^{\mathbf{x}} \subset \partial\mathcal{B}_{\mathrm{r}}$ say, and the stress vector on the remainder, $\partial\mathcal{B}_{\mathrm{r}}^{\mathbf{t}}$, so that $\partial\mathcal{B}_{\mathrm{r}}^{\mathbf{x}} \cup \partial\mathcal{B}_{\mathrm{r}}^{\mathbf{t}} = \partial\mathcal{B}_{\mathrm{r}}$ and $\partial\mathcal{B}_{\mathrm{r}}^{\mathbf{x}} \cap \partial\mathcal{B}_{\mathrm{r}}^{\mathbf{t}} = \emptyset$. We write

$$\mathbf{x} = \boldsymbol{\xi}(\mathbf{X}) \quad \text{on } \partial\mathcal{B}_{\mathrm{r}}^{\mathbf{x}}, \tag{3.205}$$

$$\mathbf{S}^{\mathrm{T}}\mathbf{N} = \mathbf{t}(\mathbf{F}, \mathbf{X}) \quad \text{on } \partial\mathcal{B}_{\mathrm{r}}^{\mathbf{t}}, \tag{3.206}$$

where $\boldsymbol{\xi}$ and $\mathbf{t}$ are specified functions. In general, $\mathbf{t}$ may depend on the deformation, and this is indicated in (3.206) by showing the dependence of $\mathbf{t}$ on the deformation gradient $\mathbf{F}$. If the surface traction defined by (3.206) is independent of $\mathbf{F}$, it is referred to as a *dead-load traction*. In the particular case in which the boundary traction in (3.206) is associated with a hydrostatic pressure, P say, so that $\boldsymbol{\sigma}\mathbf{n} = -P\mathbf{n}$, then $\mathbf{t}$ depends on the deformation in the form

$$\mathbf{t} = -JP\mathbf{F}^{-\mathrm{T}}\mathbf{N} \quad \text{on } \partial\mathcal{B}_{\mathrm{r}}^{\mathbf{t}}, \tag{3.207}$$

as noted in Sect. 3.2.3.3.

When coupled with suitable boundary conditions, either of (3.201) or (3.202) in conjunction with (3.203) or (3.204), as appropriate, forms a coupled system of three highly nonlinear second-order partial differential equations for the components of $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$.

For homogeneous deformations, of course, the equilibrium equations are satisfied automatically, and such deformations can be maintained by the application of suitable boundary tractions. For nonhomogeneous deformations, it is necessary to solve the equilibrium equations. In the case of *unconstrained materials* very few explicit solutions have been obtained for boundary-value problems involving non-homogeneous deformations, and these arise for very special choices of the form of W and for relatively simple geometries. For *incompressible materials*, on the other hand, many more explicit solutions are available. In the following sections we describe some simple examples of boundary-value problems for an incompressible isotropic elastic material in which the deformation is non-homogeneous.

3.4.1 Extension and Inflation of a Thick-Walled Tube

We consider a thick-walled circular cylindrical tube whose initial geometry is defined by

$$0 < A \le R \le B, \quad 0 \le \Theta \le 2\pi, \quad 0 \le Z \le L, \tag{3.208}$$

where A, B, L are constants and (R, Θ, Z) are cylindrical polar coordinates associated with unit basis vectors $\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_Z$. The deformed configuration is specified in terms of cylindrical polar coordinates (r, θ, z), with unit basis vectors $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$, and the position vector in the deformed configuration may be written as

$$\mathbf{x} = r\mathbf{e}_r + z\mathbf{e}_z. \tag{3.209}$$

The tube is deformed so that the circular cylindrical shape is maintained. Since the material is incompressible the deformation is described by the equations

$$r = f(R) \equiv [a^2 + \lambda_z^{-1}(R^2 - A^2)]^{1/2}, \quad \theta = \Theta, \quad z = \lambda_z Z, \tag{3.210}$$

where λ_z is the (uniform) axial stretch and a is the internal radius of the deformed tube.

Since, for this deformation, $\mathbf{e}_r = \mathbf{E}_R, \mathbf{e}_\theta = \mathbf{E}_\Theta, \mathbf{e}_z = \mathbf{E}_Z$, the deformation gradient is calculated as

$$\begin{aligned}\mathbf{F} = \operatorname{Grad}\mathbf{x} &= \frac{\partial \mathbf{x}}{\partial R} \otimes \mathbf{e}_r + \frac{1}{R}\frac{\partial \mathbf{x}}{\partial \Theta} \otimes \mathbf{e}_\theta + \frac{\partial \mathbf{x}}{\partial Z} \otimes \mathbf{e}_z \\ &= f'(R)\mathbf{e}_r \otimes \mathbf{e}_r + \frac{r}{R}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \lambda_z \mathbf{e}_z \otimes \mathbf{e}_z. \end{aligned} \tag{3.211}$$

Thus, $\mathbf{F}$ is symmetric and in spectral form with respect to the cylindrical polar axes. The principal stretches $\lambda_1, \lambda_2, \lambda_3$, which are associated, respectively, with the radial, azimuthal and axial directions, are therefore identified. Thus,

$$\lambda_1 = \lambda^{-1}\lambda_z^{-1}, \quad \lambda_2 = \frac{r}{R} = \lambda, \quad \lambda_3 = \lambda_z, \tag{3.212}$$

where the notation λ has been introduced to replace λ_2 (note that this is the same symbol as used for one of the Lamé moduli, which is not used henceforth in this chapter). It follows from (3.210) and (3.212) that

$$\lambda_a^2\lambda_z - 1 = \frac{R^2}{A^2}(\lambda^2\lambda_z - 1) = \frac{B^2}{A^2}(\lambda_b^2\lambda_z - 1), \tag{3.213}$$

where

$$\lambda_a = a/A, \quad \lambda_b = b/B, \quad b = f(B). \tag{3.214}$$

For a fixed value of λ_z, the inequalities

$$\lambda_a^2\lambda_z \geq 1, \quad \lambda_a \geq \lambda \geq \lambda_b \tag{3.215}$$

hold during inflation of the tube, with equality holding if and only if $\lambda = \lambda_z^{-1/2}$ for $A \leq R \leq B$. Note that when this latter equality holds the deformation is homogeneous and corresponds to simple tension.

We use the notation (3.176) for the strain energy but with $\lambda_2 = \lambda$ and $\lambda_3 = \lambda_z$ as the independent stretches (instead of λ_1 and λ_2), so that

$$\hat{W}(\lambda, \lambda_z) = W(\lambda^{-1}\lambda_z^{-1}, \lambda, \lambda_z). \tag{3.216}$$

Hence

$$\sigma_2 - \sigma_1 = \lambda\hat{W}_\lambda, \quad \sigma_3 - \sigma_1 = \lambda_z\hat{W}_{\lambda_z}, \tag{3.217}$$

where the subscripts on $\hat{W}$ indicate partial derivatives, and, because the material is isotropic, $\boldsymbol{\sigma}$ is coaxial with the cylindrical polar axes, and we can write

$$\boldsymbol{\sigma} = \sigma_1\mathbf{e}_r \otimes \mathbf{e}_r + \sigma_2\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_3\mathbf{e}_z \otimes \mathbf{e}_z. \tag{3.218}$$

Since the deformation depends only on the radial coordinate, it follows from (3.218) that

$$\operatorname{div}\boldsymbol{\sigma} \equiv \left[\frac{\partial\sigma_1}{\partial r} + \frac{1}{r}(\sigma_1 - \sigma_2)\right]\mathbf{e}_r,$$

and the equation of equilibrium (3.201) therefore reduces to the radial equation

$$\frac{\mathrm{d}\sigma_1}{\mathrm{d}r} + \frac{1}{r}(\sigma_1 - \sigma_2) = 0 \tag{3.219}$$

in terms of the principal Cauchy stresses. Associated with this equation we have the (radial) boundary conditions

$$\sigma_1 = \begin{cases} -P & \text{on } r = a \\ 0 & \text{on } r = b \end{cases} \tag{3.220}$$

corresponding to pressure P (≥ 0) on the inside of the tube and zero traction on the outside.

By making use of (3.210) and (3.212)–(3.214), we obtain (after some rearrangement)

$$r\frac{\mathrm{d}\lambda}{\mathrm{d}r} = -\lambda(\lambda^2\lambda_z - 1), \tag{3.221}$$

and it is convenient to use this to change the independent variable from r to λ. Then, integration of (3.219) and application of the boundary conditions (3.220) leads to

$$P = \int_{\lambda_b}^{\lambda_a} (\lambda^2\lambda_z - 1)^{-1}\hat{W}_\lambda \mathrm{d}\lambda. \tag{3.222}$$

From (3.213) we recall that λ_b depends on λ_a. Equation (3.222) therefore provides an expression for P as a function of λ_a (equivalently of the deformed radius a) when λ_z is fixed.

In order to hold λ_z fixed an axial load, N say, must be applied on the ends of the tube. This is given by

$$N = 2\pi \int_a^b \sigma_3 r \mathrm{d}r. \tag{3.223}$$

After some rearrangements and use of (3.217) and (3.219), the formula (3.223) can be expressed in the form

$$N/\pi A^2 - P\lambda_a^2 = (\lambda_a^2\lambda_z - 1)\int_{\lambda_b}^{\lambda_a} (\lambda^2\lambda_z - 1)^{-2}(2\lambda_z\hat{W}_{\lambda_z} - \lambda\hat{W}_\lambda)\lambda \mathrm{d}\lambda. \tag{3.224}$$

We note that πA^2 times the integral in (3.224), i.e. $N - P\pi a^2$, is referred to as the *reduced axial load* since it accounts for the effect of the pressure on the ends of the cylinder, it being assumed that the cylinder has closed ends. For a more detailed discussion of this problem, including an analysis of bifurcation into non-circular cylindrical modes of deformation, we refer to Haughton and Ogden (1979a,b).

In the special case in which the wall thickness of the tube is small compared with the radius, the integral in (3.222) may be approximated in the following way (this corresponds to the membrane approximation). Let $\epsilon \equiv (B - A)/A$ be a dimensionless measure of the wall thickness in the reference configuration. Then, from (3.213), we may obtain the approximation

$$\lambda_a \simeq \lambda_b + \epsilon\lambda^{-1}\lambda_z^{-1}(\lambda^2\lambda_z - 1), \tag{3.225}$$

where, to the first order in ϵ, λ may be taken as either λ_a or λ_b. On use of (3.225) we may then approximate P as

$$P \simeq \epsilon \lambda_z^{-1} \lambda^{-1} \hat{W}_\lambda(\lambda, \lambda_z), \tag{3.226}$$

so that, at fixed λ_z, P behaves like $\lambda^{-1} \hat{W}_\lambda$ as a function of λ.

3.4.2 The Azimuthal Shear Problem

Here we consider the model problem of *pure azimuthal shear* of an incompressible isotropic nonlinearly elastic thick-walled circular cylindrical tube whose cross section in its natural (unstressed) configuration is defined by

$$0 < A \le R \le B, \quad 0 \le \Theta \le 2\pi, \tag{3.227}$$

where (R, Θ) are polar coordinates. Attention is restricted to plane deformations in which there is no extension along the axis of the cylinder and the deformation of a cross section is independent of the axial coordinate, Z say. To maintain plane strain conditions, appropriate axial loading is required on the ends of the tube, but this will not be needed explicitly for our purposes here.

Pure azimuthal shear is defined by

$$r = R, \quad \theta = \Theta + g(R), \quad z = Z, \tag{3.228}$$

where (r, θ, z) are cylindrical polar coordinates associated with the deformed configuration. Then $a = A$ and $b = B$ are the unchanged internal and external radii, respectively. Let the inner boundary of the tube be fixed, and consider the outer boundary to be rotated through an angle ψ, so that

$$g(A) = 0, \quad g(B) = \psi. \tag{3.229}$$

Referred to cylindrical polar coordinates the deformation gradient tensor $\mathbf{F}$ has components, denoted F, given by

$$\mathsf{F} = \begin{bmatrix} 1 & 0 & 0 \\ rg'(r) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.230}$$

where the prime indicates differentiation with respect to $r = R$, and we use r now as the independent variable instead of R. The inverse of F is

$$\mathsf{F}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -rg'(r) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.231}$$

The principal invariants I_1, I_2 of the deformation tensor $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ are given by

$$I_1 = I_2 = 3 + r^2 g'(r)^2, \tag{3.232}$$

and, since the material is incompressible, $I_3 = 1$.

With the restriction to plane strain there is only one independent invariant I_1, and the strain energy $\bar{W}(I_1, I_2)$, with $I_2 = I_1$, is now written $\check{W}(I_1)$. The in-plane restriction of the nominal stress tensor $\mathbf{S}$ is then calculated as

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{A}} = 2\check{W}'\mathbf{A}^{\mathrm{T}} - p\mathbf{A}^{-1}, \tag{3.233}$$

where $\check{W}' = \partial \check{W}/\partial I_1$ and $\mathbf{A}$ is the in-plane restriction of $\mathbf{F}$, with components given by the leading 2×2 matrix in (3.230), and similarly for $\mathbf{F}^{-1}$. The corresponding (in-plane) Cauchy stress tensor $\boldsymbol{\sigma} = \mathbf{AS}$ is

$$\boldsymbol{\sigma} = 2\check{W}'\mathbf{b} - p\mathbf{I}, \tag{3.234}$$

where $\mathbf{I}$ is the (in-plane) identity tensor and $\mathbf{b}$ is now taken as $\mathbf{A}\mathbf{A}^{\mathrm{T}}$.

For the strain energy and the stress to vanish in the natural configuration and for compatibility with the classical (linear) theory of incompressible isotropic elasticity, we require

$$\check{W}(3) = 0, \quad \check{W}'(3) = \frac{1}{2}\mu, \tag{3.235}$$

where μ is again the shear modulus in the reference configuration.

The in-plane components of the equation of equilibrium (3.201) are

$$\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0, \quad \frac{\mathrm{d}(r^2\sigma_{r\theta})}{\mathrm{d}r} = 0, \tag{3.236}$$

and, from (3.234), the components of $\boldsymbol{\sigma}$ are

$$\sigma_{rr} = 2\check{W}' - p, \quad \sigma_{\theta\theta} = 2\check{W}'(1 + r^2 g'(r)^2) - p, \quad \sigma_{r\theta} = 2\check{W}' r g'(r). \tag{3.237}$$

The deformation represents locally a simple shear with amount of shear $rg'(r)$, which we denote by γ. Then $I_1 = 3 + \gamma^2$ and it is convenient to introduce the notation

$$w(\gamma) = \check{W}(3 + \gamma^2), \tag{3.238}$$

from which it is easy to show that

$$\sigma_{r\theta} = w'(\gamma). \tag{3.239}$$

Integration of $(3.236)_2$ then yields

$$w'(\gamma) = \frac{\tau b^2}{r^2}, \tag{3.240}$$

where τ is the value of the shear stress $\sigma_{r\theta}$ on the boundary $r = b$. We adopt the assumption that $\gamma > 0\,(< 0)$ is associated with $\tau > 0\,(< 0)$, i.e. shearing in the positive (negative) θ direction with $g(r) > 0$ ($g(r) < 0$) for $r > a$ and hence $w'(\gamma) > 0\,(< 0)$ for $\gamma > 0\,(< 0)$. Moreover, increasing shear γ corresponds to increasing shear stress provided

$$w''(\gamma) > 0, \tag{3.241}$$

and we therefore impose (3.241) for all γ. The monotonicity of $w'(\gamma)$ implied by (3.241) ensures that, in principle, (3.240) can be inverted to give γ uniquely as a function of r, and then $g(r)$ can be determined by integration of $rg'(r) = \gamma$.

The radial equation of equilibrium $(3.236)_1$ may be written as

$$r\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}r} = \gamma w'(\gamma). \tag{3.242}$$

For any given incompressible isotropic strain-energy function, (3.242) serves to determine σ_{rr} (or, equivalently, p) once γ is found from (3.240).

Equations (3.240) and (3.242) involve no restriction on the strain-energy function other than that imposed by the incompressibility constraint and the requirement of monotonicity. A class of solutions for this problem has been given by Jiang and Ogden (1998) by starting with a compressible material and considering the strain-energy functions for which the pure azimuthal shear deformation can exist. A basic example for which a closed-form solution exists is the neo-Hookean model (3.169), for which we calculate $w'(\gamma) = \mu\gamma$ and hence

$$g'(r) = \frac{\tau b^2}{\mu r^3}. \tag{3.243}$$

On integration and application of the boundary conditions, we obtain the solution

$$g(r) = \frac{\tau b^2}{2\mu}\left(\frac{1}{a^2} - \frac{1}{r^2}\right), \quad \psi = \frac{\tau b^2}{2\mu}\left(\frac{1}{a^2} - \frac{1}{b^2}\right). \tag{3.244}$$

3.4.3 The Axial Shear Problem

We now consider the tube geometry as in (3.208) but with the *pure axial shear* deformation defined by

$$r = R, \quad \theta = \Theta, \quad z = Z + h(R), \tag{3.245}$$

where $h(R)$ denotes the axial displacement. Referred to basis vectors $\mathbf{E}_R$, $\mathbf{E}_\Theta$, $\mathbf{E}_Z$ and $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$, the deformation gradient can be expanded as

$$\mathbf{F} = \mathbf{I} + h'(r)\mathbf{e}_z \otimes \mathbf{E}_R, \tag{3.246}$$

where the prime signifies differentiation with respect to $r = R$.

As for the azimuthal shear problem, the deformation here is locally a simple shear, this time with amount of shear $\gamma = h'(r)$, and the strain energy can again be written as $w(\gamma)$ and the Cauchy stress as (3.234), with $w(\gamma) = \check{W}(3 + \gamma^2)$ and $\sigma_{rz} = w'(\gamma)$. For this problem the equilibrium equations are

$$\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0, \quad \frac{\mathrm{d}(r\sigma_{rz})}{\mathrm{d}r} = 0, \tag{3.247}$$

the latter of which can be integrated to give

$$w'(\gamma) = \frac{\tau b}{r}, \tag{3.248}$$

where τ is now the axial shear stress on $r = b$.

We take the conditions to be satisfied by $h(r)$ on the boundaries $r = a$ and $r = b$ to be

$$h(a) = 0, \quad h(b) = d, \tag{3.249}$$

where d is the axial displacement of the outer boundary.

For the neo-Hookean material model we have $w'(\gamma) = \mu\gamma$ and hence

$$h'(r) = \frac{\tau b}{\mu r}, \tag{3.250}$$

which on integration and use of the boundary condition yields

$$h(r) = \frac{\tau b}{\mu}\log(r/a), \quad d = \frac{\tau b}{\mu}\log(b/a). \tag{3.251}$$

For more details of the axial shear problem in the context of compressible materials, we refer to Jiang and Ogden (2000).

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Chapter 4
Nonlinear Electroelastic Interactions

Abstract In this chapter we bring together the theory of electrostatics and the nonlinear theory of elasticity in order to build a theory of nonlinear electroelastic interactions. For this purpose it is necessary to distinguish between different measures of stress, and this is reflected in different choices of potential energy in considering the equation of energy balance. We then introduce Lagrangian counterparts of the usual Eulerian electric and electric displacement field vectors, and this facilitates the construction of electroelastic constitutive equations that have compact general forms. Material symmetry considerations lead to the definition of an isotropic electroelastic material and explicit representations for the total Cauchy stress and the electric displacement field (if the electric field is the independent electric variable), or the electric field (if the electric displacement is the independent electric variable), in terms of invariants of the independent kinematic and electric variables. In the final section of the chapter we consider the linear specialization of the general nonlinear theory and discuss briefly the linear theory of piezoelectricity.

4.1 Preliminaries

In this chapter we focus on electrostatics and the interaction between electric fields and nonlinear elastic deformations with the aim of developing general forms of constitutive law for nonlinear electroelastic materials and the equations governing the equilibrium of such materials.

We now suppose that the material is deformable and electrically polarizable and that it occupies the configuration $\mathcal{B}$ with boundary $\partial\mathcal{B}$ when deformed and that in this configuration $\mathbf{E}$, $\mathbf{D}$, $\mathbf{P}$ and ρ_f denote, respectively, the electric field, the electric displacement, the polarization and the free charge density, as introduced in Sect. 2.4.2, and are regarded as functions of $\mathbf{x}$. The fields $\mathbf{E}$ and $\mathbf{D}$ satisfy the equations

$$\mathrm{curl}\,\mathbf{E} = \mathbf{0}, \quad \mathrm{div}\,\mathbf{D} = \rho_\mathrm{f}, \tag{4.1}$$

L. Dorfmann and R.W. Ogden, *Nonlinear Theory of Electroelastic and Magnetoelastic Interactions*, DOI 10.1007/978-1-4614-9596-3_4,

which are the appropriate specializations of Maxwell's equations (2.163) and (2.164) in the absence of magnetic interactions and time dependence. The polarization vector $\mathbf{P}$ is related to $\mathbf{E}$ and $\mathbf{D}$ by the standard equation

$$\mathbf{P} = \mathbf{D} - \varepsilon_0 \mathbf{E}, \tag{4.2}$$

which reduces to $\mathbf{D} = \varepsilon_0 \mathbf{E}$ in vacuum or in non-polarizable material.

On the boundary $\partial\mathcal{B}$ the fields have to satisfy certain boundary conditions, which were derived in Sect. 2.5.1. For convenience of reference, we repeat them here in the form

$$\mathbf{n} \times [\![\mathbf{E}]\!] = \mathbf{0}, \quad \mathbf{n} \cdot [\![\mathbf{D}]\!] = \sigma_\mathrm{f} \quad \text{on } \partial\mathcal{B}, \tag{4.3}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\mathcal{B}$ and σ_f is the free surface charge per unit area, and in the alternative forms

$$\varepsilon_0 [\![\mathbf{E}]\!] = (\mathbf{n} \cdot \mathbf{P} + \sigma_\mathrm{f})\mathbf{n}, \quad [\![\mathbf{D}]\!] = (\mathbf{n} \cdot \mathbf{P} + \sigma_\mathrm{f})\mathbf{n} - \mathbf{P} \quad \text{on } \partial\mathcal{B}. \tag{4.4}$$

We assume that there are no surfaces of discontinuity within $\mathcal{B}$.

4.2 Equilibrium and Stress

For a continuum in the absence of electric and magnetic fields, the equilibrium equation has the form

$$\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{f} = \mathbf{0} \quad \text{in } \mathcal{B}, \tag{4.5}$$

which specializes (3.86) to the static situation, where we recall that $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\mathbf{f}$ is the *mechanical* body force per unit mass and ρ is the mass density of the material in the configuration $\mathcal{B}$. In the absence of couple stresses $\boldsymbol{\sigma}$ is symmetric. This equation is coupled with the traction boundary condition

$$\boldsymbol{\sigma} \mathbf{n} = \mathbf{t}_\mathrm{a} \tag{4.6}$$

on $\partial\mathcal{B}$ or on any part of the boundary $\partial\mathcal{B}$ where the mechanical traction $\mathbf{t}_\mathrm{a}$ is applied, the latter being defined per unit area of $\partial\mathcal{B}$. Here we have introduced the subscript a on $\mathbf{t}$ to indicate 'applied'.

For an electroelastic material in the presence of an electric field, both (4.5) and (4.6) require modification. There are several ways in which the modification can be applied, and we now examine how this comes about.

From Sect. 2.1.10 we know that the force and couple on a single dipole $\mathbf{p}$ in an electric field $\mathbf{E}$ are $(\mathbf{p} \cdot \operatorname{grad})\mathbf{E}$ and $\mathbf{p} \times \mathbf{E}$, respectively. These formulas translate directly to counterparts in a polarized medium as densities (per unit volume in $\mathcal{B}$):

$$(\mathbf{P} \cdot \operatorname{grad})\mathbf{E}, \quad \mathbf{P} \times \mathbf{E}. \tag{4.7}$$

As noted in Sect. 2.1.10, $(\mathbf{p}\cdot\mathrm{grad})\mathbf{E}$ can also be written $\varepsilon_0^{-1}(\mathbf{p}\cdot\mathrm{grad})\mathbf{D}$ in the case of a single dipole, but $(\mathbf{P}\cdot\mathrm{grad})\mathbf{E}$ is not equal to $\varepsilon_0^{-1}(\mathbf{P}\cdot\mathrm{grad})\mathbf{D}$ in a polarized medium. In fact, since $\mathrm{curl}\,\mathbf{E} = \mathbf{0}$, $(\mathbf{p}\cdot\mathrm{grad})\mathbf{E}$ can also be written $(\mathrm{grad}\,\mathbf{E})^{\mathrm{T}}\mathbf{p}$ or $\varepsilon_0^{-1}(\mathrm{grad}\,\mathbf{D})^{\mathrm{T}}\mathbf{p}$. It is the latter form that provides the appropriate translation to a polarized medium as $\varepsilon_0^{-1}(\mathrm{grad}\,\mathbf{D})^{\mathrm{T}}\mathbf{P}$, as we discuss below.

For a polarized medium containing free charges we may write the electric body force per unit volume as $\rho_{\mathrm{f}}\mathbf{E} + (\mathbf{P}\cdot\mathrm{grad})\mathbf{E}$ if we use the first form of the force on a dipole distribution noted above. By incorporating this body force in the equation of equilibrium (4.5), we obtain

$$\mathrm{div}\,\boldsymbol{\sigma} + \rho\mathbf{f} + \rho_{\mathrm{f}}\mathbf{E} + (\mathbf{P}\cdot\mathrm{grad})\mathbf{E} = \mathbf{0} \quad \text{in } \mathcal{B}, \tag{4.8}$$

but now the Cauchy stress is different from that in (4.5) since in general it depends on the electric field (this dependence will be made explicit in the following sections). Next we note that the electric body force can be written as the divergence of a second-order tensor in the form

$$\rho_{\mathrm{f}}\mathbf{E} + (\mathbf{P}\cdot\mathrm{grad})\mathbf{E} = \mathrm{div}\,\boldsymbol{\tau}_{\mathrm{e}}, \tag{4.9}$$

where $\boldsymbol{\tau}_{\mathrm{e}}$ is referred to as an *electrostatic Maxwell stress* and is defined by

$$\boldsymbol{\tau}_{\mathrm{e}} = \mathbf{D}\otimes\mathbf{E} - \frac{1}{2}\varepsilon_0(\mathbf{E}\cdot\mathbf{E})\mathbf{I}, \tag{4.10}$$

where $\mathbf{I}$ is the identity tensor. This can easily be established by use of (4.1). Note that this is different from the definition given in (2.54), although the two definitions are equivalent in vacuo. Thus, the equilibrium equation can be written

$$\mathrm{div}\,(\boldsymbol{\sigma} + \boldsymbol{\tau}_{\mathrm{e}}) + \rho\mathbf{f} = \mathbf{0} \quad \text{in } \mathcal{B}. \tag{4.11}$$

The rotational balance equation, which ensures that $\boldsymbol{\sigma}$ is symmetric in the absence of couple stresses, is now, since, by $(2.145)_1$, $\mathbf{P}\times\mathbf{E} = \mathbf{D}\times\mathbf{E}$, replaced by

$$\boldsymbol{\epsilon}\boldsymbol{\sigma} + \mathbf{D}\times\mathbf{E} \equiv \boldsymbol{\epsilon}(\boldsymbol{\sigma} + \boldsymbol{\tau}_{\mathrm{e}}) = \mathbf{0}, \tag{4.12}$$

where we have used the connection $\boldsymbol{\epsilon}(\mathbf{D}\otimes\mathbf{E}) = \mathbf{D}\times\mathbf{E}$ and the fact that $\boldsymbol{\epsilon}\mathbf{I} = \mathbf{0}$.

In general, neither $\boldsymbol{\sigma}$ nor $\boldsymbol{\tau}_{\mathrm{e}}$ is symmetric, but the above shows that their sum is symmetric, and this prompts the definition

$$\boldsymbol{\tau} = \boldsymbol{\sigma} + \boldsymbol{\tau}_{\mathrm{e}} \tag{4.13}$$

of the *total Cauchy stress tensor* $\boldsymbol{\tau}$, which is *symmetric*. The equilibrium equation can now be written compactly as

$$\mathrm{div}\,\boldsymbol{\tau} + \rho\mathbf{f} = \mathbf{0} \quad \text{in } \mathcal{B}, \tag{4.14}$$

with the electric body force incorporated in the stress tensor and, since $\boldsymbol{\tau}$ is symmetric, the rotational balance equation automatically satisfied.

Outside the polarized material we have the simple connection $\mathbf{D} = \varepsilon_0\mathbf{E}$, and we may also define a Maxwell stress tensor following the definition (2.54), a special case of (4.10). In this case we use the notation $\boldsymbol{\tau}_\mathrm{e}^\star$, which is symmetric:

$$\boldsymbol{\tau}_\mathrm{e}^\star = \mathbf{D}^\star \otimes \mathbf{E}^\star - \frac{1}{2}\varepsilon_0(\mathbf{E}^\star \cdot \mathbf{E}^\star)\mathbf{I} \equiv \varepsilon_0\mathbf{E}^\star \otimes \mathbf{E}^\star - \frac{1}{2}\varepsilon_0(\mathbf{E}^\star \cdot \mathbf{E}^\star)\mathbf{I}. \tag{4.15}$$

Here and henceforth we use the superscript $\star$ for quantities where the connection $\mathbf{D} = \varepsilon_0\mathbf{E}$ holds. Thus, $\mathbf{D}^\star = \varepsilon_0\mathbf{E}^\star$. Note that on use of the equations $\operatorname{curl}\mathbf{E}^\star = \mathbf{0}$ and $\operatorname{div}\mathbf{E}^\star = 0$, it follows that

$$\operatorname{div}\boldsymbol{\tau}_\mathrm{e}^\star = \mathbf{0}. \tag{4.16}$$

In order to see how the traction boundary condition is modified, we now consider the overall balance of forces on the body $\mathcal{B}$. Integrating (4.11) over the volume and using the divergence theorem, we obtain

$$\int_{\partial\mathcal{B}} (\boldsymbol{\sigma}^\mathrm{T}\mathbf{n} + \boldsymbol{\tau}_\mathrm{e}^\mathrm{T}\mathbf{n})\,\mathrm{d}s + \int_{\mathcal{B}} \rho\mathbf{f}\,\mathrm{d}v = \mathbf{0}. \tag{4.17}$$

The surface integral term must match the total surface traction on $\partial\mathcal{B}$. To proceed, we first use the connections (4.4) to show that, on $\partial\mathcal{B}$,

$$\boldsymbol{\tau}_\mathrm{e}^\mathrm{T}\mathbf{n} - \boldsymbol{\tau}_\mathrm{e}^\star\mathbf{n} = -\sigma_\mathrm{f}\mathbf{E} - \frac{1}{2}\varepsilon_0^{-1}(\mathbf{n}\cdot\mathbf{P} + \sigma_\mathrm{f})^2\mathbf{n} \equiv -\mathbf{t}_\mathrm{e}, \tag{4.18}$$

where $\mathbf{E}$ and $\mathbf{P}$ are evaluated in the material (of course, $\mathbf{P} = \mathbf{0}$ outside the material) and wherein $\mathbf{t}_\mathrm{e}$ is defined. Now, $\boldsymbol{\tau}_\mathrm{e}^\star\mathbf{n}$ represents an electrostatic force per unit area on the boundary $\partial\mathcal{B}$, i.e. the traction associated with the Maxwell stress $\boldsymbol{\tau}_\mathrm{e}^\star$ *outside* the material. We denote this by $\mathbf{t}_\mathrm{e}^\star$. Thus, the boundary traction at a point where the mechanical traction is $\mathbf{t}_\mathrm{a}$ on $\partial\mathcal{B}$ is $\mathbf{t}_\mathrm{a} + \mathbf{t}_\mathrm{e}^\star$. Thus, the remaining term in the surface integral in (4.17), i.e. $\boldsymbol{\sigma}^\mathrm{T}\mathbf{n} - \mathbf{t}_\mathrm{e}$, must balance $\mathbf{t}_\mathrm{a}$. The boundary condition for $\boldsymbol{\sigma}$, replacing (4.6) in the purely mechanical case, is therefore

$$\boldsymbol{\sigma}^\mathrm{T}\mathbf{n} = \mathbf{t}_\mathrm{a} + \mathbf{t}_\mathrm{e} \quad \text{on } \partial\mathcal{B}. \tag{4.19}$$

The corresponding boundary condition for the total Cauchy stress is simply

$$\boldsymbol{\tau}\mathbf{n} = \mathbf{t}_\mathrm{a} + \mathbf{t}_\mathrm{e}^\star \quad \text{on } \partial\mathcal{B}, \tag{4.20}$$

with

$$\mathbf{t}_\mathrm{e}^\star = \boldsymbol{\tau}_\mathrm{e}^\star\mathbf{n}, \tag{4.21}$$

$\boldsymbol{\tau}_{\mathrm{e}}^{\star}$ being evaluated on the exterior of $\partial\mathcal{B}$. In general, $\mathbf{t}_{\mathrm{a}}$ may be prescribed on only part of $\partial\mathcal{B}$ or not at all. Some other possible mechanical boundary conditions have been noted in Sect. 3.4.

Let us now consider the alternative representation for the electric body force mentioned above, namely $\rho_{\mathrm{f}}\mathbf{E} + \varepsilon_0^{-1}(\operatorname{grad}\mathbf{D})^{\mathrm{T}}\mathbf{P}$. This can also be written as the divergence of a second-order tensor, which, in this case, we denote by $\bar{\boldsymbol{\tau}}_{\mathrm{e}}$. This is another form of Maxwell stress, in general unsymmetric, which is given by

$$\bar{\boldsymbol{\tau}}_{\mathrm{e}} = \boldsymbol{\tau}_{\mathrm{e}} + \frac{1}{2}\varepsilon_0^{-1}(\mathbf{P}\cdot\mathbf{P})\mathbf{I}. \tag{4.22}$$

It can be checked directly that $\operatorname{div}\bar{\boldsymbol{\tau}}_{\mathrm{e}} = \rho_{\mathrm{f}}\mathbf{E} + \varepsilon_0^{-1}(\operatorname{grad}\mathbf{D})^{\mathrm{T}}\mathbf{P}$ by using (4.9), the connection $(\operatorname{grad}\mathbf{E})^{\mathrm{T}}\mathbf{P} = (\operatorname{grad}\mathbf{E})\mathbf{P}$, and $(2.145)_1$. The equilibrium equation is now written as

$$\operatorname{div}(\bar{\boldsymbol{\sigma}} + \bar{\boldsymbol{\tau}}_{\mathrm{e}}) + \rho\mathbf{f} = \mathbf{0} \quad \text{in } \mathcal{B}, \tag{4.23}$$

where we have defined

$$\bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \frac{1}{2}\varepsilon_0^{-1}(\mathbf{P}\cdot\mathbf{P})\mathbf{I}. \tag{4.24}$$

Then, we have

$$\bar{\boldsymbol{\sigma}} + \bar{\boldsymbol{\tau}}_{\mathrm{e}} = \boldsymbol{\sigma} + \boldsymbol{\tau}_{\mathrm{e}} = \boldsymbol{\tau}. \tag{4.25}$$

Note that these connections can be adjusted by addition of a divergence-free second-order tensor, as discussed by Bustamante et al. (2009b), but its presence is inessential.

The boundary condition for $\bar{\boldsymbol{\sigma}}$ analogous to (4.19) is

$$\bar{\boldsymbol{\sigma}}^{\mathrm{T}}\mathbf{n} = \mathbf{t}_{\mathrm{a}} + \bar{\mathbf{t}}_{\mathrm{e}}, \quad \text{on } \partial\mathcal{B}, \tag{4.26}$$

where $\bar{\mathbf{t}}_{\mathrm{e}}$ is defined by

$$\bar{\mathbf{t}}_{\mathrm{e}} = \sigma_{\mathrm{f}}\mathbf{E} + \frac{1}{2}\varepsilon_0^{-1}(\mathbf{n}\cdot\mathbf{P} + \sigma_{\mathrm{f}})^2\mathbf{n} - \frac{1}{2}\varepsilon_0^{-1}(\mathbf{P}\cdot\mathbf{P})\mathbf{n} = \mathbf{t}_{\mathrm{e}} - \frac{1}{2}\varepsilon_0^{-1}(\mathbf{P}\cdot\mathbf{P})\mathbf{n}. \tag{4.27}$$

A key message from the above analysis is that there are alternative ways in which to write the electric body force, which in each case is expressible as the divergence of a Maxwell-type stress tensor, and in turn each is associated with a different Cauchy-like stress tensor. The simplest representation involves the total Cauchy stress tensor, for which the boundary tractions have a direct interpretation in terms of the applied mechanical load and the electric surface force associated with the exterior Maxwell stress. With the total stress tensor, there is no need to define a Maxwell stress within the body, thus avoiding confusing non-uniqueness of representation. This will be discussed further in the next section.

In a deformable medium each stress tensor depends in general on the deformation, through the deformation gradient $\mathbf{F}$, and the electric field, either directly via $\mathbf{E}$ or through a constitutive law that, in addition to the connection (4.2), provides an expression for $\mathbf{D}$ or $\mathbf{P}$ in terms of $\mathbf{E}$. In the following section we develop constitutive laws of this kind but with restriction to purely electroelastic interactions with no dissipation.

4.3 Energy Balance and Constitutive Laws

From Sects. 3.2.4 and 3.3.1, we recall that for a hyperelastic material, the rate of working of the body forces and surface tractions is balanced by the rate of increase of kinetic and stored elastic energies, i.e. for a region $\mathcal{D}_t$ with boundary $\partial\mathcal{D}_t$

$$\int_{\mathcal{D}_t} \rho\mathbf{f}\cdot\mathbf{v}\,\mathrm{d}v + \int_{\partial\mathcal{D}_t} \mathbf{t}\cdot\mathbf{v}\,\mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{K}+\mathcal{W}), \tag{4.28}$$

where

$$\mathcal{W} = \int_{\mathcal{D}_\mathrm{r}} W\,\mathrm{d}V, \tag{4.29}$$

W being the strain-energy function per unit reference volume and $\mathcal{D}_\mathrm{r}$ the reference region. Here we are concerned with quasi-static deformations, so the kinetic energy term can be dropped. Instead of the velocity $\mathbf{v}$, we consider 'virtual' incremental displacements $\mathbf{u}$, and a superposed dot now represents an associated virtual change in the quantity on which it sits. The above can now be written, after use of the connection $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$, the divergence theorem and the equilibrium equation (4.5), in the form

$$\int_{\mathcal{D}_\mathrm{r}} \dot{W}\,\mathrm{d}V = \int_{\mathcal{D}} [\rho\mathbf{f}\cdot\mathbf{u} + \mathrm{div}\,(\boldsymbol{\sigma}\mathbf{u})]\,\mathrm{d}v = \int_{\mathcal{D}} \mathrm{tr}\,(\boldsymbol{\sigma}\mathbf{L})\,\mathrm{d}v, \tag{4.30}$$

where $\mathbf{L} = \mathrm{grad}\,\mathbf{u}$ is now the virtual displacement gradient (as distinct from the velocity gradient $\boldsymbol{\Gamma}$). The subscript t has now been dropped from $\mathcal{D}_t$ since we are not considering time dependence. The connection $\dot{W} = J\,\mathrm{tr}\,(\boldsymbol{\sigma}\mathbf{L})$ then follows. The strain energy is measured per unit reference volume, whereas the internal energy density, denoted U, is usually measured per unit mass. For an elastic material they are connected by $W = \rho_\mathrm{r}U$, where ρ_r is the reference density, so that $\rho_\mathrm{r}\dot{U} = J\,\mathrm{tr}\,(\boldsymbol{\sigma}\mathbf{L})$. We now extend this formulation to account for electroelastic interactions, and we work in terms of the reference volume.

For an electroelastic material we consider the virtual work of the mechanical and electric body forces $\mathbf{f}$ and $\mathbf{f}_\mathrm{e}$ and the surface tractions associated with the stress $\boldsymbol{\sigma}$, which yields

$$\int_{\mathcal{D}} (\rho \mathbf{f} \cdot \mathbf{u} + \mathbf{f}_{\mathrm{e}} \cdot \mathbf{u}) \, \mathrm{d}v + \int_{\partial \mathcal{D}} (\boldsymbol{\sigma}^{\mathrm{T}} \mathbf{n}) \cdot \mathbf{u} \, \mathrm{d}s = \int_{\mathcal{D}} \mathrm{tr}\,(\boldsymbol{\sigma} \mathbf{L}) \, \mathrm{d}v. \tag{4.31}$$

The right-hand side of this is not in this case the increment in the internal energy, for two reasons. First because the internal energy should include an electric contribution and second because, as we shall see below, the choice of 'Cauchy' stress is not unique. However, with this proviso let U again denote the internal energy density of the material, per unit mass, so that the total internal energy in the considered region is

$$\int_{\mathcal{D}} \rho U \, \mathrm{d}v = \int_{\mathcal{D}_{\mathrm{r}}} \rho_{\mathrm{r}} U \, \mathrm{d}V. \tag{4.32}$$

The internal energy depends now not just on $\mathbf{F}$ but also on some electric variable. We consider it to depend on the polarization $\mathbf{P}$ since $\mathbf{P} = \mathbf{0}$ outside the material, but we use the polarization per unit reference volume, i.e. $J\mathbf{P}$, which we denote by $\mathbf{P}_{\mathrm{r}}$, whereas Toupin (1956) adopts the polarization per unit mass. Thus, $U = U(\mathbf{F}, \mathbf{P}_{\mathrm{r}})$, and hence

$$\rho_{\mathrm{r}} \dot{U} = J \,\mathrm{tr}\,(\boldsymbol{\sigma} \mathbf{L}) + \mathbf{E} \cdot \dot{\mathbf{P}}_{\mathrm{r}}, \tag{4.33}$$

where the term $\mathbf{E} \cdot \dot{\mathbf{P}}_{\mathrm{r}}$ represents the contribution to the increase in internal energy from the work done by the electric field $\mathbf{E}$ in an arbitrary virtual increment of the polarization density (per unit reference volume). Using this in (4.31) and expressing it partially in the reference configuration, we obtain

$$\int_{\mathcal{D}_{\mathrm{r}}} (\rho_{\mathrm{r}} \mathbf{f} \cdot \mathbf{u} + J \mathbf{f}_{\mathrm{e}} \cdot \mathbf{u} + \mathbf{E} \cdot \dot{\mathbf{P}}_{\mathrm{r}}) \, \mathrm{d}V + \int_{\partial \mathcal{D}} (\boldsymbol{\sigma}^{\mathrm{T}} \mathbf{n}) \cdot \mathbf{u} \, \mathrm{d}s = \int_{\mathcal{D}_{\mathrm{r}}} \rho_{\mathrm{r}} \dot{U} \, \mathrm{d}V. \tag{4.34}$$

From U we may now calculate

$$\rho_{\mathrm{r}} \dot{U} = \rho_{\mathrm{r}} \mathrm{tr} \left(\frac{\partial U}{\partial \mathbf{F}} \dot{\mathbf{F}} \right) + \rho_{\mathrm{r}} \frac{\partial U}{\partial \mathbf{P}_{\mathrm{r}}} \cdot \dot{\mathbf{P}}_{\mathrm{r}}, \tag{4.35}$$

and since $\mathbf{L}$ and $\dot{\mathbf{P}}_{\mathrm{r}}$ are arbitrary and $\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}$, the counterpart of (3.48) for increments, comparison with (4.33) yields the formulas

$$\boldsymbol{\sigma} = \rho \mathbf{F} \frac{\partial U}{\partial \mathbf{F}}, \quad \mathbf{E} = \rho_{\mathrm{r}} \frac{\partial U}{\partial \mathbf{P}_{\mathrm{r}}} \equiv \rho \frac{\partial U}{\partial \mathbf{P}}, \tag{4.36}$$

the latter identity holding because the derivative is at fixed $\mathbf{F}$. These are constitutive equations for the stress $\boldsymbol{\sigma}$, similar to (3.113), and the electric field $\mathbf{E}$.

An alternative set of constitutive equations is obtained by defining the energy (or potential) function $\phi = \phi(\mathbf{F}, \mathbf{E})$ by means of the partial Legendre transform (assuming that there is a one-to-one relation between $\mathbf{E}$ and $\mathbf{P}_{\mathrm{r}}$)

$$\rho_{\mathrm{r}} \phi = \rho_{\mathrm{r}} U - \mathbf{E} \cdot \mathbf{P}_{\mathrm{r}}, \tag{4.37}$$

from which we obtain

$$\boldsymbol{\sigma} = \rho \mathbf{F} \frac{\partial \phi}{\partial \mathbf{F}}, \quad \mathbf{P} = -\rho \frac{\partial \phi}{\partial \mathbf{E}}, \tag{4.38}$$

giving the same Cauchy stress $\boldsymbol{\sigma}$ as in (4.36).

Yet another option, this time a function of $\mathbf{F}$ and $\mathbf{D}$, involves the function $\bar{\phi} = \bar{\phi}(\mathbf{F}, \mathbf{D})$ defined by

$$\rho_\mathrm{r} \bar{\phi} = \rho_\mathrm{r} \phi - \frac{1}{2} \varepsilon_0^{-1} J \mathbf{P} \cdot \mathbf{P}, \tag{4.39}$$

from which we obtain

$$\rho_\mathrm{r} \dot{\bar{\phi}} = J \operatorname{tr}(\bar{\boldsymbol{\sigma}} \mathbf{L}) - \varepsilon_0^{-1} \mathbf{P}_\mathrm{r} \cdot \dot{\mathbf{D}}, \tag{4.40}$$

involving the Cauchy stress $\bar{\boldsymbol{\sigma}}$ defined in (4.24), and hence

$$\bar{\boldsymbol{\sigma}} = \rho \mathbf{F} \frac{\partial \bar{\phi}}{\partial \mathbf{F}}, \quad \mathbf{P} = -\varepsilon_0 \rho \frac{\partial \bar{\phi}}{\partial \mathbf{D}}. \tag{4.41}$$

There are several other transformations between energy functions that enable switching between different pairs of dependent and independent electric variables $\mathbf{E}$, $\mathbf{D}$ and $\mathbf{P}$, but we do not detail them all here since they are not of central interest. The main ones can be found tabled in the paper by Bustamante et al. (2009b). However, for illustration, we do list several of their magnetoelastic counterparts in Chap. 6. Each gives in general a different electric body force and associated Cauchy and Maxwell stresses. We summarize them in the form of their general structure, for which the equilibrium equation is written

$$\operatorname{div} \hat{\boldsymbol{\sigma}} + \rho \mathbf{f} + \hat{\mathbf{f}}_\mathrm{e} = \mathbf{0}, \tag{4.42}$$

where $\hat{\boldsymbol{\sigma}}$ is a Cauchy-like stress tensor and $\hat{\mathbf{f}}_\mathrm{e}$ is an electric body force (defined per unit volume in $\mathcal{B}$). The latter is always expressible in the form $\hat{\mathbf{f}}_\mathrm{e} = \operatorname{div} \hat{\boldsymbol{\tau}}_\mathrm{e}$, where $\hat{\boldsymbol{\tau}}_\mathrm{e}$ is a 'Maxwell stress' within the material and $\hat{\boldsymbol{\sigma}} + \hat{\boldsymbol{\tau}}_\mathrm{e} = \boldsymbol{\tau}$. Clearly, the concepts of 'stress', 'electric body force' and 'Maxwell stress' inside the material are not uniquely defined, as is well known, and, in particular, the electric 'body force' and 'Maxwell stress' terms are different for each choice of 'stress tensor'. Moreover, the boundary conditions for each stress tensor are different since, by (4.20), $\hat{\boldsymbol{\sigma}}$ must satisfy

$$\hat{\boldsymbol{\sigma}}^\mathrm{T} \mathbf{n} = \mathbf{t}_\mathrm{a} + \boldsymbol{\tau}_\mathrm{e}^\star \mathbf{n} - \hat{\boldsymbol{\tau}}_\mathrm{e}^\mathrm{T} \mathbf{n} \quad \text{on } \partial \mathcal{B}, \tag{4.43}$$

where, it should be emphasized, $\boldsymbol{\tau}_\mathrm{e}^\star$ is the Maxwell stress calculated on the boundary from the exterior fields, while $\hat{\boldsymbol{\tau}}_\mathrm{e}$ is the Maxwell stress calculated on the boundary from the fields *inside* the material. A case in point is $\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma}$ with $\hat{\boldsymbol{\tau}}_\mathrm{e} = \boldsymbol{\tau}_\mathrm{e}$ and the boundary condition (4.19).

The distinctions between the relative contributions of the many different body forces or, equivalently, Maxwell stresses are therefore somewhat artificial. In any case, it is unlikely that experiment will be able to distinguish between different choices of Maxwell stress by direct measurement, notwithstanding the fact that for a given body force, the associated Maxwell stress is undetermined to within an additive divergence-free stress. From the mathematical point of view, the formulation based on the 'total stress' is the cleanest and avoids the need to define either an electric body force or a Maxwell stress *within* a polarizable material. Indeed, as shown by Dorfmann and Ogden (2005a), the stress $\boldsymbol{\tau}$ has a very simple expression in terms of a modified form of the energy function. This will be recalled and highlighted within the following section. The formulations discussed above are each Eulerian in character, but in the following we discuss Lagrangian formulations that lead to a very clear structure of the constitutive laws and governing equations.

4.4 Lagrangian Formulations

The equations and boundary conditions governing electroelastic equilibrium in the preceding section are expressed in Eulerian form and involve the operators div and curl. In this section we re-cast the equations in Lagrangian form using the operators Div and Curl and express the boundary conditions also in Lagrangian form. The spatial variable $\mathbf{x}$ is replaced by $\mathbf{X}$ in the Lagrangian formulation. The constitutive equations are also re-cast using Lagrangian variables, leading to a relatively simple overall structure for the governing equations.

4.4.1 Electrostatic Equations and Boundary Conditions

First we consider $(4.1)_1$ and integrate it over an arbitrary (but suitably regular) open surface $\mathcal{S}$ and then apply Stokes' theorem to obtain

$$0 = \int_{\mathcal{S}} (\operatorname{curl} \mathbf{E}) \cdot \mathbf{n}\, \mathrm{d}s = \int_{\partial \mathcal{S}} \mathbf{E} \cdot \mathrm{d}\mathbf{x}, \tag{4.44}$$

where $\partial\mathcal{S}$ is the boundary curve of $\mathcal{S}$. Since the line element $\mathrm{d}\mathbf{x}$ comes from a corresponding line element $\mathrm{d}\mathbf{X}$ in the reference configuration, we have $\mathrm{d}\mathbf{x} = \mathbf{F}\mathrm{d}\mathbf{X}$, and we may therefore convert the line integral to an integral over the reference boundary $\partial\mathcal{S}_\mathrm{r}$ of the reference surface $\mathcal{S}_\mathrm{r}$ and use the connection $\mathbf{E}\cdot(\mathbf{F}\mathrm{d}\mathbf{X}) = (\mathbf{F}^\mathrm{T}\mathbf{E})\cdot \mathrm{d}\mathbf{X}$ followed by Stokes' theorem over the reference region to obtain

$$0 = \int_{\partial\mathcal{S}_\mathrm{r}} (\mathbf{F}^\mathrm{T}\mathbf{E}) \cdot \mathrm{d}\mathbf{X} = \int_{\mathcal{S}_\mathrm{r}} [\operatorname{Curl}(\mathbf{F}^\mathrm{T}\mathbf{E})] \cdot \mathbf{N}\, \mathrm{d}S. \tag{4.45}$$

Since $\mathcal{S}_\mathrm{r}$ is arbitrary we deduce that

$$\mathrm{Curl}\,(\mathbf{F}^\mathrm{T}\mathbf{E}) = \mathbf{0}, \tag{4.46}$$

where, as usual, continuity of the integrand is assumed.

Next, we integrate $(4.1)_2$ over the arbitrary volume $\mathcal{D}$ (with suitably regular boundary $\partial\mathcal{D}$) and apply the divergence theorem to obtain

$$\int_{\mathcal{D}} \rho_\mathrm{f}\,\mathrm{d}v = \int_{\mathcal{D}} \mathrm{div}\,\mathbf{D}\,\mathrm{d}v = \int_{\partial\mathcal{D}} \mathbf{D}\cdot\mathbf{n}\,\mathrm{d}s. \tag{4.47}$$

Then, by using Nanson's formula (3.18) and the connection $\mathbf{D}\cdot(J\mathbf{F}^{-\mathrm{T}}\mathbf{N}) = (J\mathbf{F}^{-1}\mathbf{D})\cdot\mathbf{N}$, applying the divergence theorem over the reference region $\mathcal{D}_\mathrm{r}$, with boundary $\partial\mathcal{D}_\mathrm{r}$, and converting the integral on the left-hand side to an integral over $\mathcal{D}_\mathrm{r}$, we obtain

$$\int_{\partial\mathcal{D}_\mathrm{r}} (J\mathbf{F}^{-1}\mathbf{D})\cdot\mathbf{N}\,\mathrm{d}S = \int_{\mathcal{D}_\mathrm{r}} \mathrm{Div}\,(J\mathbf{F}^{-1}\mathbf{D})\,\mathrm{d}V = \int_{\mathcal{D}_\mathrm{r}} J\rho_\mathrm{f}\,\mathrm{d}V. \tag{4.48}$$

Since $\mathcal{D}_\mathrm{r}$ is arbitrary we deduce that

$$\mathrm{Div}\,(J\mathbf{F}^{-1}\mathbf{D}) = J\rho_\mathrm{f}. \tag{4.49}$$

Inspection of (4.46) and (4.49) suggests the introduction of the Lagrangian quantities defined by

$$\mathbf{E}_\mathrm{L} = \mathbf{F}^\mathrm{T}\mathbf{E}, \quad \mathbf{D}_\mathrm{L} = J\mathbf{F}^{-1}\mathbf{D}, \quad \rho_\mathrm{F} = J\rho_\mathrm{f}, \tag{4.50}$$

where the subscript L signifies 'Lagrangian'. The first two of these are 'pull-back' versions of $\mathbf{E}$ and $\mathbf{D}$ from the deformed to the reference configuration, which we refer to as the Lagrangian electric and electric displacement fields, respectively. The third is a corresponding pull-back of the free charge density and represents the free charge density per unit reference volume. With these Lagrangian quantities (4.46) and (4.49) can be written as

$$\mathrm{Curl}\,\mathbf{E}_\mathrm{L} = \mathbf{0}, \quad \mathrm{Div}\,\mathbf{D}_\mathrm{L} = \rho_\mathrm{F}, \tag{4.51}$$

which are the Lagrangian equivalents of the Eulerian field equations (4.1) provided that the deformation is sufficiently regular. Alternatively, the kinematical identities (3.17) may be used to move directly between (4.1) and (4.51).

The Lagrangian forms of the boundary conditions are entirely analogous to their Eulerian counterparts in (4.3). The boundary conditions associated with (4.51) are

$$\mathbf{N}\times[\![\mathbf{E}_\mathrm{L}]\!] = \mathbf{0}, \quad \mathbf{N}\cdot[\![\mathbf{D}_\mathrm{L}]\!] = \sigma_\mathrm{F}, \tag{4.52}$$

where $\mathbf{N}$ is the unit normal to the reference boundary $\partial\mathcal{D}_\mathrm{r}$ associated with $\mathbf{n}$ through Nanson's formula (3.18) and σ_F is the density of free surface charges per unit reference area. In order to obtain these from (4.3) we note that the first requires use of the vector identity $J\mathbf{F}^{-1}[(\mathbf{F}^{-\mathrm{T}}\mathbf{N})\times(\mathbf{F}^{-\mathrm{T}}\mathbf{E}_\mathrm{L})] = \mathbf{N}\times\mathbf{E}_\mathrm{L}$, while the second involves the transformation $(J\mathbf{F}^{-\mathrm{T}}\mathbf{N})\cdot\mathbf{D} = \mathbf{N}\cdot(J\mathbf{F}^{-1}\mathbf{D})$. We do not include the Lagrangian counterparts of the forms (4.4) of the Eulerian boundary conditions since we do not make use of a Lagrangian counterpart of $\mathbf{P}$.

4.4.2 Equilibrium Equation and Traction Boundary Condition

In Sect. 4.2 we saw that the mechanical equilibrium equation could be written in various equivalent forms. From now on we focus on the form (4.14) that involves the total Cauchy stress tensor $\boldsymbol{\tau}$ in order to translate the equilibrium equation and boundary conditions into Lagrangian form. The natural transformation of $\boldsymbol{\tau}$ introduces the *total nominal stress tensor*, denoted $\mathbf{T}$, which is the generalization to the present context of the nominal stress tensor in nonlinear elasticity theory. This is related to $\boldsymbol{\tau}$ by

$$\mathbf{T} = J\mathbf{F}^{-1}\boldsymbol{\tau}. \tag{4.53}$$

Then, the equation of equilibrium (4.14) translates into the alternative form

$$\mathrm{Div}\,\mathbf{T} + \rho_\mathrm{r}\mathbf{f} = \mathbf{0} \quad \text{in } \mathcal{D}_\mathrm{r}, \tag{4.54}$$

where we recall that $\rho_\mathrm{r} = \rho J$ is the mass density of the material in $\mathcal{D}_\mathrm{r}$.

The traction boundary condition associated with (4.54) and corresponding to (4.20) can be re-cast as

$$\mathbf{T}^\mathrm{T}\mathbf{N} = \mathbf{t}_\mathrm{A} + \mathbf{t}_\mathrm{E}^\star \quad \text{on } \partial\mathcal{D}_\mathrm{r}, \tag{4.55}$$

where $\mathbf{t}_\mathrm{A}$ is the applied mechanical load per unit reference area, while $\mathbf{t}_\mathrm{E}^\star$ is the Maxwell traction per unit reference area, given by $\mathbf{t}_\mathrm{E}^\star = \mathbf{T}_\mathrm{E}^{\star\mathrm{T}}\mathbf{N}$ with $\mathbf{T}_\mathrm{E}^\star = J\mathbf{F}^{-1}\boldsymbol{\tau}_\mathrm{e}^\star$ and we have used Nanson's formula to transform the left-hand side ($\boldsymbol{\tau}\mathbf{n}\mathrm{d}s = \mathbf{T}^\mathrm{T}\mathbf{N}\mathrm{d}S$).

4.4.3 Constitutive Equations

The energy function $\phi(\mathbf{F}, \mathbf{E})$ was introduced in Sect. 4.3. In Dorfmann and Ogden (2005a) this was re-cast with $\mathbf{E}_\mathrm{L}$ used as the independent electric variable instead of $\mathbf{E}$. Instead of doing this directly here we introduce an intermediate step by defining an energy function $\psi = \psi(\mathbf{F}, \mathbf{E})$ by

$$\rho_\mathrm{r}\psi = \rho_\mathrm{r}\phi - \frac{1}{2}\varepsilon_0 J\mathbf{E}\cdot\mathbf{E}, \tag{4.56}$$

from which we obtain

$$\rho_\mathrm{r}\mathbf{F}\frac{\partial\psi}{\partial\mathbf{F}} = J\boldsymbol{\sigma} - \frac{1}{2}\varepsilon_0 J(\mathbf{E}\cdot\mathbf{E})\mathbf{I}, \quad \rho_\mathrm{r}\frac{\partial\psi}{\partial\mathbf{E}} = -J\mathbf{D}. \tag{4.57}$$

Now we define a new function, $\Psi = \Psi(\mathbf{F}, \mathbf{E}_\mathrm{L})$, through

$$\Psi(\mathbf{F}, \mathbf{E}_\mathrm{L}) = \psi(\mathbf{F}, \mathbf{E}) = \psi(\mathbf{F}, \mathbf{F}^{-\mathrm{T}}\mathbf{E}_\mathrm{L}). \tag{4.58}$$

To obtain the required result we need to use the formula for the derivative of $\mathbf{F}^{-1}$ with respect to $\mathbf{F}$. We give this in component form as

$$\frac{\partial F^{-1}_{\beta j}}{\partial F_{i\alpha}} = -F^{-1}_{\beta i}F^{-1}_{\alpha j}, \tag{4.59}$$

where the notation $F^{-1}_{\alpha i} = (\mathbf{F}^{-1})_{\alpha i}$ has been adopted. This can be obtained by differentiating the product $\mathbf{F}^{-1}\mathbf{F} = \mathbf{I}$ in component form and then rearranging. Then we obtain

$$\rho_\mathrm{r}\frac{\partial\Psi}{\partial\mathbf{F}} = J\mathbf{F}^{-1}\boldsymbol{\tau}, \quad \rho_\mathrm{r}\frac{\partial\Psi}{\partial\mathbf{E}_\mathrm{L}} = -\mathbf{D}_\mathrm{L}, \tag{4.60}$$

where $\boldsymbol{\tau} = \boldsymbol{\sigma} + \boldsymbol{\tau}_\mathrm{e}$ and $\boldsymbol{\tau}_\mathrm{e}$ is the Maxwell stress given by (4.10). Furthermore, for simplicity, we now absorb the factor ρ_r into Ψ and define the so-called *total energy density function* $\Omega = \Omega(\mathbf{F}, \mathbf{E}_\mathrm{L})$ per unit reference volume by

$$\Omega(\mathbf{F}, \mathbf{E}_\mathrm{L}) = \rho_\mathrm{r}\Psi(\mathbf{F}, \mathbf{E}_\mathrm{L}). \tag{4.61}$$

This then allows the constitutive equations to be written in the compact Lagrangian forms

$$\mathbf{T} = \frac{\partial\Omega}{\partial\mathbf{F}}, \quad \mathbf{D}_\mathrm{L} = -\frac{\partial\Omega}{\partial\mathbf{E}_\mathrm{L}}. \tag{4.62}$$

The corresponding Eulerian expressions based on Ω are given by

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial\Omega}{\partial\mathbf{F}}, \quad \mathbf{D} = -J^{-1}\mathbf{F}\frac{\partial\Omega}{\partial\mathbf{E}_\mathrm{L}}. \tag{4.63}$$

Since $\mathbf{E}_\mathrm{L}$ satisfies $\mathrm{Curl}\,\mathbf{E}_\mathrm{L} = \mathbf{0}$, we may introduce a scalar potential function $\varphi_\mathrm{L}(\mathbf{X})$ so that $\mathbf{E}_\mathrm{L} = -\mathrm{Grad}\,\varphi_\mathrm{L}$, analogously to $\mathbf{E} = -\mathrm{grad}\,\varphi$. Moreover, since, by (3.15), $\mathbf{F}^\mathrm{T}\mathrm{grad}\,\varphi = \mathrm{Grad}\,\varphi$, we have the connection $\varphi_\mathrm{L}(\mathbf{X}) = \varphi(\mathbf{x}) = \varphi(\boldsymbol{\chi}(\mathbf{X}))$. Given the mechanical body force and the form of Ω, the relevant equations to be solved for $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ and $\varphi_\mathrm{L}(\mathbf{X})$ are the coupled nonlinear equations

$$\mathrm{Div}\,\mathbf{T} + \rho_\mathrm{r}\mathbf{f} = \mathbf{0}, \quad \mathrm{Div}\,\mathbf{D}_\mathrm{L} = 0, \tag{4.64}$$

in conjunction with the constitutive equations (4.62), the boundary conditions (4.52) and (4.55) and the solution of Laplace's equation for $\varphi(\mathbf{x})$ outside the material.

As an alternative to using $\mathbf{E}_\mathrm{L}$, the Lagrangian electric displacement vector $\mathbf{D}_\mathrm{L}$ can be used as the independent electric variable. This can be achieved in a similar way to that leading to Ψ. We define $\psi^* = \psi^*(\mathbf{F},\mathbf{D})$ by

$$\rho_\mathrm{r}\psi^* = \rho_\mathrm{r}\bar{\phi} + \frac{1}{2}\varepsilon_0^{-1}J\mathbf{D}\cdot\mathbf{D} \tag{4.65}$$

and then calculate

$$\rho_\mathrm{r}\mathbf{F}\frac{\partial\psi^*}{\partial\mathbf{F}} = J\bar{\boldsymbol{\sigma}} + \frac{1}{2}\varepsilon_0^{-1}J(\mathbf{D}\cdot\mathbf{D})\mathbf{I}, \quad \rho_\mathrm{r}\frac{\partial\psi^*}{\partial\mathbf{D}} = J\mathbf{E}. \tag{4.66}$$

We now switch the independent variable $\mathbf{D}$ to $\mathbf{D}_\mathrm{L}$ by defining $\Psi^* = \Psi^*(\mathbf{F},\mathbf{D}_\mathrm{L})$ via

$$\Psi^*(\mathbf{F},\mathbf{D}_\mathrm{L}) = \psi^*(\mathbf{F},\mathbf{D}) = \psi^*(\mathbf{F},J^{-1}\mathbf{F}\mathbf{D}_\mathrm{L}), \tag{4.67}$$

and hence

$$\rho_\mathrm{r}\frac{\partial\Psi^*}{\partial\mathbf{F}} = J\mathbf{F}^{-1}\boldsymbol{\tau}, \quad \rho_\mathrm{r}\frac{\partial\Psi^*}{\partial\mathbf{D}_\mathrm{L}} = \mathbf{E}_\mathrm{L}. \tag{4.68}$$

Again we absorb the factor ρ_r, and in this case, we define the total energy (density) function $\Omega^* = \Omega^*(\mathbf{F},\mathbf{D}_\mathrm{L})$ per unit reference volume by

$$\Omega^*(\mathbf{F},\mathbf{D}_\mathrm{L}) = \rho_\mathrm{r}\Psi^*(\mathbf{F},\mathbf{D}_\mathrm{L}) \tag{4.69}$$

and arrive at the constitutive equations

$$\mathbf{T} = \frac{\partial\Omega^*}{\partial\mathbf{F}}, \quad \mathbf{E}_\mathrm{L} = \frac{\partial\Omega^*}{\partial\mathbf{D}_\mathrm{L}}, \tag{4.70}$$

and their Eulerian counterparts

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial\Omega^*}{\partial\mathbf{F}}, \quad \mathbf{E} = \mathbf{F}^{-\mathrm{T}}\frac{\partial\Omega^*}{\partial\mathbf{D}_\mathrm{L}}. \tag{4.71}$$

Note that Ω^* and Ω are connected via the partial Legendre transformation

$$\Omega^*(\mathbf{F},\mathbf{D}_\mathrm{L}) = \Omega(\mathbf{F},\mathbf{E}_\mathrm{L}) + \mathbf{E}_\mathrm{L}\cdot\mathbf{D}_\mathrm{L}, \tag{4.72}$$

but transformation between these two formulations requires that the relationships between $\mathbf{D}_\mathrm{L}$ and $\mathbf{E}_\mathrm{L}$ given by $(4.62)_2$ and $(4.71)_2$ be monotonic.

For these 'total' energy formulations the equation of energy balance (4.34) can be expressed via

$$\int_{\mathcal{D}} \rho \mathbf{f} \cdot \mathbf{u} \mathrm{d}v + \int_{\partial \mathcal{D}} (\boldsymbol{\tau} \mathbf{n}) \cdot \mathbf{u} \mathrm{d}s = \int_{\mathcal{D}} \operatorname{tr}(\boldsymbol{\tau} \mathbf{L}) \mathrm{d}v \tag{4.73}$$

as

$$\int_{\mathcal{D}_r} (\rho_r \mathbf{f} \cdot \mathbf{u} - \mathbf{D}_L \cdot \dot{\mathbf{E}}_L) \mathrm{d}V + \int_{\partial \mathcal{D}_r} (\mathbf{T}^T \mathbf{N}) \cdot \mathbf{u} \mathrm{d}S = \int_{\mathcal{D}_r} \dot{\Omega} \mathrm{d}V, \tag{4.74}$$

and

$$\int_{\mathcal{D}_r} (\rho_r \mathbf{f} \cdot \mathbf{u} + \mathbf{E}_L \cdot \dot{\mathbf{D}}_L) \mathrm{d}V + \int_{\partial \mathcal{D}_r} (\mathbf{T}^T \mathbf{N}) \cdot \mathbf{u} \mathrm{d}S = \int_{\mathcal{D}_r} \dot{\Omega}^* \mathrm{d}V, \tag{4.75}$$

where the integrals have been cast in the reference configuration.

4.4.4 Incompressible Materials

The expressions for the various stress tensors in the foregoing section apply for a material that is not subject to any internal mechanical constraint. For an important class of materials, including electro-sensitive elastomers, it is appropriate to adopt the constraint of incompressibility, in which case the expressions for the stresses require modification.

For an incompressible material we have the constraint

$$\det \mathbf{F} \equiv 1 \tag{4.76}$$

and $\rho = \rho_r$. In accordance with the procedure outlined in Chap. 3 for an elastic material, the total nominal stress $\mathbf{T}$ and Cauchy stress $\boldsymbol{\tau}$ given by $(4.62)_1$ and $(4.63)_1$ in terms of Ω are then amended in the forms

$$\mathbf{T} = \frac{\partial \Omega}{\partial \mathbf{F}} - p \mathbf{F}^{-1}, \quad \boldsymbol{\tau} = \mathbf{F} \frac{\partial \Omega}{\partial \mathbf{F}} - p \mathbf{I}, \tag{4.77}$$

respectively, where p is a Lagrange multiplier associated with the constraint (4.76). The expressions $(4.62)_2$ and $(4.63)_2$ are unchanged except that (4.76) is in force. In terms of Ω^*, we have, instead of (4.77),

$$\mathbf{T} = \frac{\partial \Omega^*}{\partial \mathbf{F}} - p^* \mathbf{F}^{-1}, \quad \boldsymbol{\tau} = \mathbf{F} \frac{\partial \Omega^*}{\partial \mathbf{F}} - p^* \mathbf{I}, \tag{4.78}$$

where, in general, the p in (4.77) is not the same as the p^* in (4.78).

4.4.5 *Material Symmetry Considerations*

Thus far no restrictions have been placed on the forms of the energy functions other than those required by objectivity, so that considerable generality remains. Other restrictions may be physically or mathematically based. For example, physical restrictions arise from the nature of the material itself, such as its inherent symmetry. Electro-sensitive elastomers are typically isotropic in their response in the absence of an electric field, but application of an electric field endows the material with a preferred direction. Thus, the electric field vector $\mathbf{E}$ generates a preferred direction in the deformed configuration $\mathcal{B}$. However, from the point of view of constitutive law development, it is advantageous to make use of the Lagrangian field $\mathbf{E}_{\mathrm{L}}$ instead of $\mathbf{E}$ and to consider the energy function $\Omega(\mathbf{F}, \mathbf{E}_{\mathrm{L}})$. Equally, $\Omega^{*}(\mathbf{F}, \mathbf{D}_{\mathrm{L}})$ may be adopted.

4.4.5.1 Isotropic Materials

For simplicity we now consider the so-called *isotropic electro-sensitive materials*, for which the material symmetry considerations are similar to those that arise for a transversely isotropic elastic material, which possesses a preferred direction in the reference configuration. This is appropriate for fiber-reinforced materials, for which the preferred direction is the fiber direction in the reference configuration, as discussed in Sect. 3.3.3.2. The vector field $\mathbf{E}_{\mathrm{L}}$ has an analogous role in the present context, as does $\mathbf{D}_{\mathrm{L}}$.

The electroelastic material considered here is said to be *isotropic* if Ω is an isotropic function of the two tensors $\mathbf{c}$ and $\mathbf{E}_{\mathrm{L}} \otimes \mathbf{E}_{\mathrm{L}}$. Note that the latter expression is unaffected by reversal of the sign of $\mathbf{E}_{\mathrm{L}}$. Then, the form of Ω is reduced to dependence on the principal invariants I_1, I_2, I_3 of $\mathbf{c}$, which, we recall from (3.132), are defined by

$$I_1 = \mathrm{tr}\,\mathbf{c}, \quad I_2 = \frac{1}{2}\left[(\mathrm{tr}\,\mathbf{c})^2 - \mathrm{tr}\,(\mathbf{c}^2)\right], \quad I_3 = \det \mathbf{c} = J^2, \tag{4.79}$$

together with three independent invariants (sometimes referred to as quasi-invariants) that depend on $\mathbf{E}_{\mathrm{L}}$. A convenient choice of the latter is

$$I_4 = \mathbf{E}_{\mathrm{L}} \cdot \mathbf{E}_{\mathrm{L}}, \quad I_5 = (\mathbf{c}\mathbf{E}_{\mathrm{L}}) \cdot \mathbf{E}_{\mathrm{L}}, \quad I_6 = (\mathbf{c}^2\mathbf{E}_{\mathrm{L}}) \cdot \mathbf{E}_{\mathrm{L}}. \tag{4.80}$$

(We have duplicated the notation in (3.150) since the expressions for I_4 and I_5 in (3.150) will not be used henceforth.) Note that for a transversely isotropic elastic material the counterpart of the invariant I_4 would be absent since in that case the preferred direction is a unit vector.

In the following, the subscripts $1, 2, \ldots, 6$ on Ω signify differentiation with respect to $I_1, I_2, \ldots, I_6$, respectively. We require the first derivatives of these invariants with respect to $\mathbf{F}$ and $\mathbf{E}_\mathrm{L}$. The derivatives of I_1, I_2 and I_3 with respect to $\mathbf{F}$ are given by (3.135). Their derivatives with respect to $\mathbf{E}_\mathrm{L}$ vanish, and the remaining non-vanishing derivatives are

$$\frac{\partial I_5}{\partial \mathbf{F}} = 2\mathbf{E}_\mathrm{L} \otimes \mathbf{F}\mathbf{E}_\mathrm{L}, \quad \frac{\partial I_6}{\partial \mathbf{F}} = 2(\mathbf{E}_\mathrm{L} \otimes \mathbf{F}\mathbf{c}\mathbf{E}_\mathrm{L} + \mathbf{c}\mathbf{E}_\mathrm{L} \otimes \mathbf{F}\mathbf{E}_\mathrm{L}), \tag{4.81}$$

which are similar to those in (3.152), and

$$\frac{\partial I_4}{\partial \mathbf{E}_\mathrm{L}} = 2\mathbf{E}_\mathrm{L}, \quad \frac{\partial I_5}{\partial \mathbf{E}_\mathrm{L}} = 2\mathbf{c}\mathbf{E}_\mathrm{L}, \quad \frac{\partial I_6}{\partial \mathbf{E}_\mathrm{L}} = 2\mathbf{c}^2\mathbf{E}_\mathrm{L}. \tag{4.82}$$

A direct calculation based on $(4.63)_1$ then leads to

$$\begin{aligned}\boldsymbol{\tau} = {} & J^{-1}[2\Omega_1\mathbf{b} + 2\Omega_2(I_1\mathbf{b} - \mathbf{b}^2) + 2I_3\Omega_3\mathbf{I} + 2\Omega_5\mathbf{b}\mathbf{E} \otimes \mathbf{b}\mathbf{E} \\ & + 2\Omega_6(\mathbf{b}\mathbf{E} \otimes \mathbf{b}^2\mathbf{E} + \mathbf{b}^2\mathbf{E} \otimes \mathbf{b}\mathbf{E})],\end{aligned} \tag{4.83}$$

which is clearly symmetric, and $(4.63)_2$ gives

$$\mathbf{D} = -2J^{-1}(\Omega_4\mathbf{b}\mathbf{E} + \Omega_5\mathbf{b}^2\mathbf{E} + \Omega_6\mathbf{b}^3\mathbf{E}), \tag{4.84}$$

where we recall that $\mathbf{b} = \mathbf{F}\mathbf{F}^\mathrm{T}$ is the left Cauchy–Green deformation tensor. The corresponding Lagrangian forms may be obtained from the connections $\mathbf{T} = J\mathbf{F}^{-1}\boldsymbol{\tau}$ and $\mathbf{D}_\mathrm{L} = J\mathbf{F}^{-1}\mathbf{D}$.

For an incompressible material, $I_3 \equiv 1$ and (4.83) is replaced by

$$\boldsymbol{\tau} = 2\Omega_1\mathbf{b} + 2\Omega_2(I_1\mathbf{b} - \mathbf{b}^2) - p\mathbf{I} + 2\Omega_5\mathbf{b}\mathbf{E} \otimes \mathbf{b}\mathbf{E} + 2\Omega_6(\mathbf{b}\mathbf{E} \otimes \mathbf{b}^2\mathbf{E} + \mathbf{b}^2\mathbf{E} \otimes \mathbf{b}\mathbf{E}), \tag{4.85}$$

while (4.84) is unchanged in form, but with $J = 1$ and I_3 absent from Ω. Note that we have not introduced a new notation for Ω when it is treated as a function of the invariants, and this will be the case henceforth, and similarly for Ω^*.

If we work with Ω^* instead of Ω, then the invariants based on $\mathbf{E}_\mathrm{L}$ have to be changed to invariants based on $\mathbf{D}_\mathrm{L}$. These are denoted here by K_4, K_5, K_6, and possible choices for these, analogously to (4.80), are

$$K_4 = \mathbf{D}_\mathrm{L} \cdot \mathbf{D}_\mathrm{L}, \quad K_5 = (\mathbf{c}\mathbf{D}_\mathrm{L}) \cdot \mathbf{D}_\mathrm{L}, \quad K_6 = (\mathbf{c}^2\mathbf{D}_\mathrm{L}) \cdot \mathbf{D}_\mathrm{L}. \tag{4.86}$$

The associated formula for $\boldsymbol{\tau}$ is similar to that based on Ω. For an incompressible material, for example, we have

$$\boldsymbol{\tau} = 2\Omega_1^*\mathbf{b} + 2\Omega_2^*(I_1\mathbf{b} - \mathbf{b}^2) - p^*\mathbf{I} + 2\Omega_5^*\mathbf{D} \otimes \mathbf{D} + 2\Omega_6^*(\mathbf{D} \otimes \mathbf{b}\mathbf{D} + \mathbf{b}\mathbf{D} \otimes \mathbf{D}). \tag{4.87}$$

The electric field is given by $(4.71)_2$ and has the form

$$\mathbf{E} = 2(\Omega_4^*\mathbf{b}^{-1}\mathbf{D} + \Omega_5^*\mathbf{D} + \Omega_6^*\mathbf{b}\mathbf{D}). \tag{4.88}$$

The Lagrangian counterparts are obtained via $\mathbf{T} = J\mathbf{F}^{-1}\boldsymbol{\tau}$ and $\mathbf{E}_\mathrm{L} = \mathbf{F}^\mathrm{T}\mathbf{E}$. In (4.87) and (4.88), for which $\Omega^* = \Omega^*(I_1, I_2, K_4, K_5, K_6)$, Ω_i^* is defined as $\partial\Omega^*/\partial I_i$ for $i = 1, 2$ and $\partial\Omega^*/\partial K_i$ for $i = 4, 5, 6$.

When the material is not isotropic the constitutive formulation is more complicated and involves a larger number of invariants. For example, for a transversely isotropic electroelastic material, ten invariants are required (nine for an incompressible material), as shown by Bustamante (2009).

There are important differences between the formulations based on Ω and Ω^* in respect of their application to particular boundary-value problems. If $\mathbf{D}_\mathrm{L}$ is taken as the independent variable, then it has to satisfy $\mathrm{Div}\,\mathbf{D}_\mathrm{L} = 0$ and hence can be replaced by a vector potential, $\mathbf{A}_\mathrm{L}$ say, such that $\mathbf{D}_\mathrm{L} = \mathrm{Curl}\,\mathbf{A}_\mathrm{L}$. The resulting $\mathbf{E}_\mathrm{L}$, calculated from $(4.70)_2$, then has to satisfy the vector equation $\mathrm{Curl}\,\mathbf{E}_\mathrm{L} = \mathbf{0}$, which, for some problems, puts severe restrictions on the class of constitutive laws that admit the deformation in question for the considered electric displacement field $\mathbf{D}_\mathrm{L}$. On the other hand, if we start with $\mathbf{E}_\mathrm{L}$ as the independent variable, it has to satisfy $\mathrm{Curl}\,\mathbf{E}_\mathrm{L} = \mathbf{0}$, and hence we can work in terms of a scalar potential φ_L with $\mathbf{E}_\mathrm{L} = -\mathrm{Grad}\,\varphi_\mathrm{L}$, and then the resulting $\mathbf{D}_\mathrm{L}$, calculated from $(4.62)_2$, has to satisfy the scalar equation $\mathrm{Div}\,\mathbf{D}_\mathrm{L} = 0$. This also may, in some situations, put restrictions on the admissible class of constitutive laws, but they are different from and generally less severe than for the other formulation. We shall discuss the solution of representative problems in Chap. 5.

An interesting alternative approach to the construction of constitutive laws for electroelastic materials based on an implicit theory has recently been developed by Bustamante and Rajagopal (2013a) and applied to several boundary-value problems in Bustamante and Rajagopal (2013b).

4.4.5.2 Hemitropic Materials

For an isotropic material the symmetry condition

$$\Omega(\mathbf{QcQ}^\mathrm{T}, \mathbf{QE}_\mathrm{L} \otimes \mathbf{E}_\mathrm{L}\mathbf{Q}^\mathrm{T}) = \Omega(\mathbf{c}, \mathbf{E}_\mathrm{L} \otimes \mathbf{E}_\mathrm{L}) \tag{4.89}$$

must be satisfied for all orthogonal $\mathbf{Q}$, and the stress is unaffected by reversal of the direction of the electric field. A hemitropic material, on the other hand, does not possess centrosymmetry (i.e. its properties depend on the sense of $\mathbf{E}_\mathrm{L}$), and we have to consider $\mathbf{E}_\mathrm{L}$ rather than $\mathbf{E}_\mathrm{L} \otimes \mathbf{E}_\mathrm{L}$ in the argument of Ω. The appropriate symmetry condition in this case is then written

$$\Omega(\mathbf{QcQ}^\mathrm{T}, \mathbf{QE}_\mathrm{L}) = \Omega(\mathbf{c}, \mathbf{E}_\mathrm{L}) \tag{4.90}$$

for all *proper* orthogonal $\mathbf{Q}$. This introduces one extra invariant, which we denote by I_7, defined by

$$I_7 = \mathbf{E}_\mathrm{L} \cdot [(\mathbf{cE}_\mathrm{L}) \times (\mathbf{c}^2\mathbf{E}_\mathrm{L})]. \tag{4.91}$$

Note that, by means of the vector identity (A.10) in Appendix A, I_7 can also be written as $I_7 = J\mathbf{E} \cdot [(\mathbf{bE}) \times (\mathbf{b}^2\mathbf{E})]$. When I_7 is included in the invariant representation of Ω, the expressions for $\boldsymbol{\tau}$ and $\mathbf{D}$ are significantly more complicated than for the isotropic case. For completeness we record these expressions here as

$$\begin{aligned}\boldsymbol{\tau} = {} & 2J^{-1}[\Omega_1\mathbf{b} + \Omega_2(I_1\mathbf{b} - \mathbf{b}^2) + I_3\Omega_3\mathbf{I} + \Omega_5\mathbf{bE}\otimes\mathbf{bE} \\ & + \Omega_6(\mathbf{bE}\otimes\mathbf{b}^2\mathbf{E} + \mathbf{b}^2\mathbf{E}\otimes\mathbf{bE})] + \Omega_7\{[(\mathbf{b}^2\mathbf{E})\times\mathbf{E}]\otimes\mathbf{bE} \\ & + \mathbf{bE}\otimes[(\mathbf{b}^2\mathbf{E})\times\mathbf{E}] + [\mathbf{E}\times(\mathbf{bE})]\otimes\mathbf{b}^2\mathbf{E} + \mathbf{b}^2\mathbf{E}\otimes[\mathbf{E}\times(\mathbf{bE})] \\ & + \mathbf{b}[\mathbf{E}\times(\mathbf{bE})]\otimes\mathbf{bE} + \mathbf{bE}\otimes\mathbf{b}[\mathbf{E}\times(\mathbf{bE})]\},\end{aligned} \tag{4.92}$$

and

$$\begin{aligned}\mathbf{D} = {} & -2J^{-1}(\Omega_4\mathbf{bE} + \Omega_5\mathbf{b}^2\mathbf{E} + \Omega_6\mathbf{b}^3\mathbf{E}) - \Omega_7\{(\mathbf{bE})\times(\mathbf{b}^2\mathbf{E}) \\ & + \mathbf{b}[(\mathbf{b}^2\mathbf{E})\times\mathbf{E}] + \mathbf{b}^2[\mathbf{E}\times(\mathbf{bE})]\},\end{aligned} \tag{4.93}$$

respectively, where $\Omega_7 = \partial\Omega/\partial I_7$. Note that all the terms involving Ω_7 disappear if $\mathbf{E}$ is aligned with a principal axis of $\mathbf{b}$, as does I_7 itself.

4.5 Specialization to Linear Elasticity

We now relate the theory in the previous section to the linear theory of isotropic electroelasticity. We restrict attention to the theory based on Ω. Since I_4, I_5, I_6 depend quadratically on the electric field, the stress tensor (4.83) does not include a term linear in the electric field. We therefore retain terms of quadratic order in $\mathbf{E}$. For purposes of linearization in the strain we consider the infinitesimal strain tensor $\mathbf{e}$ so that, to linear order $\mathbf{b} = \mathbf{I} + 2\mathbf{e}$, $I_1 = 3 + 2e$, $I_2 = 3 + 4e$, $I_3 = 1 + 2e$, where $e = \operatorname{tr}\mathbf{e}$. To the considered order there is no need to distinguish $\mathbf{E}$ from $\mathbf{E}_\mathrm{L}$ and $I_4 = I_5 = I_6 = \mathbf{E}\cdot\mathbf{E}$.

Linearization of $\boldsymbol{\tau}$ with terms linear in $\mathbf{e}$ and $\mathbf{E}\otimes\mathbf{E}$ retained leads, after some algebra, to

$$\boldsymbol{\tau} = 2\mu\mathbf{e} + \lambda e\mathbf{I} + \nu(\mathbf{E}\cdot\mathbf{E})\mathbf{I} + \xi\mathbf{E}\otimes\mathbf{E}, \tag{4.94}$$

where λ, μ, ν and ξ are constants defined in terms of the Ω_i and the second derivatives $\Omega_{ij} = \partial^2\Omega/\partial I_i\partial I_j$ by

$$\lambda = 4(\Omega_{11} + 4\Omega_{12} + 2\Omega_{13} + 4\Omega_{22} + 4\Omega_{23} + \Omega_{33} + \Omega_2 + \Omega_3), \tag{4.95}$$

$$\mu = 2(\Omega_1 + \Omega_2), \tag{4.96}$$

$$\nu = 2(\Omega_{14} + \Omega_{15} + \Omega_{16} + 2\Omega_{24} + 2\Omega_{25} + 2\Omega_{26} + \Omega_{34} + \Omega_{35} + \Omega_{36}), \tag{4.97}$$

$$\xi = 2(\Omega_5 + 2\Omega_6), \tag{4.98}$$

all evaluated in the reference configuration where $\mathbf{F} = \mathbf{I}$ and $\mathbf{E} = \mathbf{0}$, and we have used the fact that

$$\Omega_1 + 2\Omega_2 + \Omega_3 = 0 \tag{4.99}$$

when evaluated in the reference configuration, which arises since the reference configuration is stress-free and can be obtained from (4.83) (recall (3.147) in the purely elastic case). Note also that λ and μ are the classical Lamé moduli (already mentioned in (3.148)).

The corresponding approximation for $\mathbf{D}$ is obtained from (4.84) by linearizing with respect to $\mathbf{e}$ and $\mathbf{E}$ (but retaining quadratic terms in the their product) to give

$$\mathbf{D} = \alpha\mathbf{E} - (2\nu + \alpha)e\mathbf{E} + 2(\alpha - \xi)\mathbf{eE}, \tag{4.100}$$

which introduces the additional material constant defined by

$$\alpha = -2(\Omega_4 + \Omega_5 + \Omega_6), \tag{4.101}$$

evaluated in the reference configuration.

The quadratic dependence of the stress on the electric field evident in (4.94) indicates that the material possesses centrosymmetry. Thus, if the stress vanishes, then a strain is induced by application of an electric field and is independent of the sense of the electric field. This is the phenomenon of *electrostriction*, for a general discussion of which we refer to Eringen and Maugin (1990), for example. This contrasts with the *piezoelectric effect*, which occurs in materials without a centre of symmetry, as we discuss briefly in the following section.

4.5.1 Relation to Linear Piezoelectricity

In the linear theory of piezoelectricity the stress depends linearly on the electric field and therefore changes sign when the electric field is reversed. Such a dependence is not captured by the isotropic theory in the above section since the stress depends quadratically on $\mathbf{E}$. In general, if the stress is reversed by reversal of the electric field, then

$$\frac{\partial\Omega}{\partial\mathbf{F}}(\mathbf{F}, -\mathbf{E}_\mathrm{L}) = -\frac{\partial\Omega}{\partial\mathbf{F}}(\mathbf{F}, \mathbf{E}_\mathrm{L}) \tag{4.102}$$

must be satisfied. For example, a linear term in Ω of the form $\mathbf{A}(\mathbf{c})\cdot\mathbf{E}_\mathrm{L}$ satisfies this condition, where $\mathbf{A}$ is a vector function of $\mathbf{c}$ defined in the reference configuration and therefore automatically objective.

The contribution to the total Cauchy stress from just this single term is

$$-J^{-1}\mathbf{F}(\mathbf{E}_{\mathrm{L}}\boldsymbol{\mathcal{E}})\mathbf{F}^{\mathrm{T}}, \quad -J^{-1}F_{i\alpha}E_{\mathrm{L}\gamma}\mathcal{E}_{\gamma|\alpha\beta}F_{j\beta}, \tag{4.103}$$

where

$$\boldsymbol{\mathcal{E}} = -2\frac{\partial \mathbf{A}}{\partial \mathbf{c}}, \quad \mathcal{E}_{\gamma|\alpha\beta} = -\frac{\partial A_\gamma}{\partial c_{\alpha\beta}} = \mathcal{E}_{\gamma|\beta\alpha}. \tag{4.104}$$

The minus sign has been included for consistency with the linear theory of piezoelectricity, and the vertical bar has been introduced to separate the single index associated with the vector **A** from the pair of indices associated with the tensor **c**.

The corresponding contribution to **D** is

$$-J^{-1}\mathbf{F}\mathbf{A}(\mathbf{c}), \quad -J^{-1}F_{i\alpha}A_\alpha(\mathbf{c}). \tag{4.105}$$

When the material is undeformed, the latter term should not contribute to **D**, and hence we set

$$\mathbf{A}(\mathbf{I}) = \mathbf{0}. \tag{4.106}$$

Turning now to the linear specialization (linear in both the infinitesimal strain tensor **e** and the electric field **E**), the above contributions to the stress and electric displacement reduce to

$$-\mathbf{E}\boldsymbol{\mathcal{E}}, \quad -E_k\mathcal{E}_{k|ij} \tag{4.107}$$

and

$$\boldsymbol{\mathcal{E}}\mathbf{e}, \quad \mathcal{E}_{i|jk}e_{jk}, \tag{4.108}$$

respectively, Greek indices now being replaced by Roman ones. The coefficients $\mathcal{E}_{i|jk}$ are the components of the so-called *piezoelectric tensor* $\boldsymbol{\mathcal{E}}$ (see, e.g., Eringen and Maugin 1990; Yang 2005).

For a linear isotropically elastic piezoelectric material, for example, the stress is given by

$$\boldsymbol{\tau} = 2\mu\mathbf{e} + \lambda e\mathbf{I} - \mathbf{E}\boldsymbol{\mathcal{E}}, \quad \tau_{ij} = 2\mu e_{ij} + \lambda e\delta_{ij} - E_k\mathcal{E}_{k|ij}, \tag{4.109}$$

and the electric displacement by

$$\mathbf{D} = \alpha\mathbf{E} + \boldsymbol{\mathcal{E}}\mathbf{e}, \quad D_i = \alpha E_i + \mathcal{E}_{i|jk}e_{jk}. \tag{4.110}$$

The coefficient α is the electric permittivity of the considered material.

