Intersection of a Line and a Cone

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Let the line be $\vec{X}(t) = \vec{P} + t\vec{D}$ for $t \in \mathbb{R}$. The cone has vertex $\vec{V}$, axis direction vector $\vec{A}$, and angle θ between axis and outer edge. In most applications, the cone is *acute*, that is, $\theta \in (0, \pi/2)$. This document assumes that, in fact, the cone is acute, so $\cos\theta > 0$. The cone consists of those points $\vec{X}$ for which the angle between $\vec{X} - \vec{V}$ and $\vec{A}$ is θ. Algebraically the condition is

$$\vec{A} \cdot \left(\frac{\vec{X} - \vec{V}}{|\vec{X} - \vec{V}|} \right) = \cos\theta.$$

Figure 1 shows a 2D representation of the cone. The shaded portion indicates the *inside* of the cone, a region represented algebraically by replacing $=$ in the above equation with $\geq$.

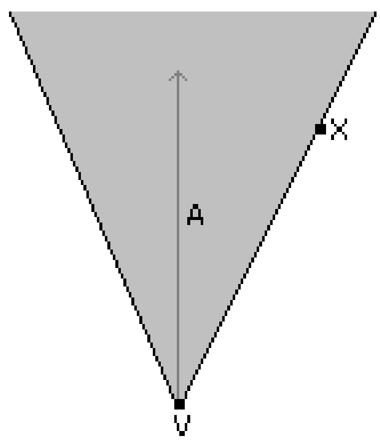

Figure 1. An acute cone. The inside region is shaded.

To avoid the square root calculation $|\vec{X} - \vec{V}|$, the cone equation may be squared to obtain the quadratic equation

$$\left(\vec{A} \cdot (\vec{X} - \vec{V}) \right)^2 = (\cos^2\theta)|\vec{X} - \vec{V}|^2.$$

However, the set of point satisfying this equation is a *double cone*. The original cone on the side of the plane $\vec{A} \cdot (\vec{X} - \vec{V}) = 0$ to which $\vec{A}$ points. The quadratic equation defines the original cone and its reflection through the plane. Specifically, if $\vec{X}$ is a solution to the quadratic equation, then its reflection through the vertex, $2\vec{V} - \vec{X}$, is also a solution. Figure 2 shows the double cone.

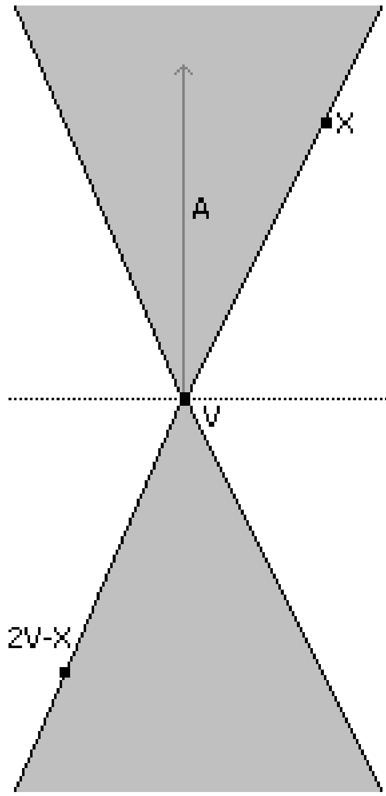

Figure 2. An acute double cone. The inside region is shaded.

