Distance Between Line and Circle or Disk in 3D

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1 Line and Circle

A circle in 3D is represented by a center $\vec{C}$, a radius R, and a plane containing the circle, $\vec{N} \cdot (\vec{X} - \vec{C}) = 0$ where $\vec{N}$ is a unit length normal to the plane. If $\vec{U}$ and $\vec{V}$ are also unit length vectors so that $\vec{U}$, $\vec{V}$, and $\vec{N}$ form a right–handed orthonormal coordinate system (the matrix with these vectors as columns is orthonormal with determinant 1), then the circle is parameterized as

$$\vec{X} = \vec{C} + R(\cos(\theta)\vec{U} + \sin(\theta)\vec{V}) =: \vec{C} + R\vec{W}(\theta)$$

for angles $\theta \in [0, 2\pi)$. Note that $|\vec{X} - \vec{C}| = R$, so the $\vec{X}$ values are all equidistant from $\vec{C}$. Moreover, $\vec{N} \cdot (\vec{X} - \vec{C}) = 0$ since $\vec{U}$ and $\vec{V}$ are perpendicular to $\vec{N}$, so the $\vec{X}$ lie in the plane.

From the paper on distance between point and line, the minimum distance between point $\vec{P}$ and circle is

$$F = R^2 + |\vec{C} - \vec{P}|^2 - 2R\vec{W} \cdot (\vec{P} - \vec{C})$$

where $\vec{W} = (\vec{Q} - \vec{C})/|\vec{Q} - \vec{C}|$ and $\vec{Q} - \vec{C} = \vec{P} - \vec{C} - (\vec{N} \cdot (\vec{P} - \vec{C}))\vec{N}$. Now if $\vec{P} = \vec{B} + t\vec{M}$, then F is a function of t. The problem is to minimize F for all $t \in \mathbb{R}$. All of $\vec{P}$, $\vec{W}$, and F depend on t.

Since $\vec{W}$ is always unit length, $\vec{W} \cdot \vec{W}' = 0$. Since $\vec{N} \cdot \vec{W} = 0$, it must be that $\vec{N} \cdot \vec{W}' = 0$. Therefore $0 = \vec{W}' \cdot (\vec{Q} - \vec{C}) = \vec{W}' \cdot (\vec{P} - \vec{C}) - (\vec{N} \cdot (\vec{P} - \vec{C}))\vec{W}' \cdot \vec{N} = \vec{W}' \cdot (\vec{P} - \vec{C})$. Using these conditions in the derivative calculation of F yields

$$F'(t) = 2(\vec{P} - \vec{C}) \cdot \vec{P}' - 2R[\vec{W} \cdot \vec{P}' + \vec{W}' \cdot (\vec{P} - \vec{C})] = 2(\vec{P} - \vec{C}) \cdot \vec{P}' - 2R\vec{W} \cdot \vec{P}'.$$

Defining $\vec{D} = \vec{B} - \vec{C}$ and setting $F'(t) = 0$ yields

$$(\vec{M} \cdot \vec{M})t + \vec{M} \cdot \vec{D} = R^2 \frac{\vec{M} \cdot (\vec{Q} - \vec{C})}{|\vec{Q} - \vec{C}|}.$$

Now consider

$$\begin{aligned} \vec{Q} - \vec{C} &= t\vec{M} + \vec{D} - (t\vec{N} \cdot \vec{M} + \vec{N} \cdot \vec{D})\vec{N} \\ &= (\vec{M} - (\vec{N} \cdot \vec{M})\vec{N})t + (\vec{D} - (\vec{N} \cdot \vec{D})\vec{N} \\ &= \vec{\alpha}t + \vec{\beta}. \end{aligned}$$

