Calculation of Curvatures and Torsions

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1 Introduction

A smooth curve in $\mathbb{R}^2$ is a 1D manifold of codimension 1 and is characterized by its curvature (modulo rigid motions). A smooth curve in $\mathbb{R}^3$ is a 1D manifold (codimension 2) and is characterized by a curvature function and a torsion function. A smooth surface in $\mathbb{R}^3$ is a 2D manifold (codimension 1) and is characterized by its metric and curvature tensors. Such a surface has two principal curvature functions associated with it.

The problem is to generalize these notions to manifolds in $\mathbb{R}^4$. Measurements for curves (manifolds of codimension 3) are similar to those for curves in $\mathbb{R}^3$. Measurements for hypersurfaces (manifolds of codimension 1) are similar to those for surfaces in $\mathbb{R}^3$. Measurements for 2D manifolds (codimension 2) are a mixture of the ideas for curves and surfaces.

2 Curves in $\mathbb{R}^2$

3 Curves in $\mathbb{R}^3$

Let $\vec{x}(s)$ be a smooth curve in $\mathbb{R}^3$ where s is the arc length parameter. The derivative vector $d\vec{x}/ds$ must always be unit length. The Frenet–Serret formulas are

$$\frac{d\vec{x}}{ds} = \vec{T}, \ \ \frac{d\vec{T}}{ds} = \kappa\vec{N}, \ \ \frac{d\vec{N}}{ds} = -\kappa\vec{T} + \tau\vec{B}\frac{d\vec{B}}{ds} = -\tau\vec{N} \tag{1}$$

where $\vec{N}$ is a unit normal and $\vec{B} = \vec{T} \times \vec{N}$. The function κ is the *curvature* and measures how the curve bends within the plane spanned by $\vec{T}$ and $\vec{N}$. The function τ is the *torsion* and measures how the curve bends out of the plane in the direction of $\vec{B}$. Curvature and torsion are easily extracted as

$$\kappa(s) = \vec{N} \cdot \frac{d\vec{T}}{ds}, \ \ \tau(s) = \vec{B} \cdot \frac{d\vec{N}}{ds}.$$

If $\vec{y}\,(t)$ is a smooth curve where t is not necessarily the arc length parameter, then a more complicated set of equations arise. Arc length s and t are related by $s' = ds/dt = |\vec{y}\,'(t)|$ where the prime symbol indicates differentiation with respect to t. Thus,

$$\vec{y}\,' = s'\vec{T}. \tag{2}$$

Differentiate with respect to t and use the Frenet–Serret equations to obtain

$$\vec{y}\,'' = (s')^2\kappa\vec{N} + s''\vec{T}\,. \tag{3}$$

Differentiate again, group terms, and use the Frenet–Serret equations to obtain

$$\vec{y}\,''' = [s''' - \kappa^2(s')^3]\vec{T} + [(s')^3(d\kappa/ds) + 3s's''\kappa]\vec{N} + [(s')^3\kappa\tau]\vec{B}\,. \tag{4}$$

The curvature is

$$\kappa(t) = \frac{\vec{N}\cdot\vec{y}\,''}{(s')^2} = \pm\frac{|\vec{y}\,'\times\vec{y}\,''|}{|\vec{y}\,'|^3}$$

where the sign depends on the orientation of the normal to the tangent. The torsion is

$$\tau(t) = \frac{\vec{B}\cdot\vec{y}\,'''}{(s')^3\kappa} = \frac{\vec{y}\,'\cdot\vec{y}\,''\times\vec{y}\,'''}{|\vec{y}\,'\times\vec{y}\,''|^2}.$$

4 Surfaces in $\mathbb{R}^3$

5 Curves in $\mathbb{R}^4$

6 Hypersurfaces in $\mathbb{R}^4$

7 Manifolds of Codimension 2 in $\mathbb{R}^4$

7.1 Measurement by Sections

To compute curvature at a point on a surface in $\mathbb{R}^3$, a tangential direction was selected and the plane containing the tangent and the surface normal was intersected with the surface. The surface curvature in the tangential direction is the curvature of the intersection curve.

