Perspective Mappings Between Cuboids

David Eberly
Magic Software, Inc.
http://www.magic-software.com

Created: March 2, 1999

1 Mapping Unit Square to Quadrilateral

The problem of mapping the unit square to a quadrilateral can be solved by considering the problem in a three dimensional setting. Translate one vertex of the quadrilateral to the origin (say this is $\vec{q}_{00}$). Label the other vertices in counterclockwise order as $\vec{q}_{10}$, $\vec{q}_{11}$, and $\vec{q}_{01}$. The plane containing these points is $z = 0$ and has normal $(0, 0, 1)$. Select an eye point $\vec{E} = (e_0, e_1, e_2)$. Rotate the plane of the quadrilateral so that its normal is $\vec{N} = (n_0, n_1, n_2)$. The quadrilateral can be projected onto the viewing plane $z = 0$ by a perspective projection. The idea is to choose an eye point $\vec{E}$ and normal $\vec{N}$ so that the projection is the unit square with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 1, 0)$.

Let $\vec{\imath} = (1, 0, 0)$, $\vec{\jmath} = (0, 1, 0)$, $\vec{k} = (0, 0, 1)$, and $\vec{\xi} = (x, y, 0)$. The perspective mapping involves finding the intersection of the line $(1 - t)\vec{\xi} + t\vec{E}$ with the plane $\vec{N} \cdot \vec{\xi} = 0$. Replacing the line equation in the plane equation, solving for t, the mapping is

$$(\bar{x}, \bar{y}, \bar{z}) = \frac{(\vec{N} \cdot \vec{\xi})\vec{E} - (\vec{N} \cdot \vec{E})\vec{\xi}}{\vec{N} \cdot (\vec{\xi} - \vec{E})}.$$

The four rays through the quadrilateral points on the plane $\vec{N} \cdot (x, y, z) = 0$ must intersect the four corners of the square. Let $\vec{p}_{10}$, $\vec{p}_{01}$, and $\vec{p}_{11}$ be the four quadrilateral points in that plane. Then

$$\begin{aligned}
\vec{p}_{10} &= \tfrac{(\vec{N}\cdot\vec{\imath})\vec{E}-(\vec{N}\cdot\vec{E})\vec{\imath}}{\vec{N}\cdot(\vec{\imath}-\vec{E})} \\
\vec{p}_{01} &= \tfrac{(\vec{N}\cdot\vec{\jmath})\vec{E}-(\vec{N}\cdot\vec{E})\vec{\jmath}}{\vec{N}\cdot(\vec{\jmath}-\vec{E})} \\
\vec{p}_{11} &= \tfrac{(\vec{N}\cdot(\vec{\imath}+\vec{\jmath}))\vec{E}-(\vec{N}\cdot\vec{E})(\vec{\imath}+\vec{\jmath})}{\vec{N}\cdot(\vec{\imath}+\vec{\jmath}-\vec{E})}
\end{aligned}$$

Also, $\vec{p}_{11} = \alpha\vec{p}_{10} + \beta\vec{p}_{01}$. The equation for $\vec{p}_{11}$ can be solved for α and β using the two equations for $\vec{p}_{10}$ and $\vec{p}_{01}$:

$$\begin{aligned}
\alpha &= \tfrac{\vec{N}\cdot(\vec{E}-\vec{\imath})}{\vec{N}\cdot(\vec{E}-\vec{\imath}-\vec{\jmath})} \\
\beta &= \tfrac{\vec{N}\cdot(\vec{E}-\vec{\jmath})}{\vec{N}\cdot(\vec{E}-\vec{\imath}-\vec{\jmath})}
\end{aligned}$$