To eliminate the reflected cone, any solutions to the quadratic equation must also satisfy $\vec{A}\cdot(\vec{X}-\vec{V}) \geq 0$. Also, the quadratic equation can be written as a quadratic form, $(\vec{X}-\vec{V})^{\mathrm{T}}M(\vec{X}-\vec{V}) = 0$ where $M = (\vec{A}\vec{A}^{\mathrm{T}} - \gamma^2 I)$ and $\gamma = \cos\theta$. Therefore, $\vec{X}$ is a point on the acute cone whenever

$$(\vec{X}-\vec{V})^{\mathrm{T}}M(\vec{X}-\vec{V}) = 0 \text{ and } \vec{A}\cdot(\vec{X}-\vec{V}) \geq 0.$$

To find the intersection points of the line and the cone, replace $\vec{X}(t)$ in the quadratic equation and simplify to obtain $c_2t^2 + 2c_1t + c_0 = 0$ where $\vec{\Delta} = \vec{P} - \vec{V}$, $c_2 = \vec{D}^{\mathrm{T}}M\vec{D}$, $c_1 = \vec{D}^{\mathrm{T}}M\vec{\Delta}$, and $c_0 = \vec{\Delta}^{\mathrm{T}}M\vec{\Delta}$. The following discussion analyzes the quadratic equation to determine points of intersection of the line with the double cone. The dot product test to eliminate points on the reflected cone must be applied to these points.

It is possible that the quadratic equation is degenerate in the sense that $c_2 = 0$. In this case the equation is linear, but even that might be degenerate in the sense that $c_1 = 0$. An implementation must take this into account when finding the intersections.

First, suppose that $c_2 \neq 0$. Define $\delta = c_1^2 - c_0c_2$. If $\delta < 0$, then the quadratic equation has no real–valued roots, in which case the line does not intersect the double cone. If $\delta = 0$, then the equation has a repeated real–valued root $t = -c_1/c_2$, in which case the line is tangent to the double cone at a single point. If $\delta > 0$, the equation has two distinct real–valued roots $t = (-c_1 \pm \sqrt{\delta})/c_2$, in which case the line penetrates the double cone at two points.

Second, suppose that $c_2 = 0$. This means $\vec{D}^{\mathrm{T}} M \vec{D} = 0$. A consequence is that the line $\vec{V} + s\vec{D}$ is on the double cone for all $s \in \mathbb{R}$. Geometrically, the line $\vec{P} + t\vec{D}$ is parallel to some line on the cone. If additionally $c_0 = 0$, this condition implies $\vec{\Delta}^{\mathrm{T}} M \vec{\Delta} = 0$. A consequence is that $\vec{P}$ is a point on the double cone. Even so, it is not necessary that the original line is completely contained by the cone (although it is possible that it is). Figure 3 shows the cases $c_2 = 0$ and $c_0 \neq 0$ or $c_0 = 0$.

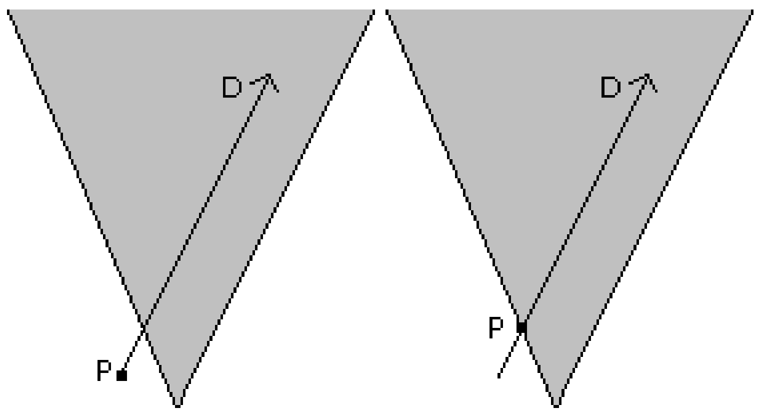

Figure 3. Case $c_2 = 0$. The left figure is $c_0 \neq 0$. The right figure is $c_0 = 0$.

Finally, if $c_i = 0$ for all i, then $\vec{P} + t\vec{D}$ is on the double cone for all $t \in \mathbb{R}$. Algebraically, when $c_1 \neq 0$, the root to the linear equation is $t = -c_2/(2c_1)$. If $c_1 = 0$ and $c_2 \neq 0$, the line does not intersect the code. If $c_1 = c_2 = 0$, then the line is contained by the double cone.