The same idea can be applied to a manifold of codimension 2, $\vec{x} : \mathbb{R}^2 \to \mathbb{R}^4$, say $\vec{x} = \vec{x}(\vec{u})$. At a point $\vec{p}$ on the manifold, let $\vec{T}_1$ and $\vec{T}_2$ be two orthonormal tangent vectors and let $\vec{N}$ and $\vec{B}$ be two orthonormal normal vectors. Let $\vec{T}$ be any unit length tangent direction. The affine hyperplane at $\vec{p}$ and spanned by $\vec{T}$, $\vec{N}$, and $\vec{B}$ intersects the manifold in a curve. This curve may be represented by $\vec{y}(s) = \vec{x}(\vec{u}(s))$ where s is the arc length parameter. While $\vec{p} = \vec{y}(0)$, and $\vec{T}$, $\vec{N}$, and $\vec{B}$ are vectors at $\vec{p}$, in the following discussion I will be loose with the notation and use the same names for the tangents and normals for any s.

While living in $\mathbb{R}^4$, the curve is restricted to a 3D space, so its bitorsion is zero. Effectively the 3D Frenet–Serret equations apply even though the vectors in the formula live in $\mathbb{R}^4$. The curve satisfies

$$\frac{d\vec{y}}{ds} = \vec{T},\quad \frac{d\vec{T}}{ds} = \kappa\vec{N},\quad \frac{d\vec{N}}{ds} = -\kappa\vec{T} + \tau\vec{N},\quad \frac{d\vec{B}}{ds} = -\tau\vec{N},$$

where κ is curvature and τ is torsion. Define $\vec{v} = d\vec{u}/ds$, $\vec{a} = d\vec{v}/ds$, and $\vec{\psi} = d\vec{a}/ds$. Differentiating $\vec{y}(s)$ yields

$$\begin{aligned} \frac{dy_i}{ds} &= x_{i,j}v_j \\ \frac{d^2y_i}{ds^2} &= x_{i,j}a_j + x_{i,jk}v_jv_k \\ \frac{d^3y_i}{ds^3} &= x_{i,j}\psi_j + 3x_{i,jk}v_ja_k + x_{i,jk\ell}v_jv_kv_\ell. \end{aligned}$$

Once again being loose with the notation, we can write these as

$$\begin{aligned} \frac{d\vec{y}}{ds} &= (D\vec{x})\vec{v} \\ \frac{d^2\vec{y}}{ds^2} &= (D\vec{x})\vec{a} + (D^2\vec{x})\vec{v}^2 \\ \frac{d^3\vec{y}}{ds^3} &= (D\vec{x})\vec{\psi} + 3(D^2\vec{x})\vec{v}\vec{a} + (D^3\vec{x})\vec{v}^3. \end{aligned}$$

7.2 Curvature

Since $\vec{T}$ is unit length, we have

$$1 = \vec{T}^t\vec{T} = \vec{v}^t(D\vec{x}^tD\vec{x})\vec{v} = \vec{v}^tG\vec{v}$$

where G is the 2×2 metric tensor for the manifold. Also, $\vec{T} = (D\vec{x})\vec{v}$ implies $\vec{v} = (D\vec{x}^tD\vec{x})^{-1}D\vec{x}^t\vec{T}$ and $[I - D\vec{x}(D\vec{x}^tD\vec{x})^{-1}D\vec{x}^t]\vec{T} = \vec{0}$. The matrix $I - D\vec{x}(D\vec{x}^tD\vec{x})^{-1}D\vec{x}^t$ is a projection operator onto the 2–dimensional normal space at each point on the manifold.