The equations for α and β can be rewritten as

$$\vec{N} \cdot ((\alpha - 1)(\vec{E} - \vec{\imath}) - \alpha\vec{\jmath}) = 0$$

$$\vec{N} \cdot ((\beta - 1)(\vec{E} - \vec{\jmath}) - \beta\vec{\imath}) = 0$$

Vector $\vec{N}$ may be selected as the cross product of the two vectors which are perpendicular to it,

$$\vec{N} = ((\beta - 1)e_2, (\alpha - 1)e_2, (\alpha + \beta - 1) + (1 - \beta)e_0 + (1 - \alpha)e_1)\,.$$

Consequently,

$$\begin{aligned} \vec{N} \cdot (\vec{E} - \vec{i}) &= \alpha e_2 \\ \vec{N} \cdot (\vec{E} - \vec{j}) &= \beta e_2 \\ \vec{N} \cdot (\vec{E} - \vec{i} - \vec{j}) &= e_2 \\ \vec{N} \cdot \vec{E} &= (\alpha + \beta - 1)e_2 \end{aligned}$$

The general mapping is

$$\begin{aligned} \vec{p} &= \tfrac{[\vec{N}\cdot(x\vec{i}+y\vec{j})]\vec{E}-(\vec{N}\cdot\vec{E})(x\vec{i}+y\vec{j})}{\vec{N}\cdot(x\vec{i}+y\vec{j}-\vec{E})} \\ &= \tfrac{x[(\vec{N}\cdot\vec{i})\vec{E}-(\vec{N}\cdot\vec{E})\vec{i}]+y[(\vec{N}\cdot\vec{j})\vec{E}-(\vec{N}\cdot\vec{E})\vec{j}]}{\vec{N}\cdot(x\vec{i}+y\vec{j}-\vec{E})} \\ &= \tfrac{x\vec{N}\cdot(\vec{i}-\vec{E})\vec{p}_{10}+y\vec{N}\cdot(\vec{j}-\vec{E})\vec{p}_{01}}{\vec{N}\cdot(x\vec{i}+y\vec{j}-\vec{E})} \\ &= \tfrac{\alpha x}{(\alpha+\beta-1)+(1-\beta)x+(1-\alpha)y}\vec{p}_{10} + \tfrac{\beta x}{(\alpha+\beta-1)+(1-\beta)x+(1-\alpha)y}\vec{p}_{01}. \end{aligned}$$

Rotating the plane back to $z = 0$ and translating the origin back to the original vertex, the mapping is

$$\vec{q} - \vec{q}_{00} = \frac{\alpha x}{(\alpha + \beta - 1) + (1 - \beta)x + (1 - \alpha)y}(\vec{q}_{10} - \vec{q}_{00}) + \frac{\beta x}{(\alpha + \beta - 1) + (1 - \beta)x + (1 - \alpha)y}(\vec{q}_{01} - \vec{q}_{00}).$$

2 Mapping Unit Cube to Cuboid

The problem of mapping the unit cube to a cuboid can be solved by considering the problem in a four dimensional setting (coordinates are (x, y, z, w)). Translate one vertex of the cuboid to the origin (say this is $\vec{q}_{000}$). Label the other vertices as $\vec{q}_{100}$, $\vec{q}_{010}$, $\vec{q}_{001}$, $\vec{q}_{110}$, $\vec{q}_{101}$, $\vec{q}_{011}$, and $\vec{q}_{111}$. The ordering of the vertices corresponds to the ordering of those in the cube whose vertices are the subscripts of the $\vec{q}$ values. The hyperplane containing these points is $z = 0$ and has normal $(0, 0, 0, 1)$. Select an eye point $\vec{E} = (e_0, e_1, e_2, e_3)$. Rotate the hyperplane of the cuboid so that its normal is $\vec{N} = (n_0, n_1, n_2, n_3)$. The cuboid can be projected onto the viewing volume $w = 0$ by a perspective projection. The idea is to choose an eye point $\vec{E}$ and normal $\vec{N}$ so that the projection is the unit cube with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$, and $(1, 1, 1)$.