4.5.2 General Linear Theory

Recalling that the Green strain tensor is defined by $\mathbf{e} = (\mathbf{c} - \mathbf{I})/2$, we may regard Ω as a function of $\mathbf{e}$ and $\mathbf{E}_\mathrm{L}$. Now we restrict attention to situations in which both $\mathbf{e}$ and $\mathbf{E}_\mathrm{L}$ are small and Ω can be expanded to quadratic terms relative to the reference configuration where both vanish. Thus,

$$\Omega = \frac{1}{2}\mathrm{tr}[(\boldsymbol{\mathcal{C}}\mathbf{e})\mathbf{e}] - (\boldsymbol{\mathcal{E}}\mathbf{e}) \cdot \mathbf{E} - \frac{1}{2}(\mathsf{K}\mathbf{E}) \cdot \mathbf{E}, \tag{4.111}$$

in components

$$\Omega = \frac{1}{2}\mathcal{C}_{ijkl}e_{ij}e_{kl} - \mathcal{E}_{k|ij}e_{ij}E_k - \frac{1}{2}\mathsf{K}_{ij}E_iE_j, \tag{4.112}$$

where, with $\Omega(\mathbf{e}, \mathbf{E})$,

$$\boldsymbol{\mathcal{C}} = \frac{\partial^2\Omega}{\partial\mathbf{e}\partial\mathbf{e}}(\mathbf{O}, \mathbf{0}), \quad \mathcal{C}_{ijkl} = \frac{\partial^2\Omega}{\partial e_{ij}\partial e_{kl}}(\mathbf{O}, \mathbf{0}), \tag{4.113}$$

$$\boldsymbol{\mathcal{E}} = -\frac{\partial^2\Omega}{\partial\mathbf{E}\partial\mathbf{e}}(\mathbf{O}, \mathbf{0}), \quad \boldsymbol{\mathcal{E}}^\mathrm{T} = -\frac{\partial^2\Omega}{\partial\mathbf{e}\partial\mathbf{E}}(\mathbf{O}, \mathbf{0}), \quad \mathcal{E}_{k|ij} = -\frac{\partial^2\Omega}{\partial E_k\partial e_{ij}}(\mathbf{O}, \mathbf{0}), \tag{4.114}$$

$$\mathsf{K} = -\frac{\partial^2\Omega}{\partial\mathbf{E}\partial\mathbf{E}}(\mathbf{O}, \mathbf{0}), \quad \mathsf{K}_{ij} = -\frac{\partial^2\Omega}{\partial E_i\partial E_j}(\mathbf{O}, \mathbf{0}), \tag{4.115}$$

$\mathbf{O}$ being the zero second-order tensor and $\mathbf{0}$ the zero vector. We note the symmetries

$$\mathcal{C}_{ijkl} = \mathcal{C}_{klij} = \mathcal{C}_{ijlk}, \quad \mathcal{E}_{k|ij} = \mathcal{E}_{k|ji} = \mathcal{E}_{ij|k}, \quad \mathsf{K}_{ij} = \mathsf{K}_{ji}. \tag{4.116}$$

Again the vertical bar is a separator used to distinguish the pair of indices that go together from the single index.

In this linear approximation there is no distinction between $\mathbf{E}_\mathrm{L}$ and $\mathbf{E}$ at the considered order of approximation, and $\mathbf{e}$ is the infinitesimal strain tensor $[\mathrm{grad}\,\mathbf{u} + (\mathrm{grad}\,\mathbf{u})^\mathrm{T}]/2$. There is also no distinction between $\mathbf{T}$ and $\boldsymbol{\tau}$ or between $\mathbf{D}_\mathrm{L}$ and $\mathbf{D}$, and we obtain

$$\boldsymbol{\tau} = \boldsymbol{\mathcal{C}}\mathbf{e} - \boldsymbol{\mathcal{E}}^\mathrm{T}\mathbf{E}, \quad \tau_{ij} = \mathcal{C}_{ijkl}e_{kl} - \mathcal{E}_{k|ij}E_k, \tag{4.117}$$

$$\mathbf{D} = \boldsymbol{\mathcal{E}}\mathbf{e} + \mathsf{K}\mathbf{E}, \quad D_i = \mathcal{E}_{i|jk}e_{jk} + \mathsf{K}_{ij}E_j. \tag{4.118}$$

These are the classical equations appropriate for an anisotropic piezoelectric material. Note, in particular, that the stress is linear in the electric field, a dependence that is not captured by the isotropic or hemitropic constitutive laws when they are specialized to the linear theory. The fourth-order tensor $\boldsymbol{\mathcal{C}}$ is the elastic modulus tensor for the case when the electric field is the independent electric variable, the third-order tensor $\boldsymbol{\mathcal{E}}$ is the piezoelectric tensor that couples the electric and mechanical effects, and the second-order tensor K is the dielectric tensor.

When $\mathbf{D}$ rather than $\mathbf{E}$ is used as the independent electric variable, we work in terms of $\Omega^*(\mathbf{e}, \mathbf{D})$ in the linear approximation, and then

$$\boldsymbol{\tau} = \boldsymbol{\mathcal{C}}^*\mathbf{e} + \boldsymbol{\mathcal{E}}^{*\mathrm{T}}\mathbf{D}, \quad \tau_{ij} = \mathcal{C}^*_{ijkl}e_{kl} + \mathcal{E}^*_{k|ij}D_k, \tag{4.119}$$

$$\mathbf{E} = \boldsymbol{\mathcal{E}}^*\mathbf{e} + \mathsf{K}^*\mathbf{D}, \quad E_i = \mathcal{E}^*_{i|jk}e_{jk} + \mathsf{K}^*_{ij}E_j, \tag{4.120}$$

where

$$\boldsymbol{\mathcal{C}}^* = \frac{\partial^2\Omega^*}{\partial\mathbf{e}\partial\mathbf{e}}(\mathbf{O}, \mathbf{0}), \quad \boldsymbol{\mathcal{E}}^* = \frac{\partial^2\Omega^*}{\partial\mathbf{D}\partial\mathbf{e}}(\mathbf{O}, \mathbf{0}), \quad \mathsf{K}^* = -\frac{\partial^2\Omega^*}{\partial\mathbf{D}\partial\mathbf{D}}(\mathbf{O}, \mathbf{0}). \tag{4.121}$$

From these relations we easily obtain

$$\boldsymbol{\mathcal{C}} - \boldsymbol{\mathcal{C}}^* = \boldsymbol{\mathcal{E}}^{\mathrm{T}}\boldsymbol{\mathcal{E}}^* = \boldsymbol{\mathcal{E}}^{*\mathrm{T}}\boldsymbol{\mathcal{E}}, \quad \mathsf{K}^* = \mathsf{K}^{-1}. \tag{4.122}$$

Note that for an isotropic or hemitropic material $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^* = \mathbf{O}$, which now denotes the zero third-order tensor.

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Chapter 5
Electroelastic Boundary-Value Problems

Abstract In this chapter we apply the theory of nonlinear electroelasticity developed in the previous chapter to several representative boundary-value problems using either the constitutive equation with the electric field as the independent electric variable or the electric displacement. First we consider two homogeneous deformations, namely pure homogeneous strain and simple shear of a slab of material with finite thickness and two parallel plane faces. We then examine three non-homogeneous deformations. For the inflation and extension of a thick-walled circular cylindrical tube subject to an axial or radial electric field, we obtain, in particular, expressions for the inflating pressure and the resultant axial load as functions of the electric field in each case. For the helical shear of a thick-walled circular cylindrical tube, we find conditions on the energy function and electric field for which the deformation is admissible. Finally, we study the problem of the radial inflation of a thick-walled spherical shell under a radial electric field and obtain an expression for the inflating pressure for a general form of energy function, and we show, by taking account of the Maxwell stress, that the electric field counteracts the effect of the pressure.

5.1 Governing Equations

We now apply the equations derived in Chap. 4 to solve representative boundary-value problems in order to illustrate the theory. Exact solutions are available only for a very limited number of problems, some of which are discussed in detail by Dorfmann and Ogden (2005, 2006). As compared to the purely elastic case, when electromechanical interactions are present, a significant problem arises in the difficulty of meeting the boundary conditions for the electric field **E** and the electric displacement **D** for bodies with finite geometry. For example, the boundary conditions on the ends of a tube of finite length are not in general compatible with those of the lateral surfaces; see Bustamante et al. (2007) for discussion of this in the magnetoelastic context, where a numerical method was used to cater for the

L. Dorfmann and R.W. Ogden, *Nonlinear Theory of Electroelastic and Magnetoelastic Interactions*, DOI 10.1007/978-1-4614-9596-3_5,

(spatially) rapidly changing fields near the ends of the tube. For this reason, here we consider boundary-value problems with selected geometries such that the boundary conditions can be satisfied exactly. Solutions of the corresponding problems for the purely elastic situation are given in Sect. 3.4.

Solutions in the present chapter are based on the use of one or other of the energy functions $\Omega(\mathbf{F},\mathbf{E}_\mathrm{L})$ and $\Omega^*(\mathbf{F},\mathbf{D}_\mathrm{L})$ introduced in Sect. 4.4. We focus attention on incompressible materials so that the constitutive equations can be summarized as

$$\boldsymbol{\tau} = \mathbf{F}\frac{\partial\Omega}{\partial\mathbf{F}} - p\mathbf{I}, \quad \mathbf{D} = -\mathbf{F}\frac{\partial\Omega}{\partial\mathbf{E}_\mathrm{L}}, \tag{5.1}$$

$$\boldsymbol{\tau} = \mathbf{F}\frac{\partial\Omega^*}{\partial\mathbf{F}} - p^*\mathbf{I}, \quad \mathbf{E} = \mathbf{F}^{-\mathrm{T}}\frac{\partial\Omega^*}{\partial\mathbf{D}_\mathrm{L}} \tag{5.2}$$

for Ω and Ω^*, respectively, in Eulerian form. We restrict attention to the Eulerian forms in this chapter.

For incompressible isotropic materials, Ω depends on the invariants $(I_1, I_2, I_4, I_5, I_6)$ and Ω^* depends on $(I_1, I_2, K_4, K_5, K_6)$, where the invariants are defined in (4.79), (4.80) and (4.86). We summarize the relevant equations from Sect. 4.4.5.1 in Eulerian form as

$$\begin{aligned}\boldsymbol{\tau} = {} & 2\Omega_1\mathbf{b} + 2\Omega_2\left(I_1\mathbf{b} - \mathbf{b}^2\right) - p\mathbf{I} + 2\Omega_5\mathbf{bE}\otimes\mathbf{bE} \\ & + 2\Omega_6\left(\mathbf{bE}\otimes\mathbf{b}^2\mathbf{E} + \mathbf{b}^2\mathbf{E}\otimes\mathbf{bE}\right),\end{aligned} \tag{5.3}$$

$$\mathbf{D} = -2(\Omega_4\mathbf{b} + \Omega_5\mathbf{b}^2 + \Omega_6\mathbf{b}^3)\mathbf{E}, \tag{5.4}$$

$$\begin{aligned}\boldsymbol{\tau} = {} & 2\Omega_1^*\mathbf{b} + 2\Omega_2^*\left(I_1\mathbf{b} - \mathbf{b}^2\right) - p^*\mathbf{I} + 2\Omega_5^*\mathbf{D}\otimes\mathbf{D} \\ & + 2\Omega_6^*\left(\mathbf{D}\otimes\mathbf{bD} + \mathbf{bD}\otimes\mathbf{D}\right),\end{aligned} \tag{5.5}$$

$$\mathbf{E} = 2(\Omega_4^*\mathbf{b}^{-1} + \Omega_5^*\mathbf{I} + \Omega_6^*\mathbf{b})\mathbf{D}. \tag{5.6}$$

The governing equations that need to be solved for any boundary-value problem are summarized next. These are

$$\operatorname{div}\mathbf{D} = 0, \quad \operatorname{curl}\mathbf{E} = \mathbf{0}, \quad \operatorname{div}\boldsymbol{\tau} = \mathbf{0}, \tag{5.7}$$

and we are now assuming there is no distributed free charge ($\rho_\mathrm{f} = 0$).

The associated boundary conditions for $\mathbf{E}$ and $\mathbf{D}$ are now written

$$\mathbf{n}\times(\mathbf{E} - \mathbf{E}^\star) = \mathbf{0}, \quad \mathbf{n}\cdot(\mathbf{D} - \mathbf{D}^\star) = 0, \tag{5.8}$$

which were derived in Sect. 2.5.1, where the vectors $\mathbf{E}^\star$ and $\mathbf{D}^\star$ represent the electric field and electric displacement outside the material, evaluated on the boundary $\partial\mathcal{B}$, and we have set the surface free charge density to zero ($\sigma_\mathrm{f} = 0$). Note the distinction between * and *.

The remaining boundary condition involves the total Cauchy stress $\boldsymbol{\tau}$, and we write this in the form

$$\boldsymbol{\tau}\mathbf{n} = \mathbf{t}_\mathrm{a} + \mathbf{t}_\mathrm{e}^\star, \tag{5.9}$$

where $\mathbf{t}_\mathrm{a}$ is the applied mechanical load per unit deformed area. The traction vector $\mathbf{t}_\mathrm{e}^\star$ represents a force per unit area in the deformed configuration associated with the Maxwell stress $\boldsymbol{\tau}_\mathrm{e}^\star$ outside the material on the boundary $\partial\mathcal{B}$ and is defined by (4.21).

To quantify traction boundary conditions in the solution of boundary-value problems, it is convenient to recall the explicit expression for the Maxwell stress tensor. Specifically, for an electroelastic material surrounded by vacuum, we have

$$\boldsymbol{\tau}_\mathrm{e}^\star = \varepsilon_0 \left[\mathbf{E}^\star \otimes \mathbf{E}^\star - \frac{1}{2} (\mathbf{E}^\star \cdot \mathbf{E}^\star)\, \mathbf{I} \right], \tag{5.10}$$

and we recall that $\mathbf{D}^\star = \varepsilon_0 \mathbf{E}^\star$.

Although we shall not use the Lagrangian forms of the governing equations and boundary conditions, we shall make use of the interconnections

$$\mathbf{D} = \mathbf{F}\mathbf{D}_\mathrm{L}, \quad \mathbf{E} = \mathbf{F}^{-\mathrm{T}}\mathbf{E}_\mathrm{L}, \tag{5.11}$$

between the Eulerian and Lagrangian fields in formulating specific problems.

5.2 Homogeneous Deformations

To illustrate the theory we consider first the simple situation in which the deformation is homogeneous and the electric field is uniform. More particularly, we consider a slab (or plate) of material of uniform thickness with faces normal to the X_2 direction and of infinite extent in the other, X_1 and X_3, directions. The infinite geometry is considered here so as to avoid inhomogeneities that would arise from the electric boundary conditions for a plate with finite dimensions. In practice, if the lateral dimensions of the plate are large compared with its thickness, end effects are negligible and the electric field can be considered as uniform.

A uniform electric field is applied in the X_2 direction. Two different deformations are examined: a pure homogeneous strain and a simple shear of the slab. Since the deformation and electric fields are uniform, the governing differential equations in (5.7) are automatically satisfied, and it is only necessary to consider the constitutive equations and boundary conditions.

5.2.1 Pure Homogeneous Strain

Pure homogeneous strain is defined in (3.32) and was used in (3.175) in considering isotropic elasticity. We write this deformation in component form as

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3, \tag{5.12}$$

where we recall that $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches. Since we are considering an incompressible material they are subject to the constraint

$$\lambda_1 \lambda_2 \lambda_3 = 1. \tag{5.13}$$

The applied electric field is $E_2^\star$ exterior to the slab, and the associated electric displacement is $D_2^\star = \varepsilon_0 E_2^\star$. By the continuity condition $(5.8)_2$, $D_2 = D_2^\star$ is also the electric displacement inside the slab (and $D_1 = D_3 = 0$). It is therefore convenient to use the constitutive formulation based on $\Omega^*(\mathbf{F}, \mathbf{D}_\mathrm{L})$, for which we obtain from (5.5) and (5.6) the non-zero components of $\boldsymbol{\tau}$ and $\mathbf{E}$ as

$$\tau_{11} = 2\Omega_1^* \lambda_1^2 + 2\Omega_2^* \lambda_1^2 (\lambda_2^2 + \lambda_3^2) - p^*, \tag{5.14}$$

$$\tau_{22} = 2\Omega_1^* \lambda_2^2 + 2\Omega_2^* \lambda_2^2 (\lambda_3^2 + \lambda_1^2) - p^* + 2\Omega_5^* D_2^2 + 4\Omega_6^* \lambda_2^2 D_2^2, \tag{5.15}$$

$$\tau_{33} = 2\Omega_1^* \lambda_3^2 + 2\Omega_2^* \lambda_3^2 (\lambda_1^2 + \lambda_2^2) - p^*, \tag{5.16}$$

and

$$E_2 = 2(\Omega_4^* \lambda_2^{-2} + \Omega_5^* + \Omega_6^* \lambda_2^2) D_2. \tag{5.17}$$

On use of the incompressibility condition (5.13), the invariants I_1 and I_2 are given in terms of independent stretches λ_1 and λ_2 by

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2}, \quad I_2 = \lambda_1^{-2} + \lambda_2^{-2} + \lambda_1^2 \lambda_2^2, \tag{5.18}$$

while K_4, K_5, K_6 reduce to

$$K_4 = D_{\mathrm{L}2}^2, \quad K_5 = \lambda_2^2 K_4, \quad I_6 = \lambda_2^4 K_4, \tag{5.19}$$

and $D_{\mathrm{L}2} = \lambda_2^{-1} D_2$ is the Lagrangian counterpart of D_2. There are now just three independent variables, λ_1, λ_2 and K_4, and we therefore introduce a reduced energy function, say ω^*, which depends on these variables. We define

$$\omega^*(\lambda_1, \lambda_2, K_4) = \Omega^*(I_1, I_2, K_4, K_5, K_6), \tag{5.20}$$

with the invariants given by (5.18) and (5.19).

Then, from (5.14)–(5.16), it is easily seen that the stress differences are simply expressed as

$$\tau_{11} - \tau_{33} = \lambda_1 \omega_1^*, \quad \tau_{22} - \tau_{33} = \lambda_2 \omega_2^*, \tag{5.21}$$

and from (5.17) the electric field (within the slab) is

$$E_2 = -2\lambda_2^{-2} \omega_4^* D_2, \tag{5.22}$$

where the subscripts $1, 2, 4$ on ω^* indicate differentiation with respect to $\lambda_1, \lambda_2, K_4$, respectively.

From (5.10) the components of the Maxwell stress are found to be

$$\tau_{e11}^{\star} = \tau_{e33}^{\star} = -\frac{1}{2}\varepsilon_0^{-1} K_4, \quad \tau_{e22}^{\star} = \frac{1}{2}\varepsilon_0^{-1} K_4. \tag{5.23}$$

If no mechanical traction is supplied to the plane faces of the slab, then the normal stress τ_{22} is continuous, and hence, from $(5.21)_2$,

$$\tau_{33} = \tau_{e22}^{\star} - \lambda_2 \omega_2^*, \quad \tau_{11} = \tau_{33} + \lambda_1 \omega_1^*, \tag{5.24}$$

which provide expressions for the stresses τ_{11} and τ_{33} required to produce the deformation in the presence of the electric field. More particularly, if we specialize to the case of equibiaxial deformation, with $\lambda_1 = \lambda_3$ and $\lambda_2 = \lambda_1^{-2}$, we obtain from $(5.21)_1$ that $\omega_1^*(\lambda_1, \lambda_1^{-2}, K_4) = 0$ and

$$\tau_{11} = \tau_{e22}^{\star} - \lambda_1^{-2} \omega_2^*(\lambda_1, \lambda_1^{-2}, K_4), \tag{5.25}$$

where the latter provides an expression for the stress τ_{11} as a function of λ_1 and K_4. If the electric field is applied and accompanied by lateral mechanical traction in such a way that there is no deformation, then the required mechanical traction is $\tau_{11} - \tau_{e11}^{\star}$, which is

$$\tau_{11} - \tau_{e11}^{\star} = \varepsilon_0 K_4 - \omega_2^*(1, 1, K_4), \tag{5.26}$$

i.e. the lateral traction needed to prevent deformation due to the electric field.

More generally, if the material is prevented from deforming when an electric field is applied, the invariants retain their reference values $I_1 = I_2 = 3$ and $K_4 = K_5 = K_6$ and the (pre-)stress in the material is $\boldsymbol{\tau}_0$, which is given by

$$\boldsymbol{\tau}_0 = [-p^* + 2(\Omega_1^* + 2\Omega_2^*)]\mathbf{I} + 2(\Omega_5^* + 2\Omega_6^*)\mathbf{D} \otimes \mathbf{D}. \tag{5.27}$$

In general, an electro-sensitive material will deform when subject to an electric field, a phenomenon referred to as *electrostriction* in the linear context in Sect. 4.5.

5.2.2 Simple Shear

Consider again the slab of material discussed in Sect. 5.2.1, but now, instead of pure homogeneous strain, it is subject to simple shear in the X_1 direction in the (X_1, X_2) plane with the amount of shear denoted γ. The matrix of Cartesian components of the deformation gradient tensor $\mathbf{F}$, denoted F, is given by (3.36) and the corresponding right Cauchy–Green deformation tensor $\mathbf{c} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ has matrix written c. This and its square are given by

$$\mathsf{c} = \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathsf{c}^2 = \begin{bmatrix} 1+\gamma^2 & \gamma(2+\gamma^2) & 0 \\ \gamma(2+\gamma^2) & 1+3\gamma^2+\gamma^4 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5.28}$$

The principal invariants I_1, I_2 defined in (4.79) then simplify to

$$I_1 = I_2 = 3 + \gamma^2. \tag{5.29}$$

Since the deformation is again homogeneous, the field equations are satisfied automatically, and there is no essential difference in the use of the electric field or the electric displacement as the independent variable in the total energy formulation. Thus, for this problem, we focus on the formulation based on $\Omega(\mathbf{F}, \mathbf{E}_{\mathrm{L}})$. We again consider an applied field with just a single component $E_2^\star$ in the X_2 direction. The corresponding field in the material is E_2. The Lagrangian electric field has components $E_{\mathrm{L}1} = E_{\mathrm{L}3} = 0, E_{\mathrm{L}2} = E_2$, as can be seen by using the connection $\mathbf{E}_{\mathrm{L}} = \mathbf{F}^{\mathrm{T}}\mathbf{E}$. From (4.80) we calculate the invariants

$$I_4 = E_2^2, \quad I_5 = (1+\gamma^2)I_4, \quad I_6 = (1+3\gamma^2+\gamma^4)I_4. \tag{5.30}$$

The resulting components of $\boldsymbol{\tau}$, obtained from (5.3), are

$$\begin{aligned} \tau_{11} = -p + 2\Omega_1(1+\gamma^2) + 2\Omega_2(2+\gamma^2) \\ + 2I_4\gamma^2\left[\Omega_5 + 2\Omega_6(2+\gamma^2)\right], \end{aligned} \tag{5.31}$$

$$\tau_{22} = -p + 2\Omega_1 + 4\Omega_2 + 2I_4\left[\Omega_5 + 2\Omega_6(1+\gamma^2)\right], \tag{5.32}$$

$$\tau_{33} = -p + 2\Omega_1 + 2\Omega_2(2+\gamma^2), \tag{5.33}$$

$$\tau_{12} = 2\gamma\left\{\Omega_1 + \Omega_2 + I_4[\Omega_5 + \Omega_6(3+2\gamma^2)]\right\}, \tag{5.34}$$

and $\tau_{13} = \tau_{23} = 0$. The components of the electric displacement field $\mathbf{D}$, from (5.4), are

$$D_1 = -2\gamma\left[\Omega_4 + \Omega_5\left(2+\gamma^2\right) + \Omega_6\left(3+4\gamma^2+\gamma^4\right)\right]E_2, \tag{5.35}$$

$$D_2 = -2\left[\Omega_4 + \Omega_5\left(1+\gamma^2\right) + \Omega_6\left(1+3\gamma^2+\gamma^4\right)\right]E_2, \tag{5.36}$$

with $D_3 = 0$.

The invariants (5.29) and (5.30) are given in terms of two independent variables, namely γ and I_4. Therefore, it is convenient to define a reduced form of the energy Ω, as a function of these two variables only. We define the appropriate specialization, denoted ω, by

$$\omega(\gamma, I_4) = \Omega(I_1, I_2, I_4, I_5, I_6), \tag{5.37}$$

with the values of the invariants specified by (5.29) and (5.30). It follows that

$$\begin{aligned}\omega_\gamma &= 2\gamma\left\{\Omega_1 + \Omega_2 + I_4[\Omega_5 + \Omega_6(3+2\gamma^2)]\right\},\\ \omega_4 &= \Omega_4 + \Omega_5(1+\gamma^2) + \Omega_6(1+3\gamma^2+\gamma^4),\end{aligned} \tag{5.38}$$

where the subscripts γ and 4 on ω indicate partial differentiation with respect to γ and I_4. This allows the shear stress τ_{12}, defined in (5.34), to be expressed in a very simple form, namely

$$\tau_{12} = \omega_\gamma, \tag{5.39}$$

just as in standard nonlinear elasticity theory. Similarly, the expression for the component D_2 in (5.36) simplifies to

$$D_2 = -2\omega_4 E_2. \tag{5.40}$$

Since $D_2 = D_2^\star = \varepsilon_0 E_2^\star$, we obtain the connection

$$\omega_4(\gamma, I_4)E_2 = -\frac{1}{2}\varepsilon_0 E_2^\star, \tag{5.41}$$

which implicitly gives E_2 in terms of the applied electric field $E_2^\star$ and the amount of shear γ for a given form of energy function.

The Maxwell stresses outside the material are again given by (5.10), which we write now as

$$\tau_{e11}^\star = \tau_{e33}^\star = -\frac{1}{2}\varepsilon_0 I_4, \quad \tau_{e22}^\star = \frac{1}{2}\varepsilon_0 I_4. \tag{5.42}$$

The difference between the stress τ_{22} and the Maxwell stress $\tau_{e22}^\star$ is the normal mechanical stress on the plane boundaries of the slab required, along with the shear stress, to maintain the deformation. Since there is no shear component of the Maxwell stress, the applied shear stress on the boundary must balance τ_{12}.

5.3 Non-homogeneous Deformations

In this section we consider three problems involving non-homogeneous deformations and in some cases non-uniform electric fields. These are the extension and inflation and the helical shear of a thick-walled circular cylindrical tube, subject to axial and/or radial electric fields, and the inflation of a spherical shell in the presence of a radial electric field.

5.3.1 Extension and Inflation of a Tube

We consider again the circular cylindrical tube of incompressible material whose geometry was defined in Sect. 3.4.1 except that we now assume that the tube is infinitely long so as to avoid difficulties with the end conditions of a finite length tube that are incompatible with conditions on the lateral surfaces when simple forms of the deformation and electric field are assumed. In general, solution of problems for tubes of finite length will require numerical solution. In terms of cylindrical polar coordinates (R, Θ, Z), the reference configuration is then defined by

$$0 < A \leq R \leq B, \quad 0 \leq \Theta \leq 2\pi, \quad -\infty < Z < \infty, \tag{5.43}$$

and the associated unit basis vectors are $\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_Z$. The deformation consists of a combined axial extension and radial expansion that maintain the circular cylindrical shape of the tube and is described using cylindrical polar coordinates (r, θ, z) with basis vectors $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$. It is effected by the combination of internal pressure, axial load and axial and radial electric fields and given by

$$r = f(R) = \left[a^2 + \lambda_z^{-1}\left(R^2 - A^2\right)\right]^{1/2}, \quad \theta = \Theta, \quad z = \lambda_z Z, \tag{5.44}$$

where λ_z is the (uniform) axial stretch. The deformation gradient is then given by

$$\mathbf{F} = \lambda_1 \mathbf{e}_r \otimes \mathbf{E}_R + \lambda_2 \mathbf{e}_\theta \otimes \mathbf{E}_\Theta + \lambda_3 \mathbf{e}_z \otimes \mathbf{E}_Z,$$

and because of the radial symmetry, $\mathbf{e}_r = \mathbf{E}_R, \mathbf{e}_\theta = \mathbf{E}_\Theta, \mathbf{e}_z = \mathbf{E}_Z$, and $\mathbf{F}$ has a diagonal matrix representation with respect to these basis vectors, λ_1, λ_2 and λ_3 being the principal stretches associated, respectively, with the radial, azimuthal and axial directions. In the notation used in Sect. 3.4.1, these are

$$\lambda_1 = \lambda^{-1}\lambda_z^{-1}, \quad \lambda = \lambda_2 = \frac{r}{R}, \quad \lambda_3 = \lambda_z. \tag{5.45}$$

We use λ and λ_z as the two independent stretches, and note that the principal invariants I_1 and I_2 defined in $(4.79)_{1,2}$ have the forms

$$I_1 = \lambda^{-2}\lambda_z^{-2} + \lambda^2 + \lambda_z^2, \quad I_2 = \lambda^2\lambda_z^2 + \lambda^{-2} + \lambda_z^{-2}. \tag{5.46}$$

5.3.1.1 Electric Field Components

For this problem we use the formulation based on Ω with the deformation gradient $\mathbf{F}$ and the electric field $\mathbf{E}_\mathrm{L}$ as the independent variables. We assume the electric field $\mathbf{E}$ in the deformed configuration, with components E_r and E_z, is known and determine the Lagrangian form $\mathbf{E}_\mathrm{L}$ by using the relation $\mathbf{E}_\mathrm{L} = \mathbf{F}^\mathrm{T}\mathbf{E}$ to obtain the components

$$E_{\mathrm{L}R} = \lambda^{-1}\lambda_z^{-1}E_r, \quad E_{\mathrm{L}Z} = \lambda_z E_z, \tag{5.47}$$

which enables the invariants I_4, I_5, I_6 defined by (4.80) to be written

$$I_4 = E_{\mathrm{L}R}^2 + E_{\mathrm{L}Z}^2, \quad I_5 = \lambda^{-2}\lambda_z^{-2}E_{\mathrm{L}R}^2 + \lambda_z^2 E_{\mathrm{L}Z}^2, \quad I_6 = \lambda^{-4}\lambda_z^{-4}E_{\mathrm{L}R}^2 + \lambda_z^4 E_{\mathrm{L}Z}^2. \tag{5.48}$$

The components of the electric displacement vector $\mathbf{D}$ are obtained using (5.4) as

$$D_r = -2\lambda^{-2}\lambda_z^{-2}\left(\Omega_4 + \Omega_5\lambda^{-2}\lambda_z^{-2} + \Omega_6\lambda^{-4}\lambda_z^{-4}\right)E_r, \tag{5.49}$$

$$D_z = -2\lambda_z^2\left(\Omega_4 + \Omega_5\lambda_z^2 + \Omega_6\lambda_z^4\right)E_z, \tag{5.50}$$

and the Lagrangian form is obtained using the connection $\mathbf{D}_\mathrm{L} = \mathbf{F}^{-1}\mathbf{D}$.

5.3.1.2 Stress Components

The total stress tensor $\boldsymbol{\tau}$ of an incompressible electroelastic material is defined by (5.3), which, when applied to the present problem, yields the components

$$\begin{aligned}\tau_{rr} = &-p + 2\lambda^{-2}\lambda_z^{-2}\left[\Omega_1 + \Omega_2\left(\lambda^2 + \lambda_z^2\right)\right] \\ &+ 2\lambda^{-4}\lambda_z^{-4}\left(\Omega_5 + 2\Omega_6\lambda^{-2}\lambda_z^{-2}\right)E_r^2,\end{aligned} \tag{5.51}$$

$$\tau_{\theta\theta} = -p + 2\lambda^2\left[\Omega_1 + \Omega_2\left(\lambda^{-2}\lambda_z^{-2} + \lambda_z^2\right)\right], \tag{5.52}$$

$$\tau_{zz} = -p + 2\lambda_z^2\left[\Omega_1 + \Omega_2\left(\lambda^{-2}\lambda_z^{-2} + \lambda^2\right)\right] + 2\lambda_z^4\left(\Omega_5 + 2\Omega_6\lambda_z^2\right)E_z^2, \tag{5.53}$$

$$\tau_{rz} = 2\lambda^{-2}\left[\Omega_5 + \Omega_6\left(\lambda_z^2 + \lambda^{-2}\lambda_z^{-2}\right)\right]E_rE_z, \tag{5.54}$$

with $\tau_{r\theta} = \tau_{\theta z} = 0$.

Outside the tube we take to be a vacuum, where the electric displacement and electric field are related by the equation $\mathbf{D}^\star = \varepsilon_0\mathbf{E}^\star$. Then, use of (5.10) gives the non-zero components of the associated Maxwell stress tensor as

$$\tau_{\mathrm{e}rr}^\star = -\tau_{\mathrm{e}zz}^\star = \frac{1}{2}\varepsilon_0\left(E_r^{\star 2} - E_z^{\star 2}\right), \quad \tau_{\mathrm{e}\theta\theta}^\star = -\frac{1}{2}\varepsilon_0\left(E_r^{\star 2} + E_z^{\star 2}\right), \tag{5.55}$$

and

$$\tau_{\mathrm{e}rz}^{\star} = \varepsilon_0 E_r^{\star} E_z^{\star}. \tag{5.56}$$

The full expressions for the components of the equilibrium equation $\operatorname{div}\boldsymbol{\tau} = \mathbf{0}$ in cylindrical polar coordinates are given in Appendix A. Their specialization to the present axisymmetric situation leaves two non-trivial component equations, specifically

$$r\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} = \tau_{\theta\theta} - \tau_{rr}, \quad \frac{\mathrm{d}}{\mathrm{d}r}(r\tau_{rz}) = 0. \tag{5.57}$$

Similarly, the full expressions for $\operatorname{curl}\mathbf{E}$ and $\operatorname{div}\mathbf{D}$ in cylindrical polar coordinates are obtained from Appendix A. When specialized to the present situation, they yield simply

$$E_z = \text{constant}, \quad rD_r = \text{constant}, \tag{5.58}$$

which apply both within and outside the material. Since, by $(5.8)_1$, the tangential component of $\mathbf{E}$ is continuous across the boundary, it follows that E_z is constant throughout space. In vacuum the solution (obtained from Laplace's equation) for the electrostatic potential is

$$\varphi^{\star} = -\frac{q}{2\pi\varepsilon_0}\log r, \tag{5.59}$$

where q is a uniform charge per unit length placed along the axis of the tube, and we deduce that

$$rD_r^{\star} = \frac{q}{2\pi}. \tag{5.60}$$

But, within the material, D_r satisfies this equation and is also given by (5.49). Hence, we obtain the condition

$$-2\lambda^{-2}\lambda_z^{-2}\left(\Omega_4 + \Omega_5\lambda^{-2}\lambda_z^{-2} + \Omega_6\lambda^{-4}\lambda_z^{-4}\right)E_r = \frac{q}{2\pi r}. \tag{5.61}$$

Given the (constant) value of E_z, this is an implicit equation for the electric field E_r inside the material in terms of E_z, the deformation (via λ and λ_z) and r for a given form of energy function.

We shall study the mechanical equilibrium equations (5.57) and associated boundary conditions in the following sections where we consider separately an axial and then a radial electric field, followed by a brief examination of their combined effect.

5.3.1.3 Axial Electric Field

We now specialize the equations derived above to the case where the electric field has only a component in the axial direction. The invariants I_1, I_2, given by (5.46), remain unchanged, while the invariants in (5.48) reduce to

$$I_4 = E_{\mathrm{LZ}}^2, \quad I_5 = \lambda_z^2 I_4, \quad I_6 = \lambda_z^4 I_4, \tag{5.62}$$

which are now constants. Since the invariants now depend only on the three independent variables, λ, λ_z and I_4, it is convenient again to adopt a reduced form of the energy function, denoted ω, given by

$$\omega\left(\lambda, \lambda_z, I_4\right) = \Omega\left(I_1, I_2, I_4, I_5, I_6\right), \tag{5.63}$$

with the values of the invariants on the right-hand side given by (5.46) and (5.62). It is straightforward then to show that, in terms of the derivatives of ω, the stress differences are given by the simple formulas

$$\tau_{\theta\theta} - \tau_{rr} = \lambda\omega_\lambda, \quad \tau_{zz} - \tau_{rr} = \lambda_z\omega_{\lambda_z}, \tag{5.64}$$

where the subscripts λ and λ_z on ω signify differentiation with respect to λ and λ_z, respectively. Equation $(5.57)_1$ can now be written in the alternative form

$$r\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} = \lambda\omega_\lambda, \tag{5.65}$$

while $(5.57)_2$ is satisfied trivially since $E_r = 0$ and hence $\tau_{rz} = 0$.

We also have $D_r = 0$, while D_z is given simply by

$$D_z = -2\lambda_z^2\omega_4 E_z, \tag{5.66}$$

where $\omega_4 = \partial\omega/\partial I_4$.

The components (5.55) of the Maxwell stress reduce to

$$\tau_{\mathrm{e}rr}^\star = \tau_{\mathrm{e}\theta\theta}^\star = -\frac{1}{2}\varepsilon_0 E_z^{\star 2}, \quad \tau_{\mathrm{e}zz}^\star = \frac{1}{2}\varepsilon_0 E_z^{\star 2}, \tag{5.67}$$

and $\tau_{\mathrm{e}r\theta}^\star = 0$.

Let the tube be subjected to a pressure P on its inner surface $r = a$ with no mechanical loads applied to the outside surface $r = b$. The two boundary conditions, including the radial Maxwell stress component $\tau_{\mathrm{e}rr}^\star$, are

$$\tau_{rr} = -P + \tau_{\mathrm{e}rr}^\star \quad \text{on} \quad r = a, \quad \tau_{rr} = \tau_{\mathrm{e}rr}^\star \quad \text{on} \quad r = b. \tag{5.68}$$

Integration of (5.65), using the boundary conditions (5.68), leads to

$$P = \int_a^b \lambda \omega_\lambda \frac{\mathrm{d}r}{r}. \tag{5.69}$$

Since, from (5.44), we have $b^2 = a^2 + \lambda_z^{-1}\left(B^2 - A^2\right)$, (5.69) provides a relationship between the pressure P and the inner deformed radius a when λ_z and E_z are known. Note that this formula does not involve the Maxwell stress since $\tau^\star_{\mathrm{e}rr}$ has the same value on each boundary.

By changing the variable of integration from r to λ using (3.221), we obtain an alternative representation for P analogous to that given in (3.222) for an elastic material, namely

$$P = \int_{\lambda_b}^{\lambda_a} \left(\lambda^2\lambda_z - 1\right)^{-1} \omega_\lambda \mathrm{d}\lambda, \tag{5.70}$$

where λ_a and λ_b are defined in (3.214). This provides an expression for P as a function of λ_a (equivalently of the deformed radius) when λ_z and E_z are fixed. Thus, (5.70) has precisely the same form as for a purely elastic material given by (3.222) except that ω here depends additionally on the invariant I_4. For a more extensive discussion in the purely elastic case, see Haughton and Ogden (1979).

The resultant axial load N on any cross section of the tube is calculated from

$$N = 2\pi \int_a^b \tau_{zz} r \mathrm{d}r = \pi \left[\int_a^b 2\left(\tau_{zz} - \tau_{rr}\right) r \mathrm{d}r + \int_a^b 2\tau_{rr} r \mathrm{d}r\right]. \tag{5.71}$$

The last integral on the right-hand side can be integrated by parts to give

$$2\int_a^b \tau_{rr} r \mathrm{d}r = a^2 P + (b^2 - a^2)\tau^\star_{\mathrm{e}rr} - \int_a^b \left(\tau_{\theta\theta} - \tau_{rr}\right) r \mathrm{d}r, \tag{5.72}$$

in which we have used the equilibrium equation $(5.57)_1$ and the boundary conditions on the inner and outer surfaces as specified by (5.68). The resultant axial force can then be written as

$$N = \pi \int_a^b \left(2\lambda_z \omega_{\lambda_z} - \lambda\omega_\lambda\right) r \mathrm{d}r + \pi a^2 P - \frac{1}{2}\pi \left(b^2 - a^2\right) \varepsilon_0 E_z^2, \tag{5.73}$$

where use has been made of the relations (5.64).

Using (3.221) together with (3.213) and the relation $r = \lambda R$, we obtain

$$r\mathrm{d}r = -\frac{\left(\lambda_a^2\lambda_z - 1\right)}{\left(\lambda^2\lambda_z - 1\right)^2} A^2 \lambda \mathrm{d}\lambda, \tag{5.74}$$

which enables the variable of integration in (5.73) to be changed from r to λ, yielding

$$N/\pi A^2 = (\lambda_a^2\lambda_z - 1)\int_{\lambda_b}^{\lambda_a} \left(\lambda^2\lambda_z - 1\right)^{-2} (2\lambda_z\omega_{\lambda_z} - \lambda\omega_\lambda)\lambda \mathrm{d}\lambda$$

$$+ \lambda_a^2 P - \frac{1}{2}\left(\lambda_b^2 \frac{B^2}{A^2} - \lambda_a^2\right) \varepsilon_0 E_z^2. \tag{5.75}$$

We are interested in evaluating the response of the tube in the presence of an axial electric field with $P = 0$. A convenient starting point is to rewrite the integrand of (5.69) in the explicit form

$$\lambda\omega_\lambda = 2\left(\Omega_1 + \lambda_z^2\Omega_2\right)\left(\lambda^2 - \lambda^{-2}\lambda_z^{-2}\right), \tag{5.76}$$

where we note that for an elastic material, the term $\Omega_1 + \lambda_z^2\Omega_2$ is usually taken to be positive. This follows from the Baker–Ericksen inequalities, which require that the stretches are ordered in the same sequence as the principal stresses (see, e.g., Truesdell and Noll 1965 for a discussion of the Baker–Ericksen inequalities). We therefore assume that $\Omega_1 + \lambda_z^2\Omega_2 > 0$ also in the present context.

But, from (3.213), we have

$$\left(\lambda_a^2\lambda_z - 1\right) A^2 = \left(\lambda^2\lambda_z - 1\right) R^2, \tag{5.77}$$

which shows that $\lambda^2\lambda_z - 1$ is either positive, negative or zero throughout the wall thickness $A \le R \le B$. Since $P = 0$, then the integrand in (5.69) must vanish, and hence, by (5.76) and Baker–Ericksen, $\lambda^2\lambda_z = 1$ for all R in the tube. This means that the deformation is homogeneous, and the radial stress must be uniform, i.e. it is equal to the Maxwell stress $\tau^\star_{\mathrm{e}rr} = -\varepsilon E_z^{\star 2}/2$. From (5.73) we find that the resulting axial load is then given by

$$N = \pi\left(b^2 - a^2\right)\left(\lambda_z\omega_{\lambda_z} - \frac{1}{2}\varepsilon_0 E_z^2\right), \tag{5.78}$$

where the term $\lambda_z\omega_{\lambda_z}$ has the explicit expression

$$\lambda_z\omega_{\lambda_z} = 2\Omega_1\left(\lambda_z^2 - \lambda_z^{-1}\right) + 2\Omega_2\left(\lambda_z - \lambda_z^{-2}\right) - 2\lambda_z^4\left(\Omega_5 + 2\lambda_z^{-2}\Omega_6\right) E_z^2, \tag{5.79}$$

when evaluated for $\lambda^2\lambda_z = 1$. Note that this result can also be applied to a solid cylinder by taking $a = 0$. For the special case when $\lambda = \lambda_z = 1$, (5.79) simplifies and (5.78) reduces to

$$N = -2\pi\left(b^2 - a^2\right)\left(\Omega_5 + 2\Omega_6 + \frac{1}{4}\varepsilon_0\right) E_z^2. \tag{5.80}$$

This is the axial load required to maintain the undeformed configuration in the presence of the applied axial electric field. If the sign of the term $\Omega_5 + 2\Omega_6 + \frac{1}{4}\varepsilon_0$ is positive (negative), then N is negative (positive) and the electric field tends to lengthen (shorten) the tube.

5.3.1.4 Radial Electric Field

We now return to the equations derived in Sects. 5.3.1.1 and 5.3.1.2 and consider an electric field with the single non-zero component $E_r = \lambda\lambda_z E_{LR}$ in the radial direction. The expressions for I_1 and I_2 remain unchanged, while the invariants in (5.48) become

$$I_4 = E_{LR}^2, \quad I_5 = \lambda^{-2}\lambda_z^{-2} I_4, \quad I_6 = \lambda^{-4}\lambda_z^{-4} I_4. \tag{5.81}$$

Similarly to the case considered in the previous section, it is again convenient to define a reduced energy function $\omega(\lambda, \lambda_z, I_4)$ by means of (5.63), but now with I_4, I_5, I_6 defined by (5.81). The expressions for the principal stress differences in (5.64) are still valid, as is the form of the equilibrium equation in the radial direction given by (5.65). Without an axial component of the electric field, we have $\tau_{rz} = 0$. Using (5.49), in combination with the reduced energy function ω, gives the single non-zero component of the electric displacement vector

$$D_r = -2\lambda^{-2}\lambda_z^{-2}\omega_4 E_r. \tag{5.82}$$

The exterior of the tube is again taken to be a vacuum, and (5.10) gives the Maxwell stress outside the tube as

$$\tau_{\mathrm{e}rr}^{\star} = \frac{1}{2}\varepsilon_0 E_r^{\star 2}, \quad \tau_{\mathrm{e}\theta\theta}^{*} = \tau_{\mathrm{e}zz}^{\star} = -\frac{1}{2}\varepsilon_0 E_r^{\star 2}, \tag{5.83}$$

where $E_r^\star = \varepsilon_0^{-1} D_r^\star$. The electric displacement field component $D_r^\star$ is again given by (5.60) and hence

$$\tau_{\mathrm{e}rr}^{\star} = \tau_{\mathrm{e}rr}^{\star}(r) = \frac{q^2}{8\pi^2\varepsilon_0 r^2}, \tag{5.84}$$

which defines the notation $\tau_{\mathrm{e}rr}^{\star}(r)$.

The specialization of (5.61) in this case is

$$-2\lambda^{-2}\lambda_z^{-2}\omega_4(\lambda, \lambda_z, I_4) E_r = \frac{q}{2\pi r}, \tag{5.85}$$

which, bearing in mind that $I_4 = \lambda^{-2}\lambda_z^{-2}E_r^2$, is an implicit equation for E_r in terms of r for a given form of energy function and deformation.

Contrary to the formula in (5.69), the Maxwell stress component $\tau_{\mathrm{e}rr}^{\star}$ influences the pressure–radius relation, as we now show. Integration of (5.65) yields, on use of the boundary conditions (5.68),

$$P - [\tau_{\mathrm{e}rr}^{\star}(a) - \tau_{\mathrm{e}rr}^{\star}(b)] = \int_a^b \lambda\omega_\lambda \frac{\mathrm{d}r}{r}, \tag{5.86}$$

with the expression for $\tau_{\mathrm{e}rr}^{\star}(r)$ given by (5.84). Clearly, from (5.84), $\tau_{\mathrm{e}rr}^{\star}(a) - \tau_{\mathrm{e}rr}^{\star}(b)$ is positive, and hence, the effect of the Maxwell stress is the opposite of that of an internal pressure.

The axial load N is obtained in a similar way to (5.73) and has the form

$$N = \pi \int_a^b \left(2\lambda_z \omega_{\lambda_z} - \lambda\omega_\lambda\right) r\mathrm{d}r + \pi a^2 P, \tag{5.87}$$

but in this case, the contributions from the Maxwell stresses on $r = a$ and $r = b$ cancel since, by (5.84), $a^2\tau^\star_{\mathrm{e}rr}(a) = b^2\tau^\star_{\mathrm{e}rr}(b)$. See Dorfmann and Ogden (2006) for a more detailed discussion.

5.3.1.5 Combined Axial and Radial Electric Fields

We now consider an applied electric field having non-zero components in the radial and axial directions. Equation (5.54) shows that the total shear stress τ_{rz} is no longer zero. Using the expression (5.54) together with $(5.57)_2$, we find that

$$\tau_{rz} = 2\lambda^{-2}\left[\Omega_5 + \Omega_6\left(\lambda_z^2 + \lambda^{-2}\lambda_z^{-2}\right)\right] E_r E_z = \frac{c_\tau}{r}, \tag{5.88}$$

where c_τ is a constant. Similarly, from (5.61),

$$D_r = -2\lambda^{-2}\lambda_z^{-2}\left(\Omega_4 + \Omega_5\lambda^{-2}\lambda_z^{-2} + \Omega_6\lambda^{-4}\lambda_z^{-4}\right) E_r = \frac{c_d}{r}, \tag{5.89}$$

where $c_d = q/2\pi$ is a constant.

If there is no mechanical shear traction applied on the inner and outer cylindrical surfaces $r = a$ and $r = b$, the continuity requirement (5.9) states that the stress τ_{rz} must be equal to the Maxwell stress $\tau^\star_{\mathrm{e}rz} = \varepsilon_0 E^\star_r E^\star_z$ given by (5.56). Since outside the material we have the connection $D^\star_r = \varepsilon_0 E^\star_r$, the Maxwell stress can equivalently be written as $\tau^\star_{\mathrm{e}rz} = D^\star_r E^\star_z = c_d E_z/r$, where we have used the continuity of E_z and $D^\star_r = c_d/r$. Hence, by applying continuity of traction on $r = a$ (or $r = b$), we obtain the connection

$$c_\tau = c_d E_z, \tag{5.90}$$

where we recall that E_z is constant.

Using this connection and combining (5.88) and (5.89), we obtain the restriction

$$\Omega_4 + \Omega_5(\lambda^{-2}\lambda_z^{-2} + \lambda_z^2) + \Omega_6(\lambda^{-4}\lambda_z^{-4} + \lambda^{-2} + \lambda_z^4) = 0. \tag{5.91}$$

This implies that only energy functions satisfying this condition can support the two-component electric field for the considered deformation. We refer to Dorfmann and Ogden (2006) for further discussion of this problem, but we remark that different invariants I_5 and I_6 were used therein.

5.3.2 Helical Shear of a Circular Cylinder

We again consider a right circular cylindrical tube of an incompressible electroelastic material with the geometry described in the previous section, but now it is subject to a helical shear deformation. Helical shear is the combination of azimuthal shear and axial shear. Both these deformations were discussed separately for a purely elastic material in Sect. 3.4.

In the reference configuration we again use cylindrical polar coordinates (R, Θ, Z) with corresponding unit basis vectors $\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_Z$. The deformed configuration is described by cylindrical polar coordinates (r, θ, z) with basis vectors $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$.

The helical shear deformation has the form

$$r = R, \quad \theta = \Theta + g(R), \quad z = Z + w(R), \tag{5.92}$$

where the functions $g(R)$ and $w(R)$ are to be determined by solving the equilibrium equations together with the boundary conditions. The deformation gradient $\mathbf{F}$, referred to the two sets of cylindrical polar coordinate axes, is represented by the matrix F, which is given by

$$\mathsf{F} = \begin{bmatrix} 1 & 0 & 0 \\ rg'(r) & 1 & 0 \\ w'(r) & 0 & 1 \end{bmatrix}, \tag{5.93}$$

where we now regard g and w as functions of r $(= R)$; correspondingly, we write $a = A$, $b = B$. The quantities $rg'(r)$ and $w'(r)$ each represent a local simple shear, respectively in the azimuthal and axial directions. For convenience we denote them by $\gamma_\theta = rg'(r)$ and $\gamma_z = w'(r)$. The combined deformation is also a local simple shear, with amount of shear $\gamma = \sqrt{\gamma_\theta^2 + \gamma_z^2}$, and we take $\gamma \geq 0$. For a detailed discussion of a circular cylinder subject to helical shear deformation in the elastic context, we refer to Ogden et al. (1973) and for a magnetoelastic material to Dorfmann and Ogden (2004).

Corresponding to F, we denote by b and c, respectively, the matrix representations of the left and right Cauchy–Green tensors $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ and $\mathbf{c} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$. These are given by

$$\mathsf{b} = \begin{bmatrix} 1 & \gamma_\theta & \gamma_z \\ \gamma_\theta & 1+\gamma_\theta^2 & \gamma_\theta\gamma_z \\ \gamma_z & \gamma_\theta\gamma_z & 1+\gamma_z^2 \end{bmatrix}, \quad \mathsf{c} = \begin{bmatrix} 1+\gamma^2 & \gamma_\theta & \gamma_z \\ \gamma_\theta & 1 & 0 \\ \gamma_z & 0 & 1 \end{bmatrix}, \tag{5.94}$$

and it follows from $(4.79)_{1,2}$ that

$$I_1 = I_2 = 3 + \gamma_\theta^2 + \gamma_z^2 = 3 + \gamma^2. \tag{5.95}$$

5.3.2.1 Stress and Field Components

For this problem we choose to use the Lagrangian electric displacement field $\mathbf{D}_\mathrm{L}$ as the independent variable and work in terms of the energy function Ω^*. We apply a field $\mathbf{D}$ in the deformed configuration with non-zero components D_r and D_z in the radial and axial directions. The components of $\mathbf{D}_\mathrm{L}$ are then obtained using $\mathbf{D}_\mathrm{L} = \mathbf{F}^{-1}\mathbf{D}$ from (5.11)$_1$. These are

$$D_{\mathrm{L}R} = D_r, \quad D_{\mathrm{L}\Theta} = -\gamma_\theta D_r, \quad D_{\mathrm{L}Z} = D_z - \gamma_z D_r, \tag{5.96}$$

and from (4.86), using (5.94)$_2$, we then calculate the invariants

$$K_4 = D_{\mathrm{L}R}^2 + D_{\mathrm{L}\Theta}^2 + D_{\mathrm{L}Z}^2, \tag{5.97}$$

$$K_5 = \left(1+\gamma_z^2\right) D_{\mathrm{L}R}^2 + 2\gamma_z D_{\mathrm{L}R} D_{\mathrm{L}Z} + D_{\mathrm{L}Z}^2, \tag{5.98}$$

$$K_6 = \left(1+3\gamma_z^2+\gamma_z^4\right) D_{\mathrm{L}R}^2 + 2\gamma_z\left(2+\gamma_z^2\right) D_{\mathrm{L}R} D_{\mathrm{L}Z} + \left(1+\gamma_z^2\right) D_{\mathrm{L}Z}^2, \tag{5.99}$$

and we notice that these are independent of γ_θ, and only K_4 depends on $D_{\mathrm{L}\Theta}$.

From (5.5) we find the components of the total stress tensor $\boldsymbol{\tau}$, which are

$$\tau_{rr} = -p + 2\left(\Omega_1^* + 2\Omega_2^*\right) + 2\Omega_5^* D_r^2 + 4\Omega_6^*\left(D_r^2 + \gamma_z D_r D_z\right), \tag{5.100}$$

$$\tau_{\theta\theta} = -p + 2\Omega_1^*\left(1+\gamma_\theta^2\right) + 2\Omega_2^*\left(2+\gamma^2\right), \tag{5.101}$$

$$\begin{aligned}\tau_{zz} = &-p + 2\Omega_1^*\left(1+\gamma_z^2\right) + 2\Omega_2^*\left(2+\gamma^2\right) + 2\Omega_5^* D_z^2 \\ &+ 4\Omega_6^*\left[\gamma_z D_r D_z + \left(1+\gamma_z^2\right) D_z^2\right],\end{aligned} \tag{5.102}$$

$$\tau_{r\theta} = 2\left(\Omega_1^* + \Omega_2^*\right)\gamma_\theta + 2\Omega_6^*\gamma_\theta\left(D_r^2 + \gamma_z D_r D_z\right), \tag{5.103}$$

$$\begin{aligned}\tau_{rz} = &\, 2\left(\Omega_1^* + \Omega_2^*\right)\gamma_z + 2\Omega_5^* D_r D_z \\ &+ 2\Omega_6^*\left[\gamma_z\left(D_r^2 + D_z^2\right) + \left(2+\gamma_z^2\right) D_r D_z\right],\end{aligned} \tag{5.104}$$

$$\tau_{\theta z} = 2\Omega_1^*\gamma_\theta\gamma_z + 2\Omega_6^*\gamma_\theta\left(\gamma_z D_z^2 + D_r D_z\right) \tag{5.105}$$

and the electric field vector $\mathbf{E}$ has components given by (5.6) as

$$E_r = 2\left[\Omega_4^*\left(1+\gamma^2\right) + \Omega_5^* + \Omega_6^*\right] D_r + 2\left(-\Omega_4^* + \Omega_6^*\right)\gamma_z D_z, \tag{5.106}$$

$$E_\theta = 2\gamma_\theta\left(-\Omega_4^* + \Omega_6^*\right) D_r + 2\Omega_6^*\gamma_\theta\gamma_z D_z, \tag{5.107}$$

$$E_z = 2\gamma_z\left(-\Omega_4^* + \Omega_6^*\right) D_r + 2\left[\Omega_4^* + \Omega_5^* + \Omega_6^*\left(1+\gamma_z^2\right)\right] D_z. \tag{5.108}$$

The requirements resulting from the equations $\operatorname{div}\mathbf{D} = 0$ and $\operatorname{curl}\mathbf{E} = \mathbf{0}$, specialized to problems for which the cylindrical configuration is maintained during deformation, are

$$rD_r = \text{constant}, \quad E_z = \text{constant}, \quad rE_\theta = \text{constant}. \tag{5.109}$$

In the first of these the constant is $q/2\pi$, as noted previously, while the second and third impose restrictions on either the deformation or the constitutive law or both. Since the third one implies that E_θ is infinite at $r = 0$, we must take the constant in this case to be zero, and hence $E_\theta = 0$. From (5.107) we will see that this condition can be satisfied in a number of special cases: (i) if $\gamma_\theta = 0$ and if $\gamma_\theta \neq 0$, (ii) $\gamma_z = 0$ and $D_r = 0$, (iii) $\gamma_z = 0$ and $\Omega_6^* = \Omega_4^*$, (iv) $D_z = 0$ and $\Omega_6^* = \Omega_4^*$, (v) $\Omega_6^* = \Omega_4^* = 0$ and (vi) $\Omega_6^* = 0$ and $D_r = 0$.

In considering E_z, we note that D_r is given and that K_4, K_5 and K_6 are independent of γ_θ so that E_z depends on γ_θ only through I_1 and I_2. Then, once γ_θ and γ_z are determined (by solving the equilibrium equations) and the constitutive law specified, the equation $E_z =$ constant determines the value of D_z inside the material (in general implicitly since K_4, K_5 and K_6 depend on D_z through the connection $D_z = D_{\mathrm{LZ}} + \gamma_z D_{\mathrm{L}R}$) and the constant in question is the value of $E_z = E_z^\star$ specified outside the material. In the case (i) noted above, the form of E_z is unchanged from (5.108) although now $I_1 = I_2 = 3 + \gamma_z^2$ within the coefficients $\Omega_4^*, \Omega_5^*, \Omega_6^*$. In cases (ii) and (iii), $E_z = 2(\Omega_4^* + \Omega_5^* + \Omega_6^*)D_z$, in case (iv) $E_z = 0$, in case (v) $E_z = 2\Omega_5^* D_z$ and in case (vi) $E_z = 2(\Omega_4^* + \Omega_5^*)D_z$.

5.3.2.2 Radial Electric Field

In an attempt to solve this problem analytically, we restrict attention to an applied electric displacement in the radial direction, so that $D_z = 0$ and the equations for the stress components specialize to

$$\tau_{rr} = -p + 2\left(\Omega_1^* + 2\Omega_2^*\right) + 2\left(\Omega_5^* + 2\Omega_6^*\right) D_r^2, \tag{5.110}$$

$$\tau_{\theta\theta} = -p + 2\Omega_1^*\left(1 + \gamma_\theta^2\right) + 2\Omega_2^*\left(2 + \gamma^2\right), \tag{5.111}$$

$$\tau_{zz} = -p + 2\Omega_1^*\left(1 + \gamma_z^2\right) + 2\Omega_2^*\left(2 + \gamma^2\right), \tag{5.112}$$

$$\tau_{r\theta} = 2\left(\Omega_1^* + \Omega_2^* + \Omega_6^* D_r^2\right)\gamma_\theta, \tag{5.113}$$

$$\tau_{rz} = 2\left(\Omega_1^* + \Omega_2^* + \Omega_6^* D_r^2\right)\gamma_z, \tag{5.114}$$

$$\tau_{\theta z} = 2\Omega_1^*\gamma_\theta\gamma_z, \tag{5.115}$$

and the components of $\mathbf{E}$ become

$$E_r = 2\left[\Omega_4^*\left(1+\gamma^2\right) + \Omega_5^* + \Omega_6^*\right] D_r, \tag{5.116}$$

$$E_\theta = 2\gamma_\theta\left(\Omega_6^* - \Omega_4^*\right) D_r, \tag{5.117}$$

$$E_z = 2\gamma_z\left(\Omega_6^* - \Omega_4^*\right) D_r. \tag{5.118}$$

Since now $D_{\mathrm{L}R} = D_r, D_{\mathrm{L}\Theta} = -\gamma_\theta D_r, D_{\mathrm{LZ}} = -\gamma_z D_r$, the invariants K_4, K_5 and K_6 are

$$K_4 = (1+\gamma^2)D_{\mathrm{L}R}^2, \quad K_5 = K_6 = D_{\mathrm{L}R}^2. \tag{5.119}$$

From (5.113), (5.114), (5.117) and (5.118), we obtain the connections

$$\gamma_z\tau_{r\theta} = \gamma_\theta\tau_{rz}, \quad \gamma_z E_\theta = \gamma_\theta E_z, \tag{5.120}$$

the second of which is satisfied trivially since $E_\theta = E_z = 0$. Satisfaction of these conditions for a non-trivial solution requires the condition $\Omega_4^* = \Omega_6^*$, which we now adopt. Then, if we introduce the reduced energy function

$$\omega^*(\gamma_\theta, \gamma_z, K_5) = \Omega^*\left(3+\gamma^2, 3+\gamma^2, (1+\gamma^2)K_5, K_5, K_5\right) \tag{5.121}$$

we obtain simple expressions for the shear stresses $\tau_{r\theta}$ and τ_{rz}, specifically

$$\tau_{r\theta} = \frac{\partial\omega^*}{\partial\gamma_\theta}, \quad \tau_{rz} = \frac{\partial\omega^*}{\partial\gamma_z}. \tag{5.122}$$

In cylindrical polar coordinates, for the deformation considered in this section, the equilibrium equation $\operatorname{div}\boldsymbol{\tau} = \mathbf{0}$ has radial, azimuthal and axial components

$$r\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} = \tau_{\theta\theta} - \tau_{rr}, \quad \frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\tau_{r\theta}\right) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}r}\left(r\tau_{rz}\right) = 0. \tag{5.123}$$

Using the expressions (5.122), $(5.123)_{2,3}$ can be integrated to give

$$\tau_{r\theta} = \frac{\partial\omega^*}{\partial\gamma_\theta} = \frac{\tau_\theta b^2}{r^2}, \quad \tau_{rz} = \frac{\partial\omega^*}{\partial\gamma_z} = \frac{\tau_z b}{r}, \tag{5.124}$$

where τ_θ is the value of $\tau_{r\theta}$ and τ_z that of τ_{rz} on the boundary $r = b$. For a given energy function Ω^* (subject to $\Omega_4^* = \Omega_6^*$) and for known D_r, (5.124) can in principle be solved for γ_θ and γ_z and, by integration, for the deformation functions $g(r)$ and $w(r)$. In fact, from $(5.120)_1$ and (5.124), we obtain the connection

$$\frac{rg'(r)}{w'(r)} \equiv \frac{\gamma_\theta}{\gamma_z} = \frac{\tau_\theta b}{\tau_z r} \tag{5.125}$$

so that the two local simple shears are not independent.

To illustrate the theory we consider a prototype form of the energy Ω^* that satisfies $\Omega_4^* = \Omega_6^*$. It is based on the neo-Hookean model for elasticity given by (3.169) and has the form

$$\Omega^* = \frac{1}{2}\mu(I_1 - 3) + \frac{1}{2}\alpha\varepsilon_0^{-1}(K_4 + K_6), \tag{5.126}$$

where α is a dimensionless constant and is linear in the sum $K_4 + K_6$ and independent of K_5, although, of course, for the considered specialization herein, both K_4 and K_6 depend on K_5 through (5.119). Hence

$$\omega^* = \frac{1}{2}(\gamma_\theta^2 + \gamma_z^2) + \frac{1}{2}\alpha\varepsilon_0^{-1}(2 + \gamma^2)K_5. \tag{5.127}$$

It follows from (5.124) that

$$\gamma_\theta(r^2 + c) = d, \quad \gamma_z(r^2 + c) = \frac{\tau_z r}{\tau_\theta b}d, \tag{5.128}$$

where

$$c = \frac{\alpha q^2}{4\pi^2\mu\varepsilon_0}, \quad d = \frac{\tau_\theta b^2}{\mu}. \tag{5.129}$$

Assuming that the inner boundary is fixed, i.e.

$$g(a) = w(a) = 0, \tag{5.130}$$

we use $\gamma_\theta = rg'(r)$ and $\gamma_z = w'(r)$ to integrate the above and obtain the explicit solutions

$$g(r) = \frac{d}{c}\left[\log\left(\frac{r}{a}\right) - \frac{1}{2}\log\left(\frac{r^2 + c}{a^2 + c}\right)\right], \quad w(r) = \frac{\tau_z d}{2\tau_\theta b}\log\left(\frac{r^2 + c}{a^2 + c}\right). \tag{5.131}$$

It is straightforward to show that both g and w are decreasing functions of c, so that either an increase in the magnitude of the electric field (as measured by q) or an increase in the value of the material constant α will stiffen the mechanical response of the material.

5.3.3 Inflation of a Spherical Shell

In this problem we follow closely the analysis in Dorfmann and Ogden (2006). A thick-walled spherical shell is conveniently described in its unloaded reference configuration by using spherical polar coordinates (R, Θ, Φ) with

$$0 < A \leq R \leq B, \quad 0 \leq \Theta \leq \pi, \quad 0 \leq \Phi \leq 2\pi, \tag{5.132}$$

where A and B denote the inner and outer radii of the sphere. The shell is now inflated by applying a pressure P on the inner surface $R = A$ to produce a spherically symmetric deformation given by $\mathbf{x} = f(R)\mathbf{X}$, where $R = |\mathbf{X}|$, with the origin at the centre of the sphere (we do not consider here the possibility of bifurcation into an aspherical mode of deformation). The deformed configuration is described by

$$0 < a \leq r \leq b, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi, \tag{5.133}$$

where (r, θ, ϕ) are again spherical polar coordinates and a and b the inner and outer radii of the deformed shell. Since we are considering incompressible materials and radial symmetry, we have

$$r^3 - R^3 = a^3 - A^3, \quad \theta = \Theta, \quad \phi = \Phi, \tag{5.134}$$

and hence, $f(R) = (R^3 + a^3 - A^3)^{1/3}/R$.

The matrix of components of the deformation gradient with respect to the spherical polar coordinate axes is diagonal and written $\mathsf{F} = \mathrm{diag}[\lambda^{-2}, \lambda, \lambda]$, where $\lambda = r/R > 1$ is the (equibiaxial) principal stretch normal to the radial direction and λ^{-2} is the principal stretch in the radial direction. The principal invariants I_1 and I_2 are therefore given by

$$I_1 = 2\lambda^2 + \lambda^{-4}, \quad I_2 = 2\lambda^{-2} + \lambda^4. \tag{5.135}$$

For later use we define the stretches at the inner and outer surface of the deformed shell as $\lambda_a = a/A$ and $\lambda_b = b/B$, respectively. Then, from (5.134) we obtain

$$\lambda_a^3 - 1 = \left(\frac{R}{A}\right)^3 (\lambda^3 - 1) = \left(\frac{B}{A}\right)^3 (\lambda_b^3 - 1), \tag{5.136}$$

and hence, in particular, λ_b and λ_a are interdependent. For inflation of the shell we have

$$\lambda_a \geq \lambda \geq \lambda_b \geq 1, \tag{5.137}$$

but if the radius of the sphere decreases, the reverse of these inequalities holds.

5.3.3.1 The Electric Field and Stress Components

Because of the spherical symmetry we consider only a radial electric field and electric displacement, with components E_r and D_r, respectively, in the material and $E_r^\star$ and $D_r^\star = \varepsilon_0 E_r^\star$ outside the material, which depend only on r. The equation $\mathrm{curl}\,\mathbf{E} = \mathbf{0}$ is then satisfied identically and $\mathrm{div}\,\mathbf{D} = 0$ reduces to the form

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2 D_r) = 0 \tag{5.138}$$

(see Appendix A for the full equation in spherical polar coordinates). If we suppose that the field is generated by a spherically symmetric charge distribution within the radius A with total charge q, then by applying Gauss's theorem (2.36), we obtain

$$D_r = \frac{q}{4\pi r^2}, \tag{5.139}$$

which holds outside the distribution. Since D_r is continuous across the boundaries of the shell, then this applies both within the material and outside (it would need to be modified *within* the charge distribution, but we are not concerned with this).

The Lagrangian field associated with $\mathbf{D}$ is $\mathbf{D}_\mathrm{L} = \mathbf{F}^{-1}\mathbf{D}$, which has only the radial component $D_{\mathrm{L}R} = \lambda^2 D_r$. We use the constitutive formulation based on Ω^* and therefore the invariants K_4, K_5, K_6, which are given by

$$K_4 = D_{\mathrm{L}R}^2, \quad K_5 = \lambda^{-4} K_4, \quad K_6 = \lambda^{-8} K_4. \tag{5.140}$$

Use of (5.5) allows us to calculate the components of the total stress tensor $\boldsymbol{\tau}$, which are

$$\tau_{rr} = -p + 2\left[\Omega_1^*\lambda^{-4} + 2\,\Omega_2^*\lambda^{-2} + (\Omega_5^*\lambda^{-4} + 2\,\Omega_6^*\lambda^{-8})K_4\right], \tag{5.141}$$

$$\tau_{\theta\theta} = -p + 2\left[\Omega_1^*\lambda^2 + \Omega_2^*\left(\lambda^4 + \lambda^{-2}\right)\right], \tag{5.142}$$

and because of radial symmetry, $\tau_{\phi\phi} = \tau_{\theta\theta}$ and all shear stress components vanish.

From (5.6) we find the expression for the electric field, which has the single non-zero component E_r given by

$$E_r = 2\left(\Omega_4^*\lambda^4 + \Omega_5^* + \Omega_6^*\lambda^{-4}\right) D_r. \tag{5.143}$$

The equilibrium equation $\operatorname{div}\boldsymbol{\tau} = \mathbf{0}$ reduces (see Appendix A), for the considered spherically symmetric deformation, to the radial component equation

$$r\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} = 2\left(\tau_{\theta\theta} - \tau_{rr}\right). \tag{5.144}$$

Since the invariants (5.135) and (5.140) depend on only two independent quantities, namely the stretch λ and the invariant K_4, it is convenient to define a reduced energy function ω^* given by

$$\omega^*\left(\lambda, K_4\right) = \Omega^*\left(I_1, I_2, K_4, K_5, K_6\right), \tag{5.145}$$

which allows the stress difference in (5.144) to be written in the alternative form

$$2\left(\tau_{\theta\theta} - \tau_{rr}\right) = \lambda\omega_\lambda^*, \tag{5.146}$$

and hence the equilibrium equation as

$$r\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} = \lambda\omega_\lambda^*, \tag{5.147}$$

where $\omega_\lambda^* = \partial\omega^*/\partial\lambda$. The constitutive equation for the radial electric field is written in the simplified form

$$E_r = 2\lambda^4\omega_4^* D_r, \tag{5.148}$$

where $\omega_4^* = \partial\omega^*/\partial K_4$.

Outside the material the non-zero components of the Maxwell stress tensor are, by use of (4.15),

$$\tau_{\mathrm{e}rr}^\star(r) = \tau_{\mathrm{e}rr}^\star = \frac{1}{2}\varepsilon_0^{-1}D_r^{\star 2}, \quad \tau_{\mathrm{e}\theta\theta}^\star = \tau_{\mathrm{e}\phi\phi}^\star = -\frac{1}{2}\varepsilon_0^{-1}D_r^{\star 2}, \tag{5.149}$$

wherein $\tau_{\mathrm{e}rr}^\star(r)$ is defined.

The boundary $r = b$ is free of mechanical load, while on the inner boundary $r = a$ the pressure P is applied. Accounting for the Maxwell stress, it follows that the boundary conditions are

$$\tau_{rr} = -P + \tau_{\mathrm{e}rr}^\star(a) \quad \text{on} \quad r = a, \quad \tau_{rr} = \tau_{\mathrm{e}rr}^\star(b) \quad \text{on} \quad r = b, \tag{5.150}$$

where $\tau_{\mathrm{e}rr}^\star(a)$ and $\tau_{\mathrm{e}rr}^\star(b)$ are the Maxwell stresses evaluated at the inner and outer boundaries, respectively.

Integration of (5.147) using these boundary conditions gives

$$P - [\tau_{\mathrm{e}rr}^\star(a) - \tau_{\mathrm{e}rr}^\star(b)] = \int_a^b \lambda\omega_\lambda^* \frac{\mathrm{d}r}{r}. \tag{5.151}$$

A change of variable in (5.151) using

$$\frac{\mathrm{d}\lambda}{\mathrm{d}r} = \frac{1}{r}\left(\lambda - \lambda^4\right), \tag{5.152}$$

which is obtained by using the definition $\lambda = r/R$ and differentiation of (5.134), results in the alternative expression

$$P - [\tau_{\mathrm{e}rr}^\star(a) - \tau_{\mathrm{e}rr}^\star(b)] = \int_{\lambda_b}^{\lambda_a} \frac{\omega_\lambda^*}{(\lambda^3 - 1)}\mathrm{d}\lambda. \tag{5.153}$$

For details of the analysis of the inflation of an elastic spherical shell and its stability, we refer to, for example, Haughton and Ogden (1978) and Ogden (1997).

From (5.139) and (5.149), we have

$$\tau^{\star}_{\mathrm{e}rr}(a) - \tau^{\star}_{\mathrm{e}rr}(b) = \frac{\left(b^4 - a^4\right) q^2}{32\pi^2 \varepsilon_0 a^4 b^4}, \tag{5.154}$$

which is positive. Thus, the effect of the electric field counteracts the effect of the inflating pressure.

From (5.141), (5.142) and (5.146) we have

$$\lambda\omega^*_\lambda = 4\lambda^{-4}\left[\left(\Omega^*_1 + \lambda^2\Omega^*_2\right)\left(\lambda^6 - 1\right) - \left(\Omega^*_5 + 2\lambda^{-4}\Omega^*_6\right) K_4\right], \tag{5.155}$$

and it follows that the reference configuration $\lambda = 1$ can be maintained by the combined action of pressure and the electric field provided that

$$P - [\tau^{\star}_{\mathrm{e}rr}(a) - \tau^{\star}_{\mathrm{e}rr}(b)] = -4\int_a^b \left(\Omega^*_5 + 2\Omega^*_6\right) K_4 \frac{\mathrm{d}r}{r}. \tag{5.156}$$

For the special case when $P = 0$, the right-hand side of (5.156) must be negative, which suggests that the inequality $\Omega^*_5 + 2\Omega^*_6 > 0$ should hold. More generally, when $P = 0$, the right-hand side of (5.151) must be negative, and we expect a reduction in the radius, so that $\lambda < 1$. Consistent with this expectation, the expression (5.155) is negative when the inequality $\Omega^*_5 + 2\Omega^*_6 > 0$ holds under the assumption that $\Omega^*_1 + \lambda^2\Omega^*_2 > 0$ also holds.

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Chapter 6
Nonlinear Magnetoelastic Interactions

Abstract This chapter begins with a summary of the equations of magnetostatics following which there is a discussion of the magnetic forces and couples acting on the material. The main aim of the chapter is to combine the nonlinear theory of elasticity with the magnetostatic theory of magnetizable materials in order to provide a general theory of nonlinear magnetoelastic interactions. Towards this end the modifications of the mechanical balance equations required to incorporate magnetic influences are derived. The balance of energy is then used to catalogue a variety of forms of the general constitutive law for a magnetoelastic material, first in Eulerian form. Lagrangian versions of the magnetic field and magnetic induction field vectors are then introduced, leading to particularly simple forms of the constitutive laws. Material symmetry is then discussed in general form, and explicit representations for the constitutive equations of a nonlinear isotropic magnetoelastic material are obtained, followed by a brief discussion of their linear specializations.

6.1 Preliminaries

In this chapter we provide a parallel development to that in Chap. 4 for the nonlinear theory of static magnetoelasticity for a highly deformable elastic material in which the mechanical and magnetic effects are fully coupled. In part we follow the development given in Ogden (2011). For a magnetizable material occupying the region $\mathcal{B}$, with boundary $\partial\mathcal{B}$, in a deformed configuration, the magnetic field, magnetic induction, magnetization and free current density vectors are again denoted $\mathbf{H}$, $\mathbf{B}$, $\mathbf{M}$ and $\mathbf{J}_\mathrm{f}$, respectively, as in Sect. 2.4.2. We summarize the equations satisfied by $\mathbf{H}$ and $\mathbf{B}$ as

$$\operatorname{curl}\mathbf{H} = \mathbf{J}_\mathrm{f}, \quad \operatorname{div}\mathbf{B} = 0, \tag{6.1}$$

L. Dorfmann and R.W. Ogden, *Nonlinear Theory of Electroelastic and Magnetoelastic Interactions*, DOI 10.1007/978-1-4614-9596-3_6,

which apply within $\mathcal{B}$ and, with $\mathbf{J}_\mathrm{f} = \mathbf{0}$, outside $\mathcal{B}$. The magnetization is given by

$$\mathbf{M} = \mu_0^{-1}\mathbf{B} - \mathbf{H} \tag{6.2}$$

in a magnetizable material, and $\mathbf{B} = \mu_0\mathbf{H}$ in vacuum or non-magnetizable material.

The boundary conditions are

$$\mathbf{n} \times [\![\mathbf{H}]\!] = \mathbf{K}_\mathrm{f}, \quad \mathbf{n} \cdot [\![\mathbf{B}]\!] = 0 \quad \text{on } \partial\mathcal{B}, \tag{6.3}$$

where $\mathbf{K}_\mathrm{f}$ is the free surface current density (per unit area). Alternatively, these are written

$$[\![\mathbf{H}]\!] = (\mathbf{n} \cdot \mathbf{M})\mathbf{n} - \mathbf{n} \times \mathbf{K}_\mathrm{f}, \quad [\![\mathbf{B}]\!] = \mu_0\mathbf{n} \times (\mathbf{n} \times \mathbf{M}) - \mu_0\mathbf{n} \times \mathbf{K}_\mathrm{f} \quad \text{on } \partial\mathcal{B}. \tag{6.4}$$

The above equations and boundary conditions are conjoined with appropriate forms of the mechanical equilibrium equation and mechanical boundary conditions. We now provide an overview of different ways in which the equations of mechanical equilibrium and the accompanying traction boundary conditions can be written in the presence of magneto-mechanical interactions, which requires consideration of several different stress tensors.

6.2 Equilibrium and Stress

6.2.1 Magnetic Forces and Couples

We recall that the magnetic Lorentz force density on a material volume V containing a distribution of current with density $\mathbf{J}$ is given by $\mathbf{J} \times \mathbf{B}$ and that from (2.92) and (2.96) the force on a single dipole $\mathbf{m}$ in a magnetic field may be written as either $(\operatorname{grad}\mathbf{B})^\mathrm{T}\mathbf{m}$ or $\mu_0(\operatorname{grad}\mathbf{H})^\mathrm{T}\mathbf{m}$ and that these two expressions are equal. In a magnetized material the appropriate form of the Lorentz force density is based on the free current density $\mathbf{J}_\mathrm{f}$ and given by $\mathbf{J}_\mathrm{f} \times \mathbf{B}$, but the counterparts of the force on a single dipole when the magnetization density is $\mathbf{M}$, namely $(\operatorname{grad}\mathbf{B})^\mathrm{T}\mathbf{M}$ and $\mu_0(\operatorname{grad}\mathbf{H})^\mathrm{T}\mathbf{M}$, are not equal and have to be treated separately. Thus, the volumetric magnetic force density in a magnetized material may be written as either

$$\mathbf{J}_\mathrm{f} \times \mathbf{B} + \mu_0(\operatorname{grad}\mathbf{H})^\mathrm{T}\mathbf{M} \quad \text{or} \quad \mathbf{J}_\mathrm{f} \times \mathbf{B} + (\operatorname{grad}\mathbf{B})^\mathrm{T}\mathbf{M}, \tag{6.5}$$

analogously to the two expressions for the volumetric force density in an electroelastic material discussed in Sect. 4.2. These two expressions are different. However, the total magnetic force on a body involves not just volumetric (body) forces but also surface forces, which, as we see below, are also different in the two cases. The total magnetic force on a body is the same whichever representation is adopted. We begin by considering the first of these, which, since

$$\mathbf{J}_\mathrm{f} \times \mathbf{B} = \operatorname{curl}\mathbf{H} \times \mathbf{B} = (\operatorname{grad}\mathbf{H})\mathbf{B} - (\operatorname{grad}\mathbf{H})^\mathrm{T}\mathbf{B} \tag{6.6}$$

can be written with the help of $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ as

$$\mathbf{J}_\mathrm{f} \times \mathbf{B} + \mu_0(\operatorname{grad}\mathbf{H})^\mathrm{T}\mathbf{M} = \operatorname{div}\boldsymbol{\tau}_\mathrm{m}, \tag{6.7}$$

where $\boldsymbol{\tau}_\mathrm{m}$ is the magnetic Maxwell stress defined by

$$\boldsymbol{\tau}_\mathrm{m} = \mathbf{B} \otimes \mathbf{H} - \frac{1}{2}\mu_0(\mathbf{H} \cdot \mathbf{H})\mathbf{I}. \tag{6.8}$$

(Here, we recall that the subscript m stands for 'magnetic' rather than 'Maxwell'.) This, which is not in general symmetric, is the counterpart for a magnetized material of the electric Maxwell stress (4.10) and plays a similar role here to that of $\boldsymbol{\tau}_\mathrm{e}$ in Sect. 4.2.

The associated total magnetic volumetric force acting on the body $\mathcal{B}$, with boundary $\partial\mathcal{B}$, may therefore be written with the help of the divergence theorem as

$$\int_{\mathcal{B}} \operatorname{div}\boldsymbol{\tau}_\mathrm{m}\,\mathrm{d}v = \int_{\partial\mathcal{B}} \boldsymbol{\tau}_\mathrm{m}^\mathrm{T}\mathbf{n}\,\mathrm{d}s. \tag{6.9}$$

Next, we wish to relate the surface traction term $\boldsymbol{\tau}_\mathrm{m}^\mathrm{T}\mathbf{n}$ calculated on $\partial\mathcal{B}$ from within the material to the corresponding expression calculated on the outside of $\partial\mathcal{B}$, where there is a vacuum or non-magnetizable material.

Outside $\mathcal{B}$, the Maxwell stress is symmetric, as defined in (2.98). Again we use a superscript $\star$ to distinguish quantities outside the material from those inside the material. Thus, $\mathbf{H}^\star$ and $\mathbf{B}^\star = \mu_0\mathbf{H}^\star$ are the fields exterior to $\mathcal{B}$, and the Maxwell stress is written as

$$\boldsymbol{\tau}_\mathrm{m}^\star = \mathbf{B}^\star \otimes \mathbf{H}^\star - \frac{1}{2}\mu_0(\mathbf{H}^\star \cdot \mathbf{H}^\star)\mathbf{I}. \tag{6.10}$$

Then, with the help of (6.4), we obtain the traction discontinuity

$$\begin{aligned}[\![\boldsymbol{\tau}_\mathrm{m}^\mathrm{T}]\!]\mathbf{n} = (\boldsymbol{\tau}_\mathrm{m}^\star - \boldsymbol{\tau}_\mathrm{m}^\mathrm{T})\mathbf{n} &= \frac{1}{2}\mu_0(\mathbf{M} \cdot \mathbf{n})^2\mathbf{n} + \mathbf{K}_\mathrm{f} \times \mathbf{B} \\ &+ \mu_0[(\mathbf{n} \times \mathbf{M}) \cdot \mathbf{K}_\mathrm{f}]\mathbf{n} - \frac{1}{2}\mu_0(\mathbf{K}_\mathrm{f} \cdot \mathbf{K}_\mathrm{f})\mathbf{n} \equiv \mathbf{t}_\mathrm{m},\end{aligned} \tag{6.11}$$

where $\mathbf{B}$ and $\mathbf{M}$ are evaluated on $\partial\mathcal{B}$ from the *inside* and wherein the notation $\mathbf{t}_\mathrm{m}$ is defined, analogously to $\mathbf{t}_\mathrm{e}$ in (4.18). It represents a mechanical traction on the boundary $\partial\mathcal{B}$ due to magnetic effects.

By using this to write $\boldsymbol{\tau}_{\mathrm{m}}^{\mathrm{T}}\mathbf{n} = \boldsymbol{\tau}_{\mathrm{m}}^{\star}\mathbf{n} - \mathbf{t}_{\mathrm{m}}$, we can rewrite (6.9) as

$$\mathbf{F}_{\mathrm{m}} \equiv \int_{\mathcal{B}} \operatorname{div} \boldsymbol{\tau}_{\mathrm{m}} \,\mathrm{d}v + \int_{\partial\mathcal{B}} \mathbf{t}_{\mathrm{m}} \,\mathrm{d}s = \int_{\partial\mathcal{B}} \boldsymbol{\tau}_{\mathrm{m}}^{\star}\mathbf{n} \,\mathrm{d}s, \tag{6.12}$$

where $\mathbf{F}_{\mathrm{m}}$ denotes the total magnetic force acting on the body. It consists of both a body force and a surface force but can also be calculated from the traction on the boundary due to the exterior Maxwell stress. We note here that the argument used to arrive at this result in Ogden (2011) is incorrect.

Now consider again the expression for $\operatorname{div} \boldsymbol{\tau}_{\mathrm{m}}$ within $\mathcal{B}$ given by (6.7). There are different ways in which it can be rewritten in terms of two or all three of $\mathbf{B}, \mathbf{H}$ and $\mathbf{M}$, starting from the definition (6.8) and using $\operatorname{div} \mathbf{B} = 0$ and the connection $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$. Here we consider one such possibility, namely

$$\operatorname{div} \boldsymbol{\tau}_{\mathrm{m}} = \mathbf{J}_{\mathrm{f}} \times \mathbf{B} + (\operatorname{grad} \mathbf{B})^{\mathrm{T}}\mathbf{M} - \tfrac{1}{2}\mu_0 \operatorname{grad}(\mathbf{M} \cdot \mathbf{M}). \tag{6.13}$$

The magnetic force $\mathbf{F}_{\mathrm{m}}$ in (6.12) is now written in two alternative ways, as either

$$\mathbf{F}_{\mathrm{m}} = \int_{\mathcal{B}} [\mathbf{J}_{\mathrm{f}} \times \mathbf{B} + \mu_0(\operatorname{grad} \mathbf{H})^{\mathrm{T}}\mathbf{M}] \,\mathrm{d}v + \int_{\partial\mathcal{B}} \mathbf{t}_{\mathrm{m}} \,\mathrm{d}s, \tag{6.14}$$

or

$$\mathbf{F}_{\mathrm{m}} = \int_{\mathcal{B}} [\mathbf{J}_{\mathrm{f}} \times \mathbf{B} + (\operatorname{grad} \mathbf{B})^{\mathrm{T}}\mathbf{M}] \,\mathrm{d}v + \int_{\partial\mathcal{B}} \bar{\mathbf{t}}_{\mathrm{m}} \,\mathrm{d}s, \tag{6.15}$$

where $\bar{\mathbf{t}}_{\mathrm{m}}$ is defined by

$$\bar{\mathbf{t}}_{\mathrm{m}} = \mathbf{t}_{\mathrm{m}} - \tfrac{1}{2}\mu_0(\mathbf{M} \cdot \mathbf{M})\mathbf{n}, \tag{6.16}$$

analogously to $\bar{\mathbf{t}}_{\mathrm{e}}$ in (4.27).

We emphasize that the associated volumetric force densities $(\operatorname{grad} \mathbf{B})^{\mathrm{T}}\mathbf{M}$ and $\mu_0(\operatorname{grad} \mathbf{H})^{\mathrm{T}}\mathbf{M}$ are analogous to the expressions for the force on a single dipole given by (2.92) and $(2.96)_1$, respectively, but, unlike the latter, they are not the same in the present continuum setting. They are also supplemented by the Lorentz force term associated with the free current density. In the present context there is a contribution to the force from the boundary term which does not arise when considering an isolated dipole in free space. The expressions (6.14) and (6.15), though, are entirely equivalent.

The analogue of the couple $\mathbf{m} \times \mathbf{B}$ on a single magnetic dipole, given by (2.95), is the couple per unit volume $\mathbf{M} \times \mathbf{B}$, which, because of the connection $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$, may also be written as $\mu_0\mathbf{M} \times \mathbf{H}$ or $\mathbf{B} \times \mathbf{H}$. The total magnetic couple about the origin, denoted $\mathbf{G}_{\mathrm{m}}$, corresponding to the representation (6.14) based on use of $\mathbf{H}$ is then

$$\mathbf{G}_{\mathrm{m}} = \int_{\mathcal{B}} (\mathbf{x} \times \operatorname{div} \boldsymbol{\tau}_{\mathrm{m}} + \mu_0\mathbf{M} \times \mathbf{H}) \,\mathrm{d}v + \int_{\partial\mathcal{B}} \mathbf{x} \times \mathbf{t}_{\mathrm{m}} \,\mathrm{d}s, \tag{6.17}$$

while that based on use of $\mathbf{B}$ in (6.15) is

$$\mathbf{G}_{\mathrm{m}} = \int_{\mathcal{B}} (\mathbf{x} \times \operatorname{div} \bar{\boldsymbol{\tau}}_{\mathrm{m}} + \mathbf{M} \times \mathbf{B})\, \mathrm{d}v + \int_{\partial\mathcal{B}} \mathbf{x} \times \bar{\mathbf{t}}_{\mathrm{m}}\, \mathrm{d}s, \tag{6.18}$$

where we have assumed that there are no intrinsic mechanical couples, and we have introduced the notation

$$\bar{\boldsymbol{\tau}}_{\mathrm{m}} = \boldsymbol{\tau}_{\mathrm{m}} + \frac{1}{2}\mu_0(\mathbf{M} \cdot \mathbf{M})\mathbf{I}, \tag{6.19}$$

and we note that

$$\operatorname{div} \bar{\boldsymbol{\tau}}_{\mathrm{m}} = \operatorname{div} \boldsymbol{\tau}_{\mathrm{m}} + \mu_0(\operatorname{grad} \mathbf{M})^{\mathrm{T}}\mathbf{M} = \mathbf{J}_{\mathrm{f}} \times \mathbf{B} + (\operatorname{grad} \mathbf{B})^{\mathrm{T}}\mathbf{M}. \tag{6.20}$$

The above derivations have been applied to the *whole* body with no magnetization exterior to it. However, the formulas (6.14), (6.15), (6.17) and (6.18) actually apply to an arbitrary material volume, as shown by Brown (1966). If there were no surface discontinuity, then the total magnetic force on a volume V with surface S within $\mathcal{B}$ is given simply by

$$\int_V \operatorname{div} \boldsymbol{\tau}_{\mathrm{m}}\, \mathrm{d}v = \int_S \boldsymbol{\tau}_{\mathrm{m}}^{\mathrm{T}}\mathbf{n}\, \mathrm{d}s, \tag{6.21}$$

where the traction term is due to the magnetic interaction with the surrounding material. If there is a discontinuity on S, then this would need to be modified in a similar way to the whole body. For completeness in what follows we assume that the region V is subject to a volumetric force corresponding to the left-hand expression above and additionally a surface force on its boundary S.

