Information About Ellipses

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This document contains various facts about ellipses in the xy–plane. The terminology used is from the book *Calculus and Analytic Geometry, 7th edition* by George B. Thomas, Jr. and Ross L. Finney, Addison–Wesley Publishing Company, Reading, Massachusetts, 1988.

Geometric Definition. An *ellipse* is the set of points in a plane whose distances from two fixed points in that plane add to a constant. One of the fixed points is called a *focal point* of the ellipse. The two together are referred to as the *foci* of the ellipse.

Standard Form. Let the foci be $(\pm c, 0)$ where $c > 0$. Let (x, y) be an ellipse point and let the sum of the distances from (x, y) to the foci be denoted $2a$ for $a > 0$. The equation that (x, y) must satisfy is

$$\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a.$$

The points (x, y), $(c, 0)$, and $(-c, 0)$ form a triangle. The sum of the lengths of two sides of a triangle must be larger than the length of the third side, so $2a > 2c$. Some algebraic manipulation of this equation leads to the *standard form* for an ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$

where $b = \sqrt{a^2 - c^2}$. The argument of the square root is positive since earlier we argued that $a > c$. Moreover, $b < a$ is guaranteed since $b = \sqrt{a^2 - c^2} < \sqrt{a^2} = a$.

The *center* of the standard form ellipse is $(0, 0)$. The *vertices* are $(\pm a, 0)$. The *major axis* is the line segment that connects the vertices. The *minor axis* is the line segment with end points $(0, \pm b)$. The number a is called the *semimajor axis* and the number b is called the *semiminor axis*. [Note: I disagree with the use of the term "axis" to denote length.] The *eccentricity* is the ratio $c/a \in [0, 1]$ and is a measure of how stretched the ellipse is from a circle. A ratio of 0 occurs for a circle. A ratio nearly 1 incidates a long and narrow ellipse.

If the foci are chosen to be $(0, \pm c)$ and the sum of distances is $2b$, the standard form is also given by Equation (1), but now $b > c$ and $a = \sqrt{b^2 - c^2} < b$. The center is still $(0, 0)$, but the vertices are now $(0, \pm b)$, the major axis is the line segment connecting the vertices, the minor axis is the line segment with end points $(\pm a, 0)$, the semimajor axis is b, the semiminor axis is a, and the eccentricity is now defined as the ratio c/b.

If $a = b$, the foci are coincident with the origin $(0, 0)$ and the ellipse is really a circle. The concepts of major and minor axes do not apply here, but the eccentricity is 0.

Area. The *area* of an ellipse in standard form is

$$A = \pi a b. \tag{2}$$

Length. The *length* of an ellipse is the total arc length of the curve. A closed form algebraic solution does not exist, but the length is given by an integral

$$L = 2 \int_{-a}^{a} \sqrt{1 + \frac{b^2 x^2}{a^2(a^2 - x^2)}}\, dx = 2 \int_{-1}^{1} \sqrt{\frac{1 - (\lambda^2 - 1)t^2}{1 - t^2}}\, dt \tag{3}$$

where $\lambda = b/a$. The integral can be approximated with a numerical integrator.

Center–Orient Form. An ellipse in the standard form given by Equation (1) can be oriented via a rotation so that the major and minor axes are not necessarily parallel to the coordinate axes. In vector/matrix form, the standard form is

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} =: \vec{X}^T D \vec{X} \tag{4}$$

where the last equality defines the 2×1 vector $\vec{X} = [x\, y]^T$, the 2×2 diagonal matrix $D = \mathrm{Diag}(1/a^2, 1/b^2)$, and superscript T denotes the transpose operation.

