Geodesics on a Parameterized Surface

David Eberly
Magic Software, Inc.
http://www.magic-software.com

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Consider a parameterized surface $\vec{x}(u,v)$. Let $\vec{x}_u$ and $\vec{x}_v$ denote the first–order partial derivatives. Let $\vec{x}_{uu}$, $\vec{x}_{uv}$, and $\vec{x}_{vv}$ denote the second–order partial derivatives. From differential geometry, the metric for the surface is

$$G = [g_{ij}] = \begin{bmatrix} \vec{x}_u \cdot \vec{x}_u & \vec{x}_u \cdot \vec{x}_v \\ \vec{x}_u \cdot \vec{x}_v & \vec{x}_v \cdot \vec{x}_v \end{bmatrix}$$

The inverse of the matrix is

$$G^{-1} = [g^{ij}] = \frac{1}{g_{11}g_{22} - g_{12}^2} \begin{bmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{bmatrix} = \frac{1}{|\vec{x}_u \times \vec{x}_v|^2} \begin{bmatrix} \vec{x}_v \cdot \vec{x}_v & -\vec{x}_u \cdot \vec{x}_v \\ -\vec{x}_u \cdot \vec{x}_v & -\vec{x}_u \cdot \vec{x}_u \end{bmatrix}$$

The Christoffel symbols of the first kind are Γ_{ijk} and are symmetric in i and j. They are

$$\begin{aligned}
\Gamma_{111} &= \vec{x}_{uu} \cdot \vec{x}_u \\
\Gamma_{121} &= \vec{x}_{uv} \cdot \vec{x}_u \\
\Gamma_{221} &= \vec{x}_{vv} \cdot \vec{x}_u \\
\Gamma_{112} &= \vec{x}_{uu} \cdot \vec{x}_v \\
\Gamma_{122} &= \vec{x}_{uv} \cdot \vec{x}_v \\
\Gamma_{222} &= \vec{x}_{vv} \cdot \vec{x}_v
\end{aligned}$$

The Christoffel symbols of the second kind are Γ^i_{jk} and are symmetric in i and j. They are

$$\begin{aligned}
\Gamma^1_{11} &= g^{11}\Gamma_{111} + g^{12}\Gamma_{112} = \tfrac{1}{|\vec{x}_u \times \vec{x}_v|^2}\left[(\vec{x}_v \cdot \vec{x}_v)\vec{x}_{uu} \cdot \vec{x}_u - (\vec{x}_u \cdot \vec{x}_v)\vec{x}_{uu} \cdot \vec{x}_v\right] \\
\Gamma^1_{12} &= g^{11}\Gamma_{121} + g^{12}\Gamma_{122} = \tfrac{1}{|\vec{x}_u \times \vec{x}_v|^2}\left[(\vec{x}_v \cdot \vec{x}_v)\vec{x}_{uv} \cdot \vec{x}_u - (\vec{x}_u \cdot \vec{x}_v)\vec{x}_{uv} \cdot \vec{x}_v\right] \\
\Gamma^1_{22} &= g^{11}\Gamma_{221} + g^{12}\Gamma_{222} = \tfrac{1}{|\vec{x}_u \times \vec{x}_v|^2}\left[(\vec{x}_v \cdot \vec{x}_v)\vec{x}_{vv} \cdot \vec{x}_u - (\vec{x}_u \cdot \vec{x}_v)\vec{x}_{vv} \cdot \vec{x}_v\right] \\
\Gamma^2_{11} &= g^{21}\Gamma_{111} + g^{22}\Gamma_{112} = \tfrac{1}{|\vec{x}_u \times \vec{x}_v|^2}\left[-(\vec{x}_u \cdot \vec{x}_v)\vec{x}_{uu} \cdot \vec{x}_u + (\vec{x}_u \cdot \vec{x}_u)\vec{x}_{uu} \cdot \vec{x}_v\right] \\
\Gamma^2_{12} &= g^{21}\Gamma_{121} + g^{22}\Gamma_{122} = \tfrac{1}{|\vec{x}_u \times \vec{x}_v|^2}\left[-(\vec{x}_u \cdot \vec{x}_v)\vec{x}_{uv} \cdot \vec{x}_u + (\vec{x}_u \cdot \vec{x}_u)\vec{x}_{uv} \cdot \vec{x}_v\right] \\
\Gamma^2_{22} &= g^{21}\Gamma_{221} + g^{22}\Gamma_{222} = \tfrac{1}{|\vec{x}_u \times \vec{x}_v|^2}\left[-(\vec{x}_u \cdot \vec{x}_v)\vec{x}_{vv} \cdot \vec{x}_u + (\vec{x}_u \cdot \vec{x}_u)\vec{x}_{vv} \cdot \vec{x}_v\right]
\end{aligned}$$

A geodesic curve must satisfy the following system of differential equations in parameter space where s is the arc length parameter,

$$\frac{d^2u}{ds} + \Gamma^1_{11}\left(\frac{du}{ds}\right)^2 + 2\Gamma^1_{12}\frac{du}{ds}\frac{dv}{ds} + \Gamma^1_{22}\left(\frac{dv}{ds}\right)^2 = 0$$
$$\frac{d^2v}{ds} + \Gamma^2_{11}\left(\frac{du}{ds}\right)^2 + 2\Gamma^2_{12}\frac{du}{ds}\frac{dv}{ds} + \Gamma^2_{22}\left(\frac{dv}{ds}\right)^2 = 0$$

The curve itself is $\vec{y}(s) = \vec{x}(u(s), v(s))$. Initial conditions specified on the surface are $\vec{y}_0 = \vec{y}(0)$ and $\vec{y}_1 = \vec{y}'(0)$ with $|\vec{y}_1| = 1$. The Jacobian matrix is the 3×2 matrix whose columns are $\vec{x}_u$ and $\vec{x}_v$. Note that $J^T J = G$. The derivative in parameter space is

$$\begin{bmatrix} \frac{du}{ds} \\ \frac{dv}{ds} \end{bmatrix} = G^{-1} J^T \vec{y}'(s).$$

Given $\vec{y}'(0)$, the corresponding initial derivative in parameter space can be computed.

The system of equations is of the form

$$u'' = F(u, v, u', v')$$
$$v'' = G(u, v, u', v')$$

where the $'$ indicate derivatives. Setting $p = u'$ and $q = v'$, this becomes a system of four equations

$$u' = p$$
$$v' = q$$
$$p' = F(u, v, p, q)$$
$$q' = G(u, v, p, q)$$

The initial value problem can be solved using standard differential equation solvers such as Runge–Kutta 4th order.