Distance Between Two Ellipses in 3D

David Eberly
Magic Software, Inc.
http://www.magic-software.com

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1 Introduction

An ellipse in 3D is represented by a center $\vec{C}$, unit–length axes $\vec{U}$ and $\vec{V}$ with corresponding axis lengths a and b, and a plane containing the ellipse, $\vec{N} \cdot (\vec{X} - \vec{C}) = 0$ where $\vec{N}$ is a unit length normal to the plane. The vectors $\vec{U}$, $\vec{V}$, and $\vec{N}$ form a right–handed orthonormal coordinate system (the matrix with these vectors as columns is orthonormal with determinant 1). The ellipse is parameterized as

$$\vec{X} = \vec{C} + a\cos(\theta)\vec{U} + b\sin(\theta)\vec{V}$$

for angles $\theta \in [0, 2\pi)$. The ellipse is also defined by the two polynomial equations

$$\begin{array}{l} \vec{N} \cdot (\vec{X} - \vec{C}) = 0 \\ (\vec{X} - \vec{C})^{\mathrm{T}} \left(\frac{\vec{U}\vec{U}^{\mathrm{T}}}{a^2} + \frac{\vec{V}\vec{V}^{\mathrm{T}}}{b^2} \right) (\vec{X} - \vec{C}) = 1 \end{array}$$

where the last equation is written as a quadratic form. The first equation defines a plane and the second equation defines an ellipsoid. The intersection of plane and ellipsoid is an ellipse.

2 Solution as Polynomial System

The two ellipses are $\vec{N}_0 \cdot (\vec{X} - \vec{C}_0) = 0$ and $(\vec{X} - \vec{C}_0)^{\mathrm{T}} A_0 (\vec{X} - \vec{C}_0) = 1$ where $A_0 = \vec{U}_0\vec{U}_0^{\mathrm{T}}/a_0^2 + \vec{V}_0\vec{V}_0^{\mathrm{T}}/b_0^2$ and $\vec{N}_1 \cdot (\vec{Y} - \vec{C}_1) = 0$ and $(\vec{Y} - \vec{C}_1)^{\mathrm{T}} A_1 (\vec{Y} - \vec{C}_1) = 1$ where $A_1 = \vec{U}_1\vec{U}_1^{\mathrm{T}}/a_1^2 + \vec{V}_1\vec{V}_1^{\mathrm{T}}/b_1^2$.

The problem is to minimize the squared distance $|\vec{Y} - \vec{Y}|^2$ subject to the four constraints mentioned above. The problem can be solved with the method of Lagrange multipliers. Introduce four new parameters, α, β, γ, and δ and minimize

$$\begin{array}{rcl} F(\vec{X}, \vec{Y}; \alpha, \beta, \gamma, \delta) & = & |\vec{X} - \vec{Y}|^2 + \alpha((\vec{X} - \vec{C}_0)^{\mathrm{T}} A_0 (\vec{X} - \vec{C}_0) - 1) + \beta(\vec{N}_0 \cdot (\vec{X} - \vec{C}_0) - 0) \\ & & +\gamma((\vec{Y} - \vec{C}_1)^{\mathrm{T}} A_1 (\vec{Y} - \vec{C}_1) - 1) + \delta(\vec{N}_1 \cdot (\vec{Y} - \vec{C}_1) - 0). \end{array}$$

Taking derivatives yields

$$
\begin{aligned}
F_{\vec{X}} &= 2(\vec{X}-\vec{Y}) + 2\alpha A_0(\vec{X}-\vec{C}_0) + \beta\vec{N}_0 \\
F_{\vec{Y}} &= -2(\vec{X}-\vec{Y}) + 2\gamma A_1(\vec{Y}-\vec{C}_1) + \delta\vec{N}_1 \\
F_\alpha &= (\vec{X}-\vec{C}_0)^{\mathrm{T}} A_0(\vec{X}-\vec{C}_0) - 1 \\
F_\beta &= \vec{N}_0 \cdot (\vec{X}-\vec{C}_0) \\
F_\gamma &= (\vec{Y}-\vec{C}_1)^{\mathrm{T}} A_1(\vec{Y}-\vec{C}_1) - 1 \\
F_\delta &= \vec{N}_1 \cdot (\vec{Y}-\vec{C}_1)
\end{aligned}
$$

