Representing a Circle or a Sphere with NURBS

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This is just a brief note on representing circles and spheres with NURBS. For more information about NURBS, a good engineering–style approach to NURBS is [1]. A more mathematically advanced presentation is [2].

1 Representing a Circle

A quadrant of a circle can be represented as a NURBS curve of degree 2. The curve is $x^2 + y^2 = 1$, $x \geq 0$, $y \geq 0$. The general parameterization is

$$(x(u), y(u)) = \frac{w_0(1-u)^2(1,0) + w_1 2u(1-u)(1,1) + w_2 u^2(0,1)}{w_0(1-u)^2 + w_1 2u(1-u) + w_2 u^2}$$

for $u \in [0,1]$. The requirement that $x^2 + y^2 = 1$ leads to the weights constraint $2w_1^2 = w_0 w_2$. The choice of weights $w_0 = 1$, $w_1 = 1$, and $w_2 = 2$ leads to a commonly mentioned parameterization

$$(x(u), y(u)) = \frac{(1-u^2, 2u)}{1+u^2}$$

If you were to tessellate the curve with an odd number of uniform samples of u, say $u_i = i/(2n)$ for $0 \leq i \leq 2n$, then the resulting polyline is not symmetric about the midpoint $u = 1/2$. To obtain a symmetric tessellation you need to choose $w_0 = w_2$. The weight constraint then implies $w_0 = w_1\sqrt{2}$. The parameterization is then

$$(x(u), y(u)) = \frac{(\sqrt{2}(1-u)^2 + 2u(1-u), 2u(1-u) + \sqrt{2}u^2)}{\sqrt{2}(1-u)^2 + 2u(1-u) + \sqrt{2}u^2}$$

In either case we have a ratio of quadratic polynomials.

An algebraic construction of the same type, but quite a bit more tedious to solve, produces a ratio of quartic polynomials. The control points and control weights are required to be symmetric to obtain a tessellation that is symmetric about its midpoint. The middle weight is chosen as $w_2 = 4$.

$$(x(u), y(u)) = \frac{(1-u)^4 w_0(1,0) + 4(1-u)^3 u w_1(x_1, y_1) + 24(1-u)^2 u^2(x_2, x_2) + 4(1-u)u^3 w_1(y_1, x_1) + u^4 w_0}{(1-u)^4 w_0 + 4(1-u)^3 u w_1 + 24(1-u)^2 u^2 + 4(1-u)u^3 w_1 + u^4 w_0}$$

The parameters are $x_1 = 1$, $y_1 = (\sqrt{3}-1)/\sqrt{3}$, $x_2 = (\sqrt{3}+1)/(2\sqrt{3})$, $w_0 = 4\sqrt{3}(\sqrt{3}-1)$, and $w_1 = 3/\sqrt{2}$.

2 Representing a Sphere

An octant of a sphere can be represented as a triangular NURBS surface patch of degree 4. A simple parameterization of $x^2 + y^2 + z^2 = 1$ can be made by setting $r^2 = x^2 + y^2$. The sphere is then $r^2 + z^2 = 1$. Now apply the parameterization for a circle,

$$(r, z) = \frac{(1 - u^2, 2u)}{1 + u^2}$$

But $(x/r)^2 + (y/r)^2 = 1$, so another application of the parameterization for a circle is

$$\frac{(x, y)}{r} = \frac{(1 - v^2, 2v)}{1 + v^2}$$

Combining these produces

$$(x(u, v), y(u, v), z(u, v)) = \frac{((1 - u^2)(1 - v^2), (1 - u^2)2v, 2u(1 + v^2))}{(1 + u^2)(1 + v^2)}$$

The components are ratios of quartic polynomials. The domain is $u \geq 0$, $v \geq 0$, and $u+v \leq 1$. In barycentric coordinates, set $w = 1 - u - v$ so that $u + v + w = 1$ with u, v, and w nonnegative. In this setting, you can think of the octant of the sphere as a mapping from the uvw–triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Although a valid parameterization, a symmetric subdivision of the the uvw–triangle does not lead to a symmetric tessellation of the sphere.