For a general region V, with boundary S, we now write the total force $\mathbf{F}_{\mathrm{m}}$ and couple $\mathbf{G}_{\mathrm{m}}$ in terms of a generic magnetic body force density and surface traction density, denoted $\hat{\mathbf{f}}_{\mathrm{m}}$ and $\hat{\mathbf{t}}_{\mathrm{m}}$, respectively, with a generic counterpart of $\boldsymbol{\tau}_{\mathrm{m}}$, denoted $\hat{\boldsymbol{\tau}}_{\mathrm{m}}$, such that $\hat{\mathbf{f}}_{\mathrm{m}} = \operatorname{div} \hat{\boldsymbol{\tau}}_{\mathrm{m}}$, which we may refer to as a generic Maxwell stress. We also introduce a corresponding generic intrinsic magnetic couple $\hat{\mathbf{g}}_{\mathrm{m}}$. Then,

$$\mathbf{F}_{\mathrm{m}} = \int_V \hat{\mathbf{f}}_{\mathrm{m}}\, \mathrm{d}v + \int_S \hat{\mathbf{t}}_{\mathrm{m}}\, \mathrm{d}s, \tag{6.22}$$

and

$$\mathbf{G}_{\mathrm{m}} = \int_V (\mathbf{x} \times \hat{\mathbf{f}}_{\mathrm{m}} + \hat{\mathbf{g}}_{\mathrm{m}})\, \mathrm{d}v + \int_S \mathbf{x} \times \hat{\mathbf{t}}_{\mathrm{m}}\, \mathrm{d}s. \tag{6.23}$$

Thus, for the two cases considered, we have either $\hat{\mathbf{f}}_{\mathrm{m}} = \operatorname{div} \boldsymbol{\tau}_{\mathrm{m}}$ with $\hat{\mathbf{t}}_{\mathrm{m}} = \mathbf{t}_{\mathrm{m}}$ or $\hat{\mathbf{f}}_{\mathrm{m}} = \operatorname{div} \bar{\boldsymbol{\tau}}_{\mathrm{m}}$ with $\hat{\mathbf{t}}_{\mathrm{m}} = \bar{\mathbf{t}}_{\mathrm{m}}$, and for each representation $\hat{\mathbf{g}}_{\mathrm{m}} = \mu_0\mathbf{M} \times \mathbf{H} = \mathbf{B} \times \mathbf{H} =$

$\mathbf{M} \times \mathbf{B}$. Moreover, we note also that $\boldsymbol{\epsilon}\bar{\boldsymbol{\tau}}_{\mathrm{m}} = \boldsymbol{\epsilon}\boldsymbol{\tau}_{\mathrm{m}} = \mathbf{B} \times \mathbf{H} = \hat{\mathbf{g}}_{\mathrm{m}}$ in each case, where we recall that $\boldsymbol{\epsilon}$ is the alternating tensor defined in Sect. 2.2.3.

The generic definitions above admit the possibility of expressions for the magnetic body and surface force densities other than the two introduced here. Such expressions may or may not have direct physical interpretations, but they may be useful from the point of view of the mathematical formulation of the governing equations. In the following subsection we incorporate the magnetic force and couple into the mechanical balance equations.

6.2.2 Mechanical Equilibrium

In equilibrium the total force and total couple acting on a part V of a body in its deformed configuration must each vanish. Let $\mathbf{f}$ be the *mechanical body force* per unit mass, ρ the mass density of the material and $\mathbf{t}_{\mathrm{a}}$ the *mechanical traction* per unit area of the boundary S of V. Then, on taking account of the magnetic force and couple given by (6.22) and (6.23), we have

$$\int_V (\rho\mathbf{f} + \hat{\mathbf{f}}_{\mathrm{m}})\,\mathrm{d}v + \int_S (\mathbf{t}_{\mathrm{a}} + \hat{\mathbf{t}}_{\mathrm{m}})\,\mathrm{d}s = \mathbf{0}, \tag{6.24}$$

and

$$\int_V [\mathbf{x} \times (\rho\mathbf{f} + \hat{\mathbf{f}}_{\mathrm{m}}) + \hat{\mathbf{g}}_{\mathrm{m}}]\,\mathrm{d}v + \int_S [\mathbf{x} \times (\mathbf{t}_{\mathrm{a}} + \hat{\mathbf{t}}_{\mathrm{m}})]\,\mathrm{d}s = \mathbf{0}. \tag{6.25}$$

Then, by a standard tetrahedron argument from continuum mechanics applied to (6.24), we deduce that there exists a second-order (Cauchy-like) stress tensor, which we denote by $\hat{\boldsymbol{\sigma}}$, defined in V, such that

$$\hat{\boldsymbol{\sigma}}^{\mathrm{T}}\mathbf{n} = \mathbf{t}_{\mathrm{a}} + \hat{\mathbf{t}}_{\mathrm{m}} \quad \text{on } S \tag{6.26}$$

and $\hat{\boldsymbol{\sigma}}$ is independent of $\mathbf{n}$. Substitution of this into (6.24) and application of the divergence theorem yields

$$\int_V (\rho\mathbf{f} + \hat{\mathbf{f}}_{\mathrm{m}} + \operatorname{div}\hat{\boldsymbol{\sigma}})\,\mathrm{d}v = \mathbf{0}. \tag{6.27}$$

Since V is arbitrary, then, provided the integrand is continuous, we may deduce the local form of the equilibrium equation, specifically

$$\operatorname{div}\hat{\boldsymbol{\sigma}} + \rho\mathbf{f} + \hat{\mathbf{f}}_{\mathrm{m}} = \mathbf{0} \quad \text{in } V. \tag{6.28}$$

Note, in particular, that $\operatorname{div}\hat{\boldsymbol{\sigma}} + \hat{\mathbf{f}}_{\mathrm{m}}$ is the same for each $(\hat{\boldsymbol{\sigma}}, \hat{\mathbf{f}}_{\mathrm{m}})$ pair, including the pairs $(\boldsymbol{\sigma}, \mathbf{f}_{\mathrm{m}})$ and $(\bar{\boldsymbol{\sigma}}, \bar{\mathbf{f}}_{\mathrm{m}})$ associated with $\boldsymbol{\tau}_{\mathrm{m}}$ and $\bar{\boldsymbol{\tau}}_{\mathrm{m}}$, respectively, where

$$\mathbf{f}_{\mathrm{m}} = \operatorname{div}\boldsymbol{\tau}_{\mathrm{m}} = \mathbf{J}_{\mathrm{f}} \times \mathbf{B} + \mu_0(\operatorname{grad}\mathbf{H})^{\mathrm{T}}\mathbf{M},$$

$$\bar{\mathbf{f}}_{\mathrm{m}} = \operatorname{div}\bar{\boldsymbol{\tau}}_{\mathrm{m}} = \mathbf{J}_{\mathrm{f}} \times \mathbf{B} + (\operatorname{grad}\mathbf{B})^{\mathrm{T}}\mathbf{M}.$$

Substitution of (6.26) into (6.25) followed by another application of the divergence theorem then leads to

$$\int_V (\boldsymbol{\epsilon}\hat{\boldsymbol{\sigma}} + \hat{\mathbf{g}}_{\mathrm{m}})\,\mathrm{d}v = \mathbf{0}, \tag{6.29}$$

which has local form

$$\boldsymbol{\epsilon}\hat{\boldsymbol{\sigma}} + \hat{\mathbf{g}}_{\mathrm{m}} = \mathbf{0} \quad \text{in } V. \tag{6.30}$$

This shows that in general $\boldsymbol{\epsilon}\hat{\boldsymbol{\sigma}} \neq \mathbf{0}$, i.e. $\hat{\boldsymbol{\sigma}}$ is not symmetric.

Now, for the two examples considered above, we have $\hat{\mathbf{g}}_{\mathrm{m}} = \boldsymbol{\epsilon}\hat{\boldsymbol{\tau}}_{\mathrm{m}}$, and hence, $\boldsymbol{\epsilon}(\hat{\boldsymbol{\sigma}} + \hat{\boldsymbol{\tau}}_{\mathrm{m}}) = \mathbf{0}$. Let us introduce the second-order tensor $\boldsymbol{\tau}$ defined by

$$\boldsymbol{\tau} = \hat{\boldsymbol{\sigma}} + \hat{\boldsymbol{\tau}}_{\mathrm{m}}, \tag{6.31}$$

so that $\boldsymbol{\epsilon}\boldsymbol{\tau} = \mathbf{0}$, i.e. $\boldsymbol{\tau}$ is *symmetric*. Since $\hat{\mathbf{f}}_{\mathrm{m}} = \operatorname{div}\hat{\boldsymbol{\tau}}_{\mathrm{m}}$, it follows from (6.28) that $\boldsymbol{\tau}$ satisfies the equilibrium equation

$$\operatorname{div}\boldsymbol{\tau} + \rho\mathbf{f} = \mathbf{0}. \tag{6.32}$$

By integrating this equation over the whole body $\mathcal{B}$ and applying the divergence theorem and then using (6.24), we obtain

$$\int_{\partial\mathcal{B}} \boldsymbol{\tau}\mathbf{n}\,\mathrm{d}s = -\int_{\mathcal{B}} \rho\mathbf{f}\,\mathrm{d}v = \int_{\mathcal{B}} \hat{\mathbf{f}}_{\mathrm{m}}\,\mathrm{d}v + \int_{\partial\mathcal{B}} (\mathbf{t}_{\mathrm{a}} + \hat{\mathbf{t}}_{\mathrm{m}})\,\mathrm{d}s. \tag{6.33}$$

A further application of the divergence theorem after replacing $\hat{\mathbf{f}}_{\mathrm{m}}$ by $\operatorname{div}\hat{\boldsymbol{\tau}}_{\mathrm{m}}$ then yields

$$\int_{\partial\mathcal{B}} \boldsymbol{\tau}\mathbf{n}\,\mathrm{d}s = \int_{\partial\mathcal{B}} (\mathbf{t}_{\mathrm{a}} + \hat{\mathbf{t}}_{\mathrm{m}} + \hat{\boldsymbol{\tau}}_{\mathrm{m}}^{\mathrm{T}}\mathbf{n})\,\mathrm{d}s. \tag{6.34}$$

This suggests that we should identify $\boldsymbol{\tau}\mathbf{n}$ with $\mathbf{t}_{\mathrm{a}} + \hat{\mathbf{t}}_{\mathrm{m}} + \hat{\boldsymbol{\tau}}_{\mathrm{m}}^{\mathrm{T}}\mathbf{n}$ on the boundary $\partial\mathcal{B}$. However, from the discontinuity condition (6.11) on $\partial\mathcal{B}$, we obtain $\hat{\mathbf{t}}_{\mathrm{m}} + \hat{\boldsymbol{\tau}}_{\mathrm{m}}^{\mathrm{T}}\mathbf{n} = \boldsymbol{\tau}_{\mathrm{m}}^{\star}\mathbf{n}$, where $\boldsymbol{\tau}_{\mathrm{m}}^{\star}$ evaluated on the exterior of $\partial\mathcal{B}$. Thus, we may consider the boundary condition for $\boldsymbol{\tau}$ to be

$$\boldsymbol{\tau}\mathbf{n} = \mathbf{t}_{\mathrm{a}} + \mathbf{t}_{\mathrm{m}}^{\star} \quad \text{on } \partial\mathcal{B}, \tag{6.35}$$

where $\mathbf{t}_{\mathrm{m}}^{\star}$ is defined by

$$\mathbf{t}_{\mathrm{m}}^{\star} = \boldsymbol{\tau}_{\mathrm{m}}^{\star}\mathbf{n}. \tag{6.36}$$

We refer to $\boldsymbol{\tau}$ as the *magnetic total Cauchy stress tensor* since it enables the magnetic body forces to be treated as stresses, as we have seen for electroelastic materials in Chap. 4. That the total stress tensor is *symmetric* proves to be an advantage in the subsequent analysis. In particular, it is interesting to note that the intrinsic magnetic couple is absorbed by use of this stress tensor, and the rotational balance equation is satisfied automatically. Its global form is simply

$$\int_{\mathcal{B}} \rho\mathbf{x} \times \mathbf{f}\,\mathrm{d}v + \int_{\partial\mathcal{B}} \mathbf{x} \times (\mathbf{t}_{\mathrm{a}} + \mathbf{t}_{\mathrm{m}}^{\star})\,\mathrm{d}s = \mathbf{0}. \tag{6.37}$$

Note that we have used the same notation $\boldsymbol{\tau}$ as in the electroelastic case since there is no danger of conflict in this chapter.

6.3 Constitutive Equations

6.3.1 Eulerian Formulations

There are three magnetic field vectors $\mathbf{B}, \mathbf{H}, \mathbf{M}$, which enjoy the connection $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$. Any one of these can be used as the independent magnetic variable in the formulation of constitutive laws for a deformable magnetizable material along with the deformation gradient tensor $\mathbf{F}$. Such constitutive laws are based on various scalar potential functions or energy functions.

We begin by providing a parallel development to that given in Sect. 4.3 for electroelastic materials. For the generic fields introduced in the above section, the virtual work done on a region $\mathcal{D}$, with boundary $\partial\mathcal{D}$, can be expressed as

$$\int_{\mathcal{D}} (\rho\mathbf{f} \cdot \mathbf{u} + \hat{\mathbf{f}}_{\mathrm{m}} \cdot \mathbf{u})\mathrm{d}v + \int_{\partial\mathcal{D}} (\hat{\boldsymbol{\sigma}}^{\mathrm{T}}\mathbf{n}) \cdot \mathbf{u}\mathrm{d}s = \int_{\mathcal{D}} \mathrm{tr}\,(\hat{\boldsymbol{\sigma}}\mathbf{L})\mathrm{d}v, \tag{6.38}$$

where $\mathbf{u}$ is again a virtual displacement and $\mathbf{L} = \mathrm{grad}\,\mathbf{u}$.

Let $\hat{\phi}$, correspondingly, be a generic energy density of the material per unit mass, and $\hat{W}_{\mathrm{m}}$ be a generic magnetic energy density per unit reference volume such that their virtual changes are related by

$$\rho_{\mathrm{r}}\dot{\hat{\phi}} = J\,\mathrm{tr}\,(\hat{\boldsymbol{\sigma}}\mathbf{L}) + \dot{\hat{W}}_{\mathrm{m}}, \tag{6.39}$$

and (6.38) can be rewritten as

$$\int_{\mathcal{D}_r}(\rho_r \mathbf{f}\cdot\mathbf{u} + J\hat{\mathbf{f}}_m\cdot\mathbf{u} + \dot{\hat{W}}_m)\mathrm{d}V + \int_{\partial\mathcal{D}}(\hat{\boldsymbol{\sigma}}^{\mathrm{T}}\mathbf{n})\cdot\mathbf{u}\mathrm{d}s = \int_{\mathcal{D}_r}\rho_r\dot{\hat{\phi}}\mathrm{d}V, \tag{6.40}$$

Thus, the combination of virtual magnetic and mechanical work is converted into virtual energy (which, depending of the choice of magnetic independent variable, may be the internal energy or the free energy).

As a first example we consider $\hat{\phi}$ to be a function of $\mathbf{F}$ and $\mathbf{M}_r = J\mathbf{M}$ following the prescription in the electroelastic case in Sect. 4.3. Thus, $\hat{\phi} = \phi^*(\mathbf{F}, \mathbf{M}_r)$, and we set $\hat{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}}$ with $\hat{\mathbf{f}}_m = \bar{\mathbf{f}}_m = \mathrm{div}\,\bar{\boldsymbol{\tau}}_m$, where $\bar{\boldsymbol{\tau}}_m$ is the Maxwell-type stress introduced in Sect. 6.2.1 and $\bar{\boldsymbol{\sigma}}$ is the associated Cauchy-type stress. We also take $\dot{\hat{W}}_m$ to be of the form $\dot{\hat{W}}_m = \mathbf{B}\cdot\dot{\mathbf{M}}_r$, which represents the virtual work done by the magnetic induction $\mathbf{B}$ in producing a virtual change in the magnetization per unit mass $\mathbf{M}/\rho$. We could equally well have taken $\dot{\hat{W}}_m$ to be $\mu_0\mathbf{H}\cdot\dot{\mathbf{M}}_r$, the direct analogy of $\mathbf{E}\cdot\dot{\mathbf{P}}_r$ in (4.33), and associated with the stress tensor $\boldsymbol{\sigma}$ rather than $\bar{\boldsymbol{\sigma}}$. It follows that

$$\bar{\boldsymbol{\sigma}} = \rho\mathbf{F}\frac{\partial\phi^*}{\partial\mathbf{F}}, \quad \mathbf{B} = \rho_r\frac{\partial\phi^*}{\partial\mathbf{M}_r} \equiv \rho\frac{\partial\phi^*}{\partial\mathbf{M}}. \tag{6.41}$$

Next, we consider a formulation based on use of the magnetic induction vector $\mathbf{B}$ as the independent magnetic variable, and we introduce the energy density function $\hat{\phi} = \bar{\phi}(\mathbf{F}, \mathbf{B})$, per unit mass, defined as the Legendre transform of ϕ^* given by

$$\rho_r\bar{\phi} = \rho_r\phi^* - \mathbf{B}\cdot\mathbf{M}_r, \tag{6.42}$$

which yields

$$\bar{\boldsymbol{\sigma}} = \rho\mathbf{F}\frac{\partial\bar{\phi}}{\partial\mathbf{F}}, \quad \mathbf{M}_r = -\rho_r\frac{\partial\bar{\phi}}{\partial\mathbf{B}} \quad \text{or equivalently} \quad \mathbf{M} = -\rho\frac{\partial\bar{\phi}}{\partial\mathbf{B}}. \tag{6.43}$$

Another example, which also involves $\mathbf{B}$ as the independent magnetic variable, makes use of the energy density function $\psi^*(\mathbf{F}, \mathbf{B})$. This is related to $\bar{\phi}$ by

$$\rho\psi^*(\mathbf{F}, \mathbf{B}) = \rho\bar{\phi}(\mathbf{F}, \mathbf{B}) + \tfrac{1}{2}\mu_0^{-1}\mathbf{B}\cdot\mathbf{B}. \tag{6.44}$$

This yields a different stress tensor, which we denote by $\boldsymbol{\sigma}^*$, and the magnetic field, specifically

$$\boldsymbol{\sigma}^* = \rho\mathbf{F}\frac{\partial\psi^*}{\partial\mathbf{F}}, \quad \mathbf{H} = \rho\frac{\partial\psi^*}{\partial\mathbf{B}}, \tag{6.45}$$

and in this case, we have

$$\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^*, \quad \hat{\boldsymbol{\tau}}_m = \mathbf{B}\otimes\mathbf{H} - (\mathbf{H}\cdot\mathbf{B})\mathbf{I}, \quad \hat{\mathbf{f}}_m = \mathbf{J}_f\times\mathbf{B} - (\mathrm{grad}\,\mathbf{B})^{\mathrm{T}}\mathbf{H}. \tag{6.46}$$

An alternative starting point is to consider the magnetic field $\mathbf{H}$ as the independent magnetic variable and to work in terms of the potential function $\phi(\mathbf{F}, \mathbf{H})$, say, where

$$\rho\phi = \rho\bar{\phi} + \frac{1}{2}\mu_0\mathbf{M}\cdot\mathbf{M}. \tag{6.47}$$

This yields the stress tensor $\boldsymbol{\sigma}$ and the magnetization $\mathbf{M}$ in the forms

$$\boldsymbol{\sigma} = \rho\mathbf{F}\frac{\partial\phi}{\partial\mathbf{F}}, \quad \mathbf{M} = -\mu_0^{-1}\rho\frac{\partial\phi}{\partial\mathbf{H}}, \tag{6.48}$$

and now we have

$$\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma}, \quad \hat{\boldsymbol{\tau}}_{\mathrm{m}} = \boldsymbol{\tau}_{\mathrm{m}}, \quad \hat{\mathbf{f}}_{\mathrm{m}} = \mathbf{f}_{\mathrm{m}} = \mathbf{J}_{\mathrm{f}} \times \mathbf{B} + \mu_0(\mathrm{grad}\,\mathbf{H})^{\mathrm{T}}\mathbf{M}, \tag{6.49}$$

where $\boldsymbol{\tau}_{\mathrm{m}}$ is the Maxwell-type stress defined in Sect. 6.2.1 and $\boldsymbol{\sigma}$ is the associated Cauchy-type stress.

The final example introduces the potential function $\psi(\mathbf{F}, \mathbf{H})$, related to ϕ by

$$\rho\psi(\mathbf{F}, \mathbf{H}) = \rho\phi(\mathbf{F}, \mathbf{H}) - \frac{1}{2}\mu_0\mathbf{H}\cdot\mathbf{H}, \tag{6.50}$$

and the associated stress, denoted $\bar{\boldsymbol{\sigma}}^*$, and the magnetic induction $\mathbf{B}$ are derived as

$$\bar{\boldsymbol{\sigma}}^* = \rho\mathbf{F}\frac{\partial\psi}{\partial\mathbf{F}}, \quad \mathbf{B} = -\rho\frac{\partial\psi}{\partial\mathbf{H}}. \tag{6.51}$$

In this case we have

$$\hat{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}}^*, \quad \hat{\boldsymbol{\tau}}_{\mathrm{m}} = \mathbf{B}\otimes\mathbf{H}, \quad \hat{\mathbf{f}}_{\mathrm{m}} = \mathbf{J}_{\mathrm{f}} \times \mathbf{B} + (\mathrm{grad}\,\mathbf{H})^{\mathrm{T}}\mathbf{B}, \tag{6.52}$$

and we note the connection

$$\rho\psi^* = \rho\psi + \mathbf{B}\cdot\mathbf{H}. \tag{6.53}$$

The above formulations provide a selection of the possible alternatives that have been used variously in the literature. For ease of reference, they are collected together in Table 6.1, together with two further energy densities for which the magnetization $\mathbf{M}$ is the independent variable. None of these options, however, allows the total stress tensor to be given directly in the form

$$\boldsymbol{\tau} = \rho\mathbf{F}\frac{\partial}{\partial\mathbf{F}}(\text{potential function}), \tag{6.54}$$

Table 6.1 Energy (potential) functions based on $\mathbf{B}$, $\mathbf{H}$, $\mathbf{M}$, and in each case the associated stress $\hat{\boldsymbol{\sigma}}$, the derived field vector, and the Maxwell stress $\hat{\boldsymbol{\tau}}_{\mathrm{m}}$

Potential	Stress $\hat{\boldsymbol{\sigma}}$	Magnetic vector	Maxwell stress $\hat{\boldsymbol{\tau}}_{\mathrm{m}}$
$\bar{\phi}(\mathbf{F},\mathbf{B})$	$\rho\mathbf{F}\dfrac{\partial\bar{\phi}}{\partial\mathbf{F}}$	$\mathbf{M}=-\rho\dfrac{\partial\bar{\phi}}{\partial\mathbf{B}}$	$\bar{\boldsymbol{\tau}}_{\mathrm{m}}$
$\psi^*(\mathbf{F},\mathbf{B})$	$\rho\mathbf{F}\dfrac{\partial\psi^*}{\partial\mathbf{F}}$	$\mathbf{H}=\rho\dfrac{\partial\psi^*}{\partial\mathbf{B}}$	$\bar{\boldsymbol{\tau}}_{\mathrm{m}}-\frac{1}{2}\mu_0^{-1}(\mathbf{B}\cdot\mathbf{B})\mathbf{I}$
$\phi(\mathbf{F},\mathbf{H})$	$\rho\mathbf{F}\dfrac{\partial\phi}{\partial\mathbf{F}}$	$\mathbf{M}=-\mu_0^{-1}\rho\dfrac{\partial\phi}{\partial\mathbf{H}}$	$\boldsymbol{\tau}_{\mathrm{m}}$
$\psi(\mathbf{F},\mathbf{H})$	$\rho\mathbf{F}\dfrac{\partial\psi}{\partial\mathbf{F}}$	$\mathbf{B}=-\rho\dfrac{\partial\psi}{\partial\mathbf{H}}$	$\boldsymbol{\tau}_{\mathrm{m}}+\frac{1}{2}\mu_0(\mathbf{H}\cdot\mathbf{H})\mathbf{I}$
$\chi(\mathbf{F},\mathbf{M})$	$\rho\mathbf{F}\dfrac{\partial\chi}{\partial\mathbf{F}}$	$\mathbf{H}=\mu_0^{-1}\rho\dfrac{\partial\chi}{\partial\mathbf{M}}$	$\boldsymbol{\tau}_{\mathrm{m}}-\mu_0(\mathbf{M}\cdot\mathbf{H})\mathbf{I}$
$\chi^*(\mathbf{F},\mathbf{M})$	$\rho\mathbf{F}\dfrac{\partial\chi^*}{\partial\mathbf{F}}$	$\mathbf{B}=\rho\dfrac{\partial\chi^*}{\partial\mathbf{M}}$	$\bar{\boldsymbol{\tau}}_{\mathrm{m}}-(\mathbf{M}\cdot\mathbf{B})\mathbf{I}$

Note that in the last two lines it is $\mathbf{M}$ rather than $\mathbf{M}_{\mathrm{r}}$ that is the independent magnetic variable

although the equilibrium equation has its simplest mathematical statement in terms of the total stress $\boldsymbol{\tau}$ and, as in the case of electroelasticity, it avoids the need to define either a Maxwell stress or a magnetic body force *within* the material. We shall return to this point shortly in deriving a formulation that allows for a potential of the desired kind.

As in the case of a polarizable electrostatic material, the notions of 'stress', 'Maxwell stress' and 'magnetic body force' *inside a magnetizable material* are not uniquely defined. On the other hand, outside a magnetizable material, the Maxwell stress *is* uniquely defined and given by (6.10), with $\mathbf{B}^\star=\mu_0\mathbf{H}^\star$. Moreover, the potentials ϕ^*, $\bar{\phi}$, ϕ, χ and χ^* vanish in vacuo (or are at most constants), as do the associated stresses $\boldsymbol{\sigma}$ and $\bar{\boldsymbol{\sigma}}$. However, ψ^* and ψ do not vanish nor do $\boldsymbol{\sigma}^*$ and $\bar{\boldsymbol{\sigma}}^*$. In fact, we have $\rho\psi^*=\frac{1}{2}\mathbf{B}^\star\cdot\mathbf{H}^\star$, which represents the magnetostatic energy density (per unit volume) outside the material (although the factor ρ has no meaning there and should be absorbed into the definition of ψ^*). The potential ψ is complementary (or dual) to ψ^* and satisfies (6.53), giving $\rho\psi=-\frac{1}{2}\mathbf{B}^\star\cdot\mathbf{H}^\star$, with the same proviso regarding ρ, while

$$\boldsymbol{\sigma}^*=\frac{1}{2}(\mathbf{B}^\star\cdot\mathbf{H}^\star)\mathbf{I}=-\bar{\boldsymbol{\sigma}}^*. \tag{6.55}$$

Note the distinction between * and *.

6.4 Lagrangian Formulations

6.4.1 Magnetostatic Equations and Boundary Conditions

The Lagrangian counterparts of $\mathbf{B}$ and $\mathbf{H}$ are derived as follows in a similar way to the Lagrangian electroelastic fields. We write the equations in (6.1) in the global forms

$$0 = \int_{\mathcal{B}} \operatorname{div} \mathbf{B} \, \mathrm{d}v = \int_{\partial\mathcal{B}} \mathbf{B} \cdot \mathbf{n} \, \mathrm{d}s, \tag{6.56}$$

for a volume $\mathcal{B}$ with boundary $\partial\mathcal{B}$, and

$$\int_{\mathcal{S}} \mathbf{J}_{\mathrm{f}} \cdot \mathbf{n} \mathrm{d}s = \int_{\mathcal{S}} (\operatorname{curl} \mathbf{H}) \cdot \mathbf{n} \, \mathrm{d}s = \int_{\partial\mathcal{S}} \mathbf{H} \cdot \mathrm{d}\mathbf{x}, \tag{6.57}$$

for an open surface $\mathcal{S}$ with closed bounding curve $\partial\mathcal{S}$.

Using Nanson's formula $\mathbf{n}\mathrm{d}s = J\mathbf{F}^{-\mathrm{T}}\mathbf{N}\mathrm{d}S$ for the transformation of area elements and the rule $\mathrm{d}\mathbf{x} = \mathbf{F}\mathrm{d}\mathbf{X}$ connecting line elements, the identities in (6.56) and (6.57) can be written in the referential forms

$$0 = \int_{\partial\mathcal{B}_{\mathrm{r}}} (J\mathbf{F}^{-1}\mathbf{B}) \cdot \mathbf{N} \, \mathrm{d}S = \int_{\mathcal{B}_{\mathrm{r}}} \operatorname{Div} (J\mathbf{F}^{-1}\mathbf{B}) \, \mathrm{d}V, \tag{6.58}$$

where the subscript r indicates the referential region and the divergence theorem has been used to obtain the latter integral, and

$$\int_{\mathcal{S}_{\mathrm{r}}} (J\mathbf{F}^{-1}\mathbf{J}_{\mathrm{f}}) \cdot \mathbf{N}\mathrm{d}S = \int_{\partial\mathcal{S}_{\mathrm{r}}} (\mathbf{F}^{\mathrm{T}}\mathbf{H}) \cdot \mathrm{d}\mathbf{X} = \int_{\mathcal{S}_{\mathrm{r}}} \operatorname{Curl} (\mathbf{F}^{\mathrm{T}}\mathbf{H}) \cdot \mathbf{N} \, \mathrm{d}S, \tag{6.59}$$

in which Stokes' theorem has been used to obtain the final integral.

By continuity we therefore obtain

$$\operatorname{Div} (J\mathbf{F}^{-1}\mathbf{B}) = 0, \quad \operatorname{Curl} (\mathbf{F}^{\mathrm{T}}\mathbf{H}) = J\mathbf{F}^{-1}\mathbf{J}_{\mathrm{f}}, \tag{6.60}$$

which suggest the introduction of the Lagrangian field variables

$$\mathbf{B}_{\mathrm{L}} = J\mathbf{F}^{-1}\mathbf{B}, \quad \mathbf{H}_{\mathrm{L}} = \mathbf{F}^{\mathrm{T}}\mathbf{H}, \tag{6.61}$$

analogous to the Lagrangian electric fields, and the Lagrangian free current density

$$\mathbf{J}_{\mathrm{F}} = J\mathbf{F}^{-1}\mathbf{J}_{\mathrm{f}}, \tag{6.62}$$

enabling the field equations to be written in the Lagrangian forms

$$\operatorname{Curl} \mathbf{H}_{\mathrm{L}} = \mathbf{J}_{\mathrm{F}}, \quad \operatorname{Div} \mathbf{B}_{\mathrm{L}} = 0. \tag{6.63}$$

These equations can also be obtained directly in differential form since, by (3.16), $J\,\mathrm{div}\,\mathbf{B} = \mathrm{Div}\,(J\mathbf{F}^{-1}\mathbf{B})$ and $J\mathbf{F}^{-1}\mathrm{curl}\,\mathbf{H} = \mathrm{Curl}\,(\mathbf{F}^{\mathrm{T}}\mathbf{H})$.

The Lagrangian forms of the boundary conditions are entirely analogous to their Eulerian counterparts in (2.167) and (2.172). Associated with (6.63) are the boundary conditions

$$\mathbf{N} \times [\![\mathbf{H}_{\mathrm{L}}]\!] = \mathbf{K}_{\mathrm{F}}, \quad \mathbf{N} \cdot [\![\mathbf{B}_{\mathrm{L}}]\!] = 0, \tag{6.64}$$

where $\mathbf{K}_{\mathrm{F}} = \mathbf{F}^{-1}\mathbf{K}_{\mathrm{f}}\mathrm{d}s/\mathrm{d}S$ is the Lagrangian free surface current, defined per unit reference area.

6.4.2 Equilibrium Equation and Traction Boundary Condition

In terms of the total Cauchy stress tensor $\boldsymbol{\tau}$ (6.32) may be converted to Lagrangian form by defining, as is done in Chap. 4, an associated *magnetic total nominal stress tensor*, denoted $\mathbf{T}$ and defined by

$$\mathbf{T} = J\mathbf{F}^{-1}\boldsymbol{\tau}. \tag{6.65}$$

Then, by using (3.16), (6.32) may be written in the alternative form

$$\mathrm{Div}\,\mathbf{T} + \rho_{\mathrm{r}}\mathbf{f} = \mathbf{0}, \tag{6.66}$$

where $\rho_{\mathrm{r}} = \rho J$ is the mass density of the material in the reference configuration $\mathcal{B}_{\mathrm{r}}$. This change to Lagrangian form is now coupled with a corresponding change in the representation of the potential functions, which leads to an elegant formulation of the constitutive law for a nonlinear magnetoelastic material with an accompanying simple structure of the governing equations. The traction boundary condition corresponding to (6.35) is

$$\mathbf{T}^{\mathrm{T}}\mathbf{N} = \mathbf{t}_{\mathrm{A}} + \mathbf{t}_{\mathrm{M}}^{\star}, \tag{6.67}$$

where $\mathbf{t}_{\mathrm{A}}$ is the applied mechanical load per unit reference area and $\mathbf{t}_{\mathrm{M}}^{\star}$ is the Maxwell traction per unit reference area, given by $\mathbf{t}_{\mathrm{M}}^{\star} = \mathbf{T}_{\mathrm{M}}^{\star\,\mathrm{T}}\mathbf{N}$ with $\mathbf{T}_{\mathrm{M}}^{\star} = J\mathbf{F}^{-1}\boldsymbol{\tau}_{\mathrm{m}}^{\star}$, similarly to the electroelastic case.

6.4.3 Constitutive Equations

The following development is based on the potential function $\psi^* = \psi^*(\mathbf{F}, \mathbf{B})$ introduced in (6.44). In view of the connection $(6.61)_1$ between $\mathbf{B}$ and $\mathbf{B}_{\mathrm{L}}$, we may regard $\psi^*(\mathbf{F}, \mathbf{B})$, equivalently, as a function of $\mathbf{F}$ and $\mathbf{B}_\mathrm{L}$, which we denote by Ψ^*. This is defined by

$$\Psi^*(\mathbf{F}, \mathbf{B}_\mathrm{L}) \equiv \psi^*(\mathbf{F}, J^{-1}\mathbf{F}\mathbf{B}_\mathrm{L}). \tag{6.68}$$

Since $\mathbf{B}_\mathrm{L}$ is a Lagrangian vector, it is indifferent to superimposed rotations $\mathbf{Q}$ in the deformed configuration, while the deformation gradient $\mathbf{F}$ changes to $\mathbf{QF}$. For Ψ^* to be objective, we must have

$$\Psi^*(\mathbf{QF}, \mathbf{B}_\mathrm{L}) = \Psi^*(\mathbf{F}, \mathbf{B}_\mathrm{L}) \tag{6.69}$$

for all proper orthogonal $\mathbf{Q}$. This requires that Ψ^* depends on $\mathbf{F}$ only through the right Cauchy–Green tensor $\mathbf{c} = \mathbf{F}^\mathrm{T}\mathbf{F}$. Thus, Ψ^* is a function of $\mathbf{c}$ and $\mathbf{B}_\mathrm{L}$.

From (6.45) we have

$$\boldsymbol{\sigma}^* = \rho\mathbf{F}\frac{\partial\psi^*}{\partial\mathbf{F}}, \quad \mathbf{H} = \rho\,\frac{\partial\psi^*}{\partial\mathbf{B}} \tag{6.70}$$

in the Eulerian formulation. When recast in terms of Ψ^* these equations become

$$\boldsymbol{\sigma}^* = \rho\mathbf{F}\frac{\partial\Psi^*}{\partial\mathbf{F}} - \mathbf{B}\otimes\mathbf{H} + (\mathbf{B}\cdot\mathbf{H})\mathbf{I}, \quad \mathbf{H} = \rho J\mathbf{F}^{-\mathrm{T}}\frac{\partial\Psi^*}{\partial\mathbf{B}_\mathrm{L}}, \tag{6.71}$$

and hence, by (6.31) and $(6.46)_{1,2}$, we obtain the simple formula

$$\boldsymbol{\tau} = \rho\mathbf{F}\frac{\partial\Psi^*}{\partial\mathbf{F}}. \tag{6.72}$$

By (6.65) and $(6.61)_2$ the corresponding Lagrangian expressions are

$$\mathbf{T} = \rho_\mathrm{r}\frac{\partial\Psi^*}{\partial\mathbf{F}}, \quad \mathbf{H}_\mathrm{L} = \rho_\mathrm{r}\frac{\partial\Psi^*}{\partial\mathbf{B}_\mathrm{L}}, \tag{6.73}$$

wherein we have used the connection $\rho_\mathrm{r} = \rho J$.

Similarly to the electroelastic case we now define the potential function $\Omega^* = \Omega^*(\mathbf{F}, \mathbf{B}_\mathrm{L})$, *per unit reference volume*, by

$$\Omega^*(\mathbf{F}, \mathbf{B}_\mathrm{L}) = \rho_\mathrm{r}\Psi^*(\mathbf{F}, \mathbf{B}_\mathrm{L}), \tag{6.74}$$

so that the formulas (6.73) become simply

$$\mathbf{T} = \frac{\partial\Omega^*}{\partial\mathbf{F}}, \quad \mathbf{H}_\mathrm{L} = \frac{\partial\Omega^*}{\partial\mathbf{B}_\mathrm{L}}. \tag{6.75}$$

We may also refer to Ω^* as a *magnetic total energy density function*. The corresponding formulas for $\boldsymbol{\tau}$ and $\mathbf{H}$ are

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial\Omega^*}{\partial\mathbf{F}}, \quad \mathbf{H} = \mathbf{F}^{-\mathrm{T}}\frac{\partial\Omega^*}{\partial\mathbf{B}_\mathrm{L}}. \tag{6.76}$$

When $\mathbf{B}_\mathrm{L}$ is used as the independent magnetic variable, $(6.75)_1$ and $(6.75)_2$ are inserted into (6.66) and $(6.63)_1$, while $\mathbf{B}_\mathrm{L}$ itself satisfies $(6.63)_2$. These coupled equations, when combined with appropriate boundary conditions, provide the equations governing the deformation $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$, with $\mathbf{F} = \mathrm{Grad}\,\mathbf{x}$, and a vector potential $\mathbf{A}_\mathrm{L}(\mathbf{X})$, with $\mathbf{B}_\mathrm{L} = \mathrm{Curl}\,\mathbf{A}_\mathrm{L}$.

If, instead of $\mathbf{B}_\mathrm{L}$, we use $\mathbf{H}_\mathrm{L}$ as the independent magnetic variable, then we can adopt the following approach. Let us now define, analogously to the definition (6.68), the potential function Ψ by

$$\Psi(\mathbf{F}, \mathbf{H}_\mathrm{L}) = \psi(\mathbf{F}, \mathbf{F}^{-\mathrm{T}}\mathbf{H}_\mathrm{L}). \tag{6.77}$$

Then, by using (6.53), the connections (6.61) and $\rho_\mathrm{r} = \rho J$, we obtain $\mathbf{B}\cdot\mathbf{H} = J^{-1}\mathbf{B}_\mathrm{L}\cdot\mathbf{H}_\mathrm{L}$, and hence

$$\rho_\mathrm{r}\Psi = \rho_\mathrm{r}\Psi^* - \mathbf{B}_\mathrm{L}\cdot\mathbf{H}_\mathrm{L}. \tag{6.78}$$

Introduction of the notation Ω, defined by

$$\Omega(\mathbf{F}, \mathbf{H}_\mathrm{L}) = \rho_\mathrm{r}\Psi(\mathbf{F}, \mathbf{H}_\mathrm{L}), \tag{6.79}$$

leads to the Legendre transformation

$$\Omega = \Omega^* - \mathbf{B}_\mathrm{L}\cdot\mathbf{H}_\mathrm{L}, \tag{6.80}$$

and in terms of Ω, we then obtain the counterparts of (6.75) as

$$\mathbf{T} = \frac{\partial\Omega}{\partial\mathbf{F}}, \quad \mathbf{B}_\mathrm{L} = -\frac{\partial\Omega}{\partial\mathbf{H}_\mathrm{L}}. \tag{6.81}$$

For the validity of the Legendre transform, we would require that $\mathbf{B}_\mathrm{L}$ be a monotonic function of $\mathbf{H}_\mathrm{L}$. However, one could avoid assuming this by starting with Ω instead of deriving it via (6.80). In this case $(6.63)_1$, with $\mathbf{J}_\mathrm{F} = \mathbf{0}$, is satisfied by taking the independent variable $\mathbf{H}_\mathrm{L}$ in the form $-\mathrm{Grad}\,\varphi_\mathrm{L}$ for some scalar function φ_L, and the remaining equations are then coupled as equations for $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ and $\varphi_\mathrm{L}(\mathbf{X})$.

On applying the 'virtual' energy balance (6.38) to the 'total' constitutive formulations, we obtain, first in Eulerian form

$$\int_\mathcal{D} \rho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}v + \int_{\partial\mathcal{D}} (\boldsymbol{\tau}\mathbf{n})\cdot\mathbf{u}\,\mathrm{d}s = \int_\mathcal{D} \mathrm{tr}\,(\boldsymbol{\tau}\mathbf{L})\,\mathrm{d}v, \tag{6.82}$$

and in Lagrangian form

$$\int_{\mathcal{D}_\mathrm{r}} (\rho_\mathrm{r}\mathbf{f}\cdot\mathbf{u} - \mathbf{B}_\mathrm{L}\cdot\dot{\mathbf{H}}_\mathrm{L})\,\mathrm{d}V + \int_{\partial\mathcal{D}_\mathrm{r}} (\mathbf{T}^\mathrm{T}\mathbf{N})\cdot\mathbf{u}\,\mathrm{d}S = \int_{\mathcal{D}_\mathrm{r}} \dot{\Omega}\,\mathrm{d}V, \tag{6.83}$$

and

$$\int_{\mathcal{D}_r} (\rho_r \mathbf{f} \cdot \mathbf{u} + \mathbf{H}_L \cdot \dot{\mathbf{B}}_L)\, dV + \int_{\partial \mathcal{D}_r} (\mathbf{T}^T \mathbf{N}) \cdot \mathbf{u}\, dS = \int_{\mathcal{D}_r} \dot{\Omega}^*\, dV. \tag{6.84}$$

These are the counterparts for magnetoelasticity of their electroelastic brothers in Sect. 4.4.3.

6.4.4 Incompressible Materials

The expressions for the various stress tensors in the foregoing apply for a material that is not subject to any internal mechanical constraint. For an important class of materials, including magneto-sensitive elastomers, it is appropriate to adopt the constraint of incompressibility, in which case the expressions for the stresses require modification.

For an incompressible material, we have the constraint

$$\det \mathbf{F} \equiv 1. \tag{6.85}$$

The total nominal and Cauchy stresses given by $(6.75)_1$ and $(6.76)_1$ in terms of Ω^* are then amended in the forms

$$\boldsymbol{\tau} = \mathbf{F}\frac{\partial \Omega^*}{\partial \mathbf{F}} - p^*\mathbf{I}, \quad \mathbf{T} = \frac{\partial \Omega^*}{\partial \mathbf{F}} - p^*\mathbf{F}^{-1}, \tag{6.86}$$

respectively, where p^* is a Lagrange multiplier associated with the constraint (6.85). The expressions $(6.75)_2$ and $(6.76)_2$ are unchanged except that (6.85) is in force. In terms of Ω we have, instead of (6.86),

$$\boldsymbol{\tau} = \mathbf{F}\frac{\partial \Omega}{\partial \mathbf{F}} - p\mathbf{I}, \quad \mathbf{T} = \frac{\partial \Omega}{\partial \mathbf{F}} - p\mathbf{F}^{-1}, \tag{6.87}$$

where, in general, the p in (6.87) is not the same as the p^* in (6.86).

6.4.5 Material Symmetry Considerations

Magneto-sensitive elastomers are typically isotropic in their response in the absence of a magnetic field, but application of a magnetic field endows the material with a preferred direction, as is the case with an electric field for electroelastic materials. For simplicity we restrict attention to so-called *isotropic magnetoelastic materials*, for which the material symmetry considerations are similar to those that arise for an isotropic electroelastic material.

The magnetoelastic material considered here is said to be *isotropic* if Ω^* is an isotropic function of the two tensors $\mathbf{c}$ and $\mathbf{B}_\mathrm{L} \otimes \mathbf{B}_\mathrm{L}$ (or if Ω is an isotropic function of $\mathbf{c}$ and $\mathbf{H}_\mathrm{L} \otimes \mathbf{H}_\mathrm{L}$). Then, the form of Ω^* is reduced to dependence on the principal invariants I_1, I_2, I_3 of $\mathbf{c}$, which we repeat here as

$$I_1 = \mathrm{tr}\,\mathbf{c}, \quad I_2 = \frac{1}{2}\left[(\mathrm{tr}\,\mathbf{c})^2 - \mathrm{tr}\,(\mathbf{c}^2)\right], \quad I_3 = \det\mathbf{c} = J^2, \tag{6.88}$$

together with three invariants that depend on $\mathbf{B}_\mathrm{L}$. We use the same notation K_4, K_5 and K_6 as in electroelasticity [see (4.86)], but now these are defined by

$$K_4 = \mathbf{B}_\mathrm{L} \cdot \mathbf{B}_\mathrm{L}, \quad K_5 = (\mathbf{c}\mathbf{B}_\mathrm{L}) \cdot \mathbf{B}_\mathrm{L}, \quad K_6 = (\mathbf{c}^2\mathbf{B}_\mathrm{L}) \cdot \mathbf{B}_\mathrm{L}. \tag{6.89}$$

In the following the subscripts $1, 2, \ldots, 6$ on Ω^* signify differentiation with respect to $I_1, I_2, I_3, K_4, K_5, K_6$, respectively. Then, we expand (6.76) to obtain

$$\begin{aligned}\boldsymbol{\tau} = J^{-1}[&2\Omega_1^*\mathbf{b} + 2\Omega_2^*(I_1\mathbf{b} - \mathbf{b}^2) + 2I_3\Omega_3^*\mathbf{I} + 2\Omega_5^*\mathbf{B} \otimes \mathbf{B} \\ &+ 2\Omega_6^*(\mathbf{B} \otimes \mathbf{bB} + \mathbf{bB} \otimes \mathbf{B})],\end{aligned} \tag{6.90}$$

which is clearly symmetric, and

$$\mathbf{H} = 2(\Omega_4^*\mathbf{b}^{-1}\mathbf{B} + \Omega_5^*\mathbf{B} + \Omega_6^*\mathbf{bB}), \tag{6.91}$$

where again $\mathbf{b} = \mathbf{F}\mathbf{F}^\mathrm{T}$ is the left Cauchy–Green deformation tensor. The corresponding Lagrangian forms may be obtained from the connections (6.65) and $(6.61)_2$.

For an incompressible material, $I_3 \equiv 1$ and (6.90) is replaced by

$$\begin{aligned}\boldsymbol{\tau} = &2\Omega_1^*\mathbf{b} + 2\Omega_2^*(I_1\mathbf{b} - \mathbf{b}^2) - p^*\mathbf{I} + 2\Omega_5^*\mathbf{B} \otimes \mathbf{B} \\ &+ 2\Omega_6^*(\mathbf{B} \otimes \mathbf{bB} + \mathbf{bB} \otimes \mathbf{B}),\end{aligned} \tag{6.92}$$

while (6.91) is unchanged in form, but with I_3 absent from Ω^*.

If we work with Ω instead of Ω^*, then the invariants based on $\mathbf{B}_\mathrm{L}$ are changed to invariants based on $\mathbf{H}_\mathrm{L}$. Again we use the same notation as for the invariants in electroelasticity, namely I_4, I_5, I_6, which are now defined by

$$I_4 = \mathbf{H}_\mathrm{L} \cdot \mathbf{H}_\mathrm{L}, \quad I_5 = (\mathbf{c}\mathbf{H}_\mathrm{L}) \cdot \mathbf{H}_\mathrm{L}, \quad I_6 = (\mathbf{c}^2\mathbf{H}_\mathrm{L}) \cdot \mathbf{H}_\mathrm{L}. \tag{6.93}$$

The associated formulas for $\boldsymbol{\tau}$ are similar to those based on Ω^*, and we give just that for an incompressible material, with $\Omega = \Omega(I_1, I_2, I_4, I_5, I_6)$, specifically

$$\begin{aligned}\boldsymbol{\tau} = &2\Omega_1\mathbf{b} + 2\Omega_2(I_1\mathbf{b} - \mathbf{b}^2) - p\mathbf{I} + 2\Omega_5\mathbf{bH} \otimes \mathbf{bH} \\ &+ 2\Omega_6(\mathbf{bH} \otimes \mathbf{b}^2\mathbf{H} + \mathbf{b}^2\mathbf{H} \otimes \mathbf{bH}).\end{aligned} \tag{6.94}$$

The magnetic induction has the form

$$\mathbf{B} = -2(\Omega_4 \mathbf{bH} + \Omega_5 \mathbf{b}^2\mathbf{H} + \Omega_6 \mathbf{b}^3\mathbf{H}). \tag{6.95}$$

In the above equations Ω_i is defined as $\partial\Omega/\partial I_i$ for $i = 1, 2, 4, 5, 6$.

In Chap. 7 we give solutions to a number of boundary-value problems based on the above formulations for problems involving either homogeneous or non-homogeneous deformations with no distributed current. When $\mathbf{B}_\mathrm{L}$ is used as the independent magnetic variable, it has to satisfy $\mathrm{Div}\,\mathbf{B}_\mathrm{L} = 0$ and it is then convenient to make use of a vector potential $\mathbf{A}_\mathrm{L}$, so that $\mathbf{B}_\mathrm{L} = \mathrm{Curl}\,\mathbf{A}_\mathrm{L}$. The resulting $\mathbf{H}_\mathrm{L}$, calculated from $(6.75)_2$, has to satisfy the vector equation $\mathrm{Curl}\,\mathbf{H}_\mathrm{L} = \mathbf{0}$. On the other hand, if we start with $\mathbf{H}_\mathrm{L}$ as the independent magnetic variable it has to satisfy $\mathrm{Curl}\,\mathbf{H}_\mathrm{L} = \mathbf{0}$, for which purpose we use a scalar potential function φ_L such that $\mathbf{H}_\mathrm{L} = -\mathrm{Grad}\,\varphi_\mathrm{L}$, and then the resulting $\mathbf{B}_\mathrm{L}$, calculated from $(6.81)_2$, must satisfy the scalar equation $\mathrm{Div}\,\mathbf{B}_\mathrm{L} = 0$.

A more general model than the isotropic model considered here has been developed by Bustamante (2010). This is a transversely isotropic model for which, in addition to the preferred direction due to the applied field, there is a second preferred direction, defined in the reference configuration, which is associated with alignment of magnetic particles during the curing process and is ‘frozen in’ to the material by the cure (see, e.g., Bellan and Bossis 2002; Varga et al. 2005, 2006). For details of this model, which involves a total of 10 invariants (9 for an incompressible material), we refer to Bustamante (2010).

6.5 Linear Magnetoelasticity and Piezomagnetism

We may linearize the constitutive equation on the same basis as for electroelasticity. Thus, for example, by linearizing (6.90) with respect to the infinitesimal strain tensor $\mathbf{e}$ and the quadratic product $\mathbf{B} \otimes \mathbf{B}$, we obtain

$$\boldsymbol{\tau} = 2\mu\mathbf{e} + \lambda e\mathbf{I} + \nu(\mathbf{B}\cdot\mathbf{B})\mathbf{I} + \xi\mathbf{B}\otimes\mathbf{B}, \tag{6.96}$$

where the constants λ, μ, ν and ξ are again given by (4.95)–(4.98), and (4.99) holds, but with Ω replaced by the Ω^* used here.

Similarly, by linearizing (6.91), we obtain

$$\mathbf{H} = \alpha\mathbf{B} + 2\nu e\mathbf{B} + 2(\alpha + \xi)\mathbf{eB}, \tag{6.97}$$

with α given by

$$\alpha = 2(\Omega_4^* + \Omega_5^* + \Omega_6^*), \tag{6.98}$$

as distinct from the α in (4.101).

The quadratic dependence of the stress on the magnetic induction means that reversal of the magnetic field does not change the stress (and hence not the strain). Thus, similarly to the effect of electrostriction in linear electroelasticity, a strain is induced by the application of a magnetic field which depends quadratically on the magnetic field. This is referred to as *magnetostriction*. This contrasts with the situation for a piezomagnetic material where the stress and strain are linear in the magnetic induction. For this case the constitutive law has to be changed in a similar way to that for piezoelectricity, although it should be mentioned that piezomagnetism is much less common than piezoelectricity. For a linear isotropically elastic piezomagnetic material, the constitutive equations can be written in the form

$$\boldsymbol{\tau} = 2\mu\mathbf{e} + \lambda e\mathbf{I} - \mathbf{B}\boldsymbol{\mathcal{E}}, \quad \tau_{ij} = 2\mu e_{ij} + \lambda e\delta_{ij} - B_k\mathcal{E}_{k|ij}, \tag{6.99}$$

and

$$\mathbf{H} = \alpha\mathbf{B} - \boldsymbol{\mathcal{E}}\mathbf{e}, \quad H_i = \alpha B_i - \mathcal{E}_{i|jk}e_{jk}, \tag{6.100}$$

similarly to (4.109) and (4.110), but now $\boldsymbol{\mathcal{E}}$ is a *piezomagnetic tensor* and α, given by (6.98), is the inverse of the magnetic permeability of the material.

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Chapter 7
Magnetoelastic Boundary-Value Problems

Abstract In this chapter we first summarize the constitutive equations of a nonlinear isotropic magnetoelastic material based on the Lagrangian magnetic field and magnetic induction along with the associated governing differential equations and boundary conditions. We then apply the theory to a number of representative boundary-value problems. We examine two problems for a slab of material involving homogeneous deformation and a uniform magnetic field normal to the faces of the slab, namely pure homogeneous strain and simple shear, using the constitutive law based on the magnetic induction as the independent magnetic variable. A simple prototype energy function is used to illustrate the effect of the magnetic field on the stress in the material for pure homogeneous strain while general theoretical results for the stress and magnetic field are obtained in the case of simple shear. We then consider two non-homogeneous deformations of a thick-walled circular cylindrical tube, with the magnetic field as the independent magnetic variable. These are the extension and inflation of a tube and the helical shear of a tube with either an axial or an azimuthal magnetic field. In the first of these problems we focus on determining the effect of the magnetic field on the pressure and axial load characteristics of the material response. For the helical shear problem, we highlight the restrictions on the constitutive law for which the considered deformation is admissible and then obtain explicit results for a specific form of energy function.

7.1 Preliminaries

In Chap. 6 we introduced alternative but equivalent formulations of the equilibrium equations for nonlinear magnetoelastic deformations of magneto-sensitive solids. The most elegant and simple formulation is based on the use of an energy density with either the Lagrangian magnetic field or magnetic induction vector as the independent magnetic variable. The differences arising from use of one or other of these formulations are quite significant. As indicated by Dorfmann and

L. Dorfmann and R.W. Ogden, *Nonlinear Theory of Electroelastic and Magnetoelastic Interactions*, DOI 10.1007/978-1-4614-9596-3_7,

Ogden (2004a), for certain problems involving non-homogeneous deformations, restrictions are placed on the class of constitutive laws admitting particular magnetic (or magnetic induction) fields. These restrictions are generally more severe when the magnetic induction is used as the independent magnetic variable rather than the magnetic field. The latter paper examined the helical shear problem of a circular cylindrical tube, which is discussed in Sect. 7.4.2, and the extension and torsion of a circular cylinder, which is not discussed herein.

In this chapter we provide solutions of some simple boundary-value problems in order to illustrate the influence of the magnetic field on the mechanical response. We begin by summarizing the basic equations in general form, together with the appropriate constitutive laws, but we confine attention to incompressible materials and also assume that there is no free volumetric or surface current (so $\mathbf{J}_\mathrm{f} = \mathbf{K}_\mathrm{f} = \mathbf{0}$). Equations based on either the magnetic induction or the magnetic field as the independent magnetic vector quantity are included. Problems involving homogeneous deformations are considered in Sect. 7.3. First, pure homogeneous deformation of a slab of material with a magnetic field normal to its faces is discussed, and this is followed by an analysis of the problem of simple shear of a slab with a magnetic field initially normal to the direction of shear and in the plane of shear.

In Sect. 7.4 we consider problems related to a thick-walled tube with circular cylindrical geometry preserved during deformation and with the applied magnetic field in either the axial or azimuthal direction. To avoid difficulties associated with compatibility of the magnetic boundary conditions on the ends of the tube and on its lateral surfaces, the tube is taken to be infinitely long.

7.2 Governing Equations

Before we summarize the relevant equations necessary to describe the magneto-mechanical behaviour of magneto-sensitive solids, we recall the connections

$$\mathbf{H} = \mathbf{F}^{-\mathrm{T}}\mathbf{H}_\mathrm{L}, \quad \mathbf{B} = \mathbf{F}\mathbf{B}_\mathrm{L} \tag{7.1}$$

between the Eulerian and Lagrangian field variables for an incompressible material. We use the Lagrangian variables in the formulation of constitutive equations, but for the simple problems considered in this chapter, we use the Eulerian forms of the governing equations and boundary conditions for their solution. For more general problems use of the Lagrangian forms may be more suitable.

If the development is based on the energy function $\Omega^*(\mathbf{F}, \mathbf{B}_\mathrm{L})$ then, bearing in mind that we are considering incompressible materials, the total Cauchy stress and the Eulerian magnetic field are given by

$$\boldsymbol{\tau} = \mathbf{F}\frac{\partial \Omega^*}{\partial \mathbf{F}} - p^*\mathbf{I}, \quad \mathbf{H} = \mathbf{F}^{-\mathrm{T}}\frac{\partial \Omega^*}{\partial \mathbf{B}_\mathrm{L}}. \tag{7.2}$$

These have to satisfy the governing equations

$$\operatorname{div} \boldsymbol{\tau} = \mathbf{0}, \quad \operatorname{curl} \mathbf{H} = \mathbf{0} \quad \text{in } \mathcal{B}, \tag{7.3}$$

which are to be solved for $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ and the vector potential $\mathbf{A}_{\mathrm{L}}(\mathbf{X})$, where $\mathbf{B}_{\mathrm{L}} = \operatorname{Curl} \mathbf{A}_{\mathrm{L}}$, which ensures that $\operatorname{Div} \mathbf{B}_{\mathrm{L}} = 0$.

For an isotropic material, the constitutive equations expand in the forms

$$\begin{aligned} \boldsymbol{\tau} = {} & 2\Omega_1^* \mathbf{b} + 2\Omega_2^* \left(I_1 \mathbf{b} - \mathbf{b}^2 \right) - p^* \mathbf{I} + 2\Omega_5^* \mathbf{B} \otimes \mathbf{B} \\ & + 2\Omega_6^* \left(\mathbf{B} \otimes \mathbf{b}\mathbf{B} + \mathbf{b}\mathbf{B} \otimes \mathbf{B} \right), \end{aligned} \tag{7.4}$$

and

$$\mathbf{H} = 2 \left(\Omega_4^* \mathbf{b}^{-1} \mathbf{B} + \Omega_5^* \mathbf{B} + \Omega_6^* \mathbf{b}\mathbf{B} \right) \tag{7.5}$$

in the notation defined in Chap. 6.

When the magnetic field $\mathbf{H}_{\mathrm{L}}$ replaces $\mathbf{B}_{\mathrm{L}}$ as the independent field variable, the constitutive relations (7.2) are replaced by

$$\boldsymbol{\tau} = \mathbf{F} \frac{\partial \Omega}{\partial \mathbf{F}} - p\mathbf{I}, \quad \mathbf{B} = -\mathbf{F} \frac{\partial \Omega}{\partial \mathbf{H}_{\mathrm{L}}}, \tag{7.6}$$

where the total energy density function is $\Omega\left(\mathbf{F}, \mathbf{H}_{\mathrm{L}}\right)$. If we work with Ω instead of Ω^*, then the relevant governing equations are

$$\operatorname{div} \boldsymbol{\tau} = \mathbf{0}, \quad \operatorname{div} \mathbf{B} = 0 \quad \text{in } \mathcal{B}, \tag{7.7}$$

which are to be solved for $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ and the scalar potential $\varphi_{\mathrm{L}}(\mathbf{X})$, where $\mathbf{H}_{\mathrm{L}} = -\operatorname{Grad} \varphi_{\mathrm{L}}$, which satisfies $\operatorname{Curl} \mathbf{H}_{\mathrm{L}} = \mathbf{0}$.

For an isotropic material,

$$\begin{aligned} \boldsymbol{\tau} = {} & 2\Omega_1 \mathbf{b} + 2\Omega_2 \left(I_1 \mathbf{b} - \mathbf{b}^2 \right) - p\mathbf{I} + 2\Omega_5 \mathbf{b}\mathbf{H} \otimes \mathbf{b}\mathbf{H} \\ & + 2\Omega_6 \left(\mathbf{b}\mathbf{H} \otimes \mathbf{b}^2\mathbf{H} + \mathbf{b}^2\mathbf{H} \otimes \mathbf{b}\mathbf{H} \right), \end{aligned} \tag{7.8}$$

and

$$\mathbf{B} = -2 \left(\Omega_4 \mathbf{b}\mathbf{H} + \Omega_5 \mathbf{b}^2 \mathbf{H} + \Omega_6 \mathbf{b}^3 \mathbf{H} \right). \tag{7.9}$$

The latter can be rearranged, if required, by using the Cayley–Hamilton theorem in the form

$$\mathbf{b}^3 = I_1 \mathbf{b}^2 - I_2 \mathbf{b} + \mathbf{I}, \tag{7.10}$$

which is appropriate for incompressible materials.

The boundary conditions associated with the magnetic field variables are collected here from (6.3), but with $\mathbf{K}_\mathrm{f} = \mathbf{0}$, as

$$\mathbf{n} \times (\mathbf{H} - \mathbf{H}^\star) = \mathbf{0}, \quad \mathbf{n} \cdot (\mathbf{B} - \mathbf{B}^\star) = 0 \quad \text{on } \partial\mathcal{B}, \tag{7.11}$$

where $\mathbf{H}^\star$ and $\mathbf{B}^\star = \mu_0\mathbf{H}^\star$ denote the magnetic field and magnetic induction outside the material, but evaluated on the surface $\partial\mathcal{B}$.

On any part of the boundary, say $\partial\mathcal{B}^\mathbf{t} \subset \partial\mathcal{B}$, where the mechanical traction $\mathbf{t}_\mathrm{a}$ is prescribed, the traction boundary condition is given by

$$\boldsymbol{\tau}\mathbf{n} = \mathbf{t}_\mathrm{a} + \mathbf{t}_\mathrm{m}^\star \quad \text{on } \partial\mathcal{B}^\mathbf{t}, \tag{7.12}$$

where $\mathbf{t}_\mathrm{m}^\star = \boldsymbol{\tau}_\mathrm{m}^\star\mathbf{n}$ and is the traction due to the Maxwell stress $\boldsymbol{\tau}_\mathrm{m}^\star$ calculated on the exterior of $\partial\mathcal{B}^\mathbf{t}$, and we recall that the Maxwell stress outside the material is given by

$$\boldsymbol{\tau}_\mathrm{m}^\star = \mathbf{B}^\star \otimes \mathbf{H}^\star - \frac{1}{2}(\mathbf{B}^\star \cdot \mathbf{H}^\star)\,\mathbf{I}, \tag{7.13}$$

with $\mathbf{B}^\star = \mu_0\mathbf{H}^\star$, which is symmetric and satisfies the equation $\operatorname{div}\boldsymbol{\tau}_\mathrm{m}^\star = \mathbf{0}$.

We now apply the theory summarized in the previous subsection in the solution of representative boundary-value problems and quantify the effect of an applied magnetic field on the coupled magneto-mechanical response of a magneto-sensitive solid. We include the boundary traction generated by the Maxwell stress calculated on the exterior of the body. We begin by considering two problems involving homogeneous deformation.

7.3 Homogeneous Deformations

7.3.1 Pure Homogeneous Strain

Consider a slab made of an incompressible isotropic magnetoelastic material of uniform thickness which, in its reference configuration $\mathcal{B}_\mathrm{r}$, has top and bottom faces normal to the X_2 direction and is of infinite extent in the X_1 and X_3 directions, where (X_1, X_2, X_3) are reference rectangular Cartesian coordinates. Outside the slab there is a vacuum in which a uniform magnetic field is applied in the X_2 direction with component $H_2^\star$, so that the corresponding magnetic induction is $B_2^\star = \mu_0 H_2^\star$. There may also be an applied mechanical traction on the faces of the slab in the normal direction. As a result the slab suffers a pure homogeneous strain, which we specify in terms of Cartesian coordinates by

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3, \tag{7.14}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the (constant) principal stretches satisfying the incompressibility condition

$$\lambda_1\lambda_2\lambda_3 = 1. \tag{7.15}$$

We write the matrix representation of the deformation gradient in the diagonal form $\mathrm{diag}[\lambda_1, \lambda_2, \lambda_1^{-1}\lambda_2^{-1}]$. If there is no applied mechanical traction, then the deformation resulting from the magnetic field alone is known as *magnetostriction*; for a discussion of which for magneto-sensitive elastomers, we refer to Guan et al. (2008).

For this specialization the Lagrangian magnetic induction, given by $(7.1)_2$ in the general case, has just a single component $B_{L2} = \lambda_2^{-1}B_2$, where, by the continuity condition $(7.11)_2$, the magnetic induction in the slab is $B_2 = B_2^\star$. The associated invariants are calculated in terms of the two independent principal stretches λ_1 and λ_2 and B_{L2} as

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2}, \quad I_2 = \lambda_1^{-2} + \lambda_2^{-2} + \lambda_1^2\lambda_2^2, \tag{7.16}$$

$$K_4 = B_{L2}^2, \quad K_5 = \lambda_2^2 K_4, \quad K_6 = \lambda_2^4 K_4, \tag{7.17}$$

which involve three independent quantities in all, namely λ_1 and λ_2 and K_4. Therefore, the general energy function Ω^* can be replaced by a modified form as a function of $\lambda_1, \lambda_2, K_4$, and we write

$$\omega^*(\lambda_1, \lambda_2, K_4) = \Omega^*(I_1, I_2, K_4, K_5, K_6), \tag{7.18}$$

with the invariants on the right-hand side given by (7.16) and (7.17).

Then, on specializing (7.4), we obtain the simple expressions

$$\tau_{11} - \tau_{33} = \lambda_1\omega_1^*, \quad \tau_{22} - \tau_{33} = \lambda_2\omega_2^* \tag{7.19}$$

for the stress differences, where the subscripts 1 and 2 on ω^* indicate differentiation with respect to λ_1 and λ_2, respectively.

The magnetic field vector $\mathbf{H}$ within the material is given by (7.5), which yields just the single component

$$H_2 = 2\lambda_2^{-2}\omega_4^* B_2, \tag{7.20}$$

where the subscript 4 on ω^* indicates differentiation with respect to K_4. Because the deformation and field are uniform all the governing differential equations are satisfied trivially.

Outside the slab the Maxwell stress components are calculated from (7.13) with the help of $(7.17)_1$ as

$$\tau_{m22}^\star = \frac{1}{2}\mu_0^{-1}\lambda_2^2 K_4, \quad \tau_{m11}^\star = \tau_{m33}^\star = -\frac{1}{2}\mu_0^{-1}\lambda_2^2 K_4. \tag{7.21}$$

It is of interest to consider the symmetric situation in which $\lambda_3 = \lambda_1 = \lambda_2^{-1/2}$ and $\omega_1^*(\lambda_2^{-1/2}, \lambda_2, K_4) = 0$. The mechanical traction t_{a2} supplied to the plane faces of the slab must balance the difference between the normal stress τ_{22} and the Maxwell stress $\tau^\star_{m22}$, and in this symmetric case we therefore have, by specializing the general traction boundary condition (6.35),

$$t_{a2} = \lambda_2\omega_2^* + \tau_{11} - \tau^\star_{m22}. \tag{7.22}$$

Without loss of generality, since the material is incompressible, we may adjust the value of the hydrostatic stress p^* so that τ_{11} has any desired value. In particular, it may be taken to be zero or to have the value $\tau^\star_{m11} = -\tau^\star_{m22}$ to match the lateral Maxwell stress. In these two cases the above equation becomes

$$t_{a2} = \lambda_2\omega_2^* - \frac{1}{2}\alpha\mu_0^{-1}\lambda_2^2 K_4, \tag{7.23}$$

where $\alpha = 1$ or 2. When evaluated in the undeformed configuration this reduces to

$$t_{a2} = \omega_2^*(1, 1, K_4) - \frac{1}{2}\alpha\mu_0^{-1}K_4, \tag{7.24}$$

which gives the value of the applied mechanical load needed to prevent the material from deforming in the presence of the magnetic field.

If $t_{a2} > (<)\, 0$, then the material will want to contract (expand) in the X_2 direction when the magnetic field is applied.

Suppose next that no mechanical load is applied. Then,

$$\lambda_2\omega_2^*(\lambda_2^{-1/2}, \lambda_2, K_4) = \frac{1}{2}\alpha\mu_0^{-1}\lambda_2^2 K_4, \tag{7.25}$$

which is an equation determining λ_2 (in general implicitly) in terms of K_4 for any given form of ω^*.

7.3.2 Illustration

For purposes of illustration of the above results, we now choose a specific form of ω^*, namely

$$\omega^*(\lambda_1, \lambda_2, K_4) = \frac{1}{2}\mu(K_4)(\lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2} - 3) + \nu(K_4) + \frac{1}{2}\beta\mu_0^{-1}K_5, \tag{7.26}$$

with $K_5 = \lambda_2^2 K_4$, where μ and ν are functions of K_4 and β is a dimensionless parameter. This is a slight modification of a model used by Dorfmann and Ogden

(2005). In the absence of a magnetic field $K_4 = 0$, and then $\mu(0)$ is just the shear modulus of the material in the reference configuration and (7.26) is the classical neo-Hookean strain-energy function provided, for compatibility, we set $\nu(0) = 0$.

From (7.26) we then obtain

$$\lambda_2 \omega_2^* = \mu(K_4)(\lambda_2^2 - \lambda_2^{-1}) + \beta \mu_0^{-1} \lambda_2^2 K_4 \tag{7.27}$$

for $\lambda_3 = \lambda_1$. Equation (7.24) then gives

$$t_{a2} = \mu_0^{-1}(\beta - \frac{1}{2}\alpha) K_4. \tag{7.28}$$

Thus, for $\beta > \alpha/2$, $t_{a2} > 0$ and the material tends to contract (at least initially) when a magnetic field is applied, whereas if $\beta < \alpha/2$, the reverse is true.

Equation (7.25) becomes

$$\mu(K_4)(\lambda_2^2 - \lambda_2^{-1}) + \mu_0^{-1}(\beta - \frac{1}{2}\alpha)\lambda_2^2 K_4 = 0, \tag{7.29}$$

so that, assuming the shear modulus is positive, $\beta > \alpha/2$ ($< \alpha/2$) is consistent with contraction $\lambda_2 < 1$ (expansion $\lambda_2 > 1$).

For definiteness we choose $\mu(K_4) = \mu_1 + \epsilon \mu_0^{-1} K_4$, where $\mu_1 = \mu(0)$ and ϵ is a dimensionless material parameter. Then, in dimensionless form, (7.27) becomes

$$\bar{\tau}_{22} \equiv \tau_{22}/\mu_1 = (1 + \epsilon m)(\lambda_2^2 - \lambda_2^{-1}) + m\beta\lambda_2^2, \tag{7.30}$$

where m is another dimensionless parameter, defined by $m = \mu_0^{-1} K_4/\mu_1$ and reflecting the magnitude of the initial magnetic (induction) field. Note that this expression is not affected by the function ν.

In Fig. 7.1 we plot the dimensionless applied stress $\bar{t}_{a2} \equiv t_{a2}/\mu_1$ using (7.23) appropriately specialized against λ_2 for different values of the parameter m with ϵ taken to be 0.2, and with $\alpha = 1$ and two different values of β. Clearly, the stiffness of the material increases with the value of the magnetic field strength. We emphasize that this result is for a very simple prototype example of constitutive law, but it does nevertheless reflect the limited available data for MS-elastomers (for references and discussion, see Dorfmann and Ogden 2004a,b). There is some limited evidence suggested by Jolly et al. (1996) that the stiffness of the material reaches a maximum as the magnetic field strength increases, but, in using a linear form for $\mu(K_4)$, we have not attempted to model this effect although it could easily be accommodated if more comprehensive data indicate that this is indeed an important characteristic of the considered materials.

The corresponding component of the magnetization is obtained by using (6.2) to give

$$M_2 = (\mu_0^{-1} - 2\lambda_2^{-2}\omega_4^*) B_2. \tag{7.31}$$

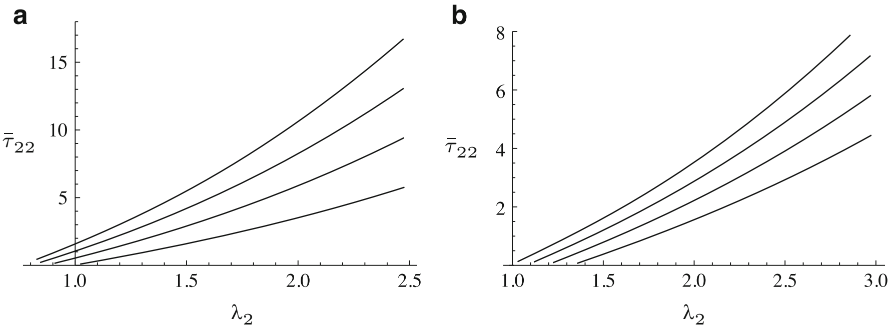

Fig. 7.1 Plot of the dimensionless applied stress $\bar{t}_{a2}$ against stretch λ_2 for $\epsilon = 0.2$ and the following values of the parameter m: $0, 0.5, 1, 1.5$ (reading from *bottom to top*). (**a**) $\beta = 2$; (**b**) $\beta = 0.5$

When evaluated in the undeformed configuration for the model (7.26) this gives

$$M_2 = [\mu_0^{-1}(1-\beta) - 2\nu'(K_4)]B_2, \tag{7.32}$$

which allows $\nu'(K_4)$ to be interpreted as a measure of magnetization.

Thus, we can interpret the function $\nu'(K_4)$ as characterizing the magnetization in the absence of deformation. For further discussion of magnetization for this model with $\beta = 1/2$, we refer to Dorfmann and Ogden (2005).

7.3.3 Simple Shear

Consider the slab of material discussed in Sect. 7.3.1 but now, instead of pure homogeneous deformation, subject to a simple shear deformation in the X_1 direction in the (X_1, X_2) plane with amount of shear γ. For the magnetoelastic case, this deformation was discussed in Dorfmann and Ogden (2005) and Ogden and Dorfmann (2005). As in Sect. 5.2.2 the matrix of Cartesian components F of the deformation gradient tensor is

$$\mathsf{F} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{7.33}$$

The corresponding matrices b and c of the left and right Cauchy–Green deformation tensors are

$$\mathsf{b} = \begin{bmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathsf{c} = \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{7.34}$$

while the components of $\mathbf{b}^2$ and $\mathbf{b}^{-1}$, required in (7.4) and (7.5), are given by the matrices

$$\mathsf{b}^2 = \begin{bmatrix} 1+3\gamma^2+\gamma^4 & \gamma(2+\gamma^2) & 0 \\ \gamma(2+\gamma^2) & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathsf{b}^{-1} = \begin{bmatrix} 1 & -\gamma & 0 \\ -\gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{7.35}$$

The principal invariants I_1, I_2 are

$$I_1 = I_2 = 3 + \gamma^2. \tag{7.36}$$

For this problem we again take $\mathbf{B}$ to be in the X_2 direction with component $B_2 = B_2^\star = \mu_0 H_2^\star$ and $H_1^\star = H_3^\star = 0$. The corresponding magnetic field $\mathbf{H}$ similarly has a single component H_2. Then, with $\mathbf{B}_\mathrm{L} = \mathbf{F}^{-1}\mathbf{B}$ we obtain

$$B_{\mathrm{L}1} = -\gamma B_2, \quad B_{\mathrm{L}2} = B_2, \quad B_{\mathrm{L}3} = 0, \tag{7.37}$$

and the invariants K_4, K_5 and K_6 are calculated from (6.89) as

$$K_4 = (1+\gamma^2)K_5, \quad K_5 = K_6 = B_{\mathrm{L}2}^2. \tag{7.38}$$

The components of $\mathbf{bB}$ required in the expressions (7.4) for $\boldsymbol{\tau}$ and (7.5) for $\mathbf{H}$ are given by

$$\mathsf{bB} = [\gamma, 1, 0]^\mathrm{T}\, B_2, \tag{7.39}$$

and $\mathbf{b}^{-1}\mathbf{B}$, which is needed to determine $\mathbf{H}$, has the component form

$$\mathsf{b}^{-1}\mathsf{B} = \left[-\gamma, (1+\gamma^2), 0\right]^\mathrm{T} B_2. \tag{7.40}$$

The resulting components of $\boldsymbol{\tau}$ and $\mathbf{H}$ are

$$\begin{aligned} \tau_{11} &= -p^* + 2\Omega_1^*(1+\gamma^2) + 2\Omega_2^*(2+\gamma^2), \\ \tau_{22} &= -p^* + 2\Omega_1^* + 4\Omega_2^* + 2(\Omega_5^* + 2\Omega_6^*)K_5, \qquad (7.41) \\ \tau_{33} &= -p^* + 2\Omega_1^* + 2\Omega_2^*(2+\gamma^2), \\ \tau_{12} &= 2\gamma\left(\Omega_1^* + \Omega_2^* + \Omega_6^* K_5\right), \qquad (7.42) \end{aligned}$$

with $\tau_{13} = \tau_{23} = 0$, and

$$H_1 = 2\gamma(\Omega_6^* - \Omega_4^*)B_2, \tag{7.43}$$

$$H_2 = 2\left[(1+\gamma^2)\Omega_4^* + \Omega_5^* + \Omega_6^*\right] B_2, \tag{7.44}$$

with $H_3 = 0$.

Since the component of $\mathbf{H}$ tangential to the boundary is continuous and $H_1^\star = 0$ we deduce from (7.43) that for the simple shear deformation to be maintained in the presence of the given magnetic field, the material must be subject to the restriction

$$\Omega_4^* = \Omega_6^*. \tag{7.45}$$

In view of (7.36) and (7.38) there remain two independent variables, namely γ and K_5. It is again convenient to use a reduced form of the energy function Ω^*, as a function of these two variables only. We define the appropriate specialization, denoted ω^*, by

$$\omega^*(\gamma, K_5) = \Omega^*(I_1, I_2, K_4, K_5, K_6), \tag{7.46}$$

with (7.36) and (7.38).

It follows that

$$\omega_\gamma^* = 2\gamma\left(\Omega_1^* + \Omega_2^* + K_5\Omega_4^*\right), \tag{7.47}$$

and hence, in view of (7.45), we obtain the very simple formula

$$\tau_{12} = \omega_\gamma^* \tag{7.48}$$

for the shear stress. Similarly, we obtain

$$\omega_5^* = (1+\gamma^2)\Omega_4^* + \Omega_5^* + \Omega_6^*, \tag{7.49}$$

where the subscripts γ and 5 on ω^* indicate partial differentiation with respect to γ and K_5, and hence the expression for H_2 can be simplified as well as

$$H_2 = 2\omega_5^* B_2. \tag{7.50}$$

Outside the material the relevant in-plane components of the Maxwell stress are calculated as

$$\tau_{\mathrm{m}22}^\star = \frac{1}{2}\mu_0^{-1}K_5, \quad \tau_{\mathrm{m}11}^\star = -\frac{1}{2}\mu_0^{-1}K_5, \quad \tau_{\mathrm{m}12}^\star = 0. \tag{7.51}$$

Since there is no shear component of the Maxwell stress a shear stress has to be applied to balance the shear stress τ_{12} within the material. The normal component of the applied stress is given by the difference between the components τ_{22} and $\tau^{\star}_{\mathrm{m}22}$, which we write as

$$\tau_{22} - \tau^{\star}_{\mathrm{m}22} = -2(\Omega_1^* + \Omega_2^*)\gamma^2 + 2(\Omega_5^* + 2\Omega_6^*)K_5 - \frac{1}{2}\mu_0^{-1}K_5 + \tau_{11}. \quad (7.52)$$

7.4 Application to Circular Cylindrical Geometry

We now specialize the field equations to problems for which the cylindrical configuration is maintained during deformation. For these problems it is convenient to work in terms of cylindrical polar coordinates, which in the deformed configuration are defined by unit vectors pointing in the (r, θ, z) directions. From Appendix A.2.2 we obtain the component form of $(7.7)_2$, i.e.

$$\frac{\partial B_r}{\partial r} + \frac{B_r}{r} + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0, \quad (7.53)$$

and of $(7.3)_2$

$$\frac{1}{r}\frac{\partial H_z}{\partial \theta} - \frac{\partial H_\theta}{\partial z} = 0, \quad \frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = 0, \quad \frac{1}{r}\frac{\partial (rH_\theta)}{\partial r} - \frac{1}{r}\frac{\partial H_r}{\partial \theta} = 0, \quad (7.54)$$

where we recall that the free current density $\mathbf{J}_\mathrm{f}$ has been set to $\mathbf{0}$.

We have assumed that the components of the magnetic field and magnetic induction are independent of the coordinates θ and z, in which case the above equations are solved to give

$$rB_r = \text{constant}, \quad H_z = \text{constant}, \quad rH_\theta = \text{constant}, \quad (7.55)$$

but no restrictions are imposed on the components B_θ, B_z and H_r (as functions of r). Recall that $(7.11)_2$ requires that the normal component of $\mathbf{B}$ be continuous across a boundary. Since $(7.55)_1$ shows that B_r is proportional to $1/r$, then to avoid a singularity at $r = 0$ we must set $B_r = 0$. Under the same cylindrically symmetric condition the components of $(7.7)_1$ simplify to

$$r\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} = \tau_{\theta\theta} - \tau_{rr}, \quad \frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\tau_{r\theta}\right) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}r}(r\tau_{rz}) = 0, \quad (7.56)$$

which are identical to those in the electroelastic case in Chap. 5.

We now consider problems with circular cylindrical geometry in which the reference configuration is described in terms of cylindrical polar coordinates

(R, Θ, Z) and the deformed configuration by (r, θ, z). Note that if the vector $\mathbf{B}_\mathrm{L}$ is used as the independent variable, the magnetic field $\mathbf{H}$ is given by the constitutive law (7.5), and its components must satisfy (7.54). On the other hand, if $\mathbf{H}_\mathrm{L}$ is chosen as the independent variable, the magnetic induction $\mathbf{B}$ is given by (7.9) and its components must satisfy (7.53). It is somewhat easier to work with $\mathbf{H}_\mathrm{L}$ as the independent variable since this leaves only a single scalar equation to be satisfied, namely (7.53).

7.4.1 Extension and Inflation of a Tube

We now consider an infinitely long tube with reference geometry given by the inequalities

$$A \le R \le B, \quad 0 \le \Theta \le 2\pi, \quad -\infty < Z < \infty, \tag{7.57}$$

where A and B denote the inner and outer radii, respectively. The tube is deformed by combining axial extension and radial expansion according to the equations

$$r = \left[a^2 + \lambda_z^{-1}(R^2 - A^2)\right]^{1/2}, \quad \theta = \Theta, \quad z = \lambda_z Z, \tag{7.58}$$

where λ_z is the (uniform) axial stretch, a is the internal radius of the inflated tube, and

$$b = \left[a^2 + \lambda_z^{-1}(B^2 - A^2)\right]^{1/2} \tag{7.59}$$

is the corresponding external radius. The deformation gradient is diagonal with respect to the cylindrical coordinate axes with principal stretches in the radial, azimuthal and axial directions given by

$$\lambda_1 = \lambda^{-1}\lambda_z^{-1}, \quad \lambda_2 = \lambda = \frac{r}{R}, \quad \lambda_3 = \lambda_z, \tag{7.60}$$

wherein the incompressibility constraint $\lambda_1\lambda_2\lambda_3 \equiv 1$ has been used to give λ_1 in terms of the independent stretches λ and λ_z and the notation λ has been defined.

7.4.1.1 Axial Magnetic Field

In this case H_z is constant, and since, in the absence of surface currents, the tangential component of $\mathbf{H}$ must be continuous across an interface, it follows that H_z is continuous across the surfaces $r = a$ and $r = b$ and is therefore spatially uniform in the present problem. Outside the material (assumed to be vacuum), the magnetic induction is given by $\mu_0 H_z$, and there the components of the Maxwell stress are given by

$$\tau^\star_{\mathrm{m}rr} = \tau^\star_{\mathrm{m}\theta\theta} = -\frac{1}{2}\mu_0 H_z^2 = -\tau^\star_{\mathrm{m}zz}. \tag{7.61}$$

In particular, $\tau^\star_{\mathrm{m}rr}$ contributes to the (radial) traction on the boundaries $r = a$ and $r = b$, but has the same value on each boundary.

Suppose that we adopt $\mathbf{H}_\mathrm{L}$ as the independent magnetic variable within the material. Then, from (7.9) it follows that the only non-zero component of $\mathbf{B}$ is B_z, which is not in general constant and is given by

$$B_z = -2(\Omega_4 + \Omega_5\lambda_z^2 + \Omega_6\lambda_z^4)\lambda_z^2 H_z. \tag{7.62}$$

For this problem the Lagrangian component $H_{\mathrm{L}Z}$ of the magnetic field is related to H_z by $H_{\mathrm{L}Z} = \lambda_z H_z$, and the invariants I_4, I_5, I_6, from their definitions in (6.93), are simply

$$I_4 = H_{\mathrm{L}Z}^2, \quad I_5 = \lambda_z^2 I_4, \quad I_6 = \lambda_z^4 I_4. \tag{7.63}$$

The relevant set of invariants is completed with

$$I_1 = \lambda^2 + \lambda_z^2 + \lambda^{-2}\lambda_z^{-2}, \quad I_2 = \lambda^{-2} + \lambda_z^{-2} + \lambda^2\lambda_z^2. \tag{7.64}$$

There are now just three independent quantities for this problem, namely λ, λ_z and I_4, so it is convenient to introduce a reduced form of the energy function Ω, defined by

$$\omega(\lambda, \lambda_z, I_4) = \Omega(I_1, I_2, I_4, I_5, I_6) \tag{7.65}$$

with the invariants on the right-hand side given above. It then follows from (7.62) that

$$B_z = -2\omega_4\lambda_z^2 H_z, \tag{7.66}$$

while from (7.8) the normal (and hence principal for the considered geometry) stress differences are calculated as

$$\tau_{\theta\theta} - \tau_{rr} = \lambda\omega_\lambda, \quad \tau_{zz} - \tau_{rr} = \lambda_z\omega_{\lambda_z}, \tag{7.67}$$

in which the subscripts 4, λ and λ_z signify differentiation with respect to I_4, λ and λ_z, respectively.

The equations of equilibrium (7.56) reduce to the single equation

$$r\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} = \tau_{\theta\theta} - \tau_{rr}. \tag{7.68}$$

Suppose that inflation of the tube is achieved by application of an internal pressure P with no mechanical load applied on the exterior boundary. Then appropriate boundary conditions are

$$\tau_{rr} = -P - \frac{1}{2}\mu_0 H_z^2 \quad \text{on} \quad r = a, \quad \tau_{rr} = -\frac{1}{2}\mu_0 H_z^2 \quad \text{on} \quad r = b. \tag{7.69}$$

Integration of (7.68), use of the relation $(7.67)_1$ and application of the boundary conditions (7.69) leads to

$$P = \int_a^b \lambda\omega_\lambda \frac{\mathrm{d}r}{r}, \tag{7.70}$$

and we note, in particular, that this does not involve the exterior Maxwell stress. The corresponding expression for the resultant axial load N on a cross section of the cylinder is calculated from

$$N = 2\pi \int_a^b \tau_{zz} r\,\mathrm{d}r, \tag{7.71}$$

using (7.67), (7.68) and (7.70), as

$$N = \pi \int_a^b (2\lambda_z\omega_{\lambda_z} - \lambda\omega_\lambda) r\,\mathrm{d}r + \pi a^2 P - \frac{1}{2}\pi(b^2 - a^2)\mu_0 H_z^2. \tag{7.72}$$

The expression for P is identical in form to the corresponding equation in the purely elastic theory in Sect. 3.4.1, while that for N differs by the inclusion of the final term involving H_z (see also Ogden 1997, pp. 289–291), but now ω includes the influence of an axial magnetic field through I_4. The expressions for P and N above are counterparts of the corresponding expressions in the electroelastic case considered in Sect. 5.3.1.3 for an axial electric field.

For illustration, we now consider for ω the analogue of the model (7.26) used for ω^*. This is

$$\omega(\lambda, \lambda_z, I_4) = \frac{1}{2}\mu(I_4)(\lambda^2 + \lambda_z^2 + \lambda^{-2}\lambda_z^{-2} - 3) + \nu(I_4) + \frac{1}{2}\beta\mu_0 I_5, \tag{7.73}$$

with $I_5 = \lambda_z^2 I_4$, $\mu(I_4)$ and $\nu(I_4)$ functions of I_4 such that $\nu(0) = 0$ and $\mu(0)$ is the shear modulus of the parent neo-Hookean elastic material, and β is a dimensionless constant. Then the expression for P takes the form

$$P = \mu(I_4) \int_a^b (\lambda^2 - \lambda^{-2}\lambda_z^{-2}) \frac{\mathrm{d}r}{r}, \tag{7.74}$$

which differs from the corresponding expression in the purely elastic case (with the neo-Hookean strain-energy function) only by virtue of the dependence of μ on I_4. Thus, if, for example, μ is an increasing function of I_4, then the pressure–radius response stiffens as a result of the presence of the axial magnetic field.

A similar comment applies to N in that it is influenced by the dependence of μ on I_4, but there is also an additional contribution from the 'Maxwell' term in the energy function. Thus, we can write

$$N = N_{\mathrm{nh}} + \frac{1}{2}\pi\mu_0(b^2 - a^2)(\beta\lambda_z^4 - 1)H_z^2, \tag{7.75}$$

where N_{nh} is the expression for N obtained for the neo-Hookean material but with μ dependent on I_4.

If no pressure is applied, then $\lambda^2\lambda_z = 1$, and it follows that

$$N = \frac{1}{2}\pi(b^2 - a^2)[2(\lambda_z^2 - \lambda_z^{-1}) + \mu_0(\beta\lambda_z^4 - 1)H_z^2]. \tag{7.76}$$

For $N = 0$ this has a (positive) solution for λ_z if either $1 < \lambda_z < \beta^{-1/4}$ ($\beta < 1$) or $\beta^{-1/4} < \lambda_z < 1$ ($\beta > 1$). If $\beta = 1$ then necessarily $\lambda_z = 1$. On the other hand, if the reference deformation $\lambda = \lambda_z = 1$ is to be maintained for $P = 0$, we have

$$N = \frac{1}{2}\pi\mu_0(b^2 - a^2)(\beta - 1)H_z^2, \tag{7.77}$$

so whether the magnetic field tends to shorten or lengthen the tube depends on whether $\beta > 1$ or $\beta < 1$.

7.4.1.2 Azimuthal Magnetic Field

For this problem, by contrast with that for an electric field in Sect. 5.3.1, we consider an azimuthal magnetic field, with component H_θ, which for the considered geometry has the form

$$H_\theta = \frac{c}{r}, \tag{7.78}$$

where c is a constant. Such a magnetic field can be generated by a current flowing along the core of the tube or a surface current along its inner boundary, so there is no difficulty associated with a possible singularity on $r = 0$ in this example. Assuming that there is no surface current, the boundary conditions require that H_θ is continuous across the cylindrical surfaces $r = a$ and $r = b$. This is the only component of the magnetic field, and there is, correspondingly, only a single component B_θ of the magnetic induction throughout space.

