Particle Motion on a Height Field

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A height field is specified by $z = f(x, y)$ for points (x, y) in the xy–plane. The height at (x, y) is the z–value. The three–dimensional position of a particle of mass m constrained to lie on the surface is

$$\mathbf{r} = (x, y, f(x, y)).$$

Assuming that the surface is frictionless and the only force on the particle is gravitational, $\mathbf{F} = -mg(0, 0, 1)$ where $g > 0$ is a constant, this document shows how to construct the equations of motion for the particle using Lagrangian dynamics. Derivatives with respect to time are denoted by dots placed over the variables. The velocity of the particle is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{df(x, y)}{dt}\right) = (\dot{x}, \dot{y}, f_x\dot{x} + f_y\dot{y})$$

where $f_x = \partial f/\partial x$ and $f_y = \partial f/\partial y$ are the first–order partial derivatives of the function $f(x, y)$. The last equality in the equation follows by applying the chain rule from calculus. The kinetic energy of the particle is

$$K = \frac{m}{2}|\mathbf{v}|^2 = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + (f_x\dot{x} + f_y\dot{y})^2\right).$$

In the construction below second derivative quantities are $\ddot{x} = d^2x/dt^2$, $\ddot{y} = d^2y/dt^2$, $f_{xx} = \partial^2 f/\partial x^2$, $f_{xy} = \partial f^2/\partial x\partial y$, and $f_{yy} = \partial^2 f/\partial y^2$.

We have two Lagrangian equations of motion since the surface has two constraint variables, x and y. Formally these equations are

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}}\right) - \frac{\partial K}{\partial x} = \mathbf{F}\cdot\frac{\partial \mathbf{r}}{\partial x} \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{y}}\right) - \frac{\partial K}{\partial y} = \mathbf{F}\cdot\frac{\partial \mathbf{r}}{\partial y}$$

The partial derivatives of K are computed assuming x, y, $\dot{x}$, and $\dot{y}$ are independent variables. That is the kinetic energy is a function $K(x, y, \dot{x}, \dot{y})$. The derivatives with respect to x and y are

$$\frac{\partial K}{\partial x} = m\left(f_x\dot{x} + f_y\dot{y}\right)\left(f_{xx}\dot{x} + f_{xy}\dot{y}\right) \quad \text{and} \quad \frac{\partial K}{\partial y} = m\left(f_x\dot{x} + f_y\dot{y}\right)\left(f_{xy}\dot{x} + f_{yy}\dot{y}\right)$$

The derivatives with respect to $\dot{x}$ and $\dot{y}$ are

$$\frac{\partial K}{\partial \dot{x}} = m\left(\dot{x} + \left(f_x\dot{x} + f_y\dot{y}\right)f_x\right) \quad \text{and} \quad \frac{\partial K}{\partial \dot{y}} = m\left(\dot{y} + \left(f_x\dot{x} + f_y\dot{y}\right)f_y\right)$$

The time derivatives consider all of x, y, $\dot{x}$, and $\dot{y}$ to be functions of time. The first such derivative is

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}}\right) = m\{\ddot{x} + (f_x\dot{x} + f_y\dot{y})(f_{xx}\dot{x} + f_{xy}\dot{y}) + [f_x\ddot{x} + (f_{xx}\dot{x} + f_{xy}\dot{y})\dot{x} + f_y\ddot{y} + (f_{xy}\dot{x} + f_{yy}\dot{y})\dot{y}]f_x\}$$

The second such derivative is

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{y}}\right) \quad = \quad m\{\ddot{y} + (f_x\dot{x} + f_y\dot{y})(f_{xy}\dot{x} + f_{yy}\dot{y}) + [f_x\ddot{x} + (f_{xx}\dot{x} + f_{xy}\dot{y})\dot{x} + f_y\ddot{y} + (f_{xy}\dot{x} + f_{yy}\dot{y})\dot{y}]f_y\}$$

The left–hand side of the first Lagrangian equation is

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}}\right) - \frac{\partial K}{\partial x} = m\{(1+f_x^2)\ddot{x} + f_xf_y\ddot{y} + f_x[(f_{xx}\dot{x} + f_{xy}\dot{y})\dot{x} + (f_{xy}\dot{x} + f_{yy}\dot{y})\dot{y}]\}$$

The left–hand side of the second Lagrangian equation is

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{y}}\right) - \frac{\partial K}{\partial y} = m\{f_xf_y\ddot{x} + (1+f_y^2)\ddot{y} + f_y[(f_{xx}\dot{x} + f_{xy}\dot{y})\dot{x} + (f_{xy}\dot{x} + f_{yy}\dot{y})\dot{y}]\}$$

The right–hand sides of the Lagrangian equations are $\mathbf{F}\cdot\partial\mathbf{r}/\partial x = (0,0,-mg)\cdot(1,0,f_x) = -mgf_x$ and $\mathbf{F}\cdot\partial\mathbf{r}/\partial y = (0,0,-mg)\cdot(0,1,f_y) = -mgf_y$, respectively. The equations of motion are therefore,

$$\begin{aligned}(1+f_x^2)\ddot{x} + f_xf_y\ddot{y} + f_x[(f_{xx}\dot{x} + f_{xy}\dot{y})\dot{x} + (f_{xy}\dot{x} + f_{yy}\dot{y})\dot{y}] &= -gf_x \\ f_xf_y\ddot{x} + (1+f_y^2)\ddot{y} + f_y[(f_{xx}\dot{x} + f_{xy}\dot{y})\dot{x} + (f_{xy}\dot{x} + f_{yy}\dot{y})\dot{y}] &= -gf_y\end{aligned}$$

The second derivative terms appear implicitly. You can write these equations in matrix form,

$$\begin{bmatrix} 1+f_x^2 & f_xf_y \\ f_xf_y & 1+f_y^2 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = -\left(\begin{bmatrix} \dot{x} & \dot{y} \end{bmatrix}\begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + g\right)\begin{bmatrix} f_x \\ f_y \end{bmatrix}$$

The inverse of the matrix on the left–hand side is

$$\begin{bmatrix} 1+f_x^2 & f_xf_y \\ f_xf_y & 1+f_y^2 \end{bmatrix}^{-1} = \frac{1}{1+f_x^2+f_y^2}\begin{bmatrix} 1+f_y^2 & -f_xf_y \\ -f_xf_y & 1+f_x^2 \end{bmatrix}$$

It is easily checked that

$$\begin{bmatrix} 1+f_y^2 & -f_xf_y \\ -f_xf_y & 1+f_x^2 \end{bmatrix}\begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \end{bmatrix}$$

Multiplying the Lagrangian equations by the inverse matrix and using the identity in the last displayed equation leads to

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = -\left(\frac{\begin{bmatrix} \dot{x} & \dot{y} \end{bmatrix}\begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + g}{1+f_x^2+f_y^2}\right)\begin{bmatrix} f_x \\ f_y \end{bmatrix}$$

Now the second derivatives of x and y appear explicitly in the equations. You can procede, as usual, in converting to a first–order system of four equations in the variables x, y, $u = \dot{x}$, and $v = \dot{y}$, followed by applying your favorite solver for systems of ordinary differential equations.