Let $\vec{i} = (1, 0, 0, 0)$, $\vec{j} = (0, 1, 0, 0)$, $\vec{k} = (0, 0, 1, 0)$, $\vec{\ell} = (0, 0, 0, 1)$, and $\vec{\xi} = (x, y, z, 0)$. The perspective mapping involves finding the intersection of the line $(1-t)\vec{\xi}+t\vec{E}$ with the plane $\vec{N} \cdot \vec{\xi} = 0$. Replacing the line equation in the plane equation, solving for t, the mapping is

$$(\bar{x}, \bar{y}, \bar{z}, \bar{w}) = \frac{(\vec{N} \cdot \vec{\xi})\vec{E} - (\vec{N} \cdot \vec{E})\vec{\xi}}{\vec{N} \cdot (\vec{\xi} - \vec{E})}.$$

The eight rays through the cuboid points on the hyperplane $\vec{N} \cdot (x, y, z, w) = 0$ must intersect the eight corners of the cube. Let $\vec{p}_{ijk}$ be the eight cuboid points in the hyperplane. Then

$$\vec{p}_{ijk} = \frac{(\vec{N} \cdot \vec{b}_{ijk})\vec{E} - (\vec{N} \cdot \vec{E})\vec{b}_{ijk}}{\vec{N} \cdot (\vec{b}_{ijk} - \vec{E})}$$

where $\vec{b}_{ijk}$ are the eight vertices of the cube.

Also, $\vec{p}_{111} = \alpha\vec{p}_{100} + \beta\vec{p}_{010} + \gamma\vec{p}_{001}$. This equations can be solved using the other equations

$$\begin{aligned}\alpha &= \tfrac{\vec{N}\cdot(\vec{E}-\vec{i})}{\vec{N}\cdot(\vec{E}-\vec{i}-\vec{j}-\vec{k})} \\ \beta &= \tfrac{\vec{N}\cdot(\vec{E}-\vec{j})}{\vec{N}\cdot(\vec{E}-\vec{i}-\vec{j}-\vec{k})} \\ \gamma &= \tfrac{\vec{N}\cdot(\vec{E}-\vec{k})}{\vec{N}\cdot(\vec{E}-\vec{i}-\vec{j}-\vec{k})}\end{aligned}$$

The equations for α, β, and γ can be rewritten as

$$\vec{N} \cdot ((\alpha-1)(\vec{E}-\vec{i}) - \alpha\vec{j} - \alpha\vec{k} = 0$$
$$\vec{N} \cdot ((\beta-1)(\vec{E}-\vec{j}) - \beta\vec{i} - \beta\vec{k}) = 0$$
$$\vec{N} \cdot ((\gamma-1)(\vec{E}-\vec{k}) - \gamma\vec{i} - \gamma\vec{j} = 0$$

Vector $\vec{N}$ may be selected as the generalized cross product of the three vectors which are perpendicular to it,

$$n_0 = (-1-\alpha+\beta+\gamma)e_3,$$
$$n_1 = (-1+\alpha-\beta+\gamma)e_3,$$
$$n_2 = (-1+\alpha+\beta-\gamma)e_3,$$
$$n_3 = (\alpha+\beta+\gamma-1) + (1+\alpha-\beta-\gamma)e_0 + (1-\alpha+\beta-\gamma)e_1 + (1-\alpha-\beta+\gamma)e_2.$$

Consequently,

$$\begin{aligned}\vec{N}\cdot(\vec{E}-\vec{i}) &= 2\alpha e_3 \\ \vec{N}\cdot(\vec{E}-\vec{j}) &= 2\beta e_3 \\ \vec{N}\cdot(\vec{E}-\vec{k}) &= 2\gamma e_3 \\ \vec{N}\cdot(\vec{E}-\vec{i}-\vec{j}) &= (1+\alpha+\beta-\gamma)e_3 \\ \vec{N}\cdot(\vec{E}-\vec{i}-\vec{k}) &= (1+\alpha-\beta+\gamma)e_3 \\ \vec{N}\cdot(\vec{E}-\vec{j}-\vec{k}) &= (1-\alpha+\beta+\gamma)e_3 \\ \vec{N}\cdot(\vec{E}-\vec{i}-\vec{j}-\vec{k}) &= 2e_3 \\ \vec{N}\cdot\vec{E} &= (\alpha+\beta+\gamma-1)e_3\end{aligned}$$