In the space surrounding the tube, where $B_\theta^\star = \mu_0 H_\theta^\star = \mu_0 H_\theta$, the components of the Maxwell stress tensor may again be obtained by specializing (7.13). The only non-zero components are

$$\tau_{\mathrm{m}rr}^\star = \tau_{\mathrm{m}zz}^\star = -\frac{1}{2}\mu_0 H_\theta^2, \quad \tau_{\mathrm{m}\theta\theta}^\star = \frac{1}{2}\mu_0 H_\theta^2, \tag{7.79}$$

which depend on the radius r. For later use, we define the notation

$$\tau_{\mathrm{m}}^\star(r) = \frac{1}{2}\mu_0 H_\theta^2 = \frac{1}{2}\mu_0\frac{c^2}{r^2}. \tag{7.80}$$

In the material $H_{\mathrm{L}\Theta} = \lambda H_\theta = c/R$, and the invariants I_4, I_5, I_6 now assume the values

$$I_4 = H_{\mathrm{L}\Theta}^2, \quad I_5 = \lambda^2 I_4, \quad I_6 = \lambda^4 I_4. \tag{7.81}$$

Similarly to the previous section, this allows a reduced form of the energy function Ω to be defined. Again we use the notation ω defined by

$$\omega(\lambda, \lambda_z, I_4) = \Omega(I_1, I_2, I_4, I_5, I_6), \tag{7.82}$$

with I_1 and I_2 as in the previous subsection and I_4, I_5, I_6 now given by (7.81).

The magnetic induction B_θ is then given by

$$B_\theta = -2\lambda^2 \omega_4 H_\theta, \tag{7.83}$$

and the equation $\operatorname{div}\mathbf{B} = 0$ is automatically satisfied.

The components of $\boldsymbol{\tau}$ within the material are obtained from the specialization of (7.8). The only non-zero components are τ_{rr}, $\tau_{\theta\theta}$, and τ_{zz}, and it is straightforward to show, by using (7.81) and (7.82), that

$$\tau_{\theta\theta} - \tau_{rr} = \lambda\omega_\lambda, \quad \tau_{zz} - \tau_{rr} = \lambda_z \omega_{\lambda_z}. \tag{7.84}$$

For the considered deformation, (7.68) is again the only non-trivial equation to be satisfied. On substituting from $(7.84)_1$ we may integrate this equation in the form

$$\tau_{rr} = \int_a^r \lambda\omega_\lambda \frac{\mathrm{d}r}{r} + \tau_{rr}(a), \tag{7.85}$$

where $\tau_{rr}(a)$ is the value of τ_{rr} on $r = a$. For the traction boundary conditions, we suppose that the inner boundary $r = a$ is subjected to a pressure P while the outer boundary $r = b$ is free of mechanical load. Thus, the boundary conditions are

$$\tau_{rr}(a) = -P - \tau_{\mathrm{m}}^\star(a), \quad \tau_{rr}(b) = -\tau_{\mathrm{m}}^\star(b), \tag{7.86}$$

and by using these in (7.85) we obtain

$$P = \int_a^b \lambda\omega_\lambda \frac{\mathrm{d}r}{r} + \tau_{\mathrm{m}}^\star(b) - \tau_{\mathrm{m}}^\star(a), \tag{7.87}$$

which involves the Maxwell stress $\tau_{\mathrm{m}}^\star(r)$ at the inner and outer surfaces. Since

$$\tau_{\mathrm{m}}^\star(b) - \tau_{\mathrm{m}}^\star(a) = \frac{1}{2}\mu_0 c^2 \left(\frac{1}{b^2} - \frac{1}{a^2}\right) \tag{7.88}$$

is negative, this means that the resultant effect of the magnetic field is similar to that of the pressure in that it can cause inflation of the tube at fixed axial extension. In particular, the magnetic field can induce inflation in the absence of internal pressure.

The resultant axial force, again denoted N, on any cross section of the tube is

$$N = 2\pi \int_a^b \tau_{zz} r \mathrm{d}r = 2\pi \left[\int_a^b (\tau_{zz} - \tau_{rr}) r \mathrm{d}r + \int_a^b \tau_{rr} r \mathrm{d}r \right]. \tag{7.89}$$

By integrating the second integral on the right-hand side by parts, using (7.68) and the fact that $a^2 \tau_{\mathrm{m}}^{\star}(a) = b^2 \tau_{\mathrm{m}}^{\star}(b)$, we obtain the formula

$$N = \pi \int_a^b (2\tau_{zz} - \tau_{rr} - \tau_{\theta\theta}) r \mathrm{d}r + \pi a^2 P. \tag{7.90}$$

Then, use of (7.84) allows N to be rewritten in terms of the reduced energy function ω as

$$N = \pi \int_a^b (2\lambda_z \omega_{\lambda_z} - \lambda \omega_{\lambda}) r \mathrm{d}r + \pi a^2 P. \tag{7.91}$$

Thus, when Ω, and hence ω, is given both the pressure and the axial load can be calculated. Note the contrast between this formula, which does not involve the exterior Maxwell stress and (7.72), which does. For further discussion of the extension/inflation problem, see Dorfmann and Ogden (2005).

7.4.2 Helical Shear of a Tube

We consider again the tube whose geometry was described in (7.57), but now the tube is subjected to the helical shear deformation described by the equations

$$r = R, \quad \theta = \Theta + g(R), \quad z = Z + w(R), \tag{7.92}$$

where $g(R)$ and $w(R)$ are functions of R to be determined by solving the governing equations and applying the boundary conditions. The deformation defined by (7.92) is a combination of a pure axial shear deformation, which was analyzed in Dorfmann and Ogden (2004b) and Ogden and Dorfmann (2005) for magnetoelastic materials, and a pure azimuthal shear deformation. This combination is locally a simple shear deformation. Since $r = R$, we use r as the variable in the functions g and w, and we also set $a = A$ and $b = B$. We assume that the inner boundary of the tube is fixed and let the outer boundary be subject to an azimuthal rotation through an angle β and an axial displacement d, so that

$$g(a) = 0, \quad g(b) = \beta, \quad w(a) = 0, \quad w(b) = d. \tag{7.93}$$

Referred to the two sets of cylindrical polar coordinate axes, the components of the deformation gradient $\mathbf{F}$ are represented by the matrix F, which is given by

$$\mathsf{F} = \begin{bmatrix} 1 & 0 & 0 \\ rg'(r) & 1 & 0 \\ w'(r) & 0 & 1 \end{bmatrix}, \tag{7.94}$$

where the prime indicates differentiation with respect to r. For convenience, we use the notations

$$\gamma_\theta = rg'(r), \quad \gamma_z = w'(r), \quad \gamma^2 = \gamma_\theta^2 + \gamma_z^2, \tag{7.95}$$

and, without loss of generality, we take γ_θ, γ_z and γ to be positive, γ being (locally) a simple shear.

The matrices of the left and right Cauchy–Green tensors $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ and $\mathbf{c} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$, written b and c, respectively, are given by

$$\mathsf{b} = \begin{bmatrix} 1 & \gamma_\theta & \gamma_z \\ \gamma_\theta & 1+\gamma_\theta^2 & \gamma_\theta\gamma_z \\ \gamma_z & \gamma_\theta\gamma_z & 1+\gamma_z^2 \end{bmatrix}, \quad \mathsf{c} = \begin{bmatrix} 1+\gamma^2 & \gamma_\theta & \gamma_z \\ \gamma_\theta & 1 & 0 \\ \gamma_z & 0 & 1 \end{bmatrix}, \tag{7.96}$$

and the matrix corresponding to $\mathbf{b}^2$ is

$$\mathsf{b}^2 = \begin{bmatrix} 1+\gamma^2 & (2+\gamma^2)\,\gamma_\theta & (2+\gamma^2)\,\gamma_z \\ (2+\gamma^2)\,\gamma_\theta & 1+(3+\gamma^2)\,\gamma_\theta^2 & (3+\gamma^2)\,\gamma_\theta\gamma_z \\ (2+\gamma^2)\,\gamma_z & (3+\gamma^2)\,\gamma_\theta\gamma_z & 1+(3+\gamma^2)\,\gamma_z^2 \end{bmatrix}. \tag{7.97}$$

A similar expression for c^2 can be written down if required.

The principal invariants I_1, I_2 are then obtained as

$$I_1 = I_2 = 3 + \gamma^2, \tag{7.98}$$

just as for simple shear in (7.36) except that here γ depends on r. The invariants I_4, I_5, I_6 depend on the choice of magnetic field and will be quantified in what follows.

7.4.2.1 Axial Magnetic Field

We work in terms of the energy function Ω and assume that a uniform axial magnetic field $\mathbf{H}_{\mathrm{L}}$, with components $(0, 0, H_{\mathrm{L}Z})$, is applied prior to deformation. In the deformed configuration, the magnetic field vector is given by $\mathbf{H} = \mathbf{F}^{-\mathrm{T}}\mathbf{H}_{\mathrm{L}}$ and has components

$$H_r = -\gamma_z H_{\mathrm{L}Z}, \quad H_\theta = 0, \quad H_z = H_{\mathrm{L}Z}. \tag{7.99}$$

The invariants I_4, I_5, I_6 are then calculated from (6.93), using (7.96), as

$$I_4 = H_{LZ}^2, \quad I_5 = I_4, \quad I_6 = I_4(1 + \gamma_z^2). \tag{7.100}$$

Thus, the only remaining kinematic and magnetic variables are γ_θ, γ_z and I_4. Application of H_{LZ} in the reference configuration requires accompanying stresses to maintain the reference geometry, and then further stresses are needed to generate the helical shear deformation.

Equation (7.9) is then used to determine the corresponding components of the magnetic induction vector $\mathbf{B}$ as

$$B_r = -2[\Omega_5 + (2 + \gamma^2)\Omega_6]\gamma_z H_z, \tag{7.101}$$

$$B_\theta = -2[\Omega_5 + (3 + \gamma^2)\Omega_6]\gamma_\theta \gamma_z H_z, \tag{7.102}$$

$$B_z = -2\{\Omega_4 + \Omega_5 + \Omega_6 + [\Omega_5 + (3 + \gamma^2)\Omega_6]\gamma_z^2\}H_z. \tag{7.103}$$

Equation (7.101) shows that the radial component B_r is not in general zero. However, because B_r depends only on r, the equation $\operatorname{div}\mathbf{B} = 0$ reduces to $\mathrm{d}(rB_r)/\mathrm{d}r = 0$, and hence, $B_r = c/r$ for some constant c. But, the normal component must be continuous across the inner boundary of the tube, and therefore, to avoid a singularity at $r = 0$, we must have $c = 0$. Thus, $B_r = 0$ everywhere. This implies that for a circular cylindrical tube subject to helical shear with $\gamma_z \neq 0$ and an applied axial field the constitutive constraint

$$\Omega_5 + \Omega_6(2 + \gamma^2) = 0 \tag{7.104}$$

must be satisfied. There are many ways to satisfy this requirement. An immediate example is by using an energy function that does not depend on I_5 and I_6. Another possibility is an energy function depending on the variables I_1, I_4 and the combination $(I_1 - 1)I_5 - I_6 = I_4(1 + \gamma_\theta^2)$.

Since $B_r = 0$ everywhere, then $H_r^\star = 0$ outside the material, while H_r is given by $(7.99)_1$ inside the material. Moreover, $(7.11)_1$ requires that the tangential component of $\mathbf{H}$ be continuous across the inner and outer cylindrical boundaries of the tube. Thus, the component H_z of the magnetic field, which is tangential to the boundaries $r = a$ and $r = b$ of the tube and is constant, must be continuous across those boundaries, i.e. $H_z^\star = H_z$, and also $H_\theta^\star = 0$.

Outside the material the Maxwell stress components are obtained from (7.13) as

$$\tau^\star_{\mathrm{m}rr} = \tau^\star_{\mathrm{m}\theta\theta} = -\tfrac{1}{2}\mu_0 H_z^2, \quad \tau^\star_{\mathrm{m}zz} = \tfrac{1}{2}\mu_0 H_z^2, \tag{7.105}$$

with $\tau^\star_{\mathrm{m}r\theta} = 0$, $\tau^\star_{\mathrm{m}rz} = 0$ and $\tau^\star_{\mathrm{m}\theta z} = 0$.

The total stress tensor $\boldsymbol{\tau}$ inside the material is given by (7.8), and its components are

$$\tau_{rr} = -p + 2(\Omega_1 + 2\Omega_2), \tag{7.106}$$

$$\tau_{\theta\theta} = -p + 2\Omega_1(1 + \gamma_\theta^2) + 2\Omega_2(2 + \gamma^2), \tag{7.107}$$

$$\tau_{zz} = -p + 2\Omega_1(1 + \gamma_z^2) + 2\Omega_2(2 + \gamma^2) + 2I_4[\Omega_5 + 2\Omega_6(1 + \gamma_z^2)], \tag{7.108}$$

$$\tau_{r\theta} = 2(\Omega_1 + \Omega_2)\gamma_\theta, \tag{7.109}$$

$$\tau_{rz} = 2(\Omega_1 + \Omega_2)\gamma_z + 2\Omega_6 I_4 \gamma_z, \tag{7.110}$$

$$\tau_{\theta z} = 2\Omega_1\gamma_\theta\gamma_z + 2\Omega_6 I_4 \gamma_\theta\gamma_z. \tag{7.111}$$

Since the original five invariants now depend on just three independent quantities, namely γ_θ, γ_z and I_4, we may define a reduced energy function, again denoted ω but now given by

$$\omega(\gamma_\theta, \gamma_z, I_4) = \Omega(I_1, I_2, I_4, I_5, I_6), \tag{7.112}$$

with (7.95), (7.98) and (7.100). Then, the use of the chain rule enables (7.109) and (7.110) to be reduced to the simple forms

$$\tau_{r\theta} = \frac{\partial\omega}{\partial\gamma_\theta}, \quad \tau_{rz} = \frac{\partial\omega}{\partial\gamma_z}. \tag{7.113}$$

Equations $(7.56)_{2,3}$ may now be integrated and combined with (7.113) to give

$$\tau_{r\theta} = \frac{\partial\omega}{\partial\gamma_\theta} = \frac{\tau_\theta b^2}{r^2}, \quad \tau_{rz} = \frac{\partial\omega}{\partial\gamma_z} = \frac{\tau_z b}{r}, \tag{7.114}$$

where τ_θ is the value of $\tau_{r\theta}$ on the boundary $r = b$ and τ_z that of τ_{rz}, as in the electroelastic case in (5.124).

Since the components of the exterior Maxwell stress corresponding to $\tau_{r\theta}$ and τ_{rz} are zero, τ_θ and τ_z are the values of the externally applied mechanical traction components in the azimuthal and axial directions, respectively, that are required to achieve the helical shear deformation with the specified β and d in (7.93). By definition of the deformation there is no change in the radius so a radial traction boundary condition is not required, and $(7.56)_1$ may be used simply to calculate the radial mechanical traction required to maintain the deformation, if needed. If τ_θ and τ_z are given, (7.114) can in principle be used to determine γ_θ and γ_z and then by integration the displacement functions $g(r)$ and $w(r)$, subject to $(7.93)_{1,3}$.

7.4.2.2 Illustration

We now apply the equations in the above subsection to a particular material model, with Ω depending only on the invariants I_1 and I_4, so that (7.104) is satisfied.

Specifically, and generalizing to the magnetoelastic situation a model introduced by Jiang and Ogden (1998) for elasticity, we set

$$\Omega(I_1, I_4) = \frac{\mu(I_4)}{k}\left[\left(\frac{I_1 - 1}{2}\right)^k - 1\right] + \nu(I_4), \tag{7.115}$$

where μ and ν are functions of I_4 and k is a constant such that $k \geq 1/2$. In particular, μ is such that $\mu(0)\,(> 0)$ is the shear modulus of the material in the undeformed configuration in the absence of a magnetic field. The term $\nu(I_4)$ represents the magnetic energy in the material in the absence of deformation and requires that $\nu(0) = 0$. In the purely elastic case explicit solutions were obtained in Jiang and Ogden (1998) for several values of k, and such solutions can be carried over to the magnetoelastic case considered here.

Using (7.114) and the function (7.115), we obtain the shear stresses as

$$\tau_{r\theta} = \mu(I_4)\gamma_\theta\left(\frac{2+\gamma^2}{2}\right)^{k-1} = \frac{\tau_\theta b^2}{r^2}, \tag{7.116}$$

$$\tau_{rz} = \mu(I_4)\gamma_z\left(\frac{2+\gamma^2}{2}\right)^{k-1} = \frac{\tau_z b}{r}. \tag{7.117}$$

The dependence of the shear stresses τ_θ and τ_z on the rotation angle β and the axial displacement d can in principle be determined by integration of (7.116) and (7.117) via $\gamma_\theta = rg'(r)$ and $\gamma_z = w'(r)$ together with (7.93). We illustrate the results for the special case $k = 1$, for which explicit results can be obtained for $g(r)$ and $w(r)$. These are

$$g(r) = \frac{\tau_\theta b^2}{2\mu(I_4)}\left(\frac{1}{a^2} - \frac{1}{r^2}\right), \quad w(r) = \frac{\tau_\theta b}{\mu(I_4)}\log\left(\frac{r}{a}\right). \tag{7.118}$$

Hence

$$\beta = \frac{\tau_\theta b^2}{2\mu(I_4)}\left(\frac{1}{a^2} - \frac{1}{b^2}\right), \quad d = \frac{\tau_\theta b}{\mu(I_4)}\log\left(\frac{b}{a}\right), \tag{7.119}$$

and it can be seen that the stiffness of the mechanical response of the tube to either azimuthal or axial shearing increases with $\mu(I_4)$, and hence with the magnitude of the applied magnetic field if μ is an increasing function of I_4.

These expressions do not involve the function $\nu(I_4)$, which can be interpreted in terms of the magnetization of the material in a similar way to the interpretation of $\nu(K_4)$ in Sect. 7.3.2. Since $B_r = 0$ the components of the magnetization vector are given by

$$M_r = -H_r, \quad M_\theta = \mu_0^{-1}B_\theta, \quad M_z = \mu_0^{-1}B_z - H_z. \tag{7.120}$$

In particular, in the undeformed configuration, where $\gamma_\theta = \gamma_z = 0$, the only non-zero component of the magnetic field is $H_z = H_{\mathrm{L}Z}$, and, by (7.103), $B_z = -2\Omega_4 H_z$ in respect of the model (7.115). The only non-vanishing component of the magnetization is then given by $(7.120)_3$ with (7.115) as

$$M_z = -[1 + 2\mu_0^{-1}\nu'(I_4)]H_z. \tag{7.121}$$

Thus, while the function $\mu(I_4)$ characterizes the dependence of the shear response of the material on the magnetic field, this shows that $\nu(I_4)$, through its derivative, characterizes the magnetization in the undeformed configuration. In the deformed configuration, the magnetization depends on both functions.

7.4.2.3 Azimuthal Magnetic Field

Now we assume that the magnetic field is purely azimuthal prior to application of the helical shear, with component $H_{\mathrm{L}\Theta}$. Then, by using $\mathbf{H} = \mathbf{F}^{-\mathrm{T}}\mathbf{H}_{\mathrm{L}}$ we find that the components of $\mathbf{H}$ are

$$H_r = -\gamma_\theta H_\theta, \quad H_\theta = H_{\mathrm{L}\Theta}, \quad H_z = 0. \tag{7.122}$$

The invariants I_1 and I_2 are again given by (7.98), but now we have

$$I_4 = I_5 = H_\theta^2, \quad I_6 = (1 + \gamma_\theta^2)I_4, \tag{7.123}$$

with H_θ $(= c/r)$, where c is a constant.

It then follows that

$$B_r = -2[\Omega_5 + (2 + \gamma^2)\Omega_6]\gamma_\theta H_\theta, \tag{7.124}$$

$$B_\theta = -2\{\Omega_4 + \Omega_5 + \Omega_6 + [\Omega_5 + (3 + \gamma^2)\Omega_6]\gamma_\theta^2\}H_\theta, \tag{7.125}$$

$$B_z = -2[\Omega_5 + (3 + \gamma^2)\Omega_6]\gamma_\theta\gamma_z H_\theta. \tag{7.126}$$

If $\gamma_\theta = 0$ then $B_r = 0$ and no restriction need be placed of the form of Ω. But if $\gamma_\theta \neq 0$ then the restriction (7.104) must again be in place for the combination of helical shear with an azimuthal magnetic field to be possible.

We now consider, as in the previous section, Ω to be independent of I_5 and I_6. Then, from (7.125), as well as $B_r = B_z = 0$, we have simply

$$B_\theta = -2\Omega_4 H_\theta. \tag{7.127}$$

For the model (7.115) with $k = 1$, (7.114) yields

$$\mu(I_4)\gamma_\theta = \frac{\tau_\theta b^2}{r^2}, \quad \mu(I_4)\gamma_z = \frac{\tau_z b}{r}, \tag{7.128}$$

but in contrast to the previous section, I_4 depends on r. Integration of these using $\gamma_\theta = rg'(r)$ and $\gamma_z = w'(r)$ leads to

$$\beta = \tau_\theta b^2 \int_a^b \frac{\mathrm{d}r}{\mu(I_4)r^3}, \quad d = \tau_z b \int_a^b \frac{dr}{\mu(I_4)r}, \tag{7.129}$$

where $I_4 = H_\theta^2 = c^2/r^2$, c being a measure of the strength of the magnetic field. These provide uncoupled linear relations between τ_θ and β and between τ_z and d, which, as for the axial field, show stiffening of the response if $\mu(I_4)$ is an increasing function of I_4. The decoupling of the axial and azimuthal responses is a consequence of the very simple choice of energy function and is not to be expected in general.

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Chapter 8
Variational Formulations in Electroelasticity and Magnetoelasticity

Abstract In this chapter we set the equations of nonlinear electroelasticity and magnetoelasticity within a variational framework for a material body and its exterior. First, for the electroelastic case we establish a variational statement for a functional for which the energy density is expressed in terms of the electric field. The independent variables in the functional are the deformation function and the scalar electric potential. The first variation of the functional vanishes if and only if the derived stress and electric displacement satisfy the appropriate field equations and boundary conditions both within and outside the material. Next, we consider a functional for which the energy density depends on the electric displacement and the independent variables are the deformation function and a vector potential. A similar conclusion follows from the vanishing of the first variation of the functional. In each case the mechanical body force is taken to be conservative and the mechanical traction to be of dead-load type. We then provide parallel results for a magnetoelastic material body, and in the case of a scalar magnetoelastic potential, by way of illustration of a connection that applies to all the functionals considered in this chapter, we show that the functional can be derived by starting from the energy balance equation discussed in earlier chapters.

8.1 Variational Formulations in Electroelasticity

A variational formulation of the nonlinear theory of elastic dielectrics was introduced in the classic paper of Toupin (1956), while Ericksen (2007), in revisiting this theory, has also constructed a variational principle, essentially equivalent to that of Toupin. Recent work on the formulation of the equations of nonlinear electroelasticity includes that of McMeeking et al. (2007), who constructed a principle of virtual work, as distinct from a variational principle. Of course, one motivation for the development of variational principles is their use in the finite element formulation and numerical solution of the governing equations. Such formulations have been provided by, for example, Landis (2002) and Vu et al. (2007).

L. Dorfmann and R.W. Ogden, *Nonlinear Theory of Electroelastic and Magnetoelastic Interactions*, DOI 10.1007/978-1-4614-9596-3_8,

For extensive references we refer to the recent work by Vogel et al. (2012). In considering variational principles here we follow the work of Bustamante et al. (2009a), but with some minor differences, including differences in notation, for consistency with the notations used in the earlier chapters in this volume.

8.1.1 Formulation in Terms of the Electrostatic Scalar Potential

We start with the expression for the 'free energy' E given by Bustamante et al. (2009a), which we write in the form

$$E = \int_{\mathcal{B}} \rho\psi \, \mathrm{d}v - \tfrac{1}{2}\varepsilon_0 \int_{\mathcal{B}'} \mathbf{E}^\star \cdot \mathbf{E}^\star \, \mathrm{d}v - \int_{\partial\mathcal{B}^\infty} \varphi \mathbf{D}_\mathrm{a} \cdot \mathbf{n} \, \mathrm{d}s, \tag{8.1}$$

where $\varphi(\mathbf{x})$ is the (Eulerian) scalar electrostatic potential, so that $\mathbf{E} = -\mathrm{grad}\,\varphi$; $\psi = \psi(\mathbf{F}, \mathbf{E})$ is the potential function introduced in (4.56); $\mathbf{F}$ is the deformation gradient Grad $\mathbf{x}$, with the deformation given by $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$; and $\mathbf{n} \cdot \mathbf{D}_\mathrm{a}$ is a prescribed ('applied') normal component of the electric displacement field. In view of the boundary condition $(2.167)_2$, we could think of this as a prescription of a surface charge density distribution on $\partial\mathcal{B}^\infty$.

We take φ and its tangential derivatives to be continuous across the boundary $\partial\mathcal{B}$ so that, in particular, the jump condition $(4.3)_1$ is satisfied (alternatively, we can start with the jump condition and deduce the stated conditions on φ). We do not use a separate notation for φ outside the body (in $\mathcal{B}'$), and we therefore also have $\mathbf{E}^\star = -\mathrm{grad}\,\varphi$ in $\mathcal{B}'$.

The first integral is over the body $\mathcal{B}$ in the deformed configuration, the second integral over the exterior of the body, and the third integral over the boundary at 'infinity,' which we take to be fixed. Clearly E can be thought of as a functional of $\boldsymbol{\chi}$ and φ. As shown in Bustamante et al. (2009a) the free energy E is equivalent to that introduced by Toupin (1956) and also essentially equivalent to the energy used by Ericksen (2007). Unlike the potentials in the latter two papers, the E defined above does not include separate 'applied' and 'self' fields.

It is now convenient to re-cast E in (partial) Lagrangian form, and we write

$$E\{\boldsymbol{\chi}, \varphi_\mathrm{L}\} = \int_{\mathcal{B}_\mathrm{r}} \Omega(\mathbf{F}, \mathbf{E}_\mathrm{L}) \, \mathrm{d}V - \frac{1}{2}\varepsilon_0 \int_{\mathcal{B}'_\mathrm{r}} J(\mathbf{F}^{-\mathrm{T}}\mathbf{E}^\star_\mathrm{L}) \cdot (\mathbf{F}^{-\mathrm{T}}\mathbf{E}^\star_\mathrm{L}) \, \mathrm{d}V$$
$$- \int_{\partial\mathcal{B}^\infty} \varphi_\mathrm{L} \mathbf{D}_\mathrm{a} \cdot \mathbf{n} \, \mathrm{d}s, \tag{8.2}$$

where $\mathcal{B}_\mathrm{r}$ is the reference configuration of the body, $\mathcal{B}'_\mathrm{r}$ is the exterior of $\mathcal{B}_\mathrm{r} \cup \partial\mathcal{B}_\mathrm{r}$, $\Omega(\mathbf{F}, \mathbf{E}_\mathrm{L})$ is the total energy density introduced in Sect. 4.4.3, $\mathbf{E}_\mathrm{L} = -\mathrm{Grad}\,\varphi_\mathrm{L}$, where $\varphi_\mathrm{L}(\mathbf{X}) = \varphi(\boldsymbol{\chi}(\mathbf{X}))$, $\mathbf{E}^\star_\mathrm{L} = \mathbf{F}^\mathrm{T}\mathbf{E}^\star$, and we have left the integral at infinity in the

Eulerian form since the boundary $\partial\mathcal{B}^\infty$ is fixed. In (8.2), following Toupin (1956), we have introduced a fictitious 'deformation' in $\mathcal{B}'_\mathrm{r}$ that is a smooth extension of the deformation in $\mathcal{B}$ into $\mathcal{B}'$ (with the deformation gradient $\mathbf{F}$ also continuous). This is a convenient device for aiding the establishment of the variational principle, and the result is independent of this extension.

In order to construct a variational principle we introduce the functional Π, which depends on the deformation $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ and the scalar potential $\varphi_\mathrm{L}(\mathbf{X})$, defined by

$$\Pi\{\mathbf{x}, \varphi_\mathrm{L}\} = E\{\mathbf{x}, \varphi_\mathrm{L}\} - L\{\mathbf{x}\}, \tag{8.3}$$

where L is the contribution to the total energy of the system due to the *mechanical* loads, i.e. the mechanical body forces and boundary tractions. In (8.3) we have used $\mathbf{x}$ rather than $\boldsymbol{\chi}$, and we do this henceforth. To form a true variational principle, rather than a principle of virtual work, the body force $\mathbf{f}$ is required to be conservative and can therefore be written as $\mathbf{f} = -\mathrm{grad}\,\mathsf{V}$, where $\mathsf{V} = \mathsf{V}(\mathbf{x})$ is the associated potential, a function of $\mathbf{x}$ only. Equally, the boundary traction is required to be of a special type, and we take it here to be a dead load, so that $\mathbf{t}_\mathrm{a}\mathrm{d}s \equiv \mathbf{t}_\mathrm{A}\mathrm{d}S$, with $\mathbf{t}_\mathrm{A}$ depending only on $\mathbf{X}$ (not on the deformation). Thus,

$$L\{\mathbf{x}\} = -\int_{\mathcal{B}} \rho\mathsf{V}(\mathbf{x})\mathrm{d}v + \int_{\partial\mathcal{B}} \mathbf{t}_\mathrm{a}\cdot\mathbf{x}\mathrm{d}s, \tag{8.4}$$

with $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$, or in Lagrangian form

$$L\{\mathbf{x}\} = -\int_{\mathcal{B}_\mathrm{r}} \rho_\mathrm{r}\mathsf{V}(\mathbf{x})\mathrm{d}V + \int_{\partial\mathcal{B}_\mathrm{r}} \mathbf{t}_\mathrm{A}\cdot\mathbf{x}\mathrm{d}S. \tag{8.5}$$

If the traction is prescribed on only part of the boundary $\partial\mathcal{B}_\mathrm{r}$ and $\mathbf{x}$ is prescribed on the remainder of $\partial\mathcal{B}_\mathrm{r}$, then the integration in the latter integral must be restricted to the part of $\partial\mathcal{B}_\mathrm{r}$ where the traction is prescribed, but this possibility is left implicit in what follows.

We now have

$$\begin{aligned}\Pi\{\mathbf{x}, \varphi_\mathrm{L}\} &= E\{\mathbf{x}, \varphi_\mathrm{L}\} - L\{\mathbf{x}\}\\ &= \int_{\mathcal{B}_\mathrm{r}} \Omega(\mathbf{F}, \mathbf{E}_\mathrm{L})\,\mathrm{d}V - \frac{1}{2}\varepsilon_0\int_{\mathcal{B}'_\mathrm{r}} J(\mathbf{F}^{-\mathrm{T}}\mathbf{E}^\star_\mathrm{L})\cdot(\mathbf{F}^{-\mathrm{T}}\mathbf{E}^\star_\mathrm{L})\,\mathrm{d}V\\ &\quad - \int_{\partial\mathcal{B}^\infty} \varphi_\mathrm{L}\mathbf{D}_\mathrm{a}\cdot\mathbf{n}\,\mathrm{d}s + \int_{\mathcal{B}_\mathrm{r}} \rho_\mathrm{r}\mathsf{V}(\mathbf{x})\mathrm{d}V - \int_{\partial\mathcal{B}_\mathrm{r}} \mathbf{x}\cdot\mathbf{t}_\mathrm{A}\mathrm{d}S.\end{aligned} \tag{8.6}$$

In the following we will obtain an expression for the first variation of Π, but first we focus on the first variation of E. Variations are represented by a superposed dot. Thus, $\dot{\varphi}_\mathrm{L}$ and $\dot{\mathbf{x}}$ are variations in φ_L and $\mathbf{x}$ (i.e. $\boldsymbol{\chi}$), respectively, and we require $\dot{\varphi}_\mathrm{L}$ and $\dot{\mathbf{x}}$ to be continuous everywhere and continuously differentiable in each of $\mathcal{B}_\mathrm{r}$ and $\mathcal{B}'_\mathrm{r}$. We also note that when $\mathbf{x}$ is prescribed on part of $\partial\mathcal{B}_\mathrm{r}$, we take $\dot{\mathbf{x}} = \mathbf{0}$

there, but we do not use this boundary condition explicitly here. We take $\dot{\mathbf{x}} = \mathbf{0}$ on the fixed boundary $\partial\mathcal{B}^\infty$. Variations in $\mathbf{x}$ and φ_L are independent, but some care is needed since φ depends on $\mathbf{x}$ and thus a variation in $\mathbf{x}$ will induce a variation in φ independently of the variation in φ at fixed $\mathbf{x}$. Accordingly, we write

$$\dot{\varphi}_\mathrm{L} = \dot{\varphi} + \dot{\varphi}_\mathrm{ind}, \tag{8.7}$$

where $\dot{\varphi}_\mathrm{ind} = (\mathrm{grad}\,\varphi) \cdot \dot{\mathbf{x}}$ is the variation in φ induced by a variation in $\mathbf{x}$, whereas $\dot{\varphi}$ is the variation in φ at fixed $\mathbf{x}$. We do not make use of this decomposition in the following since we work in terms of $\dot{\varphi}_\mathrm{L}$, whereas the decomposition was used in Bustamante et al. (2009a).

We also note the variations

$$\dot{\mathbf{E}}_\mathrm{L} = -\mathrm{Grad}\,\dot{\varphi}_\mathrm{L}, \quad \dot{\mathbf{F}} = \mathrm{Grad}\,\dot{\mathbf{x}} = \mathbf{L}\mathbf{F}, \tag{8.8}$$

which are required for obtaining the variation $\dot{E}$, where $\mathbf{L} = \mathrm{grad}\,\dot{\mathbf{x}}$ (essentially the same as the $\mathbf{L}$ introduced in Sect. 4.3 in which a virtual displacement was considered), and

$$\dot{J} = J\,\mathrm{tr}\,\mathbf{L}, \quad \dot{\overline{(\mathbf{F}^{-\mathrm{T}})}} = -\mathbf{L}^\mathrm{T}\mathbf{F}^{-\mathrm{T}}, \quad \mathrm{grad}\,\dot{\varphi}_\mathrm{L} = \mathbf{F}^{-\mathrm{T}}\mathrm{Grad}\,\dot{\varphi}_\mathrm{L}, \tag{8.9}$$

the first two of which may be obtained by replacing the velocity $\mathbf{v}$ by the variation $\dot{\mathbf{x}}$, $\boldsymbol{\Gamma}$ by $\mathbf{L}$ and the time derivative by a variation in (3.53) and (3.49), respectively; the third equation follows from $(3.15)_1$.

In view of the form of the constitutive law in (4.62), we also use the notations

$$\mathbf{T} = \frac{\partial\Omega}{\partial\mathbf{F}}, \quad \boldsymbol{\tau} = J^{-1}\mathbf{F}\mathbf{T}, \quad \mathbf{D}_\mathrm{L} = -\frac{\partial\Omega}{\partial\mathbf{E}_\mathrm{L}}, \quad \mathbf{D} = J^{-1}\mathbf{F}\mathbf{D}_\mathrm{L} \tag{8.10}$$

in $\mathcal{B}$ and $\mathbf{D}^\star = \varepsilon_0\mathbf{E}^\star$ in $\mathcal{B}'$.

Then,

$$\begin{aligned}\dot{\Omega} &= \mathrm{tr}\,(\mathbf{T}\dot{\mathbf{F}}) - \mathbf{D}_\mathrm{L} \cdot \dot{\mathbf{E}}_\mathrm{L} = J[\mathrm{tr}\,(\boldsymbol{\tau}\mathbf{L}) + \mathbf{D} \cdot \mathrm{grad}\,\dot{\varphi}_\mathrm{L}] \\ &= J[\mathrm{div}\,(\boldsymbol{\tau}\dot{\mathbf{x}}) - (\mathrm{div}\,\boldsymbol{\tau}) \cdot \dot{\mathbf{x}} + \mathrm{div}\,(\dot{\varphi}_\mathrm{L}\mathbf{D}) - \dot{\varphi}_\mathrm{L}\mathrm{div}\,\mathbf{D}].\end{aligned} \tag{8.11}$$

The variation of

$$-\frac{1}{2}\varepsilon_0 J(\mathbf{F}^{-\mathrm{T}}\mathbf{E}_\mathrm{L}^\star) \cdot (\mathbf{F}^{-\mathrm{T}}\mathbf{E}_\mathrm{L}^\star)$$

is, with the help of (8.9) and the connection $\mathbf{E}_\mathrm{L}^\star = \mathbf{F}^\mathrm{T}\mathbf{E}^\star$,

$$\begin{aligned}&\frac{1}{2}\varepsilon_0 J[2\mathbf{E}^\star \cdot (\mathbf{L}^\mathrm{T}\mathbf{E}^\star) - (\mathbf{E}^\star \cdot \mathbf{E}^\star)\mathrm{tr}\,\mathbf{L} + 2\mathbf{E}^\star \cdot \mathrm{grad}\,\dot{\varphi}_\mathrm{L}] \\ &= J\,\mathrm{tr}\,(\boldsymbol{\tau}_\mathrm{e}^\star\mathbf{L}) + \varepsilon_0 J\mathbf{E}^\star \cdot \mathrm{grad}\,\dot{\varphi}_\mathrm{L} \\ &= J[\mathrm{div}\,(\boldsymbol{\tau}_\mathrm{e}^\star\dot{\mathbf{x}}) - (\mathrm{div}\,\boldsymbol{\tau}_\mathrm{e}^\star) \cdot \dot{\mathbf{x}} + \mathrm{div}\,(\dot{\varphi}_\mathrm{L}\mathbf{D}^\star) - \dot{\varphi}_\mathrm{L}\mathrm{div}\,\mathbf{D}^\star],\end{aligned} \tag{8.12}$$

where $\boldsymbol{\tau}_\mathrm{e}^\star$ is the Maxwell stress defined in (5.10) and we have set $\varepsilon_0\mathbf{E}^\star = \mathbf{D}^\star$.

On use of the divergence theorem, applied separately to $\mathcal{B}$ and $\mathcal{B}'$, the first variation of E can be arranged as

$$\begin{aligned}\dot{E} = &-\int_{\mathcal{B}}[(\operatorname{div}\boldsymbol{\tau})\cdot\dot{\mathbf{x}} + \dot{\varphi}_{\mathrm{L}}\operatorname{div}\mathbf{D}]\,\mathrm{d}v + \int_{\partial\mathcal{B}}[(\boldsymbol{\tau}\mathbf{n})\cdot\dot{\mathbf{x}} + \dot{\varphi}_{\mathrm{L}}\mathbf{D}\cdot\mathbf{n}]\,\mathrm{d}s \\ &-\int_{\mathcal{B}'}[(\operatorname{div}\boldsymbol{\tau}_{\mathrm{e}}^{\star})\cdot\dot{\mathbf{x}} + \dot{\varphi}_{\mathrm{L}}\operatorname{div}\mathbf{D}^{\star}]\,\mathrm{d}v + \int_{\partial\mathcal{B}'}[(\boldsymbol{\tau}_{\mathrm{e}}^{\star}\mathbf{n})\cdot\dot{\mathbf{x}} + \dot{\varphi}_{\mathrm{L}}\mathbf{D}^{\star}\cdot\mathbf{n}]\,\mathrm{d}s \\ &+\int_{\partial\mathcal{B}^{\infty}}(\mathbf{D}^{\star}-\mathbf{D}_{\mathrm{a}})\cdot\mathbf{n}\dot{\varphi}_{\mathrm{L}}\,\mathrm{d}s + \int_{\partial\mathcal{B}^{\infty}}(\boldsymbol{\tau}_{\mathrm{e}}^{\star}\mathbf{n})\cdot\dot{\mathbf{x}}\,\mathrm{d}s. \end{aligned}\tag{8.13}$$

Since $\dot{\mathbf{x}} = \mathbf{0}$ on $\partial\mathcal{B}^{\infty}$, the final term in the above can therefore be dropped. We also note that $\dot{\varphi}_{\mathrm{L}} = \dot{\varphi}$ on $\partial\mathcal{B}^{\infty}$, which has been used implicitly in arriving at the penultimate term in the above. Furthermore, the direction of the unit normal on $\partial\mathcal{B}'$ is opposite to that on $\partial\mathcal{B}$, so we may rewrite the above as

$$\begin{aligned}\dot{E} = &-\int_{\mathcal{B}}[(\operatorname{div}\boldsymbol{\tau})\cdot\dot{\mathbf{x}} + \dot{\varphi}_{\mathrm{L}}\operatorname{div}\mathbf{D}]\,\mathrm{d}v + \int_{\partial\mathcal{B}}(\boldsymbol{\tau}\mathbf{n}-\boldsymbol{\tau}_{\mathrm{e}}^{\star}\mathbf{n})\cdot\dot{\mathbf{x}}\,\mathrm{d}s \\ &-\int_{\partial\mathcal{B}}\dot{\varphi}_{\mathrm{L}}(\mathbf{D}^{\star}-\mathbf{D})\cdot\mathbf{n}\,\mathrm{d}s - \int_{\mathcal{B}'}[(\operatorname{div}\boldsymbol{\tau}_{\mathrm{e}}^{\star})\cdot\dot{\mathbf{x}} + \dot{\varphi}_{\mathrm{L}}\operatorname{div}\mathbf{D}^{\star}]\,\mathrm{d}v \\ &+\int_{\partial\mathcal{B}^{\infty}}(\mathbf{D}^{\star}-\mathbf{D}_{\mathrm{a}})\cdot\mathbf{n}\dot{\varphi}_{\mathrm{L}}\,\mathrm{d}s. \end{aligned}\tag{8.14}$$

Turning now to the variation of L, we obtain

$$\dot{L} = -\int_{\mathcal{B}_{\mathrm{r}}}\rho_{\mathrm{r}}\dot{\mathsf{V}}\,\mathrm{d}V + \int_{\partial\mathcal{B}_{\mathrm{r}}}\mathbf{t}_{\mathrm{A}}\cdot\dot{\mathbf{x}}\,\mathrm{d}S = \int_{\mathcal{B}}\rho\mathbf{f}\cdot\dot{\mathbf{x}}\,\mathrm{d}v + \int_{\partial\mathcal{B}}\mathbf{t}_{\mathrm{a}}\cdot\dot{\mathbf{x}}\,\mathrm{d}s. \tag{8.15}$$

Thus, $\dot{\Pi} = \dot{E} - \dot{L}$ is given by

$$\begin{aligned}\dot{\Pi} = &-\int_{\mathcal{B}}[(\operatorname{div}\boldsymbol{\tau} + \rho\mathbf{f})\cdot\dot{\mathbf{x}} + \dot{\varphi}_{\mathrm{L}}\operatorname{div}\mathbf{D}]\,\mathrm{d}v + \int_{\partial\mathcal{B}}(\boldsymbol{\tau}\mathbf{n}-\mathbf{t}_{\mathrm{a}}-\mathbf{t}_{\mathrm{e}}^{\star})\cdot\dot{\mathbf{x}}\,\mathrm{d}s \\ &-\int_{\partial\mathcal{B}}\dot{\varphi}_{\mathrm{L}}[\![\mathbf{D}]\!]\cdot\mathbf{n}\,\mathrm{d}s - \int_{\mathcal{B}'}[(\operatorname{div}\boldsymbol{\tau}_{\mathrm{e}}^{\star})\cdot\dot{\mathbf{x}} + \dot{\varphi}_{\mathrm{L}}\operatorname{div}\mathbf{D}^{\star}]\,\mathrm{d}v \\ &+\int_{\partial\mathcal{B}^{\infty}}(\mathbf{D}^{\star}-\mathbf{D}_{\mathrm{a}})\cdot\mathbf{n}\dot{\varphi}_{\mathrm{L}}\,\mathrm{d}s, \end{aligned}\tag{8.16}$$

where $\mathbf{t}_{\mathrm{e}}^{\star} = \boldsymbol{\tau}_{\mathrm{e}}^{\star}\mathbf{n}$ and $[\![\mathbf{D}]\!] = \mathbf{D}^{\star} - \mathbf{D}$, evaluated on $\partial\mathcal{B}$. This expression for $\dot{\Pi}$ could also be arrived at from (4.74) by appropriate rearrangement and manipulation. To illustrate this point, we shall carry out the corresponding rearrangement explicitly in the magnetoelastic context in Sect. 8.2.1.

Given the constitutive laws in $\mathcal{B}$, and the relation $\mathbf{D}^\star = \varepsilon_0 \mathbf{E}^\star$ in $\mathcal{B}'$, then, by the fundamental theorem of the calculus of variations, Π is stationary with respect to independent variations in $\mathbf{x}$ and φ_L such that $\dot{\mathbf{x}} = \mathbf{0}$ on $\partial\mathcal{B}^\infty$ if and only if the following hold:

(i)

$$\operatorname{div} \boldsymbol{\tau} + \rho \mathbf{f} = \mathbf{0} \quad \text{in} \quad \mathcal{B},$$

(ii)

$$\operatorname{div} \mathbf{D} = 0 \quad \text{in} \quad \mathcal{B},$$

(iii)

$$\boldsymbol{\tau}\mathbf{n} = \mathbf{t}_\mathrm{a} + \mathbf{t}_\mathrm{e}^\star \quad \text{on} \quad \partial\mathcal{B},$$

(iv)

$$[\![\mathbf{D}]\!] \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{B},$$

(v)

$$\operatorname{div} \boldsymbol{\tau}_\mathrm{e}^\star = \mathbf{0} \quad \text{in} \quad \mathcal{B}',$$

(vi)

$$\operatorname{div} \mathbf{D}^\star = 0 \quad \text{in} \quad \mathcal{B}',$$

(vii)

$$(\mathbf{D}^\star - \mathbf{D}_\mathrm{a}) \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{B}^\infty.$$

Note that (v) follows from $\operatorname{curl} \mathbf{E}^\star = \mathbf{0}$, equivalently $\operatorname{grad} \mathbf{E}^\star = (\operatorname{grad} \mathbf{E}^\star)^\mathrm{T}$, and (vi). The above variational statement can easily be extended to the situation where there are volumetric and surface charges, as shown by Bustamante et al. (2009a). The final result (vii) can be dropped if we assume that $\mathbf{E}^\star \cdot \mathbf{n} = \varepsilon_0^{-1} \mathbf{D}_\mathrm{a} \cdot \mathbf{n}$ on $\partial\mathcal{B}^\infty$ from the outset.

8.1.2 Formulation in Terms of the Electrostatic Vector Potential

In this section, along with the deformation $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ and its gradient $\mathbf{F}$, we work in terms of the vector potential $\mathbf{A}$ instead of the scalar potential. Then $\mathbf{D} = \operatorname{curl} \mathbf{A}$ in $\mathcal{B}$, with its counterpart $\mathbf{D}^\star = \operatorname{curl} \mathbf{A}$ in $\mathcal{B}'$. As for φ, we do not adopt a separate notation for $\mathbf{A}$ in $\mathcal{B}$ and $\mathcal{B}'$, and we take $\mathbf{A}$ to be continuous everywhere and its first derivatives such that $\mathbf{n}\cdot\mathrm{curl}\,\mathbf{A}$ is continuous across $\partial\mathcal{B}$, so that the jump condition $(5.8)_2$ is satisfied.

Following the pattern of Sect. 8.1.1, we start with the 'free energy' from Bustamante et al. (2009a), again with some differences of notation and in the subsequent analysis. We denote this by E^* and write it first in the Eulerian form

$$E^* = \int_{\mathcal{B}} \rho\psi^*\,\mathrm{d}v + \tfrac{1}{2}\varepsilon_0^{-1}\int_{\mathcal{B}'} \mathbf{D}^\star\cdot\mathbf{D}^\star\,\mathrm{d}v + \int_{\partial\mathcal{B}^\infty} (\mathbf{E}_\mathrm{a}\times\mathbf{A})\cdot\mathbf{n}\,\mathrm{d}s, \tag{8.17}$$

where $\psi^* = \psi^*(\mathbf{F},\mathbf{D})$, as defined in (4.65), and $\mathbf{E}_\mathrm{a}$ is an 'applied' field whose tangential component is prescribed on $\partial\mathcal{B}^\infty$. Note the difference between the asterisk on ψ and the star on $\mathbf{D}$.

We now define the Lagrangian version of $\mathbf{A}$ as $\mathbf{A}_\mathrm{L} = \mathbf{F}^\mathrm{T}\mathbf{A}$, so that, by using the identity $(3.16)_2$, we obtain

$$\mathbf{D}_\mathrm{L} = \mathrm{Curl}\,\mathbf{A}_\mathrm{L}. \tag{8.18}$$

Again we introduce a fictitious deformation field in $\mathcal{B}'$ so that we may define a Lagrangian version of $\mathbf{D}^\star$, namely $\mathbf{D}_\mathrm{L}^\star = J\mathbf{F}^{-1}\mathbf{D}^\star$, and write E^* as

$$E^*\{\mathbf{x},\mathbf{A}_\mathrm{L}\} = \int_{\mathcal{B}_\mathrm{r}} \Omega^*(\mathbf{F},\mathbf{D}_\mathrm{L})\,\mathrm{d}V + \tfrac{1}{2}\varepsilon_0^{-1}\int_{\mathcal{B}'_\mathrm{r}} J^{-1}(\mathbf{F}\mathbf{D}_\mathrm{L}^\star)\cdot(\mathbf{F}\mathbf{D}_\mathrm{L}^\star)\,\mathrm{d}V + \int_{\partial\mathcal{B}^\infty} (\mathbf{E}_\mathrm{a}\times\mathbf{A})\cdot\mathbf{n}\,\mathrm{d}s. \tag{8.19}$$

The final term is again left in Eulerian form since the boundary at infinity is fixed, but note that it contains $\mathbf{A}$ rather than $\mathbf{A}_\mathrm{L}$ in contrast to the situation in the preceding section where φ and φ_L could be used interchangeably on $\partial\mathcal{B}^\infty$ since $\dot{\mathbf{x}}$ is zero thereon. As we shall see shortly, this presents no problem.

We adopt $\Omega^*(\mathbf{F},\mathbf{D}_\mathrm{L})$ as the total energy density in $\mathcal{B}$, with $\mathbf{D}_\mathrm{L} = \mathrm{Curl}\,\mathbf{A}_\mathrm{L}$, and the constitutive laws

$$\mathbf{T} = \frac{\partial\Omega^*}{\partial\mathbf{F}}, \quad \boldsymbol{\tau} = J^{-1}\mathbf{F}\mathbf{T}, \quad \mathbf{E}_\mathrm{L} = \frac{\partial\Omega^*}{\partial\mathbf{D}_\mathrm{L}}, \quad \mathbf{E} = \mathbf{F}^{-\mathrm{T}}\mathbf{E}_\mathrm{L} \tag{8.20}$$

in $\mathcal{B}$, and $\mathbf{E}^\star = \varepsilon_0^{-1}\mathbf{D}^\star$ in $\mathcal{B}'$.

The variations $\dot{\mathbf{x}}$ and $\dot{\mathbf{A}}_\mathrm{L}$ are assumed to have sufficient regularity for the following analysis. In forming the variation of E^*, we shall use

$$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}, \quad \dot{J} = J\,\mathrm{tr}\,\mathbf{L}, \quad \dot{\mathbf{D}}_\mathrm{L} = \mathrm{Curl}\,\dot{\mathbf{A}}_\mathrm{L}, \quad \dot{\mathbf{A}}_\mathrm{L} = \mathbf{F}^\mathrm{T}(\dot{\mathbf{A}} + \mathbf{L}^\mathrm{T}\mathbf{A}), \tag{8.21}$$

where $\dot{\mathbf{A}}$ is the variation in the vector potential $\mathbf{A}$ at fixed $\mathbf{x}$ and $\mathbf{L}^\mathrm{T}\mathbf{A}$ is its variation induced by a variation in $\mathbf{x}$.

Similarly to Sect. 8.1.1, we obtain

$$
\begin{aligned}
\dot{\Omega}^* &= \mathrm{tr}\,(\mathbf{T}\dot{\mathbf{F}}) + \mathbf{E}_\mathrm{L} \cdot \mathrm{Curl}\,\dot{\mathbf{A}}_\mathrm{L} = J\,\mathrm{tr}\,(\boldsymbol{\tau}\mathbf{L}) + \mathbf{E} \cdot (\mathbf{F}\mathrm{Curl}\,\dot{\mathbf{A}}_\mathrm{L}) \\
&= J\,[\mathrm{tr}\,(\boldsymbol{\tau}\mathbf{L}) + \mathbf{E} \cdot \mathrm{curl}\,(\dot{\mathbf{A}} + \mathbf{L}^\mathrm{T}\mathbf{A})] \\
&= J\,\{\mathrm{div}\,(\boldsymbol{\tau}\dot{\mathbf{x}}) - (\mathrm{div}\,\boldsymbol{\tau}) \cdot \dot{\mathbf{x}} + (\dot{\mathbf{A}} + \mathbf{L}^\mathrm{T}\mathbf{A}) \cdot \mathrm{curl}\,\mathbf{E} - \mathrm{div}\,[\mathbf{E} \times (\dot{\mathbf{A}} + \mathbf{L}^\mathrm{T}\mathbf{A})]\},
\end{aligned} \tag{8.22}
$$

where we have used, respectively, $\mathbf{E}_\mathrm{L} = \mathbf{F}^\mathrm{T}\mathbf{E}$, the identity $(3.16)_2$ and the identity (A.17) from Appendix A.2.

Next, we take the variation of

$$
\frac{1}{2}\varepsilon_0^{-1} J^{-1} (\mathbf{F}\mathbf{D}_\mathrm{L}^\star) \cdot (\mathbf{F}\mathbf{D}_\mathrm{L}^\star) \tag{8.23}
$$

to give, on use of (8.21) and the connection $\mathbf{D}^\star = J^{-1}\mathbf{F}\mathbf{D}_\mathrm{L}^\star$,

$$
\begin{aligned}
&-\frac{1}{2}\varepsilon_0^{-1} J (\mathbf{D}^\star \cdot \mathbf{D}^\star)\mathrm{tr}\,\mathbf{L} + \varepsilon_0^{-1} J \mathbf{D}^\star \cdot (\mathbf{L}\mathbf{D}^\star) + \varepsilon_0^{-1}\mathbf{D}^\star \cdot (\mathbf{F}\mathrm{Curl}\,\dot{\mathbf{A}}_\mathrm{L}) \\
&= J\,[\mathrm{tr}\,(\boldsymbol{\tau}_\mathrm{e}^\star\mathbf{L}) + \varepsilon_0^{-1}\mathbf{D}^\star \cdot \mathrm{curl}\,(\dot{\mathbf{A}} + \mathbf{L}^\mathrm{T}\mathbf{A})] \\
&= J\,\{\mathrm{tr}\,(\boldsymbol{\tau}_\mathrm{e}^\star\mathbf{L}) + (\dot{\mathbf{A}} + \mathbf{L}^\mathrm{T}\mathbf{A}) \cdot \mathrm{curl}\,\mathbf{E}^\star - \mathrm{div}\,[\mathbf{E}^\star \times (\dot{\mathbf{A}} + \mathbf{L}^\mathrm{T}\mathbf{A})]\} \\
&= J\,\{\mathrm{div}\,(\boldsymbol{\tau}_\mathrm{e}^\star\dot{\mathbf{x}}) - (\mathrm{div}\,\boldsymbol{\tau}_\mathrm{e}^\star) \cdot \dot{\mathbf{x}} + (\dot{\mathbf{A}} + \mathbf{L}^\mathrm{T}\mathbf{A}) \cdot \mathrm{curl}\,\mathbf{E}^\star \\
&\qquad - \mathrm{div}\,[\mathbf{E}^\star \times (\dot{\mathbf{A}} + \mathbf{L}^\mathrm{T}\mathbf{A})]\},
\end{aligned} \tag{8.24}
$$

where we have set $\mathbf{E}^\star = \varepsilon_0^{-1}\mathbf{D}^\star$ and again used the identities $(3.16)_2$ and (A.17).

Substituting the expressions from (8.22) and (8.24) into the variation of E^*, we obtain, after an application of the divergence theorem to each of $\mathcal{B}$ and $\mathcal{B}'$,

$$
\begin{aligned}
\dot{E}^* = &- \int_{\mathcal{B}} [(\mathrm{div}\,\boldsymbol{\tau}) \cdot \dot{\mathbf{x}} - (\dot{\mathbf{A}} + \mathbf{L}^\mathrm{T}\mathbf{A}) \cdot \mathrm{curl}\,\mathbf{E}]\,\mathrm{d}v \\
&- \int_{\mathcal{B}'} [\mathrm{div}\,\boldsymbol{\tau}_\mathrm{e}^\star) \cdot \dot{\mathbf{x}} - (\dot{\mathbf{A}} + \mathbf{L}^\mathrm{T}\mathbf{A}) \cdot \mathrm{curl}\,\mathbf{E}^\star]\,\mathrm{d}v \\
&+ \int_{\partial\mathcal{B}} [(\boldsymbol{\tau}\mathbf{n}) \cdot \dot{\mathbf{x}} - (\dot{\mathbf{A}} + \mathbf{L}^\mathrm{T}\mathbf{A}) \cdot (\mathbf{n} \times \mathbf{E})]\,\mathrm{d}s \\
&+ \int_{\partial\mathcal{B}'} [(\boldsymbol{\tau}_\mathrm{e}^\star\mathbf{n}) \cdot \dot{\mathbf{x}} - (\dot{\mathbf{A}} + \mathbf{L}^\mathrm{T}\mathbf{A}) \cdot (\mathbf{n} \times \mathbf{E}^\star)]\,\mathrm{d}s \\
&- \int_{\partial\mathcal{B}^\infty} [\mathbf{n} \times (\mathbf{E}^\star - \mathbf{E}_\mathrm{a})] \cdot \dot{\mathbf{A}}\,\mathrm{d}s \\
&- \int_{\partial\mathcal{B}^\infty} (\mathbf{n} \times \mathbf{E}^\star) \cdot (\mathbf{L}^\mathrm{T}\mathbf{A})\,\mathrm{d}s + \int_{\partial\mathcal{B}^\infty} (\boldsymbol{\tau}_\mathrm{e}^\star\mathbf{n}) \cdot \dot{\mathbf{x}}\,\mathrm{d}s.
\end{aligned} \tag{8.25}
$$

The final term vanishes since, as before, we have $\dot{\mathbf{x}} = \mathbf{0}$ on $\partial\mathcal{B}^\infty$. The penultimate term also vanishes, as we now show.

Material line elements $\mathrm{d}\mathbf{X}$ and $\mathrm{d}\mathbf{x}$ in the reference and deformed configurations are connected by $\mathrm{d}\mathbf{x} = \mathbf{F}\mathrm{d}\mathbf{X}$. Taking the variation of this, we obtain $\dot{\overline{\mathrm{d}\mathbf{x}}} = \dot{\mathbf{F}}\mathrm{d}\mathbf{X} = \mathbf{L}\mathrm{d}\mathbf{x}$. Since the boundary $\partial\mathcal{B}^\infty$ is fixed, any line element $\mathrm{d}\mathbf{x}$ lying in $\partial\mathcal{B}^\infty$ has zero variation, and hence, $\mathbf{L}\mathrm{d}\mathbf{x} = \mathbf{0}$ for all line elements in $\partial\mathcal{B}^\infty$. This implies that $\mathbf{L}$ has the form $\mathbf{L} = \alpha\mathbf{n} \otimes \mathbf{n} + \beta\mathbf{n}^\perp \otimes \mathbf{n}$, for some scalars α and β, where $\mathbf{n}$ is a unit normal to $\partial\mathcal{B}^\infty$ and $\mathbf{n}^\perp$ is a vector perpendicular to $\mathbf{n}$. Hence, $\mathbf{L}^{\mathrm{T}}\mathbf{A} = [\alpha(\mathbf{n}\cdot\mathbf{A}) + \beta(\mathbf{n}^\perp\cdot\mathbf{A})]\mathbf{n}$, and taking the scalar product of this with $\mathbf{n}\times\mathbf{E}^\star$, we see that the integrand in the penultimate term vanishes.

Thus, on converting the surface integrals over $\partial\mathcal{B}'$ to integrals over $\partial\mathcal{B}$, we obtain

$$\begin{aligned}
\dot{E}^* = &-\int_{\mathcal{B}}[(\operatorname{div}\boldsymbol{\tau})\cdot\dot{\mathbf{x}} - (\dot{\mathbf{A}} + \mathbf{L}^{\mathrm{T}}\mathbf{A})\cdot\operatorname{curl}\mathbf{E}]\,\mathrm{d}v \\
&-\int_{\mathcal{B}'}[\operatorname{div}\boldsymbol{\tau}_{\mathrm{e}}^\star)\cdot\dot{\mathbf{x}} - (\dot{\mathbf{A}} + \mathbf{L}^{\mathrm{T}}\mathbf{A})\cdot\operatorname{curl}\mathbf{E}^\star]\,\mathrm{d}v \\
&+\int_{\partial\mathcal{B}}[(\boldsymbol{\tau}\mathbf{n} - \mathbf{t}_{\mathrm{e}}^\star)\cdot\dot{\mathbf{x}} + (\dot{\mathbf{A}} + \mathbf{L}^{\mathrm{T}}\mathbf{A})\cdot(\mathbf{n}\times[\![\mathbf{E}]\!])]\,\mathrm{d}s \\
&-\int_{\partial\mathcal{B}^\infty}[\mathbf{n}\times(\mathbf{E}^\star - \mathbf{E}_{\mathrm{a}})]\cdot\dot{\mathbf{A}}\,\mathrm{d}s,
\end{aligned} \tag{8.26}$$

where $\mathbf{t}_{\mathrm{e}}^\star = \boldsymbol{\tau}_{\mathrm{e}}^\star\mathbf{n}$ and $[\![\mathbf{E}]\!] = \mathbf{E}^\star - \mathbf{E}$, evaluated on $\partial\mathcal{B}$.

Accounting for the contribution of the mechanical loads, we define the total energy functional, denoted Π^*, by

$$\Pi^*\{\mathbf{x}, \mathbf{A}_{\mathrm{L}}\} = E^*\{\mathbf{x}, \mathbf{A}_{\mathrm{L}}\} - L\{\mathbf{x}\}, \tag{8.27}$$

and its variation is obtained as

$$\begin{aligned}
\dot{\Pi}^* = &-\int_{\mathcal{B}}[(\operatorname{div}\boldsymbol{\tau} + \rho\mathbf{f})\cdot\dot{\mathbf{x}} - (\dot{\mathbf{A}} + \mathbf{L}^{\mathrm{T}}\mathbf{A})\cdot\operatorname{curl}\mathbf{E}]\,\mathrm{d}v \\
&-\int_{\mathcal{B}'}[\operatorname{div}\boldsymbol{\tau}_{\mathrm{e}}^\star)\cdot\dot{\mathbf{x}} - (\dot{\mathbf{A}} + \mathbf{L}^{\mathrm{T}}\mathbf{A})\cdot\operatorname{curl}\mathbf{E}^\star]\,\mathrm{d}v \\
&+\int_{\partial\mathcal{B}}[(\boldsymbol{\tau}\mathbf{n} - \mathbf{t}_{\mathrm{a}} - \mathbf{t}_{\mathrm{e}}^\star)\cdot\dot{\mathbf{x}} + (\dot{\mathbf{A}} + \mathbf{L}^{\mathrm{T}}\mathbf{A})\cdot(\mathbf{n}\times[\![\mathbf{E}]\!])]\,\mathrm{d}s \\
&-\int_{\partial\mathcal{B}^\infty}[\mathbf{n}\times(\mathbf{E}^\star - \mathbf{E}_{\mathrm{a}})]\cdot\dot{\mathbf{A}}\,\mathrm{d}s.
\end{aligned} \tag{8.28}$$

Note that this could have been obtained by starting with (4.75) and manipulating it into the above form.

Given the constitutive laws in $\mathcal{B}$ and the relation $\mathbf{D}^\star = \varepsilon_0 \mathbf{E}^\star$ in $\mathcal{B}'$, we conclude that Π^* is stationary with respect to independent variations in $\mathbf{x}$ and $\mathbf{A}$, such that $\dot{\mathbf{x}} = \mathbf{0}$ on $\partial\mathbf{B}^\infty$, if and only if

(i)
$$\operatorname{div} \boldsymbol{\tau} + \rho \mathbf{f} = \mathbf{0} \quad \text{in} \quad \mathcal{B},$$

(ii)
$$\operatorname{curl} \mathbf{E} = \mathbf{0} \quad \text{in} \quad \mathcal{B},$$

(iii)
$$\boldsymbol{\tau}\mathbf{n} = \mathbf{t}_\mathrm{a} + \mathbf{t}_\mathrm{e}^\star \quad \text{on} \quad \partial\mathcal{B},$$

(iv)
$$\mathbf{n} \times [\![\mathbf{E}]\!] = \mathbf{0} \quad \text{on} \quad \partial\mathcal{B},$$

(v)
$$\operatorname{div} \boldsymbol{\tau}_\mathrm{e}^\star = \mathbf{0} \quad \text{in} \quad \mathcal{B}',$$

(vi)
$$\operatorname{curl} \mathbf{E}^\star = \mathbf{0} \quad \text{in} \quad \mathcal{B}',$$

(vii)
$$\mathbf{n} \times (\mathbf{E}^\star - \mathbf{E}_\mathrm{a}) = \mathbf{0} \quad \text{on} \quad \partial\mathcal{B}^\infty.$$

Note that since $\operatorname{div} \mathbf{D}^\star = \mathbf{0}$, (v) follows from (vi). Similarly to the previous subsection, if we assume $\mathbf{n} \times \mathbf{D}^\star = \varepsilon_0 \mathbf{n} \times \mathbf{E}_\mathrm{a}$ on $\partial\mathcal{B}^\infty$, the final result (vii) can be omitted.

8.2 Variational Formulations in Magnetoelasticity

In the context of nonlinear magnetoelastic interactions a variational principle based on the use of the magnetization as the independent magnetic variable along with the deformation function was provided by Brown (1965); see also Brown (1966). More recently this was developed by Kankanala and Triantafyllidis (2004), who use a magnetic vector potential as a third variable. A variational principle equivalent to that of Brown but based on the magnetic field rather than the magnetization was given by Steigmann (2004), and a different but essentially equivalent formulation is contained in the work of Ericksen (2006). An extensive list of references is contained in the recent paper by Vogel et al. (2013).

In this section, in parallel with the previous section, we develop the two variational principles for magnetoelastic problems following Bustamante et al. (2008) but with a somewhat different approach and some differences of notation. These are based on the magnetostatic scalar and vector potentials, and we deliberately do not use the magnetization vector.

8.2.1 Formulation in Terms of the Magnetostatic Scalar Potential

First we consider the energy E, which is a functional of the deformation $\boldsymbol{\chi}$ and the magnetic scalar potential φ, such that the magnetic field $\mathbf{H}$ is given by $\mathbf{H} = -\mathrm{grad}\,\varphi$. We use the same notations E and φ as for the electrostatic case since there is no danger of conflict in this section. Then, as given in Bustamante et al. (2008), E is defined as

$$E = \int_{\mathcal{B}} \rho\psi \,\mathrm{d}v - \frac{1}{2}\mu_0 \int_{\mathcal{B}'} \mathbf{H}^\star \cdot \mathbf{H}^\star \,\mathrm{d}v - \int_{\partial\mathcal{B}^\infty} \varphi \mathbf{B}_\mathrm{a} \cdot \mathbf{n}\,\mathrm{d}s, \tag{8.29}$$

where $\psi = \psi(\mathbf{F}, \mathbf{H})$ is the 'energy' potential defined in (6.50), $\mathcal{B}$ is the deformed configuration of the body, $\mathcal{B}'$ its exterior, $\partial\mathcal{B}^\infty$ the boundary at infinity, and $\mathbf{B}_\mathrm{a}$ is an applied magnetic induction whose normal component is prescribed on the latter boundary (equivalently, we may prescribe $\mathbf{H}_\mathrm{a} = \mu_0^{-1}\mathbf{B}_\mathrm{a}$). We take φ to be continuous across the boundary $\partial\mathcal{B}$ and continuously differentiable in each of $\mathcal{B}$ and $\mathcal{B}'$. Then, since $\mathrm{curl}\,\mathbf{H} = \mathbf{0}$ everywhere, $\mathbf{n} \times [\![\mathbf{H}]\!] = \mathbf{0}$, and hence the tangential derivative of φ is continuous across $\partial\mathcal{B}$.

Next we consider the Lagrangian version of (8.29) by introducing the scalar potential $\varphi_\mathrm{L}(\mathbf{X}) = \varphi(\boldsymbol{\chi}(\mathbf{X}))$, the Lagrangian magnetic field $\mathbf{H}_\mathrm{L} = \mathbf{F}^\mathrm{T}\mathbf{H}$, such that

$$\mathbf{H}_\mathrm{L} = -\mathrm{Grad}\,\varphi_\mathrm{L}, \tag{8.30}$$

and extending the deformation function $\boldsymbol{\chi}$ smoothly into the exterior region $\mathcal{B}'$, as in the previous section.

We now rewrite (8.29) with the first two integrals in Lagrangian form as

$$E\{\mathbf{x}, \varphi_\mathrm{L}\} = \int_{\mathcal{B}_\mathrm{r}} \Omega(\mathbf{F}, \mathbf{H}_\mathrm{L})\,\mathrm{d}V - \frac{1}{2}\mu_0 \int_{\mathcal{B}_\mathrm{r}'} J(\mathbf{F}^{-\mathrm{T}}\mathbf{H}_\mathrm{L}^\star) \cdot (\mathbf{F}^{-\mathrm{T}}\mathbf{H}_\mathrm{L}^\star)\,\mathrm{d}V \\ - \int_{\partial\mathcal{B}^\infty} \varphi \mathbf{B}_\mathrm{a} \cdot \mathbf{n}\,\mathrm{d}s, \tag{8.31}$$

where $\mathcal{B}'_{\mathrm{r}}$ is the exterior of $\mathcal{B}_{\mathrm{r}} \cup \partial\mathcal{B}_{\mathrm{r}}$. The boundary $\partial\mathcal{B}^\infty$ is fixed so that the final integral need not be converted to Lagrangian form.

At this point we show that this functional can be derived from the energy balance (6.83), which we write here as

$$\int_{\mathcal{B}_{\mathrm{r}}} (\rho_{\mathrm{r}}\mathbf{f}\cdot\dot{\mathbf{x}} - \mathbf{B}_{\mathrm{L}}\cdot\dot{\mathbf{H}}_{\mathrm{L}})\,\mathrm{d}V + \int_{\partial\mathcal{B}} (\boldsymbol{\tau}\mathbf{n})\cdot\dot{\mathbf{x}}\,\mathrm{d}s = \int_{\mathcal{B}_{\mathrm{r}}} \dot{\Omega}\,\mathrm{d}V, \tag{8.32}$$

where the virtual displacement $\mathbf{u}$ has been replaced by the variation $\dot{\mathbf{x}}$ and $\mathbf{T}^{\mathrm{T}}\mathbf{N}\,\mathrm{d}S$ by $\boldsymbol{\tau}\mathbf{n}\,\mathrm{d}s$. Next, we make use of the boundary condition (6.35) and assume again that the mechanical loads have the special forms embodied in (8.5), which enables the above to be arranged as

$$\int_{\mathcal{B}_{\mathrm{r}}} \dot{\Omega}\,\mathrm{d}V - \dot{L} + \int_{\mathcal{B}_{\mathrm{r}}} \mathbf{B}_{\mathrm{L}}\cdot\dot{\mathbf{H}}_{\mathrm{L}}\,\mathrm{d}V - \int_{\partial\mathcal{B}} \boldsymbol{\tau}_{\mathrm{m}}^{\star}\mathbf{n}\,\mathrm{d}s = 0, \tag{8.33}$$

where $\boldsymbol{\tau}_{\mathrm{m}}^{\star}$ is the Maxwell stress (6.10) on the outside of $\partial\mathcal{B}$. We now consider the second and third integrals in the above separately. For the following derivation we assume that all the equations of magnetoelastostatics are satisfied within both $\mathcal{B}_{\mathrm{r}}$ and $\mathcal{B}'_{\mathrm{r}}$ with the boundary conditions satisfied across $\partial\mathcal{B}_{\mathrm{r}}$.

By using $\dot{\mathbf{H}}_{\mathrm{L}} = -\mathrm{Grad}\,\dot{\varphi}_{\mathrm{L}}$ and $\mathrm{Div}\,\mathbf{B}_{\mathrm{L}} = 0$, applying the divergence theorem to convert the integral over $\mathcal{B}_{\mathrm{r}}$ to an integral over $\partial\mathcal{B}_{\mathrm{r}}$ and then using continuity of $\dot{\varphi}_{\mathrm{L}}$ and $\mathbf{B}_{\mathrm{L}}\cdot\mathbf{N}$ to write the surface integral as an integral over $\partial\mathcal{B}'_{\mathrm{r}}$, we obtain

$$\int_{\mathcal{B}_{\mathrm{r}}} \mathbf{B}_{\mathrm{L}}\cdot\dot{\mathbf{H}}_{\mathrm{L}}\,\mathrm{d}V = \int_{\partial\mathcal{B}'_{\mathrm{r}}} \dot{\varphi}_{\mathrm{L}}\mathbf{B}_{\mathrm{L}}^{\star}\cdot\mathbf{N}\,\mathrm{d}S. \tag{8.34}$$

Now, applying the divergence theorem on $\mathcal{B}'_{\mathrm{r}}$, writing $\dot{\mathbf{H}}^{\star} = -\mathrm{Grad}\,\dot{\varphi}_{\mathrm{L}}$ in $\mathcal{B}'_{\mathrm{r}}$ and using $\mathrm{Div}\,\mathbf{B}_{\mathrm{L}}^{\star} = 0$, the above becomes

$$\int_{\mathcal{B}_{\mathrm{r}}} \mathbf{B}_{\mathrm{L}}\cdot\dot{\mathbf{H}}_{\mathrm{L}}\,\mathrm{d}V = -\int_{\mathcal{B}'_{\mathrm{r}}} \mathbf{B}_{\mathrm{L}}^{\star}\cdot\dot{\mathbf{H}}_{\mathrm{L}}^{\star}\,\mathrm{d}V - \int_{\partial\mathcal{B}^\infty} \dot{\varphi}_{\mathrm{L}}\mathbf{B}^{\star}\cdot\mathbf{n}\,\mathrm{d}s. \tag{8.35}$$

The latter term has been put in Eulerian form, and we note that $\mathbf{B}^{\star}\cdot\mathbf{n} = \mathbf{B}_{\mathrm{a}}\cdot\mathbf{n}$ on $\partial\mathcal{B}^\infty$. Writing the relation $\mathbf{B}^{\star} = \mu_0\mathbf{H}^{\star}$ in terms of the Lagrangian fields, we obtain $\mathbf{B}_{\mathrm{L}}^{\star} = \mu_0 J\mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}}\mathbf{H}_{\mathrm{L}}^{\star}$, and the integrand in the first integral on the right-hand side of the above can be written

$$-\mathbf{B}_{\mathrm{L}}^{\star}\cdot\dot{\mathbf{H}}_{\mathrm{L}}^{\star} = -\mu_0 J(\mathbf{F}^{-\mathrm{T}}\mathbf{H}_{\mathrm{L}}^{\star})\cdot(\mathbf{F}^{-\mathrm{T}}\dot{\mathbf{H}}_{\mathrm{L}}^{\star}). \tag{8.36}$$

Turning next to the final integral in (8.33), we can write it as an integral over $\partial\mathcal{B}'$ since $\boldsymbol{\tau}_{\mathrm{m}}^{\star}$ is evaluated outside $\partial\mathcal{B}$, then apply the divergence theorem on $\mathcal{B}'$, use the fact that $\mathrm{div}\,\boldsymbol{\tau}_{\mathrm{m}}^{\star} = \mathbf{0}$ and take $\dot{\mathbf{x}} = \mathbf{0}$ on $\partial\mathcal{B}^\infty$ to arrive at

$$\int_{\mathcal{B}'} \mathrm{tr}\,(\boldsymbol{\tau}_{\mathrm{m}}^{\star}\mathbf{L})\,\mathrm{d}v = \int_{\mathcal{B}'_{\mathrm{r}}} J\,\mathrm{tr}\,(\boldsymbol{\tau}_{\mathrm{m}}^{\star}\mathbf{L})\,\mathrm{d}V. \tag{8.37}$$

On using the definition (6.10) of $\boldsymbol{\tau}_{\mathrm{m}}^{\star}$, the latter integrand becomes

$$J\,\mathrm{tr}\,(\boldsymbol{\tau}_{\mathrm{m}}^{\star}\mathbf{L}) = \mu_0 J\mathbf{H}^{\star}\cdot(\mathbf{L}^{\mathrm{T}}\mathbf{H}^{\star}) - \frac{1}{2}\mu_0 J(\mathbf{H}^{\star}\cdot\mathbf{H}^{\star})\mathrm{tr}\,\mathbf{L}, \tag{8.38}$$

and then by applying the formulas $\dot{J} = J\,\mathrm{tr}\,\mathbf{L}$ and $\mathbf{L}^{\mathrm{T}} = -\dot{\overline{\mathbf{F}^{-\mathrm{T}}}}\mathbf{F}^{\mathrm{T}}$ and the relation $\mathbf{H}_{\mathrm{L}}^{\star} = \mathbf{F}^{\mathrm{T}}\mathbf{H}^{\star}$, it can be arranged as

$$J\,\mathrm{tr}\,(\boldsymbol{\tau}_{\mathrm{m}}^{\star}\mathbf{L}) = -\mu_0 J(\mathbf{F}^{-\mathrm{T}}\mathbf{H}_{\mathrm{L}}^{\star})\cdot(\dot{\overline{\mathbf{F}^{-\mathrm{T}}}}\mathbf{H}_{\mathrm{L}}^{\star}) - \frac{1}{2}\mu_0\dot{J}(\mathbf{F}^{-\mathrm{T}}\mathbf{H}_{\mathrm{L}}^{\star})\cdot(\mathbf{F}^{-\mathrm{T}}\mathbf{H}_{\mathrm{L}}^{\star}). \tag{8.39}$$

The sum of the terms (8.36) and (8.39) can now be written

$$-\frac{1}{2}\mu_0\dot{\overline{J(\mathbf{F}^{-\mathrm{T}}\mathbf{H}_{\mathrm{L}}^{\star})\cdot(\mathbf{F}^{-\mathrm{T}}\mathbf{H}_{\mathrm{L}}^{\star})}} \tag{8.40}$$

and, noting that $\mathbf{B}^{\star}\cdot\mathbf{n} = \mathbf{B}_{\mathrm{a}}\cdot\mathbf{n}$ on $\partial\mathcal{B}^{\infty}$, and, since $\dot{\mathbf{x}} = \mathbf{0}$ on $\partial\mathcal{B}^{\infty}$, then $\dot{\varphi} = \dot{\varphi}_{\mathrm{L}}$ thereon, (8.33) can be written as $\dot{E} - \dot{L} = 0$, where E is given by (8.31).

This prompts the definition of the functional Π, analogously to (8.3) in the electroelasticity case (and in the same notation), as

$$\Pi\{\mathbf{x},\varphi_{\mathrm{L}}\} = E\{\mathbf{x},\varphi_{\mathrm{L}}\} - L\{\mathbf{x}\}. \tag{8.41}$$

This is now taken as the functional for the variational principle, with $\mathbf{x}$ and φ_{L} as the independent functions, and with $\mathbf{H}_{\mathrm{L}} = -\mathrm{Grad}\,\varphi_{\mathrm{L}}$. To establish the variational principle, the steps leading from (8.33) to the statement $\dot{E} - \dot{L} = 0$ are essentially reversed, but without the assumption that the divergence of the magnetic induction and that of the Maxwell stress vanish or that the normal component of the magnetic induction is continuous across the material boundary. Following the procedure that is essentially identical to that for electroelasticity, we obtain the variation

$$\begin{aligned}\dot{\Pi} = &-\int_{\mathcal{B}}[(\mathrm{div}\,\boldsymbol{\tau} + \rho\mathbf{f})\cdot\dot{\mathbf{x}} + \dot{\varphi}_{\mathrm{L}}\mathrm{div}\,\mathbf{B}]\,\mathrm{d}v + \int_{\partial\mathcal{B}}(\boldsymbol{\tau}\mathbf{n} - \mathbf{t}_{\mathrm{a}} - \mathbf{t}_{\mathrm{m}}^{\star})\cdot\dot{\mathbf{x}}\,\mathrm{d}s\\ &-\int_{\partial\mathcal{B}}\dot{\varphi}_{\mathrm{L}}[\![\mathbf{B}]\!]\cdot\mathbf{n}\,\mathrm{d}s - \int_{\mathcal{B}'}[(\mathrm{div}\,\boldsymbol{\tau}_{\mathrm{m}}^{\star})\cdot\dot{\mathbf{x}} + \dot{\varphi}_{\mathrm{L}}\mathrm{div}\,\mathbf{B}^{\star}]\,\mathrm{d}v\\ &+\int_{\partial\mathcal{B}^{\infty}}(\mathbf{B}^{\star} - \mathbf{B}_{\mathrm{a}})\cdot\mathbf{n}\dot{\varphi}_{\mathrm{L}}\,\mathrm{d}s,\end{aligned} \tag{8.42}$$

where $\boldsymbol{\tau} = J^{-1}\mathbf{F}\partial\Omega/\partial\mathbf{F}$, $\mathbf{B} = -J^{-1}\mathbf{F}\partial\Omega/\partial\mathbf{H}_{\mathrm{L}}$, $\mathbf{B}^{\star} = \mu_0\mathbf{H}^{\star}$ and $\mathbf{t}_{\mathrm{m}}^{\star} = \boldsymbol{\tau}_{\mathrm{m}}^{\star}\mathbf{n}$ with $\boldsymbol{\tau}_{\mathrm{m}}^{\star}$ given by (6.10).

We conclude that Π is stationary with respect to independent variations in $\mathbf{x}$ and φ_{L} such that $\dot{\mathbf{x}} = \mathbf{0}$ on $\partial\mathcal{B}^{\infty}$ if and only if

(i)

$$\operatorname{div}\boldsymbol{\tau} + \rho\mathbf{f} = \mathbf{0} \quad \text{in} \quad \mathcal{B},$$

(ii)

$$\operatorname{div}\mathbf{B} = 0 \quad \text{in} \quad \mathcal{B},$$

(iii)

$$\boldsymbol{\tau}\mathbf{n} = \mathbf{t}_\mathrm{a} + \mathbf{t}_\mathrm{m}^\star \quad \text{on} \quad \partial\mathcal{B},$$

(iv)

$$[\![\mathbf{B}]\!] \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{B},$$

(v)

$$\operatorname{div}\boldsymbol{\tau}_\mathrm{m}^\star = \mathbf{0} \quad \text{in} \quad \mathcal{B}',$$

(vi)

$$\operatorname{div}\mathbf{B}^\star = 0 \quad \text{in} \quad \mathcal{B}',$$

(vii)

$$(\mathbf{B}^\star - \mathbf{B}_\mathrm{a}) \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{B}^\infty.$$

Similarly to the electroelastic case, (v) follows from (vi) since $\operatorname{curl}\mathbf{H}^\star = \mathbf{0}$, and (vii) can be omitted if $\mathbf{H}^\star$ is assumed to satisfy $\mathbf{H}^\star \cdot \mathbf{n} = \mu_0^{-1}\mathbf{B}_\mathrm{a} \cdot \mathbf{n}$ on $\partial\mathcal{B}^\infty$.

8.2.2 Formulation in Terms of the Magnetostatic Vector Potential

In a similar way to the previous subsection, when we adopt the vector potential as our independent variable, we could start with the 'virtual' energy balance (6.84) in the form

$$\int_{\mathcal{B}_\mathrm{r}} (\rho_\mathrm{r}\mathbf{f} \cdot \dot{\mathbf{x}} + \mathbf{H}_\mathrm{L} \cdot \dot{\mathbf{B}}_\mathrm{L})\,\mathrm{d}V + \int_{\partial\mathcal{B}} (\boldsymbol{\tau}\mathbf{n}) \cdot \dot{\mathbf{x}}\,\mathrm{d}s = \int_{\mathcal{B}_\mathrm{r}} \dot{\Omega}^*\,\mathrm{d}V \tag{8.43}$$

and arrive at the counterpart of the functional (8.29) given by Bustamante et al. (2008); in the present notation, this is

$$E^* = \int_{\mathcal{B}} \rho\psi^* \,\mathrm{d}v + \frac{1}{2}\mu_0^{-1} \int_{\mathcal{B}'} \mathbf{B}\cdot\mathbf{B}\,\mathrm{d}v + \int_{\partial\mathcal{B}^\infty} (\mathbf{H}_\mathrm{a}\times\mathbf{A})\cdot\mathbf{n}\,\mathrm{d}s, \tag{8.44}$$

where $\psi^* = \psi^*(\mathbf{F},\mathbf{B})$ was defined in (6.44) and $\mathbf{x}$ and $\mathbf{A}$ are the independent functions, with $\mathbf{B} = \operatorname{curl}\mathbf{A}$, and the tangential component $\mathbf{n}\times\mathbf{H}_\mathrm{a}$ of the magnetic field is prescribed on $\partial\mathcal{B}^\infty$. We choose $\mathbf{A}$ so that $\mathbf{n}\cdot[\![\mathbf{B}]\!] = 0$ on $\partial\mathcal{B}$.

The Lagrangian equivalent of (8.44) is

$$E^*\{\mathbf{x},\mathbf{A}_\mathrm{L}\} = \int_{\mathcal{B}_\mathrm{r}} \Omega^*(\mathbf{F},\mathbf{B}_\mathrm{L})\,\mathrm{d}V + \frac{1}{2}\mu_0^{-1}\int_{\mathcal{B}'_\mathrm{r}} J^{-1}(\mathbf{F}\mathbf{B}_\mathrm{L})\cdot(\mathbf{F}\mathbf{B}_\mathrm{L})\,\mathrm{d}V + \int_{\partial\mathcal{B}^\infty}(\mathbf{H}_\mathrm{a}\times\mathbf{A})\cdot\mathbf{n}\,\mathrm{d}s, \tag{8.45}$$

where $\mathbf{x}$ and $\mathbf{A}_\mathrm{L} = \mathbf{F}^\mathrm{T}\mathbf{A}$ are the independent functions, with $\mathbf{B}_\mathrm{L} = \operatorname{Curl}\mathbf{A}_\mathrm{L}$. The final term is left in Eulerian form since the boundary at infinity is fixed.

From here on we follow exactly the same pattern as for the electroelastic case, so we do not repeat the details, just state the final result. We again define the functional required in the variational principle by $\Pi^*\{\mathbf{x},\mathbf{A}_\mathrm{L}\}$, so that

$$\Pi^*\{\mathbf{x},\mathbf{A}_\mathrm{L}\} = E^*\{\mathbf{x},\mathbf{A}_\mathrm{L}\} - L\{\mathbf{x}\}, \tag{8.46}$$

and hence,

$$\begin{aligned}\dot{\Pi}^* = &-\int_{\mathcal{B}}[(\operatorname{div}\boldsymbol{\tau} + \rho\mathbf{f})\cdot\dot{\mathbf{x}} - (\dot{\mathbf{A}} + \mathbf{L}^\mathrm{T}\mathbf{A})\cdot\operatorname{curl}\mathbf{H}]\,\mathrm{d}v\\ &-\int_{\mathcal{B}'}[\operatorname{div}\boldsymbol{\tau}_\mathrm{m}^\star)\cdot\dot{\mathbf{x}} - (\dot{\mathbf{A}} + \mathbf{L}^\mathrm{T}\mathbf{A})\cdot\operatorname{curl}\mathbf{H}^\star]\,\mathrm{d}v\\ &+\int_{\partial\mathcal{B}}[(\boldsymbol{\tau}\mathbf{n} - \mathbf{t}_\mathrm{a} - \mathbf{t}_\mathrm{m}^\star)\cdot\dot{\mathbf{x}} + (\dot{\mathbf{A}} + \mathbf{L}^\mathrm{T}\mathbf{A})\cdot(\mathbf{n}\times[\![\mathbf{H}]\!])]\,\mathrm{d}s\\ &-\int_{\partial\mathcal{B}^\infty}[\mathbf{n}\times(\mathbf{H}^\star - \mathbf{H}_\mathrm{a})]\cdot\dot{\mathbf{A}}\,\mathrm{d}s.\end{aligned} \tag{8.47}$$

We conclude that Π^* is stationary with respect to independent variations in $\mathbf{x}$ and $\mathbf{A}_\mathrm{L}$ if and only if

(i)

$$\operatorname{div}\boldsymbol{\tau} + \rho\mathbf{f} = \mathbf{0} \quad \text{in} \quad \mathcal{B},$$

(ii)

$$\operatorname{curl}\mathbf{H} = \mathbf{0} \quad \text{in} \quad \mathcal{B},$$

(iii)

$$\boldsymbol{\tau}\mathbf{n} = \mathbf{t}_{\mathrm{a}} + \mathbf{t}_{\mathrm{m}}^{\star} \quad \text{on} \quad \partial\mathcal{B},$$

(iv)

$$\mathbf{n} \times [\![\mathbf{H}]\!] = \mathbf{0} \quad \text{on} \quad \partial\mathcal{B},$$

(v)

$$\operatorname{div} \boldsymbol{\tau}_{\mathrm{m}}^{\star} = \mathbf{0} \quad \text{in} \quad \mathcal{B}',$$

(vi)

$$\operatorname{curl} \mathbf{H}^{\star} = \mathbf{0} \quad \text{in} \quad \mathcal{B}',$$

(vii)

$$\mathbf{n} \times (\mathbf{H}^{\star} - \mathbf{H}_{\mathrm{a}}) = \mathbf{0} \quad \text{on} \quad \partial\mathcal{B}^{\infty}.$$

Note that, since $\operatorname{div} \mathbf{B}^{\star} = \mathbf{0}$, (v) follows from (vi), and again (vii) may be omitted if initially we set $\mathbf{n} \times \mathbf{B}^{\star} = \mu_0 \mathbf{n} \times \mathbf{B}_{\mathrm{a}}$ on $\partial\mathcal{B}^{\infty}$.

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Chapter 9
Incremental Equations

Abstract In this chapter we focus on the equations governing linearized incremental deformations and electric or magnetic fields superimposed on a known finitely deformed configuration in which there is also a known electric or magnetic field. First we derive the general incremental equations based on Lagrangian variables and then specialize to the situation in which the underlying configuration is time independent, separately for the two cases of an accompanying electric field and an accompanying magnetic field. For the electroelastic theory, the formulation of the equations is based on either the electric field or the electric displacement, and for the magnetoelastic theory they are based on either the magnetic field or magnetic induction. In each case we derive expressions for the associated 'moduli' tensors (which depend in general on the underlying configuration) that provide the coupling between the incremental mechanical and electromagnetic effects. These consist of fourth-order tensors that generalize the elasticity tensor in conventional nonlinear elasticity, third-order tensors coupling the increments in the deformation and the electric (or magnetic) field increments and second-order tensors which generalize the permittivity and permeability tensors of the classical linear theory.

9.1 Continuum Electrodynamics

In this chapter the discussion focuses on the non-relativistic Galilean approximation of the governing equations of the motion of a continuum, for which the particle velocity has a magnitude that is negligible compared with the speed of light. The first part of this chapter is partly based on the papers by Otténio et al. (2008), Ogden (2009) and Dorfmann and Ogden (2010a).

Let the body be subjected to a *time-dependent deformation* under the combined action of mechanical loads and an electromagnetic field, so that at time t the particle $\mathbf{X}$ is located at position $\mathbf{x}$, which is given by $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$, where the vector

L. Dorfmann and R.W. Ogden, *Nonlinear Theory of Electroelastic and Magnetoelastic Interactions*, DOI 10.1007/978-1-4614-9596-3_9,

function $\boldsymbol{\chi}$ describes the motion and $\mathbf{x} \in \mathcal{B}_t$, where $\mathcal{B}_t$ is the image of the reference configuration $\mathcal{B}_\mathrm{r}$ under the motion. For each time t, $\boldsymbol{\chi}$ is a one-to-one mapping that satisfies suitable regularity requirements.

The velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ of the material particle $\mathbf{X}$ are defined by (3.39) and (3.40); we repeat them here for convenience as

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{x}_{,t}, \quad \mathbf{a}(\mathbf{x}, t) = \mathbf{v}_{,t} = \mathbf{x}_{,tt}, \tag{9.1}$$

and we recall that a subscript t following a comma signifies the material time derivative, i.e. the time derivative at fixed $\mathbf{X}$. The notation $\partial/\partial t$ will be reserved for the spatial time derivative, i.e. the time derivative at fixed $\mathbf{x}$.

