Principal Curvatures of Surfaces

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1 Maxima of Quadratic Forms

Let A be an $n \times n$ symmetric matrix. The function $Q : \mathbb{R}^n \to \mathbb{R}$ defined by $Q(\vec{v}) = \vec{v}^t A\vec{v}$ for $|\vec{v}| = 1$ is called a *quadratic form*. Since Q is defined on the unit sphere in $\mathbb{R}^n$ (a compact set) and since Q is continuous, it must have a maximum and a minimum on this set. What is the maximum and at which vector is the maximum attained?

Let $\vec{v} = \sum_{i=1}^n c_i\vec{v}_i$ where $A\vec{v}_i = \lambda_i\vec{v}_i$, $\lambda_1 \le \cdots \le \lambda_n$, and $\sum_{i=1}^n c_i^2 = 1$. Then

$$Q(\vec{v}) = \left(\sum_{i=1}^n c_i\vec{v}_i^t\right) A \left(\sum_{j=1}^n c_j\vec{v}_j\right) = \sum_{i=1}^n\sum_{j=1}^n c_ic_j\vec{v}_i^t A\vec{v}_j = \sum_{k=1}^n \lambda_k c_k^2.$$

The right–most summation is a convex combination of the eigenvalues of A, so its maximum is λ_n and occurs when $c_n = 1$. Consequently, $\max Q(\vec{v}) = \lambda_n = Q(\vec{v}_n)$.

1.1 Maxima of Restricted Quadratic Forms

In some applications it is desirable to find the maximum of a quadratic form defined on the unit hypersphere S^{n-1}, but restricted to the intersection of this hypersphere with a hyperplane $\vec{N} \cdot \vec{x} = 0$ for some special normal vector $\vec{N}$. Let A be an $n \times n$ symmetric matrix. Let $\vec{N} \in \mathbb{R}^n$ be a unit length vector. Define $Q : \{\vec{N}\}^\perp \to \mathbb{R}$ (the domain is the orthogonal complement of $\vec{N}$) by $Q(\vec{v}) = \vec{v}^t A\vec{v}$ where $|\vec{v}| = 1$. Now Q is defined on the unit sphere in the $(n-1)$–dimensional space $\{\vec{N}\}^\perp$, so it must have a maximum and a minimum. What is the maximum and at which vector is this maximum attained?

Let $\vec{v}_1$ through $\vec{v}_{n-1}$ be an orthonormal basis for $\{\vec{N}\}^\perp$. Let $\vec{v} = \sum_{i=1}^{n-1} c_i\vec{v}_i$ where $\sum_{i=1}^n c_i^2 = 1$. Let $A\vec{v}_i = \sum_{j=1}^{n-1} \alpha_{ji}\vec{v}_j + \alpha_{ni}\vec{N}$ where $\alpha_{ji} = \vec{v}_j^t A\vec{v}_i$ for $1 \le i \le n-1$ and $1 \le j \le n-1$, and where $\alpha_{ni} = \vec{N}^t A\vec{v}_i$ for $1 \le i \le n-1$. Then

$$Q(\vec{v}) = \left(\sum_{i=1}^{n-1} c_i\vec{v}_i^t\right) A \left(\sum_{j=1}^{n-1} c_j\vec{v}_j\right) = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} c_ic_j\alpha_{ij} = \vec{c}^t\bar{A}\vec{c} =: P(\vec{c})$$

where quadratic form $P : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfies the conditions for the maximization in the previous sections. Thus, $\max Q(\vec{v}) = \max P(\vec{c})$ which occurs for $\vec{c}$ and λ such that $\bar{A}\vec{c} = \lambda\vec{c}$ and λ is the maximum eigenvalue of $\bar{A}$. So

$$
\begin{aligned}
&\sum_{j=1}^{n-1} \alpha_{ij} c_j = \lambda c_i \\
&\sum_{j=1}^{n-1} c_j \vec{v}_i = \lambda c_i \vec{v}_i \\
&\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \alpha_{ij} c_j \vec{v}_i = \lambda \sum_{i=1}^{n-1} c_i \vec{v}_i \\
&\sum_{j=1} \left(\sum_{i=1}^{n-1} \alpha_{ij} \vec{v}_i \right) c_j = \lambda \vec{v} \\
&\sum_{j=1}^{n-1} \left(A\vec{v}_j - \alpha_{nj} \vec{N} \right) c_j = \lambda \vec{v} \\
&A \left(\sum_{j=1}^{n-1} c_j \vec{v}_j \right) - \left(\sum_{j=1}^{n-1} \alpha_{nj} c_j \right) \vec{N} = \lambda \vec{v} \\
&A\vec{v} - \left(\vec{N}^t A \vec{v} \right) \vec{N} = \lambda \vec{v} \\
&(I - \vec{N}\vec{N}^t) A \vec{v} = \lambda \vec{v}.
\end{aligned}
$$