Define $\vec{Q} = (D^2\vec{x})\vec{v}^2 = x_{i,jk}v_jv_k$, a vector of quadratic forms in $\vec{v}$. Then $\kappa\vec{N} = (D\vec{x})\vec{a} + \vec{Q}$. Note that the columns of $D\vec{x}$ are tangent vectors to the manifold, so $0 = \kappa D\vec{x}^t\vec{N} = (D\vec{x}^tD\vec{x})\vec{a} + D\vec{x}^t\vec{Q}$, and $\vec{a} = -(D\vec{x}^tD\vec{x})^{-1}D\vec{x}^t\vec{Q}$. Thus,

$$\kappa\vec{N} = [I - D\vec{x}(D\vec{x}^tD\vec{x})^{-1}D\vec{x}^t]\vec{Q}$$

The projection of $\vec{Q}$ onto the normal space has no $\vec{B}$ component, so $\vec{B}^t\vec{Q} = 0$. This also follows directly from $0 = \kappa\vec{B}^t\vec{N} = \vec{B}^t(D\vec{x})\vec{a} + \vec{B}^t\vec{Q} = \vec{B}^t\vec{Q}$ since $\vec{B}$ is orthogonal to the columns of $D\vec{x}$ which are tangent vectors. Now we have

$$\kappa = \kappa\vec{N}^t\vec{N} = \vec{N}^t(D\vec{x})\vec{a} + \vec{N}^t\vec{Q} = \vec{N}^t\vec{Q}$$

since $\vec{N}$ is orthogonal to the columns of $D\vec{x}$. Also,

$$\kappa^2 = \kappa\vec{N}^t\vec{Q} = \vec{Q}^t[I - D\vec{x}(D\vec{x}^tD\vec{x})^{-1}D\vec{x}^t]\vec{Q}.$$

As a function of $\vec{v}$, κ^2 is continuous on the ellipse $\vec{v}^tG\vec{v} = 1$, a compact set, so it attains a minimum and a maximum on that set. Let $\vec{w}$ be a unit length vector and define $\vec{v} = \vec{w}/\sqrt{\vec{w}^tG\vec{w}}$; then $\vec{v}^tG\vec{v} = 1$. The squared curvature is therefore

$$\kappa^2 = \frac{[(D^2\vec{x})\vec{w}^2]^t[I - D\vec{x}(D\vec{x}^tD\vec{x})^{-1}D\vec{x}^t][(D^2\vec{x})\vec{w}^2]}{[\vec{w}^t(D\vec{x}^tD\vec{x})\vec{w}]^2}. \tag{5}$$

Extrema are computed using standard calculus methods applied to κ^2 as a function of $\theta \in [0, 2\pi]$ where $\vec{w} = (\cos\theta, \sin\theta)$.

7.3 Torsion

The equation $\vec{B}^t\vec{Q} = 0$ implies $\vec{B}^t d\vec{Q}/ds + \vec{Q}^t d\vec{B}/ds = 0$. Using the Frenet–Serret equations yields $\vec{B}^t d\vec{Q}/ds = -\vec{Q}^t d\vec{B}/ds = \tau\kappa$.

Differentiating the equation $d^2\vec{y}/ds^2 = \kappa\vec{N}$ and using the Frenet–Serret equations yields

$$\frac{d^3\vec{y}}{ds} = \kappa\frac{d\vec{N}}{ds} + \frac{d\kappa}{ds}\vec{N} = -\kappa^2\vec{T} + \tau\kappa\vec{B} + \frac{d\kappa}{ds}\vec{N}.$$

Using the original equation for the third derivative, we have

$$-\kappa^2\vec{T} + \tau\kappa\vec{B} + \frac{d\kappa}{ds}\vec{N} = (D\vec{x})\vec{\psi} + 3(D^2\vec{x})\vec{v}\vec{a} + (D^3\vec{x})\vec{v}^3.$$