We also recall from (3.46) that $\boldsymbol{\Gamma} = \operatorname{grad} \mathbf{v}$ denotes the velocity gradient and that the following standard kinematic identities hold:

$$\mathbf{F}_{,t} = \boldsymbol{\Gamma}\mathbf{F}, \quad \left(\mathbf{F}^{-1}\right)_{,t} = -\mathbf{F}^{-1}\boldsymbol{\Gamma}, \quad J_{,t} = J \operatorname{tr} \boldsymbol{\Gamma} = J \operatorname{div} \mathbf{v}. \tag{9.2}$$

9.1.1 Maxwell's Equations

The full Maxwell equations of electrodynamics in material matter are given in Eulerian form by (2.163) and (2.164). For convenience of reference we include them here as

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \operatorname{curl} \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}_\mathrm{f}, \quad \operatorname{div} \mathbf{D} = \rho_\mathrm{f}, \quad \operatorname{div} \mathbf{B} = 0, \tag{9.3}$$

where $\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}, \mathbf{J}_\mathrm{f}$ and ρ_f are now in general functions of both $\mathbf{x}$ and t. From $(9.3)_{2,3}$ the free charge conservation equation is obtained in the form

$$\operatorname{div} \mathbf{J}_\mathrm{f} + \frac{\partial \rho_\mathrm{f}}{\partial t} = 0. \tag{9.4}$$

We now aim to express these equations in Lagrangian form in terms of the Lagrangian fields given previously as

$$\mathbf{E}_\mathrm{L} = \mathbf{F}^\mathrm{T}\mathbf{E}, \quad \mathbf{H}_\mathrm{L} = \mathbf{F}^\mathrm{T}\mathbf{H}, \quad \mathbf{D}_\mathrm{L} = J\mathbf{F}^{-1}\mathbf{D}, \quad \mathbf{B}_\mathrm{L} = J\mathbf{F}^{-1}\mathbf{B}. \tag{9.5}$$

First, from $(9.3)_{3,4}$, we obtain

$$\operatorname{Div} \mathbf{D}_\mathrm{L} = \rho_\mathrm{F}, \quad \operatorname{Div} \mathbf{B}_\mathrm{L} = 0, \tag{9.6}$$

as in (4.51) and (6.63), respectively, where we recall that $\rho_\mathrm{F} = J\rho_\mathrm{f}$. Next, we re-cast $(9.3)_{1,2}$ as

$$\operatorname{curl}(\mathbf{E}+\mathbf{v}\times\mathbf{B}) = -\frac{\partial\mathbf{B}}{\partial t} + \operatorname{curl}(\mathbf{v}\times\mathbf{B}), \tag{9.7}$$

$$\operatorname{curl}(\mathbf{H}-\mathbf{v}\times\mathbf{D}) = \frac{\partial\mathbf{D}}{\partial t} - \operatorname{curl}(\mathbf{v}\times\mathbf{D}) + (\operatorname{div}\mathbf{D})\mathbf{v} + \mathbf{J}_{\mathrm{f}} - \rho_{\mathrm{f}}\mathbf{v}, \tag{9.8}$$

and take note of the identities

$$\operatorname{curl}(\mathbf{v}\times\mathbf{B}) = \boldsymbol{\Gamma}\mathbf{B} - (\operatorname{div}\mathbf{v})\mathbf{B} - (\mathbf{v}\cdot\operatorname{grad})\mathbf{B}, \tag{9.9}$$

$$\operatorname{curl}(\mathbf{v}\times\mathbf{D}) = \boldsymbol{\Gamma}\mathbf{D} - (\operatorname{div}\mathbf{v})\mathbf{D} - (\mathbf{v}\cdot\operatorname{grad})\mathbf{D} + (\operatorname{div}\mathbf{D})\mathbf{v}, \tag{9.10}$$

in the first of which $(9.3)_1$ has been used, and the formula

$$\mathbf{B}_{\mathrm{L},t} = J\mathbf{F}^{-1}[\mathbf{B}_{,t} - \boldsymbol{\Gamma}\mathbf{B} + (\operatorname{div}\mathbf{v})\mathbf{B}], \tag{9.11}$$

with a corresponding formula for $\mathbf{D}_{\mathrm{L},t}$. Equations (9.7) and (9.8) may now be rewritten as

$$\operatorname{curl}(\mathbf{E}+\mathbf{v}\times\mathbf{B}) = -J^{-1}\mathbf{F}\mathbf{B}_{\mathrm{L},t}, \tag{9.12}$$

$$\operatorname{curl}(\mathbf{H}-\mathbf{v}\times\mathbf{D}) = J^{-1}\mathbf{F}\mathbf{D}_{\mathrm{L},t} + \mathbf{J}_{\mathrm{f}} - \rho_{\mathrm{f}}\mathbf{v}. \tag{9.13}$$

We are now in a position to obtain boundary conditions analogous to those obtained in Sect. 2.5 for the case in which the boundary is fixed in space. Now we allow the boundary to be moving with the material. Then, the same method as used in Sect. 2.5 can be applied to (9.12) and (9.13), which leads to

$$\mathbf{n}\times[\![\mathbf{E}+\mathbf{v}\times\mathbf{B}]\!] = \mathbf{0}, \quad \mathbf{n}\times[\![\mathbf{H}-\mathbf{v}\times\mathbf{D}]\!] = \mathbf{K}_{\mathrm{f}} - \sigma_{\mathrm{f}}\mathbf{v}_{\mathrm{s}}, \tag{9.14}$$

while $(9.3)_{3,4}$ yield

$$\mathbf{n}\cdot[\![\mathbf{D}]\!] = \sigma_{\mathrm{f}}, \quad \mathbf{n}\cdot[\![\mathbf{B}]\!] = 0, \tag{9.15}$$

as previously, where we have introduced the notation $\mathbf{v}_{\mathrm{s}}$ for the value of the velocity $\mathbf{v}$ on the surface, and we recall that $\mathbf{K}_{\mathrm{f}}$ and σ_{f} are the free surface current and charge densities.

We now express the left-hand sides of these equations in terms of Lagrangian variables, by first introducing a Lagrangian counterpart $\mathbf{V}$ of $\mathbf{v}$, defined by $\mathbf{V} = \mathbf{F}^{-1}\mathbf{v}$. This requires the vector identity

$$\mathbf{F}^{\mathrm{T}}(\mathbf{v}\times\mathbf{B}) = J^{-1}\mathbf{F}^{\mathrm{T}}[(\mathbf{F}\mathbf{V})\times(\mathbf{F}\mathbf{B}_{\mathrm{L}})] = \mathbf{V}\times\mathbf{B}_{\mathrm{L}}, \tag{9.16}$$

and similarly for $\mathbf{D}$, followed by the identity

$$\operatorname{curl}(\mathbf{v}\times\mathbf{B}) = J^{-1}\mathbf{F}\operatorname{Curl}\left[\mathbf{F}^{\mathrm{T}}(\mathbf{v}\times\mathbf{B})\right], \tag{9.17}$$

obtained from $(3.16)_2$ with $\mathbf{A} = \mathbf{F}^{\mathrm{T}}(\mathbf{v} \times \mathbf{B})$, and a counterpart expression for $\mathbf{D}$. Equations (9.12) and (9.13) can then be re-cast entirely in terms of Lagrangian variables as

$$\mathrm{Curl}\,(\mathbf{E}_{\mathrm{L}} + \mathbf{V} \times \mathbf{B}_{\mathrm{L}}) = -\mathbf{B}_{\mathrm{L},t}, \tag{9.18}$$

$$\mathrm{Curl}\,(\mathbf{H}_{\mathrm{L}} - \mathbf{V} \times \mathbf{D}_{\mathrm{L}}) = \mathbf{D}_{\mathrm{L},t} + \mathbf{J}_{\mathrm{F}}, \tag{9.19}$$

respectively, where we have introduced the notation $\mathbf{J}_{\mathrm{F}}$ defined by

$$\mathbf{J}_{\mathrm{F}} = J\mathbf{F}^{-1}(\mathbf{J}_{\mathrm{f}} - \rho_{\mathrm{f}}\mathbf{v}) = J\mathbf{F}^{-1}\mathbf{J}_{\mathrm{f}} - \rho_{\mathrm{F}}\mathbf{V}, \tag{9.20}$$

which generalizes the $\mathbf{J}_{\mathrm{F}}$ introduced in (6.62) to the dynamic situation.

It can then be shown that charge conservation (9.4) takes on the Lagrangian form

$$\mathrm{Div}\,\mathbf{J}_{\mathrm{F}} + \rho_{\mathrm{F},t} = 0. \tag{9.21}$$

We now summarize the Lagrangian forms of Maxwell's equations as

$$\mathrm{Curl}\,(\mathbf{E}_{\mathrm{L}} + \mathbf{V} \times \mathbf{B}_{\mathrm{L}}) = -\mathbf{B}_{\mathrm{L},t}, \quad \mathrm{Div}\,\mathbf{D}_{\mathrm{L}} = \rho_{\mathrm{F}}, \tag{9.22}$$

$$\mathrm{Curl}\,(\mathbf{H}_{\mathrm{L}} - \mathbf{V} \times \mathbf{D}_{\mathrm{L}}) = \mathbf{D}_{\mathrm{L},t} + \mathbf{J}_{\mathrm{F}}, \quad \mathrm{Div}\,\mathbf{B}_{\mathrm{L}} = 0. \tag{9.23}$$

In various notations these Lagrangian forms of the equations have been given in many publications, dating back at least to McCarthy (1967, 1968); see also Lax and Nelson (1976), Maugin (1988) and Eringen and Maugin (1990), for example.

The boundary conditions associated with (9.18) and (9.19) are obtained following steps similar to those outlined in Sect. 2.5 or directly from the Eulerian counterparts. These are

$$\mathbf{N} \times [\![\mathbf{E}_{\mathrm{L}} + \mathbf{V} \times \mathbf{B}_{\mathrm{L}}]\!] = \mathbf{0}, \tag{9.24}$$

$$\mathbf{N} \times [\![\mathbf{H}_{\mathrm{L}} - \mathbf{V} \times \mathbf{D}_{\mathrm{L}}]\!] = \mathbf{K}_{\mathrm{F}} - \sigma_{\mathrm{F}}\mathbf{V}_{\mathrm{s}}, \tag{9.25}$$

where $\mathbf{K}_{\mathrm{F}} = \mathbf{F}^{-1}\mathbf{K}_{\mathrm{f}}\mathrm{d}s/\mathrm{d}S$, i.e. the surface current per unit reference area. Similarly, $\sigma_{\mathrm{F}} = \sigma_{\mathrm{f}}\mathrm{d}s/\mathrm{d}S$ with σ_{f} given by $(2.167)_2$ and $\mathbf{V}_{\mathrm{s}} = \mathbf{F}^{-1}\mathbf{v}_{\mathrm{s}}$.

The boundary conditions associated with (9.6) are

$$\mathbf{N} \cdot [\![\mathbf{D}_{\mathrm{L}}]\!] = \sigma_{\mathrm{F}}, \quad \mathbf{N} \cdot [\![\mathbf{B}_{\mathrm{L}}]\!] = 0. \tag{9.26}$$

9.1.2 Equations of Motion

The equation governing the motion of a continuous material in the presence of an electromagnetic field can be expressed in many different forms, depending on the particular model used for the electromagnetic force, as can be seen from Chaps. 4

and 6 in the electroelastic and magnetoelastic cases, respectively. The equation of motion can always be written in the general form

$$\operatorname{div}\boldsymbol{\sigma} + \rho\mathbf{f} + \mathbf{f}^{\mathrm{em}} = \rho\mathbf{a}, \tag{9.27}$$

where $\boldsymbol{\sigma}$ is a Cauchy-like stress tensor, ρ is the mass density of the material (in the current configuration), $\mathbf{f}$ is the mechanical body force (per unit mass), $\mathbf{f}^{\mathrm{em}}$ is the electromagnetic 'body force' (per unit volume in $\mathcal{B}_t$) and $\mathbf{a}$ denotes the acceleration. For comparisons of alternative models and the associated governing equations, we refer to, for example, Pao (1978), Maugin (1988) and Hutter et al. (2006). Unlike its static counterparts $\mathbf{f}^{\mathrm{em}}$ cannot in general be written as the divergence of a second-order (stress) tensor, but it can be written as

$$\mathbf{f}^{\mathrm{em}} = \operatorname{div}\boldsymbol{\sigma}^{\mathrm{em}} - \frac{\partial \mathbf{G}}{\partial t}, \tag{9.28}$$

where $\boldsymbol{\sigma}^{\mathrm{em}}$ is an electromagnetic stress tensor and $\mathbf{G}$ is an electromagnetic momentum vector (see, e.g., Maugin and Eringen 1977).

We shall not require the most general form of the equation of motion in what follows since we shall consider mainly situations in which one of the electric and magnetic fields is absent. Then, we may take $\partial\mathbf{G}/\partial t = \mathbf{0}$ here and write (9.27) in the simple form

$$\operatorname{div}\boldsymbol{\tau} + \rho\mathbf{f} = \rho\mathbf{a}, \tag{9.29}$$

where $\boldsymbol{\tau} = \boldsymbol{\sigma} + \boldsymbol{\sigma}^{\mathrm{em}}$ is the total stress tensor, which, as in the static situation, is symmetric in the absence of intrinsic mechanical couples. The corresponding Lagrangian form of the equation is

$$\operatorname{Div}\mathbf{T} + \rho_{\mathrm{r}}\mathbf{f} = \rho_{\mathrm{r}}\mathbf{a}, \tag{9.30}$$

where $\rho_{\mathrm{r}} = \rho J$ is the mass density in $\mathcal{B}_{\mathrm{r}}$ and, as for the static situation, $\mathbf{T}$ may be defined via the connection $\mathbf{T} = J\mathbf{F}^{-1}\boldsymbol{\tau}$.

The associated traction boundary conditions are

$$[\![\boldsymbol{\tau}]\!]\mathbf{n} = \mathbf{0}, \quad [\![\mathbf{T}^{\mathrm{T}}]\!]\mathbf{N} = \mathbf{0}, \tag{9.31}$$

which include both the applied mechanical traction and electromagnetic tractions acting on the exterior boundary of the body.

9.2 Incremental Equations

Suppose that the motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ is known, along with the electromagnetic fields. Now let an incremental motion be superimposed on $\boldsymbol{\chi}(\mathbf{X}, t)$, which we denote by $\dot{\mathbf{x}} = \dot{\boldsymbol{\chi}}(\mathbf{X}, t)$, where the displacement $\dot{\mathbf{x}}$ is 'small'. Here and henceforth a superimposed dot represents an increment in the quantity concerned.

Through the motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ we may identify $\dot{\mathbf{x}}$ with its Eulerian equivalent, which we denote by $\mathbf{u}(\mathbf{x}, t)$, i.e. $\mathbf{u}(\boldsymbol{\chi}(\mathbf{X}, t), t) = \dot{\mathbf{x}}(\mathbf{X}, t)$. Then, $\mathbf{u}_{,t} = \dot{\mathbf{x}}_{,t} = \dot{\mathbf{v}}$, where $\mathbf{v}$ is the particle velocity. Recalling $(9.2)_1$, we then obtain the formulas

$$\dot{\mathbf{F}} = \mathbf{LF}, \quad \dot{\mathbf{F}}_{,t} = (\operatorname{grad}\dot{\mathbf{v}})\mathbf{F}, \quad \operatorname{grad}\dot{\mathbf{v}} = \dot{\boldsymbol{\Gamma}} + \boldsymbol{\Gamma}\mathbf{L} = \mathbf{L}_{,t} + \mathbf{L}\boldsymbol{\Gamma}, \tag{9.32}$$

where $\mathbf{L} = \operatorname{grad}\mathbf{u}$; see Chap. 3 for a discussion of the motion of a continuum. Note that it follows from the final equality in (9.32) that $\operatorname{tr}(\dot{\boldsymbol{\Gamma}}) = \operatorname{tr}(\mathbf{L}_{,t})$. We also require the formulas

$$\overline{(\mathbf{F}^{-1})}^{\boldsymbol{\cdot}} = -\mathbf{F}^{-1}\mathbf{L}, \quad \dot{J} = J\operatorname{div}\mathbf{u}. \tag{9.33}$$

With $\mathbf{V} = \mathbf{F}^{-1}\mathbf{v}$ we then obtain

$$\dot{\mathbf{V}} = \mathbf{F}^{-1}(\dot{\mathbf{v}} - \mathbf{L}\mathbf{v}), \tag{9.34}$$

while the material time derivative of $(9.33)_2$ leads to

$$\dot{J}_{,t} = J\operatorname{div}\dot{\mathbf{v}} + J(\operatorname{div}\mathbf{u})(\operatorname{div}\mathbf{v}) - J\operatorname{tr}(\boldsymbol{\Gamma}\mathbf{L}). \tag{9.35}$$

For an incompressible material

$$\operatorname{div}\mathbf{u} = 0, \quad \operatorname{div}\mathbf{v} = 0 \tag{9.36}$$

and

$$\operatorname{div}\dot{\mathbf{v}} = \operatorname{tr}(\boldsymbol{\Gamma}\mathbf{L}). \tag{9.37}$$

By taking the increments of (9.18), (9.19) and (9.6) we obtain the incremental forms of Maxwell's equations. These are

$$\operatorname{Curl}(\dot{\mathbf{E}}_{\mathrm{L}} + \mathbf{V}\times\dot{\mathbf{B}}_{\mathrm{L}} + \dot{\mathbf{V}}\times\mathbf{B}_{\mathrm{L}}) = -\dot{\mathbf{B}}_{\mathrm{L},t}, \tag{9.38}$$

$$\operatorname{Curl}(\dot{\mathbf{H}}_{\mathrm{L}} - \mathbf{V}\times\dot{\mathbf{D}}_{\mathrm{L}} - \dot{\mathbf{V}}\times\mathbf{D}_{\mathrm{L}}) = \dot{\mathbf{D}}_{\mathrm{L},t} + \dot{\mathbf{J}}_{\mathrm{F}}, \tag{9.39}$$

$$\operatorname{Div}\dot{\mathbf{D}}_{\mathrm{L}} = \dot{\rho}_{\mathrm{F}}, \quad \operatorname{Div}\dot{\mathbf{B}}_{\mathrm{L}} = 0. \tag{9.40}$$

The incremental form of (9.21) follows as

$$\operatorname{Div}\dot{\mathbf{J}}_{\mathrm{F}} + \dot{\rho}_{\mathrm{F},t} = 0. \tag{9.41}$$

Similarly, by forming the increment of (9.30), we obtain the incremental equation of motion

$$\operatorname{Div}\dot{\mathbf{T}} + \rho_r\dot{\mathbf{f}} = \rho_r\mathbf{u}_{,tt}. \tag{9.42}$$

For the development that follows, it is convenient to collect the relations (4.50)$_{1,2}$, (4.53) (equivalently (6.65)) and (6.61) in the form

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\mathbf{T}, \quad \mathbf{B} = J^{-1}\mathbf{F}\mathbf{B}_{\mathrm{L}}, \quad \mathbf{D} = J^{-1}\mathbf{F}\mathbf{D}_{\mathrm{L}}, \quad \mathbf{E} = \mathbf{F}^{-\mathrm{T}}\mathbf{E}_{\mathrm{L}}, \quad \mathbf{H} = \mathbf{F}^{-\mathrm{T}}\mathbf{H}_{\mathrm{L}}, \tag{9.43}$$

which are push forward versions of the Lagrangian variables $\mathbf{T}$, $\mathbf{B}_{\mathrm{L}}$, $\mathbf{D}_{\mathrm{L}}$, $\mathbf{E}_{\mathrm{L}}$, $\mathbf{H}_{\mathrm{L}}$, respectively. The corresponding push forward versions of the increments $\dot{\mathbf{T}}, \dot{\mathbf{B}}_{\mathrm{L}}, \dot{\mathbf{D}}_{\mathrm{L}}, \dot{\mathbf{E}}_{\mathrm{L}}, \dot{\mathbf{H}}_{\mathrm{L}}$ are then defined by

$$\dot{\mathbf{T}}_0 = J^{-1}\mathbf{F}\dot{\mathbf{T}}, \quad \dot{\mathbf{B}}_{\mathrm{L}0} = J^{-1}\mathbf{F}\dot{\mathbf{B}}_{\mathrm{L}}, \quad \dot{\mathbf{D}}_{\mathrm{L}0} = J^{-1}\mathbf{F}\dot{\mathbf{D}}_{\mathrm{L}},$$
$$\dot{\mathbf{E}}_{\mathrm{L}0} = \mathbf{F}^{-\mathrm{T}}\dot{\mathbf{E}}_{\mathrm{L}}, \quad \dot{\mathbf{H}}_{\mathrm{L}0} = \mathbf{F}^{-\mathrm{T}}\dot{\mathbf{H}}_{\mathrm{L}}, \tag{9.44}$$

where the subscript 0 is used to indicate the push forward operation. Equally, $\dot{\mathbf{T}}_0$ can be thought of as the value of $\dot{\mathbf{T}}$ when the reference configuration is updated from $\mathcal{B}_{\mathrm{r}}$ to $\mathcal{B}_t$ after the increments are formed and similarly for the other terms in (9.44). We also note that $\dot{\mathbf{F}}_0 = \dot{\mathbf{F}}\mathbf{F}^{-1} = \mathbf{L}$.

The updated versions of (9.38)–(9.40) and (9.42) are

$$\operatorname{curl}[\dot{\mathbf{E}}_{\mathrm{L}0} + \mathbf{v} \times \dot{\mathbf{B}}_{\mathrm{L}0} + (\dot{\mathbf{v}} - \mathbf{L}\mathbf{v}) \times \mathbf{B}] = -\dot{\mathbf{B}}_{\mathrm{L},t0}, \tag{9.45}$$

$$\operatorname{curl}[\dot{\mathbf{H}}_{\mathrm{L}0} - \mathbf{v} \times \dot{\mathbf{D}}_{\mathrm{L}0} - (\dot{\mathbf{v}} - \mathbf{L}\mathbf{v}) \times \mathbf{D}] = \dot{\mathbf{D}}_{\mathrm{L},t0} + \dot{\mathbf{J}}_{\mathrm{F}0}, \tag{9.46}$$

$$\operatorname{div}\dot{\mathbf{B}}_{\mathrm{L}0} = 0, \quad \operatorname{div}\dot{\mathbf{D}}_{\mathrm{L}0} = \dot{\rho}_{\mathrm{F}0}, \tag{9.47}$$

and

$$\operatorname{div}\dot{\mathbf{T}}_0 + \rho\dot{\mathbf{f}} = \rho\mathbf{u}_{,tt}, \tag{9.48}$$

respectively. For an incompressible material the restrictions (9.36) and (9.37) hold.

Note that in (9.45) the term $\dot{\mathbf{B}}_{\mathrm{L},t0}$ is in general different from $\dot{\mathbf{B}}_{\mathrm{L}0,t}$, i.e. updating (pushing forward) and material time differentiation do not in general commute. Indeed, it is easy to show that

$$\dot{\mathbf{B}}_{\mathrm{L}0,t} - \dot{\mathbf{B}}_{\mathrm{L},t0} = \boldsymbol{\Gamma}\dot{\mathbf{B}}_{\mathrm{L}0} - (\operatorname{div}\mathbf{v})\dot{\mathbf{B}}_{\mathrm{L}0}, \tag{9.49}$$

with a similar formula pertaining to $\mathbf{D}$ in (9.46). Note that the right-hand side of (9.49) vanishes if the underlying configuration $\mathcal{B}_t$ is purely static, so that, in particular, $\mathbf{v} = \mathbf{0}$, $\boldsymbol{\Gamma} = \mathbf{O}$ and $J_{,t} = 0$, and the simplifications

$$\operatorname{grad}\dot{\mathbf{v}} = \dot{\boldsymbol{\Gamma}}, \quad \dot{J}_{,t} = J\operatorname{div}\dot{\mathbf{v}}, \quad \dot{\mathbf{v}} = \mathbf{F}\dot{\mathbf{V}} \tag{9.50}$$

follow from (9.32), (9.34) and (9.35). This is the special case on which we focus from now on.

9.3 Incremental Deformations

Henceforth we consider the specialization in which the underlying fields and deformation are purely static, so that $\mathbf{E}, \mathbf{D}, \mathbf{H}, \mathbf{B}$ are independent of t, while $\mathbf{v} = \mathbf{0}$ and $\boldsymbol{\Gamma} = \mathbf{O}$. We also assume that there are no free charges or currents in this configuration, so that $\rho_{\mathrm{f}} = 0$ and $\mathbf{J}_{\mathrm{f}} = \mathbf{0}$.

Incremental forms of the governing equations are needed, in particular, for analyzing the stability of equilibrium configurations subject to electric and/or magnetic fields and to investigate infinitesimal vibrations and the propagation of small amplitude waves. In the remainder of this chapter we restrict attention to the static theory governing the equilibrium of an incremental deformation combined with an increment in the electric or the magnetic field superimposed on the underlying configuration with the aim of establishing the incremental equations, boundary conditions and constitutive laws. To avoid undue complication we deal separately with the cases in which a finite deformation is accompanied by either an electrostatic field or a magnetostatic field, but not both combined. First we consider the electroelastic coupling.

9.3.1 Incremental Electroelasticity

In Chap. 4 we summarized some of the energy functionals that can be used to derive constitutive equations for electroelastic materials. Each gives a different electric body force and associated Cauchy and Maxwell stresses, which can be included in the equilibrium equations.

We showed that the total nominal stress tensor $\mathbf{T}$ and the Lagrangian electric displacement have very simple expressions based on the energy function $\Omega = \Omega(\mathbf{F}, \mathbf{E}_{\mathrm{L}})$. For an unconstrained material, we have

$$\mathbf{T} = \frac{\partial \Omega}{\partial \mathbf{F}}, \quad \mathbf{D}_{\mathrm{L}} = -\frac{\partial \Omega}{\partial \mathbf{E}_{\mathrm{L}}}. \tag{9.51}$$

On the other hand, if we select $\mathbf{D}_{\mathrm{L}}$ as the independent electric variable the energy formulation $\Omega = \Omega(\mathbf{F}, \mathbf{E}_{\mathrm{L}})$ may be replaced by $\Omega^* = \Omega^*(\mathbf{F}, \mathbf{D}_{\mathrm{L}})$, and then we have

$$\mathbf{T} = \frac{\partial \Omega^*}{\partial \mathbf{F}}, \quad \mathbf{E}_{\mathrm{L}} = \frac{\partial \Omega^*}{\partial \mathbf{D}_{\mathrm{L}}}. \tag{9.52}$$

For an incompressible material the constraint $\det \mathbf{F} = 1$ must be satisfied, and the expressions for the total nominal stress given by $(9.51)_1$ and $(9.52)_1$ in terms of Ω and Ω^*, respectively, are amended as in Sect. 4.4.4; for convenience, we include their expressions here as

$$\mathbf{T} = \frac{\partial \Omega}{\partial \mathbf{F}} - p\mathbf{F}^{-1}, \quad \mathbf{T} = \frac{\partial \Omega^*}{\partial \mathbf{F}} - p^*\mathbf{F}^{-1}, \tag{9.53}$$

where the values of p and p^* are in general different. The expressions $(9.51)_2$ and $(9.52)_2$ are applicable to incompressible materials as well and do not need to be modified, except that $\det \mathbf{F} = 1$.

We now derive the explicit expressions for the incremental forms of the constitutive equations, the electric and mechanical boundary conditions and the increment of the electric field exterior to the material. Let $\dot{\mathbf{T}}$, $\dot{\mathbf{D}}_{\mathrm{L}}$ and $\dot{\mathbf{E}}_{\mathrm{L}}$ be time-independent increments in $\mathbf{T}$, $\mathbf{D}_{\mathrm{L}}$ and $\mathbf{E}_{\mathrm{L}}$, respectively. We assume that there is no free charge and no mechanical body force. Then the incremental forms of (9.38), (9.40) and (9.42) reduce to

$$\operatorname{Curl} \dot{\mathbf{E}}_{\mathrm{L}} = \mathbf{0}, \quad \operatorname{Div} \dot{\mathbf{D}}_{\mathrm{L}} = 0, \quad \operatorname{Div} \dot{\mathbf{T}} = \mathbf{0}, \tag{9.54}$$

and their updated versions to

$$\operatorname{curl} \dot{\mathbf{E}}_{\mathrm{L}0} = \mathbf{0}, \quad \operatorname{div} \dot{\mathbf{D}}_{\mathrm{L}0} = 0, \quad \operatorname{div} \dot{\mathbf{T}}_0 = \mathbf{0}. \tag{9.55}$$

9.3.1.1 Exterior Incremental Fields

The relation $\mathbf{D}^\star = \varepsilon_0 \mathbf{E}^\star$, which is valid in vacuum and non-polarizable materials, has the incremental form

$$\dot{\mathbf{D}}^\star = \varepsilon_0 \dot{\mathbf{E}}^\star, \tag{9.56}$$

where $\dot{\mathbf{D}}^\star$ and $\dot{\mathbf{E}}^\star$ are the increments of $\mathbf{D}^\star$ and $\mathbf{E}^\star$, respectively. The associated incremental Maxwell equations are

$$\operatorname{div} \dot{\mathbf{D}}^\star = 0, \quad \operatorname{curl} \dot{\mathbf{E}}^\star = \mathbf{0}, \tag{9.57}$$

and the increment of the Maxwell stress defined by (4.15) becomes

$$\dot{\boldsymbol{\tau}}_{\mathrm{e}}^\star = \varepsilon_0[\dot{\mathbf{E}}^\star \otimes \mathbf{E}^\star + \mathbf{E}^\star \otimes \dot{\mathbf{E}}^\star - (\mathbf{E}^\star \cdot \dot{\mathbf{E}}^\star)\mathbf{I}], \tag{9.58}$$

which satisfies the equation $\operatorname{div} \dot{\boldsymbol{\tau}}_{\mathrm{e}}^\star = \mathbf{0}$.

9.3.1.2 Incremental Boundary Conditions

On the bounding surface $\partial\mathcal{B}$ of the material occupying the current configuration $\mathcal{B}$, in addition to any applied mechanical traction $\mathbf{t}_{\mathrm{a}}$, there will in general be an additional contribution due to the Maxwell stress according to (4.20). This additional contribution is given by a force $\boldsymbol{\tau}_{\mathrm{e}}^\star \mathbf{n}$ per unit current area, where $\boldsymbol{\tau}_{\mathrm{e}}^\star$ is the Maxwell stress (4.15) in the surrounding non-polarizable space and $\mathbf{n}$ is the unit outward normal on the surface $\partial\mathcal{B}$. Use of Nanson's formula allows the boundary condition to be defined with respect to the reference configuration $\mathcal{B}_\mathrm{r}$ as

$$\mathbf{T}^\mathrm{T}\mathbf{N} = \mathbf{t}_\mathrm{A} + J\boldsymbol{\tau}_\mathrm{e}^\star\mathbf{F}^{-\mathrm{T}}\mathbf{N} \quad \text{on } \partial\mathcal{B}_\mathrm{r}, \tag{9.59}$$

where $\mathbf{N}$ is the outward normal on the surface $\partial\mathcal{B}_\mathrm{r}$; see (4.55). On taking the increment of this equation, we obtain

$$\dot{\mathbf{T}}^\mathrm{T}\mathbf{N} = \dot{\mathbf{t}}_\mathrm{A} + J\dot{\boldsymbol{\tau}}_\mathrm{e}^\star\mathbf{F}^{-\mathrm{T}}\mathbf{N} - J\boldsymbol{\tau}_\mathrm{e}^\star\mathbf{F}^{-\mathrm{T}}\dot{\mathbf{F}}^\mathrm{T}\mathbf{F}^{-\mathrm{T}}\mathbf{N} + \dot{J}\boldsymbol{\tau}_\mathrm{e}^\star\mathbf{F}^{-\mathrm{T}}\mathbf{N} \quad \text{on } \partial\mathcal{B}_\mathrm{r} \tag{9.60}$$

or, when updated to Eulerian form,

$$\dot{\mathbf{T}}_0^\mathrm{T}\mathbf{n} = \dot{\mathbf{t}}_{\mathrm{A}0} + \dot{\boldsymbol{\tau}}_\mathrm{e}^\star\mathbf{n} - \boldsymbol{\tau}_\mathrm{e}^\star\mathbf{L}^\mathrm{T}\mathbf{n} + (\operatorname{div}\mathbf{u})\boldsymbol{\tau}_\mathrm{e}^\star\mathbf{n} \quad \text{on } \partial\mathcal{B}, \tag{9.61}$$

where we have used the standard formula $\dot{J} = J\operatorname{div}\mathbf{u}$ given in $(9.33)_2$.

We now rewrite the boundary conditions (4.52) for the electric field variables in the equivalent forms

$$(\mathbf{D}_\mathrm{L} - J\mathbf{F}^{-1}\mathbf{D}^\star)\cdot\mathbf{N} = 0, \quad (\mathbf{E}_\mathrm{L} - \mathbf{F}^\mathrm{T}\mathbf{E}^\star)\times\mathbf{N} = \mathbf{0} \quad \text{on } \partial\mathcal{B}_\mathrm{r}, \tag{9.62}$$

where we have set to zero the free surface charge σ_F. On incrementing these conditions and then updating, we obtain

$$[\dot{\mathbf{D}}_{\mathrm{L}0} - \dot{\mathbf{D}}^\star + \mathbf{L}\mathbf{D}^\star - (\operatorname{div}\mathbf{u})\mathbf{D}^\star]\cdot\mathbf{n} = 0 \quad \text{on } \partial\mathcal{B} \tag{9.63}$$

and

$$(\dot{\mathbf{E}}_{\mathrm{L}0} - \dot{\mathbf{E}}^\star - \mathbf{L}^\mathrm{T}\mathbf{E}^\star)\times\mathbf{n} = \mathbf{0} \quad \text{on } \partial\mathcal{B}. \tag{9.64}$$

9.3.1.3 Formulation Based on $\boldsymbol{\Omega}(\mathbf{F}, \mathbf{E}_\mathrm{L})$

Consider an incremental deformation, denoted $\dot{\mathbf{F}}$, superimposed on the current configuration $\mathcal{B}$ combined with an increment in the electric field $\dot{\mathbf{E}}_\mathrm{L}$. To evaluate the corresponding increment of the total nominal stress $\dot{\mathbf{T}}$ and of the electric displacement $\dot{\mathbf{D}}_\mathrm{L}$ within the material, the incremental forms of the constitutive laws (9.51) are required. These are given by

$$\dot{\mathbf{T}} = \boldsymbol{\mathcal{A}}\dot{\mathbf{F}} + \mathbb{A}\dot{\mathbf{E}}_\mathrm{L}, \quad \dot{\mathbf{D}}_\mathrm{L} = -\mathbb{A}^\mathrm{T}\dot{\mathbf{F}} - \mathsf{A}\dot{\mathbf{E}}_\mathrm{L}, \tag{9.65}$$

where $\boldsymbol{\mathcal{A}}$, $\mathbb{A}$ and A are, respectively, fourth-, third- and second-order tensors, which we refer to as *electroelastic moduli tensors*. For an incompressible material the total nominal stress $\mathbf{T}$ in terms of Ω is given by $(9.53)_1$. The corresponding incremental form, using the connections defined by $(9.32)_1$ and $(9.33)_1$, is given by

$$\dot{\mathbf{T}} = \boldsymbol{\mathcal{A}}\dot{\mathbf{F}} + \mathbb{A}\dot{\mathbf{E}}_\mathrm{L} - \dot{p}\mathbf{F}^{-1} + p\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1}, \tag{9.66}$$

which replaces (9.65)$_1$. The formulation of the increment of $\dot{\mathbf{D}}_\mathrm{L}$, given by (9.65)$_2$, is valid for unconstrained as well as incompressible materials.

The electroelastic moduli tensors are defined by

$$\boldsymbol{\mathcal{A}} = \frac{\partial^2 \Omega}{\partial \mathbf{F} \partial \mathbf{F}}, \quad \mathbb{A} = \frac{\partial^2 \Omega}{\partial \mathbf{F} \partial \mathbf{E}_\mathrm{L}}, \quad \mathbb{A}^\mathrm{T} = \frac{\partial^2 \Omega}{\partial \mathbf{E}_\mathrm{L} \partial \mathbf{F}}, \quad \mathsf{\mathbf{A}} = \frac{\partial^2 \Omega}{\partial \mathbf{E}_\mathrm{L} \partial \mathbf{E}_\mathrm{L}}, \tag{9.67}$$

which for ease of future reference are expressed in component form as

$$\mathcal{A}_{\alpha i \beta j} = \frac{\partial^2 \Omega}{\partial F_{i\alpha} \partial F_{j\beta}}, \quad \mathbb{A}_{\alpha i|\beta} = \frac{\partial^2 \Omega}{\partial F_{i\alpha} \partial E_{\mathrm{L}\beta}}, \quad \mathsf{A}_{\alpha\beta} = \frac{\partial^2 \Omega}{\partial E_{\mathrm{L}\alpha} \partial E_{\mathrm{L}\beta}}, \tag{9.68}$$

where the vertical bar between the indices on $\mathbb{A}$ is a separator introduced to distinguish the single subscript from the pair of subscripts that always go together. The moduli tensors have the symmetries

$$\mathcal{A}_{\alpha i \beta j} = \mathcal{A}_{\beta j \alpha i}, \quad \mathsf{A}_{\alpha\beta} = \mathsf{A}_{\beta\alpha}. \tag{9.69}$$

Note that $\mathbb{A}$ has no corresponding indicial symmetry, although the order of the mixed derivatives in (9.68)$_2$ may be reversed. Then, the equations in (9.65) are defined, in component form, by

$$\dot{T}_{\alpha i} = \mathcal{A}_{\alpha i \beta j} \dot{F}_{j\beta} + \mathbb{A}_{\alpha i|\beta} \dot{E}_{\mathrm{L}\beta}, \quad \dot{D}_{\mathrm{L}\alpha} = -\mathbb{A}_{\beta i|\alpha} \dot{F}_{i\beta} - \mathsf{A}_{\alpha\beta} \dot{E}_{\mathrm{L}\beta}. \tag{9.70}$$

For an isotropic electroelastic material with no mechanical constraint, Ω is a function of the three principal invariants of $\mathbf{c}$ together with three invariants that depend on $\mathbf{E}_\mathrm{L}$. These have been defined previously in (4.79) and (4.80) and, for convenience of reference, are reproduced here:

$$I_1 = \mathrm{tr}\,\mathbf{c}, \quad I_2 = \frac{1}{2}\left[(\mathrm{tr}\,\mathbf{c})^2 - \mathrm{tr}\,(\mathbf{c}^2)\right], \quad I_3 = \det \mathbf{c}, \tag{9.71}$$

$$I_4 = \mathbf{E}_\mathrm{L} \cdot \mathbf{E}_\mathrm{L}, \quad I_5 = (\mathbf{c}\mathbf{E}_\mathrm{L}) \cdot \mathbf{E}_\mathrm{L}, \quad I_6 = (\mathbf{c}^2\mathbf{E}_\mathrm{L}) \cdot \mathbf{E}_\mathrm{L}. \tag{9.72}$$

The expressions for the electroelastic moduli tensors (9.68) can then be expanded in the forms

$$\begin{aligned}
\mathcal{A}_{\alpha i \beta j} &= \sum_{m=1, m\neq 4}^{6} \sum_{n=1, n\neq 4}^{6} \Omega_{mn} \frac{\partial I_m}{\partial F_{i\alpha}} \frac{\partial I_n}{\partial F_{j\beta}} + \sum_{n=1, n\neq 4}^{6} \Omega_n \frac{\partial^2 I_n}{\partial F_{i\alpha} \partial F_{j\beta}}, \\
\mathbb{A}_{\alpha i|\beta} &= \sum_{m=4}^{6} \sum_{n=1, n\neq 4}^{6} \Omega_{mn} \frac{\partial I_m}{\partial E_{\mathrm{L}\beta}} \frac{\partial I_n}{\partial F_{i\alpha}} + \sum_{n=5}^{6} \Omega_n \frac{\partial^2 I_n}{\partial F_{i\alpha} \partial E_{\mathrm{L}\beta}}, \\
\mathsf{A}_{\alpha\beta} &= \sum_{m=4}^{6} \sum_{n=4}^{6} \Omega_{mn} \frac{\partial I_m}{\partial E_{\mathrm{L}\alpha}} \frac{\partial I_n}{\partial E_{\mathrm{L}\beta}} + \sum_{n=4}^{6} \Omega_n \frac{\partial^2 I_n}{\partial E_{\mathrm{L}\alpha} \partial E_{\mathrm{L}\beta}},
\end{aligned} \tag{9.73}$$

where $\Omega_n = \partial\Omega/\partial I_n$, $\Omega_{mn} = \partial^2\Omega/\partial I_m \partial I_n$. To obtain the explicit expressions for (9.73), the first- and second-order derivatives of $I_n, n = 1, \ldots, 6$, with respect to $\mathbf{F}$ and $\mathbf{E}_{\mathrm{L}}$ are needed. We begin with the first derivatives. In component form the non-zero ones are given by

$$\frac{\partial I_1}{\partial F_{i\alpha}} = 2F_{i\alpha}, \quad \frac{\partial I_2}{\partial F_{i\alpha}} = 2(c_{\gamma\gamma}F_{i\alpha} - c_{\alpha\gamma}F_{i\gamma}), \quad \frac{\partial I_3}{\partial F_{i\alpha}} = 2I_3 F^{-1}_{\alpha i},$$
$$\frac{\partial I_5}{\partial F_{i\alpha}} = 2E_{\mathrm{L}\alpha}(F_{i\gamma}E_{\mathrm{L}\gamma}), \quad \frac{\partial I_6}{\partial F_{i\alpha}} = 2(c_{\alpha\beta}E_{\mathrm{L}\beta}F_{i\gamma}E_{\mathrm{L}\gamma} + E_{\mathrm{L}\alpha}F_{i\gamma}c_{\gamma\beta}E_{\mathrm{L}\beta}),$$
$$\frac{\partial I_4}{\partial E_{\mathrm{L}\alpha}} = 2E_{\mathrm{L}\alpha}, \quad \frac{\partial I_5}{\partial E_{\mathrm{L}\alpha}} = 2c_{\alpha\beta}E_{\mathrm{L}\beta}, \quad \frac{\partial I_6}{\partial E_{\mathrm{L}\alpha}} = 2c^2_{\alpha\beta}E_{\mathrm{L}\beta}, \tag{9.74}$$

where we note that I_4 is independent of the deformation $\mathbf{F}$ and we have adopted the notation $c^2_{\alpha\beta} = (\mathbf{c}^2)_{\alpha\beta}$ as well as the notation $F^{-1}_{\alpha i} = (\mathbf{F}^{-1})_{\alpha i}$ introduced in Sect. 4.4.3. The non-zero second derivatives of the invariants with respect to $\mathbf{F}$ have the explicit forms

$$\frac{\partial^2 I_1}{\partial F_{i\alpha}\partial F_{j\beta}} = 2\delta_{ij}\delta_{\alpha\beta},$$
$$\frac{\partial^2 I_2}{\partial F_{i\alpha}\partial F_{j\beta}} = 2(2F_{i\alpha}F_{j\beta} - F_{i\beta}F_{j\alpha} + c_{\gamma\gamma}\delta_{ij}\delta_{\alpha\beta} - b_{ij}\delta_{\alpha\beta} - c_{\alpha\beta}\delta_{ij}),$$
$$\frac{\partial^2 I_3}{\partial F_{i\alpha}\partial F_{j\beta}} = 2I_3(2F^{-1}_{\alpha i}F^{-1}_{\beta j} - F^{-1}_{\alpha j}F^{-1}_{\beta i}), \quad \frac{\partial^2 I_5}{\partial F_{i\alpha}\partial F_{j\beta}} = 2\delta_{ij}E_{\mathrm{L}\alpha}E_{\mathrm{L}\beta},$$
$$\begin{aligned}\frac{\partial^2 I_6}{\partial F_{i\alpha}\partial F_{j\beta}} = {} & 2[\delta_{ij}(c_{\alpha\gamma}E_{\mathrm{L}\gamma}E_{\mathrm{L}\beta} + c_{\beta\gamma}E_{\mathrm{L}\gamma}E_{\mathrm{L}\alpha}) + \delta_{\alpha\beta}F_{i\gamma}E_{\mathrm{L}\gamma}F_{j\delta}E_{\mathrm{L}\delta} \\ & + F_{i\gamma}E_{\mathrm{L}\gamma}F_{j\alpha}E_{\mathrm{L}\beta} + F_{j\gamma}E_{\mathrm{L}\gamma}F_{i\beta}E_{\mathrm{L}\alpha} + b_{ij}E_{\mathrm{L}\alpha}E_{\mathrm{L}\beta}].\end{aligned} \tag{9.75}$$

The second derivatives of I_4, I_5, I_6 with respect to $\mathbf{E}_{\mathrm{L}}$ are

$$\frac{\partial^2 I_4}{\partial E_{\mathrm{L}\alpha}\partial E_{\mathrm{L}\beta}} = 2\delta_{\alpha\beta}, \quad \frac{\partial^2 I_5}{\partial E_{\mathrm{L}\alpha}\partial E_{\mathrm{L}\beta}} = 2c_{\alpha\beta}, \quad \frac{\partial^2 I_6}{\partial E_{\mathrm{L}\alpha}\partial E_{\mathrm{L}\beta}} = 2c^2_{\alpha\beta}. \tag{9.76}$$

The mixed derivatives of I_1, I_2, I_3 and I_4 with respect to $\mathbf{F}$ and $\mathbf{E}_{\mathrm{L}}$ vanish, and the two remaining invariants have the mixed derivatives

$$\frac{\partial^2 I_5}{\partial F_{i\alpha}\partial E_{\mathrm{L}\beta}} = 2\delta_{\alpha\beta}F_{i\gamma}E_{\mathrm{L}\gamma} + 2E_{\mathrm{L}\alpha}F_{i\beta},$$
$$\frac{\partial^2 I_6}{\partial F_{i\alpha}\partial E_{\mathrm{L}\beta}} = 2F_{i\beta}c_{\alpha\gamma}E_{\mathrm{L}\gamma} + 2F_{i\gamma}E_{\mathrm{L}\gamma}c_{\alpha\beta} + 2F_{i\gamma}c_{\gamma\beta}E_{\mathrm{L}\alpha} + 2\delta_{\alpha\beta}F_{i\gamma}c_{\gamma\delta}E_{\mathrm{L}\delta}. \tag{9.77}$$

9.3.1.4 Updated Reference Configuration

The updated incremental governing equations are given by (9.55), and the corresponding updated incremental constitutive equations (9.65) can be re-cast in the forms

$$\dot{\mathbf{T}}_0 = \boldsymbol{\mathcal{A}}_0\mathbf{L} + \mathbb{A}_0\dot{\mathbf{E}}_{L0}, \quad \dot{\mathbf{D}}_{L0} = -\mathbb{A}_0^{\mathrm{T}}\mathbf{L} - \mathsf{A}_0\dot{\mathbf{E}}_{L0}. \tag{9.78}$$

For an unconstrained material, the tensors $\boldsymbol{\mathcal{A}}_0$, $\mathbb{A}_0$ and A_0 are defined in index notation by

$$\mathcal{A}_{0jilk} = J^{-1}F_{j\alpha}F_{l\beta}\mathcal{A}_{\alpha i\beta k}, \tag{9.79}$$

$$\mathbb{A}_{0ji|k} = J^{-1}F_{j\alpha}F_{k\beta}\mathbb{A}_{\alpha i|\beta}, \tag{9.80}$$

$$\mathsf{A}_{0ij} = J^{-1}F_{i\alpha}F_{j\beta}\mathsf{A}_{\alpha\beta}. \tag{9.81}$$

For an incompressible material $J = 1$ in the above, and (9.78) is replaced by

$$\dot{\mathbf{T}}_0 = \boldsymbol{\mathcal{A}}_0\mathbf{L} + \mathbb{A}_0\dot{\mathbf{E}}_{L0} + p\mathbf{L} - \dot{p}\mathbf{I}, \quad \dot{\mathbf{D}}_{L0} = -\mathbb{A}_0^{\mathrm{T}}\mathbf{L} - \mathsf{A}_0\dot{\mathbf{E}}_{L0}, \tag{9.82}$$

and $\mathbf{u}$ satisfies the incremental incompressibility condition

$$\operatorname{div}\mathbf{u} = 0. \tag{9.83}$$

We note that the symmetries of $\boldsymbol{\mathcal{A}}$ and A carry over to $\boldsymbol{\mathcal{A}}_0$ and A_0 in the form

$$\mathcal{A}_{0jilk} = \mathcal{A}_{0lkji}, \quad \mathsf{A}_{0ij} = \mathsf{A}_{0ji}, \tag{9.84}$$

while $\mathbb{A}_0$ has now acquired the symmetry

$$\mathbb{A}_{0ij|k} = \mathbb{A}_{0ji|k}, \tag{9.85}$$

which can be established by using the incremental form of the symmetry condition $\mathbf{FT} = (\mathbf{FT})^{\mathrm{T}}$. In updated form this is

$$\dot{\mathbf{T}}_0 + \mathbf{L}\boldsymbol{\tau} = \dot{\mathbf{T}}_0^{\mathrm{T}} + \boldsymbol{\tau}\mathbf{L}^{\mathrm{T}}. \tag{9.86}$$

This also yields the connections

$$\mathcal{A}_{0jilk} - \mathcal{A}_{0ijlk} = \tau_{jl}\delta_{ik} - \tau_{il}\delta_{jk} \tag{9.87}$$

between the components of the tensors $\boldsymbol{\mathcal{A}}_0$ and $\boldsymbol{\tau}$ for an unconstrained material. The corresponding connections for an incompressible material are

$$\mathcal{A}_{0jilk} - \mathcal{A}_{0ijlk} = \big(\tau_{jl} + p\delta_{jl})\delta_{ik} - (\tau_{il} + p\delta_{il}\big)\,\delta_{jk}. \tag{9.88}$$

The latter two equations generalize to the electroelastic situation results that hold for a purely elastic material given by Chadwick and Ogden (1971) and Chadwick (1997) and are identical to corresponding formulas for a magnetoelastic material as reported in Otténio et al. (2008).

We now give explicit expressions for $\mathcal{A}_{0jilk}$, $\mathbb{A}_{0ji|k}$ and A_{0ij}, and for compactness of representation, we introduce the notations $\bar{\mathbf{b}} = I_1\mathbf{b} - \mathbf{b}^2$, $\mathbf{b}^{(2)} = \mathbf{b}^2$, $\mathbf{b}^{(3)} = \mathbf{b}^3$, $\mathbf{E}^{(1)} = \mathbf{b}\mathbf{E}$, $\mathbf{E}^{(2)} = \mathbf{b}^2\mathbf{E}$ and $\mathbf{E}^{(3)} = \mathbf{b}^3\mathbf{E}$. Thus,

$$
\begin{aligned}
J\mathcal{A}_{0jilk} = {} & 4\{\Omega_{11}b_{ij}b_{kl} + \Omega_{12}(b_{ij}\bar{b}_{kl} + b_{kl}\bar{b}_{ij}) + \Omega_{22}\bar{b}_{ij}\bar{b}_{kl} + I_3^2\Omega_{33}\delta_{ij}\delta_{kl} \\
& + I_3\Omega_{13}(\delta_{ij}b_{kl} + \delta_{kl}b_{ij}) + I_3\Omega_{23}(\delta_{ij}\bar{b}_{kl} + \delta_{kl}\bar{b}_{ij}) \\
& + \Omega_{15}(b_{ij}E_k^{(1)}E_l^{(1)} + b_{kl}E_i^{(1)}E_j^{(1)}) + \Omega_{25}(E_i^{(1)}E_j^{(1)}\bar{b}_{kl} + E_k^{(1)}E_l^{(1)}\bar{b}_{ij}) \\
& + I_3\Omega_{35}[\delta_{ij}E_k^{(1)}E_l^{(1)} + E_i^{(1)}E_j^{(1)}\delta_{kl}] + \Omega_{55}E_i^{(1)}E_j^{(1)}E_k^{(1)}E_l^{(1)} \\
& + \Omega_{16}[b_{ij}(E_k^{(1)}E_l^{(2)} + E_k^{(2)}E_l^{(1)}) + b_{kl}(E_i^{(1)}E_j^{(2)} + E_i^{(2)}E_j^{(1)})] \\
& + \Omega_{26}[\bar{b}_{ij}(E_k^{(1)}E_l^{(2)} + E_k^{(2)}E_l^{(1)}) + \bar{b}_{kl}(E_i^{(1)}E_j^{(2)} + E_i^{(2)}E_j^{(1)})] \\
& + I_3\Omega_{36}[\delta_{ij}(E_k^{(1)}E_l^{(2)} + E_k^{(2)}E_l^{(1)}) + \delta_{kl}(E_i^{(1)}E_j^{(2)} + E_i^{(2)}E_j^{(1)})] \\
& + \Omega_{56}(E_i^{(2)}E_j^{(1)}E_k^{(1)}E_l^{(1)} + E_j^{(2)}E_i^{(1)}E_k^{(1)}E_l^{(1)} + E_k^{(2)}E_i^{(1)}E_j^{(1)}E_l^{(1)} \\
& + E_l^{(2)}E_i^{(1)}E_j^{(1)}E_k^{(1)}) + \Omega_{66}(E_i^{(1)}E_j^{(2)} + E_i^{(2)}E_j^{(1)})(E_k^{(1)}E_l^{(2)} + E_k^{(2)}E_l^{(1)})\} \\
& + 2\{\Omega_1\delta_{ik}b_{jl} + \Omega_2(2b_{ij}b_{kl} - b_{il}b_{jk} + \delta_{ik}\bar{b}_{jl} - b_{ik}b_{jl}) \\
& + I_3\Omega_3(2\delta_{ij}\delta_{kl} - \delta_{jk}\delta_{il}) + \Omega_5\delta_{ik}E_j^{(1)}E_l^{(1)} + \Omega_6[\delta_{ik}(E_j^{(1)}E_l^{(2)} + E_j^{(2)}E_l^{(1)}) \\
& + b_{ik}E_j^{(1)}E_l^{(1)} + b_{il}E_j^{(1)}E_k^{(1)} + b_{jk}E_i^{(1)}E_l^{(1)} + b_{jl}E_i^{(1)}E_k^{(1)}]\},
\end{aligned}
\tag{9.89}
$$

$$
\begin{aligned}
J\mathbb{A}_{0ji|k} = {} & 4[b_{ij}(\Omega_{14}E_k^{(1)} + \Omega_{15}E_k^{(2)} + \Omega_{16}E_k^{(3)}) \\
& + \bar{b}_{ij}(\Omega_{24}E_k^{(1)} + \Omega_{25}E_k^{(2)} + \Omega_{26}E_k^{(3)}) \\
& + I_3\delta_{ij}(\Omega_{34}E_k^{(1)} + \Omega_{35}E_k^{(2)} + \Omega_{36}E_k^{(3)}) \\
& + E_i^{(1)}E_j^{(1)}(\Omega_{45}E_k^{(1)} + \Omega_{55}E_k^{(2)} + \Omega_{56}E_k^{(3)}) \\
& + (E_i^{(1)}E_j^{(2)} + E_i^{(2)}E_j^{(1)})(\Omega_{46}E_k^{(1)} + \Omega_{56}E_k^{(2)} + \Omega_{66}E_k^{(3)})] \\
& + 2[\Omega_5(E_i^{(1)}b_{jk} + E_j^{(1)}b_{ik}) + \Omega_6(E_i^{(1)}b_{jk}^{(2)} + E_j^{(1)}b_{ik}^{(2)} + E_i^{(2)}b_{jk} + E_j^{(2)}b_{ik})],
\end{aligned}
\tag{9.90}
$$

$$
\begin{aligned}
J\mathsf{A}_{0ij} = {} & 4[\Omega_{44}E_i^{(1)}E_j^{(1)} + \Omega_{55}E_i^{(2)}E_j^{(2)} + \Omega_{66}E_i^{(3)}E_j^{(3)} \\
& + \Omega_{45}(E_i^{(1)}E_j^{(2)} + E_i^{(2)}E_j^{(1)}) + \Omega_{46}(E_i^{(1)}E_j^{(3)} + E_i^{(3)}E_j^{(1)}) \\
& + \Omega_{56}(E_i^{(2)}E_j^{(3)} + E_i^{(3)}E_j^{(2)})] + 2(\Omega_4 b_{ij} + \Omega_5 b_{ij}^{(2)} + \Omega_6 b_{ij}^{(3)}). \quad (9.91)
\end{aligned}
$$

These formulas apply for an unconstrained material. If we set $J = 1$ and omit the terms involving derivatives of Ω with respect to I_3, then they also apply for an incompressible material.

9.3.1.5 Formulation Based on $\Omega^*(\mathbf{F}, \mathbf{D}_\mathrm{L})$

Following the development outlined in the previous subsection, we now consider an incremental deformation $\dot{\mathbf{F}}$ combined with an increment in the electric displacement $\dot{\mathbf{D}}_\mathrm{L}$ superimposed on the current configuration $\mathcal{B}$. To obtain the corresponding increment in the total stress $\dot{\mathbf{T}}$ and in the electric field $\dot{\mathbf{E}}_\mathrm{L}$ we require the incremental forms of the constitutive laws (9.52). Following the derivations described in Sect. 9.3.1.3 we have

$$
\dot{\mathbf{T}} = \boldsymbol{\mathcal{A}}^*\dot{\mathbf{F}} + \mathbb{A}^*\dot{\mathbf{D}}_\mathrm{L}, \quad \dot{\mathbf{E}}_\mathrm{L} = \mathbb{A}^{*\mathrm{T}}\dot{\mathbf{F}} + \mathsf{A}^*\dot{\mathbf{D}}_\mathrm{L}, \quad (9.92)
$$

where $\boldsymbol{\mathcal{A}}^*, \mathbb{A}^*, \mathsf{A}^*$ denote the electroelastic moduli tensors associated with the total energy $\Omega^* = \Omega^*(\mathbf{F}, \mathbf{D}_\mathrm{L})$. These are again fourth-, third- and second-order tensors defined analogously to (9.67), with Ω replaced by Ω^* and $\mathbf{E}_\mathrm{L}$ by $\mathbf{D}_\mathrm{L}$. Their components are given by

$$
\mathcal{A}^*_{\alpha i\beta j} = \frac{\partial^2\Omega^*}{\partial F_{i\alpha}\partial F_{j\beta}}, \quad \mathbb{A}^*_{\alpha i|\beta} = \frac{\partial^2\Omega^*}{\partial F_{i\alpha}\partial D_{\mathrm{L}\beta}}, \quad \mathsf{A}^*_{\alpha\beta} = \frac{\partial^2\Omega^*}{\partial D_{\mathrm{L}\alpha}\partial D_{\mathrm{L}\beta}}, \quad (9.93)
$$

where we note that the symmetries specified by (9.69) again apply. The component forms of (9.92) are

$$
\dot{T}_{\alpha i} = \mathcal{A}^*_{\alpha i\beta j}\dot{F}_{j\beta} + \mathbb{A}^*_{\alpha i|\beta}\dot{D}_{\mathrm{L}\beta}, \quad \dot{E}_{\mathrm{L}\alpha} = \mathbb{A}^*_{\beta i|\alpha}\dot{F}_{i\beta} + \mathsf{A}^*_{\alpha\beta}\dot{D}_{\mathrm{L}\beta}. \quad (9.94)
$$

For isotropic electroelastic materials, the moduli tensors (9.93) can be expanded in terms of the principal invariants of $\mathbf{c}$ together with three invariants based on $\mathbf{D}_\mathrm{L}$. In what follows, to allow for a compact notation, we use K_1, K_2, K_3 instead of the I_1, I_2, I_3 used previously to denote the invariants of $\mathbf{c}$ and K_4, K_5, K_6 to denote those involving $\mathbf{D}_\mathrm{L}$. Thus, we have

$$
K_1 = \operatorname{tr}\mathbf{c}, \quad K_2 = \frac{1}{2}\left[(\operatorname{tr}\mathbf{c})^2 - \operatorname{tr}(\mathbf{c}^2)\right], \quad K_3 = \det\mathbf{c}, \quad (9.95)
$$

$$
K_4 = \mathbf{D}_\mathrm{L}\cdot\mathbf{D}_\mathrm{L}, \quad K_5 = (\mathbf{c}\mathbf{D}_\mathrm{L})\cdot\mathbf{D}_\mathrm{L}, \quad K_6 = (\mathbf{c}^2\mathbf{D}_\mathrm{L})\cdot\mathbf{D}_\mathrm{L}. \quad (9.96)
$$

The expressions for the electroelastic moduli tensors (9.93) can then be written in the forms

$$\mathcal{A}^*_{\alpha i\beta j} = \sum_{m=1,m\neq 4}^{6} \sum_{n=1,n\neq 4}^{6} \Omega^*_{mn} \frac{\partial K_m}{\partial F_{i\alpha}} \frac{\partial K_n}{\partial F_{j\beta}} + \sum_{n=1,n\neq 4}^{6} \Omega^*_n \frac{\partial^2 K_n}{\partial F_{i\alpha} \partial F_{j\beta}},$$

$$\mathbb{A}^*_{\alpha i|\beta} = \sum_{m=4}^{6} \sum_{n=1,n\neq 4}^{6} \Omega^*_{mn} \frac{\partial K_m}{\partial D_{\mathrm{L}\beta}} \frac{\partial K_n}{\partial F_{i\alpha}} + \sum_{n=5}^{6} \Omega^*_n \frac{\partial^2 K_n}{\partial F_{i\alpha} \partial D_{\mathrm{L}\beta}},$$

$$\mathsf{A}^*_{\alpha\beta} = \sum_{m=4}^{6} \sum_{n=4}^{6} \Omega^*_{mn} \frac{\partial K_m}{\partial D_{\mathrm{L}\alpha}} \frac{\partial K_n}{\partial D_{\mathrm{L}\beta}} + \sum_{n=4}^{6} \Omega^*_n \frac{\partial^2 K_n}{\partial D_{\mathrm{L}\alpha} \partial D_{\mathrm{L}\beta}}, \tag{9.97}$$

where $\Omega^*_n = \partial\Omega^*/\partial K_n$, $\Omega^*_{mn} = \partial^2\Omega^*/\partial K_m \partial K_n$, $m, n \in \{1, \ldots, 6\}$. To expand the formulations (9.97) we need the first and second derivatives of $K_n, n = 1, \ldots, 6$, with respect to $\mathbf{F}$ and $\mathbf{D}_{\mathrm{L}}$. In component form the non-zero first derivatives are

$$\frac{\partial K_1}{\partial F_{i\alpha}} = 2F_{i\alpha}, \quad \frac{\partial K_2}{\partial F_{i\alpha}} = 2(c_{\gamma\gamma} F_{i\alpha} - c_{\alpha\gamma} F_{i\gamma}), \quad \frac{\partial K_3}{\partial F_{i\alpha}} = 2K_3 F^{-1}_{\alpha i},$$

$$\frac{\partial K_5}{\partial F_{i\alpha}} = 2D_{\mathrm{L}\alpha}(F_{i\gamma} D_{\mathrm{L}\gamma}), \quad \frac{\partial K_6}{\partial F_{i\alpha}} = 2(c_{\alpha\beta} D_{\mathrm{L}\beta} F_{i\gamma} D_{\mathrm{L}\gamma} + D_{\mathrm{L}\alpha} F_{i\gamma} c_{\gamma\beta} D_{\mathrm{L}\beta}),$$

$$\frac{\partial K_4}{\partial D_{\mathrm{L}\alpha}} = 2D_{\mathrm{L}\alpha}, \quad \frac{\partial K_5}{\partial D_{\mathrm{L}\alpha}} = 2c_{\alpha\beta} D_{\mathrm{L}\beta}, \quad \frac{\partial K_6}{\partial D_{\mathrm{L}\alpha}} = 2c^2_{\alpha\beta} D_{\mathrm{L}\beta}, \tag{9.98}$$

where we note that (9.98) can be obtained directly from (9.74) by replacing I_n by $K_n, n = 1, \ldots, 6$, and $E_{\mathrm{L}\alpha}$ by $D_{\mathrm{L}\alpha}$.

The non-zero second derivatives with respect to $\mathbf{F}$ are

$$\frac{\partial^2 K_1}{\partial F_{i\alpha} \partial F_{j\beta}} = 2\delta_{ij} \delta_{\alpha\beta},$$

$$\frac{\partial^2 K_2}{\partial F_{i\alpha} \partial F_{j\beta}} = 2(2F_{i\alpha} F_{j\beta} - F_{i\beta} F_{j\alpha} + c_{\gamma\gamma} \delta_{ij} \delta_{\alpha\beta} - b_{ij} \delta_{\alpha\beta} - c_{\alpha\beta} \delta_{ij}),$$

$$\frac{\partial^2 K_3}{\partial F_{i\alpha} \partial F_{j\beta}} = 2K_3(2F^{-1}_{\alpha i} F^{-1}_{\beta j} - F^{-1}_{\alpha j} F^{-1}_{\beta i}), \quad \frac{\partial^2 K_5}{\partial F_{i\alpha} \partial F_{j\beta}} = 2\delta_{ij} D_{\mathrm{L}\alpha} D_{\mathrm{L}\beta},$$

$$\begin{aligned}\frac{\partial^2 K_6}{\partial F_{i\alpha} \partial F_{j\beta}} &= 2[\delta_{ij}(c_{\alpha\gamma} D_{\mathrm{L}\gamma} D_{\mathrm{L}\beta} + c_{\beta\gamma} D_{\mathrm{L}\gamma} D_{\mathrm{L}\alpha}) + \delta_{\alpha\beta} F_{i\gamma} D_{\mathrm{L}\gamma} F_{j\delta} D_{\mathrm{L}\delta} \\ &\quad + F_{i\gamma} D_{\mathrm{L}\gamma} F_{j\alpha} D_{\mathrm{L}\beta} + F_{j\gamma} D_{\mathrm{L}\gamma} F_{i\beta} D_{\mathrm{L}\alpha} + b_{ij} D_{\mathrm{L}\alpha} D_{\mathrm{L}\beta}].\end{aligned} \tag{9.99}$$

The second derivatives of K_4, K_5, K_6 with respect to $\mathbf{D}_\mathrm{L}$ are

$$\frac{\partial^2 K_4}{\partial D_{\mathrm{L}\alpha}\partial D_{\mathrm{L}\beta}} = 2\delta_{\alpha\beta}, \quad \frac{\partial^2 K_5}{\partial D_{\mathrm{L}\alpha}\partial D_{\mathrm{L}\beta}} = 2c_{\alpha\beta}, \quad \frac{\partial^2 K_6}{\partial D_{\mathrm{L}\alpha}\partial D_{\mathrm{L}\beta}} = 2c^2_{\alpha\beta}. \tag{9.100}$$

The mixed derivatives of K_1, K_2, K_3 and K_4 with respect to $\mathbf{F}$ and $\mathbf{D}_\mathrm{L}$ vanish. For K_5, K_6, we have

$$\frac{\partial^2 K_5}{\partial F_{i\alpha}\partial D_{\mathrm{L}\beta}} = 2\delta_{\alpha\beta}F_{i\gamma}D_{\mathrm{L}\gamma} + 2D_{\mathrm{L}\alpha}F_{i\beta},$$

$$\frac{\partial^2 K_6}{\partial F_{i\alpha}\partial D_{\mathrm{L}\beta}} = 2F_{i\beta}c_{\alpha\gamma}D_{\mathrm{L}\gamma} + 2F_{i\gamma}D_{\mathrm{L}\gamma}c_{\alpha\beta} + 2F_{i\gamma}c_{\gamma\beta}D_{\mathrm{L}\alpha} + 2\delta_{\alpha\beta}F_{i\gamma}c_{\gamma\delta}D_{\mathrm{L}\delta}.$$

For an incompressible material, the incremental form of the total nominal stress is given by

$$\dot{\mathbf{T}} = \boldsymbol{\mathcal{A}}^*\dot{\mathbf{F}} + \mathbb{A}^*\dot{\mathbf{D}}_\mathrm{L} + p^*\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1} - \dot{p}^*\mathbf{F}^{-1}, \tag{9.101}$$

which replaces (9.66) for the considered variables. The increment in the electric field, which for a compressible material is given by $(9.92)_2$, applies also to the incompressible case.

The updated versions of (9.92) are

$$\dot{\mathbf{T}}_0 = \boldsymbol{\mathcal{A}}_0^*\mathbf{L} + \mathbb{A}_0^*\dot{\mathbf{D}}_{\mathrm{L}0}, \quad \dot{\mathbf{E}}_{\mathrm{L}0} = \mathbb{A}^{*\mathrm{T}}_0\mathbf{L} + \mathsf{A}_0^*\dot{\mathbf{D}}_{\mathrm{L}0}, \tag{9.102}$$

and, for incompressible materials, $(9.102)_1$ is replaced by

$$\dot{\mathbf{T}}_0 = \boldsymbol{\mathcal{A}}_0^*\mathbf{L} + \mathbb{A}_0^*\dot{\mathbf{D}}_{\mathrm{L}0} + p^*\mathbf{L} - \dot{p}^*\mathbf{I}. \tag{9.103}$$

In index notation the tensors $\boldsymbol{\mathcal{A}}_0^*, \mathbb{A}_0^*$ and A_0^* are defined by

$$\mathcal{A}^*_{0\,jilk} = J^{-1}F_{j\alpha}F_{l\beta}\mathcal{A}^*_{\alpha i\beta k}, \tag{9.104}$$

$$\mathbb{A}^*_{0ji|k} = F_{j\alpha}F^{-1}_{\beta k}\mathbb{A}^*_{\alpha i|\beta}, \tag{9.105}$$

$$\mathsf{A}^*_{0\,ij} = JF^{-1}_{\alpha i}F^{-1}_{\beta j}\mathsf{A}^*_{\alpha\beta}, \tag{9.106}$$

which are valid for compressible and incompressible materials, for the latter using $J \equiv 1$. Note that, apart from the asterisk here, the forms of (9.105) and (9.106) are slightly different from (9.80) and (9.81). The symmetry conditions specified by (9.84)–(9.88) carry over to the present case and are therefore not repeated here.

Explicit expressions for the components of $\boldsymbol{\mathcal{A}}_0^*, \mathbb{A}_0^*$ and A_0^* are obtained in a similar way to those of $\boldsymbol{\mathcal{A}}_0, \mathbb{A}_0$ and A_0. We use the notation $\bar{\mathbf{b}} = K_1\mathbf{b} - \mathbf{b}^2$, and we also adopt the notations $\mathbf{D}^{(1)} = \mathbf{b}\mathbf{D}$ and $\mathbf{D}^{(-1)} = \mathbf{b}^{-1}\mathbf{D}$. Then we obtain

$$
\begin{aligned}
J\mathcal{A}^*_{0piqj} = {} & 4\{\Omega^*_{11} b_{ip} b_{jq} + \Omega^*_{12}(b_{ip}\bar{b}_{jq} + b_{jq}\bar{b}_{ip}) + \Omega^*_{22}\bar{b}_{ip}\bar{b}_{jq} + K_3^2\Omega^*_{33}\delta_{ip}\delta_{jq} \\
& + K_3\Omega^*_{13}(\delta_{ip} b_{jq} + \delta_{jq} b_{ip}) + K_3\Omega^*_{23}(\delta_{ip}\bar{b}_{jq} + \delta_{jq}\bar{b}_{ip}) + K_3^2\Omega^*_{55} D_i D_j D_p D_q \\
& + K_3\Omega^*_{15}(b_{ip} D_j D_q + b_{jq} D_i D_p) + K_3\Omega^*_{25}(D_i D_p \bar{b}_{jq} + D_j D_q \bar{b}_{ip}) \\
& + K_3^2\Omega^*_{35}(D_i D_p \delta_{jq} + D_j D_q \delta_{ip}) \\
& + K_3^2\Omega^*_{66}(D^{(1)}_i D_p + D^{(1)}_p D_i)(D^{(1)}_j D_q + D^{(1)}_q D_j) \\
& + K_3\Omega^*_{16}[b_{ip}(D^{(1)}_j D_q + D^{(1)}_q D_j) + b_{jq}(D^{(1)}_i D_p + D^{(1)}_p D_i)] \\
& + K_3\Omega^*_{26}[\bar{b}_{ip}(D^{(1)}_j D_q + D^{(1)}_q D_j) + \bar{b}_{jq}(D^{(1)}_i D_p + D^{(1)}_p D_i)] \\
& + K_3^2\Omega^*_{36}[\delta_{ip}(D^{(1)}_j D_q + D^{(1)}_q D_j) + \delta_{jq}(D^{(1)}_i D_p + D^{(1)}_p D_i)] \\
& + K_3^2\Omega^*_{56}(D^{(1)}_p D_i D_q D_j + D^{(1)}_i D_p D_q D_j + D^{(1)}_q D_p D_i D_j + D^{(1)}_j D_p D_i D_q)\} \\
& + 2\{\Omega^*_1 \delta_{ij} b_{pq} + \Omega^*_2[2 b_{ip} b_{jq} - b_{iq} b_{jp} + \delta_{ij}\bar{b}_{pq} - b_{ij} b_{pq}] \\
& + K_3\Omega^*_3(2\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) + K_3\Omega^*_5\delta_{ij} D_p D_q \\
& + K_3\Omega^*_6[\delta_{ij}(D^{(1)}_p D_q + D^{(1)}_q D_p) \\
& + b_{pq} D_i D_j + b_{jp} D_i D_q + b_{iq} D_j D_p + b_{ij} D_p D_q]\},
\end{aligned}
\tag{9.107}
$$

$$
\begin{aligned}
J^{-1}\mathbb{A}^*_{0pi|q} = {} & 4[\Omega^*_{14} b_{ip} D^{(-1)}_q + \Omega^*_{24}\bar{b}_{ip} D^{(-1)}_q + K_3\Omega^*_{34}\delta_{ip} D^{(-1)}_q \\
& + K_3\Omega^*_{45} D_i D_p D^{(-1)}_q + K_3\Omega^*_{46}(D^{(1)}_i D_p + D^{(1)}_p D_i) D^{(-1)}_q \\
& + \Omega^*_{15} b_{ip} D_q + \Omega^*_{25}\bar{b}_{ip} D_q + K_3\Omega^*_{35}\delta_{ip} D_q + K_3\Omega^*_{55} D_i D_p D_q \\
& + K_3\Omega^*_{56}(D^{(1)}_p D_i D_q + D^{(1)}_i D_p D_q + D^{(1)}_q D_i D_p) + \Omega^*_{16} b_{ip} D^{(1)}_q \\
& + \Omega^*_{26}\bar{b}_{ip} D^{(1)}_q + K_3\Omega^*_{36}\delta_{ip} D^{(1)}_q + K_3\Omega^*_{66}(D^{(1)}_i D_p + D^{(1)}_p D_i) D^{(1)}_q] \\
& + 2[\Omega^*_5(\delta_{pq} D_i + \delta_{iq} D_p) + \Omega^*_6(\delta_{iq} D^{(1)}_p + \delta_{pq} D^{(1)}_i + b_{pq} D_i + b_{iq} D_p)],
\end{aligned}
\tag{9.108}
$$

$$
\begin{aligned}
J^{-1}\mathsf{A}^*_{0ij} = {} & 4K_3[\Omega^*_{44} D^{(-1)}_i D^{(-1)}_j + \Omega^*_{55} D_i D_j + \Omega^*_{66} D^{(1)}_i D^{(1)}_j \\
& + \Omega^*_{45}(D^{(-1)}_i D_j + D^{(-1)}_j D_i) + \Omega^*_{46}(D^{(-1)}_i D^{(1)}_j + D^{(-1)}_j D^{(1)}_i) \\
& + \Omega^*_{56}(D^{(1)}_i D_j + D^{(1)}_j D_i)] + 2(\Omega^*_4 b^{(-1)}_{ij} + \Omega^*_5\delta_{ij} + \Omega^*_6 b_{ij}),
\end{aligned}
\tag{9.109}
$$

for an unconstrained material. Again, to obtain the formulas for an incompressible material we set $J = K_3 = 1$ in the above and omit the terms involving derivatives of Ω^* with respect to K_3.

Finally in this section we note that by combining the incremental forms of the constitutive laws based on the two different formulations given in (9.78) and (9.102), we obtain the connections

$$\boldsymbol{\mathcal{A}}_0^* = \boldsymbol{\mathcal{A}}_0 + \mathbb{A}_0 \mathbb{A}_0^{*\mathrm{T}} = \boldsymbol{\mathcal{A}}_0 + \mathbb{A}_0^* \mathbb{A}_0^{\mathrm{T}}, \tag{9.110}$$

$$\mathbb{A}_0 \mathbf{A}_0^* = \mathbb{A}_0^*, \quad \mathbb{A}_0 = -\mathbb{A}_0^* \mathbf{A}_0, \tag{9.111}$$

from which may be deduced

$$\mathbb{A}_0 \mathbb{A}_0^{*\mathrm{T}} = \mathbb{A}_0^* \mathbb{A}_0^{\mathrm{T}}, \quad \mathbf{A}_0 \mathbf{A}_0^* = \mathbf{A}_0^* \mathbf{A}_0 = -\mathbf{I}. \tag{9.112}$$

These are analogous to the relations given in Sect. 4.5.2 for the general linear theory. In this specialization moduli tensors given here reduce to their counterparts in the linear theory with the identifications (in the isotropic case)

$$\boldsymbol{\mathcal{A}}_0 \to \boldsymbol{\mathcal{C}}, \quad \mathbb{A}_0 \to -\boldsymbol{\mathcal{E}}^{\mathrm{T}}, \quad \mathbf{A}_0 \to -\mathbf{K} \tag{9.113}$$

and

$$\boldsymbol{\mathcal{A}}_0^* \to \boldsymbol{\mathcal{C}}^*, \quad \mathbb{A}_0^* \to \boldsymbol{\mathcal{E}}^{*\mathrm{T}}, \quad \mathbf{A}_0^* \to \mathbf{K}^*. \tag{9.114}$$

9.3.2 Incremental Magnetoelasticity

In this section the equations governing the deformations of incremental (infinitesimal) disturbances superimposed on finite static deformation fields involving magnetic and elastic interactions are presented. We adopt the formulation presented in Chap. 6 as the starting point for the derivation of the incremental equations of magnetoelastic materials. This involves a *total stress tensor* and a *total energy function*, which enable the constitutive law for the stress to be written in a form very similar to that in standard nonlinear elasticity theory and in nonlinear electroelasticity. The coupled governing equations then have a relatively simple structure. We use the same notation Ω for the energy function as in the electroelastic case, but with $\mathbf{E}_{\mathrm{L}}$ replaced by $\mathbf{H}_{\mathrm{L}}$ and Ω^* with $\mathbf{D}_{\mathrm{L}}$ replaced by $\mathbf{B}_{\mathrm{L}}$. Thus,

$$\Omega = \Omega(\mathbf{F}, \mathbf{H}_{\mathrm{L}}), \quad \Omega^* = \Omega^*(\mathbf{F}, \mathbf{B}_{\mathrm{L}}), \tag{9.115}$$

and then many of the formulas in the electroelastic case carry over to the magnetoelastic situation merely by changing the electric independent variable to the corresponding magnetic variable.

We consider the initial configuration to be purely static and subject only to magnetic and mechanical effects, so that $\mathbf{v} = \mathbf{0}$, $\mathbf{E} = \mathbf{0}$, $\mathbf{D} = \mathbf{0}$ and $\sigma_\mathrm{f} = \rho_\mathrm{f} = 0$, and no mechanical body forces ($\mathbf{f} = \mathbf{0}$). In addition, we assume that there are no free volume or surface currents, so that $\mathbf{J}_\mathrm{f} = \mathbf{K}_\mathrm{f} = \mathbf{0}$, with $\mathbf{H}$ and $\mathbf{B}$ being independent of time. The updated incremental governing equations (9.45)–(9.48) then specialize to

$$\operatorname{curl}\dot{\mathbf{H}}_{\mathrm{L}0} = \mathbf{0}, \quad \operatorname{div}\dot{\mathbf{B}}_{\mathrm{L}0} = 0, \quad \operatorname{div}\dot{\mathbf{T}}_0 = \mathbf{0}. \tag{9.116}$$

Outside the material, which may be vacuum or a non-magnetizable material, we use a superscript $\star$ to indicate the corresponding field quantities. Thus, $\mathbf{H}^\star$ and $\mathbf{B}^\star$, respectively, are the magnetic field and magnetic induction, which are connected by the simple relation $\mathbf{B}^\star = \mu_0\mathbf{H}^\star$, where μ_0 denotes the vacuum permeability; see (2.144). The relevant specializations of Maxwell's equations are

$$\operatorname{curl}\mathbf{H}^\star = \mathbf{0}, \quad \operatorname{div}\mathbf{B}^\star = 0. \tag{9.117}$$

We denote the increments of $\mathbf{H}^\star$ and $\mathbf{B}^\star$ by $\dot{\mathbf{H}}^\star$ and $\dot{\mathbf{B}}^\star$, respectively. These are connected by $\dot{\mathbf{B}}^\star = \mu_0\dot{\mathbf{H}}^\star$ and satisfy Maxwell's equations

$$\operatorname{curl}\dot{\mathbf{H}}^\star = \mathbf{0}, \quad \operatorname{div}\dot{\mathbf{B}}^\star = 0. \tag{9.118}$$

Note that these increments are at fixed $\mathbf{x}$, whereas the increments inside the material are at fixed $\mathbf{X}$. Within the material the connection between an increment at fixed $\mathbf{X}$ and one at fixed $\mathbf{x}$ of some quantity $(\bullet)$ is given by

$$(\dot{\bullet})_\mathbf{X} = (\dot{\bullet})_\mathbf{x} + \dot{\mathbf{x}} \cdot \operatorname{grad}(\bullet) \tag{9.119}$$

similarly to an induced variation in Chap. 8. In general we do not have to consider this difference, which is also relevant in the electroelastic case.

9.3.3 Incremental Boundary Conditions

From $(2.172)_2$, the Eulerian form of the boundary condition satisfied by the magnetic induction vector $\mathbf{B}$ on the bounding surface $\partial\mathcal{B}$ is given by $(\mathbf{B} - \mathbf{B}^\star) \cdot \mathbf{n} = 0$. Its Lagrangian counterpart has the form

$$\left(\mathbf{B}_\mathrm{L} - J\mathbf{F}^{-1}\mathbf{B}^\star\right) \cdot \mathbf{N} = 0 \quad \text{on} \quad \partial\mathcal{B}_\mathrm{r}, \tag{9.120}$$

where $J\mathbf{F}^{-1}$ is evaluated on the boundary as it is approached from the inside, whereas $\mathbf{B}^\star$ is the value of the exterior field on the boundary. Since deformation is not defined outside the material, $J\mathbf{F}^{-1}\mathbf{B}^\star$ does not have the meaning of the Lagrangian form of the magnetic induction (although we introduced a fictitious Lagrangian magnetic induction for mathematical convenience in Chap. 8).

On taking the increment of the above equation, using the incompressibility condition (9.37)$_1$ and updating the reference configuration from $\mathcal{B}_r$ to $\mathcal{B}$, we obtain the incremental boundary condition

$$(\dot{\mathbf{B}}_{L0} - \dot{\mathbf{B}}^\star + \mathbf{L}\mathbf{B}^\star) \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{B}. \tag{9.121}$$

Since we have assumed that the free current density $\mathbf{K}_f = \mathbf{0}$, it follows from (2.172)$_1$ that the boundary condition for $\mathbf{H}$ is $(\mathbf{H} - \mathbf{H}^\star) \times \mathbf{n} = \mathbf{0}$. Its Lagrangian counterpart is (6.64), but since there is no Lagrangian form of the magnetic field outside the material, we write it in the form

$$(\mathbf{H}_L - \mathbf{F}^T\mathbf{H}^\star) \times \mathbf{N} = \mathbf{0} \quad \text{on} \quad \partial\mathcal{B}_r. \tag{9.122}$$

Taking the increment of this equation and then updating to $\mathcal{B}$ we obtain

$$\left(\dot{\mathbf{H}}_{L0} - \mathbf{L}^T\mathbf{H}^\star - \dot{\mathbf{H}}^\star\right) \times \mathbf{n} = \mathbf{0} \quad \text{on} \quad \partial\mathcal{B}. \tag{9.123}$$

To derive the incremental traction boundary condition we first recall the expression (6.10) for the Maxwell stress outside the material, which we now write in the form

$$\boldsymbol{\tau}_m^\star = \mu_0^{-1}\left[\mathbf{B}^\star \otimes \mathbf{B}^\star - \frac{1}{2}(\mathbf{B}^\star \cdot \mathbf{B}^\star)\,\mathbf{I}\right]. \tag{9.124}$$

Its increment is given by

$$\dot{\boldsymbol{\tau}}_m^\star = \mu_0^{-1}\left[\dot{\mathbf{B}}^\star \otimes \mathbf{B}^\star + \mathbf{B}^\star \otimes \dot{\mathbf{B}}^\star - \left(\dot{\mathbf{B}}^\star \cdot \mathbf{B}^\star\right)\mathbf{I}\right]. \tag{9.125}$$

On the boundary $\partial\mathcal{B}_r$ the total traction vector is composed of the mechanical part $\mathbf{t}_A$ and the magnetic contribution $\mathbf{t}_M^\star$. The latter is obtained by writing the Lagrangian form of the Maxwell stress that gives the magnetic traction per unit reference area as

$$\mathbf{t}_M^\star = J\boldsymbol{\tau}_m^\star\mathbf{F}^{-T}\mathbf{N} \quad \text{on} \quad \partial\mathcal{B}_r. \tag{9.126}$$

On taking an increment of this equation, we obtain

$$\dot{\mathbf{t}}_M^\star = J\dot{\boldsymbol{\tau}}_m^\star\mathbf{F}^{-T}\mathbf{N} - J\boldsymbol{\tau}_m^\star\mathbf{F}^{-T}\dot{\mathbf{F}}^T\mathbf{F}^{-T}\mathbf{N} + J\,(\operatorname{div}\mathbf{u})\,\boldsymbol{\tau}_m^\star\mathbf{F}^{-T}\mathbf{N}, \tag{9.127}$$

which, on pushing forward to the current configuration and using the incompressibility condition $\operatorname{div}\mathbf{u} = 0$, gives

$$\dot{\mathbf{t}}_{M0}^\star = \dot{\boldsymbol{\tau}}_m^\star\mathbf{n} - \boldsymbol{\tau}_m^\star\mathbf{L}^T\mathbf{n} \quad \text{on} \quad \partial\mathcal{B}, \tag{9.128}$$

where $\dot{\mathbf{t}}_{M0}^\star$ is the push forward of $\dot{\mathbf{t}}_M^\star$.

When there is also a mechanical traction $\mathbf{t}_A$, with increment $\dot{\mathbf{t}}_A$, the incremental traction boundary condition on $\partial\mathcal{B}$ has the form

$$\dot{\mathbf{T}}_0^T\mathbf{n} = \dot{\mathbf{t}}_{A0} + \dot{\mathbf{t}}_{M0}^\star, \tag{9.129}$$

where $\dot{\mathbf{t}}_{A0}$ is the push forward of $\dot{\mathbf{t}}_A$.

9.3.4 Constitutive Relations

9.3.4.1 Formulation Based on $\Omega(\mathbf{F}, \mathbf{H}_L)$

For convenience, following the development outlined in Chap. 6, we review briefly the main aspects of the constitutive equations of magnetoelastic materials based on the use of the independent variables $\mathbf{F}$ and $\mathbf{H}_L$ and the energy function $\Omega = \Omega(\mathbf{F}, \mathbf{H}_L)$. From (6.81) the total nominal stress and the Lagrangian form of the magnetic induction are given by

$$\mathbf{T} = \frac{\partial\Omega}{\partial\mathbf{F}}, \quad \mathbf{B}_L = -\frac{\partial\Omega}{\partial\mathbf{H}_L}. \tag{9.130}$$

Their Eulerian counterparts are

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial\Omega}{\partial\mathbf{F}}, \quad \mathbf{B} = -J^{-1}\mathbf{F}\frac{\partial\Omega}{\partial\mathbf{H}_L}. \tag{9.131}$$

In the case of an incompressible magnetoelastic material the expressions for the total stress, in the reference and current configurations, respectively, are modified to

$$\mathbf{T} = \frac{\partial\Omega}{\partial\mathbf{F}} - p\mathbf{F}^{-1}, \quad \boldsymbol{\tau} = \mathbf{F}\frac{\partial\Omega}{\partial\mathbf{F}} - p\mathbf{I}, \tag{9.132}$$

where p is again a Lagrange multiplier associated with the incompressibility constraint $J = 1$.

For an unconstrained isotropic magnetoelastic material, the energy function Ω is reduced to dependence on six invariants. We take these to be I_1, I_2, I_3, as given in (6.88), and I_4, I_5, I_6, as defined in (6.93).

We then write the energy function in the form $\Omega = \Omega(I_1, I_2, I_3, I_4, I_5, I_6)$, which allows the total nominal stress tensor $\mathbf{T}$ and the Lagrangian magnetic induction vector $\mathbf{B}_L$ to be written as

$$\mathbf{T} = \sum_{n=1\, n\neq 4}^{6} \Omega_n \frac{\partial I_n}{\partial\mathbf{F}}, \quad \mathbf{B}_L = -\sum_{n=4}^{6} \Omega_n \frac{\partial I_n}{\partial\mathbf{H}_L}, \tag{9.133}$$

where $\Omega_n = \partial\Omega/\partial I_n$, $n \in \{1, \dots, 6\}$, and we recall that I_4 is independent of $\mathbf{F}$ and I_1, I_2, I_3 are independent of $\mathbf{H}_\mathrm{L}$.

The required derivatives of the invariants I_1, I_2, I_3, I_5, I_6 with respect to $\mathbf{F}$ are given by (9.74), but with the components of $\mathbf{E}_\mathrm{L}$ replaced by those of $\mathbf{H}_\mathrm{L}$. Thus,

$$\frac{\partial I_1}{\partial F_{i\alpha}} = 2F_{i\alpha}, \quad \frac{\partial I_2}{\partial F_{i\alpha}} = 2(c_{\gamma\gamma}F_{i\alpha} - c_{\alpha\gamma}F_{i\gamma}), \quad \frac{\partial I_3}{\partial F_{i\alpha}} = 2I_3 F_{\alpha i}^{-1}, \tag{9.134}$$

$$\frac{\partial I_5}{\partial F_{i\alpha}} = 2H_{\mathrm{L}\alpha}(F_{i\gamma}H_{\mathrm{L}\gamma}), \quad \frac{\partial I_6}{\partial F_{i\alpha}} = 2(F_{i\gamma}H_{\mathrm{L}\gamma}c_{\alpha\beta}H_{\mathrm{L}\beta} + F_{i\gamma}c_{\gamma\beta}H_{\mathrm{L}\beta}H_{\mathrm{L}\alpha}). \tag{9.135}$$

Similarly, the derivatives of I_4, I_5, I_6 with respect to $\mathbf{H}_\mathrm{L}$ are given in component form by

$$\frac{\partial I_4}{\partial H_{\mathrm{L}\alpha}} = 2H_{\mathrm{L}\alpha}, \quad \frac{\partial I_5}{\partial H_{\mathrm{L}\alpha}} = 2c_{\alpha\beta}H_{\mathrm{L}\beta}, \quad \frac{\partial I_6}{\partial H_{\mathrm{L}\alpha}} = 2c_{\alpha\beta}^2 H_{\mathrm{L}\beta}. \tag{9.136}$$

For an incompressible material, for example, the expression for the total stress tensor $\boldsymbol{\tau}$ does not depend on I_3 and is given by

$$\begin{aligned}\boldsymbol{\tau} = -p\mathbf{I} + 2\Omega_1\mathbf{b} + 2\Omega_2\left(I_1\mathbf{b} - \mathbf{b}^2\right) + 2\Omega_5\mathbf{bH}\otimes\mathbf{bH} \\ + 2\Omega_6\left(\mathbf{bH}\otimes\mathbf{b}^2\mathbf{H} + \mathbf{b}^2\mathbf{H}\otimes\mathbf{bH}\right),\end{aligned} \tag{9.137}$$

as given in (6.94). The Eulerian form of the magnetic induction vector, obtained using $(9.133)_2$ combined with (9.136), becomes

$$\mathbf{B} = -2\left(\Omega_4\mathbf{bH} + \Omega_5\mathbf{b}^2\mathbf{H} + \Omega_6\mathbf{b}^3\mathbf{H}\right), \tag{9.138}$$

as given previously in (6.95).