Setting the last four equations to zero yields the four original constraints. Setting the first equation to the zero vector and multiplying by $(\vec{X}-\vec{C}_0)^{\mathrm{T}}$ yields

$$\alpha = -2(\vec{X}-\vec{C}_0)^{\mathrm{T}}(\vec{X}-\vec{Y}).$$

Setting the first equation to the zero vector and multiplying by $\vec{N}_0^{\mathrm{T}}$ yields

$$\beta = -2\vec{N}_0^{\mathrm{T}}(\vec{X}-\vec{Y}).$$

Similar manipulations of the second equation yield

$$\gamma = 2(\vec{Y}-\vec{C}_1)^{\mathrm{T}}(\vec{X}-\vec{Y})$$

and

$$\delta = 2\vec{N}_1^{\mathrm{T}}(\vec{X}-\vec{Y}).$$

The first two derivative equations become

$$
\begin{aligned}
M_0(\vec{X}-\vec{Y}) = \left(\vec{N}_0\vec{N}_0^{\mathrm{T}} + A_0(\vec{X}-\vec{C}_0)(\vec{X}-\vec{C}_0)^{\mathrm{T}} - I\right)(\vec{X}-\vec{Y}) = \vec{0} \\
M_1(\vec{X}-\vec{Y}) = \left(\vec{N}_1\vec{N}_1^{\mathrm{T}} + A_1(\vec{Y}-\vec{C}_1)(\vec{Y}-\vec{C}_1)^{\mathrm{T}} - I\right)(\vec{X}-\vec{Y}) = \vec{0}
\end{aligned}
$$

Observe that $M_0\vec{N}_0 = \vec{0}$, $M_0A_0(\vec{X}-\vec{C}_0) = \vec{0}$, and $M_0(\vec{N}_0 \times (\vec{X}-\vec{C}_0)) = -\vec{N}_0 \times (\vec{X}-\vec{C}_0)$. Therefore, $M_0 = -\vec{W}_0\vec{W}_0^{\mathrm{T}}/|\vec{W}_0|^2$ where $\vec{W}_0 = \vec{N}_0 \times (\vec{X}-\vec{C}_0)$. Similarly, $M_1 = -\vec{W}_1\vec{W}_1^{\mathrm{T}}/|\vec{W}_1|^2$ where $\vec{W}_1 = \vec{N}_1 \times (\vec{Y}-\vec{C}_1)$. The previous displayed equations are equivalent to $\vec{W}_0^{\mathrm{T}}(\vec{X}-\vec{Y}) = 0$ and $\vec{W}_1^{\mathrm{T}}(\vec{X}-\vec{Y}) = 0$.

The points $\vec{X} = (x_0, x_1, x_2)$ and $\vec{Y} = (y_0, y_1, y_2)$ that attain minimum distance between the two ellipses are solutions to the six quadratic equations in six unknowns,

$$
\begin{aligned}
p_0(x_0,x_1,x_2) &= \vec{N}_0 \cdot (\vec{X}-\vec{C}_0) = 0, \\
p_1(x_0,x_1,x_2) &= (\vec{X}-\vec{C}_0)^{\mathrm{T}} A_0(\vec{X}-\vec{C}_0) = 1, \\
p_2(x_0,x_1,x_2,y_0,y_1,y_2) &= (\vec{X}-\vec{Y}) \cdot \vec{N}_0 \times (\vec{X}-\vec{C}_0) = 0, \\
q_0(y_0,y_1,y_2) &= \vec{N}_1 \cdot (\vec{Y}-\vec{C}_1) = 0, \\
q_1(y_0,y_1,y_2) &= (\vec{Y}-\vec{C}_1)^{\mathrm{T}} A_1(\vec{Y}-\vec{C}_1) = 1, \\
q_2(x_0,x_1,x_2,y_0,y_1,y_2) &= (\vec{X}-\vec{Y}) \cdot \vec{N}_1 \times (\vec{Y}-\vec{C}_1) = 0.
\end{aligned}
$$