Therefore,

$$\begin{aligned} \tau\kappa &= \vec{B}^t(D\vec{x})\vec{\psi} + 3\vec{B}^t(D^2\vec{x})\vec{v}\vec{a} + \vec{B}^t(D^3\vec{x})\vec{v}^3 \\ &= 3\vec{B}^t(D^2\vec{x})\vec{v}\vec{a} + \vec{B}^t(D^3\vec{x})\vec{v}^3. \end{aligned}$$

From the definition $\vec{Q} = (D^2\vec{x})\vec{v}^2$, taking a derivative yields

$$\tau\kappa = \vec{B}^t\frac{d\vec{Q}}{ds} = 2\vec{B}^t(D^2\vec{x})\vec{v}\vec{a} + \vec{B}^t(D^3\vec{x})\vec{v}^3.$$

The first equality in this equation comes from the derivation in the first paragraph of this section.

Subtracting the two displayed formulas yields $\vec{B}^t(D^2\vec{x})\vec{v}\vec{a} = 0$. Therefore, $\tau\kappa = \vec{B}^t(D^3\vec{x})\vec{v}^3$. The remaining problem is to replace $\vec{B}$ by an equivalent vector which depends only on $\vec{x}$ and its derivatives.

Let $\vec{\xi}_1 = x_{i,1}$ and $\vec{\xi}_2 = x_{i,2}$, the two columns of $D\vec{x}$. Define $\vec{T}_1 = \vec{\xi}_1/|\vec{\xi}_1|$ and $\vec{T}_2 = \vec{\xi}_2 - (\vec{T}_1^t\vec{\xi}_2)\vec{T}_1$. These two vectors and $\vec{N}$ are orthonormal tangent vectors, so $\vec{B}$ is the generalized cross product of the vectors, $B_i = e_{ijk\ell}T_{1j}T_{2k}N_\ell$, say $\vec{B} = \text{Cross}\,(\vec{T}_1, \vec{T}_2, \vec{N})$. Multiplying by κ and using the formula for $\vec{N}$ in terms of $\vec{Q}$, we have

$$\tau = \frac{\text{Cross}\,(\vec{T}_1, \vec{T}_2, [I - D\vec{x}(D\vec{x}^t D\vec{x})^{-1}D\vec{x}^t]\vec{Q})^t(D^3\vec{x})\vec{v}^3}{\kappa^2}.$$

In terms of the unit length vector $\vec{w}$,

$$\tau = \frac{\text{Cross}\,(\vec{T}_1, \vec{T}_2, [I - D\vec{x}(D\vec{x}^t D\vec{x})^{-1}D\vec{x}^t](D^2\vec{x})\vec{w}^2)^t(D^3\vec{x})\vec{w}^3}{\kappa^2[\vec{w}^t(D\vec{x}^t D\vec{x})\vec{w}]^{5/2}} \tag{6}$$

This can also be optimized by standard calculus techniques where τ is treated as a function of $\theta \in [0, 2\pi]$ where $\vec{w} = (\cos\theta, \sin\theta)$.

Example. Consider the manifold $\vec{x}(u,v) = (u, v, f(u,v), g(u,v))$ where f, g, f_u, f_v, g_u, and g_v are all zero at $(u,v) = (0,0)$. The orthonormal tangents are $\vec{T}_1 = (1,0,0,0)$ and $\vec{T}_2 = (0,1,0,0)$. The metric tensor G is the 2×2 identity matrix and the projection matrix $I - D\vec{x}(D\vec{x}^t D\vec{x})^{-1}D\vec{x}^t = \text{diag}\,(0,0,1,1)$. Define $Qf = (D^2 f)\vec{v}^2$, $Qg = (D^2 g)\vec{v}^2$, $Cf = (D^3 f)\vec{v}^3$, and $Cg = (D^3 g)\vec{v}^3$. At $(0,0)$ the curvature is

$$\kappa^2 = (Qf)^2 + (Qg)^2$$

and the torsion is

$$\tau = \frac{Qf\,Cg - Qg\,Cf}{(Qf)^2 + (Qg)^2}.$$