9.3.4.2 Magnetoelastic Moduli Tensors Based on $\Omega(\mathbf{F}, \mathbf{H}_\mathrm{L})$

We now examine the governing equations describing the linearized response of a magnetoelastic solid superimposed on a state of finite deformation in the presence of a magnetic field for independent incremental changes in the deformation and the magnetic field. By taking the increments of (9.130) we obtain the equations

$$\dot{\mathbf{T}} = \boldsymbol{\mathcal{A}}\dot{\mathbf{F}} + \mathbb{A}\dot{\mathbf{H}}_\mathrm{L}, \quad \dot{\mathbf{B}}_\mathrm{L} = -\mathbb{A}^\mathrm{T}\dot{\mathbf{F}} - \mathsf{A}\dot{\mathbf{H}}_\mathrm{L}, \tag{9.139}$$

which are valid within the material, where the magnetoelastic moduli tensors $\boldsymbol{\mathcal{A}}$, $\mathbb{A}$ and A, respectively fourth-, third- and second-order tensors, are defined by

$$\boldsymbol{\mathcal{A}} = \frac{\partial^2\Omega}{\partial\mathbf{F}\partial\mathbf{F}}, \quad \mathbb{A} = \frac{\partial^2\Omega}{\partial\mathbf{F}\partial\mathbf{H}_\mathrm{L}}, \quad \mathbb{A}^\mathrm{T} = \frac{\partial^2\Omega}{\partial\mathbf{H}_\mathrm{L}\partial\mathbf{F}}, \quad \mathsf{A} = \frac{\partial^2\Omega}{\partial\mathbf{H}_\mathrm{L}\partial\mathbf{H}_\mathrm{L}}. \tag{9.140}$$

Note that we have used the same notation as in the electroelastic case, so these expressions are obtained from their electroelastic counterparts by replacing $\mathbf{E}_\mathrm{L}$ by $\mathbf{H}_\mathrm{L}$. In component form these are

$$\mathcal{A}_{\alpha i\beta j} = \frac{\partial^2\Omega}{\partial F_{i\alpha}\partial F_{j\beta}}, \quad \mathbb{A}_{\alpha i|\beta} = \frac{\partial^2\Omega}{\partial F_{i\alpha}\partial H_{\mathrm{L}\beta}}, \quad \mathsf{A}_{\alpha\beta} = \frac{\partial^2\Omega}{\partial H_{\mathrm{L}\alpha}\partial H_{\mathrm{L}\beta}}, \tag{9.141}$$

and the products in (9.139) are defined by

$$\begin{aligned}(\boldsymbol{\mathcal{A}}\dot{\mathbf{F}})_{\alpha i} &= \mathcal{A}_{\alpha i\beta j}\dot{F}_{j\beta}, \quad (\mathbb{A}\dot{\mathbf{H}}_\mathrm{L})_{\alpha i} = \mathcal{A}_{\alpha i|\beta}\dot{H}_{\mathrm{L}\beta},\\ (\mathbb{A}^\mathrm{T}\dot{\mathbf{F}})_{\beta} &= \mathcal{A}_{\beta|\alpha i}\dot{F}_{i\alpha}, \quad (\mathsf{\mathbf{A}}\dot{\mathbf{H}}_\mathrm{L})_{\alpha} = \mathsf{A}_{\alpha\beta}\dot{H}_{\mathrm{L}\beta}.\end{aligned} \tag{9.142}$$

The symmetries of $\boldsymbol{\mathcal{A}}$ and $\mathsf{\mathbf{A}}$ are as given in the electroelastic case by (9.69).

For unconstrained isotropic magnetoelastic materials, expressions for the magnetoelastic moduli tensors can be expanded as in the case of the electroelastic moduli tensors by replacing $\mathbf{E}_\mathrm{L}$ by $\mathbf{H}_\mathrm{L}$ to give

$$\begin{aligned}\mathcal{A}_{\alpha i\beta j} &= \sum_{m=1,m\neq 4}^{6}\sum_{n=1,n\neq 4}^{6}\Omega_{mn}\frac{\partial I_m}{\partial F_{i\alpha}}\frac{\partial I_n}{\partial F_{j\beta}} + \sum_{n=1,n\neq 4}^{6}\Omega_n\frac{\partial^2 I_n}{\partial F_{i\alpha}\partial F_{j\beta}},\\ \mathbb{A}_{\alpha i|\beta} &= \sum_{m=4}^{6}\sum_{n=1,n\neq 4}^{6}\Omega_{mn}\frac{\partial I_m}{\partial H_{\mathrm{L}\beta}}\frac{\partial I_n}{\partial F_{i\alpha}} + \sum_{n=5}^{6}\Omega_n\frac{\partial^2 I_n}{\partial F_{i\alpha}\partial H_{\mathrm{L}\beta}},\\ \mathsf{A}_{\alpha\beta} &= \sum_{m=4}^{6}\sum_{n=4}^{6}\Omega_{mn}\frac{\partial I_m}{\partial H_{\mathrm{L}\alpha}}\frac{\partial I_n}{\partial H_{\mathrm{L}\beta}} + \sum_{n=4}^{6}\Omega_n\frac{\partial^2 I_n}{\partial H_{\mathrm{L}\alpha}\partial H_{\mathrm{L}\beta}},\end{aligned} \tag{9.143}$$

where $\Omega_n = \partial\Omega/\partial I_n$, $\Omega_{mn} = \partial^2\Omega/\partial I_m\partial I_n$.

For an incompressible material, $(9.139)_1$ is replaced by

$$\dot{\mathbf{T}} = \boldsymbol{\mathcal{A}}\dot{\mathbf{F}} + \mathbb{A}\dot{\mathbf{H}}_\mathrm{L} - \dot{p}\mathbf{F}^{-1} + p\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1}, \tag{9.144}$$

subject to $\operatorname{div}\mathbf{u} = 0$, while $(9.139)_2$ is valid for both compressible and incompressible materials.

We have given the expressions for the first derivatives of the invariants with respect to $\mathbf{F}$ and $\mathbf{H}_\mathrm{L}$. The second derivatives are the same as those given for the electroelastic case in (9.75), (9.76) and (9.77) except that $\mathbf{E}_\mathrm{L}$ is replaced by $\mathbf{H}_\mathrm{L}$. We do not therefore repeat these here but refer to (9.75), (9.76) and (9.77).

To evaluate the effects of an incremental deformation and/or an increment in the applied magnetic field, it is convenient to update the reference configuration from $\mathcal{B}_\mathrm{r}$ to coincide with the current configuration $\mathcal{B}$. This is accomplished by using the 'push forward' operations in (9.44) specialized to the magnetoelastic case. Specifically, these are

$$\dot{\mathbf{T}}_0 = J^{-1}\mathbf{F}\dot{\mathbf{T}}, \quad \dot{\mathbf{B}}_{\mathrm{L}0} = J^{-1}\mathbf{F}\dot{\mathbf{B}}_\mathrm{L}, \quad \dot{\mathbf{H}}_{\mathrm{L}0} = \mathbf{F}^{-\mathrm{T}}\dot{\mathbf{H}}_\mathrm{L}. \tag{9.145}$$

Applying the transformation laws (9.145) to (9.139) we obtain

$$\dot{\mathbf{T}}_0 = \boldsymbol{\mathcal{A}}_0\mathbf{L} + \mathbb{A}_0\dot{\mathbf{H}}_{L0}, \quad \dot{\mathbf{B}}_{L0} = -\mathbb{A}_0^{T}\mathbf{L} - \mathsf{A}_0\dot{\mathbf{H}}_{L0}, \tag{9.146}$$

where we recall from (9.32) that $\dot{\mathbf{F}} = \mathbf{LF}$ with $\mathbf{L} = \operatorname{grad}\mathbf{u}$. The increment of the total nominal stress for an incompressible material is given by (9.144), which after updating the reference configuration is modified to

$$\dot{\mathbf{T}}_0 = \boldsymbol{\mathcal{A}}_0\mathbf{L} + \mathbb{A}_0\dot{\mathbf{H}}_{L0} - \dot{p}\mathbf{I} + p\mathbf{L}, \tag{9.147}$$

thus replacing $(9.146)_1$ in this case.

The components of the tensors $\boldsymbol{\mathcal{A}}_0$, $\mathbb{A}_0$ and A_0 are given in terms of those of $\boldsymbol{\mathcal{A}}$, $\mathbb{A}$ and A in exactly the same way as in electroelasticity, i.e. by (9.79)–(9.81), with $J = 1$ for an incompressible material. They also have the symmetry properties given by (9.84), (9.85), (9.87) and (9.88). Furthermore, the components of the magnetoelastic moduli tensors are given by the same formulas as in the electroelastic case, except that $\mathbf{E}_L$ is replaced by $\mathbf{H}_L$.

In summary, the governing equations for the incremental (infinitesimal) disturbances $\dot{\mathbf{H}}_{L0}$, $\dot{\mathbf{B}}_{L0}$ and $\dot{\mathbf{T}}_0$ superimposed on a finitely deformed magnetoelastic body in the presence of a magnetic field are given by (9.116), and after the substitution of the constitutive laws, (9.146) yield equations for the displacement $\mathbf{u}$ and $\dot{\mathbf{H}}_{L0}$, namely

$$\operatorname{curl}\dot{\mathbf{H}}_{L0} = \mathbf{0}, \quad \operatorname{div}(\mathbb{A}_0^{T}\mathbf{L} + \mathsf{A}_0\dot{\mathbf{H}}_{L0}) = 0, \quad \operatorname{div}(\boldsymbol{\mathcal{A}}_0\mathbf{L} + \mathbb{A}_0\dot{\mathbf{H}}_{L0}) = \mathbf{0}. \tag{9.148}$$

For an incompressible material, the latter equation is replaced by

$$\operatorname{div}(\boldsymbol{\mathcal{A}}_0\mathbf{L} + \mathbb{A}_0\dot{\mathbf{H}}_{L0}) - \operatorname{grad}\dot{p} + \mathbf{L}^{T}\operatorname{grad}p = \mathbf{0} \tag{9.149}$$

and the displacement satisfies the incompressibility condition $\operatorname{div}\mathbf{u} = 0$.

In either case $\dot{\mathbf{H}}_{L0}$ can be written as $-\operatorname{grad}\dot{\varphi}$, where $\dot{\varphi}$ is a scalar function, which, along with $\mathbf{u}$, is to be determined by the remaining two equations (one scalar equation and one vector equation).

9.3.4.3 Formulation Based on $\Omega^*(\mathbf{F}, \mathbf{B}_L)$

In this formulation we use the deformation gradient tensor $\mathbf{F}$ and the Lagrangian form of the magnetic induction vector $\mathbf{B}_L$ as the independent variables and the energy function $\Omega^* = \Omega^*(\mathbf{F}, \mathbf{B}_L)$. The total nominal stress $\mathbf{T}$ and the Lagrangian form of the magnetic field $\mathbf{H}_L$ for an unconstrained material are given by the simple relations

$$\mathbf{T} = \frac{\partial\Omega^*}{\partial\mathbf{F}}, \quad \mathbf{H}_L = \frac{\partial\Omega^*}{\partial\mathbf{B}_L}, \tag{9.150}$$

and the corresponding Eulerian forms of the total stress and magnetic field are

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial \Omega^*}{\partial \mathbf{F}}, \quad \mathbf{H} = \mathbf{F}^{-\mathrm{T}}\frac{\partial \Omega^*}{\partial \mathbf{B}_{\mathrm{L}}}. \tag{9.151}$$

In the case of an incompressible magnetoelastic material the expressions for the total nominal and Cauchy stresses are modified to

$$\mathbf{T} = \frac{\partial \Omega^*}{\partial \mathbf{F}} - p^*\mathbf{F}^{-1}, \quad \boldsymbol{\tau} = \mathbf{F}\frac{\partial \Omega^*}{\partial \mathbf{F}} - p^*\mathbf{I}, \tag{9.152}$$

where p^* is again a Lagrange multiplier associated with the constraint $J = 1$.

For an unconstrained isotropic magnetoelastic material, the energy function Ω^* depends on six invariants. For consistency with the electroelastic case we use the notation K_1, K_2, K_3 instead of I_1, I_2, I_3 for the principal invariants of the right Cauchy–Green tensor $\mathbf{c}$ and K_4, K_5, K_6 for the invariants based on $\mathbf{B}_{\mathrm{L}}$, obtained by replacing $\mathbf{D}_{\mathrm{L}}$ by $\mathbf{B}_{\mathrm{L}}$ in (9.96). Thus,

$$K_1 = \mathrm{tr}\,\mathbf{c}, \quad K_2 = \frac{1}{2}\left[(\mathrm{tr}\,\mathbf{c})^2 - \mathrm{tr}\,(\mathbf{c}^2)\right], \quad K_3 = \det\mathbf{c}, \tag{9.153}$$

$$K_4 = \mathbf{B}_{\mathrm{L}}\cdot\mathbf{B}_{\mathrm{L}}, \quad K_5 = (\mathbf{c}\mathbf{B}_{\mathrm{L}})\cdot\mathbf{B}_{\mathrm{L}}, \quad K_6 = (\mathbf{c}^2\mathbf{B}_{\mathrm{L}})\cdot\mathbf{B}_{\mathrm{L}}. \tag{9.154}$$

The energy function is then written $\Omega^* = \Omega^*(K_1, K_2, K_3, K_4, K_5, K_6)$, and we write the total nominal stress tensor $\mathbf{T}$ and the Lagrangian magnetic field vector $\mathbf{H}_{\mathrm{L}}$ in the expanded forms

$$\mathbf{T} = \sum_{n=1, n\neq 4}^{6} \Omega_n^* \frac{\partial K_n}{\partial \mathbf{F}}, \quad \mathbf{H}_{\mathrm{L}} = \sum_{n=4}^{6} \Omega_n^* \frac{\partial K_n}{\partial \mathbf{B}_{\mathrm{L}}}, \tag{9.155}$$

where $\Omega_n^* = \partial\Omega^*/\partial K_n$, $n \in \{1, \ldots, 6\}$. The derivatives of K_n are obtained from (9.98) with $\mathbf{D}_{\mathrm{L}}$ replaced by $\mathbf{B}_{\mathrm{L}}$ and not repeated here.

The total Cauchy stress tensor $\boldsymbol{\tau}$ is then given by

$$J\boldsymbol{\tau} = 2\Omega_1^*\mathbf{b} + 2\Omega_2^*\left(K_1\mathbf{b} - \mathbf{b}^2\right) + 2K_3\Omega_3^* + 2\Omega_5^*\mathbf{B}\otimes\mathbf{B} + 2\Omega_6^*(\mathbf{B}\otimes\mathbf{b}\mathbf{B} + \mathbf{b}\mathbf{B}\otimes\mathbf{B}) \tag{9.156}$$

for an unconstrained material and

$$\boldsymbol{\tau} = -p^*\mathbf{I} + 2\Omega_1^*\mathbf{b} + 2\Omega_2^*\left(K_1\mathbf{b} - \mathbf{b}^2\right) + 2\Omega_5^*\mathbf{B}\otimes\mathbf{B} + 2\Omega_6^*(\mathbf{B}\otimes\mathbf{b}\mathbf{B} + \mathbf{b}\mathbf{B}\otimes\mathbf{B}) \tag{9.157}$$

for an incompressible material. The Eulerian form of the magnetic field vector, obtained from $(9.151)_2$, becomes

$$\mathbf{H} = 2(\Omega_4^*\mathbf{b}^{-1}\mathbf{B} + \Omega_5^*\mathbf{B} + \Omega_6^*\mathbf{b}\mathbf{B}), \tag{9.158}$$

as in (6.91).

9.3.4.4 Magnetoelastic Moduli Tensors Based on $\Omega^*(\mathbf{F}, \mathbf{B}_\mathrm{L})$

We now examine the governing equations describing the linearized response of a magnetoelastic solid superimposed on a state of finite deformation in the presence of a magnetic field for independent incremental changes in the magnetic induction and the deformation. By taking the increments of (9.150) we obtain the equations

$$\dot{\mathbf{T}} = \boldsymbol{\mathcal{A}}^*\dot{\mathbf{F}} + \mathbb{A}^*\dot{\mathbf{B}}_\mathrm{L}, \quad \dot{\mathbf{H}}_\mathrm{L} = \mathbb{A}^{*\mathrm{T}}\dot{\mathbf{F}} + \mathsf{A}^*\dot{\mathbf{B}}_\mathrm{L} \tag{9.159}$$

within the material body.

For an incompressible material, $(9.159)_1$ is replaced by

$$\dot{\mathbf{T}} = \boldsymbol{\mathcal{A}}^*\dot{\mathbf{F}} + \mathbb{A}^*\dot{\mathbf{B}}_\mathrm{L} - \dot{p}^*\mathbf{F}^{-1} + p^*\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1}, \tag{9.160}$$

subject to $\operatorname{div}\mathbf{u} = 0$. Equation $(9.159)_2$ for the increment of the magnetic field is valid for both compressible and incompressible materials.

The magnetoelastic moduli tensors $\boldsymbol{\mathcal{A}}^*$, $\mathbb{A}^*$ and A^* are defined by

$$\boldsymbol{\mathcal{A}}^* = \frac{\partial^2\Omega^*}{\partial\mathbf{F}\partial\mathbf{F}}, \quad \mathbb{A}^* = \frac{\partial^2\Omega^*}{\partial\mathbf{F}\partial\mathbf{B}_\mathrm{L}}, \quad \mathbb{A}^{*\mathrm{T}} = \frac{\partial^2\Omega^*}{\partial\mathbf{B}_\mathrm{L}\partial\mathbf{F}}, \quad \mathsf{A}^* = \frac{\partial^2\Omega^*}{\partial\mathbf{B}_\mathrm{L}\partial\mathbf{B}_\mathrm{L}} \tag{9.161}$$

in the same way as for an electroelastic material except that $\mathbf{D}_\mathrm{L}$ is replaced by $\mathbf{B}_\mathrm{L}$. In component form we have the representations

$$\mathcal{A}^*_{\alpha i\beta j} = \frac{\partial^2\Omega^*}{\partial F_{i\alpha}\partial F_{j\beta}}, \quad \mathbb{A}^*_{\alpha i|\beta} = \frac{\partial^2\Omega^*}{\partial F_{i\alpha}\partial B_{\mathrm{L}\beta}}, \quad \mathsf{A}^*_{\alpha\beta} = \frac{\partial^2\Omega^*}{\partial B_{\mathrm{L}\alpha}\partial B_{\mathrm{L}\beta}}. \tag{9.162}$$

The products in (9.159) are defined as previously, and the symmetries from before also follow here.

For unconstrained isotropic magnetoelastic materials, the expressions for the magnetoelastic moduli tensors (9.162) can be expanded in the forms

$$\begin{aligned}
\mathcal{A}^*_{\alpha i\beta j} &= \sum_{m=1,m\neq4}^{6}\sum_{n=1,n\neq4}^{6}\Omega^*_{mn}\frac{\partial K_m}{\partial F_{i\alpha}}\frac{\partial K_n}{\partial F_{j\beta}} + \sum_{n=1,n\neq4}^{6}\Omega^*_n\frac{\partial^2 K_n}{\partial F_{i\alpha}\partial F_{j\beta}},\\
\mathbb{A}^*_{\alpha i|\beta} &= \sum_{m=4}^{6}\sum_{n=1,n\neq4}^{6}\Omega^*_{mn}\frac{\partial K_m}{\partial B_{\mathrm{L}\beta}}\frac{\partial K_n}{\partial F_{i\alpha}} + \sum_{n=5}^{6}\Omega^*_n\frac{\partial^2 K_n}{\partial F_{i\alpha}\partial B_{\mathrm{L}\beta}},\\
\mathsf{A}^*_{\alpha\beta} &= \sum_{m=4}^{6}\sum_{n=4}^{6}\Omega^*_{mn}\frac{\partial K_m}{\partial B_{\mathrm{L}\alpha}}\frac{\partial K_n}{\partial B_{\mathrm{L}\beta}} + \sum_{n=4}^{6}\Omega^*_n\frac{\partial^2 K_n}{\partial B_{\mathrm{L}\alpha}\partial B_{\mathrm{L}\beta}},
\end{aligned} \tag{9.163}$$

where $\Omega^*_n = \partial\Omega^*/\partial K_n$, $\Omega^*_{mn} = \partial^2\Omega^*/\partial K_m\partial K_n$, $m, n \in \{1, \ldots, 6\}$.

To obtain the explicit expressions for (9.163), the first and second derivatives of $K_n, n = 1, \ldots, 6$, with respect to $\mathbf{F}$ and $\mathbf{B}_\mathrm{L}$ are needed. The first derivatives have been given above, while the second derivatives are identical to those for Ω^* in the electroelastic case except that $\mathbf{D}_\mathrm{L}$ is replaced by $\mathbf{B}_\mathrm{L}$.

From (9.159) and (9.145) we obtain the updated incremental equations

$$\dot{\mathbf{T}}_0 = \boldsymbol{\mathcal{A}}_0^*\mathbf{L} + \mathbb{A}_0^*\dot{\mathbf{B}}_{\mathrm{L}0}, \quad \dot{\mathbf{H}}_{\mathrm{L}0} = \mathbb{A}_0^{*\mathrm{T}}\mathbf{L} + \mathsf{A}_0^*\dot{\mathbf{B}}_{\mathrm{L}0}, \tag{9.164}$$

where $\boldsymbol{\mathcal{A}}_0^*$, $\mathbb{A}_0^*$ and A_0^* are related to $\boldsymbol{\mathcal{A}}^*$, $\mathbb{A}^*$ and A^* as in the electroelastic case by (9.104)–(9.106). For an incompressible material, the first of these is replaced by

$$\dot{\mathbf{T}}_0 = \boldsymbol{\mathcal{A}}_0^*\mathbf{L} + \mathbb{A}_0^*\dot{\mathbf{B}}_{\mathrm{L}0} - \dot{p}^*\mathbf{I} + p^*\mathbf{L}. \tag{9.165}$$

The tensors $\boldsymbol{\mathcal{A}}_0^*$, $\mathbb{A}_0^*$ and A_0^* have the same symmetry properties as in the electroelastic case.

The equations to be satisfied by the increments are again (9.116), and when the incremental constitutive equations (9.164) are substituted, the complete set of equations for the displacement $\mathbf{u}$ and the incremental magnetic induction are

$$\operatorname{div}\dot{\mathbf{B}}_{\mathrm{L}0} = 0, \quad \operatorname{curl}(\mathbb{A}_0^{*\mathrm{T}}\mathbf{L} + \mathsf{A}_0^*\dot{\mathbf{B}}_{\mathrm{L}0}) = \mathbf{0}, \quad \operatorname{div}(\boldsymbol{\mathcal{A}}_0^*\mathbf{L} + \mathbb{A}_0^*\dot{\mathbf{B}}_{\mathrm{L}0}) = \mathbf{0}. \tag{9.166}$$

For incompressible materials the latter equation is replaced by

$$\operatorname{div}(\boldsymbol{\mathcal{A}}_0^*\mathbf{L} + \mathbb{A}_0^*\dot{\mathbf{B}}_{\mathrm{L}0}) - \operatorname{grad}\dot{p}^* + \mathbf{L}^\mathrm{T}\operatorname{grad}p^* = \mathbf{0}, \tag{9.167}$$

which is accompanied by the incremental incompressibility condition $\operatorname{div}\mathbf{u} = 0$.

The components of $\boldsymbol{\mathcal{A}}_0^*$, $\mathbb{A}_0^*$ and A_0^* are given by the same formulas, namely (9.107), (9.108) and (9.109), as in the electroelastic case except that $\mathbf{D}_\mathrm{L}$ is replaced by $\mathbf{B}_\mathrm{L}$, so we do not repeat the formulas here. However, it is also useful to provide expressions for these components referred to the principal axes of the left Cauchy–Green tensor $\mathbf{b}$. Then, in terms of the principal stretches $\lambda_1, \lambda_2, \lambda_3$ and the components (B_1, B_2, B_3) of the magnetic induction vector $\mathbf{B}$, we list the different components of the moduli tensors $\boldsymbol{\mathcal{A}}_0^*, \mathbb{A}_0^*$ and A_0^* as follows. In the following expressions the indices are such that $i \neq j \neq k \neq i$, and we list them differently compared with the electroelastic case:

$$\begin{aligned}
J\mathcal{A}_{0iiii}^* = {} & 2\lambda_i^2[\Omega_1^* + (\lambda_j^2 + \lambda_k^2)\Omega_2^* + \lambda_j^2\lambda_k^2\Omega_3^* + \lambda_j^2\lambda_k^2 B_i^2(\Omega_5^* + 6\lambda_i^2\Omega_6^*)] \\
& + 4\lambda_i^4\{\Omega_{11}^* + 2(\lambda_j^2 + \lambda_k^2)\Omega_{12}^* + (\lambda_j^2 + \lambda_k^2)^2\Omega_{22}^* \\
& + \lambda_j^2\lambda_k^2[2\Omega_{13}^* + 2(\lambda_j^2 + \lambda_k^2)\Omega_{23}^* + \lambda_j^2\lambda_k^2\Omega_{33}^*] + 2\lambda_j^2\lambda_k^2 B_i^2[\Omega_{15}^* + 2\lambda_i^2\Omega_{16}^* \\
& + (\lambda_j^2 + \lambda_k^2)\Omega_{25}^* + 2\lambda_i^2(\lambda_j^2 + \lambda_k^2)\Omega_{26}^* + \lambda_j^2\lambda_k^2\Omega_{35}^* + 2I_3\Omega_{36}^*] \\
& + \lambda_j^4\lambda_k^4 B_i^4(\Omega_{55}^* + 4\lambda_i^2\Omega_{56}^* + 4\lambda_i^4\Omega_{66}^*)\},
\end{aligned}$$

$$\begin{aligned} J\mathcal{A}^*_{0iiij} &= 4B_iB_jK_3\lambda_i^2\{\Omega_6^* + \Omega_{15}^* + (\lambda_j^2 + \lambda_k^2)\Omega_{25}^* + \lambda_j^2\lambda_k^2\Omega_{35}^* \\ &+ (\lambda_i^2 + \lambda_j^2)[\Omega_{16}^* + (\lambda_j^2 + \lambda_k^2)\Omega_{26}^* + \lambda_j^2\lambda_k^2\Omega_{36}^*] \\ &+ \lambda_j^2\lambda_k^2B_i^2[\Omega_{55}^* + (3\lambda_i^2 + \lambda_j^2)\Omega_{56}^* + 2\lambda_i^2(\lambda_i^2 + \lambda_j^2)\Omega_{66}^*]\}, \end{aligned}$$

$$\begin{aligned} J\mathcal{A}^*_{0iiji} &= 2B_iB_jK_3\{\Omega_5^* + (\lambda_j^2 + 3\lambda_i^2)\Omega_6^* \\ &+ 2\lambda_i^2[\Omega_{15}^* + (\lambda_j^2 + \lambda_k^2)\Omega_{25}^* + \lambda_j^2\lambda_k^2\Omega_{35}^*] \\ &+ 2\lambda_i^2(\lambda_i^2 + \lambda_j^2)[\Omega_{16}^* + (\lambda_j^2 + \lambda_k^2)\Omega_{26}^* + \lambda_j^2\lambda_k^2\Omega_{36}^*] \\ &+ 2K_3B_i^2[\Omega_{55}^* + (3\lambda_i^2 + \lambda_j^2)\Omega_{56}^* + 2\lambda_i^2(\lambda_i^2 + \lambda_j^2)\Omega_{66}^*]\}, \end{aligned}$$

$$\begin{aligned} J\mathcal{A}^*_{0iijj} &= 4\lambda_i^2\lambda_j^2\{\Omega_2^* + \lambda_k^2\Omega_3^* + \Omega_{11}^* + (K_1 + \lambda_k^2)\Omega_{12}^* + (K_2 + \lambda_k^4)\Omega_{22}^* \\ &+ \lambda_k^2[(\lambda_i^2 + \lambda_j^2)\Omega_{13}^* + (K_2 + \lambda_i^2\lambda_j^2)\Omega_{23}^* + K_3\Omega_{33}^*] \\ &+ \lambda_k^2(\lambda_j^2B_i^2 + \lambda_i^2B_j^2)(\Omega_{15}^* + \lambda_k^2\Omega_{25}^*) \\ &+ 2K_3(\lambda_i^2B_i^2 + \lambda_j^2B_j^2)(\Omega_{26}^* + \lambda_k^2\Omega_{36}^*) \\ &+ K_3(B_i^2 + B_j^2)(2\Omega_{16}^* + \Omega_{25}^* + 2\lambda_k^2\Omega_{26}^* + \lambda_k^2\Omega_{35}^*) \\ &+ K_3\lambda_k^2B_i^2B_j^2[\Omega_{55}^* + 2(\lambda_i^2 + \lambda_j^2)\Omega_{56}^* + 4\lambda_i^2\lambda_j^2\Omega_{66}^*]\}, \end{aligned}$$

$$\begin{aligned} J\mathcal{A}^*_{0ijij} &= 2\lambda_i^2\{\Omega_1^* + \lambda_k^2\Omega_2^* + B_i^2\lambda_j^2\lambda_k^2\Omega_5^* + \lambda_j^2\lambda_k^2(2B_i^2\lambda_i^2 + B_i^2\lambda_j^2 + B_j^2\lambda_i^2)\Omega_6^* \\ &+ 2B_i^2B_j^2K_3\lambda_j^2\lambda_k^2[\Omega_{55}^* + 2(\lambda_i^2 + \lambda_j^2)\Omega_{56}^* + (\lambda_i^2 + \lambda_j^2)^2\Omega_{66}^*]\}, \end{aligned}$$

$$\begin{aligned} J\mathcal{A}^*_{0ijji} &= 2\lambda_i^2\lambda_j^2\{-\Omega_2^* - \lambda_k^2\Omega_3^* + \lambda_k^2(\lambda_j^2B_i^2 + \lambda_i^2B_j^2)\Omega_6^* \\ &+ 2B_i^2B_j^2K_3\lambda_k^2[\Omega_{55}^* + 2(\lambda_i^2 + \lambda_j^2)\Omega_{56}^* + (\lambda_i^2 + \lambda_j^2)^2\Omega_{66}^*]\}, \end{aligned}$$

$$\begin{aligned} J\mathcal{A}^*_{0iijk} &= 4B_jB_kK_3\lambda_i^2\{\Omega_{15}^* + (\lambda_j^2 + \lambda_k^2)(\Omega_{25}^* + \Omega_{16}^*) + (\lambda_j^2 + \lambda_k^2)^2\Omega_{26}^* \\ &+ \lambda_j^2\lambda_k^2\Omega_{35}^* + \lambda_j^2\lambda_k^2(\lambda_j^2 + \lambda_k^2)\Omega_{36}^* + B_i^2\lambda_j^2\lambda_k^2[\Omega_{55}^* + (K_1 + \lambda_i^2)\Omega_{56}^* \\ &+ 2\lambda_i^2(\lambda_j^2 + \lambda_k^2)\Omega_{66}^*]\}, \end{aligned}$$

$$J\mathcal{A}^*_{0ijki} = J\mathcal{A}_{0ijik} = 2B_jB_kK_3\{\lambda_i^2\Omega_6^* + 2B_i^2K_3[\Omega^*_{55} + (K_1+\lambda_i^2)\Omega^*_{56} + (K_2+\lambda_i^4)\Omega^*_{66}]\},$$

$$J\mathcal{A}^*_{0jiki} = 2B_jB_kK_3\{\Omega_5^* + K_1\Omega_6^* + 2B_i^2K_3[\Omega^*_{55} + (K_1+\lambda_i^2)\Omega^*_{56} + (K_2+\lambda_i^4)\Omega^*_{66}]\},$$

$$\begin{aligned} J^{-1}\mathbb{A}^*_{0ii|i} = 4B_i\{&\Omega_5^* + 2\lambda_i^2\Omega_6^* + \Omega^*_{14} + \lambda_i^2\Omega^*_{15} + \lambda_i^4\Omega^*_{16} \\ &+ (\lambda_j^2+\lambda_k^2)(\Omega^*_{24} + \lambda_i^2\Omega^*_{25} + \lambda_i^4\Omega^*_{26}) + \lambda_j^2\lambda_k^2(\Omega^*_{34} + \lambda_i^2\Omega^*_{35} + \lambda_i^4\Omega^*_{36}) \\ &+ B_i^2\lambda_j^2\lambda_k^2[\Omega^*_{45} + \lambda_i^2\Omega^*_{55} + \lambda_i^4\Omega^*_{56} + 2\lambda_i^2(\Omega^*_{46} + \lambda_i^2\Omega^*_{56} + \lambda_i^4\Omega^*_{66})]\}, \end{aligned}$$

$$\begin{aligned} J^{-1}\mathbb{A}^*_{0ii|j} = 4B_j\lambda_i^2\lambda_j^{-2}\{&\Omega^*_{14} + \lambda_j^2\Omega^*_{15} + \lambda_j^4\Omega^*_{16} + (\lambda_j^2+\lambda_k^2)(\Omega^*_{24} + \lambda_j^2\Omega^*_{25} \\ &+ \lambda_j^4\Omega^*_{26}) + \lambda_j^2\lambda_k^2(\Omega^*_{34} + \lambda_j^2\Omega^*_{35} + \lambda_j^4\Omega^*_{36}) + B_i^2\lambda_j^2\lambda_k^2[\Omega^*_{45} + \lambda_j^2\Omega^*_{55} \\ &+ \lambda_j^4\Omega^*_{56} + 2\lambda_i^2(\Omega^*_{46} + \lambda_j^2\Omega^*_{56} + \lambda_j^4\Omega^*_{66})]\}, \end{aligned}$$

$$\begin{aligned} J^{-1}\mathbb{A}^*_{0ij|i} = 2B_j\{&\Omega_5^* + (\lambda_i^2+\lambda_j^2)\Omega_6^* + 2B_i^2\lambda_j^2\lambda_k^2[\Omega^*_{45} + \lambda_i^2\Omega^*_{55} + \lambda_i^4\Omega^*_{56} \\ &+ (\lambda_i^2+\lambda_j^2)(\Omega^*_{46} + \lambda_i^2\Omega^*_{56} + \lambda_i^4\Omega^*_{66})]\}, \end{aligned}$$

$$\begin{aligned} J^{-1}\mathbb{A}^*_{0ij|k} = 4B_iB_jB_k\lambda_i^2\lambda_j^2[&\Omega^*_{45} + \lambda_k^2\Omega^*_{55} + \lambda_k^4\Omega^*_{56} \\ &+ (\lambda_i^2+\lambda_j^2)(\Omega^*_{46} + \lambda_k^2\Omega^*_{56} + \lambda_k^4\Omega^*_{66})], \end{aligned}$$

$$\begin{aligned} J^{-1}\mathsf{A}^*_{0ii} = 2\lambda_i^{-2}\{&\Omega_4^* + \lambda_i^2\Omega_5^* + \lambda_i^4\Omega_6^* + 2B_i^2\lambda_j^2\lambda_k^2[\Omega^*_{44} + \lambda_i^2\Omega^*_{45} + \lambda_i^4\Omega^*_{46} \\ &+ \lambda_i^2(\Omega^*_{45} + \lambda_i^2\Omega^*_{55} + \lambda_i^4\Omega^*_{56}) + \lambda_i^4(\Omega^*_{46} + \lambda_i^2\Omega^*_{56} + \lambda_i^4\Omega^*_{66}]\}, \end{aligned}$$

$$\begin{aligned} J^{-1}\mathsf{A}^*_{0ij} = 4B_iB_j\lambda_k^2[&\Omega^*_{44} + \lambda_i^2\Omega^*_{45} + \lambda_i^4\Omega^*_{46} + \lambda_j^2(\Omega^*_{45} + \lambda_i^2\Omega^*_{55} + \lambda_i^4\Omega^*_{56}) \\ &+ \lambda_j^4(\Omega^*_{46} + \lambda_i^2\Omega^*_{56} + \lambda_i^4\Omega^*_{66})]. \end{aligned}$$

For an incompressible material, these relations remain valid by omitting all terms with derivatives of Ω^* with respect to K_3 and setting $J = K_3 = 1$.

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Chapter 10
Electroelastic Stability

Abstract In this chapter the incremental equations governing static electroelastic interactions are used to analyze the stability of a half-space of an electroelastic material subject to a pure homogeneous strain and an electric field normal to its plane boundary. The analysis is formulated for a general isotropic electroelastic energy function and for plane strain. The results are illustrated for a simple neo-Hookean electroelastic model. In particular, the critical stretch corresponding to loss of stability is obtained as a function of the electric field for a series of values of the material parameters included in the model. In general the half-space is more unstable when an electric field is applied compared with the classical problem in which the stability of a half-space is lost under compression parallel to its boundary, but for a range of values of the electric field and the material parameters, stability is enhanced. The problem of a plate of finite thickness is then analyzed and the stability characteristics determined in terms of the plate thickness for a plate with or without flexible electrodes.

10.1 Preliminary Remarks

In Chap. 9 we derived the incremental Lagrangian forms of Maxwell's equations, of the charge conservation equation and of the incremental equation of motion. We also derived the corresponding expressions for the incremental constitutive equations, the incremental electromagnetic and mechanical boundary conditions and the incremental changes in the fields surrounding the material. Updated incremental variables were introduced to derive Eulerian counterparts of these equations. In this chapter we focus on the stability of the equilibrium configuration of a finitely deformed electroelastic body subject to static fields, while in Chap. 11 by way of contrast we apply the incremental equations of motion in a study of magnetoelastic waves propagating in a deformed magnetoelastic material. The stability analysis in this chapter is based partly on the paper by Dorfmann and Ogden (2010a), which has also been used in the analysis of the stability of layered dielectric

L. Dorfmann and R.W. Ogden, *Nonlinear Theory of Electroelastic and Magnetoelastic Interactions*, DOI 10.1007/978-1-4614-9596-3_10,

materials in Bertoldi and Gei (2011) and Rudykh and deBotton (2011). For a parallel study of electroelastic waves, we refer to Dorfmann and Ogden (2010b), and for various different aspects of wave propagation in electroelastic materials, we mention Carroll (1972), Baumhauer and Tiersten (1973), Paria (1973), Paria (1974), Sinha and Tiersten (1979), Pouget and Maugin (1981), Chai and Wu (1996), Li (1996), Simionescu-Panait (2002), Baesu et al. (2003), Liu et al. (2003), Hu et al. (2004), Yang and Hu (2004), Yang (2001), Collet et al. (2006), Singh (2009), Chen and Dai (2012), Shmuel et al. (2012), and Shmuel and deBotton (2013) as representative examples.

10.2 Governing Equations

In this section we examine the effect of an incremental deformation combined with an increment in the electric field or the electric displacement superimposed on an underlying finitely deformed configuration in which there is also an electric field. We focus the attention on the incremental governing equations of electroelastic materials, which can be summarized as

$$\mathrm{Div}\,\dot{\mathbf{T}} = \mathbf{0}, \quad \mathrm{Curl}\,\dot{\mathbf{E}}_\mathrm{L} = \mathbf{0}, \quad \mathrm{Div}\,\dot{\mathbf{D}}_\mathrm{L} = 0, \tag{10.1}$$

where $\dot{\mathbf{T}}$, $\dot{\mathbf{E}}_\mathrm{L}$ and $\dot{\mathbf{D}}_\mathrm{L}$ denote the increments of the total nominal stress tensor and of the Lagrangian form of the electric field and electric displacement, respectively. The Eulerian forms of these increments are denoted by $\dot{\mathbf{T}}_0$, $\dot{\mathbf{E}}_{\mathrm{L}0}$ and $\dot{\mathbf{D}}_{\mathrm{L}0}$ and are given by

$$\dot{\mathbf{T}}_0 = J^{-1}\mathbf{F}\dot{\mathbf{T}}, \quad \dot{\mathbf{E}}_{\mathrm{L}0} = \mathbf{F}^{-\mathrm{T}}\dot{\mathbf{E}}_\mathrm{L}, \quad \dot{\mathbf{D}}_{\mathrm{L}0} = J^{-1}\mathbf{F}\dot{\mathbf{D}}_\mathrm{L}, \tag{10.2}$$

which must satisfy the Eulerian counterparts of (10.1), namely

$$\mathrm{div}\,\dot{\mathbf{T}}_0 = \mathbf{0}, \quad \mathrm{curl}\,\dot{\mathbf{E}}_{\mathrm{L}0} = \mathbf{0}, \quad \mathrm{div}\,\dot{\mathbf{D}}_{\mathrm{L}0} = 0. \tag{10.3}$$

In a vacuum or non-polarizable material the increments of the electric field $\dot{\mathbf{E}}^\star$ and electric displacement $\dot{\mathbf{D}}^\star$ satisfy the incremental forms of Maxwell's equations

$$\mathrm{curl}\,\dot{\mathbf{E}}^\star = \mathbf{0}, \quad \mathrm{div}\,\dot{\mathbf{D}}^\star = 0, \tag{10.4}$$

which replace $(10.3)_{2,3}$.

The incremental boundary conditions have been given in Sect. 9.3.1.2 and are included here for completeness. The Lagrangian and Eulerian forms of the incremental traction boundary conditions are, respectively, given by

$$\dot{\mathbf{T}}^\mathrm{T}\mathbf{N} = \dot{\mathbf{t}}_\mathrm{A} + J\dot{\boldsymbol{\tau}}^\star_\mathrm{e}\mathbf{F}^{-\mathrm{T}}\mathbf{N} - J\boldsymbol{\tau}^\star_\mathrm{e}\mathbf{F}^{-\mathrm{T}}\dot{\mathbf{F}}^\mathrm{T}\mathbf{F}^{-\mathrm{T}}\mathbf{N} + \dot{J}\boldsymbol{\tau}^\star_\mathrm{e}\mathbf{F}^{-\mathrm{T}}\mathbf{N}, \tag{10.5}$$

$$\dot{\mathbf{T}}_0^\mathrm{T}\mathbf{n} = \dot{\mathbf{t}}_{\mathrm{A}0} + \dot{\boldsymbol{\tau}}^\star_\mathrm{e}\mathbf{n} - \boldsymbol{\tau}^\star_\mathrm{e}\mathbf{L}^\mathrm{T}\mathbf{n} + (\mathrm{div}\,\mathbf{u})\boldsymbol{\tau}^\star_\mathrm{e}\mathbf{n}, \tag{10.6}$$

where $\dot{\boldsymbol{\tau}}_{\mathrm{e}}^{\star}$ is the increment of the Maxwell stress defined by (9.58) and $\mathbf{u}$ denotes the incremental displacement $\dot{\mathbf{x}}$ treated as a function of $\mathbf{x}$. We also recall that $\mathbf{L}$ denotes the gradient of the incremental displacement vector $\mathbf{u}$.

The incremental boundary conditions for the electric field variables superimposed on the underlying configuration were derived in (9.63) and (9.64) and are included here as well. These are

$$[\dot{\mathbf{D}}_{\mathrm{L}0} - \dot{\mathbf{D}}^{\star} + \mathbf{L}\mathbf{D}^{\star} - (\operatorname{div}\mathbf{u})\mathbf{D}^{\star}] \cdot \mathbf{n} = 0, \tag{10.7}$$

and

$$(\dot{\mathbf{E}}_{\mathrm{L}0} - \dot{\mathbf{E}}^{\star} - \mathbf{L}^{\mathrm{T}}\mathbf{E}^{\star}) \times \mathbf{n} = \mathbf{0}. \tag{10.8}$$

10.2.1 Incremental Formulation Based on Ω

Following the development in Chap. 9, we first provide a summary for the situation in which an increment $\dot{\mathbf{F}}$ in the deformation gradient combined with an increment in the electric field $\dot{\mathbf{E}}_{\mathrm{L}}$ is superimposed on the underlying configuration $\mathcal{B}$. Equation (9.65) shows that for an unconstrained material the increments in total nominal stress $\dot{\mathbf{T}}$ and electric displacement $\dot{\mathbf{D}}_{\mathrm{L}}$ are given by

$$\dot{\mathbf{T}} = \boldsymbol{\mathcal{A}}\dot{\mathbf{F}} + \mathbb{A}\dot{\mathbf{E}}_{\mathrm{L}}, \quad \dot{\mathbf{D}}_{\mathrm{L}} = -\mathbb{A}^{\mathrm{T}}\dot{\mathbf{F}} - \mathsf{A}\dot{\mathbf{E}}_{\mathrm{L}}. \tag{10.9}$$

The expressions for the electroelastic moduli tensors $\boldsymbol{\mathcal{A}}$, $\mathbb{A}$ and A are given, for an isotropic electroelastic material, by (9.73). For an incompressible material we have

$$\operatorname{div}\mathbf{u} = 0, \tag{10.10}$$

and $(10.9)_1$ is replaced by

$$\dot{\mathbf{T}} = \boldsymbol{\mathcal{A}}\dot{\mathbf{F}} + \mathbb{A}\dot{\mathbf{E}}_{\mathrm{L}} - \dot{p}\mathbf{F}^{-1} + p\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1}. \tag{10.11}$$

Using (10.2) we can express the equations in Eulerian form from (10.9) and (10.11). These are

$$\dot{\mathbf{T}}_0 = \boldsymbol{\mathcal{A}}_0\mathbf{L} + \mathbb{A}_0\dot{\mathbf{E}}_{\mathrm{L}0}, \quad \dot{\mathbf{D}}_{\mathrm{L}0} = -\mathbb{A}_0^{\mathrm{T}}\mathbf{L} - \mathsf{A}_0\dot{\mathbf{E}}_{\mathrm{L}0}, \tag{10.12}$$

and

$$\dot{\mathbf{T}}_0 = \boldsymbol{\mathcal{A}}_0\mathbf{L} + \mathbb{A}_0\dot{\mathbf{E}}_{\mathrm{L}0} + p\mathbf{L} - \dot{p}\mathbf{I}, \tag{10.13}$$

which must satisfy (10.3). The connections between Eulerian and Lagrangian forms of the electroelastic moduli tensors are given by (9.79)–(9.81).

10.2.2 Incremental Formulation Based on Ω^*

To find a solution of the incremental equations it is sometimes more convenient to use the formulation based on Ω^* for which the electric field $\mathbf{E}_\mathrm{L}$ is replaced by the electric displacement $\mathbf{D}_\mathrm{L}$ as the independent electric variable. The increments of the total nominal stress $\mathbf{T}$ and of the Lagrangian form of the electric field $\mathbf{E}_\mathrm{L}$ are given by (9.92) and are included here for ease of reference. These are

$$\dot{\mathbf{T}} = \boldsymbol{\mathcal{A}}^*\dot{\mathbf{F}} + \mathbb{A}^*\dot{\mathbf{D}}_\mathrm{L}, \quad \dot{\mathbf{E}}_\mathrm{L} = \mathbb{A}^{*\mathrm{T}}\dot{\mathbf{F}} + \mathsf{\mathbf{A}}^*\dot{\mathbf{D}}_\mathrm{L}, \tag{10.14}$$

with expressions for the corresponding electroelastic moduli tensors given by (9.97). For an incompressible material the incremental form of the total nominal stress $(10.14)_1$ is replaced by

$$\dot{\mathbf{T}} = \boldsymbol{\mathcal{A}}^*\dot{\mathbf{F}} + \mathbb{A}^*\dot{\mathbf{D}}_\mathrm{L} + p^*\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1} - \dot{p}^*\mathbf{F}^{-1}. \tag{10.15}$$

The Eulerian forms of the incremental equations (10.14) and (10.15) are given by (9.102) and (9.103) and are included here for later reference. These are

$$\dot{\mathbf{T}}_0 = \boldsymbol{\mathcal{A}}_0^*\mathbf{L} + \mathbb{A}_0^*\dot{\mathbf{D}}_{\mathrm{L}0}, \quad \dot{\mathbf{E}}_{\mathrm{L}0} = \mathbb{A}^{*\mathrm{T}}_0\mathbf{L} + \mathsf{\mathbf{A}}_0^*\dot{\mathbf{D}}_{\mathrm{L}0}, \tag{10.16}$$

and

$$\dot{\mathbf{T}}_0 = \boldsymbol{\mathcal{A}}_0^*\mathbf{L} + \mathbb{A}_0^*\dot{\mathbf{D}}_{\mathrm{L}0} + p^*\mathbf{L} - \dot{p}^*\mathbf{I}. \tag{10.17}$$

The Eulerian forms $\boldsymbol{\mathcal{A}}_0^*, \mathbb{A}_0^*, \mathsf{\mathbf{A}}_0^*$ of the electroelastic moduli tensors $\boldsymbol{\mathcal{A}}^*, \mathbb{A}^*, \mathsf{\mathbf{A}}^*$ are given by (9.104)–(9.106).

10.3 The Electroelastic Half-Space

In this section we specialize the equations and boundary conditions from Sect. 10.2 to those associated with the pure homogeneous strain of an electroelastic half-space in the presence of an electric field normal to its boundary. We follow closely the analysis in Dorfmann and Ogden (2010a).

With reference to rectangular Cartesian coordinates (X_1, X_2, X_3) we suppose that the undeformed half-space occupies the region $\mathcal{B}_\mathrm{r}$ defined by $X_2 \geq 0$ and $-\infty < X_1 < \infty, -\infty < X_3 < \infty$ and has boundary identified by $X_2 = 0$. In what follows, we consider the material to be incompressible and subject to a pure homogeneous plane strain in the (X_1, X_2) plane. The deformation may then be defined by

$$x_1 = \lambda X_1, \quad x_2 = \lambda^{-1}X_2, \quad x_3 = X_3, \tag{10.18}$$

where λ is a constant, which is a specialization of (3.32), which defines pure homogeneous strain with constant values of $\lambda_1, \lambda_2, \lambda_3$. The components of the deformation gradient tensor $\mathbf{F}$ and the right Cauchy–Green tensor $\mathbf{c}$, written F and c respectively, are given by

$$\mathsf{F} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathsf{c} = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^{-2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{10.19}$$

where $\lambda_1 = \lambda, \lambda_2 = \lambda^{-1}, \lambda_3 = 1$ are the principal stretches.

For this problem we consider the electric displacement vector as the independent variable and use the formulation based on $\Omega^*(\mathbf{F}, \mathbf{D}_\mathrm{L})$ to evaluate the stability of the equilibrium configuration. The invariants K_1 and K_2 defined by (9.95) are then simply

$$K_1 = K_2 = 1 + \lambda^2 + \lambda^{-2}, \tag{10.20}$$

where we recall that the notation of the invariants has been changed from I_1, I_2 to K_1, K_2 for consistency with the theory developed in Sect. 9.3.

The electric displacement vector $\mathbf{D}$ is taken to be in the x_2 direction and to be independent of x_1 and x_3. With the free charge density $\rho_\mathrm{f} = 0$, $(2.163)_1$ reduces to $\operatorname{div}\mathbf{D} = 0$, and it follows that component D_2 is constant. Thus,

$$D_1 = 0, \quad D_2 \neq 0, \quad D_3 = 0. \tag{10.21}$$

The corresponding Lagrangian field $\mathbf{D}_\mathrm{L} = \mathbf{F}^{-1}\mathbf{D}$ has components

$$D_{\mathrm{L}1} = 0, \quad D_{\mathrm{L}2} = \lambda D_2, \quad D_{\mathrm{L}3} = 0, \tag{10.22}$$

and the invariants K_4, K_5, K_6 are obtained from (9.96) as

$$K_4 = D_{\mathrm{L}2}^2, \quad K_5 = \lambda^{-2} K_4, \quad K_6 = \lambda^{-4} K_4. \tag{10.23}$$

From (10.20) and (10.23) it follows that Ω^* depends on the above five invariants; thus, $\Omega^* = \Omega^*(K_1, K_2, K_4, K_5, K_6)$, and in the following Ω_i^* is defined as $\partial\Omega^*/\partial K_i$ for $i = 1, 2, 4, 5, 6$. From (4.87), (10.19), (10.20) and (10.23) we obtain the components of the stress tensor $\boldsymbol{\tau}$. For the electroelastic half-space the only non-zero components are

$$\begin{aligned} \tau_{11} &= 2\Omega_1^*\lambda^2 + 2\Omega_2^*(1 + \lambda^2) - p^*, \\ \tau_{22} &= 2\Omega_1^*\lambda^{-2} + 2\Omega_2^*(1 + \lambda^{-2}) - p^* + 2\Omega_5^*\lambda^{-2}K_4 + 4\Omega_6^*\lambda^{-4}K_4, \\ \tau_{33} &= 2\Omega_1^* + 2\Omega_2^*(\lambda^2 + \lambda^{-2}) - p^*. \end{aligned} \tag{10.24}$$

Similarly, from (4.88), the only non-zero component of the electric field $\mathbf{E}$ is

$$E_2 = 2(\Omega_4^* + \lambda^{-2}\Omega_5^* + \lambda^{-4}\Omega_6^*)\lambda D_{L2}. \tag{10.25}$$

Since D_{L2} and λ are constant, all the fields are uniform and the equilibrium equations and Maxwell's equations are satisfied automatically.

Because of the specialization it can be seen from (10.20) and (10.23) that there remain only two independent invariants, which can be expressed simply in terms of λ and K_4. This enables us to define a reduced form of Ω^*, denoted $\omega^*(\lambda, K_4)$, that depends only on these two variables:

$$\omega^*(\lambda, K_4) = \Omega^*(1 + \lambda^2 + \lambda^{-2}, 1 + \lambda^2 + \lambda^{-2}, K_4, \lambda^{-2}K_4, \lambda^{-4}K_4). \tag{10.26}$$

We then obtain

$$\begin{aligned} \omega_\lambda^* &= 2\lambda^{-1}[(\lambda^2 - \lambda^{-2})(\Omega_1^* + \Omega_2^*) - \lambda^{-2}K_4\Omega_5^* - 2\lambda^{-4}K_4\Omega_6^*], \\ \omega_4^* &= \Omega_4^* + \lambda^{-2}\Omega_5^* + \lambda^{-4}\Omega_6^*, \end{aligned} \tag{10.27}$$

where $\omega_\lambda^* = \partial\omega^*/\partial\lambda, \omega_4^* = \partial\omega^*/\partial K_4$, and the simple formulas

$$\tau_{11} - \tau_{22} = \lambda\omega_\lambda^*, \quad E_2 = 2\lambda D_{L2}\omega_4^* \tag{10.28}$$

follow.

10.3.1 Exterior Electric Field

We assume that on the material boundary $X_2 = 0$ the surface charge $\sigma_F = 0$. The continuity conditions (4.3), applied to the boundary $x_2 = 0$, give $D_2^\star = D_2$ and $E_1 = E_1^\star$, $E_3 = E_3^\star = 0$, and, using the relation

$$\mathbf{D}^\star = \varepsilon_0\mathbf{E}^\star, \tag{10.29}$$

we obtain $D_1^\star = D_3^\star = 0$ and $E_2^\star = \varepsilon_0^{-1}D_2^\star = \varepsilon_0^{-1}D_2$. Outside the material the electric field is uniform and equal to its value at $x_2 = 0$. Then, Maxwell's equations are satisfied identically, and $\mathbf{D}^\star$ and $\mathbf{E}^\star$ have components

$$D_1^\star = 0, \quad D_2^\star = D_2 = \lambda^{-1}D_{L2}, \quad D_3^\star = 0, \tag{10.30}$$

and

$$E_1^\star = 0, \quad E_2^\star = \varepsilon_0^{-1}D_2 = \varepsilon_0^{-1}\lambda^{-1}D_{L2}, \quad E_3^\star = 0, \tag{10.31}$$

respectively. We deduce that the non-zero components of the Maxwell stress (4.15) are given by

$$\tau^{\star}_{e11} = \tau^{\star}_{e33} = -\tau^{\star}_{e22} = -\tfrac{1}{2}\varepsilon_0^{-1} D_2^2 = -\tfrac{1}{2}\varepsilon_0^{-1}\lambda^{-2} K_4, \tag{10.32}$$

where use has been made of $(10.31)_2$. From (4.20) it follows that, for the general case where $\tau_{22} \neq \tau^{\star}_{e22}$, an applied mechanical traction $\tau_{22} - \tau^{\star}_{e22}$ is required on $x_2 = 0$ in addition to a normal stress in the x_3 direction in order to maintain the considered plane strain deformation.

10.3.2 Incremental Fields and Equations

We now use the incremental equations summarized in Sect. 10.2.2 to evaluate the effect of small increments in the deformation and in the electric displacement on the stability of the electroelastic half-space.

We consider, for consistency with the underlying deformation, that the incremental displacement has a plane strain character in the (x_1, x_2) plane. Thus, the incremental displacement vector $\mathbf{u}$ has in-plane components u_1, u_2 that depend only on x_1 and x_2 and an out-of-plane component $u_3 = 0$. In the plane strain specialization the incremental incompressibility condition (10.10) becomes

$$u_{1,1} + u_{2,2} = 0, \tag{10.33}$$

where the subscript following a comma indicates partial differentiation with respect to x_i, $i = 1, 2$. From (10.33) we deduce the existence of a function $\psi = \psi(x_1, x_2)$ such that

$$u_1 = \psi_{,2}, \quad u_2 = -\psi_{,1}. \tag{10.34}$$

Similarly, we consider an increment in the electric displacement field $\dot{\mathbf{D}}_{L0}$ with components $\dot{D}_{L01}, \dot{D}_{L02}$ that depend on x_1, x_2 only and with the out-of-plane component $\dot{D}_{L03} = 0$. These must satisfy $(10.3)_3$, which, for the considered specialization, reduces to

$$\dot{D}_{L01,1} + \dot{D}_{L02,2} = 0. \tag{10.35}$$

Equation (10.35) is satisfied by defining a function $\varphi = \varphi(x_1, x_2)$ such that

$$\dot{D}_{L01} = \varphi_{,2}, \quad \dot{D}_{L02} = -\varphi_{,1}. \tag{10.36}$$

The gradient $\mathbf{L}$ of the incremental displacement vector $\mathbf{u}$ and the increment in the electric displacement $\dot{\mathbf{D}}_{L0}$ induce a change in the total stress $\dot{\mathbf{T}}_0$, which must satisfy $(10.3)_1$. For the incremental plane strain considered, this equation specializes to the component equations

$$\dot{T}_{011,1} + \dot{T}_{021,2} = 0, \quad \dot{T}_{012,1} + \dot{T}_{022,2} = 0, \tag{10.37}$$

with the components of $\dot{\mathbf{T}}_0$ obtained from (10.17). Since $F_{ij} = 0$ for $i \neq j$ and $D_{\mathrm{L}1} = D_{\mathrm{L}3} = 0$ the components of the tensors $\boldsymbol{\mathcal{A}}_0^*$, $\mathbb{A}_0^*$, A_0^* simplify significantly. It is easy to show that the incremental total nominal stress tensor, in Eulerian form, has the non-zero components

$$\begin{aligned}
\dot{T}_{011} &= (\mathcal{A}_{01111}^* + p^*)u_{1,1} + \mathcal{A}_{01122}^* u_{2,2} + \mathbb{A}_{011|2}^* \dot{D}_{\mathrm{L}02} - \dot{p}^*, \\
\dot{T}_{021} &= (\mathcal{A}_{02112}^* + p^*)u_{2,1} + \mathcal{A}_{02121}^* u_{1,2} + \mathbb{A}_{021|1}^* \dot{D}_{\mathrm{L}01}, \\
\dot{T}_{012} &= (\mathcal{A}_{01221}^* + p^*)u_{1,2} + \mathcal{A}_{01212}^* u_{2,1} + \mathbb{A}_{012|1}^* \dot{D}_{\mathrm{L}01}, \\
\dot{T}_{022} &= (\mathcal{A}_{02222}^* + p^*)u_{2,2} + \mathcal{A}_{02211}^* u_{1,1} + \mathbb{A}_{022|2}^* \dot{D}_{\mathrm{L}02} - \dot{p}^*.
\end{aligned} \tag{10.38}$$

In addition to $\dot{\mathbf{T}}_0$, an increment in the deformation and in the electric displacement induces a change in the electric field $\dot{\mathbf{E}}_{\mathrm{L}0}$, which is given by $(10.16)_2$. The corresponding non-zero components are

$$\begin{aligned}
\dot{E}_{\mathrm{L}01} &= \mathbb{A}_{021|1}^*(u_{1,2} + u_{2,1}) + \mathsf{A}_{011}^* \dot{D}_{\mathrm{L}01}, \\
\dot{E}_{\mathrm{L}02} &= \mathbb{A}_{011|2}^* u_{1,1} + \mathbb{A}_{022|2}^* u_{2,2} + \mathsf{A}_{022}^* \dot{D}_{\mathrm{L}02}
\end{aligned} \tag{10.39}$$

and must satisfy $(10.3)_2$, which reduces to the single equation

$$\dot{E}_{\mathrm{L}01,2} - \dot{E}_{\mathrm{L}02,1} = 0. \tag{10.40}$$

In terms of the functions ψ and φ, (10.37) and (10.40) become

$$\begin{aligned}
&(\mathcal{A}_{01111}^* - \mathcal{A}_{01122}^* - \mathcal{A}_{01221}^*)\psi_{,112} + \mathcal{A}_{02121}^*\psi_{,222} \\
&\qquad\qquad - \mathbb{A}_{011|2}^*\varphi_{,11} + \mathbb{A}_{012|1}^*\varphi_{,22} = \dot{p}_{,1}^*, \\
&(\mathcal{A}_{02222}^* - \mathcal{A}_{02211}^* - \mathcal{A}_{01221}^*)\psi_{,122} + \mathcal{A}_{01212}^*\psi_{,111} \\
&\qquad\qquad - (\mathbb{A}_{012|1}^* - \mathbb{A}_{022|2}^*)\varphi_{,12} = -\dot{p}_{,2}^*, \\
&(\mathbb{A}_{022|2}^* - \mathbb{A}_{011|2}^* - \mathbb{A}_{021|1}^*)\psi_{,112} + \mathbb{A}_{021|1}^*\psi_{,222} \\
&\qquad\qquad + \mathsf{A}_{022}^*\varphi_{,11} + \mathsf{A}_{011}^*\varphi_{,22} = 0.
\end{aligned} \tag{10.41}$$

The elimination of $\dot{p}^*$ from the first two equations by cross-differentiation yields two coupled equations for ψ and φ, which can be written in compact form as

$$a\psi_{,1111} + 2b\psi_{,1122} + c\psi_{,2222} + (e - d)\varphi_{,112} + d\varphi_{,222} = 0 \tag{10.42}$$

and

$$d\psi_{,222}+(e-d)\psi_{,112}+f\varphi_{,22}+g\varphi_{,11}=0, \tag{10.43}$$

in which the coefficients are defined by

$$\begin{aligned}
&a=\mathcal{A}^*_{01212},\quad 2b=\mathcal{A}^*_{01111}+\mathcal{A}^*_{02222}-2\mathcal{A}^*_{01221}-2\mathcal{A}^*_{01122},\quad c=\mathcal{A}^*_{02121},\\
&d=\mathbb{A}^*_{021|1},\quad e=\mathbb{A}^*_{022|2}-\mathbb{A}^*_{011|2},\quad f=\mathsf{A}^*_{011},\quad g=\mathsf{A}^*_{022},
\end{aligned} \tag{10.44}$$

with their explicit forms obtained by using (9.107)–(9.109) as

$$\begin{aligned}
a&=2\lambda^2(\Omega^*_1+\Omega^*_2)+2K_4\Omega^*_6,\\
b&=(\lambda^2+\lambda^{-2})(\Omega^*_1+\Omega^*_2)+K_4[\lambda^{-2}\Omega^*_5+(6\lambda^{-4}-2)\Omega^*_6]\\
&\quad+2(\lambda^4+\lambda^{-4}-2)(\Omega^*_{11}+2\Omega^*_{12}+\Omega^*_{22})\\
&\quad+4K_4(\lambda^{-4}-1)[\Omega^*_{15}+\Omega^*_{25}+2\lambda^{-2}(\Omega^*_{16}+\Omega^*_{26})]\\
&\quad+2K_4^2\lambda^{-4}(\Omega^*_{55}+8\lambda^{-2}\Omega^*_{56}+4\lambda^{-4}\Omega^*_{66}),\\
c&=2\lambda^{-2}(\Omega^*_1+\Omega^*_2)+2K_4[\lambda^{-2}\Omega^*_5+(2\lambda^{-4}+1)\Omega^*_6],\\
d&=2D_{L2}\lambda[\lambda^{-2}\Omega^*_5+(\lambda^{-4}+1)\Omega^*_6],\\
e&=4D_{L2}\lambda^{-1}[\Omega^*_5+2\lambda^{-2}\Omega^*_6+(1-\lambda^4)(\Omega^*_{14}+\Omega^*_{24})\\
&\quad+(\lambda^{-2}-\lambda^2)(\Omega^*_{15}+\Omega^*_{25})+(\lambda^{-4}-1)(\Omega^*_{16}+\Omega^*_{26})\\
&\quad+K_4(\Omega^*_{54}+\lambda^{-2}\Omega^*_{55}+2\lambda^{-2}\Omega^*_{46}+3\lambda^{-4}\Omega^*_{56}+2\lambda^{-6}\Omega^*_{66})],\\
f&=2(\lambda^{-2}\Omega^*_4+\Omega^*_5+\lambda^2\Omega^*_6),\\
g&=2(\lambda^2\Omega^*_4+\Omega^*_5+\lambda^{-2}\Omega^*_6)+4K_4(\lambda^2\Omega^*_{44}+2\Omega^*_{45}+2\lambda^{-2}\Omega^*_{46}\\
&\quad+\lambda^{-2}\Omega^*_{55}+2\lambda^{-4}\Omega^*_{56}+\lambda^{-6}\Omega^*_{66}).
\end{aligned} \tag{10.45}$$

In terms of the function $\omega^*(\lambda, K_4)$ we also have

$$\begin{aligned}
&a-c=\lambda\omega^*_\lambda,\quad 2(b+c)=\lambda^2\omega^*_{\lambda\lambda},\\
&e=-2D_{L2}\lambda^2\omega^*_{\lambda 4},\quad g=2\lambda^2(\omega^*_4+2K_4\omega^*_{44}),
\end{aligned} \tag{10.46}$$

where $\omega^*_{\lambda\lambda}=\partial^2\omega^*/\partial\lambda^2$, $\omega^*_{\lambda 4}=\partial^2\omega^*/\partial\lambda\partial K_4$ and $\omega^*_{44}=\partial^2\omega^*/\partial K_4^2$.

10.3.3 Exterior Equations

Outside the material, Maxwell's equations (10.4) hold for $\dot{\mathbf{E}}^\star$ and $\dot{\mathbf{D}}^\star$. From the first equation, and the assumption that all fields depend only on x_1 and x_2, we deduce the existence of a scalar function $\varphi^\star = \varphi^\star(x_1, x_2)$ such that

$$\dot{E}_1^\star = -\varphi^\star_{,1}, \quad \dot{E}_2^\star = -\varphi^\star_{,2}, \quad \dot{E}_3^\star = 0. \tag{10.47}$$

The incremental fields outside the material are related by $\dot{\mathbf{D}}^\star = \varepsilon_0 \dot{\mathbf{E}}^\star$. Thus,

$$\dot{D}_1^\star = -\varepsilon_0\varphi^\star_{,1}, \quad \dot{D}_2^\star = -\varepsilon_0\varphi^\star_{,2}, \quad \dot{D}_3^\star = 0, \tag{10.48}$$

and from $(10.4)_2$ we obtain Laplace's equation

$$\varphi^\star_{,11} + \varphi^\star_{,22} = 0 \tag{10.49}$$

for $\varphi^\star$. Finally, the incremental Maxwell stress tensor (9.58) has non-zero components

$$\begin{aligned}\dot{\tau}^\star_{\text{e}11} &= \varepsilon_0 E_2^\star \varphi^\star_{,2} = \lambda^{-1} D_{\text{L}2}\varphi^\star_{,2} = \dot{\tau}^\star_{\text{e}33} = -\dot{\tau}^\star_{\text{e}22},\\ \dot{\tau}^\star_{\text{e}12} &= -\varepsilon_0 E_2^\star \varphi^\star_{,1} = -\lambda^{-1} D_{\text{L}2}\varphi^\star_{,1} = \dot{\tau}^\star_{\text{e}21}.\end{aligned} \tag{10.50}$$

10.3.4 Boundary Conditions

Next we specialize the incremental boundary conditions (10.6) to the present situation. We set the incremental mechanical traction $\dot{\mathbf{t}}_\text{A}$ to zero, and the incremental traction boundary condition (10.6) reduces to the component equations

$$\dot{T}_{021} + \tau^\star_{\text{e}11} u_{2,1} - \dot{\tau}^\star_{\text{e}12} = 0, \quad \dot{T}_{022} + \tau^\star_{\text{e}22} u_{2,2} - \dot{\tau}^\star_{\text{e}22} = 0, \tag{10.51}$$

on $x_2 = 0$. By using (9.107), (10.32), (10.34), (10.36), (10.38), (10.44), (10.47) and (10.50), we rewrite the equations in (10.51) in terms of ψ, φ and $\varphi^\star$ as

$$(\tau_{22} + \tfrac{1}{2}\varepsilon_0^{-1}\lambda^{-2}K_4 - c)\,\psi_{,11} + c\,\psi_{,22} + d\,\varphi_{,2} + \lambda^{-1}D_{\text{L}2}\,\varphi^\star_{,1} = 0, \tag{10.52}$$

and

$$\begin{aligned}(2b + c - \tau_{22} + \tfrac{1}{2}\varepsilon_0^{-1}\lambda^{-2}K_4)\psi_{,112} + c\psi_{,222}&\\ + e\varphi_{,11} + d\varphi_{,22} - \lambda^{-1}D_{\text{L}2}\varphi^\star_{,12} &= 0,\end{aligned} \tag{10.53}$$

which apply on $x_2 = 0$. The derivation of the latter requires differentiation of $(10.51)_2$ with respect to x_1 and use of $(10.41)_1$.

The incremental electric boundary conditions (10.7) and (10.8) specialize to

$$\dot{D}_{L02} - \dot{D}_2^\star + D_2^\star u_{2,2} = 0, \quad \dot{E}_{L01} - \dot{E}_1^\star - E_2^\star u_{2,1} = 0 \tag{10.54}$$

on $x_2 = 0$, and these may also be re-cast in terms of ψ, φ and $\varphi^\star$ as

$$\lambda^{-1} D_{L2}\psi_{,12} + \varphi_{,1} - \varepsilon_0 \varphi^\star_{,2} = 0, \tag{10.55}$$

and

$$(\varepsilon_0^{-1}\lambda^{-1}D_{L2} - d)\psi_{,11} + d\psi_{,22} + f\varphi_{,2} + \varphi^\star_{,1} = 0. \tag{10.56}$$

10.3.5 Solution

In order to solve the incremental boundary-value problem we seek small amplitude solutions in the half-space $x_2 > 0$ of the form

$$\psi = A\mathrm{e}^{-ksx_2}\mathrm{e}^{\mathrm{i}kx_1}, \quad \varphi = kB\mathrm{e}^{-ksx_2}\mathrm{e}^{\mathrm{i}kx_1}, \tag{10.57}$$

where A and B are constants, $k > 0$ is the "wave number" of the perturbation and s is to be determined subject to the restriction

$$\mathrm{Re}(s) > 0, \tag{10.58}$$

which ensures that the displacement decays away from the boundary with increasing x_2.

Substitution of (10.57) into (10.42) and (10.43) yields

$$\begin{aligned} &(cs^4 - 2bs^2 + a)A - s(ds^2 + d - e)B = 0, \\ &-s(ds^2 + d - e)A + (fs^2 - g)B = 0. \end{aligned} \tag{10.59}$$

For non-trivial solutions to exist, the determinant of coefficients of A and B must vanish. This leads to the equation

$$(cf - d^2)s^6 - [2bf + cg + 2(d-e)d]s^4 + [2bg + af - (d-e)^2]s^2 - ag = 0, \tag{10.60}$$

which is cubic in s^2. Let s_1, s_2, s_3 denote the three roots with positive real part (the other three being $-s_1, -s_2, -s_3$). The general solution for the half-space satisfying the decay condition (10.58) can then be written as

$$\psi = \sum_{j=1}^{3} A_j \mathrm{e}^{-ks_j x_2}\mathrm{e}^{\mathrm{i}kx_1}, \quad \varphi = k\sum_{j=1}^{3} B_j \mathrm{e}^{-ks_j x_2}\mathrm{e}^{\mathrm{i}kx_1}, \tag{10.61}$$

where $A_j,\ B_j,\ j = 1, 2, 3$, are constants.

The constants A_j and B_j are not independent and are connected using (10.59) via

$$s_j(ds_j^2 + d - e)A_j - (fs_j^2 - g)B_j = 0, \quad j = 1, 2, 3; \text{ no summation.} \tag{10.62}$$

For the region $x_2 < 0$ exterior to the material, we take the solution $\varphi^\star$ of (10.49) that decays as $x_2 \to -\infty$ to be

$$\varphi^\star = \mathrm{i}kC^\star \mathrm{e}^{kx_2}\mathrm{e}^{\mathrm{i}kx_1}, \tag{10.63}$$

where $C^\star$ is constant.

The traction boundary conditions (10.52) and (10.53) yield

$$\sum_{j=1}^{3}[(\tau_{22} + \tfrac{1}{2}\varepsilon_0^{-1}D_2^2 - c - cs_j^2)A_j + ds_jB_j] + D_2C^\star = 0, \tag{10.64}$$

$$\sum_{j=1}^{3}[(2b + c - cs_j^2 - \tau_{22} + \tfrac{1}{2}\varepsilon_0^{-1}D_2^2)s_jA_j - (e - ds_j^2)B_j] + D_2C^\star = 0, \tag{10.65}$$

on $x_2 = 0$, where, to simplify the expressions, we have made use of the connections $D_2 = \lambda^{-1}D_{L2}$, $K_4 = D_{L2}^2$.

To complete the required set of equations, we have the two electric boundary conditions (10.55) and (10.56), which give

$$\sum_{j=1}^{3}(D_2s_jA_j - B_j) + \varepsilon_0C^\star = 0, \tag{10.66}$$

$$\sum_{j=1}^{3}[(\varepsilon_0^{-1}D_2 - d - ds_j^2)A_j + fs_jB_j] + C^\star = 0. \tag{10.67}$$

Together, there are seven homogeneous linear equations for the seven unknowns A_j, B_j, $j = 1, 2, 3$ and $C^\star$ and for a non-trivial solution to be possible, the resulting determinant of coefficients must vanish. The system can be reduced to four equations in four unknowns by using (10.60) to eliminate the B_js in favour of the A_js, for example.

For definiteness and for purposes of illustration we specialize the form of Ω^* in the following subsection.

10.3.6 An Electroelastic Neo-Hookean Solid

To evaluate the effect of an applied electric field on the stability of an electroelastic half-space, we consider the simple form of the energy function Ω^* given by

$$\Omega^* = \frac{1}{2}\mu(K_1 - 3) + \frac{1}{2}\varepsilon_0^{-1}(\alpha K_4 + \beta K_5). \tag{10.68}$$

The first term in (10.68) corresponds to the strain-energy function of an incompressible neo-Hookean material from rubber elasticity, K_1 being the first invariant of a Cauchy–Green tensor, which reduces to (10.20) for the present specialization, and μ (> 0) being the shear modulus of the material in the absence of an electric field. This is supplemented by terms linear in the invariants K_4 and K_5 in order to incorporate in a simple way the effect of the electric displacement. These involve two dimensionless material constants α and β that serve as electroelastic coupling parameters.

From (10.24) the stress components are simply obtained as

$$\tau_{11} = \mu\lambda^2 - p^*, \quad \tau_{22} = \mu\lambda^{-2} - p^* + \varepsilon_0^{-1}\beta\lambda^{-2}K_4, \quad \tau_{33} = \mu - p^*, \tag{10.69}$$

and we note, in particular, that the parameter α does not affect the stress. On the other hand, the parameter β has the effect of stiffening (respectively softening) the material in the direction of the electric field if positive (respectively negative) compared with the situation in the absence of an electric field. If $\alpha = 0$ then the energy function (10.68) corresponds to the so-called ideal dielectric elastomer favoured by Zhao and Suo (2007).

From (4.88) the expression for the electric field simplifies to

$$\mathbf{E} = \varepsilon_0^{-1}(\alpha\mathbf{b}^{-1}\mathbf{D} + \beta\mathbf{D}), \tag{10.70}$$

with the components of $\mathbf{E}$ given by

$$E_1 = 0, \quad E_2 = \varepsilon_0^{-1}(\alpha\lambda^2 + \beta)D_2, \quad E_3 = 0. \tag{10.71}$$

This indicates that α is a measure of how the electric properties of the material are influenced by the deformation through λ. If $\alpha = 0$, however, then (10.70) is not influenced by the deformation. It is therefore clear that mutual coupling requires inclusion of both constants. When there is no deformation, i.e. when $\lambda = 1$, we have $D_2 = \varepsilon_0 E_2/(\alpha + \beta)$ so that in the linear specialization $\alpha + \beta = 1/\varepsilon_{\mathrm{r}}$, where ε_{r} is the *relative dielectric permittivity* of the material (see Sect. 2.4.2) values of which range from 1 (for vacuum) upwards. Thus, we shall take $\alpha + \beta \leq 1$ in our subsequent calculations.

In respect of the model (10.68) the material parameters defined in (10.45) are simply evaluated in the form

$$a = \mu\lambda^2, \quad c = \mu\lambda^{-2} + \varepsilon_0^{-1}\beta\lambda^{-2}K_4, \quad 2b = a + c, \tag{10.72}$$

$$d = \varepsilon_0^{-1}\beta\lambda^{-1}D_{\mathrm{L}2}, \quad e = 2d, \quad f = \varepsilon_0^{-1}(\alpha\lambda^{-2} + \beta), \quad g = \varepsilon_0^{-1}(\alpha\lambda^2 + \beta). \tag{10.73}$$

For the energy function (10.68), the bi-cubic in (10.60) factorizes to give

$$(s^2 - 1)(s^2 - \lambda^4)[\alpha\lambda^4 + \beta\lambda^2 - (\alpha + \beta\lambda^2 + \alpha\beta\lambda^2\hat{D}_2^2)s^2] = 0, \tag{10.74}$$

where we have introduced the dimensionless variable

$$\hat{D}_2 = D_2/\sqrt{\mu\varepsilon_0}. \tag{10.75}$$

We label the positive roots as

$$s_1 = 1, \quad s_2 = \lambda^2, \quad s_3 = \lambda\sqrt{\frac{\alpha\lambda^2 + \beta}{\alpha + \beta\lambda^2 + \alpha\beta\lambda^2\hat{D}_2^2}}. \tag{10.76}$$

In general s_3 may be real or pure imaginary, but to satisfy (10.58) it must be real and positive for all $\hat{D}_2$ and $\lambda > 0$. To ensure that this is the case we take $\alpha \geq 0$ and $\beta \geq 0$, with $\alpha + \beta > 0$. Note that if $\alpha = 0$, then $s_3 = s_1$ while if $\beta = 0$, then $s_3 = s_2$.

For definiteness we assume that there is no applied mechanical traction on the boundary $x_2 = 0$ in the underlying configuration, and hence

$$\tau_{22} = \tau^{\star}_{\mathrm{e}22} = \tfrac{1}{2}\varepsilon_0^{-1}D_2^2. \tag{10.77}$$

The incremental boundary conditions (10.64)–(10.67) on $x_2 = 0$ now take on the forms

$$\sum_{j=1}^{3}\{[\hat{D}_2^2 - (\lambda^{-2} + \beta\hat{D}_2^2)(1 + s_j^2)]A_j + \beta\hat{D}_2 s_j\hat{B}_j\} + \hat{D}_2\hat{C}^{\star} = 0, \tag{10.78}$$

$$\sum_{j=1}^{3}\{[\lambda^2+\lambda^{-2}+\beta\hat{D}_2^2+(\lambda^{-2}+\beta\hat{D}_2^2)(1-s_j^2)]s_jA_j-\beta\hat{D}_2(2-s_j^2)\hat{B}_j\}+\hat{D}_2\hat{C}^{\star} = 0, \tag{10.79}$$

$$\sum_{j=1}^{3}(\hat{D}_2 s_j A_j - \hat{B}_j) + \hat{C}^{\star} = 0, \tag{10.80}$$

$$\sum_{j=1}^{3}\{[\hat{D}_2 - \beta\hat{D}_2(1 + s_j^2)]A_j + (\alpha\lambda^{-2} + \beta)s_j\hat{B}_j\} + \hat{C}^{\star} = 0, \tag{10.81}$$

wherein we have introduced the notations

$$\hat{B}_j = B_j/\sqrt{\mu\varepsilon_0}, \quad \hat{C}^{\star} = \varepsilon_0 C^{\star}/\sqrt{\mu\varepsilon_0}, \tag{10.82}$$

and (10.62) yields the connections

$$s_j(s_j^2 - 1)\beta\hat{D}_2A_j - [(\alpha\lambda^{-2} + \beta)s_j^2 - (\alpha\lambda^2 + \beta)]\hat{B}_j = 0, \quad j = 1, 2, 3, \tag{10.83}$$

with no summations over j.

Apart from μ we have two material parameters α and β and, in addition, the variables λ, $\hat{D}_2$. For a non-trivial solution, the determinant of coefficients in (10.78)–(10.81) and (10.83) must vanish. This provides a connection between the four quantities λ, $\hat{D}_2$, α and β, which is the *bifurcation equation*. This has been obtained in an explicit form by using a computer algebra software, but it is a very lengthy expression and is not reproduced here. Instead, we illustrate its implications with some numerical examples. We fix the values of α and β and find the critical stretch λ_{cr} as a function of $\hat{D}_2$. In particular, for $\hat{D}_2 = 0$, we recover the well-known critical compression stretch for buckling instability of an elastic half-space, namely $\lambda_{\mathrm{cr}} = 0.5437$, as originally obtained by Biot; for details, see Biot (1965) and the more recent treatment by Dowaikh and Ogden (1990).

The plots in Fig. 10.1 show the critical stretch as a function of $\hat{D}_2$ for a range of values of the parameters α and β consistent with the inequality $\alpha + \beta \leq 1$ discussed earlier in this section. The plot in Fig. 10.1a is for $\alpha = 0$ with four curves corresponding to $\beta = 0.1, 0.3, 0.5, 1$ (e.g., relative permittivity decreasing from 10 to 1, which is a range of values appropriate for electroactive polymers). In particular, the curve for $\beta = 1$ is a horizontal straight line, which is the limiting case in which the (fictitious) material has permittivity equal to the vacuum permittivity and the critical value λ_{cr} is unaffected by the electric field. For $\alpha = 0$ the electric field has a destabilizing effect, i.e. the half-space becomes unstable at a value of the stretch closer to 1 than in the classical case without an electric field, but increasing the value of β moderates this influence. Figure 10.1b–d correspond to the increasing values of α, namely 0.1, 0.3 and 0.5, with a range of values of β in each case, and therefore showing the effect of α on the stability of the half-space. It tends to enhance stability, and for the larger values of β, the half-space becomes more stable than in the absence of an electric field, at least for small values of $\hat{D}_2$. For sufficiently large values of $\hat{D}_2$, the undeformed configuration becomes unstable, i.e. $\lambda_{\mathrm{cr}} > 1$, and an extension needs to be applied in the x_1 direction to prevent loss of stability. Although not shown in the plots, for even larger values of $\hat{D}_2$, the gain in stability for the larger values of β is lost, and the corresponding curves have a similar trend to those for smaller values of β.

10.4 An Electroelastic Plate

Many of the equations derived in Sect. 10.3 can be used to evaluate the stability of an electroelastic slab (or plate), which we consider next. We define the reference configuration $\mathcal{B}_{\mathrm{r}}$ of a plate in terms of rectangular Cartesian coordinates (X_1, X_2, X_3). We assume that the plate is made of an incompressible isotropic electroelastic material, has a uniform thickness H in $\mathcal{B}_{\mathrm{r}}$ with top and bottom faces normal to the X_2 direction and is of finite extent in the X_1 and X_3 directions, but such that H is small compared with the lateral dimensions so that end effects can be neglected. This configuration is then defined by

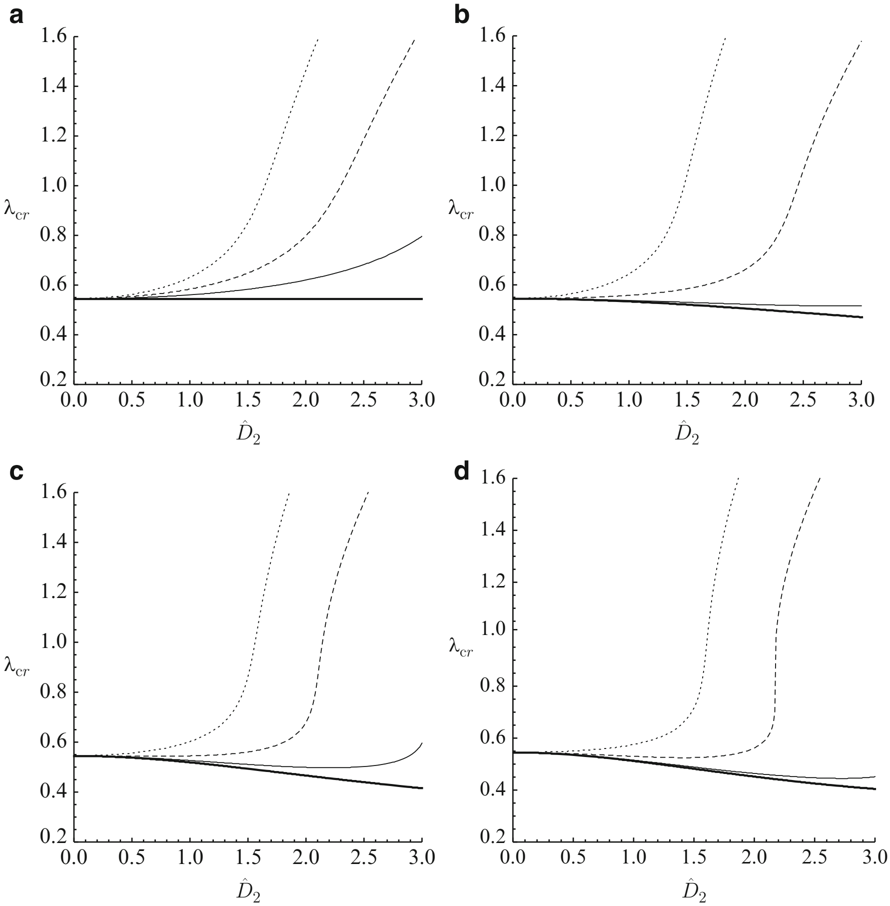

Fig. 10.1 Plot of the critical stretch $\lambda_{\rm cr}$ for an electroelastic neo-Hookean half-space as a function of the dimensionless measure $\hat{D}_2$ of the applied electric displacement field D_2. Results show that the electric field has a highly nonlinear influence on the stability. Depending on the values of the coupling parameters α and β, the electric field stabilizes or destabilizes the electroelastic half-space. The values of α and β are as follows: (**a**) $\alpha = 0, \beta = 0.1, 0.3, 0.5, 1$; (**b**) $\alpha = 0.1$, $\beta = 0, 0.3, 0.6, 0.9$; (**c**) $\alpha = 0.3, \beta = 0, 0.2, 0.4, 0.7$; (**d**) $\alpha = 0.5, \beta = 0, 0.2, 0.4, 0.5$. The values of β running from the smallest to the largest correspond, respectively, to the *dotted, dashed, continuous* and *thick continuous curves* in each of (**a**)–(**d**)

$$-L_1 < X_1 < L_1, \quad 0 \le X_2 \le H, \quad -L_3 < X_3 < L_3, \tag{10.84}$$

with $H \ll L_1$ and $H \ll L_3$.

The plate is surrounded by vacuum and subject to a uniform electric displacement $D_2^\star$ (equal to D_2 in the plate by continuity) in the X_2 direction with the Lagrangian component denoted by $D_{\rm L2}$ in the material and the other components zero. We consider the plate to be subjected to plane strain in the (X_1, X_2) plane with the corresponding stretches given by

$$\lambda_1 = \lambda, \quad \lambda_2 = \lambda^{-1}, \quad \lambda_3 = 1. \tag{10.85}$$

The deformation gradient and right Cauchy–Green tensor have components given by (10.19), and the invariants K_1, K_2, K_4, K_5 and K_6 are given by (10.20) and (10.23), with $D_{L2} = \lambda D_2 = \lambda D_2^\star$.

The deformed configuration of the plate, with uniform thickness h, is defined by

$$-l_1 < x_1 < l_1, \quad 0 \le x_2 \le h, \quad -L_3 < x_3 < L_3, \tag{10.86}$$

where $l_1 = \lambda L_1$ and $h = \lambda^{-1} H$.

The components of the total stress tensor and of the electric field vector are identical to the components derived for the electroelastic half-space and are given in (10.24) and (10.25), while the Maxwell stress outside the plate has components given by (10.32).

We again consider an incremental deformation and electric field confined to the (x_1, x_2) plane, as for the half-space. Thus, the formulas given in Sects. 10.3.2–10.3.4 carry over to the current geometry and are therefore not given again here. In particular, (10.42) and (10.43) apply now in the region (10.86), while the boundary conditions (10.52)–(10.56) now apply on both $x_2 = 0$ and $x_2 = h$.

10.4.1 Solution

Similarly to the half-space we now assume that ψ and φ have the forms

$$\psi = A\mathrm{e}^{-ksx_2}\mathrm{e}^{\mathrm{i}kx_1}, \quad \varphi = kB\mathrm{e}^{-ksx_2}\mathrm{e}^{\mathrm{i}kx_1} \tag{10.87}$$

within the plate. These must satisfy (10.42) and (10.43). This again leads to the bi-cubic (10.59), but for the plate we require all six values of s, which we denote by $s_1, \ldots, s_6$, with $s_4 = -s_1$, $s_5 = -s_2$, $s_6 = -s_3$. The resulting general expressions for ψ and φ are therefore given by

$$\psi = \sum_{j=1}^{6} A_j \mathrm{e}^{-ks_j x_2}\mathrm{e}^{\mathrm{i}kx_1}, \quad \varphi = k\sum_{j=1}^{6} B_j \mathrm{e}^{-ks_j x_2}\mathrm{e}^{\mathrm{i}kx_1}, \tag{10.88}$$

for $0 \le x_2 \le h$, where A_j and B_j, $j = 1, \ldots, 6$, are constants.

We again use $(10.59)_2$ to obtain a connection between the constants A_j and B_j, which is now given by

$$s_j(ds_j^2 + d - e)A_j - (fs_j^2 - g)B_j = 0, \quad j = 1, \ldots, 6; \text{ no summation.} \tag{10.89}$$

The function $\varphi^\star$ applies to the exterior of the material and we define $\varphi_+^\star$ and $\varphi_-^\star$ for the space bounded by the top and bottom surfaces, respectively. These are given by

$$\varphi_+^\star = \mathrm{i}kC_+^\star \mathrm{e}^{-kx_2}\mathrm{e}^{\mathrm{i}kx_1}, \quad \text{for} \quad x_2 > h, \tag{10.90}$$

$$\varphi_-^\star = \mathrm{i}kC_-^\star \mathrm{e}^{kx_2}\mathrm{e}^{\mathrm{i}kx_1}, \quad \text{for} \quad x_2 < 0, \tag{10.91}$$

which involve the additional constants $C_+^\star$ and $C_-^\star$.