On a computer algebra system that supports the resultant operation for eliminating polynomial variables, the following set of operations leads to a polynomial in one variable. Let Resultant$[P, Q, z]$ denote the resultant of polynomials P and Q where the variable z is eliminated,

$$
\begin{aligned}
r_0(x_0, x_1, y_0, y_1, y_2) &= \text{Resultant}[p_0, p_2, x_2] \\
r_1(x_0, x_1) &= \text{Resultant}[p_1, p_2, x_2] \\
r_2(x_0, x_1, y_0, y_1) &= \text{Resultant}[r_0, q_2, y_2] \\
s_0(x_0, x_1, x_2, y_0, y_1) &= \text{Resultant}[q_0, q_2, y_2] \\
s_1(y_0, y_1) &= \text{Resultant}[q_1, q_2, y_2] \\
s_2(x_0, x_1, y_0, y_1) &= \text{Resultant}[s_0, p_2, x_2] \\
r_3(x_0, y_0, x_1) &= \text{Resultant}[r_2, r_1, x_1] \\
r_4(x_0, y_0) &= \text{Resultant}[r_3, s_1, y_1] \\
s_3(x_0, x_1, y_0) &= \text{Resultant}[s_2, s_1, y_1] \\
s_4(x_0, y_0) &= \text{Resultant}[s_3, r_1, x_1] \\
\phi(x_0) &= \text{Resultant}[r_4, s_4, y_0]
\end{aligned}
$$

For two circles, the degree of ϕ is 8. For a circle and an ellipse, the degree of ϕ is 12. For two ellipses, the degree of ϕ is 16.

3 Solution using Trigonometric Approach

Let the two ellipses be

$$
\begin{aligned}
\vec{X} &= \vec{C}_0 + a_0 \cos(\theta)\vec{U}_0 + b_0 \sin(\theta)\vec{V}_0 \\
\vec{Y} &= \vec{C}_1 + a_1 \cos(\phi)\vec{U}_1 + b_1 \sin(\phi)\vec{V}_1
\end{aligned}
$$

for $\theta \in [0, 2\pi)$ and $\phi \in [0, 2\pi)$. The squared distance between any two points on the ellipses is $F(\theta, \phi) = |\vec{X}(\theta) - \vec{Y}(\phi)|^2$. The problem is to minimize $F(\theta, \phi)$.

Define $c_0 = \cos(\theta)$, $s_0 = \sin(\theta)$, $c_1 = \cos(\phi)$, and $s_1 = \sin(\phi)$. Compute derivatives, $F_\theta = (\vec{X}(\theta) - \vec{Y}(\phi)) \cdot \vec{X}'(\theta)$ and $F_\phi = -(\vec{X}(\theta) - \vec{Y}(\phi)) \cdot \vec{Y}'(\phi)$. Setting these equal to zero leads to the two polynomial equations in c_0, s_0, c_1, and s_1. The two polynomial constraints for the sines and cosines are also listed.

$$
\begin{aligned}
p_0 &= (a_0^2 - b_0^2)s_0 c_0 + a_0(\alpha_{00} + \alpha_{01}s_1 + \alpha_{02}c_1)s_0 + b_0(\beta_{00} + \beta_{01}s_1 + \beta_{02}c_1)c_0 = 0 \\
p_1 &= (a_1^2 - b_1^2)s_1 c_1 + a_1(\alpha_{10} + \alpha_{11}s_0 + \alpha_{12}c_0)s_1 + b_1(\beta_{10} + \beta_{11}s_0 + \beta_{12}c_0)c_1 = 0 \\
q_0 &= s_0^2 + c_0^2 - 1 = 0 \\
q_1 &= s_1^2 + c_1^2 - 1 = 0
\end{aligned}
$$