Therefore, $\max Q(\vec{v}) = \lambda_{n-1} = Q(\vec{v}_{n-1})$ where λ_{n-1} is the maximum eigenvalue corresponding to the eigenvector $\vec{v}_{n-1}$ of $(I - \vec{N}\vec{N}^t)A$. Note that $n-1$ of the eigenvectors are in $\{\vec{N}\}^\perp$. The remaining eigenvector is $\vec{v}_n = A^{\text{adj}}\vec{N}$ where $AA^{\text{adj}} = (\det A)I$ and $\lambda_n = 0$.

2 Curvatures of Implicit Surfaces (1)

Given an n–dimensional surface implicitly defined by $F(\vec{x}) \equiv 0$ where $F : \mathbb{R}^{n+1} \to \mathbb{R}$ and $\nabla F(\vec{x}) \neq \vec{0}$, find the curvature of the normal section determined by normal vector $\vec{N} = \nabla F / |\nabla F|$ and tangent vector $\vec{T}$ at $\vec{x}$.

Let $\vec{y}(t) = \vec{x} + t\vec{T} + a(t)\vec{N}$ where $a(0) = 0$ and $a'(0) = 0$. Then $\vec{y}'(t) = \vec{T} + a'(t)\vec{N}$ and $\vec{y}''(t) = a''(t)\vec{N}$. The curvature is the normal coefficient $\kappa_{\vec{T}}(\vec{x}) = a''(0)$. Differentiating $F(\vec{y}(t)) \equiv 0$ yields $\vec{y}'(t) \cdot \nabla F(\vec{y}(t)) \equiv 0$, so at $t = 0$, $\vec{T} \cdot \nabla F(\vec{x}) = 0$. Differentiating again yields

$$\vec{y}'(t) \cdot D^2 F(\vec{y}(t)) \vec{y}'(t) + \vec{y}''(t) \cdot \nabla F(\vec{y}(t)) = 0.$$

Evaluating at $t = 0$ yields

$$0 = \vec{T}^t D^2 F(\vec{x}) \vec{T} + \vec{y}''(0) \cdot \nabla F(\vec{x}) = \vec{T}^t D^2 N(\vec{x}) \vec{T} + \kappa_{\vec{T}}(\vec{x}) \vec{N} \cdot \nabla F(\vec{x}),$$

so

$$\kappa_{\vec{T}}(\vec{x}) = -\vec{T}^t \frac{D^2 F(\vec{x})}{|\nabla F(\vec{x})|} \vec{T}.$$

To find the maximal curvature, note that $\kappa_{\vec{T}}$ is a quadratic form restricted to the orthogonal complement of $\vec{N}$, so the results of the previous section apply. The maximum curvature is the maximum eigenvalue for the eigensystem

$$-\left(I - \frac{\nabla F}{|\nabla F|} \frac{\nabla F}{|\nabla F|}^t \right) \frac{D^2 F}{|\nabla F|} \vec{v} = \kappa \vec{v}.$$

The maximum is attained at the corresponding eigenvector $\vec{v}$.

3 Curvatures of Implicit Surfaces (2)

In standard differential geometry textbooks, hypersurfaces are described by a parameterization which is used in obtaining principal curvatures and principal directions. In computational vision applications, typically one obtains a surface as a collection of points with no underlying parameterization. Such surfaces are assumed to be implicitly defined, so we also want to construct principal curvatures and principal directions for surfaces defined as level sets of functions $F : \mathbb{R}^{n+1} \to \mathbb{R}$. Assume that F is a C^4 function for which $\nabla F \neq 0$. The normal vectors to the surface are $N = \nabla F/|\nabla F|$.

