Distance from Linear Component to Tetrahedron

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Let $\vec{V}_i$, $0 \le i \le 3$ be the vertices of the tetrahedron. The linear component is $\vec{P} + t\vec{D}$ where $\vec{D}$ is a unit length vector and $t \in \mathbb{R}$ (line), $t \ge 0$ (ray), or $t \in [0, T]$ (segment). The construction can be modified slightly to handle $\vec{D}$ that is not unit length. The tetrahedron can be parameterized by $\vec{V}_0 + s_1\vec{E}_1 + s_2\vec{E}_2 + s_3\vec{E}_3$ where $\vec{E}_i = \vec{V}_i - \vec{V}_0$, $s_i \ge 0$, and $s_1 + s_2 + s_3 \le 1$.

1 Line and Tetrahedron

1.1 Distance

Translate the tetrahedron and line by subtracting $\vec{P}$. The tetrahedron vertices are now $\vec{U}_i = \vec{V}_i - \vec{P}$ for all i. The line becomes $t\vec{D}$. Project onto the plane containing the origin $\vec{0}$ and having normal $\vec{D}$. The projected line is the single point $\vec{0}$. The projected tetrahedron vertices are $\vec{W}_i = (I - \vec{D}\vec{D}^T)\vec{U}_i$ for all i. The boundary of the projected solid tetrahedron is a convex polygon, either a triangle or a quadrilateral. Figure 1 shows the line, tetrahedron, and projections.

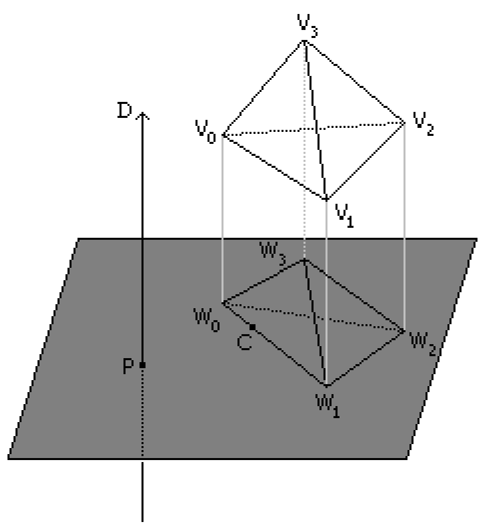

Figure 1. Line, tetrahedron, and projections onto a plane perpendicular to the line.

If the convex polygon contains $\vec{0}$, the distance from the line to the tetrahedron is zero. Otherwise, the distance from the line to the tetrahedron is the distance from $\vec{0}$ to the convex polygon. The projected values are in a plane in 3D and can be projected into 2D with the standard technique of eliminating the coordinate corresponding to the maximum absolute component of $\vec{D}$. The distance between a point and convex polygon can be computed in 2D. This value must be adjusted to account for the 3D–to–2D projection. For example, if $\vec{D} = (d_0, d_1, d_2)$ with $|d_2| = \max_i\{|d_i|\}$ and r is the computed 2D distance, then the 3D distance is r/d_2.

1.2 Closest Points

The set of tetrahedron points closest to the line in many cases consists of a single point. In other cases, the set can consist of a line segment of points. For example, consider the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. The line $(1/4,1/4,0)+t(0,0,1)$ intersects the tetrahedron for $t \in [0,1/2]$, so the corresponding points are zero units of distance from the tetrahedron. The line $(-1,-1,1/2)+t(0,0,1)$ is $\sqrt{2}$ units of distance from the tetrahedron. The closest points on the line are generated by $t \in [0,1/2]$ and the closest points on the tetrahedron are $(0,0,t)$ for the same interval of t values. The line $(1/2,-1/2,0)+t(0,0,1)$ is $1/2$ units of distance from the tetrahedron. The closest points on the line are generated by $t \in [0,1/2]$ and the closest points on the tetrahedron are $(1/2,0,t)$ for the same interval of t values.