The explicit expressions for the boundary and continuity conditions on the surface $x_2 = 0$ are

$$\begin{aligned}
&\sum_{j=1}^{6}[(\tau_{22} + \tfrac{1}{2}\varepsilon_0^{-1}D_2^2 - c - cs_j^2)A_j + ds_jB_j] + D_2C_-^\star = 0,\\
&\sum_{j=1}^{6}[(2b + c - cs_j^2 - \tau_{22} + \tfrac{1}{2}\varepsilon_0^{-1}D_2^2)s_jA_j - (e - ds_j^2)B_j] + D_2C_-^\star = 0,\\
&\sum_{j=1}^{6}(D_2s_jA_j - B_j) + \varepsilon_0C_-^\star = 0,\\
&\sum_{j=1}^{6}[(\varepsilon_0^{-1}D_2 - d - ds_j^2)A_j + fs_jB_j] + C_-^\star = 0,
\end{aligned} \tag{10.92}$$

and on the surface $x_2 = h$ we have

$$\begin{aligned}
&\sum_{j=1}^{6}[(\tau_{22} + \tfrac{1}{2}\varepsilon_0^{-1}D_2^2 - c - cs_j^2)A_j + ds_jB_j]\mathrm{e}^{-ks_jh} + D_2C_+^\star\mathrm{e}^{-kh} = 0,\\
&\sum_{j=1}^{6}[(2b + c - cs_j^2 - \tau_{22} + \tfrac{1}{2}\varepsilon_0^{-1}D_2^2)s_jA_j - (e - ds_j^2)B_j]\mathrm{e}^{-ks_jh}\\
&\qquad\qquad\qquad\qquad + D_2C_+^\star\mathrm{e}^{-kh} = 0,\\
&\sum_{j=1}^{6}(D_2s_jA_j - B_j)\mathrm{e}^{-ks_jh} + \varepsilon_0C_+^\star\mathrm{e}^{-kh} = 0,\\
&\sum_{j=1}^{6}[(\varepsilon_0^{-1}D_2 - d - ds_j^2)A_j + fs_jB_j]\mathrm{e}^{-ks_jh} + C_+^\star\mathrm{e}^{-kh} = 0.
\end{aligned} \tag{10.93}$$

The 14 homogeneous linear equations (10.89), (10.92) and (10.93) involve 14 unknown constants $A_j,\ B_j,\ j = 1,\dots,6$, and $C_-^\star$ and $C_+^\star$. For a non-trivial solution, the resulting determinant of these coefficients must vanish. We first use (10.89) to express the constants B_j in terms of A_j for $j = 1,\dots,6$, which reduces the number of independent unknowns to eight. Using the boundary and continuity conditions on $x_2 = 0$ and $x_2 = h$, the solution is now reduced to finding the roots of the determinant of these remaining coefficients. To illustrate the theory we specialize the form of Ω^* to a neo-Hookean electroelastic material and evaluate the stability of the plate for increasing values of the electric displacement.

While the "wave number" k does not influence the solution (which is non-dispersive) for a half-space, it strongly affects the solution for the plate, but it always appears in the combination kh and need not be specified separately. It can be determined in terms of l_1 by applying appropriate boundary conditions on the faces $x_1 = \pm l_1$, but here it is not necessary to specify such conditions. In the purely nonlinear elastic context an example of the specification of such boundary conditions has been given by Ogden and Roxburgh (1993).

10.4.2 Application to the Neo-Hookean Electroelastic Solid

In this section we apply the theory for the plate to the neo-Hookean electroelastic material with energy function given by (10.68), with the underlying stresses and electric field given by (10.69) and (10.71), respectively. We assume that there is no applied mechanical traction on the faces $x_2 = 0$, $x_2 = h$ so that the stress τ_{22} in the material is equal to the corresponding component of the Maxwell stress according to (10.77). Combining this with $(10.69)_{1,2}$ we obtain

$$\tau_{11} = \mu(\lambda^2 - \lambda^{-2}) + (0.5 - \beta)\varepsilon_0^{-1}\lambda^{-2}K_4, \tag{10.94}$$

which gives an expression for the in-plane lateral traction required to support a given value of the electric field and lateral stretch. The positive Maxwell stress has a tendency to cause the plate to thicken and to be compressed in the lateral direction, and this is consistent with a negative value of τ_{11}. However, a value of λ less than 1 may also be consistent with positive τ_{11} if $\beta < 0.5$. In the following stability analysis we shall illustrate the results for the case $\beta = 0.5$.

For the incremental deformation, the relevant moduli are (10.72) and (10.73). The appropriate values of the solutions of the bi-cubic are (10.76), i.e.

$$s_1 = 1, \quad s_2 = \lambda^2, \quad s_3 = \lambda\sqrt{\frac{\alpha\lambda^2 + \beta}{\alpha + \beta\lambda^2 + \alpha\beta\lambda^2\hat{D}_2^2}}, \tag{10.95}$$

together with

$$s_4 = -1, \quad s_5 = -\lambda^2, \quad s_6 = -\lambda\sqrt{\frac{\alpha\lambda^2 + \beta}{\alpha + \beta\lambda^2 + \alpha\beta\lambda^2\hat{D}_2^2}}. \tag{10.96}$$

We again use the definitions (10.75) and (10.82) for $\hat{D}_2$ and $\hat{B}_j$, respectively, and the connections (10.83) between the $\hat{B}_j$s and the A_js with $j = 1, \dots, 6$. We also assume once more that there is no applied mechanical traction on either $x_2 = 0$ or $x_2 = h$, so that the surfaces $x_2 = 0$ and $x_2 = h$ are subject only to the Maxwell stress given by (10.77).

On the lower boundary $x_2 = 0$ the incremental boundary conditions are

$$\sum_{j=1}^{6}\{[\hat{D}_2^2 - (\lambda^{-2} + \beta\hat{D}_2^2)(1 + s_j^2)]A_j + \beta\hat{D}_2 s_j\hat{B}_j\} + \hat{D}_2\hat{C}_-^\star = 0,$$

$$\sum_{j=1}^{6}\{[\lambda^2 + \lambda^{-2} + \beta\hat{D}_2^2 + (\lambda^{-2} + \beta\hat{D}_2^2)(1 - s_j^2)]s_j A_j - \beta\hat{D}_2(2 - s_j^2)\hat{B}_j\} + \hat{D}_2\hat{C}_-^\star = 0,$$

$$\sum_{j=1}^{6}(\hat{D}_2 s_j A_j - \hat{B}_j) + \hat{C}_-^\star = 0,$$

$$\sum_{j=1}^{6}\{[\hat{D}_2 - \beta\hat{D}_2(1 + s_j^2)]A_j + (\alpha\lambda^{-2} + \beta)s_j\hat{B}_j\} + \hat{C}_-^\star = 0, \tag{10.97}$$

where $\hat{C}_-^\star = \varepsilon_0 C_-^\star/\sqrt{\mu\varepsilon_0}$. On the upper boundary $x_2 = h$ we have

$$\sum_{j=1}^{6}\{[\hat{D}_2^2 - (\lambda^{-2} + \beta\hat{D}_2^2)(1 + s_j^2)]A_j + \beta\hat{D}_2 s_j\hat{B}_j\}e^{-ks_jh} + \hat{D}_2\hat{C}_+^\star e^{-kh} = 0,$$

$$\sum_{j=1}^{6}\{[\lambda^2 + \lambda^{-2} + \beta\hat{D}_2^2 + (\lambda^{-2} + \beta\hat{D}_2^2)(1 - s_j^2)]s_j A_j - \beta\hat{D}_2(2 - s_j^2)\hat{B}_j\}e^{-ks_jh} + \hat{D}_2\hat{C}_+^\star e^{-kh} = 0,$$

$$\sum_{j=1}^{6}(\hat{D}_2 s_j A_j - \hat{B}_j)e^{-ks_jh} + \hat{C}_+^\star e^{-kh} = 0,$$

$$\sum_{j=1}^{6}\{[\hat{D}_2 - \beta\hat{D}_2(1 + s_j^2)]A_j + (\alpha\lambda^{-2} + \beta)s_j\hat{B}_j\}e^{-ks_jh} + \hat{C}_+^\star e^{-kh} = 0, \tag{10.98}$$

where $\hat{C}_+^\star = \varepsilon_0 C_+^\star/\sqrt{\mu\varepsilon_0}$.

In addition to the eight remaining constants A_j, $j = 1, \ldots, 6$, $C_-^\star$ and $C_+^\star$, (10.97) and (10.98) involve the material parameters α and β, the wave number of the disturbance k, the thickness h (but only as the product kh), the dimensionless variable $\hat{D}_2$ and the principal stretch λ. As a representative example we now consider the electroelastic neo-Hookean material with dimensionless constants

$\alpha = 0.5$ and $\beta = 0.5$. Vanishing of the resulting determinant of the 8×8 matrix of coefficients in (10.97) and (10.98) provides a connection between the critical stretch λ_{cr}, the applied field $D_2^\star = \sqrt{\mu\varepsilon_0}\hat{D}_2$ and the dimensionless quantity $kH = \lambda_{cr}kh$, H being the reference thickness of the plate.

The results are illustrated graphically in Fig. 10.2a–f for values of $\hat{D}_2 = 0, 1, 1.5, 2, 2.5$ and 3, respectively. Figure 10.2a provides a reference point since it corresponds to the classical case of instability of an elastic plate. The upper curve is associated with a flexural (antisymmetric) mode of buckling while the lower curve corresponds to a barrelling mode. Clearly, the flexural mode occurs first during compression of the plate (λ decreasing from 1) for all values of kH, and the two curves merge for very large values of kH, for which the plate effectively becomes a half-space and the half-space value of λ_{cr} is recovered (compare with the thick continuous curve in Fig. 10.1d, which corresponds to $\alpha = \beta = 0.5$, as used here).

The main conclusion to draw from the other plots in Fig. 10.2 is that an electric field stabilizes the plate, i.e. stability is lost for a smaller value of $\lambda < 1$ than in the absence of an electric field, and the critical value of λ decreases with the magnitude of the electric field. The electric field also disturbs the pattern that distinguishes flexural and barrelling modes in the case $\hat{D}_2 = 0$ for which the equations that govern flexural and barrelling modes decouple (see, e.g., Ogden and Roxburgh 1993). The curves in Fig. 10.2b–f correspond to neither flexural nor barrelling modes in general, although the upper curve in each of Fig. 10.2d–f is very similar to the corresponding curve in Fig. 10.2a.

When $\lambda > 1$ the plate is always stable in the presence of an electric field for the chosen values of the parameters α and β, but these values were selected only to illustrate one possibility. However, the stability of the plate is strongly influenced by the values of α and β for the particular material model adopted here, as is clearly the case for the half-space, and, in particular, calculations carried out for other choices of α and β show that the plate can be unstable when $\lambda > 1$. More generally, results for different choices of material model can also vary significantly.

10.5 An Electroelastic Plate with Electrodes

It is of interest to investigate the stability of an electroelastic plate when the electric field in the material is generated by surface charges on flexible electrodes that are attached to the top and bottom faces and there is no field outside the material. The electric boundary condition therefore differs from the one considered in Sect. 10.4 where the plate is surrounded by a vacuum and subject to a uniform electric field normal to the top and bottom faces.

The reference configuration of the plate is again defined by a uniform thickness H, by top and bottom faces that are normal to the X_2 direction and the geometry is defined by (10.84). The deformed configuration has uniform thickness h as specified by (10.86). We maintain the plane strain condition in the (X_1, X_2) plane and write

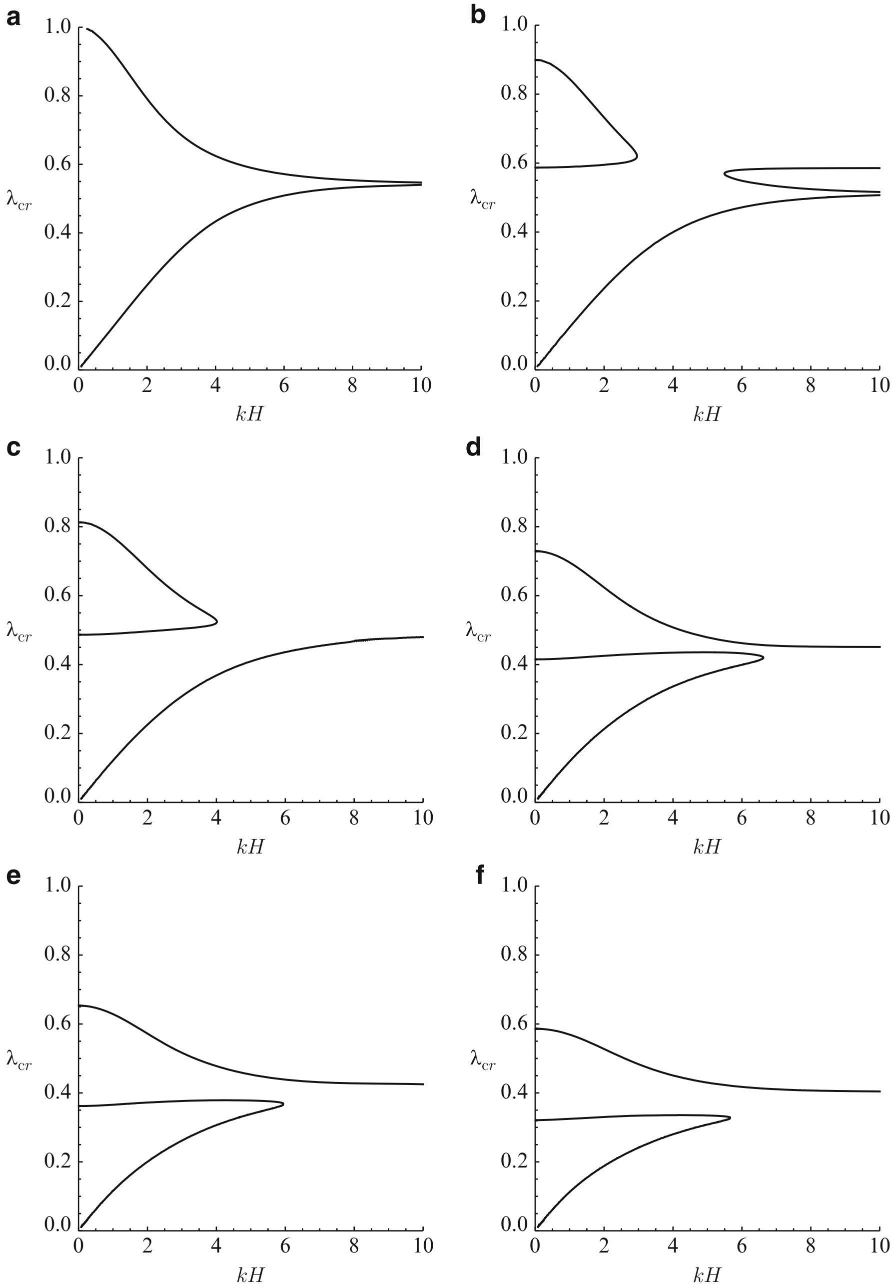

Fig. 10.2 Plots of the critical stretch $\lambda_{\rm cr}$ as a function of the dimensionless parameter kH for a neo-Hookean electroelastic plate subject to an externally applied electric field with $\hat{D}_2 = 0, 1, 1.5, 2, 2.5$ and 3 in (**a**)–(**f**), respectively, and $\alpha = 0.5$ and $\beta = 0.5$ in each case

the corresponding principal stretches as

$$\lambda_1 = \lambda, \quad \lambda_2 = \lambda^{-1}, \quad \lambda_3 = 1, \tag{10.99}$$

where again the incompressibility constraint $\lambda_1\lambda_2\lambda_3 = 1$ has been adopted. Consistent with the notation used in (4.3) and (4.52), we denote the free surface charge per unit area in the deformed configuration by σ_f and the corresponding surface charge per unit reference area by σ_F. The electric displacement field inside the material is then uniform with its Eulerian and Lagrangian forms given in terms of the surface charges as

$$D_2 = \sigma_\mathrm{f}, \quad D_{\mathrm{L}2} = \sigma_\mathrm{F}. \tag{10.100}$$

There is no electric field outside the material (again with the proviso that end effects are neglected). Inside the material the field is uniform with the component in the X_2 direction given in terms of a potential difference, say V, between the electrodes by

$$E_2 = \frac{V}{h}, \quad E_{\mathrm{L}2} = \frac{V}{H} = \lambda_2 E_2. \tag{10.101}$$

The invariants, defined by (9.95) and (9.96), are

$$K_1 = K_2 = 1 + \lambda^2 + \lambda^{-2}, \quad K_4 = D_{\mathrm{L}2}^2, \quad K_5 = \lambda^{-2}K_4, \quad K_6 = \lambda^{-4}K_4, \tag{10.102}$$

which are equal to those used in Sect. 10.3 for the plane strain of an electroelastic half-space. With $\Omega^* = \Omega^*(K_1, K_2, K_4, K_5, K_6)$, the stress components are again given by (10.24) and the electric field component E_2 by (10.25). Here, however, we assume that there is no mechanical traction on the faces $x_2 = 0$, $x_2 = h$, so that $\tau_{22} = 0$. Then, in terms of the reduced energy function $\omega^*(\lambda, K_4)$ the remaining in-plane stress τ_{11} and E_2 from (10.28) are given by

$$\tau_{11} = \lambda\omega_\lambda^*, \quad E_2 = 2\lambda D_{\mathrm{L}2}\omega_4^*. \tag{10.103}$$

10.5.1 Incremental Equations

The derivations of the incremental equations follow precisely the steps in Sect. 10.3.2, resulting in identical expressions. Briefly, the in-plane components of the incremental displacement vector $\mathbf{u}$ are given in terms of a function $\psi = \psi(x_1, x_2)$ by

$$u_1 = \psi_{,2}, \quad u_2 = -\psi_{,1}. \tag{10.104}$$

Similarly, the in-plane components $\dot{D}_{\mathrm{L}01}, \dot{D}_{\mathrm{L}02}$ of the incremental electric displacement field $\dot{\mathbf{D}}_{\mathrm{L}0}$ are given by

$$\dot{D}_{\mathrm{L}01} = \varphi_{,2}, \quad \dot{D}_{\mathrm{L}02} = -\varphi_{,1}, \tag{10.105}$$

where $\varphi = \varphi(x_1, x_2)$. The equilibrium equation $\operatorname{div}\dot{\mathbf{T}}_0 = \mathbf{0}$ and the field equation $\operatorname{curl}\dot{\mathbf{E}}_{\mathrm{L}0} = \mathbf{0}$ become

$$a\psi_{,1111} + 2b\psi_{,1122} + c\psi_{,2222} + (e-d)\varphi_{,112} + d\varphi_{,222} = 0 \tag{10.106}$$

and

$$d\psi_{,222} + (e-d)\psi_{,112} + f\varphi_{,22} + g\varphi_{,11} = 0, \tag{10.107}$$

with the coefficients defined in (10.44) with (10.45).

10.5.2 Incremental Traction Boundary Conditions

If there is no incremental mechanical traction applied on the faces $x_2 = 0$, $x_2 = h$, then, since there is no external electric field, the incremental traction boundary condition (10.6) reduces to the component equations $\dot{T}_{021} = 0$ and $\dot{T}_{022} = 0$. Following the steps outlined in Sect. 10.3.4, but now with $\tau_{22} = 0$, we obtain the explicit expressions

$$c\,(\psi_{,22} - \psi_{,11}) + d\,\varphi_{,2} = 0, \tag{10.108}$$

and

$$(2b + c)\psi_{,112} + c\psi_{,222} + e\varphi_{,11} + d\varphi_{,22} = 0, \tag{10.109}$$

which apply on $x_2 = 0$ and $x_2 = h$.

10.5.3 Incremental Electric Boundary Conditions

Since there is no field outside the material, the electric boundary conditions (9.26) reduce to

$$\mathbf{N} \times \mathbf{E}_{\mathrm{L}} = \mathbf{0}, \quad \mathbf{N} \cdot \mathbf{D}_{\mathrm{L}} = \sigma_{\mathrm{F}}, \tag{10.110}$$

where $\mathbf{N}$ is the unit normal to the reference boundary and σ_{F} is the density of surface charges per unit reference area. The incremental forms of (10.110) are

$$\mathbf{N} \times \dot{\mathbf{E}}_{\mathrm{L}} = \mathbf{0}, \quad \mathbf{N} \cdot \dot{\mathbf{D}}_{\mathrm{L}} = \dot{\sigma}_{\mathrm{F}}. \tag{10.111}$$

There are two options for the incremental electric boundary conditions. For the underlying configuration we can specify either the potential or the surface charge. In the first case the first component of the incremental electric field is taken to vanish on the boundaries $x_2 = 0$, $x_2 = h$, while for the second case the second component of the incremental electric displacement field vanishes.

In the first case the relevant boundary condition is (10.111)$_1$, the Eulerian form of which reduces to

$$\dot{E}_{L01} = 0. \tag{10.112}$$

From the specialization of (10.56) to the present situation it follows that

$$d(\psi_{,22} - \psi_{,11}) + f\varphi_{,2} = 0 \quad \text{on } x_2 = 0, h. \tag{10.113}$$

In the second case it follows from (10.111)$_2$, again in Eulerian form, and (10.105)$_2$ that

$$\varphi_{,1} = 0 \quad \text{on } x_2 = 0, h. \tag{10.114}$$

In the following we use the second of these options. Calculations based on the first option lead to almost identical results.

The solution procedure is similar to the one used for the plate with an external applied field, which was analyzed in Sect. 10.4. For the plate with electrodes, since there is no outside field, the number of unknown functions is reduced to two and we need to consider only

$$\psi = \sum_{j=1}^{6} A_j \mathrm{e}^{-ks_j x_2} \mathrm{e}^{\mathrm{i}kx_1}, \quad \varphi = k \sum_{j=1}^{6} B_j \mathrm{e}^{-ks_j x_2} \mathrm{e}^{\mathrm{i}kx_1}, \tag{10.115}$$

for $0 < x_2 < h$, where A_j and B_j, $j = 1, \ldots, 6$, are constants, and, for the neo-Hookean model (10.68), $s_1, \ldots, s_6$ are given by (10.95) and (10.96). The constants A_j and B_j are connected by (10.89).

There are three boundary conditions to be satisfied on the surface $x_2 = 0$, namely

$$\begin{aligned}
&\sum_{j=1}^{6} [c(1 + s_j^2)A_j - ds_j B_j] = 0, \\
&\sum_{j=1}^{6} [(2b + c - cs_j^2)s_j A_j - (e - ds_j^2)B_j] = 0, \\
&\sum_{j=1}^{6} B_j = 0,
\end{aligned} \tag{10.116}$$

and three on the surface $x_2 = h$:

$$\sum_{j=1}^{6}[c(1+s_j^2)A_j - ds_j B_j]\mathrm{e}^{-ks_j h} = 0,$$

$$\sum_{j=1}^{6}[(2b+c-cs_j^2)s_j A_j - (e-ds_j^2)B_j]\mathrm{e}^{-ks_j h} = 0,$$

$$\sum_{j=1}^{6} B_j \mathrm{e}^{-ks_j h} = 0. \tag{10.117}$$

We use (10.89) to reduce the number of unknowns from 12 to 6, and the solution is then reduced to finding the roots of a 6×6 determinant. To compare the effects of an applied field and flexible electrodes on the electroelastic response of a plate, we again consider the electroelastic neo-Hookean material (10.68) with $\alpha = 0.5$ and $\beta = 0.5$.

The critical values of λ are plotted in Fig. 10.3a–f as functions of kH for $\hat{D}_2 = 0, 1, 1.5, 2, 2.5$ and 3, respectively, and can be compared and contrasted with the results for an electroelastic plate subject to an externally applied electric field shown in Fig. 10.2a–f for the same values of α and β.

Again, the first figure corresponds to the purely elastic case. For increasing values of $\hat{D}_2$, the curves all have the same shape, but the plate becomes increasingly unstable as $\hat{D}_2$ increases, especially for the smaller values of kH. The stable region lies above the upper curve in each case. Thus, for the smaller values of kH, in particular for very thin plates, a sufficiently large lateral stretch needs to be applied to prevent the plate becoming unstable when the electric field is applied. This is consistent with the findings based on the so-called Hessian approach used to examine stability of an equibiaxially stretched thin film (see, e.g., Zhao and Suo 2007 and Díaz-Calleja et al. 2009). However, the Hessian approach neither takes account of the plate thickness nor allows for non-homogeneous deformations of the kind considered here. Indeed, for the plane strain considered here, it is straightforward to show that, after simplifying by multiplying by positive factors, the determinant of the Hessian

$$\begin{bmatrix} \dfrac{\partial^2 \omega^*}{\partial \lambda^2} & \dfrac{\partial^2 \omega^*}{\partial \lambda \partial D_{\mathrm{L}2}} \\[2ex] \dfrac{\partial^2 \omega^*}{\partial \lambda \partial D_{\mathrm{L}2}} & \dfrac{\partial^2 \omega^*}{\partial D_{\mathrm{L}2} \partial D_{\mathrm{L}2}} \end{bmatrix} \tag{10.118}$$

has the form

$$(\alpha\lambda^2 + \beta)(3 + \lambda^4) + \beta(3\alpha\lambda^2 - \beta)\lambda^2 \hat{D}_2^2. \tag{10.119}$$

It is easy to show that the diagonal terms in the Hessian are positive. Then, it is clear that for $\alpha = \beta = 0.5$, the Hessian is positive definite for $\lambda^2 > 1/3$ irrespective of

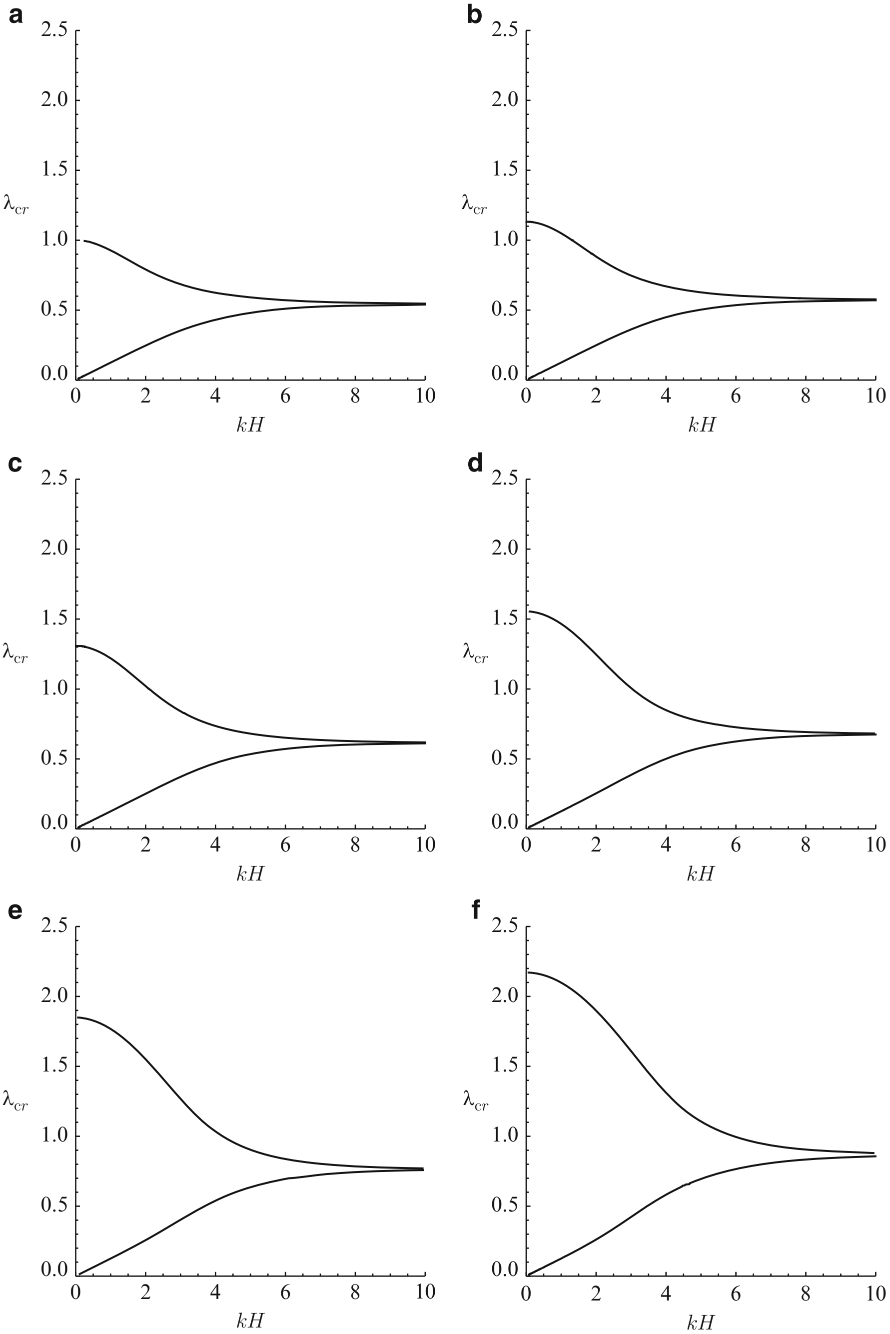

Fig. 10.3 Plots of the critical stretch $\lambda_{\rm cr}$ as a function of the dimensionless parameter kH for a neo-Hookean electroelastic plate with flexible electrodes for $\hat{D}_2 = 0, 1, 1.5, 2, 2.5$ and 3 in (**a**)–(**f**), respectively, and $\alpha = 0.5$ and $\beta = 0.5$ in each case

the value of $\hat{D}_2$, which is a very different stability criterion from that derived here. Thus, the evaluation of stability on the basis of the Hessian must be viewed with caution.

The results in Fig. 10.3 are also very different from those in Fig. 10.2, which is not surprising in view of the very different boundary conditions involved.

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Chapter 11
Magnetoelastic Wave Propagation

Abstract In this chapter we analyze incremental small amplitude motions and magnetic fields superimposed on an underlying finite deformation and magnetic field with a particular interest in the effect of the underlying configuration on the propagation of small amplitude waves. First we examine the propagation of homogeneous plane waves in an infinite medium of an incompressible magnetoelastic material where the underlying configuration corresponds to a homogenous deformation with a uniform magnetic field; no restriction is placed on the material symmetry. This involves an extension of the notion of strong ellipticity to the magnetoelastic context. The theory is then applied to a prototype model of a neo-Hookean magnetoelastic material. Following this we specialize to increments in two dimensions in a principal plane of an isotropic magnetoelastic material. This specialization is then applied to the study of surface wave propagation, first for Rayleigh-type waves on a half-space with the magnetic field either parallel to or perpendicular to the surface and then to Love-type waves with a layer of different material bonded to the half-space. Finally, we investigate the propagation of Bleustein–Gulyaev-type waves on a half-space without a layer. For each type of surface wave numerical results are obtained for the speed of wave propagation in terms of parameters associated with the underlying configuration and illustrated graphically, while for the Bleustein–Gulyaev-type waves some closed-form expressions are obtained for the wave speed in particular cases.

11.1 Preliminaries

In Chap. 10 we were concerned with the stability analysis of electroelastic materials. A parallel analysis can be carried through for magnetoelastic materials following the development in Otténio et al. (2008), which has been applied in Rudykh and Bertoldi (2013) to the stability of a micro-structured magnetoelastic material. However, in this chapter we focus on small amplitude wave propagation rather than directly on

L. Dorfmann and R.W. Ogden, *Nonlinear Theory of Electroelastic and Magnetoelastic Interactions*, DOI 10.1007/978-1-4614-9596-3_11,

stability analysis, although we note during the analysis that vanishing of the speed of propagation corresponds to loss of stability in the considered mode of deformation.

Useful general background on magnetoacoustic waves can be found in the representative papers by Maugin (1981) and Abd-Alla and Maugin (1987) and the book by Maugin (1988). One-dimensional bulk waves are treated in Hefni et al. (1995a) and different aspects of surface wave propagation in Maugin and Hakmi (1985), Abd-Alla and Maugin (1990), Lee and Its (1992) and Hefni et al. (1995b,c).

Here we consider the possibility of incremental motions superimposed on a statically deformed configuration in the presence of a magnetic field, with no accompanying electric field. Thus, $\mathbf{v} = \mathbf{0}, \mathbf{D} = \mathbf{E} = \mathbf{0}$ and hence $\rho_\mathrm{f} = 0$ and $\sigma_\mathrm{f} = 0$. We also assume that the material is non-conducting, so that $\mathbf{J}_\mathrm{f} = \mathbf{0}$, that there are no surface currents ($\mathbf{K}_\mathrm{f} = \mathbf{0}$) and no mechanical body forces ($\mathbf{f} = \mathbf{0}$). The resulting reduced forms of the incremental Maxwell equations are, from (9.45)–(9.47),

$$\mathrm{curl}\,(\dot{\mathbf{E}}_{\mathrm{L}0} + \dot{\mathbf{v}} \times \mathbf{B}) = -\dot{\mathbf{B}}_{\mathrm{L},t0}, \quad \mathrm{div}\,\dot{\mathbf{D}}_{\mathrm{L}0} = 0, \tag{11.1}$$

$$\mathrm{curl}\,\dot{\mathbf{H}}_{\mathrm{L}0} = \dot{\mathbf{D}}_{\mathrm{L},t0}, \quad \mathrm{div}\,\dot{\mathbf{B}}_{\mathrm{L}0} = 0, \tag{11.2}$$

where $\dot{\mathbf{v}} = \mathbf{u}_{,t}$, $\mathbf{u}$ being the displacement, and the incremental equation of motion (9.48) becomes

$$\mathrm{div}\,\dot{\mathbf{T}}_0 = \rho \mathbf{u}_{,tt}. \tag{11.3}$$

11.2 The Quasimagnetostatic Approximation

We now apply the so-called *quasimagnetostatic approximation* or *magnetoacoustic approximation*, which is based on the fact that the speed of acoustic waves is very small compared with the speed of light. In the present (incremental) context, this allows the increments $\dot{\mathbf{E}}_{\mathrm{L}0}$ and $\dot{\mathbf{D}}_{\mathrm{L}0}$ to be neglected. When the approximation is applied, the remaining equations, coupling magnetic and mechanical effects, are

$$\mathrm{curl}\,\dot{\mathbf{H}}_{\mathrm{L}0} = \mathbf{0}, \quad \mathrm{div}\,\dot{\mathbf{B}}_{\mathrm{L}0} = 0, \quad \mathrm{div}\,\dot{\mathbf{T}}_0 = \rho_r \mathbf{u}_{,tt}. \tag{11.4}$$

These are the equations we use in the rest of this chapter for the interior of the material.

Outside the material, which may be vacuum or a non-magnetizable (and non-polarizable) material, we use a superscript * to indicate field quantities. Thus, $\mathbf{H}^\star$ and $\mathbf{B}^\star$, respectively, are the magnetic field and magnetic induction, which are in the simple relation $\mathbf{B}^\star = \mu_0 \mathbf{H}^\star$, where μ_0 is the vacuum permeability. Then the magnetostatic equations are

$$\mathrm{div}\,\mathbf{B}^\star = 0, \quad \mathrm{curl}\,\mathbf{H}^\star = \mathbf{0}, \tag{11.5}$$

and in the quasimagnetostatic approximation their incremental counterparts are

$$\operatorname{div}\dot{\mathbf{B}}^{\star} = 0, \quad \operatorname{curl}\dot{\mathbf{H}}^{\star} = \mathbf{0}, \tag{11.6}$$

with $\dot{\mathbf{B}}^{\star} = \mu_0\dot{\mathbf{H}}^{\star}$.

Henceforth we use the notations $\mathcal{B}$ and $\partial\mathcal{B}$ for the (time-independent) initial configuration upon which the infinitesimal motion is superimposed, and we confine attention to incompressible materials. Thus, $J = 1$ and the incremental incompressibility condition is $\operatorname{div}\mathbf{u} = 0$.

We use the formulation based on $\Omega^*(\mathbf{F}, \mathbf{B}_\mathrm{L})$ although the formulation based on $\Omega(\mathbf{F}, \mathbf{H}_\mathrm{L})$ could equally well be used. The required incremental constitutive equations in $\mathcal{B}$ are therefore, from (9.165) and $(9.164)_2$, respectively,

$$\dot{\mathbf{T}}_0 = \boldsymbol{\mathcal{A}}_0^*\mathbf{L} + \mathbb{A}_0^*\dot{\mathbf{B}}_{\mathrm{L}0} - \dot{p}^*\mathbf{I} + p^*\mathbf{L}, \tag{11.7}$$

$$\dot{\mathbf{H}}_{\mathrm{L}0} = \mathbb{A}_0^{*\mathrm{T}}\mathbf{L} + \mathsf{A}_0^*\dot{\mathbf{B}}_{\mathrm{L}0}. \tag{11.8}$$

On substituting these into $(11.4)_{1,3}$ we obtain

$$\operatorname{div}(\boldsymbol{\mathcal{A}}_0^*\mathbf{L} + \mathbb{A}_0^*\dot{\mathbf{B}}_{\mathrm{L}0}) - \operatorname{grad}\dot{p}^* + \mathbf{L}^\mathrm{T}\operatorname{grad}p^* = \rho\mathbf{u}_{,tt}, \tag{11.9}$$

$$\operatorname{curl}(\mathbb{A}_0^{*\mathrm{T}}\mathbf{L} + \mathsf{A}_0^*\dot{\mathbf{B}}_{\mathrm{L}0}) = \mathbf{0}, \tag{11.10}$$

which, together with

$$\operatorname{div}\mathbf{u} = 0, \quad \operatorname{div}\dot{\mathbf{B}}_{\mathrm{L}0} = 0, \tag{11.11}$$

govern the variables $\mathbf{u}$ and $\dot{\mathbf{B}}_{\mathrm{L}0}$. Note that $\dot{p}^*$ could be eliminated by taking the curl of (11.9).

The unknowns are the six components of $\mathbf{u}$ and $\dot{\mathbf{B}}_{\mathrm{L}0}$ together with $\dot{p}^*$, which are governed by the eight equations (11.9), (11.10) and (11.11).

Finally in this section, we take the underlying deformation to be homogeneous and the field vectors $\mathbf{H}$ and $\mathbf{B}$ to be uniform. Then, the tensors $\boldsymbol{\mathcal{A}}_0^*, \mathbb{A}_0^*, \mathsf{A}_0^*$ are constant, as is p^*. Equations (11.9) and (11.10) may then be written in slightly more compact forms as

$$\operatorname{div}(\boldsymbol{\mathcal{A}}_0^*\mathbf{L} + \mathbb{A}_0^*\mathbf{w}) - \operatorname{grad}\dot{p}^* = \rho\mathbf{u}_{,tt}, \tag{11.12}$$

and

$$\operatorname{curl}(\mathbb{A}_0^{*\mathrm{T}}\mathbf{L} + \mathsf{A}_0^*\mathbf{w}) = \mathbf{0}, \tag{11.13}$$

respectively, where, for convenience, we have set $\dot{\mathbf{B}}_{\mathrm{L}0} = \mathbf{w}$. Equations (11.11) become

$$\operatorname{div}\mathbf{u} = 0, \quad \operatorname{div}\mathbf{w} = 0. \tag{11.14}$$

11.3 Incremental Homogeneous Plane Waves

As a first illustration we consider infinitesimal homogenous plane waves propagating with speed v in the direction of the unit vector $\mathbf{n}$ in the form

$$\mathbf{u} = \mathbf{m} f(\mathbf{n}\cdot\mathbf{x} - vt), \quad \dot{\mathbf{B}}_{\mathrm{L}0} = \mathbf{w} = \mathbf{q} g(\mathbf{n}\cdot\mathbf{x} - vt), \quad \dot{p}^* = h(\mathbf{n}\cdot\mathbf{x} - vt), \tag{11.15}$$

where $\mathbf{m}$ and $\mathbf{q}$ are constant (polarization) unit vectors in the directions of the incremental displacement and magnetic induction, respectively, and f, g and h are appropriately regular functions of the argument $\mathbf{n}\cdot\mathbf{x} - vt$. Substituting these expressions into (11.12)–(11.14) we obtain

$$\mathbf{Q}^*(\mathbf{n})\mathbf{m} f'' + \mathbf{R}^*(\mathbf{n})\mathbf{q} g' - h'\mathbf{n} = \rho v^2 \mathbf{m} f'', \tag{11.16}$$

$$\mathbf{n} \times \{\mathbf{R}^*(\mathbf{n})^{\mathrm{T}}\mathbf{m} f'' + \mathsf{A}_0^*\mathbf{q} g'\} = \mathbf{0}, \quad \mathbf{q}\cdot\mathbf{n} = 0, \quad \mathbf{m}\cdot\mathbf{n} = 0, \tag{11.17}$$

where $\mathbf{Q}^*(\mathbf{n})$, the *acoustic tensor*, and $\mathbf{R}^*(\mathbf{n})$, the *magnetoacoustic tensor*, are given by

$$\left[\mathbf{Q}^*(\mathbf{n})\right]_{ij} = \mathcal{A}^*_{0piqj} n_p n_q, \quad \left[\mathbf{R}^*(\mathbf{n})\right]_{ij} = \mathbb{A}^*_{0ip|j} n_p, \tag{11.18}$$

and a prime signifies differentiation with respect to the argument $\mathbf{n}\cdot\mathbf{x} - vt$. Note that $\mathbf{Q}^*(\mathbf{n})$ is symmetric, but in general $\mathbf{R}^*(\mathbf{n})$ is not.

Let $\hat{\mathbf{I}}(\mathbf{n}) = \mathbf{I} - \mathbf{n}\otimes\mathbf{n}$ denote the symmetric projection tensor onto the plane with normal $\mathbf{n}$. Then, following Destrade and Ogden (2011), we define the projections of $\mathbf{Q}^*(\mathbf{n})$, $\mathbf{R}^*(\mathbf{n})$ and $\mathsf{A}_0^*(\mathbf{n})$ onto the plane normal to $\mathbf{n}$ by

$$\hat{\mathbf{Q}}^*(\mathbf{n}) = \hat{\mathbf{I}}(\mathbf{n})\mathbf{Q}^*(\mathbf{n})\hat{\mathbf{I}}(\mathbf{n}), \quad \hat{\mathbf{R}}^*(\mathbf{n}) = \hat{\mathbf{I}}(\mathbf{n})\mathbf{R}^*(\mathbf{n})\hat{\mathbf{I}}(\mathbf{n}), \quad \hat{\mathsf{A}}_0^*(\mathbf{n}) = \hat{\mathbf{I}}(\mathbf{n})\mathsf{A}_0^*(\mathbf{n})\hat{\mathbf{I}}(\mathbf{n}), \tag{11.19}$$

respectively.

Using $(11.17)_3$ we obtain from (11.16)

$$h' = [\mathbf{Q}^*(\mathbf{n})\mathbf{m}]\cdot\mathbf{n} f'' + [\mathbf{R}^*(\mathbf{n})\mathbf{q}]\cdot\mathbf{n} g', \tag{11.20}$$

and substitution of this back into (11.16) enables the latter to be written

$$\hat{\mathbf{Q}}^*(\mathbf{n})\mathbf{m} f'' + \hat{\mathbf{R}}^*(\mathbf{n})\mathbf{q} g' = \rho v^2 \mathbf{m} f''. \tag{11.21}$$

Similarly, from $(11.17)_1$ we deduce that

$$\mathbf{R}^*(\mathbf{n})^{\mathrm{T}}\mathbf{m} f'' + \mathsf{A}_0^*\mathbf{q} g' = \{[\mathbf{R}^*(\mathbf{n})^{\mathrm{T}}\mathbf{m}]\cdot\mathbf{n} f'' + (\mathsf{A}_0^*\mathbf{q})\cdot\mathbf{n} g'\}\mathbf{n}, \tag{11.22}$$

which can be written more compactly as

$$\hat{\mathbf{R}}^*(\mathbf{n})^{\mathrm{T}}\mathbf{m} f'' + \hat{\mathsf{A}}_0^*\mathbf{q} g' = \mathbf{0}. \tag{11.23}$$

As in Destrade and Ogden (2011) we assume that $\hat{\mathsf{A}}_0^*$ is non-singular as an operator restricted to the plane normal to $\mathbf{n}$ and also positive definite in view of its interpretation as the inverse of the incremental permeability tensor (which is *minus* A_0), and this is positive definite (at least in the linear theory). Indeed A^* itself is defined by (9.161)$_4$, while its updated version A_0^* appears in the incremental constitutive equation (11.8). This is connected to the corresponding updated version of the incremental permeability tensor $-\mathsf{A}_0$ by $\mathsf{A}_0\mathsf{A}_0^* = \mathsf{A}_0^*\mathsf{A}_0 = -\mathbf{I}$. The derivation of these relations follows the same pattern as in the electroelastic case at the end of Sect. 9.3.1.5.

We then obtain $\mathbf{q}g' = -\hat{\mathsf{A}}_0^{*-1}\hat{\mathbf{R}}^*(\mathbf{n})^{\mathrm{T}}\mathbf{m}f''$, and substitution into (11.21) and elimination of $f'' \neq 0$ yields the *propagation condition* for acoustic waves under the influence of a magnetic field, explicitly

$$\hat{\mathbf{P}}^*(\mathbf{n})\mathbf{m} \equiv \hat{\mathbf{Q}}^*(\mathbf{n})\mathbf{m} - \hat{\mathbf{R}}^*(\mathbf{n})\hat{\mathsf{A}}_0^{*-1}\hat{\mathbf{R}}^*(\mathbf{n})^{\mathrm{T}}\mathbf{m} = \rho v^2\mathbf{m}, \tag{11.24}$$

wherein the generalized acoustic tensor $\hat{\mathbf{P}}^*$ (otherwise known as the Christoffel tensor) is defined as $\hat{\mathbf{Q}}^*(\mathbf{n}) - \hat{\mathbf{R}}^*(\mathbf{n})\hat{\mathsf{A}}_0^{*-1}\hat{\mathbf{R}}^*(\mathbf{n})^{\mathrm{T}}$, which is symmetric. Equation (11.24) is a generalization of the propagation condition for homogeneous plane waves in an incompressible elastic solid in the absence of a magnetic field. This prompts a corresponding generalization of the *strong ellipticity condition* in the form

$$\mathbf{m} \cdot [\hat{\mathbf{P}}^*(\mathbf{n})\mathbf{m}] > 0, \tag{11.25}$$

for all unit vectors $\mathbf{m}$ and $\mathbf{n}$ such that $\mathbf{m} \cdot \mathbf{n} = 0$, as given in Destrade and Ogden (2011). This guarantees that homogeneous plane waves have real wave speeds. In component form, which will be useful later, the generalized strong ellipticity inequality (11.25) can be written as

$$(\mathcal{A}^*_{0piqj} - \mathbb{A}^*_{0ip|k}\hat{\mathsf{A}}^{*-1}_{0kl}\mathbb{A}^*_{0jq|l})m_i m_j n_p n_q > 0. \tag{11.26}$$

11.3.1 Isotropic Magnetoelasticity

We now specialize the constitutive law to that of an incompressible isotropic magnetoelastic material described in Chap. 6 in terms of the energy function $\Omega^*(\mathbf{F}, \mathbf{B}_{\mathrm{L}})$ using the invariants K_1, K_2, K_4, K_5, K_6 defined in (9.153) and (9.154). From (6.92) we have

$$\boldsymbol{\tau} = -p^*\mathbf{I} + 2\Omega_1^*\mathbf{b} + 2\Omega_2^*(K_1\mathbf{b} - \mathbf{b}^2) + 2\Omega_5^*\mathbf{B} \otimes \mathbf{B} + 2\Omega_6^*(\mathbf{B} \otimes \mathbf{bB} + \mathbf{bB} \otimes \mathbf{B}), \tag{11.27}$$

and from (6.91)

$$\mathbf{H} = 2(\Omega_4^*\mathbf{b}^{-1}\mathbf{B} + \Omega_5^*\mathbf{B} + \Omega_6^*\mathbf{bB}). \tag{11.28}$$

Note that we used K_1 and K_2 instead of I_1 and I_2 in Chap. 9 when using $\mathbf{B}_{\mathrm{L}}$ as the independent magnetic variable.

Referred to the principal axes of the left Cauchy–Green tensor **b**, which then has the diagonal representation diag$[\lambda_1^2, \lambda_2^2, \lambda_3^2]$, where $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches, (11.27) and (11.28) have component forms

$$\tau_{ii} = -p^* + 2\lambda_i^2 \Omega_1^* + 2\lambda_i^2(\lambda_j^2 + \lambda_k^2)\Omega_2^* + 2\Omega_5^* B_i^2 + 4\lambda_i^2 \Omega_6^* B_i^2, \quad (11.29)$$

$$\tau_{ij} = 2[\Omega_5^* + (\lambda_i^2 + \lambda_j^2)\Omega_6^*] B_i B_j, \quad (11.30)$$

where (i, j, k) is a permutation of $(1, 2, 3)$, and

$$H_i = 2(\lambda_i^{-2}\Omega_4^* + \Omega_5^* + \lambda_i^2 \Omega_6^*) B_i, \quad i = 1, 2, 3. \quad (11.31)$$

For explicit expressions for the components of the moduli tensors $\boldsymbol{\mathcal{A}}_0^*$, $\mathbb{A}_0^*$ and $\mathsf{\mathbf{A}}_0^*$, we refer to Sect. 9.3.4.4.

11.3.2 Example: A Prototype Magnetoelastic Solid

To illustrate the above results we consider a simple prototype model of a magnetoelastic solid based on the neo-Hookean model from rubber elasticity that is the counterpart of the model (10.68) used in Chap. 10. This has energy function Ω^* given by

$$\Omega^* = \frac{1}{2}\mu(K_1 - 3) + \frac{1}{2}\mu_0^{-1}(\alpha K_4 + \beta K_5), \quad (11.32)$$

where $K_1 = \mathrm{tr}\,\mathbf{c}$ is the first principal invariant of $\mathbf{c}$, $K_4 = \mathbf{B}_\mathrm{L} \cdot \mathbf{B}_\mathrm{L}$ and $K_5 = \mathbf{B}_\mathrm{L} \cdot (\mathbf{c}\mathbf{B}_\mathrm{L})$, μ is the shear modulus of the material in the absence of a magnetic field, and α and β are dimensionless material constants that couple the mechanical and magnetic effects. Note the distinction between μ and μ_0, the latter, we recall, being the magnetic permeability of free space. In a slightly different notation this model was used in Otténio et al. (2008), wherein the interpretation of the coupling constants was discussed.

It follows from (11.27) and (11.28) that

$$\boldsymbol{\tau} = -p^*\mathbf{I} + \mu\mathbf{b} + \mu_0^{-1}\beta\mathbf{B} \otimes \mathbf{B}, \quad \mathbf{H} = \mu_0^{-1}(\alpha\mathbf{b}^{-1}\mathbf{B} + \beta\mathbf{B}). \quad (11.33)$$

From the definitions (11.18) and the expressions in Sect. 9.3.4.4, we also obtain

$$\mathbf{Q}^*(\mathbf{n}) = [\mu(\mathbf{bn}) \cdot \mathbf{n} + \mu_0^{-1}\beta(\mathbf{B} \cdot \mathbf{n})^2]\mathbf{I}, \quad (11.34)$$

$$\mathbf{R}^*(\mathbf{n}) = \mu_0^{-1}\beta[(\mathbf{B} \cdot \mathbf{n})\mathbf{I} + \mathbf{B} \otimes \mathbf{n}], \quad (11.35)$$

and

$$\mathbf{A}_0^* = \mu_0^{-1}(\alpha\mathbf{b}^{-1} + \beta\mathbf{I}). \tag{11.36}$$

Since we are taking $\mathbf{A}_0^*$ to be positive definite, we should have $\alpha \geq 0$ and $\beta \geq 0$ with $\alpha + \beta > 0$, as for the electroelastic model.

Equation (11.16) becomes

$$[\mu(\mathbf{bn}) \cdot \mathbf{n} + \mu_0^{-1}\beta(\mathbf{B} \cdot \mathbf{n})^2 - \rho v^2]\mathbf{m}f'' + \mu_0^{-1}\beta(\mathbf{B} \cdot \mathbf{n})\mathbf{q}g' = h'\mathbf{n}, \tag{11.37}$$

from which, with the help of $(11.17)_{2,3}$, we deduce that $h' = 0$. For $\mathbf{B} \cdot \mathbf{n} = 0$ this gives

$$\rho v^2 = \mu(\mathbf{bn}) \cdot \mathbf{n}, \tag{11.38}$$

which is independent of $\mathbf{B}$. Thus, if $\mathbf{B}$ is perpendicular to the direction of propagation, then the wave speed is unaffected by the magnetic field. In view of the inequalities satisfied by α and β, it follows that $g' = 0$, and hence there is no disturbance to the underlying magnetic field by the (transverse) mechanical wave.

If $\mathbf{B} \cdot \mathbf{n} \neq 0$, then the wave speed is given by

$$\rho v^2 = \mu(\mathbf{bn}) \cdot \mathbf{n} + \frac{\mu_0^{-1}\alpha\beta}{\beta + \alpha(\mathbf{b}^{-1}\mathbf{m}) \cdot \mathbf{m}}(\mathbf{B} \cdot \mathbf{n})^2(\mathbf{b}^{-1}\mathbf{m}) \cdot \mathbf{m}, \tag{11.39}$$

which recovers the result for $\mathbf{B} \cdot \mathbf{n} = 0$ in this specialization. Clearly, the squared wave speed is increased by the presence of the magnetic field compared with that in the absence of the field. If the underlying magnetic field is supported (by appropriate stresses) in the absence of deformation (so that $\mathbf{b} = \mathbf{I}$), then (11.39) simplifies to

$$\rho v^2 = \mu + \frac{\mu_0^{-1}\alpha\beta}{\alpha + \beta}(\mathbf{B} \cdot \mathbf{n})^2. \tag{11.40}$$

11.3.3 Two-Dimensional Specialization

We now restrict attention to two-dimensional incremental motions, in the (X_1, X_2) plane but without specializing the form of the (isotropic) constitutive law. Let the underlying deformation correspond to a uniform stretch λ_3 in the X_3 direction and let λ_1 and λ_2 be the principal stretches in the (X_1, X_2) plane, also uniform, but in general the corresponding principal directions of strain need not coincide with the X_1 and X_2 axes. Let the initial magnetic induction be uniform with components $(B_1, B_2, 0)$ within the material and $(B_1^\star, B_2^\star, 0)$ outside the material. In the notation used in Sect. 11.2, we take $\mathbf{u}$ and $\mathbf{w}$ to have components $(u_1, u_2, 0)$ and $(w_1, w_2, 0)$, respectively, with u_1, u_2, w_1, w_2 functions of x_1, x_2, t only.

The third component of the equation of motion (11.12) and the first two components of (11.13) are then satisfied trivially, and the remaining equations are

$$\begin{aligned}
&\mathcal{A}^*_{01111}u_{1,11} + 2\mathcal{A}^*_{01121}u_{1,12} + \mathcal{A}^*_{02121}u_{1,22} + \mathcal{A}^*_{01112}u_{2,11} \\
&\quad + \left(\mathcal{A}^*_{01122} + \mathcal{A}^*_{01221}\right)u_{2,12} + \mathcal{A}^*_{02122}u_{2,22} + \mathbb{A}^*_{011|1}w_{1,1} \\
&\quad + \mathbb{A}^*_{021|1}w_{1,2} + \mathbb{A}^*_{011|2}w_{2,1} + \mathbb{A}^*_{021|2}w_{2,2} - \dot{p}^*_{,1} = \rho u_{1,tt},
\end{aligned} \tag{11.41}$$

$$\begin{aligned}
&\mathcal{A}^*_{01211}u_{1,11} + \left(\mathcal{A}^*_{01221} + \mathcal{A}^*_{01122}\right)u_{1,12} + \mathcal{A}^*_{02221}u_{1,22} \\
&\quad + \mathcal{A}^*_{01212}u_{2,11} + 2\mathcal{A}^*_{01222}u_{2,12} + \mathcal{A}^*_{02222}u_{2,22} + \mathbb{A}^*_{012|1}w_{1,1} \\
&\quad + \mathbb{A}^*_{022|1}w_{1,2} + \mathbb{A}^*_{012|2}w_{2,1} + \mathbb{A}^*_{022|2}w_{2,2} - \dot{p}^*_{,2} = \rho u_{2,tt},
\end{aligned} \tag{11.42}$$

$$\begin{aligned}
&\mathbb{A}^*_{011|2}u_{1,11} + (\mathbb{A}^*_{021|2} - \mathbb{A}^*_{011|1})u_{1,12} - \mathbb{A}^*_{021|1}u_{1,22} \\
&\quad + \mathbb{A}^*_{012|2}u_{2,11} + (\mathbb{A}^*_{022|2} - \mathbb{A}^*_{012|1})u_{2,12} - \mathbb{A}^*_{022|1}u_{2,22} \\
&\quad + \mathsf{A}^*_{012}w_{1,1} - \mathsf{A}^*_{011}w_{1,2} + \mathsf{A}^*_{022}w_{2,1} - \mathsf{A}^*_{012}w_{2,2} = 0.
\end{aligned} \tag{11.43}$$

Elimination of $\dot{p}^*$ from (11.41) and (11.42) by cross-differentiation and subtraction yields

$$\begin{aligned}
&\mathcal{A}^*_{01211}u_{1,111} + (\mathcal{A}^*_{01221} + \mathcal{A}^*_{01122} - \mathcal{A}^*_{01111})u_{1,112} + (\mathcal{A}^*_{02221} - 2\mathcal{A}^*_{01121})u_{1,122} \\
&\quad - \mathcal{A}^*_{02121}u_{1,222} + \mathcal{A}^*_{01212}u_{2,111} + (2\mathcal{A}^*_{01222} - \mathcal{A}^*_{01112})u_{2,112} \\
&\quad - (\mathcal{A}^*_{01122} + \mathcal{A}^*_{01221} - \mathcal{A}^*_{02222})u_{2,122} - \mathcal{A}^*_{02122}u_{2,222} + \mathbb{A}^*_{012|1}w_{1,11} \\
&\quad + (\mathbb{A}^*_{022|1} - \mathbb{A}^*_{011|1})w_{1,12} - \mathbb{A}^*_{021|1}w_{1,22} + \mathbb{A}^*_{012|2}w_{2,11} \\
&\quad + (\mathbb{A}^*_{022|2} - \mathbb{A}^*_{011|2})w_{2,12} - \mathbb{A}^*_{021|2}w_{2,22} = \rho(u_{2,1} - u_{1,2})_{,tt}.
\end{aligned} \tag{11.44}$$

Next, from (11.14) we are able to introduce two scalar functions $\psi(x_1, x_2, t)$ and $\varphi(x_1, x_2, t)$ such that

$$u_1 = \psi_{,2}, \quad u_2 = -\psi_{,1}, \quad w_1 = \varphi_{,2}, \quad w_2 = -\varphi_{,1}. \tag{11.45}$$

Here we are using the same notation as in Chap. 10 for the counterpart quantities. Substituting these expressions into (11.44) and (11.43), these two coupled equations for ψ and φ can be written compactly as

$$a\psi_{,1111} + b_1\psi_{,1112} + 2b\psi_{,1122} + b_2\psi_{,1222} + c\psi_{,2222}$$
$$+ d\varphi_{,111} + e\varphi_{,112} + \bar{e}\varphi_{,122} + \bar{d}\varphi_{,222} = \rho\left(\psi_{,11} + \psi_{,22}\right)_{,tt}, \quad (11.46)$$

and

$$d\psi_{,111} + e\psi_{,112} + \bar{e}\psi_{,122} + \bar{d}\psi_{,222} + \mathsf{A}^*_{011}\varphi_{,22} + \mathsf{A}^*_{022}\varphi_{,11} - 2\mathsf{A}^*_{012}\varphi_{,12} = 0, \quad (11.47)$$

respectively, where we have introduced the notations

$$a = \mathcal{A}^*_{01212}, \quad 2b = \mathcal{A}^*_{01111} + \mathcal{A}^*_{02222} - 2\mathcal{A}^*_{01122} - 2\mathcal{A}^*_{01221}, \quad (11.48)$$

$$c = \mathcal{A}^*_{02121}, \quad b_1 = 2(\mathcal{A}^*_{01222} - \mathcal{A}^*_{01211}), \quad b_2 = 2(\mathcal{A}^*_{01121} - \mathcal{A}^*_{02221}), \quad (11.49)$$

$$d = \mathbb{A}^*_{012|2}, \quad e = \mathbb{A}^*_{022|2} - \mathbb{A}^*_{011|2} - \mathbb{A}^*_{012|1}, \quad (11.50)$$

$$\bar{e} = \mathbb{A}^*_{011|1} - \mathbb{A}^*_{022|1} - \mathbb{A}^*_{021|2}, \quad \bar{d} = \mathbb{A}^*_{021|1}. \quad (11.51)$$

Equations (11.46) and (11.47) for ψ and φ apply in the material. Outside the material we have

$$\dot{B}^\star_{1,1} + \dot{B}^\star_{2,2} = 0, \quad \dot{B}^\star_{2,1} - \dot{B}^\star_{1,2} = 0, \quad (11.52)$$

and introduction of a scalar function $\varphi^\star$ such that

$$\dot{B}^\star_1 = \varphi^\star_{,2}, \quad \dot{B}^\star_2 = -\varphi^\star_{,1} \quad (11.53)$$

enables us to obtain a single equation, namely Laplace's equation, which is applicable outside the material:

$$\varphi^\star_{,11} + \varphi^\star_{,22} = 0. \quad (11.54)$$

The components of $\boldsymbol{\mathcal{A}}^*_0$, $\mathbb{A}^*_0$ and A^*_0 can be read off from the general expressions given in Sect. 9.3.4.4, which are referred to the principal axes of $\mathbf{b}$. Note that no component with an index 3 appears in the above equations. If either $B_1 = 0$ or $B_2 = 0$, then $b_1 = b_2 = 0$; if $B_1 = 0$, then $d = \bar{e} = 0$; and if $B_2 = 0$, then $e = \bar{d} = 0$. Henceforth we take the principal axes of $\mathbf{b}$ to coincide with the Cartesian coordinate axes.

Consider again plane waves, this time in the form

$$\psi = \psi_0 f(\mathbf{n}\cdot\mathbf{x} - vt), \quad \varphi = \varphi_0 f(\mathbf{n}\cdot\mathbf{x} - vt), \quad (11.55)$$

where ψ_0 and φ_0 are constants and f is a function of $\mathbf{n}\cdot\mathbf{x} - vt$, where $\mathbf{n}$ and $\mathbf{x}$ are vectors lying in the (x_1, x_2) plane. Substitution into (11.46) and (11.47) yields

$$
\begin{aligned}
&(an_1^4 + b_1 n_1^3 n_2 + 2bn_1^2 n_2^2 + b_2 n_1 n_2^3 + cn_2^4 - \rho v^2)\psi_0 \\
&\qquad + (dn_1^3 + en_1^2 n_2 + \bar{e} n_1 n_2^2 + \bar{d} n_2^3)\varphi_0 = 0,
\end{aligned} \tag{11.56}
$$

$$
\begin{aligned}
&(dn_1^3 + en_1^2 n_2 + \bar{e} n_1 n_2^2 + \bar{d} n_2^3)\psi_0 \\
&\qquad + (\mathsf{A}^*_{022} n_1^2 - 2\mathsf{A}^*_{012} n_1 n_2 + \mathsf{A}^*_{011} n_2^2)\varphi_0 = 0.
\end{aligned} \tag{11.57}
$$

For a non-trivial solution for ψ_0 and φ_0, we set the determinant of coefficients to zero and obtain

$$
\begin{aligned}
\rho v^2 = {} & an_1^4 + b_1 n_1^3 n_2 + 2bn_1^2 n_2^2 + b_2 n_1 n_2^3 + cn_2^4 \\
& - \frac{(dn_1^3 + en_1^2 n_2 + \bar{e} n_1 n_2^2 + \bar{d} n_2^3)^2}{(\mathsf{A}^*_{022} n_1^2 - 2\mathsf{A}^*_{012} n_1 n_2 + \mathsf{A}^*_{011} n_2^2)},
\end{aligned} \tag{11.58}
$$

which is the *propagation condition* that determines the wave speed for any direction of propagation $\mathbf{n} = (n_1, n_2, 0)$. Note that the denominator in the second line of this expression is strictly positive since $\mathbf{A}^*_0$ is positive definite.

Suppose that the direction of propagation is along one of the principal axes; specifically, we take $n_1 = 1, n_2 = 0$. Equation (11.58) then simplifies to

$$
\rho v^2 = a - d^2/\mathsf{A}^*_{022}. \tag{11.59}
$$

If $B_1 = 0$ then $d = 0$ and $\rho v^2 = a$, where

$$
a = \mathcal{A}^*_{01212} = 2\lambda_1^2(\Omega_1^* + \lambda_3^2 \Omega_2^*) + 2\lambda_1^2 \Omega_6^* B_2^2. \tag{11.60}
$$

Thus, the effect of Ω_6^* is to increase (decrease) the wave speed if $\Omega_6^* > 0\,(< 0)$ for a magnetic field transverse to the direction of propagation, bearing in mind that Ω_1^* and Ω_2^* may also depend on the magnetic field. Moreover, it follows from (11.57) that $\varphi_0 = 0$, so the mechanical wave is not accompanied by a disturbance in the underlying magnetic field. If $\Omega_6^* = 0$ then the wave speed is affected by the magnetic field only by any dependence of Ω_1^* and Ω_2^* on the magnetic field.

If, instead, $B_2 = 0$ then

$$
d = 2[\Omega_5^* + (\lambda_1^2 + \lambda_2^2)\Omega_6^*]B_1, \quad \mathsf{A}^*_{022} = 2(\lambda_2^{-2}\Omega_4^* + \Omega_5^* + \lambda_2^2 \Omega_6^*), \tag{11.61}
$$

and

$$
a = \mathcal{A}^*_{01212} = 2\lambda_1^2(\Omega_1^* + \lambda_3^2 \Omega_2^*) + 2[\Omega_5^* + (2\lambda_1^2 + \lambda_2^2)\Omega_6^*]B_1^2, \tag{11.62}
$$

and the wave speed depends on B_1 and the constitutive functions as well as the underlying deformation in a fairly complicated way. For the prototype model (11.32), however, (11.59) reduces to

$$
\rho v^2 = \mu \lambda_1^2 + \frac{\mu_0^{-1}\alpha\beta}{\alpha + \beta\lambda_2^2} B_1^2. \tag{11.63}
$$

11.4 Surface Waves

In this section we consider the propagation of Rayleigh-type and Bleustein–Gulyaev-type waves in a half-space of a magnetoelastic material which is subject to a finite homogeneous deformation and a uniform magnetic field, and Love-type waves in a half-space with a uniform layer of different magnetoelastic material. This is based primarily on the papers by Saxena and Ogden (2011, 2012). Specifically, we consider that in its undeformed reference configuration, the half-space is defined by $X_2 < 0$ and that its boundary $X_2 = 0$ has unit outward normal $\mathbf{N}$ with components $(0, 1, 0)$. The half-space is subject to the pure homogeneous strain defined by

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3, \tag{11.64}$$

and a uniform magnetic field with magnetic induction components $(B_1, B_2, 0)$. The deformed half-space then occupies the region $x_2 < 0$, and the unit outward normal $\mathbf{n}$ to its boundary $x_2 = 0$ has components $(0, 1, 0)$. Outside the material the magnetic induction has components $(B_1^\star, B_2^\star, 0)$, and by continuity $B_2^\star = B_2$ and $\mu_0^{-1} B_1^\star = H_1^\star = H_1$, with H_1 and H_2 given by the constitutive law (11.28) as

$$H_1 = 2(\Omega_4^* \lambda_1^{-2} + \Omega_5^* + \Omega_6^* \lambda_1^2) B_1, \quad H_2 = 2(\Omega_4^* \lambda_2^{-2} + \Omega_5^* + \Omega_6^* \lambda_2^2) B_2, \tag{11.65}$$

with $H_3 = 0$. Assuming that the outside field components $B_1^\star$ and $B_2^\star$ are given the first of these determines B_1 (in general implicitly) in terms of $H_1 = \mu_0^{-1} B_1^\star$ and $B_2 = B_2^\star$, and the second then gives H_2 in terms of B_1 and B_2.

The Maxwell stress outside the material is given by (9.124) and has components

$$\tau_{\mathrm{m}11}^\star = \frac{1}{2}\mu_0^{-1}(B_1^{\star 2} - B_2^{\star 2}), \quad \tau_{\mathrm{m}22}^\star = -\tau_{\mathrm{m}11}^\star, \tag{11.66}$$

$$\tau_{\mathrm{m}12}^\star = \mu_0^{-1} B_1^\star B_2^\star, \quad \tau_{\mathrm{m}33}^\star = -\frac{1}{2}\mu_0^{-1}(B_1^{\star 2} + B_2^{\star 2}). \tag{11.67}$$

The components that contribute to the traction on $x_2 = 0$ are $\tau_{\mathrm{m}22}^\star$ and $\tau_{\mathrm{m}12}^\star$. The corresponding components of stress calculated inside the material are obtained from (11.27) as

$$\tau_{22} = -p^* + 2\Omega_1^* \lambda_2^2 + 2\Omega_2^* \lambda_2^2(\lambda_1^2 + \lambda_3^2) + 2(\Omega_5^* + 2\Omega_6^* \lambda_2^2) B_2^2, \tag{11.68}$$

$$\tau_{12} = 2[\Omega_5^* + \Omega_6^*(\lambda_1^2 + \lambda_2^2)] B_1 B_2. \tag{11.69}$$

The differences $\tau_{22} - \tau_{\mathrm{m}22}^\star$ and $\tau_{12} - \tau_{\mathrm{m}12}^\star$ constitute the normal and shear components of the mechanical traction required to maintain the deformed configuration.

Upon this deformed configuration and magnetic field we now superimpose an incremental motion and an accompanying incremental magnetic field. In the body we take the incremental displacement to have components $(u_1, u_2, 0)$ and the incremental components of the updated Lagrangian magnetic induction to have components $(w_1, w_2, 0)$, as in Sect. 11.3.3, and to depend only on x_1, x_2 and t. Equations (11.46) and (11.47) are therefore applicable in the present context inside the material, while (11.54) applies outside in $x_2 > 0$, with the incremental components of the magnetic induction $\dot{B}_1^\star$ and $\dot{B}_2^\star$ given by (11.53). Now we need, in addition, appropriate boundary conditions on $x_2 = 0$, as well as suitable decay conditions as $x_2 \to \pm\infty$.

The incremental Maxwell stress is given by (9.125) and has components

$$\dot{\tau}^\star_{\text{m}11} = \mu_0^{-1}(B_1^\star \dot{B}_1^\star - B_2^\star \dot{B}_2^\star), \quad \dot{\tau}^\star_{\text{m}22} = -\dot{\tau}^\star_{\text{m}11}, \tag{11.70}$$

$$\dot{\tau}^\star_{\text{m}12} = \mu_0^{-1}(\dot{B}_1^\star B_2^\star + B_1^\star \dot{B}_2^\star), \quad \dot{\tau}^\star_{\text{m}33} = -\mu_0^{-1}(B_1^\star \dot{B}_1^\star + B_2^\star \dot{B}_2^\star). \tag{11.71}$$

The updated incremental versions of the constitutive laws are given by (11.7) and (11.8) for $\dot{\mathbf{T}}_0$ and $\dot{\mathbf{H}}_{\text{L}0}$, respectively.

The incremental traction boundary condition is given by (9.129) and with the help of (9.128) leads to the two boundary conditions

$$\dot{T}_{021} = \dot{t}_{\text{A}01} + \dot{\tau}^\star_{\text{m}12} - \tau^\star_{\text{m}11} u_{2,1} - \tau^\star_{\text{m}12} u_{2,2} \quad \text{on } x_2 = 0, \tag{11.72}$$

$$\dot{T}_{022} = \dot{t}_{\text{A}02} + \dot{\tau}^\star_{\text{m}22} - \tau^\star_{\text{m}12} u_{2,1} - \tau^\star_{\text{m}22} u_{2,2} \quad \text{on } x_2 = 0, \tag{11.73}$$

where $\dot{t}_{\text{A}01}$ and $\dot{t}_{\text{A}02}$ are the components of the incremental mechanical traction applied on $x_2 = 0$.

The corresponding components of the incremental magnetic boundary conditions are obtained from (9.121) and (9.123) as

$$w_2 - \dot{B}_2^\star + B_1^\star u_{2,1} + B_2^\star u_{2,2} = 0 \quad \text{on } x_2 = 0, \tag{11.74}$$

$$\dot{H}_{\text{L}01} - \dot{H}_1^\star - H_1^\star u_{1,1} - H_2^\star u_{2,1} = 0 \quad \text{on } x_2 = 0. \tag{11.75}$$

Henceforth, for simplicity, we restrict attention to two separate cases, corresponding first to $B_1 = 0$ with $B_2 \neq 0$ and then $B_1 \neq 0$ with $B_2 = 0$.

11.4.1 The Case $B_1 = 0$

For $B_1 = 0$, $b_1 = b_2 = 0$ and $d = \bar{e} = 0$, and (11.46) and (11.47), respectively, simplify to

$$a\psi_{,1111} + 2b\psi_{,1122} + c\psi_{,2222} + e\varphi_{,112} + \bar{d}\varphi_{,222} = \rho\,(\psi_{,11} + \psi_{,22})_{,tt}\,, \tag{11.76}$$

$$e\psi_{,112} + \bar{d}\psi_{,222} + f\varphi_{,22} + g\varphi_{,11} = 0, \tag{11.77}$$

where, since $\mathsf{A}^*_{012} = 0$, we have now adopted the notation $f = \mathsf{A}^*_{011}$ and $g = \mathsf{A}^*_{022}$ from (10.44).

If we now assume there is no incremental mechanical boundary traction then the incremental traction boundary conditions (11.72) and (11.73) reduce to

$$\dot{T}_{021} - \mu_0^{-1} B_2 \dot{B}_1^\star - \frac{1}{2}\mu_0^{-1} B_2^2 u_{2,1} = 0 \quad \text{on } x_2 = 0, \tag{11.78}$$

$$\dot{T}_{022} - \mu_0^{-1} B_2 \dot{B}_2^\star + \frac{1}{2}\mu_0^{-1} B_2^2 u_{2,2} = 0 \quad \text{on } x_2 = 0. \tag{11.79}$$

Similarly, (11.74) and (11.75) reduce to

$$w_2 - \dot{B}_2^\star + B_2 u_{2,2} = 0, \quad \dot{H}_{L01} - \dot{H}_1^\star - \mu_0^{-1} B_2 u_{2,1} = 0 \quad \text{on } x_2 = 0. \tag{11.80}$$

By substituting the updated incremented constitutive equations (11.7) and (11.8), appropriately specialized, into the incremental boundary conditions (11.78) and (11.80) and making use of the connection

$$\mathcal{A}^*_{01221} + \tau_{22} + p^* = \mathcal{A}^*_{02121}, \tag{11.81}$$

which comes from the counterpart of (9.88) for the present magnetoelastic case, we obtain

$$(\mathcal{A}^*_{02121} - \tau_{22} - \frac{1}{2}\mu_0^{-1} B_2^2) u_{2,1} + \mathcal{A}^*_{02121} u_{1,2} + \mathbb{A}^*_{021|1} w_1 - \mu_0^{-1} B_2 \dot{B}_1^\star = 0, \tag{11.82}$$

$$\mathcal{A}^*_{01122} u_{1,1} + (\mathcal{A}^*_{02222} + p^* + \frac{1}{2}\mu_0^{-1} B_2^2) u_{2,2} + \mathbb{A}^*_{022|2} w_2 - \dot{p}^* - \mu_0^{-1} B_2 \dot{B}_2^\star = 0, \tag{11.83}$$

$$\mathbb{A}^*_{012|1} u_{2,1} + \mathbb{A}^*_{021|1} u_{1,2} + \mathsf{A}^*_{011} w_1 - \mu_0^{-1} B_2 u_{2,1} - \mu_0^{-1} \dot{B}_1^\star = 0, \tag{11.84}$$

each holding on $x_2 = 0$.

It is now convenient to eliminate the term involving $\dot{p}^*$ in (11.83). For this purpose we differentiate (11.83) with respect to x_1 and then eliminate $\dot{p}^*_{,1}$ by using of (11.41). The potentials ψ, φ and $\varphi^\star$ are then substituted into the resulting (11.80), (11.82) and (11.84). Next we suppose that there is no mechanical traction applied on the boundary $x_2 = 0$ in the underlying configuration. Then the normal stress τ_{22} in the material must balance the Maxwell stress $\tau^\star_{m22}$ on $x_2 = 0$, i.e.

$$\tau_{22} = \tau^\star_{m22} = \frac{1}{2}\mu_0^{-1} B_2^2. \tag{11.85}$$

On use of this and the notations (11.48)–(11.51), the boundary conditions become

$$(c - 2\tau^\star_{m22})\psi_{,11} - c\psi_{,22} - \bar{d}\varphi_{,2} + \mu_0^{-1} B_2 \varphi^\star_{,2} = 0, \tag{11.86}$$

$$(2b + c)\psi_{,112} + c\psi_{,222} + (e + \bar{d})\varphi_{,11} + \bar{d}\varphi_{,22} - \mu_0^{-1} B_2 \varphi^\star_{,11} - \rho\psi_{,2tt} = 0, \tag{11.87}$$

$$B_2\psi_{,12} + \varphi_{,1} - \varphi^\star_{,1} = 0, \tag{11.88}$$

$$\bar{d}(\psi_{,11} - \psi_{,22}) - f\varphi_{,2} - \mu_0^{-1}B_2\psi_{,11} + \mu_0^{-1}\varphi^\star_{,2} = 0, \tag{11.89}$$

each of which applies on $x_2 = 0$.

The surface wave problem therefore requires solution of (11.76) and (11.77) in $x_2 < 0$ and (11.54) in $x_2 > 0$ subject to the boundary conditions (11.86)–(11.89) on $x_2 = 0$ and appropriate decay behaviour as $x_2 \to \pm\infty$.

11.4.1.1 Rayleigh-Type Surface Wave Propagation

We consider harmonic surface wave solutions of the form

$$\psi = P\exp(skx_2 + \mathrm{i}kx_1 - \mathrm{i}\omega t) \quad \text{in } x_2 < 0, \tag{11.90}$$

$$\varphi = kQ\exp(skx_2 + \mathrm{i}kx_1 - \mathrm{i}\omega t) \quad \text{in } x_2 < 0, \tag{11.91}$$

$$\varphi^\star = kR\exp(s^\star kx_2 + \mathrm{i}kx_1 - \mathrm{i}\omega t) \quad \text{in } x_2 > 0, \tag{11.92}$$

corresponding to propagation parallel to the surface $x_2 = 0$ and in the x_1 direction, where P, Q, R are constants, k is the wave number, ω the angular frequency, and s and $s^\star$ are initially unknown and are to be determined by substitution of the above expressions for ψ, φ and $\varphi^\star$ into (11.76), (11.77) and (11.54). This leads to two equations involving s, namely

$$[a - 2bs^2 + cs^4 + \rho v^2(s^2 - 1)]P + (\bar{d}s^2 - e)sQ = 0, \tag{11.93}$$

$$(\bar{d}s^2 - e)sP + (fs^2 - g)Q = 0, \tag{11.94}$$

where the wave speed is $v = \omega/k$, and a single equation for $s^{\star 2}$:

$$s^{\star 2} = 1. \tag{11.95}$$

For the overall solution to qualify as a surface wave, it is required that $\mathrm{Re}(s) > 0$ and $\mathrm{Re}(s^\star) < 0$, conditions which ensure that ψ and φ decay as $x_2 \to -\infty$ and $\varphi^\star$ decays as $x_2 \to \infty$. From (11.95) we therefore deduce that $s^\star = -1$.

For non-trivial solutions for P and Q from (11.93) and (11.94), the determinant of coefficients must vanish. This results in a cubic equation for s^2, specifically

$$\begin{aligned}(cf - \bar{d}^2)s^6 &+ [f(\rho v^2 - 2b) - cg + 2e\bar{d}]s^4 \\ &+ [g(2b - \rho v^2) + f(a - \rho v^2) - e^2]s^2 + (\rho v^2 - a)g = 0.\end{aligned} \tag{11.96}$$

By setting $n_1 = 0, n_2 = 1, m_1 = 1, m_2 = 0$ in the strong ellipticity condition (11.26) and using the definitions (11.48)–(11.51) with $d = 0$, it follows that $c - \bar{d}^2/f$ is positive, and since A_0^* is positive definite, $f = \mathsf{A}_{011}^* > 0$. The coefficient $cf - \bar{d}^2$ of s^6 is therefore non-zero. Thus, in general, (11.96) has three solutions satisfying $\mathrm{Re}(s) > 0$. Exceptionally, $s = 0$ is a solution if $\rho v^2 = a$, but this corresponds to a homogeneous plane wave which cannot in general be supported by the boundary conditions. Therefore, we let s_1, s_2, s_3 be the three solutions with positive real part, and by P_1, P_2, P_3 and Q_1, Q_2, Q_3 we denote the corresponding coefficients, which from (11.94) are related by

$$Q_i = \frac{(e - \bar{d} s_i^2) s_i}{f s_i^2 - g} P_i, \quad i = 1, 2, 3. \tag{11.97}$$

The general solutions of the considered form that meet the decay conditions are therefore given by

$$\psi = (P_1 \mathrm{e}^{s_1 k x_2} + P_2 \mathrm{e}^{s_2 k x_2} + P_3 \mathrm{e}^{s_3 k x_2}) \mathrm{e}^{\mathrm{i}(k x_1 - \omega t)}, \tag{11.98}$$

$$\varphi = k(Q_1 \mathrm{e}^{s_1 k x_2} + Q_2 \mathrm{e}^{s_2 k x_2} + Q_3 \mathrm{e}^{s_3 k x_2}) \mathrm{e}^{\mathrm{i}(k x_1 - \omega t)}, \tag{11.99}$$

$$\varphi^\star = k R \mathrm{e}^{-k x_2 + \mathrm{i}(k x_1 - \omega t)}. \tag{11.100}$$

Substitution of (11.98)–(11.100) into the boundary conditions (11.86)–(11.89) then yields the four equations

$$(c - 2\tau_{\mathrm{m}22}^\star) \sum_{j=1}^{3} P_j + c \sum_{j=1}^{3} s_j^2 P_j + \bar{d} \sum_{j=1}^{3} s_j Q_j + \mu_0^{-1} B_2 R = 0, \tag{11.101}$$

$$(2b + c - \rho v^2) \sum_{j=1}^{3} s_j P_j - c \sum_{j=1}^{3} s_j^3 P_j + (e + \bar{d}) \sum_{j=1}^{3} Q_j - \bar{d} \sum_{j=1}^{3} s_j^2 Q_j - \mu_0^{-1} B_2 R = 0, \tag{11.102}$$

$$B_2 \sum_{j=1}^{3} s_j P_j + \sum_{j=1}^{3} Q_j - R = 0, \tag{11.103}$$

$$\bar{d} \sum_{j=1}^{3} (s_j^2 + 1) P_j + f \sum_{j=1}^{3} s_j Q_j - \mu_0^{-1} B_2 \sum_{j=1}^{3} P_j + \mu_0^{-1} R = 0, \tag{11.104}$$

which, together with (11.97), form a system of seven linear equations for the seven unknowns $P_1, P_2, P_3, Q_1, Q_2, Q_3$ and R. Equivalently, if we substitute for Q_i from (11.97), we obtain four linear equations for P_1, P_2, P_3 and R.

The secular equation, which gives the wave speed v in terms of the initial deformation, the material properties and the initial magnetic induction B_2 is obtained by setting to zero the determinant of coefficients in either case. We recall from (11.85) that the Maxwell stress component $\tau_{\mathrm{m}22}^\star$ in (11.101) depends on B_2.

11.4.1.2 Reduction to the Pure Elasticity Results

When the magnetic field is absent we recover known results for incompressible nonlinear elasticity. In this case, $\mathbb{A}_0^*$ vanishes, and hence, $d = e = \bar{e} = \bar{d} = 0$, and (11.96) factorizes in the form

$$(s^2 f - g)[cs^4 - (2b - \rho v^2)s^2 + a - \rho v^2] = 0. \tag{11.105}$$

Only the second factor corresponds to the purely elastic effects, and its solutions s_1^2 and s_2^2, say, have the properties

$$c(s_1^2 + s_2^2) = 2b - \rho v^2, \quad cs_1^2 s_2^2 = a - \rho v^2. \tag{11.106}$$

We select the solutions s_1 and s_2 to have positive real part, as required for a surface wave. The strong ellipticity condition in this specialization imposes the inequality $c > 0$. Since s_1 and s_2 are then either real or complex conjugates, it follows that $\rho v^2 \leq a$, as discussed in Dowaikh and Ogden (1990). As indicated earlier, the upper limit $\rho v^2 = a$ does not correspond to a surface wave, so that the strict inequality $\rho v^2 < a$ must hold.

The boundary conditions (11.86)–(11.89) now yield the two equations

$$(c - \tau_{22})\psi_{,11} - c\psi_{,22} = 0 \quad \text{on } x_2 = 0, \tag{11.107}$$

$$(2b + c - \tau_{22})\psi_{,112} + c\psi_{,222} - \rho\psi_{,2tt} = 0 \quad \text{on } x_2 = 0, \tag{11.108}$$

where now a normal traction τ_{22} is considered to be applied on $x_2 = 0$.

The general solution for ψ now has the reduced form

$$\psi = (P_1 e^{s_1 k x_2} + P_2 e^{s_2 k x_2}) e^{i(kx_1 - \omega t)}, \tag{11.109}$$

and substitution of this into the boundary conditions (11.82) and (11.83) then yields

$$(c - \tau_{22} + cs_1^2)P_1 + (c - \tau_{22} + cs_2^2)P_2 = 0, \tag{11.110}$$

$$(2b + c - \tau_{22} - \rho v^2 - cs_1^2)s_1 P_1 + (2b + c - \tau_{22} - \rho v^2 - cs_2^2)s_2 P_2 = 0. \tag{11.111}$$

By using (11.106) and the fact that the determinant of coefficients vanishes, we obtain the *secular equation*:

$$c(a - \rho v^2) + (2b + 2c - 2\tau_{22} - \rho v^2)\sqrt{c(a - \rho v^2)} = (c - \tau_{22})^2. \tag{11.112}$$

Apart from some minor differences of notation, this agrees with the formula (5.17) obtained by Dowaikh and Ogden (1990) that characterizes the speed of surface waves within a principal plane of a deformed half-space of an incompressible isotropic elastic material.

11.4.1.3 Illustration

For purposes of illustration we consider again the prototype energy function defined by (11.32). Then, from the formulas in Sect. 9.3.4.4, the components of the magnetoelastic tensors have simple forms. The only non-zero ones are

$$\mathcal{A}^*_{01111} = \mathcal{A}^*_{01212} = \mu\lambda_1^2, \quad \mathcal{A}^*_{02222} = \mathcal{A}^*_{02121} = \mu\lambda_2^2 + \mu_0^{-1}\beta B_2^2, \tag{11.113}$$

$$\mathbb{A}^*_{022|2} = 2\mathbb{A}^*_{012|1} = 2\mu_0^{-1}\beta B_2, \tag{11.114}$$

$$\mathsf{A}^*_{011} = \mu_0^{-1}(\lambda_1^{-2}\alpha + \beta), \quad \mathsf{A}^*_{022} = \mu_0^{-1}(\lambda_2^{-2}\alpha + \beta). \tag{11.115}$$

In particular, it follows from the notation defined in (11.48)–(11.51) that

$$2b = a + c, \quad e = \bar{d}. \tag{11.116}$$

In this case (11.96) factorizes as

$$(s^2 - 1)\{(cf - \bar{d}^{\,2})s^4 - [cg + (a - \rho v^2)f - \bar{d}^{\,2}]s^2 + (a - \rho v^2)g\} = 0. \tag{11.117}$$

Other specializations in which the bi-cubic factorizers have been noted by Saxena and Ogden (2011). We denote the solutions of (11.96) with positive real part by s_1, s_2 and s_3. Then, by setting $s_1 = 1$, s_2 and s_3 satisfy

$$s_2^2 + s_3^2 = \frac{cg + (a - \rho v^2)f - \bar{d}^{\,2}}{cf - \bar{d}^{\,2}}, \quad s_2^2 s_3^2 = \frac{(a - \rho v^2)g}{cf - \bar{d}^{\,2}}. \tag{11.118}$$

From the strong ellipticity condition and the positive definiteness of $\mathbf{A}_0^*$, we see that the factor multiplying $a - \rho v^2$ in $(11.118)_2$ is positive. Then, by the same argument as used in the purely elastic case, we deduce that

$$\rho v^2 \le a = \rho v_s^2, \tag{11.119}$$

where v_s is the speed of a homogeneous plane shear wave propagating in the x_1 direction in an infinite medium. For the prototype model, this is given by $\rho v_s^2 = \mu\lambda_1^2$, which is identical to that in the purely elastic case and hence independent of the magnetic field.

The secular equation is obtained by setting to zero the determinant of coefficients in the boundary conditions (11.101)–(11.104) after substituting for s_1, s_2 and s_3. This is a rather lengthy equation and not therefore reproduced here. The results are illustrated numerically using the standard value $1.257 \times 10^{-6}\,\mathrm{N\,A^{-2}}$ of μ_0, the value $2.6 \times 10^5\,\mathrm{N\,m^{-2}}$ of μ that was adopted by Otténio et al. (2008) based on the data for an elastomer filled with 10 % by volume of iron particles from Jolly et al. (1996) and

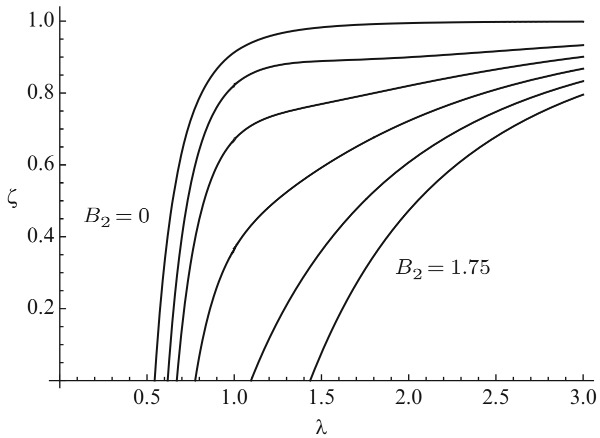

Fig. 11.1 Plot of $\zeta = v^2/v_s^2$ vs. $\lambda_1 = \lambda$ with $\lambda_3 = 1$ for $B_2 = 0, 0.75, 1, 1.25, 1.5, 1.75$ T (*curves reading from left to right*) and material parameters $\alpha = 0.2$ and $\beta = 0.3$

representative values of the material constants α and β. Note that in the reference configuration $1/(\alpha + \beta)$ is interpreted as the relative (magnetic) permeability of the material. The magnetic induction is measured in Tesla (T).

For definiteness we now take the underlying deformation to correspond to plane strain in the (x_1, x_2) plane with $\lambda_1 = \lambda$, $\lambda_2 = \lambda^{-1}$, $\lambda_3 = 1$. We introduce the notation $\zeta = v^2/v_s^2$ and plot ζ (which has an upper bound 1) as a function of λ in Fig. 11.1 for a range of values of B_2 and for the material parameters $\alpha = 0.2$ and $\beta = 0.3$. The curve for $B_2 = 0$ corresponds to the purely elastic case and provides a point of reference. It cuts the λ axis at $\lambda = \lambda_{\rm cr} \simeq 0.5437$, which agrees with the classical result for the critical value $\lambda_{\rm cr}$ corresponding to loss of stability of the half-space under compression for the neo-Hookean model; see Biot (1965) and Dowaikh and Ogden (1990) for details.