Another parameterization is provided in [2]. This one chooses symmetric control points and symmetric weights,

$$(x(u, v), y(u, v), z(u, v)) = \frac{\sum_{i=0}^{4} \sum_{j=0}^{4-i} w_{i,j,4-i-j} \mathbf{P}_{i,j,4-i-j} B_{i,j}(u, v)}{\sum_{i=0}^{4} \sum_{j=0}^{4-i} w_{i,j,4-i-j} B_{i,j}(u, v)}$$

where

$$B_{i,j}(u, v) = \frac{4!}{i!j!(4 - i - j)!} u^i v^j (1 - u - v)^{4-i-j}, \quad u \geq 0, \quad v \geq 0, \quad u + v \leq 1$$

are the Bernstein polynomials. The control points $\mathbf{P}_{i,j,k}$ are defined in terms of three constants $a_0 = (\sqrt{3} - 1)/\sqrt{3}$, $a_1 = (\sqrt{3} + 1)/(2\sqrt{3})$, and $a_2 = 1 - (5 - \sqrt{2})(7 - \sqrt{3})/46$,

$$\begin{array}{lllll} \mathbf{P}_{040} & & & & \\ \mathbf{P}_{031} & \mathbf{P}_{130} & & & \\ \mathbf{P}_{022} & \mathbf{P}_{121} & \mathbf{P}_{220} & & \\ \mathbf{P}_{013} & \mathbf{P}_{112} & \mathbf{P}_{211} & \mathbf{P}_{310} & \\ \mathbf{P}_{004} & \mathbf{P}_{103} & \mathbf{P}_{202} & \mathbf{P}_{301} & \mathbf{P}_{400} \end{array} = \begin{array}{lllll} (0, 1, 0) & & & & \\ (0, 1, a_0) & (a_0, 1, 0) & & & \\ (0, a_1, a_1) & (a_2, 1, a_2) & (a_1, a_1, 0) & & \\ (0, a_0, 1) & (a_2, a_2, 1) & (1, a_2, a_2) & (1, a_0, 0) & \\ (0, 0, 1) & (a_0, 0, 1) & (a_1, 0, a_1) & (1, 0, a_0) & (1, 0, 0) \end{array}$$

The control weights $w_{i,j,k}$ are defined in terms of four constants, $b_0 = 4\sqrt{3}(\sqrt{3} - 1)$, $b_1 = 3\sqrt{2}$, $b_2 = 4$, and

$b_3 = \sqrt{2}(3 + 2\sqrt{2} - \sqrt{3})/\sqrt{3}$,

$$
\begin{array}{ccccc}
w_{040} & & & & \\
w_{031} & w_{130} & & & \\
w_{022} & w_{121} & w_{220} & & \\
w_{013} & w_{112} & w_{211} & w_{310} & \\
w_{004} & w_{103} & w_{202} & w_{301} & w_{400}
\end{array}
=
\begin{array}{ccccc}
b_0 & & & & \\
b_1 & b_1 & & & \\
b_2 & b_3 & b_2 & & \\
b_1 & b_3 & b_3 & b_1 & \\
b_0 & b_1 & b_2 & b_1 & b_0
\end{array}
$$

Both the numerator and denominator polynomial are quartic polynomials. Notice that each boundary curve of the triangle patch is a quartic polynomial of one variable that is exactly what was shown earlier for a quadrant of a circle.

References

[1] David F. Rogers, *An Introduction to NURBS with Historical Perspective*, Morgan Kaufmann Publishers, San Francisco, CA, 2001

[2] *Curves and Surfaces for Computer Aided Geometric Design: A Practical Guide*, Academic Press, San Diego, CA, 1990