Perspective Mapping Between Two Convex Quadrilaterals

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This is just a brief note about the equations for mapping one convex quadrilateral to another.

Let $\vec{p}_{00}$, $\vec{p}_{10}$, $\vec{p}_{11}$, and $\vec{p}_{01}$ be the vertices of the first quadrilateral listed in counterclockwise order. Let $\vec{q}_{00}$, $\vec{q}_{10}$, $\vec{q}_{11}$, and $\vec{q}_{01}$ be the vertices of the second quadrilateral listed in counterclockwise order. Define $\vec{P}_{10} = \vec{p}_{10} - \vec{p}_{00}$, $\vec{P}_{11} = \vec{p}_{11} - \vec{p}_{00}$, $\vec{P}_{01} = \vec{p}_{01} - \vec{p}_{00}$, $\vec{Q}_{10} = \vec{q}_{10} - \vec{q}_{00}$, $\vec{Q}_{11} = \vec{q}_{11} - \vec{q}_{00}$, and $\vec{Q}_{01} = \vec{q}_{01} - \vec{q}_{00}$. Compute a and b so that $\vec{Q}_{11} = a\vec{Q}_{10} + b\vec{Q}_{01}$. This is a set of two linear equations in two unknowns. Similarly, compute c and d so that $\vec{P}_{11} = c\vec{P}_{10} + d\vec{P}_{01}$. It turns out that $a \geq 0$, $b \geq 0$, $a + b > 1$, $c \geq 0$, $d \geq 0$, and $c + d > 1$.

Any point $\vec{p}$ in the $\langle \vec{p}_{00}, \vec{p}_{10}, \vec{p}_{11}, \vec{p}_{01} \rangle$ quadrilateral can be written as $\vec{p} = (1 - x - y)\vec{p}_{00} + x\vec{p}_{10} + y\vec{p}_{01}$ where $x \geq 0$, $y \geq 0$, $(1 - d)x + c(y - 1) \leq 0$, and $d(x - 1) + (1 - c)y \leq 0$. Any point $\vec{q}$ in the $\langle \vec{q}_{00}, \vec{q}_{10}, \vec{q}_{11}, \vec{q}_{01} \rangle$ quadrilateral can be written as $\vec{q} = (1 - u - v)\vec{q}_{00} + u\vec{q}_{10} + v\vec{q}_{01}$ where $u \geq 0$, $v \geq 0$, $(1 - b)u + a(v - 1) \leq 0$, and $b(u - 1) + (1 - a)v \leq 0$. The perspective mapping relating (u, v) to (x, y) is

$$u = \frac{m_0 x}{n_0 + n_1 x + n_2 y} \text{ and } v = \frac{m_1 y}{n_0 + n_1 x + n_2 y}$$

where $m_0 = ad(1-c-d)$, $m_1 = bc(1-c-d)$, $n_0 = cd(1-a-b)$, $n_1 = d(a-c+bc-ad)$, and $n_2 = c(b-d-bc+ad)$.

Effectively the p–quadrilateral is mapped to a "standard" one, $\langle (0,0), (1,0), (0,1), (c,d) \rangle$ and the q–quadrilateral is mapped to $\langle (0,0), (1,0), (0,1), (a,b) \rangle$. The (x, y) to (u, v) mapping relates these two. You can verify that

- $(x, y) = (0, 0)$ is mapped to $(u, v) = (0, 0)$
- $(x, y) = (1, 0)$ is mapped to $(u, v) = (1, 0)$
- $(x, y) = (0, 1)$ is mapped to $(u, v) = (0, 1)$
- $(x, y) = (c, d)$ is mapped to $(u, v) = (a, b)$