Construction With Parameterization. Let the surface be parameterized by position $x : \mathbb{R}^n \to \mathbb{R}^{n+1}$, say $x = x(u)$. Define $J = \partial x/\partial u$, an $(n+1)\times n$ matrix which has rank n and satisfies the property $N^t J = 0$. That is, the columns of J are a basis of the tangent space and are orthogonal to N at position $x(u)$. The first and second fundamental forms are given by the $n \times n$ matrices $\mathbf{I} = J^t J$ and $\mathbf{II} = -J^t \partial N/\partial u$, respectively. The matrix representing the shape operator on the tangent space is $S = \mathbf{I}^{-1}\mathbf{II}$. Consider the eigenvector problem $Sp = \kappa p$. Each eigenvector p is a principal direction. The corresponding eigenvalue κ is a principal curvature. The vector p is an n–vector given in terms of tangent space coordinates, but its representation in $\mathbb{R}^{n+1}$ is $\xi = Jp$.

Construction without Parameterization. Define $W = -\partial N/\partial x$, an $(n+1)\times(n+1)$ matrix. We claim that if $Sp = \kappa p$, then $\xi = Jp$ satisfies $W\xi = \kappa\xi$. Firstly, we have $\mathbf{I} = J^t J$. Secondly, by the chain rule we have $\partial N/\partial u = (\partial N/\partial x)J$, so $\mathbf{II} = J^t W J$. The eigenvector problem $(\mathbf{II} - \kappa\mathbf{I})p = 0$ is therefore transformed to $J^t(W - \kappa E)\xi = 0$, where E is the $(n+1)\times(n+1)$ identity matrix and where $\xi = Jp$.

Since J^t has full rank n, its generalized inverse is given by $(J^t)^+ = J(J^t J)^{-1}$. If p is a principal direction, that is $Sp = \kappa p$, then $WJp = JSp = \kappa JP$, so $\xi = Jp$ is an eigenvector of W with corresponding eigenvalue κ. Conversely, if ξ is an eigenvector of W, that is $W\xi = \kappa\xi$, and ξ is a tangent vector, say $\xi = Jp$, then $J^+\xi = p$, $JJ^+\xi = JP = \xi$, and

$$SJ^+\xi = (J^+WJ)J^+\xi = J^+W\xi = \kappa J^+\xi,$$

so $J^+\xi$ is a principal directions with corresponding principal curvature κ. Additionally, W has an identically zero eigenvalue, but the corresponding eigenvector is not a tangent vector. This follows from the identity $W = (E - NN^t)\,\mathrm{Hess}\,(F)/|\nabla F|$ which can be derived by explicitly computing $\partial N_i/\partial x_j$ for $N = \nabla F/|\nabla F|$. The eigenvector is $\mathrm{adj}(\,\mathrm{Hess}\,(F))\nabla F$ where adj indicates the adjoint of a matrix. A short computation shows that $W\,\mathrm{adj}(\,\mathrm{Hess}\,(f))\nabla F = 0$.

4 Curvatures of Graphs

Let $f : \mathbb{R}^n \to \mathbb{R}$, say $f = f(\vec{x})$ where $\vec{x} \in \mathbb{R}^n$. The graph of the function is parameterized by

$$\vec{y} = (\vec{x}, f(\vec{x})) \in \mathbb{R}^{n+1}.$$

Tangents to the graph are the derivatives of position,

$$\vec{T}_i = \frac{\partial \vec{y}}{\partial x_i} = \left(\vec{e}_i, \frac{\partial f}{\partial x_i}\right)$$

where $\vec{e}_i$ is the vector whose i^{th} component is 1 and all other components are 0. The normal to the graph is

$$\vec{N} = \frac{(\nabla f, -1)}{\sqrt{1 + |\nabla f|^2}}$$

where ∇f is the vector of first derivatives of f. Also note that

$$\frac{\partial^2 \vec{y}}{\partial x_i \partial x_j} = \left(\vec{0}, \frac{\partial^2 f}{\partial x_i \partial x_j}\right).$$

The metric tensor for the graph is

$$M_1 = \left[\vec{y}_{x_i} \cdot \vec{y}_{x_j}\right] = \left[\vec{e}_i \cdot \vec{e}_j + f_{x_i} f_{x_j}\right] = I + \nabla f \nabla f^t$$

where I is the $n \times n$ identity matrix. The curvature tensor for the graph is

$$M_2 = \left[-\vec{N} \cdot \vec{y}_{x_i x_j}\right] = \left[f_{x_i x_j} / \sqrt{1 + |\nabla f|^2}\right].$$

Principal curvatures and principal directions are the solutions to the eigensystem

$$M_2 \vec{v} = \kappa M_1 \vec{v}.$$

For the graph of f, the system is therefore

$$\frac{D^2 f}{\sqrt{1 + |\nabla f|^2}} \vec{v} = \kappa (I + \nabla f \nabla f^t) \vec{v}.$$