Because the object is a cuboid, there are some restrictions on its vertices. If the other vertices are defined as

$$\begin{aligned}\vec{p}_{110} &= v_0\vec{p}_{100} + v_1\vec{p}_{010} \\ \vec{p}_{101} &= u_0\vec{p}_{100} + u_2\vec{p}_{001} \\ \vec{p}_{011} &= w_1\vec{p}_{010} + w_2\vec{p}_{001}\end{aligned}$$

Then

$$\vec{N}\cdot((v_0-1)(\vec{E}-\vec{i})-v_0\vec{j})=0$$
$$\vec{N}\cdot((v_1-1)(\vec{E}-\vec{j})-v_1\vec{i})=0$$
$$\vec{N}\cdot((u_0-1)(\vec{E}-\vec{i})-u_0\vec{k})=0$$
$$\vec{N}\cdot((u_2-1)(\vec{E}-\vec{k})-u_2\vec{i})=0$$
$$\vec{N}\cdot((w_1-1)(\vec{E}-\vec{j})-w_1\vec{k})=0$$
$$\vec{N}\cdot((w_2-1)(\vec{E}-\vec{k})-w_2\vec{j})=0$$

which implies the conditions

$$v_0=\tfrac{2\alpha}{1+\alpha+\beta-\gamma}$$
$$v_1=\tfrac{2\beta}{1+\alpha+\beta-\gamma}$$
$$u_0=\tfrac{2\alpha}{1+\alpha-\beta+\gamma}$$
$$u_2=\tfrac{2\gamma}{1+\alpha-\beta+\gamma}$$
$$w_1=\tfrac{2\beta}{1-\alpha+\beta+\gamma}$$
$$w_2=\tfrac{2\gamma}{1-\alpha+\beta+\gamma}$$

The general mapping is

$$\begin{aligned}\vec{p} &= \tfrac{[\vec{N}\cdot(x\vec{i}+y\vec{j}+z\vec{k})]\vec{E}-(\vec{N}\cdot\vec{E})(x\vec{i}+y\vec{j}+z\vec{k})}{\vec{N}\cdot(x\vec{i}+y\vec{j}+z\vec{k}-\vec{E})}\\ &= \tfrac{x\vec{N}\cdot(\vec{i}-\vec{E})\vec{p}_{100}+y\vec{N}\cdot(\vec{j}-\vec{E})\vec{p}_{010}+z\vec{N}\cdot(\vec{k}-\vec{E})\vec{p}_{001}}{\vec{N}\cdot(x\vec{i}+y\vec{j}+z\vec{k}-\vec{E})}\\ &= \tfrac{2\alpha x}{\Delta}\vec{p}_{100}+\tfrac{2\beta y}{\Delta}\vec{p}_{010}+\tfrac{2\gamma z}{\Delta}\vec{p}_{001}\end{aligned}$$

where

$$\Delta=(\alpha+\beta+\gamma-1)+(1+\alpha-\beta-\gamma)x+(1-\alpha+\beta-\gamma)y+(1-\alpha-\beta+\gamma)z$$

Rotating the plane back to $z=0$ and translating the origin back to the original vertex, the mapping is

$$\vec{q}-\vec{q}_{000}=\frac{2\alpha x}{\Delta}(\vec{q}_{100}-\vec{q}_{000})+\frac{2\beta y}{\Delta}(\vec{q}_{010}-\vec{q}_{000})+\frac{2\gamma z}{\Delta}(\vec{q}_{001}-\vec{q}_{000}).$$