The ellipse may be rotated to a different orientation by a 2×2 rotation matrix

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The major axis direction $(1,0)$ is rotated to $(\cos\theta, \sin\theta)$ and the minor axis direction $(0,1)$ is rotated to $(-\sin\theta, \cos\theta)$. The general transformation is $\vec{Y} = R\vec{X}$ with inverse $\vec{X} = R^T \vec{Y}$. Substituting this into Equation (4) leads to

$$\vec{Y}^T R D R^T \vec{Y} = 1. \tag{5}$$

After orientation the ellipse can be additionally translated so that its old center, the origin $\vec{0}$, is mapped to a new center $\vec{K}$. The general transformation is $\vec{Y} = R\vec{X} + \vec{K}$; the rotation R is applied first, followed by the translation $\vec{K}$. Equation (5) is modified to include the translation,

$$(\vec{Y} - \vec{K})^T R D R^T (\vec{Y} - \vec{K}) = 1. \tag{6}$$

General Quadratic Form. When the Equation (6) is expanded and all terms are grouped on the left–hand side of the equation, the resulting polynomial has x, y, x^2, xy, and y^2 terms. The general quadratic equation for an ellipse is

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + b_1 x + b_2 y + c = 0 \tag{7}$$

or in vector/matrix form,

$$\vec{Y}^T A \vec{Y} + \vec{B}^T \vec{Y} + c = 0 \tag{8}$$

where

$$\vec{Y} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}, \quad \text{and } \vec{B} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

All conic sections are represented by these equations. The ellipses are those for which $a_{11}a_{22} - a_{12}^2 > 0$. Observe that this condition states the determinant of A is positive, so A is an invertible matrix with inverse

denoted by A^{-1}. The matrix A and its inverse A^{-1} are both symmetric matrices since $A^T = A$ and $A^{-T} = (A^T)^{-1} = A^{-1}$.

A typical problem is to start with the general quadratic form and convert to the center–orient form. This can be done by first completing the square on the equation. Consider that

$$
\begin{aligned}
(\vec{Y} - \vec{K})^T A(\vec{Y} - \vec{K}) &= \vec{Y}^T A\vec{Y} - 2\vec{K}^T A\vec{Y} + \vec{K}^T A\vec{K} \\
&= (\vec{Y}^T A\vec{Y} + \vec{B}^T\vec{Y} + c) - (2A\vec{K} + \vec{B})^T\vec{Y} + (\vec{K}^T A\vec{K} - c) \\
&= -(2A\vec{K} + \vec{B})^T\vec{Y} + (\vec{K}^T A\vec{K} - c).
\end{aligned}
$$

If you set $\vec{K} = -A^{-1}\vec{B}/2$, then $\vec{K}^T A\vec{K} = \vec{B}^T A^{-1}\vec{B}/4$ and

$$(\vec{Y} - \vec{K})^T A(\vec{Y} - \vec{K}) = \vec{B}^T A^{-1}\vec{B}/4 - c.$$

Dividing by the scalar on the right–hand side of the last equation and setting $M = A/(\vec{B}^T A^{-1}\vec{B}/4 - c)$ produces
$$(\vec{Y} - \vec{K})^T M(\vec{Y} - \vec{K}) = 1.$$

Finally, M can be factored using an eigendecomposition into $M = RDR^T$ where R is a rotation matrix and D is a diagonal matrix whose diagonal entries are positive. The final equation obtained by substituting the factorization for M is exactly Equation (6).

For a 2×2 matrix, the eigendecomposition can be done symbolically. An eigenvector $\vec{V}$ of M corresponding to an eigenvalue λ is a nonzero vector such that $M\vec{V} = \lambda\vec{V}$. The eigenvalues are solutions to the quadratic equation $\det(M - \lambda I) = 0$ where I is the identity matrix. Since M is a symmetric matrix, the eigenvalues must be real numbers. For each eigenvalue, a corresponding eigenvector $\vec{V}$ is a nonzero solution to $(M - \lambda I)\vec{V} = \vec{0}$. Let $M = [m_{ij}]$. The quadratic equation is

$$
\begin{aligned}
0 &= \det(M - \lambda I) \\
&= \det \begin{bmatrix} m_{11} - \lambda & m_{12} \\ m_{12} & m_{22} - \lambda \end{bmatrix} \\
&= (m_{11} - \lambda)(m_{22} - \lambda) - m_{12}^2 \\
&= \lambda^2 - (m_{11} + m_{22})\lambda + (m_{11}m_{22} - m_{12}^2).
\end{aligned}
$$

The roots are

$$\lambda = \frac{(m_{11} + m_{22}) \pm \sqrt{(m_{11} + m_{22})^2 - 4(m_{11}m_{22} - m_{12}^2)}}{2} = \frac{(m_{11} + m_{22}) \pm \sqrt{(m_{11} - m_{22})^2 + 4m_{12}^2}}{2}. \quad (9)$$

The argument of the square root is nonnegative, so the roots must be real–valued. The only way for the roots to be equal is if $m_{11} = m_{22}$ and $m_{12} = 0$, in which case M must have been a scalar multiple of the identity matrix (the ellipse is really a circle). I assume for the remainder of the construction that the two eigenvalues are different.

