# Minimum Area Rectangle Containing a Convex Polygon

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## 1 Introduction

Given a convex polygon with ordered vertices $\vec{P}_i$ for $0 \le i < N$, the problem is to construct the minimum area rectangle that contains the polygon. The rectangle is not required to be axis–aligned with the coordinate system axes. It is the case that at least one of the edges of the convex polygon must be contained by an edge of the minimum area rectangle. Given this is so, an algorithm for computing the minimum area rectangle need only compute the tightest fitting bounding rectangles whose orientations are determined by the polygon edges.

## 2 Proof of Edge Containment

The proof is by contradiction. Suppose that in fact no edge of the convex polygon is contained by an edge of the minimum area rectangle. The rectangle must be supported by four vertices of the convex polygon, as illustrated by Figure 1.

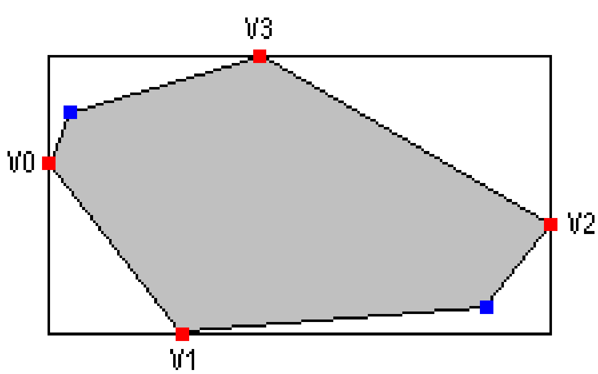


Figure 1. Minimum area rectangle that has no coincident polygon edges.

The supporting vertices are drawn in red and labeled $\vec{V}_0$ through $\vec{V}_3$. Other polygon vertices are drawn in blue. For the sake of the argument, rotate the convex polygon so that the axes of this rectangle are $(1, 0)$

and $(0,1)$ as shown in the figure.

Define $\vec{U}_0(\theta) = (\cos\theta, \sin\theta)$ and $\vec{U}_1(\theta) = (-\sin\theta, \cos\theta)$. There exists a value $\varepsilon > 0$ such that the $\vec{V}_i$ are always the supporting vertices of the bounding rectangle with axes $\vec{U}_0(\theta)$ and $\vec{U}_1(\theta)$ for all angles $\theta$ satisfying the condition $|\theta| <= \varepsilon$. To compute the bounding rectangle area, the supporting vertices are projected onto the axis lines $\vec{V}_0 + s\vec{U}_0(\theta)$ and $\vec{V}_0 + t\vec{U}_1(\theta)$. The intervals of projection are $[0, s_1]$ and $[t_0, t_1]$ where $s_1 = \vec{U}_0(\theta) \cdot (\vec{V}_2 - \vec{V}_0)$, $t_0 = \vec{U}_1(\theta) \cdot (\vec{V}_1 - \vec{V}_0)$, and $t_1 = \vec{U}_1(\theta) \cdot (\vec{V}_3 - \vec{V}_0)$.

Define $\vec{K}_0 = (x_0, y_0) = \vec{V}_2 - \vec{V}_0$ and $\vec{K}_1 = (x_1, y_1) = \vec{V}_3 - \vec{V}_1$. From Figure 1 it is clear that $x_0 > 0$ and $y_1 > 0$. The area of the rectangle for $|\theta| \le \varepsilon$ is

$$A(\theta) = s_1(t_1 - t_0) = [\vec{K}_0 \cdot \vec{U}_0(\theta)][\vec{K}_1 \cdot \vec{U}_1(\theta)].$$

In particular, $A(0) = x_0 y_1 > 0$.

Since $A(\theta)$ is differentiable on its domain and since $A(0)$ is assumed to be the global minimum, it must be that $A'(0) = 0$. Generally,

$$\begin{aligned} A'(\theta) &= [\vec{K}_0 \cdot \vec{U}_0(\theta)][\vec{K}_1 \cdot \vec{U}_1'(\theta)] + [\vec{K}_0 \cdot \vec{U}_0'(\theta)][\vec{K}_1 \cdot \vec{U}_1(\theta)] \\ &= -[\vec{K}_0 \cdot \vec{U}_0(\theta)][\vec{K}_1 \cdot \vec{U}_0(\theta)] + [\vec{K}_0 \cdot \vec{U}_1(\theta)][\vec{K}_1 \cdot \vec{U}_1(\theta)] \end{aligned}$$

Therefore, $0 = A'(0) = -x_0 x_1 + y_0 y_1$, or $x_0 x_1 = y_0 y_1$. Since $x_0 > 0$ and $y_1 > 0$, it must be that $\text{Sign}(x_1) = \text{Sign}(y_0)$. Moreover, since $A(0)$ is assumed to be the global minimum, it must be that $A''(0) \ge 0$. Generally,

$$\begin{aligned} A''(\theta) &= -[\vec{K}_0 \cdot \vec{U}_0(\theta)][\vec{K}_1 \cdot \vec{U}_0'(\theta)] - [\vec{K}_0 \cdot \vec{U}_0'(\theta)][\vec{K}_1 \cdot \vec{U}_0(\theta)] \\ &\quad +[\vec{K}_0 \cdot \vec{U}_1(\theta)][\vec{K}_1 \cdot \vec{U}_1'(\theta)] + [\vec{K}_0 \cdot \vec{U}_1'(\theta)][\vec{K}_1 \cdot \vec{U}_1(\theta)] \\ &= -[\vec{K}_0 \cdot \vec{U}_0(\theta)][\vec{K}_1 \cdot \vec{U}_1(\theta)] - [\vec{K}_0 \cdot \vec{U}_1(\theta)][\vec{K}_1 \cdot \vec{U}_0(\theta)] \\ &\quad -[\vec{K}_0 \cdot \vec{U}_1(\theta)][\vec{K}_1 \cdot \vec{U}_0(\theta)] - [\vec{K}_0 \cdot \vec{U}_0(\theta)][\vec{K}_1 \cdot \vec{U}_1(\theta)] \\ &= -2\left\{[\vec{K}_0 \cdot \vec{U}_0(\theta)][\vec{K}_1 \cdot \vec{U}_1(\theta)] + [\vec{K}_0 \cdot \vec{U}_1(\theta)][\vec{K}_1 \cdot \vec{U}_0(\theta)]\right\} \end{aligned}$$

In particular, $A''(0) = -2(x_0 y_1 + x_1 y_0) \ge 0$. However, note that $x_0 y_1 > 0$ since $A(0) > 0$ and $x_1 y_0 > 0$ since $\text{Sign}(x_1) = \text{Sign}(y_0)$, which implies that $A''(0) < 0$, a contradiction.

# 3 The Algorithm

Pseudocode for the algorithm is given below.

```
ordered vertices P[0] through P[N-1];
define P[N] = P[0];

minimumArea = infinity;
for (i = 1; i <= N; i++)
{
    U0 = P[i] - P[i-1];
    U1 = (-U0.x,U0.y);
    s0 = t0 = s1 = t1 = 0;
    for (j = 1; j < N; j++)
    {
        D = P[j] - P[0];
        test = Dot(U0,D);
        if ( test < s0 ) s0 = test; else if ( test > s1 ) s1 = test;
        test = Dot(U1,D);
        if ( test < t0 ) t0 = test; else if ( test > t1 ) t1 = test;
    }
    area = (s1-s0)*(t1-t0);
    if ( area < minimumArea )
        minimumArea = area;
}
```