Case 1. Let $\vec{0}$ be strictly inside the convex polygon. In this case, the line intersects the tetrahedron in an interval of points. Let $E = [\vec{E}_1\ \vec{E}_2\ \vec{E}_3]$ be the matrix whose columns are the specified edge vectors of the tetrahedron. Let $\vec{s}$ be the 3×1 vector whose components are the s_i parameters. The line segment of intersection is $t\vec{D} + \vec{P} = E\vec{s} + \vec{V}_0$ for $t \in [t_{\min}, t_{\max}]$. The problem now is to compute the t–interval. The edge vectors of the tetrahedron are linearly independent, so E is invertible. Multiplying the vector equation by the inverse and solving for the tetrahedron parameters yields

$$\vec{s} = E^{-1}\left(t\vec{D} + \vec{P} - \vec{V}_0\right) = \vec{A}t + \vec{B}$$

where $\vec{A} = (a_1,a_2,a_3) = E^{-1}\vec{D}$ and $\vec{B} = (b_1,b_2,b_3) = E^{-1}(\vec{P} - \vec{V}_0)$. The parameters $\vec{s}$ must satisfy the inequality constraints for the tetrahedron. The parameter t is therefore constrained by the four inequalities:

$$a_1t + b_1 \geq 0,\ a_2t + b_2 \geq 0,\ a_3t + b_3 \geq 0,\ (a_1+a_2+a_3)t + (b_1+b_2+b_3) \leq 1.$$

Each of these inequalities defines a semi–infinite interval of the form $[\bar{t}, \infty)$ or $(-\infty, \bar{t}]$. In this particular case, we know the intersection of the four intervals must be nonempty and of the form $[t_{\min}, t_{\max}]$.

The division required to compute E^{-1} can be avoided. Let us assume that the tetrahedron is oriented so that $\det(E) > 0$. Multiply by the adjoint E^{adj} to obtain

$$\det(E)\vec{s} = E^{\text{adj}}\left(t\vec{D} + \vec{P} - \vec{V}_0\right) = \vec{\alpha}t + \vec{\beta}.$$

The four t–inequalities are of the same form as earlier, but where a_i refers to the components of $\vec{\alpha}$, b_i refers to the components of $\vec{\beta}$, and the last inequality becomes a comparison to $\det(E)$ instead of to 1.

Case 2. Let $\vec{0}$ be on the convex polygon boundary or outside the polygon. Let $\vec{C}$ be the closest polygon point (in 3D) to $\vec{0}$. The line $t\vec{D} + \vec{C}$ intersects the tetrahedron with $\vec{U}_i$ vertices either in a single point or in an interval of points. The method in case 1 may be used again, but now you need to be careful with the interval construction when using floating point arithmetic. If the intersection is a single point, theoretically $t_{\min} = t_{\max}$, but numerically you might wind up with an empty intersection. It is not difficult to trap this and handle appropriately. Observe that cases 1 and 2 are handled by the same code since in case 1 you can choose $\vec{C} = \vec{0}$.

2 Ray and Tetrahedron

Use the line–tetrahedron algorithm for computing the closest line points with parameters $I = [t_{\min}, t_{\max}]$ (with possibly $t_{\min} = t_{\max}$). Define $J = I \cap [0, \infty)$. If $J \neq \emptyset$, the ray–tetrahedron distance is the same as the line–tetrahedron distance. The closest ray points are determined by J. If $J = \emptyset$, the ray origin $\vec{P}$ is closest to the tetrahedron.

3 Segment and Tetrahedron

Use the line–tetrahedron algorithm for computing the closest line points with parameters $[t_{\min}, t_{\max}]$ (with possibly $t_{\min} = t_{\max}$). Define $J = I \cap [0, T]$. If $J \neq \emptyset$, the segment–tetrahedron distance is the same as the line–tetrahedron distance. The closest segment points are determined by J. If $J = \emptyset$, the closest segment point is $\vec{P}$ when $t_{\max} < 0$ or $\vec{P} + T\vec{D}$ when $t_{\min} > T$.