Where the curves cut the axis, $\zeta = 0$ corresponds, for each value of B_2, to the critical value of λ at which the underlying configuration loses stability. From the figure it can be seen that as B_2 increases, the critical value moves from 0.5437 towards 1 and then passes to $\lambda > 1$. This means that the magnetic field destabilizes the material, i.e. instability occurs at a compression closer to the undeformed configuration where $\lambda = 1$, and the configuration where $\lambda = 1$ itself becomes unstable. Thus, for each value of B_2 there is a critical value of λ beyond which a surface wave exists, and the wave speed increases with λ consistently with the upper bound (11.119). To prevent instability at the larger values of B_2, a sufficiently large lateral stretch λ needs to be applied. For any given value of λ, the wave speed decreases as the magnitude of the magnetic induction increases.

11.4.2 The Case $B_2 = 0$

Now we take B_1 to be the active component of $\mathbf{B}$ with the underlying deformation the same as in Sect. 11.4.1. The magnetic field in the material then has only a single component H_1, which is obtained from (11.28) as

$$H_1 = 2(\Omega_4^* \lambda_1^{-2} + \Omega_5^* + \Omega_6^* \lambda_1^2) B_1. \tag{11.120}$$

For the prototype model (11.32) this reduces to $H_1 = \mu_0^{-1}(\alpha\lambda_1^{-2} + \beta)B_1$. The magnetic boundary conditions on $x_2 = 0$ require that $H_1^\star = H_1$. Hence $B_1^\star = \mu_0 H_1^\star$ is related to B_1 by

$$B_1^\star = (\alpha\lambda_1^{-2} + \beta)B_1. \tag{11.121}$$

For the present specialization, $e = \bar{d} = 0$ and (11.46) and (11.47) therefore reduce to

$$a\psi_{,1111} + 2b\psi_{,1122} + c\psi_{,2222} + d\varphi_{,111} + \bar{e}\varphi_{,122} = \rho(\psi_{,11} + \psi_{,22})_{,tt}, \tag{11.122}$$

$$d\psi_{,111} + \bar{e}\psi_{,122} + g\varphi_{,11} + f\varphi_{,22} = 0, \tag{11.123}$$

respectively, for $x_2 < 0$, while again (11.54) holds for $x_2 > 0$.

Next we assume that there is no incremental mechanical traction on the surface $x_2 = 0$. Then, the components of the incremental traction on the boundary that are not satisfied identically are obtained from (9.128) as

$$\dot{T}_{021} - \mu_0^{-1} B_1^\star \dot{B}_2^\star + \frac{1}{2}\mu_0^{-1} B_1^{\star 2} u_{2,1} = 0 \quad \text{on } x_2 = 0, \tag{11.124}$$

$$\dot{T}_{022} + \mu_0^{-1} B_1^\star \dot{B}_1^\star - \frac{1}{2}\mu_0^{-1} B_1^{\star 2} u_{2,2} = 0 \quad \text{on } x_2 = 0, \tag{11.125}$$

in which we have made use of the component forms of $\boldsymbol{\tau}_\mathrm{m}^\star$ and $\dot{\boldsymbol{\tau}}_\mathrm{m}^\star$ from (11.66)–(11.71) appropriately specialized with $B_2 = 0$.

The incremental magnetic boundary conditions are obtained from (11.74) and (11.75) as

$$w_2 - \dot{B}_2^\star + B_1^\star u_{2,1} = 0, \quad \dot{H}_{l01} - \dot{H}_1^\star - H_1^\star u_{1,1} = 0 \quad \text{on } x_2 = 0. \tag{11.126}$$

The updated incremental constitutive equations (11.7) and (11.8) are then introduced into (11.124)–(11.126), and the same procedure as in the previous section is used to eliminate $\dot{p}^*$ with the help of (11.81) and the boundary condition $\tau_{22} = \tau_{\mathrm{m}22}^\star$, with $\tau_{\mathrm{m}22}^\star = -B_1^{\star 2}/2\mu_0$ from (11.66). The four boundary conditions are then formed as

$$(c - 2\tau^\star_{m22})\psi_{,11} - c\psi_{,22} + d\varphi_{,1} - \mu_0^{-1}B_1^\star\varphi^\star_{,1} = 0 \quad \text{on } x_2 = 0, \tag{11.127}$$

$$(2b + c)\psi_{,112} + c\psi_{,222} - \rho\psi_{,2tt} + \bar{e}\varphi_{,12} - \mu_0^{-1}B_1^\star\varphi^\star_{,12} = 0 \quad \text{on } x_2 = 0, \tag{11.128}$$

$$B_1^\star\psi_{,11} + \varphi_{,1} - \varphi^\star_{,1} = 0 \quad \text{on } x_2 = 0, \tag{11.129}$$

$$(\bar{e} + d - \mu_0^{-1}B_1^\star)\psi_{,12} - \mu_0^{-1}\varphi^\star_{,2} = 0 \quad \text{on } x_2 = 0. \tag{11.130}$$

11.4.2.1 Illustration

As in Sect. 11.4.1, we consider surface wave solutions of the form (11.90)–(11.92). When these are substituted into (11.122) and (11.123), we obtain

$$[cs^4 - (2b - \rho v^2)s^2 + a - \rho v^2]P + \mathrm{i}(\bar{e}s^2 - d)Q = 0, \tag{11.131}$$

$$\mathrm{i}(\bar{e}s^2 - d)P + (fs^2 - g)Q = 0, \tag{11.132}$$

and again from (11.54) we have $s^{\star 2} = 1$, and hence we must take $s^\star = -1$. The wave speed is again given by $v = \omega/k$.

For non-trivial solutions for P and Q the determinant of coefficients should vanish, which again leads to a bi-cubic equation for s, specifically

$$\begin{aligned} &cfs^6 - [f(2b - \rho v^2) + cg - \bar{e}^2]s^4 \\ &\quad + [f(a - \rho v^2) + g(2b - \rho v^2) - 2dc\bar{e}]s^2 - g(a - \rho v^2) + d^2 = 0, \end{aligned} \tag{11.133}$$

which is different from (11.96).

For the material model (11.32), the only non-zero components of the magnetoelastic tensors are obtained from the general formulas in Sect. 9.3.4.4 as

$$\mathcal{A}^*_{01111} = \mathcal{A}^*_{01212} = \mu\lambda_1^2 + 2\beta B_1^2, \tag{11.134}$$

$$\mathcal{A}^*_{02222} = \mathcal{A}^*_{02121} = \mu\lambda_2^2, \tag{11.135}$$

$$\mathbb{A}^*_{011|1} = 2\mathbb{A}^*_{012|2} = 2\mu_0^{-1}\beta B_1, \tag{11.136}$$

$$\mathsf{A}^*_{011} = \mu_0^{-1}(\alpha\lambda_1^{-2} + \beta), \quad \mathsf{A}^*_{022} = (\alpha\lambda_2^{-2} + \beta). \tag{11.137}$$

From the notation defined in (11.48)–(11.51), we then obtain

$$2b = a + c, \quad \bar{e} = d. \tag{11.138}$$

With these specializations (11.133) factorizes to give

$$(s^2 - 1)\{cfs^4 - [cg + (a - \rho v^2)f - d^2]s^2 + (a - \rho v^2)g - d^2\} = 0. \tag{11.139}$$

To satisfy the decay condition $\mathrm{Re}(s) > 0$ we set $s_1 = 1$ and take s_2 and s_3 to be the solutions with positive real part corresponding to the second factor. As in the previous section we require $s_2^2 s_3^2 \geq 0$, which, after noting that $c > 0$, $f > 0$ and $g > 0$ and specializing the generalized strong ellipticity condition as in Sect. 11.4.1, gives

$$\rho v^2 \leq a - d^2/g, \tag{11.140}$$

the right-hand side of which is positive. As distinct from (11.119), the upper bound in (11.140) depends on the magnetic field. For the considered model, the inequality (11.140) is now expressed in the form

$$v^2/v_s^2 \leq 1 + \frac{\alpha\beta B_1^2}{\mu_0\mu\lambda^2(\alpha + \beta\lambda^2)}. \tag{11.141}$$

Next we substitute the solutions (11.98)–(11.100) for ψ, φ and $\varphi^\star$ into the boundary conditions (11.127)–(11.130). On use of (11.138), the boundary conditions become

$$(c - 2\tau^\star_{\mathrm{m}22}) \sum_{j=1}^{3} P_j + c \sum_{j=1}^{3} s_j^2 P_j - \mathrm{i}d \sum_{j=1}^{3} Q_j + \mathrm{i}\mu_0^{-1} B_1^\star R = 0 \quad \text{on } x_2 = 0, \tag{11.142}$$

$$(a+2c-\rho v^2) \sum_{j=1}^{3} s_j P_j - c \sum_{j=1}^{3} s_j^3 P_j - \mathrm{i}d \sum_{j=1}^{3} s_j Q_j - \mathrm{i}\mu_0^{-1} B_1^\star R = 0 \quad \text{on } x_2 = 0, \tag{11.143}$$

$$B_1^\star \sum_{j=1}^{3} P_j - \mathrm{i} \sum_{j=1}^{3} Q_j + \mathrm{i}R = 0 \quad \text{on } x_2 = 0, \tag{11.144}$$

$$(2d - \mu_0^{-1} B_1^\star) \sum_{j=1}^{3} s_j P_j - \mathrm{i}\mu_0^{-1} R = 0 \quad \text{on } x_2 = 0. \tag{11.145}$$

From (11.132) we also have the connections

$$\mathrm{i}d(1 - s_i^2)P_i + (g - f s_i^2)Q_i = 0 \quad i = 1, 2, 3. \tag{11.146}$$

As was the case in Sect. 11.4.1.3, there are seven linear equations in P_1, P_2, P_3, Q_1, Q_2, Q_3 and R, and the solution has been obtained numerically. Results for $\lambda_1 = \lambda$, $\lambda_2 = \lambda^{-1}$, $\lambda_3 = 1$ are shown in Fig. 11.2 where $\zeta = v^2/v_s^2$ is plotted against λ for several values of B_1 using the material parameters $\alpha = 0.2$ and $\beta = 0.3$, as in Fig. 11.1. The results are quite different from those shown in Fig. 11.1 although the magnetic field does again have a destabilizing effect. This effect is not as strong as for the B_2 results. In contrast to the results in Fig. 11.1, the B_1 component has the effect of increasing the wave speed apart from a small range of values of λ close to the critical value for loss of stability in which vicinity the curves for different B_1 intersect.

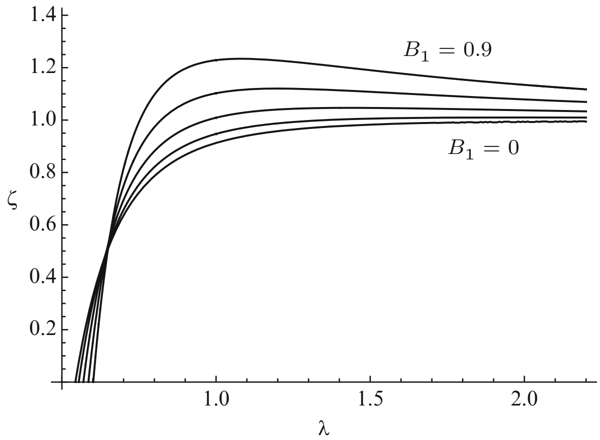

Fig. 11.2 Plot of $\zeta = v^2/v_s^2$ vs. $\lambda_1 = \lambda$ with $\lambda_3 = 1$ for $B_1 = 0, 0.3, 0.5, 0.7, 0.9\,\mathrm{T}$ (*curves reading from bottom to top*) and material parameters $\alpha = 0.2$ and $\beta = 0.3$

The results are consistent with the upper bound given in (11.141) which clearly depends on the magnitude of B_1. For large values of the stretch λ, all the curves tend to the asymptotic value 1.

11.4.3 Love-Type Surface Waves

Once more we consider the pure homogeneous strain deformation of the previous sections with the magnetic induction vector having components either $(B_1, B_2, 0)$ or $(0, 0, B_3)$ and correspondingly $(B_1^\star, B_2^\star, 0)$ or $(0, 0, B_3^\star)$ outside the material. Unlike in Sect. 11.4 here we take the displacement components u_1 and u_2 to vanish and consider the displacement component u_3 and seek solutions depending on the in-plane variables x_1 and x_2, together with t: $u_3 = u_3(x_1, x_2, t)$. The incremental incompressibility condition $\operatorname{div}\mathbf{u} = 0$ is then automatically satisfied and, with all incremental quantities independent of x_3, $\dot{p}^*_{,3} = 0$. From $(11.14)_2$ we have

$$w_{1,1} + w_{2,2} = 0 \tag{11.147}$$

since we are taking w_3 to be independent of x_3.

The component forms of (11.9) are then

$$\begin{aligned}\mathcal{A}^*_{01113}u_{3,11} &+ (\mathcal{A}^*_{01123} + \mathcal{A}^*_{02113})u_{3,12} + \mathcal{A}^*_{02123}u_{3,22} \\ &+ \mathbb{A}^*_{011|1}w_{1,1} + \mathbb{A}^*_{011|2}w_{2,1} + \mathbb{A}^*_{021|1}w_{1,2} + \mathbb{A}^*_{021|2}w_{2,2} \\ &+ \mathbb{A}^*_{011|3}w_{3,1} + \mathbb{A}^*_{021|3}w_{3,2} - \dot{p}^*_{,1} = 0,\end{aligned} \tag{11.148}$$

$$\begin{aligned}\mathcal{A}^*_{01213}u_{3,11} &+ (\mathcal{A}^*_{01223} + \mathcal{A}^*_{02213})u_{3,12} + \mathcal{A}^*_{02223}u_{3,22}\\ &+ \mathbb{A}^*_{012|1}w_{1,1} + \mathbb{A}^*_{012|2}w_{2,1} + \mathbb{A}^*_{022|1}w_{1,2} + \mathbb{A}^*_{022|2}w_{2,2}\\ &+ \mathbb{A}^*_{012|3}w_{3,1} + \mathbb{A}^*_{022|3}w_{3,2} - \dot{p}^*_{,2} = 0,\end{aligned} \tag{11.149}$$

$$\begin{aligned}\mathcal{A}^*_{01313}u_{3,11} &+ 2\mathcal{A}^*_{01323}u_{3,12} + \mathcal{A}^*_{02323}u_{3,22} + \mathbb{A}^*_{013|1}w_{1,1}\\ &+ \mathbb{A}^*_{013|2}w_{2,1} + \mathbb{A}^*_{023|1}w_{1,2} + \mathbb{A}^*_{023|2}w_{2,2}\\ &+ \mathbb{A}^*_{013|3}w_{3,1} + \mathbb{A}^*_{023|3}w_{3,2} = \rho u_{3,tt},\end{aligned} \tag{11.150}$$

and of (11.10)

$$(\mathbb{A}^*_{013|3}u_{3,1} + \mathbb{A}^*_{023|3}u_{3,2} + \mathsf{A}^*_{013}w_1 + \mathsf{A}^*_{023}w_2 + \mathsf{A}^*_{033}w_3)_{,2} = 0, \tag{11.151}$$

$$(\mathbb{A}^*_{013|3}u_{3,1} + \mathbb{A}^*_{023|3}u_{3,2} + \mathsf{A}^*_{013}w_1 + \mathsf{A}^*_{023}w_2 + \mathsf{A}^*_{033}w_3)_{,1} = 0, \tag{11.152}$$

$$\begin{aligned}\mathbb{A}^*_{013|2}u_{3,11} &+ (\mathbb{A}^*_{023|2} - \mathbb{A}^*_{013|1})u_{3,12} - \mathbb{A}^*_{023|1}u_{3,22} + \mathsf{A}^*_{012}w_{1,1} + \mathsf{A}^*_{022}w_{2,1}\\ &+ \mathsf{A}^*_{023}w_{3,1} - \mathsf{A}^*_{011}w_{1,2} - \mathsf{A}^*_{012}w_{2,2} - \mathsf{A}^*_{013}w_{3,2} = 0.\end{aligned} \tag{11.153}$$

We now specialize by considering the underlying magnetic (induction) field to be parallel to the (x_1, x_2) plane with components $(B_1, B_2, 0)$. Again we consider the material to consist of a half-space defined by $X_2 < 0$ in the undeformed configuration and subject to the pure homogeneous strain (11.64) so that in the deformed configuration the half-space is defined by $x_2 < 0$. A layer of different magnetoelastic material with uniform thickness H in the undeformed configuration is placed above the half-space and bonded to it and subject to the same pure homogeneous strain as the half-space. Let h be the thickness of the layer in the deformed configuration so that the deformed layer occupies the region $0 < x_2 < h$. Quantities in the layer are distinguished by an overbar.

11.4.3.1 In-Plane Magnetic Induction: $\mathbf{B} = (B_1, B_2, 0)$

In this case we have $B_3 = \bar{B}_3 = B^\star_3 = 0$. The Maxwell stress components are again given by (11.66) and (11.67), while the increment of the Maxwell stress has components

$$\dot{\tau}^\star_{\mathrm{m}11} = \mu_0^{-1}(B^\star_1\dot{B}^\star_1 - B^\star_2\dot{B}^\star_2), \quad \dot{\tau}^\star_{\mathrm{m}22} = -\dot{\tau}^\star_{\mathrm{m}11}, \quad \dot{\tau}^\star_{\mathrm{m}12} = \mu_0^{-1}(\dot{B}^\star_1 B^\star_2 + B^\star_1\dot{B}^\star_2) \tag{11.154}$$

$$\dot{\tau}^\star_{\mathrm{m}13} = \mu_0^{-1}B^\star_1\dot{B}^\star_3, \quad \dot{\tau}^\star_{\mathrm{m}23} = \mu_0^{-1}B^\star_2\dot{B}^\star_3, \quad \dot{\tau}^\star_{\mathrm{m}33} = -\mu_0^{-1}(B^\star_1\dot{B}^\star_1 + B^\star_2\dot{B}^\star_2). \tag{11.155}$$

For the half-space, (11.148)–(11.153) simplify to

$$\mathbb{A}^*_{011|1}w_{1,1} + \mathbb{A}^*_{021|1}w_{1,2} + \mathbb{A}^*_{021|2}w_{2,2} + \mathbb{A}^*_{011|2}w_{2,1} - \dot{p}^*_{,1} = 0, \tag{11.156}$$

$$\mathbb{A}^*_{012|1}w_{1,1} + \mathbb{A}^*_{012|2}w_{2,1} + \mathbb{A}^*_{022|2}w_{2,2} + \mathbb{A}^*_{022|1}w_{1,2} - \dot{p}^*_{,2} = 0, \tag{11.157}$$

$$\mathcal{A}^*_{01313}u_{3,11} + 2\mathcal{A}^*_{01323}u_{3,12} + \mathcal{A}^*_{02323}u_{3,22} + \mathbb{A}^*_{013|3}w_{3,1} + \mathbb{A}^*_{023|3}w_{3,2} = \rho u_{3,tt}, \tag{11.158}$$

$$(\mathbb{A}^*_{013|3}u_{3,1} + \mathbb{A}^*_{023|3}u_{3,2} + \mathsf{A}^*_{033}w_3)_{,2} = 0, \tag{11.159}$$

$$(\mathbb{A}^*_{013|3}u_{3,1} + \mathbb{A}^*_{023|3}u_{3,2} + \mathsf{A}^*_{033}w_3)_{,1} = 0, \tag{11.160}$$

$$\mathsf{A}^*_{022}w_{2,1} + \mathsf{A}^*_{012}w_{1,1} - \mathsf{A}^*_{012}w_{2,2} - \mathsf{A}^*_{011}w_{1,2} = 0, \tag{11.161}$$

with corresponding equations for the layer, which we do not write explicitly at this point. Note that w_1 and w_2 are coupled through (11.147), (11.156), (11.157) and (11.161), equations that do not involve u_3 or w_3, which themselves are coupled through (11.158)–(11.160). In fact, w_1 and w_2 are overdetermined by the four equations, which can in general only be satisfied by $w_1 = w_2 = 0$, and hence, $\dot{p}^*$ depends on t only but does not feature in the ensuing analysis. Thus, we now set $w_1 = w_2 = 0$. It then follows from (11.8) and the components of $\mathbb{A}^*_0$ and A^*_0 given in Sect. 9.3.4.4 that $\dot{H}_{L01} = \dot{H}_{L02} = 0$, while (11.159) and (11.160), since there is no dependence on x_3, show that $\dot{H}_{L03}$ is a function of t only, say $f(t)$.

Each of (11.159) and (11.160) is satisfied by the single equation

$$\mathsf{A}^*_{013|3}u_{3,1} + \mathsf{A}^*_{023|3}u_{3,2} + \mathsf{A}^*_{033}w_3 = f(t), \tag{11.162}$$

with a similar equation for the layer. However, in anticipation of the functional dependence of u_3 and w_3 on x_1 and x_2 as well as t, we find that this equation can only be satisfied for $f(t) = 0$. It follows that $\dot{\mathbf{H}}_{L0} = \mathbf{0}$, and from the incremental boundary conditions (9.121) and (9.123) that $\dot{B}^\star_2 = 0$, and hence $\dot{H}^\star_2 = 0$, and that $\dot{H}^\star_1 = 0$ and $\dot{H}^\star_3 = 0$. Thus, any incremental field outside the material has no influence on the boundary of the material and we therefore set $\dot{\mathbf{B}}^\star = \mathbf{0}$ for $x_2 \geq h$.

The equations to be satisfied are therefore, in the half-space,

$$\mathcal{A}^*_{01313}u_{3,11} + 2\mathcal{A}^*_{01323}u_{3,12} + \mathcal{A}^*_{02323}u_{3,22} + \mathbb{A}^*_{013|3}w_{3,1} + \mathbb{A}^*_{023|3}w_{3,2} = \rho u_{3,tt}, \tag{11.163}$$

$$\mathbb{A}^*_{013|3}u_{3,1} + \mathbb{A}^*_{023|3}u_{3,2} + \mathsf{A}^*_{033}w_3 = 0, \tag{11.164}$$

and in the layer

$$\bar{\mathcal{A}}^*_{01313}\bar{u}_{3,11} + 2\bar{\mathcal{A}}^*_{01323}\bar{u}_{3,12} + \bar{\mathcal{A}}^*_{02323}\bar{u}_{3,22} + \bar{\mathbb{A}}^*_{013|3}\bar{w}_{3,1} + \bar{\mathbb{A}}^*_{023|3}\bar{w}_{3,2} = \bar{\rho}\bar{u}_{3,tt}, \tag{11.165}$$

$$\bar{\mathbb{A}}^*_{013|3}\bar{u}_{3,1} + \bar{\mathbb{A}}^*_{023|3}\bar{u}_{3,2} + \bar{\mathsf{A}}^*_{033}\bar{w}_3 = 0. \tag{11.166}$$

The incremental magnetic boundary conditions on $x_2 = 0$ and $x_2 = h$ are now automatically satisfied, and it remains to consider the incremental traction boundary conditions (9.129) with (9.128). By taking the incremental mechanical traction to vanish, we find that the only non-trivial components of the incremental traction are $\dot{T}_{022}$ and $\dot{T}_{023}$. But, from (11.7) and the expressions for the components of $\boldsymbol{\mathcal{A}}_0^*$ and $\mathbb{A}_0^*$ given in Sect. 9.3.4.4 and the fact that the only components of $\mathbf{L}$ are $L_{31} = u_{3,1}$ and $L_{32} = u_{3,2}$, it follows that $\dot{T}_{022} = -\dot{p}^*$. Since there is now no incremental Maxwell traction in this case, we have $\dot{p}^* = 0$ throughout the material. At the layer–vacuum boundary the remaining boundary condition yields

$$\bar{\mathcal{A}}^*_{02313}\bar{u}_{3,1} + \bar{\mathcal{A}}^*_{02323}\bar{u}_{3,2} + \bar{\mathbb{A}}^*_{023|3}\bar{w}_3 = 0 \quad \text{on } x_2 = h, \tag{11.167}$$

and at the layer–half-space interface, $\dot{T}_{023} = \dot{\bar{T}}_{023}$, and hence

$$\begin{aligned} &\mathcal{A}^*_{02313}u_{3,1} + \mathcal{A}^*_{02323}u_{3,2} + \mathbb{A}^*_{023|3}w_3 \\ &\qquad = \bar{\mathcal{A}}^*_{02313}\bar{u}_{3,1} + \bar{\mathcal{A}}^*_{02323}\bar{u}_{3,2} + \bar{\mathbb{A}}^*_{023|3}\bar{w}_3 \quad \text{on } x_2 = 0. \end{aligned} \tag{11.168}$$

The set of boundary conditions is completed by setting the displacement to be continuous at the interface. Thus,

$$u_3 = \bar{u}_3 \quad \text{on } x_2 = 0. \tag{11.169}$$

The problem of Love-type wave propagation therefore consists of solving (11.163) and (11.164) in $x_2 < 0$ and (11.165) and (11.166) in $0 < x_2 < h$ subject to the boundary conditions (11.167)–(11.169). In the following we derive the secular equation governing the speed of such waves, harmonic in character and propagating in the x_1 direction.

The Secular Equation We consider harmonic solutions of the form

$$u_3 = P\exp(skx_2 + \mathrm{i}kx_1 - \mathrm{i}\omega t), \quad x_2 < 0, \tag{11.170}$$

$$w_3 = kQ\exp(skx_2 + \mathrm{i}kx_1 - \mathrm{i}\omega t), \quad x_2 < 0, \tag{11.171}$$

$$\bar{u}_3 = \bar{P}\exp[\mathrm{i}(\bar{s}kx_2 + kx_1 - \omega t)], \quad 0 < x_2 < h, \tag{11.172}$$

$$\bar{w}_3 = k\bar{Q}\exp[\mathrm{i}(\bar{s}kx_2 + kx_1 - \omega t)], \quad 0 < x_2 < h, \tag{11.173}$$

where $P, Q, \bar{P}$ and $\bar{Q}$ are constants and again k is the wave number and ω is the angular frequency. The factors s and $\bar{s}$ in the exponents are determined below subject to the condition $\mathrm{Re}(s) > 0$, which is required to ensure that the displacement decays away from the surface in the half-space.

Substitution of (11.170) and (11.171) into (11.163) and (11.164) leads to the two equations

$$(\rho v^2 - \mathcal{A}^*_{01313} + 2\mathrm{i}s\mathcal{A}^*_{01323} + s^2\mathcal{A}^*_{02323})P + (\mathrm{i}\mathbb{A}^*_{013|3} + s\mathbb{A}^*_{023|3})Q = 0, \quad (11.174)$$

$$(\mathrm{i}\mathbb{A}^*_{013|3} + s\mathbb{A}^*_{023|3})P + \mathsf{A}^*_{033}Q = 0 \quad (11.175)$$

for the half-space, where $v = \omega/k$ is the wave speed.

Setting to zero the determinant of coefficients of P and Q yields a quadratic for s, which we write compactly as

$$As^2 + 2\mathrm{i}Bs + \rho v^2 - C = 0, \quad (11.176)$$

within which we have introduced the notations defined by

$$A = \mathcal{A}^*_{02323} - \frac{\mathbb{A}^{*2}_{023|3}}{\mathsf{A}^*_{033}}, \quad B = \mathcal{A}^*_{01323} - \frac{\mathbb{A}^*_{013|3}\mathbb{A}^*_{023|3}}{\mathsf{A}^*_{033}}, \quad C = \mathcal{A}^*_{01313} - \frac{\mathbb{A}^{*2}_{013|3}}{\mathsf{A}^*_{033}}. \quad (11.177)$$

By selecting appropriate values of the components of **m** and **n** in the generalized strong ellipticity condition (11.26), it can be seen that $A > 0$, $B > 0$ and $C > 0$. There is a solution of (11.176) for s with $\mathrm{Re}(s) > 0$ provided

$$\rho v^2 < (AC - B^2)/A. \quad (11.178)$$

For a real wave speed to exist, this requires, in particular, $AC > B^2$. Since only one value of s is required, the solution for the half-space remains in the form (11.170) for u_3 and (11.171) for w_3, with P and Q related by (11.175).

For the layer, $\bar{s}$ satisfies the quadratic

$$\bar{A}\bar{s}^2 + 2\bar{B}\bar{s} + \bar{C} - \bar{\rho}v^2 = 0, \quad (11.179)$$

where

$$\bar{A} = \bar{\mathcal{A}}^*_{02323} - \frac{\bar{\mathbb{A}}^{*2}_{023|3}}{\bar{\mathsf{A}}^*_{033}}, \quad \bar{B} = \bar{\mathcal{A}}^*_{01323} - \frac{\bar{\mathbb{A}}^*_{013|3}\bar{\mathbb{A}}_{023|3}}{\bar{\mathsf{A}}^*_{033}}, \quad \bar{C} = \bar{\mathcal{A}}^*_{01313} - \frac{\bar{\mathbb{A}}^{*2}_{013|3}}{\bar{\mathsf{A}}^*_{033}}. \quad (11.180)$$

Just as for A, B and C, the generalized strong ellipticity condition requires that $\bar{A} > 0$, $\bar{B} > 0$ and $\bar{C} > 0$.

We denote by $\bar{s}_1$ and $\bar{s}_2$ the two solutions of (11.179). Then the general solution for $\bar{u}_3$ and $\bar{w}_3$ of the considered form is

$$\bar{u}_3 = (\bar{P}_1 \mathrm{e}^{\mathrm{i}\bar{s}_1 k x_2} + \bar{P}_2 \mathrm{e}^{\mathrm{i}\bar{s}_2 k x_2}) \exp[\mathrm{i}(kx_1 - \omega t)], \quad (11.181)$$

$$\bar{w}_3 = k(\bar{Q}_1 \mathrm{e}^{\mathrm{i}\bar{s}_1 k x_2} + \bar{Q}_2 \mathrm{e}^{\mathrm{i}\bar{s}_2 k x_2}) \exp[\mathrm{i}(kx_1 - \omega t)]. \quad (11.182)$$

The coefficients $\bar{P}_j$ and $\bar{Q}_j$, $j = 1, 2$, are related by

$$\bar{Q}_j = -\frac{\mathrm{i}(\bar{\mathbb{A}}^*_{013|3} + \bar{s}_j \bar{\mathbb{A}}^*_{023|3})}{\bar{\mathsf{A}}^*_{033}} \bar{P}_j, \quad j = 1, 2, \quad (11.183)$$

which is obtained by replacing s by $\mathrm{i}\bar{s}$ in (11.175) and placing bars over the other quantities.

The next step is to substitute (11.170), (11.171), (11.181) and (11.182) into the boundary conditions (11.167)–(11.169) to obtain

$$\begin{aligned}\mathrm{i}(\bar{\mathcal{A}}^*_{02313} + \bar{s}_1\bar{\mathcal{A}}^*_{02323})\bar{P}_1\mathrm{e}^{\mathrm{i}\bar{s}_1kh} + \mathrm{i}(\bar{\mathcal{A}}^*_{02313} + \bar{s}_2\bar{\mathcal{A}}^*_{02323})\bar{P}_2\mathrm{e}^{\mathrm{i}\bar{s}_2kh} \\ + \bar{\mathbb{A}}^*_{023|3}(\bar{Q}_1\mathrm{e}^{\mathrm{i}\bar{s}_1kh} + \bar{Q}_2\mathrm{e}^{\mathrm{i}\bar{s}_2kh}) = 0,\end{aligned} \tag{11.184}$$

$$\begin{aligned}\mathrm{i}(\bar{\mathcal{A}}^*_{02313} + \bar{s}_1\bar{\mathcal{A}}^*_{02323})\bar{P}_1 + \mathrm{i}(\bar{\mathcal{A}}^*_{02313} + \bar{s}_2\bar{\mathcal{A}}^*_{02323})\bar{P}_2 \\ + \bar{\mathbb{A}}^*_{023|3}(\bar{Q}_1 + \bar{Q}_2) = (\mathrm{i}\mathcal{A}^*_{02313} + s\mathcal{A}^*_{02323})P + \mathbb{A}^*_{023|3}Q,\end{aligned} \tag{11.185}$$

$$\bar{P}_1 + \bar{P}_2 = P. \tag{11.186}$$

Then the relations (11.175) and (11.183) are used to eliminate Q, $\bar{Q}_1$ and $\bar{Q}_2$ to obtain three linear equations for $\bar{P}_1$, $\bar{P}_2$ and P. With the help of the notations defined by (11.177) and (11.180), they are expressed as

$$(\bar{s}_1\bar{A} + \bar{B})\mathrm{e}^{\mathrm{i}\bar{s}_1kh}\bar{P}_1 + (\bar{s}_2\bar{A} + \bar{B})\mathrm{e}^{\mathrm{i}\bar{s}_2kh}\bar{P}_2 = 0, \tag{11.187}$$

$$(\bar{s}_1\bar{A} + \bar{B})\bar{P}_1 + (\bar{s}_2\bar{A} + \bar{B})\bar{P}_2 + (\mathrm{i}sA - B)P = 0, \tag{11.188}$$

$$\bar{P}_1 + \bar{P}_2 - P = 0. \tag{11.189}$$

For non-trivial solutions for $\bar{P}_1$, $\bar{P}_2$ and P, the determinant of their coefficients must vanish, leading to the secular equation for the wave speed, specifically

$$\begin{aligned}[(\bar{s}_1\bar{A} + \bar{B})(\bar{s}_2\bar{A} + \bar{B}) + (\mathrm{i}sA - B)\bar{B}](\mathrm{e}^{\mathrm{i}\bar{s}_2kh} - \mathrm{e}^{\mathrm{i}\bar{s}_1kh}) \\ + (\mathrm{i}sA - B)\bar{A}(\bar{s}_2\mathrm{e}^{\mathrm{i}\bar{s}_2kh} - \bar{s}_1\mathrm{e}^{\mathrm{i}\bar{s}_1kh}) = 0,\end{aligned} \tag{11.190}$$

where $\bar{s}_1$ and $\bar{s}_2$ are the solutions of (11.179) and s is the solution of (11.176) with positive real part. This equation can be shown to give a real solution for the wave speed in only two special cases. One case arises if $\bar{s}_2 = -\bar{s}_1$ is real and s is real, and this requires $B = \bar{B} = 0$, and a second case if s is real (and $B = 0$) but $\bar{s}_2$ and $\bar{s}_1$ are complex conjugates. We illustrate the first case below but first we show that the formula reduces to known results for a purely elastic material.

For this purpose we now take the magnetic field to vanish, and hence, $\mathbb{A}^*$ and $\bar{\mathbb{A}}^*$ also vanish, and (11.176) and (11.179) reduce to

$$s^2 = \frac{\mathcal{A}^*_{01313} - \rho v^2}{\mathcal{A}^*_{02323}}, \quad \bar{s}^2 = \frac{\bar{\rho}v^2 - \bar{\mathcal{A}}^*_{01313}}{\bar{\mathcal{A}}^*_{02323}}. \tag{11.191}$$

With these simplifications, the secular equation (11.190) becomes

$$\tan(\bar{s}kh) = \frac{s\mathcal{A}^*_{02323}}{\bar{s}\bar{\mathcal{A}}^*_{02323}}. \tag{11.192}$$

This has no real solutions for the wave speed if $\bar{s}^2 < 0$. We therefore take $\bar{s}^2 > 0$, which, by $(11.191)_2$, places an upper bound on the squared wave speed. Without loss of generality we may therefore take $\bar{s} > 0$. Since, from $(11.191)_1$, s^2 is real, it must be positive in order to ensure that the decay condition is satisfied, which then requires that $s > 0$. These two inequalities put both upper and lower bounds on the squared wave speed, specifically

$$\bar{\mathcal{A}}^*_{01313}/\bar{\rho} < v^2 < \mathcal{A}^*_{01313}/\rho. \tag{11.193}$$

Note that the limiting case $\bar{s} = 0$ entails $s = 0$ also, and hence, the outer two expressions in (11.193) must be equal. The corresponding wave speed does not correspond to a surface wave but to a homogeneous plane shear wave with equal speeds in the two materials. Such a wave can propagate in an infinite medium but is not possible in a half-space with a layer for the given boundary conditions.

For linear isotropic elasticity, $\mathcal{A}^*_{01313} = \mathcal{A}^*_{02323} = \mu$ and $\bar{\mathcal{A}}^*_{01313} = \bar{\mathcal{A}}^*_{02323} = \bar{\mu}$, where μ and $\bar{\mu}$ are the shear moduli of the half-space and layer, respectively. If the transverse body wave speed is denoted by $v_T = (\mu/\rho)^{1/2}$ in the half-space and $\bar{v}_T = (\bar{\mu}/\bar{\rho})^{1/2}$ in the layer, then the secular equation reduces to

$$\tan\left[\left(\frac{v^2}{\bar{v}_T^2} - 1\right)^{\frac{1}{2}} kh\right] = \frac{\mu}{\bar{\mu}}\frac{\left[1 - (v/v_T)^2\right]^{\frac{1}{2}}}{\left[(v/\bar{v}_T)^2 - 1\right]^{\frac{1}{2}}}, \quad \bar{v}_T < v < v_T, \tag{11.194}$$

thus recovering the well-known dispersion relation for Love waves in linear elasticity (see, e.g., Achenbach 1973).

Illustration For purposes of illustration we again use the form of energy function (11.32) for the half-space and its counterpart with overbars for the layer, and we consider the magnetic field to be either parallel or perpendicular to the boundary (in the (x_1, x_2) plane). In either case $\mathcal{A}^*_{01323} = 0$ and $\mathbb{A}^*_{013|3}\mathbb{A}^*_{023|3} = 0$, with

$$\mathbb{A}^*_{013|3} = 0 \text{ if } B_1 = 0, \quad \mathbb{A}^*_{023|3} = 0 \text{ if } B_2 = 0. \tag{11.195}$$

Hence $B = 0$ and similarly $\bar{B} = 0$. Equations (11.176) and (11.179) then simplify to

$$s^2 = (C - \rho v^2)/A, \quad \bar{s}^2 = (\bar{\rho}v^2 - \bar{C})/\bar{A}, \tag{11.196}$$

and, similarly to the linear theory, (11.190) becomes

$$\tan(\bar{s}kh) = \frac{sA}{\bar{s}\bar{A}}, \tag{11.197}$$

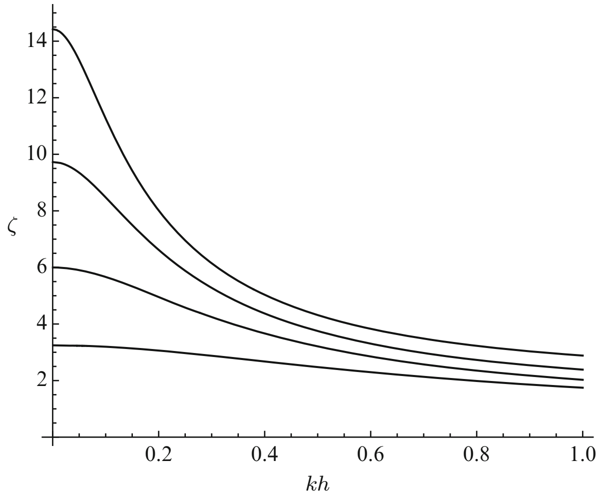

Fig. 11.3 First mode dispersion curves $\zeta = \bar{\rho}v^2/\bar{\mu}$ vs. kh under finite deformation in the absence of a magnetic field: *reading from bottom to top* $\lambda_1 = 0.7, 1, 1.3, 1.6$, with $\lambda_2 = \lambda_1^{-1}, \lambda_3 = 1$

where $\bar{s} > 0$ and $s > 0$. In this case the bounds (11.193) on the wave speed are given by

$$\bar{C}/\bar{\rho} < v^2 < C/\rho. \tag{11.198}$$

The inequality $\bar{C}/\bar{\rho} < C/\rho$ imposes restrictions on the deformation, the magnetic field and the energy functions for the layer and the half-space that are necessary for the existence of Love-type waves. The upper limit corresponds to the wave speed in the half-space ($kh \to 0$) of a bulk shear wave propagating in the x_1 direction, and the lower limit to the corresponding shear wave in the layer ($kh \to \infty$).

There are infinitely many propagation modes, but all modes have the same character, and we therefore only illustrate the results for the mode with the smallest wave speed in each example. This is the only mode existing in the limit $kh \to 0$. Each of the other modes is 'cut off' for values of kh below a certain value, a value which increases with the mode number.

For the mechanical properties of the materials, we take

$$\bar{\mu} = 2.6 \times 10^5\,\mathrm{N\,m^{-2}}, \quad \mu = 2\bar{\mu}, \tag{11.199}$$

the first value having been used in Sect. 11.4.1.3, and the density ratio $\bar{\rho}/\rho = 3$.

First, for the purely elastic problem (no magnetic field) with a finite initial deformation, the results are shown in Fig. 11.3, in which the dimensionless squared wave speed $\zeta = \bar{\rho}v^2/\bar{\mu}$ is plotted against the dimensionless wave number kh for several values of the stretch λ, where we are considering the plane strain situation with $\lambda_1 = \lambda, \lambda_2 = \lambda^{-1}, \lambda_3 = 1$. (Note that the definition of ζ here is different from that used in Sects. 11.4.1.3 and 11.4.2.1.) For the linearly elastic case ($\lambda_1 = \lambda_2 = \lambda_3 = 1$), the curves intersect the ζ axis at $\mu\bar{\rho}/\bar{\mu}\rho$, which agrees with the classical solution obtained by taking the limit $kh \to 0$ in (11.194) and is equal to 6 for the values adopted here.

Next, we consider the effect of the magnetic field, for which purpose we adopt the following values of the coupling parameters for the half-space (the same values of α and β used in Sect. 11.4.1.3) and slightly different values of $\bar{\alpha}$ and $\bar{\beta}$ for the layer. Thus, we take

$$\alpha = 0.2, \ \beta = 0.3, \ \bar{\alpha} = 0.4, \ \bar{\beta} = 0.2, \tag{11.200}$$

along with the standard value $\mu_0 = 4\pi \times 10^{-7}$ N/A^2. In the following two examples we consider the material to be undeformed.

For $B_2 = 0$ we have

$$A = \mu, \quad C = \mu + \frac{\mu_0^{-1}\alpha\beta}{\alpha + \beta} B_1^2, \quad \bar{A} = \bar{\mu}, \quad \bar{C} = \bar{\mu} + \frac{\mu_0^{-1}\bar{\alpha}\bar{\beta}}{\bar{\alpha} + \bar{\beta}} \bar{B}_1^2, \tag{11.201}$$

and from the continuity $\bar{H}_1 = H_1$, we obtain $(\alpha + \beta)B_1 = (\bar{\alpha} + \bar{\beta})\bar{B}_1$.

On the other hand, for $B_1 = 0$ we have

$$A = \mu + \frac{\mu_0^{-1}\alpha\beta}{\alpha + \beta} B_2^2, \quad C = \mu, \quad \bar{A} = \bar{\mu} + \frac{\mu_0^{-1}\bar{\alpha}\bar{\beta}}{\bar{\alpha} + \bar{\beta}} \bar{B}_2^2, \quad \bar{C} = \bar{\mu} \tag{11.202}$$

and the continuity $\bar{B}_2 = B_2$.

In Fig. 11.4 we plot the dimensionless squared wave speed $\zeta = \bar{\rho}v^2/\bar{\mu}$ against kh to illustrate the influence of the magnetic induction field on the wave speed. As can be seen in Fig. 11.4a the wave speed increases as the magnetic induction field ($\bar{B}_1$) parallel to the boundary increases for all values of kh, but the effect is strongest for the smaller values of kh. The effect of a magnetic induction field (B_2) perpendicular to the boundary, shown in Fig. 11.4b, is similar but the increases in wave speed are not so marked. In particular, as $kh \to 0$, the effect of B_2 vanishes.

When there is an initial finite deformation, the results are qualitatively similar to those for the undeformed case and are not therefore shown separately. For an underlying pure homogeneous plane strain with $\lambda < 1$, the wave speeds are reduced while for $\lambda > 1$ they are increased. More details are given in Saxena and Ogden (2012).

The problem in which the underlying magnetic (induction) field has components $(0, 0, B_3)$ and the incremental field has components $(\dot{B}_{L01}, \dot{B}_{L02}, 0)$ in the half-space and corresponding components in the layer and outside the material is also of interest, but it is in general rather more complicated than the problem dealt with above and is not considered here. Details can be found in Saxena and Ogden (2012).

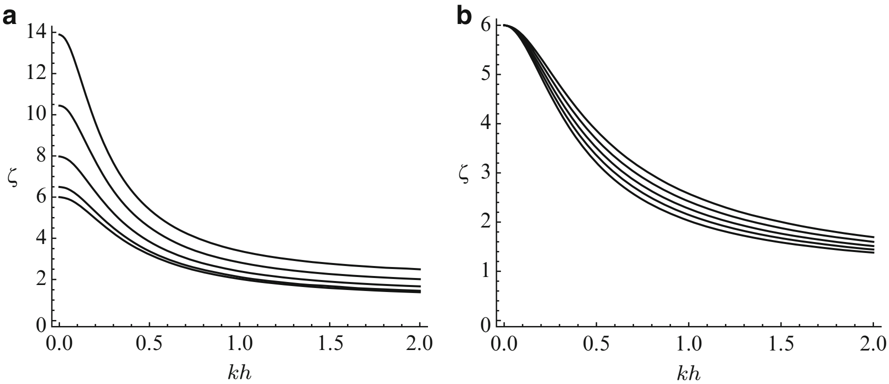

Fig. 11.4 First mode dispersion curves: $\zeta = \bar{\rho}v^2/\bar{\mu}$ vs. kh for the linear elastic case in the presence of a magnetic field (*reading from bottom to top* in each case) (**a**) $B_2 = 0$, $\bar{B}_1 = 0, 0.4, 0.8, 1.2, 1.6\,\mathrm{T}$; (**b**) $B_1 = 0$, $B_2 = 0, 0.8, 1.2, 1.6, 2\,\mathrm{T}$

11.4.3.2 Bleustein–Gulyaev-Type Waves

Shear horizontal waves, as exemplified by the Love waves considered above, cannot be supported in a half-space of an elastic material without a layer. However, in the presence of an electric field such waves are possible. These are known as Bleustein–Gulyaev waves after Bleustein (1968) and Gulyaev et al. (1997), and they have counterparts in the present context where a magnetic field is present (Parekh 1969a,b). Here, we therefore consider the same magnetoelastic half-space as in Sect. 11.4.3 *without* a layer and study Bleustein–Gulyaev-type waves. First, we note that such waves do not exist if the underlying magnetic field is in-plane, as shown by Saxena and Ogden (2012). We therefore restrict attention to the case in which the underlying field has just an out-of-plane magnetic field component H_3 and magnetic induction component B_3 in the half-space, which, from (11.28), are related by

$$H_3 = 2(\lambda_3^{-2}\Omega_4^* + \Omega_5^* + \lambda_3^2\Omega_6^*)B_3. \tag{11.203}$$

By continuity, the magnetic field outside the half-space is given by $H_3^\star = H_3$, and hence we have the connection

$$H_3^\star = 2(\lambda_3^{-2}\Omega_4^* + \Omega_5^* + \lambda_3^2\Omega_6^*)B_3. \tag{11.204}$$

For the incremental solution, we consider the displacement component u_3 and in-plane components of the incremental magnetic induction, denoted $w_1 = \dot{B}_{L01}$ and $w_2 = \dot{B}_{L02}$. Then, from (11.148)–(11.153) there remain just two equations, which reduce to

$$\mathcal{A}^*_{01313}u_{3,11} + \mathcal{A}^*_{02323}u_{3,22} + \mathbb{A}^*_{013|1}w_{1,1} + \mathbb{A}^*_{023|2}w_{2,2} = \rho u_{3,tt}, \tag{11.205}$$

and

$$(\mathbb{A}^*_{023|2} - \mathbb{A}^*_{013|1})u_{3,12} + \mathsf{A}^*_{022}w_{2,1} - \mathsf{A}^*_{011}w_{1,2} = 0, \tag{11.206}$$

which hold in $x_2 < 0$ with $w_1 = \varphi_{,2}$ and $w_2 = -\varphi_{,1}$. In $x_2 > 0$, (11.54) is again the relevant equation.

For the half-space, we consider solutions of the form

$$u_3 = P\exp(skx_2 + \mathrm{i}kx_1 - \mathrm{i}\omega t) \quad \text{in } x_2 < 0, \tag{11.207}$$

$$\varphi = Q\exp(skx_2 + \mathrm{i}kx_1 - \mathrm{i}\omega t) \quad \text{in } x_2 < 0, \tag{11.208}$$

with the condition $\mathrm{Re}(s) > 0$ required for the solutions to decay as $x_2 \to -\infty$, P and Q being constants. Substituting these into (11.205) and (11.206), we obtain

$$(-\mathcal{A}^*_{01313} + s^2\mathcal{A}^*_{02323} + \rho' v^2)P + \mathrm{i}s(\mathbb{A}^*_{013|1} - \mathbb{A}^*_{023|2})Q = 0, \tag{11.209}$$

$$\mathrm{i}s(\mathbb{A}^*_{023|2} - \mathbb{A}^*_{013|1})P + (\mathsf{A}^*_{022} - s^2\mathsf{A}^*_{011})Q = 0, \tag{11.210}$$

wherein the wave speed v is defined as $v = \omega/k$.

For non-trivial solutions the determinant of the coefficients of P and Q must be zero, which gives the bi-quadratic

$$\begin{aligned}\mathcal{A}^*_{02323}\mathsf{A}^*_{011}s^4 &+ \{\mathsf{A}^*_{011}(\rho v^2 - \mathcal{A}^*_{01313}) - \mathsf{A}^*_{022}\mathcal{A}^*_{02323} + (\mathbb{A}^*_{023|2} - \mathbb{A}^*_{013|1})^2\}s^2 \\ &- \mathsf{A}^*_{022}(\rho v^2 - \mathcal{A}^*_{01313}) = 0\end{aligned} \tag{11.211}$$

for s. Let s_1 and s_2 be the two solutions of this equation satisfying the condition $\mathrm{Re}(s) > 0$. Then, as before, we note that the condition $s_1^2 s_2^2 \geq 0$ gives an upper bound on the wave speed, in this case

$$\rho v^2 \leq \mathcal{A}^*_{01313}. \tag{11.212}$$

With the two possible values of s, the relevant general solutions for u_3 and φ are

$$u_3 = (P_1\mathrm{e}^{s_1kx_2} + P_2\mathrm{e}^{s_2kx_2})\exp[\mathrm{i}(kx_1 - \omega t)], \tag{11.213}$$

$$\varphi = (Q_1\mathrm{e}^{s_1kx_2} + Q_2\mathrm{e}^{s_2kx_2})\exp[\mathrm{i}(kx_1 - \omega t)], \tag{11.214}$$

where P_j and Q_j are constants related by (11.210) as

$$(\mathsf{A}^*_{022} - s_j^2\mathsf{A}^*_{011})Q_j = -\mathrm{i}s_j(\mathbb{A}^*_{023|2} - \mathbb{A}^*_{013|1})P_j, \quad j = 1, 2. \tag{11.215}$$

The corresponding solution in $x_2 > 0$ is

$$\varphi^\star = R\exp(-kx_2 + \mathrm{i}kx_1 - \mathrm{i}\omega t), \tag{11.216}$$

as in (11.100) but without the factor k multiplying R.

Supposing again that there is no incremental mechanical traction on the boundary $x_2 = 0$, it follows on specializing (9.129) with (9.128) and (9.125) that the incremental traction boundary conditions are

$$\dot{T}_{022} = 0, \quad \dot{T}_{023} = \dot{\tau}^{\star}_{\mathrm{m}23} = \mu_0^{-1}\dot{B}_2^{\star}B_3^{\star} \quad \text{on } x_2 = 0. \tag{11.217}$$

The corresponding incremental magnetic boundary conditions are obtained by specializing (9.121) and (9.123). This gives

$$w_2 = \dot{B}_2^{\star}, \quad \dot{H}_{\mathrm{L}01} - \dot{H}_1^{\star} - H_3^{\star}u_{3,1} = 0 \quad \text{on } x_2 = 0. \tag{11.218}$$

From the incremental constitutive equation (11.7) we have $\dot{T}_{011} = \dot{T}_{022} = -\dot{p}^*$ and $\dot{T}_{012} = \dot{T}_{021} = 0$, and since there is no dependence on x_3, it follows on application of the equilibrium equations that $\dot{p}^*$ is a constant, which, by the boundary condition $\dot{T}_{022} = 0$, is zero throughout the material.

It also follows from the appropriate specialization of the incremental constitutive equations (11.7) and (11.8) that

$$\dot{T}_{023} = \mathcal{A}^*_{02323}u_{3,2} + \mathbb{A}^*_{023|2}w_2, \quad \dot{H}_{\mathrm{L}01} = \mathbb{A}^*_{013|1}u_{3,1} + \mathsf{A}^*_{011}w_1, \tag{11.219}$$

and substitution into the boundary conditions gives

$$\mathcal{A}^*_{02323}(s_1P_1 + s_2P_2) - \mathrm{i}\mathbb{A}^*_{023|2}(Q_1 + Q_2) + \mathrm{i}H_3^*R = 0, \tag{11.220}$$

$$Q_1 + Q_2 - R = 0, \tag{11.221}$$

$$\mathrm{i}(\mathbb{A}^*_{013|1} - H_3^*)(P_1 + P_2) + \mathsf{A}^*_{011}(s_1Q_1 + s_2Q_2) + \mu_0^{-1}R = 0, \tag{11.222}$$

with Q_j given in terms of P_j, $j = 1, 2$, by (11.210).

Illustration In order to illustrate the results we consider a generalization of the neo-Hookean prototype energy function (11.32) given by

$$\Omega^* = \frac{1}{2}\mu(1 + \xi K_4)(I_1 - 3) + \frac{1}{2}\mu_0^{-1}(\alpha K_4 + \beta K_5 + \gamma K_6), \tag{11.223}$$

where α and β are the parameters used in (11.32), while $\mu(K_4) = \mu(1 + \xi K_4)$ is a shear modulus that varies with the magnetic field, ξ here being a material constant, and γ is an additional magnetoelastic coupling parameter. In different notation, this is a slight modification of the energy function used in Dorfmann and Ogden (2005b). For this function, the relevant components of the moduli tensors are

$$\begin{aligned}
\mathcal{A}^*_{01313} &= \mu\lambda_1^2(1 + \xi K_4) + \mu_0^{-1}\gamma\lambda_1^2B_3^2, \\
\mathcal{A}^*_{02323} &= \mu\lambda_2^2(1 + \xi K_4) + \mu_0^{-1}\gamma\lambda_2^2B_3^2, \\
\mathbb{A}^*_{013|1} &= \mu_0^{-1}B_3[\beta + \gamma(\lambda_1^2 + \lambda_3^2)],
\end{aligned}$$

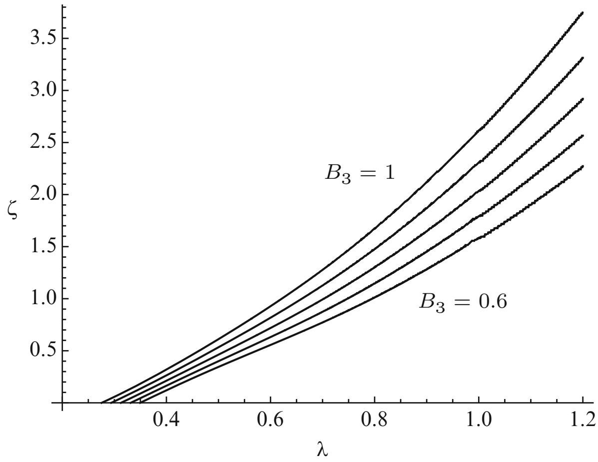

Fig. 11.5 Plots of the dimensionless squared wave speed $\zeta = \rho v^2/\mu$ vs. the stretch λ for a Bleustein–Gulyaev-type wave in a magnetoelastic neo-Hookean type solid subject to plane strain with $\lambda_1 = \lambda = \lambda_2^{-1}, \lambda_3 = 1$ and magnetic induction fields $B_3 = 0.6, 0.7, 0.8, 0.9, 1$ T (reading from bottom to top)

$$\begin{aligned}
\mathbb{A}^*_{023|2} &= \mu_0^{-1} B_3[\beta + \gamma(\lambda_2^2 + \lambda_3^2)], \\
\mathsf{A}^*_{011} &= \lambda_1^{-2}\left[\mu\xi(I_1 - 3) + \mu_0^{-1}\alpha\right] + \mu_0^{-1}(\alpha + \gamma\lambda_1^2), \\
\mathsf{A}^*_{022} &= \lambda_2^{-2}\left[\mu\xi(I_1 - 3) + \mu_0^{-1}\alpha\right] + \mu_0^{-1}(\alpha + \gamma\lambda_2^2).
\end{aligned} \tag{11.224}$$

For this model, we study the effect of changes in the underlying magnetic field and deformation on the dimensionless squared wave speed $\zeta = \rho v^2/\mu$. The underlying deformation corresponds to plane strain with $\lambda_3 = 1$ and $\lambda_1 = \lambda$ and $\lambda_2 = \lambda^{-1}$. In Fig. 11.5, results are shown for ζ plotted against λ for values of B_3 running through 0.6, 0.7, 0.8, 0.9, 1 T. The values of the material constants used are

$$\mu = 5.2 \times 10^5\,\mathrm{N\,m^{-2}}, \quad \xi = 1\,\mathrm{T^{-2}}, \quad \alpha = 0.2, \quad \beta = 0.3, \quad \gamma = 0.2. \tag{11.225}$$

For $B_3 = 0$ no wave can propagate. As B_3 increases from zero, the general characteristics are as shown in Fig. 11.5. For $\lambda < 1$ the wave speed vanishes in each case at a critical value of λ depending on B_3. This corresponds to instability of the half-space under compression in the considered mode (which we may refer to as a Bleustein–Gulyaev instability). However, the half-space becomes unstable at a prior value of λ under the Biot-type surface buckling, so Bleustein–Gulyaev instabilities are unlikely to be observed in practice. As can be seen from the figure, an increase in the stretch parallel to the direction of wave propagation tends to increase the wave speed. Increasing B_3 also increases the wave speed, consistently with the upper bound (11.212), which in this case has the form

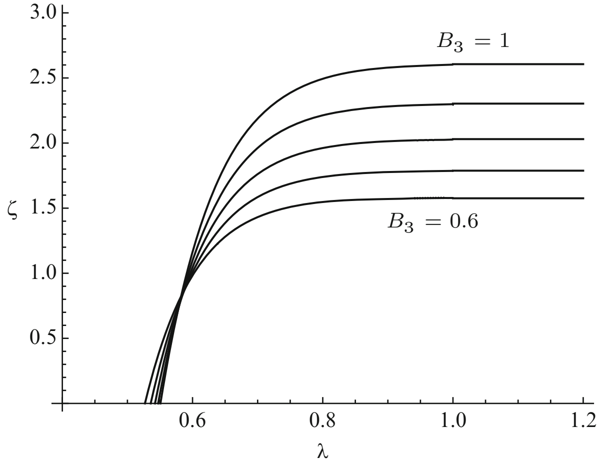

Fig. 11.6 Plots of the dimensionless squared wave speed $\zeta = \rho v^2/\mu$ vs. the stretch λ for a Bleustein–Gulyaev- type wave in a magnetoelastic neo-Hookean type solid subject to plane strain with $\lambda_1 = 1$, $\lambda_3 = \lambda_2^{-1} = \lambda$ and magnetic induction fields $B_3 = 0.6, 0.7, 0.8, 0.9, 1$ T (reading from bottom to top)

$$\zeta \equiv \frac{\rho v^2}{\mu} = \lambda^2(1 + \xi B_3^2) + \frac{\gamma \lambda^2 B_3^2}{\mu \mu_0}. \tag{11.226}$$

Next, in Fig. 11.6, we illustrate results for an underlying plane strain deformation for which there is no compression or extension parallel to the direction of wave propagation ($\lambda_1 = 1$), with $\lambda_3 = \lambda = \lambda_2^{-1}$. In this case we use the same parameters as in (11.225), except for α and β, which are taken to be $\alpha = 0.5$ and $\beta = 0.04$. These values have been selected to demonstrate the 'cross-over' effect, which is observed in Fig. 11.6, i.e. there is a value of $\lambda = \lambda_{\mathrm{cr}}$ for which the wave speed becomes independent of the underlying magnetic field B_3. When $\lambda < \lambda_{\mathrm{cr}}$, the wave speed decreases with an increase in B_3 while in the region $\lambda > \lambda_{\mathrm{cr}}$ the wave speed increases with an increase in B_3.

For large values of compression (smaller λ), ζ goes to zero, which, as in Fig. 11.5, corresponds to the onset of instability of the half-space. The wave speed increases with an increase in λ and tends to an asymptotic value dependent on the underlying magnetic field B_3 consistently with the upper bound (11.212), which in this case is

$$\zeta \equiv \frac{\rho v^2}{\mu} = 1 + \xi \lambda^{-2} B_3^2 + \frac{\gamma B_3^2}{\mu \mu_0}. \tag{11.227}$$

The cross-over effect is a function of the particular parameters chosen and does not appear for other values of the parameters, including those in (11.225). Otherwise, the general characteristics are the same for each choice of parameter values.

We now consider the special case for which the half-space is undeformed. Then, for the considered model $\mathbb{A}^*_{013|1} = \mathbb{A}^*_{023|2}$ and $\mathsf{A}^*_{011} = \mathsf{A}^*_{022}$, thus allowing (11.211) to be factorized to obtain $s_1^2 = 1 - \rho v^2/\mathcal{A}^*_{01313}$ and $s_2^2 = 1$, with (11.206) simplifying to $\varphi_{,11} + \varphi_{,22} = 0$. The equations are now uncoupled and the coefficients in (11.210) are identically zero. Thus, instead of (11.207) and (11.208), we take the solutions for u_3 and φ to have the forms

$$u_3 = P \exp(s_1 k x_2 + \mathrm{i} k x_1 - \mathrm{i}\omega t), \quad x_2 < 0, \tag{11.228}$$

$$\varphi = Q \exp(k x_2 + \mathrm{i} k x_1 - \mathrm{i}\omega t), \quad x_2 < 0, \tag{11.229}$$

with P and Q now independent. We retain $\varphi^\star$ as

$$\varphi^\star = R \exp(-k x_2 + \mathrm{i} k x_1 - \mathrm{i}\omega t), \quad x_2 > 0. \tag{11.230}$$

The boundary conditions (11.220)–(11.222) may now be applied with $P_1 = P, P_2 = 0, Q_1 = 0, Q_2 = Q$, which yields $R = Q$ and, since $\mathbb{A}^*_{013|1} = \mathbb{A}^*_{023|2}$, $\mathcal{A}^*_{02323} = \mathcal{A}^*_{01313}$ and $s_2 = 1$,

$$\mathcal{A}^*_{01313} s_1 P + \mathrm{i}(H_3^\star - \mathbb{A}^*_{013|1}) Q = 0, \tag{11.231}$$

$$-\mathrm{i}(H_3^\star - \mathbb{A}^*_{013|1}) P + (\mu_0^{-1} + \mathsf{A}^*_{011}) Q = 0. \tag{11.232}$$

For a non-trivial solution for P and Q, we set the determinant of their coefficients to zero, which gives an explicit formula for the wave speed, specifically

$$\rho v^2 = \mathcal{A}^*_{01313} - \frac{(H_3^\star - \mathbb{A}^*_{013|1})^4}{(\mu_0^{-1} + \mathsf{A}^*_{011})^2 \mathcal{A}^*_{01313}}. \tag{11.233}$$

The predictions of this formula are consistent with the results illustrated in Figs. 11.5 and 11.6.

If we set $\xi = \gamma = 0$ in the energy function (11.223), then we recover the prototype neo-Hookean model (11.32). A consideration of Bleustein–Gulyaev waves for this material requires an analysis similar to that in the case of a linearly elastic material discussed above and again leads to an explicit formula. Noting that $s_1^2 = (\mathcal{A}^*_{01313} - \rho v^2)/\mathcal{A}^*_{02323}$ and $s_2^2 = \mathsf{A}^*_{022}/\mathsf{A}^*_{011}$, we obtain

$$\rho v^2 = \mathcal{A}^*_{01313} - \frac{(H_3^\star - \mathbb{A}^*_{013|1})^4}{(\mu_0^{-1} + \sqrt{\mathsf{A}^*_{011}\mathsf{A}^*_{022}})^2 \mathcal{A}^*_{02323}}. \tag{11.234}$$

For an underlying plane strain deformation with $\lambda_3 = 1$ and $\lambda_1 = \lambda = \lambda_2^{-1}$, substitution of the specific values of the components of the moduli tensors in the above, with the help of the connection $H_3^\star = \mu_0^{-1}(\alpha + \beta) B_3$, leads to

$$\rho v^2 = \mu\lambda^2 - \frac{\alpha^4 \lambda^2 B_3^4}{\mu\mu_0^2[1 + \sqrt{\alpha^2 + \beta^2 + \alpha\beta(\lambda^2 + \lambda^{-2})}]^2}. \tag{11.235}$$

From the above formulas it may be deduced that there is an upper bound on the underlying magnetic field for which the wave speed is real. When evaluated for $\lambda = 1$, (11.235) reduces to

$$\rho v^2 = \mu - \frac{\alpha^4 B_3^4}{\mu \mu_0^2 (1 + \alpha + \beta)^2}. \tag{11.236}$$

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Appendix A
Basic Vector and Tensor Operations

This appendix provides a reference to some basic vector and tensor operations that are used in deriving some of the results presented in this text. In the first section, we list some standard formulas involving products of vectors and tensors in three-dimensional Euclidean space.

A.1 Vector and Tensor Identities

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ be vectors. Then, the following identities involving their scalar and vector products are noted:

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = -\mathbf{a}\cdot(\mathbf{c}\times\mathbf{b}), \tag{A.1}$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}, \tag{A.2}$$

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}, \tag{A.3}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}). \tag{A.4}$$

Let $\mathbf{I}$ be the identity tensor, which has Cartesian components δ_{ij}, the Kronecker delta. In this section "components" means "Cartesian components". Then, for any vector $\mathbf{a}$, $\mathbf{Ia} = \mathbf{a}$, or in components, $\delta_{ij}a_j = a_i$. Here and elsewhere in this appendix the Einstein summation convention for repeated indices is used.

The tensor product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is written as $\mathbf{a}\otimes\mathbf{b}$ and defined by the identity

$$(\mathbf{a}\otimes\mathbf{b})\mathbf{c} = (\mathbf{b}\cdot\mathbf{c})\mathbf{a}, \tag{A.5}$$

where $\mathbf{c}$ is an arbitrary vector, or, in components, $(\mathbf{a}\otimes\mathbf{b})_{ij} = a_i b_j$.

L. Dorfmann and R.W. Ogden, *Nonlinear Theory of Electroelastic and Magnetoelastic Interactions*, DOI 10.1007/978-1-4614-9596-3,

Let $\boldsymbol{\epsilon}$ denote the alternating tensor, which has components ϵ_{ijk}. Then, the cross product of two vectors $\mathbf{a}$ and $\mathbf{b}$ may be written in tensor and component form, respectively, as

$$\mathbf{a} \times \mathbf{b} = \boldsymbol{\epsilon}(\mathbf{a} \otimes \mathbf{b}), \quad (\mathbf{a} \times \mathbf{b})_i = [\boldsymbol{\epsilon}(\mathbf{a} \otimes \mathbf{b})]_i = \epsilon_{ijk} a_j b_k. \tag{A.6}$$

Correspondingly, we define the product of $\boldsymbol{\epsilon}$ with a second-order tensor $\mathbf{A}$, written as $\boldsymbol{\epsilon}\mathbf{A}$, via the component form

$$(\boldsymbol{\epsilon}\mathbf{A})_i = \epsilon_{ijk} A_{jk}, \tag{A.7}$$

not as $\epsilon_{ijk} A_{kj}$, which is a definition sometimes used in the literature. The standard connection between the alternating symbol and the Kronecker delta has the component form

$$\epsilon_{kij}\epsilon_{kpq} = \epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}. \tag{A.8}$$

The alternating symbol is useful in expressions for the determinant of a second-order tensor. For example, if $\mathbf{A}$ denotes a second-order tensor with components A_{ij}, then

$$\epsilon_{ijk} A_{ip} A_{jq} A_{kr} = \epsilon_{ijk} A_{pi} A_{qj} A_{rk} = (\det \mathbf{A})\epsilon_{pqr}. \tag{A.9}$$

This may be used to establish the following identity involving a second-order tensor $\mathbf{A}$ and two vectors $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{A}^{\mathrm{T}}(\mathbf{A}\mathbf{a} \times \mathbf{A}\mathbf{b}) = (\det \mathbf{A})\mathbf{a} \times \mathbf{b}. \tag{A.10}$$

A.2 Differential Operations on Scalar, Vector and Tensor Fields

Let $\mathbf{x}$ denote the position vector in our three-dimensional Euclidean space, and let $|\mathbf{x}|$ denote its magnitude. We denote by grad, div and curl the standard differential operators on this space, by ψ a scalar field and by $\mathbf{a}$ and $\mathbf{b}$ vector fields. Then, the following identities apply:

$$\operatorname{grad}|\mathbf{x}| = \frac{\mathbf{x}}{|\mathbf{x}|}, \quad \operatorname{grad}\frac{1}{|\mathbf{x}|} = -\frac{\mathbf{x}}{|\mathbf{x}|^3}, \quad \operatorname{grad}\mathbf{x} = \mathbf{I}, \tag{A.11}$$

$$\operatorname{curl}\mathbf{x} = \mathbf{0}, \quad \operatorname{div}\mathbf{x} = 3, \tag{A.12}$$

$$\operatorname{curl}(\operatorname{grad}\psi) = \mathbf{0}, \quad \operatorname{div}(\operatorname{curl}\mathbf{a}) = 0, \tag{A.13}$$

$$\operatorname{curl}(\operatorname{curl}\mathbf{a}) = \operatorname{grad}(\operatorname{div}\mathbf{a}) - \nabla^2\mathbf{a}, \tag{A.14}$$

$$\operatorname{div}(\psi\mathbf{a}) = \mathbf{a}\cdot\operatorname{grad}\psi + \psi\operatorname{div}\mathbf{a}, \tag{A.15}$$

$$\operatorname{curl}(\psi\mathbf{a}) = \operatorname{grad}\psi \times \mathbf{a} + \psi\operatorname{curl}\mathbf{a}, \tag{A.16}$$

$$\operatorname{div}(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot\operatorname{curl}\mathbf{a} - \mathbf{a}\cdot\operatorname{curl}\mathbf{b}, \tag{A.17}$$

$$(\operatorname{curl}\mathbf{a})\times\mathbf{b} = (\operatorname{grad}\mathbf{a})\mathbf{b} - (\operatorname{grad}\mathbf{a})^{\mathrm{T}}\mathbf{b}, \tag{A.18}$$

$$\operatorname{curl}(\mathbf{a}\times\mathbf{b}) = (\operatorname{div}\mathbf{b})\mathbf{a} - (\operatorname{div}\mathbf{a})\mathbf{b} + (\operatorname{grad}\mathbf{a})\mathbf{b} - (\operatorname{grad}\mathbf{b})\mathbf{a}, \tag{A.19}$$

where $\nabla^2 = \operatorname{div}\operatorname{grad}$ is the Laplace operator.

The above formulas, and those that follow in this section, may be applied to either Lagrangian or Eulerian fields in the continuum mechanics setting. However, we do not make the distinction here and work in terms of the position vector $\mathbf{x}$ and its associated operators.

A.2.1 Rectangular Cartesian Coordinates

Let the position vector $\mathbf{x}$ relative to an arbitrarily chosen origin have the representation

$$\mathbf{x} = x_i\mathbf{e}_i, \tag{A.20}$$

with respect to (orthonormal) Cartesian basis vectors $\mathbf{e}_i$, $i = 1, 2, 3$, x_i being the components of $\mathbf{x}$. The components of vector and tensor fields are represented similarly.

The gradient operations on the scalar ψ, vector $\mathbf{a}$ and second-order tensor fields $\mathbf{A}$ are here defined by

$$\operatorname{grad}\psi = \frac{\partial\psi}{\partial x_i}\mathbf{e}_i, \tag{A.21}$$

$$\operatorname{grad}\mathbf{a} = \frac{\partial a_i}{\partial x_j}\mathbf{e}_i\otimes\mathbf{e}_j, \tag{A.22}$$

$$\operatorname{grad}\mathbf{A} = \frac{\partial A_{ij}}{\partial x_k}\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k, \tag{A.23}$$

and the divergence operator applied to vector and tensor fields yields

$$\operatorname{div}\mathbf{a} = \frac{\partial a_i}{\partial x_j}\mathbf{e}_i\cdot\mathbf{e}_j = \frac{\partial a_i}{\partial x_i}, \tag{A.24}$$

$$\operatorname{div}\mathbf{A} = \frac{\partial A_{ij}}{\partial x_k}\mathbf{e}_j\,(\mathbf{e}_i\cdot\mathbf{e}_k) = \frac{\partial A_{ij}}{\partial x_i}\mathbf{e}_j, \tag{A.25}$$

where we have used the property $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$. Note that we have used the convention that the divergence of a second-order tensor involves contraction on the first index of $\mathbf{A}$, whereas in the literature it is sometimes defined on the second index.

The curl operations on $\mathbf{a}$ and $\mathbf{A}$ are defined using the alternating symbol ϵ_{ijk} by

$$\operatorname{curl}\mathbf{a} = \boldsymbol{\epsilon}(\operatorname{grad}\mathbf{a})^{\mathrm{T}} = \epsilon_{ijk}\frac{\partial a_k}{\partial x_j}\mathbf{e}_i, \tag{A.26}$$

$$\operatorname{curl}\mathbf{A} = \epsilon_{ijk}\frac{\partial A_{kl}}{\partial x_j}\mathbf{e}_i \otimes \mathbf{e}_l. \tag{A.27}$$

Explicit expressions for the Laplace operator acting on scalar and vector fields ψ and $\mathbf{a}$ are

$$\nabla^2\psi = \operatorname{div}\operatorname{grad}\psi = \frac{\partial^2\psi}{\partial x_i \partial x_i} = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}, \tag{A.28}$$

$$\nabla^2\mathbf{a} = (\operatorname{div}\operatorname{grad})\mathbf{a} = \frac{\partial}{\partial x_i}\frac{\partial a_j}{\partial x_k}(\mathbf{e}_j \otimes \mathbf{e}_k)\cdot\mathbf{e}_i = \frac{\partial^2 a_j}{\partial x_i \partial x_i}\mathbf{e}_j, \tag{A.29}$$

and similarly for the Laplacian of a tensor.

A.2.2 Cylindrical Polar Coordinates

In cylindrical polar coordinates (r, θ, z) the position vector $\mathbf{x}$ can be written in component form as

$$\mathbf{x} = r\mathbf{e}_r + z\mathbf{e}_z, \tag{A.30}$$

where r and z are the coordinates corresponding to the basis vectors $\mathbf{e}_r$ and $\mathbf{e}_z$, and the third basis vector is $\mathbf{e}_\theta = \mathbf{e}_z \times \mathbf{e}_r$. The basis vectors are given in terms of the fixed Cartesian basis by

$$\mathbf{e}_r = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2, \quad \mathbf{e}_\theta = -\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2, \quad \mathbf{e}_z = \mathbf{e}_3, \tag{A.31}$$

from which it follows that

$$\frac{\partial\mathbf{e}_r}{\partial\theta} = \mathbf{e}_\theta, \quad \frac{\partial\mathbf{e}_\theta}{\partial\theta} = -\mathbf{e}_r. \tag{A.32}$$

To derive the component form of the differential operators we use the gradient operator in the form

$$\operatorname{grad} = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_z\frac{\partial}{\partial z}. \tag{A.33}$$

When applied to the scalar and vector fields ψ and $\mathbf{a} = a_r\mathbf{e}_r + a_\theta\mathbf{e}_\theta + a_z\mathbf{e}_z$, this yields

$$\operatorname{grad}\psi = \frac{\partial\psi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\mathbf{e}_\theta + \frac{\partial\psi}{\partial z}\mathbf{e}_z, \tag{A.34}$$

and, according to our convention,

$$\begin{aligned}\operatorname{grad}\mathbf{a} &= \frac{\partial}{\partial r}(a_r\mathbf{e}_r + a_\theta\mathbf{e}_\theta + a_z\mathbf{e}_z)\otimes\mathbf{e}_r\\ &\quad + \frac{1}{r}\frac{\partial}{\partial\theta}(a_r\mathbf{e}_r + a_\theta\mathbf{e}_\theta + a_z\mathbf{e}_z)\otimes\mathbf{e}_\theta\\ &\quad + \frac{\partial}{\partial z}(a_r\mathbf{e}_r + a_\theta\mathbf{e}_\theta + a_z\mathbf{e}_z)\otimes\mathbf{e}_z,\end{aligned} \tag{A.35}$$

which expands, with the help of (A.32), into the form

$$\begin{aligned}\operatorname{grad}\mathbf{a} &= \frac{\partial a_r}{\partial r}\mathbf{e}_r\otimes\mathbf{e}_r + \frac{1}{r}\left(\frac{\partial a_r}{\partial\theta} - a_\theta\right)\mathbf{e}_r\otimes\mathbf{e}_\theta + \frac{\partial a_r}{\partial z}\mathbf{e}_r\otimes\mathbf{e}_z\\ &\quad + \frac{\partial a_\theta}{\partial r}\mathbf{e}_\theta\otimes\mathbf{e}_r + \frac{1}{r}\left(a_r + \frac{\partial a_\theta}{\partial\theta}\right)\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \frac{\partial a_\theta}{\partial z}\mathbf{e}_\theta\otimes\mathbf{e}_z\\ &\quad + \frac{\partial a_z}{\partial r}\mathbf{e}_z\otimes\mathbf{e}_r + \frac{1}{r}\frac{\partial a_z}{\partial\theta}\mathbf{e}_z\otimes\mathbf{e}_\theta + \frac{\partial a_z}{\partial z}\mathbf{e}_z\otimes\mathbf{e}_z.\end{aligned} \tag{A.36}$$

We do not include the corresponding gradient operation on a second-order tensor since it is not needed in this text (and it has 27 components).

The divergence operator applied to vector and tensor fields gives

$$\operatorname{div}\mathbf{a} = \frac{\partial a_r}{\partial r} + \frac{a_r}{r} + \frac{1}{r}\frac{\partial a_\theta}{\partial\theta} + \frac{\partial a_z}{\partial z}, \tag{A.37}$$

$$\begin{aligned}\operatorname{div}\mathbf{A} &= \left[\frac{\partial A_{rr}}{\partial r} + \frac{1}{r}\left(\frac{\partial A_{\theta r}}{\partial\theta} + A_{rr} - A_{\theta\theta}\right) + \frac{\partial A_{zr}}{\partial z}\right]\mathbf{e}_r\\ &\quad + \left[\frac{\partial A_{r\theta}}{\partial r} + \frac{1}{r}\left(\frac{\partial A_{\theta\theta}}{\partial\theta} + A_{r\theta} + A_{\theta r}\right) + \frac{\partial A_{z\theta}}{\partial z}\right]\mathbf{e}_\theta\\ &\quad + \left[\frac{\partial A_{rz}}{\partial r} + \frac{1}{r}\left(\frac{\partial A_{\theta z}}{\partial\theta} + A_{rz}\right) + \frac{\partial A_{zz}}{\partial z}\right]\mathbf{e}_z,\end{aligned}$$

where we have again used the convention that the divergence operates on the first index of the tensor $\mathbf{A}$.

Using (A.6) and (A.26) together with (A.36) we obtain

$$\operatorname{curl}\mathbf{a} = \left(\frac{1}{r}\frac{\partial a_z}{\partial\theta} - \frac{\partial a_\theta}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r}\right)\mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial(ra_\theta)}{\partial r} - \frac{\partial a_r}{\partial\theta}\right)\mathbf{e}_z. \tag{A.38}$$

In cylindrical coordinates the Laplace operator acting on the scalar and vector fields ψ and $\mathbf{a}$ has the forms

$$\nabla^2\psi = \operatorname{div}\operatorname{grad}\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2}, \tag{A.39}$$

and

$$\begin{aligned}\nabla^2\mathbf{a} = (\operatorname{div}\operatorname{grad})\mathbf{a} &= \left(\nabla^2 a_r - \frac{2}{r^2}\frac{\partial a_\theta}{\partial\theta} - \frac{a_r}{r^2}\right)\mathbf{e}_r \\ &\quad + \left(\nabla^2 a_\theta + \frac{2}{r^2}\frac{\partial a_r}{\partial\theta} - \frac{a_\theta}{r^2}\right)\mathbf{e}_\theta + (\nabla^2 a_z)\mathbf{e}_z. \end{aligned}\tag{A.40}$$

A.2.3 Spherical Polar Coordinates

We now use spherical polar coordinates (r, θ, ϕ) with corresponding unit basis vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$. The location $\mathbf{x}$ of a point in spherical polar coordinates is simply $\mathbf{x} = r\mathbf{e}_r$. Then, in terms of the background Cartesian basis vectors we have

$$\begin{aligned}\mathbf{e}_r &= \sin\theta\cos\phi\,\mathbf{e}_1 + \sin\theta\sin\phi\,\mathbf{e}_2 + \cos\theta\,\mathbf{e}_3, \\ \mathbf{e}_\theta &= \cos\theta\cos\phi\,\mathbf{e}_1 + \cos\theta\sin\phi\,\mathbf{e}_2 - \sin\theta\,\mathbf{e}_3, \\ \mathbf{e}_\phi &= -\sin\phi\,\mathbf{e}_1 + \cos\phi\,\mathbf{e}_2,\end{aligned}\tag{A.41}$$

from which we obtain

$$\begin{aligned}&\frac{\partial\mathbf{e}_r}{\partial\theta} = \mathbf{e}_\theta, \quad \frac{\partial\mathbf{e}_\theta}{\partial\theta} = -\mathbf{e}_r, \\ &\frac{\partial\mathbf{e}_r}{\partial\phi} = \sin\theta\,\mathbf{e}_\phi, \quad \frac{\partial\mathbf{e}_\theta}{\partial\phi} = \cos\theta\,\mathbf{e}_\phi, \quad \frac{\partial\mathbf{e}_\phi}{\partial\phi} = -\sin\theta\,\mathbf{e}_r - \cos\theta\,\mathbf{e}_\theta.\end{aligned}\tag{A.42}$$

In spherical coordinates the gradient operator is

$$\operatorname{grad} = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}, \tag{A.43}$$

and the gradient of the scalar field ψ is

$$\operatorname{grad}\psi = \nabla\psi = \frac{\partial\psi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\mathbf{e}_\phi. \tag{A.44}$$

For a vector field $\mathbf{a} = a_r \mathbf{e}_r + a_\theta \mathbf{e}_\theta + a_\phi \mathbf{e}_\phi$, we have

$$\begin{aligned}
\operatorname{grad}\mathbf{a} &= \frac{\partial}{\partial r}\left(a_r \mathbf{e}_r + a_\theta \mathbf{e}_\theta + a_\phi \mathbf{e}_\phi\right) \otimes \mathbf{e}_r \\
&\quad + \frac{1}{r}\frac{\partial}{\partial \theta}\left(a_r \mathbf{e}_r + a_\theta \mathbf{e}_\theta + a_\phi \mathbf{e}_\phi\right) \otimes \mathbf{e}_\theta \\
&\quad + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(a_r \mathbf{e}_r + a_\theta \mathbf{e}_\theta + a_\phi \mathbf{e}_\phi\right) \otimes \mathbf{e}_\phi ,
\end{aligned}$$

which, with the help of (A.42), expands out as

$$\begin{aligned}
&\frac{\partial a_r}{\partial r}\mathbf{e}_r \otimes \mathbf{e}_r + \frac{1}{r}\left(\frac{\partial a_r}{\partial \theta} - a_\theta\right)\mathbf{e}_r \otimes \mathbf{e}_\theta + \frac{1}{r\sin\theta}\left(\frac{\partial a_r}{\partial \phi} - a_\phi \sin\theta\right)\mathbf{e}_r \otimes \mathbf{e}_\phi \\
&+\frac{\partial a_\theta}{\partial r}\mathbf{e}_\theta \otimes \mathbf{e}_r + \frac{1}{r}\left(\frac{\partial a_\theta}{\partial \theta} + a_r\right)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \frac{1}{r\sin\theta}\left(\frac{\partial a_\theta}{\partial \phi} - a_\phi \cos\theta\right)\mathbf{e}_\theta \otimes \mathbf{e}_\phi \\
&+\frac{\partial a_\phi}{\partial r}\mathbf{e}_\phi \otimes \mathbf{e}_r + \frac{1}{r}\frac{\partial a_\phi}{\partial \theta}\mathbf{e}_\phi \otimes \mathbf{e}_\theta + \frac{1}{r\sin\theta}\left(\frac{\partial a_\phi}{\partial \phi} + a_\theta \cos\theta + a_r \sin\theta\right)\mathbf{e}_\phi \otimes \mathbf{e}_\phi .
\end{aligned}$$

The divergence operator applied to $\mathbf{a}$ is

$$\operatorname{div}\mathbf{a} = \operatorname{tr}(\operatorname{grad}\mathbf{a}) = \frac{1}{r^2}\frac{\partial(r^2 a_r)}{\partial r} + \frac{1}{r}\frac{\partial a_\theta}{\partial \theta} + \frac{\cot\theta}{r}a_\theta + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial \phi}. \tag{A.45}$$

The divergence operation on a second-order tensor $\mathbf{A}$ yields

$$\begin{aligned}
&\operatorname{div}\mathbf{A} \\
&= \frac{1}{r}\left[r\frac{\partial A_{rr}}{\partial r} + \frac{\partial A_{\theta r}}{\partial \theta} + 2A_{rr} - A_{\theta\theta} - A_{\phi\phi} + \frac{1}{\sin\theta}\left(\frac{\partial A_{\phi r}}{\partial \phi} + A_{\theta r}\cos\theta\right)\right]\mathbf{e}_r \\
&+\frac{1}{r}\left[r\frac{\partial A_{r\theta}}{\partial r} + \frac{\partial A_{\theta\theta}}{\partial \theta} + 2A_{r\theta} + A_{\theta r} + \frac{1}{\sin\theta}\left(\frac{\partial A_{\phi\theta}}{\partial \phi} + \left(A_{\theta\theta} - A_{\phi\phi}\right)\cos\theta\right)\right]\mathbf{e}_\theta \\
&+\frac{1}{r}\left[r\frac{\partial A_{r\phi}}{\partial r} + \frac{\partial A_{\theta\phi}}{\partial \theta} + 2A_{r\phi} + A_{\phi r} + \frac{1}{\sin\theta}\left(\frac{\partial A_{\phi\phi}}{\partial \phi} + \left(A_{\theta\phi} + A_{\phi\theta}\right)\cos\theta\right)\right]\mathbf{e}_\phi .
\end{aligned}$$

To evaluate the curl of $\mathbf{a}$ we use (A.6) and (A.26) with the expression for $\operatorname{grad}\mathbf{a}$ above to obtain, with the help of (A.42),

$$\begin{aligned}
\operatorname{curl}\mathbf{a} &= \frac{1}{r\sin\theta}\left(\frac{\partial(a_\phi \sin\theta)}{\partial \theta} - \frac{\partial a_\theta}{\partial \phi}\right)\mathbf{e}_r \\
&\quad + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial a_r}{\partial \phi} - \frac{\partial (r a_\phi)}{\partial r}\right)\mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial (r a_\theta)}{\partial r} - \frac{\partial a_r}{\partial \theta}\right)\mathbf{e}_\phi .
\end{aligned}$$

In spherical coordinates the Laplace operator acting on the scalar field ψ is

$$\nabla^2\psi = \operatorname{div}\operatorname{grad}\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}.$$

The corresponding result for a vector $\mathbf{a}$ is

$$\begin{aligned}\nabla^2\mathbf{a} = &\left(\nabla^2 a_r - \frac{2a_r}{r^2} - \frac{2}{r^2}\frac{\partial a_\theta}{\partial\theta} - \frac{2\cot\theta\, a_\theta}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial a_\phi}{\partial\phi}\right)\mathbf{e}_r\\ &+\left(\nabla^2 a_\theta + \frac{2}{r^2}\frac{\partial a_r}{\partial\theta} - \frac{(1+\cot^2\theta)\,a_\theta}{r^2} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial a_\phi}{\partial\phi}\right)\mathbf{e}_\theta\\ &+\left(\nabla^2 a_\phi + \frac{2}{r^2\sin\theta}\frac{\partial a_r}{\partial\phi} + \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial a_\theta}{\partial\phi} - \frac{a_\phi}{r^2\sin^2\theta}\right)\mathbf{e}_\phi.\end{aligned} \tag{A.46}$$

A.3 Integral Theorems

Here we summarize, using coordinate-free notation, some of the most important integral theorems that are essential to the theory presented in this text.

Consider a volume $\mathcal{V}$ within our three-dimensional Euclidean space and let $\partial\mathcal{V}$ denote its boundary. Also, let $\mathcal{S}$ denote an open surface and $\partial\mathcal{S}$ its bounding closed curve. In each case we assume sufficient regularity for applicability of the theorems. We again denote a vector field by $\mathbf{a}$. The standard forms of the divergence and Stokes' theorems are then

$$\int_{\mathcal{V}} \operatorname{div}\mathbf{a}\,\mathrm{d}v = \int_{\partial\mathcal{V}} \mathbf{a}\cdot\mathbf{n}\,\mathrm{d}s, \tag{A.47}$$

and

$$\int_{\mathcal{S}} \mathbf{n}\cdot(\operatorname{curl}\mathbf{a})\,\mathrm{d}s = \int_{\partial\mathcal{S}} \mathbf{a}\cdot\mathrm{d}\mathbf{x}, \tag{A.48}$$

respectively, where $\mathbf{n}$ is an outward pointing unit vector on the surface $\partial\mathcal{V}$, while on $\mathcal{S}$ it is related to the direction of traversal of $\partial\mathcal{S}$ by the right-hand screw rule. Equivalently, these can also be written in tensor form as

$$\int_{\mathcal{V}} \operatorname{grad}\mathbf{a}\,\mathrm{d}v = \int_{\partial\mathcal{V}} \mathbf{a}\otimes\mathbf{n}\,\mathrm{d}s, \tag{A.49}$$

and

$$\int_{\mathcal{S}} (\mathbf{n}\times\operatorname{grad})\,\mathbf{a}\,\mathrm{d}s = \int_{\partial\mathcal{S}} \mathrm{d}\mathbf{x}\otimes\mathbf{a}. \tag{A.50}$$

Equally, the vector field **a** may be replaced by a tensor field **A** to give

$$\int_{\mathcal{V}} \operatorname{grad} \mathbf{A} \, dv = \int_{\partial \mathcal{V}} \mathbf{A} \otimes \mathbf{n} \, ds, \tag{A.51}$$

and

$$\int_{\mathcal{S}} (\mathbf{n} \times \operatorname{grad}) \, \mathbf{A} \, ds = \int_{\partial \mathcal{S}} d\mathbf{x} \otimes \mathbf{A}. \tag{A.52}$$

Equations (A.51) and (A.52) can be specialized by contracting over the first index of the tensor **A**. Contraction of (A.51) results in the standard divergence theorem for a second-order tensor

$$\int_{\mathcal{V}} \operatorname{div} \mathbf{A} \, dv = \int_{\partial \mathcal{V}} \mathbf{A}^{\mathrm{T}} \mathbf{n} \, ds, \tag{A.53}$$

where contraction of the first index on the left-hand side requires the use of the transpose of **A** on the right-hand side. Applying the same procedure to (A.52) gives

$$\int_{\mathcal{S}} \mathbf{n} \cdot (\operatorname{curl} \mathbf{A}) \, ds = \int_{\partial \mathcal{S}} \mathbf{A}^{\mathrm{T}} d\mathbf{x}. \tag{A.54}$$

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L. Dorfmann and R.W. Ogden, *Nonlinear Theory of Electroelastic and Magnetoelastic Interactions*, DOI 10.1007/978-1-4614-9596-3,