This is a system of four quadratic polynomial equations in four unknowns and can be solved with resultants:

$$
\begin{aligned}
r_0(s_0, s_1, c_1) &= \text{Resultant}[p_0, q_0, c_0] \\
r_1(s_0, s_1, c_0) &= \text{Resultant}[p_1, q_1, c_1] \\
r_2(s_0, s_1) &= \text{Resultant}[r_0, q_1, c_1] \\
r_3(s_0, s_1) &= \text{Resultant}[r_1, q_0, c_0] \\
\phi(s_0) &= \text{Resultant}[r_2, r_3, s_1]
\end{aligned}
$$

Alternatively, we can use the simple nature of q_0 and q_1 to do some of the elimination. Let $p_0 = \alpha_0 s_0 + \beta_0 c_0 + \gamma_0 s_0 c_0$ where α_0 and β_0 are linear polynomials in s_1 and c_1. Similarly, $p_1 = \alpha_1 s_1 + \beta_1 c_1 + \gamma_1 s_1 c_1$ where α_1 and β_1 are linear polynomials in s_0 and c_0. Solving for c_0 in $p_0 = 0$ and c_1 in $p_1 = 0$, squaring, and using the q_i constraints leads to

$$
\begin{gathered}
r_0 = (1 - s_0^2)(\gamma_0 s_0 + \beta_0)^2 - \alpha_0^2 s_0^2 = 0 \\
r_1 = (1 - s_1^2)(\gamma_1 s_1 + \beta_1)^2 - \alpha_1^2 s_1^2 = 0
\end{gathered}
$$

Using the q_i constraints, write $r_i = r_i 0 + r_i 1 c_{1-i}$, $i = 0, 1$, where the r_{ij} are polynomials in s_0 and s_1. The terms $r_i 0$ are degree 4 and the terms r_{i1} is degree 3. Solving for c_0 in $r_1 = 0$ and c_1 in $r_0 = 0$, squaring, and using the q_i constraints leads to

$$
\begin{gathered}
w_0 = (1 - s_1^2) r_{01}^2 - r_{00}^2 = \textstyle\sum_{j=0}^{8} w_{0j} s_0^j = 0 \\
w_1 = (1 - s_0^2) r_{11}^2 - r_{10}^2 = \textstyle\sum_{j=0}^{4} w_{1j} s_0^j = 0
\end{gathered}
$$

The coefficients w_{ij} are polynomials in s_1. The degrees of w_{00} through w_{08} respectively are 4, 3, 4, 3, 4, 3, 2, 1, and 0. The degree of w_{1j} is $8 - j$. Total degree for each of w_i is 8.

The final elimination can be computed using a Bezout determinant, $\phi(s_1) = \det[e_{ij}]$, where the underlying matrix is $8x8$ and the entry is

$$
e_{ij} = \sum_{k=\max(9-i, 9-j)}^{\min(8, 17-i-j)} v_{k, 17-i-j-k}
$$

where $v_{i,j} = w_{0i} w_{1j} - w_{0j} w_{1i}$. If the i or j index is out of range in the w terms, then the term is assumed to be zero. The solutions to $\phi = 0$ are the candidate points for s_1. For each s_1, two c_1 values are computed using $s_1^2 + c_1^2 = 1$. For each s_1, the roots of the polynomial $w_1(s_0)$ are computed. For each s_0, two c_0 values are computed using $s_0^2 + c_0^2 = 1$. Out of all such candidates, $|\vec{X} - \vec{Y}|^2$ can be computed and the minimum value is selected.

4 Numerical Solution

Neither algebraic method above seems reasonable. Each looks very slow to compute and you have the usual numerical problems with polynomials of large degree. I have not implemented the following, but my guess is it is an alternative to consider. Implement a distance calculator for point to ellipse (in 3D). This is a function of a single parameter, say $F(\theta)$ for $\theta \in [0, 2\pi]$. Use a numerical minimizer that does not require derivative calculation (Powell's method for example) and minimize F on the interval $[0, 2\pi]$. The scheme is iterative and hopefully converges rapidly to the solution.