So

$$|\vec{Q} - \vec{C}|^2 = (\vec{\alpha} \cdot \vec{\alpha})t^2 + 2(\vec{\alpha} \cdot \vec{\beta})t + (\vec{\beta} \cdot \vec{\beta})$$

and

$$\vec{M} \cdot (\vec{Q} - \vec{C}) = (\vec{M} \cdot \vec{\alpha})t + (\vec{M} \cdot \vec{\beta}).$$

Squaring the equation obtained by setting $F'(t) = 0$ yields

$$[(\vec{M} \cdot \vec{M})t + \vec{M} \cdot \vec{D}]^2 = R^2 \frac{[(\vec{M} \cdot \vec{\alpha})t + (\vec{M} \cdot \vec{\beta})]^2}{(\vec{\alpha} \cdot \vec{\alpha})t^2 + 2(\vec{\alpha} \cdot \vec{\beta})t + (\vec{\beta} \cdot \vec{\beta})}.$$

Further computations show that

$$\begin{aligned}
\vec{M} \cdot \vec{\alpha} &= \vec{M} \cdot \vec{M} - (\vec{N} \cdot \vec{M})^2 \\
\vec{M} \cdot \vec{\beta} &= \vec{M} \cdot \vec{D} - (\vec{N} \cdot \vec{M})(\vec{N} \cdot \vec{D}) \\
\vec{\alpha} \cdot \vec{\alpha} &= \vec{M} \cdot \vec{\alpha} \\
\vec{\alpha} \cdot \vec{\beta} &= \vec{M} \cdot \vec{\beta} \\
\vec{\beta} \cdot \vec{\beta} &= \vec{D} \cdot \vec{D} - (\vec{N} \cdot \vec{D})^2
\end{aligned}$$

Setting $a_0 = \vec{M} \cdot \vec{D}$, $a_1 = \vec{M} \cdot \vec{M}$, $b_0 = \vec{M} \cdot \vec{D} - (\vec{N} \cdot \vec{M})(\vec{N} \cdot \vec{D})$, $b_1 = \vec{M} \cdot \vec{M} - (\vec{N} \cdot \vec{M})^2$, $c_0 = \vec{D} \cdot \vec{D} - (\vec{N} \cdot \vec{D})^2$, $c_1 = b_0$, and $c_2 = b_1$, the zero derivative condition becomes

$$(a_0 + a_1 t)^2 (c_0 + 2c_1 t + c_2 t^2) = R^2 (b_0 + b_1 t)^2.$$

Expanding and collecting terms yields

$$d_0 + d_1 t + d_2 t^2 + d_3 t^3 + d_4 t^4 = 0$$

where

$$\begin{aligned}
d_0 &= a_0^2 c_0 - b_0^2 R^2 \\
d_1 &= 2(a_0 a_1 c_0 + a_0^2 c_1 - b_0 b_1 R^2) \\
d_2 &= a_1^2 c_0 + 4a_0 a_1 c_1 + a_0^2 c_2 - b_1^2 R^2 \\
d_3 &= 2(a_1^2 c_1 + a_0 a_1 c_2) \\
d_4 &= a_1^2 c_2
\end{aligned}$$

The roots of this quartic are found by an standard root finder. For each root $\bar{t}$, the corresponding distance $F(\bar{t})$ is calculated. The minimum over all roots is the distance between line and circle.

2 Line and Disk

TO BE COMPLETED. The idea is to project the line onto the plane of the disk. If the line does not intersect disk, then use the line–to–circle distance algorithm. If the line does intersect disk, then the line is partitioned into two rays outside the disk and one line segment inside the disk. The minimum distance between the disk and that portion of the line whose projection is the line segment is just the minimum of the two distances measured from end points (on original line) and circle. These distances are just measured as distances from point to plane. The minimum distance between the disk and the rays uses the line–to–circle algorithm, except that if the minimum line point is not on the currently considered ray, then you also need to compute distance from ray origin to circle and compare to the distances obtained from the roots of the quartic. You get three minima, two for the rays and one for the segment. Choose the minimum of those minima.