Define λ_1 to be the eigenvalue in Equation (9) that uses the plus sign and define λ_2 to be the one that uses the minus sign. It is the case that $\lambda_1 > \lambda_2$. An eigenvector corresponding to λ_1 is $\vec{V}_1 = (\lambda_1 - m_{22}, m_{12})$. It turns out that any eigenvector corresponding to λ_2 must be perpendicular to $\vec{V}_1$, so we can choose an eigenvector

$\vec{V}_2 = (-m_{12}, \lambda_1 - m_{22})$. It is more convenient to use unit–length eigenvectors, so define $\vec{U}_i = \vec{V}_i/|\vec{V}_i|$ for $i = 1, 2$. By definition of eigenvectors, $M\vec{U}_1 = \lambda_1\vec{U}_1$ and $M\vec{U}_2 = \lambda_2\vec{U}_2$. We can write the two equations jointly by using a matrix $R = [\vec{U}_1\ \vec{U}_2]$ whose columns are the unit–length eigenvectors. The columns are unit length and perpendicular to each other, so R is an orthogonal matrix. In fact, by the choice of $\vec{U}_2$, R happens to be a rotation matrix (no reflection component so to speak). The joint equation is $MR = RD$ where $D = \text{Diag}(\lambda_1, \lambda_2)$. Multiplying on the right by R^T leads to the decomposition $M = RDR^T$.

In summary, for an ellipse specified as $a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + b_1x + b_2y + c = 0$, first verify that $a_{11}a_{22} - a_{12}^2 > 0$ (you really do have an ellipse). Then

1. The center is
$$\vec{K} = (k_1, k_2) = \frac{(a_{22}b_1 - a_{12}b_2, a_{11}b_2 - a_{12}b_1)}{2(a_{12}^2 - a_{11}a_{22})}. \tag{10}$$

2. Set $\mu = 1/(\vec{K}^T A\vec{K} - c) = 1/(a_{11}k_1^2 + 2a_{12}k_1k_2 + a_{22}k_2^2 - c)$ and define $m_{11} = \mu a_{11}$, $m_{12} = \mu a_{12}$, and $m_{22} = \mu a_{22}$.

3. Set $\lambda_1 = ((m_{11} + m_{22}) + \sqrt{(m_{11} - m_{22})^2 + 4m_{12}^2})/2$. The semimajor axis of the ellipse is
$$a = \frac{1}{\sqrt{\lambda_1}}. \tag{11}$$

   Set $\lambda_2 = ((m_{11} + m_{22}) - \sqrt{(m_{11} - m_{22})^2 + 4m_{12}^2})/2$. The semiminor axis of the ellipse is
$$b = \frac{1}{\sqrt{\lambda_2}}. \tag{12}$$

4. Set $\omega = 1/\sqrt{(\lambda_1 - m_{22})^2 + m_{12}^2}$. The major axis direction of the ellipse is
$$\vec{U}_1 = \omega(\lambda_1 - m_{22}, m_{12}). \tag{13}$$

   The minor axis direction of the ellipse is
$$\vec{U}_2 = \omega(-m_{12}, \lambda_1 - m_{22}). \tag{14}$$

5. For $R = [\vec{U}_1\ \vec{U}_2]$ where $\vec{U}_1$ and $\vec{U}_2$ are written as columns and $D = \text{Diag}(1/a^2, 1/b^2)$, the ellipse is represented by the factored form
$$(\vec{Y} - \vec{K})^T RDR^T(\vec{Y} - \vec{K}) = (\vec{Y} - \vec{K})^T \left(\frac{1}{a^2}\vec{U}_1\vec{U}_1^T + \frac{1}{b^2}\vec{U}_2\vec{U}_2^T\right)(\vec{Y} - \vec{K}) = 1. \tag{15}$$

6. Observe that $\vec{Y} = \vec{K} + R\vec{X} = \vec{K} + x\vec{U}_1 + y\vec{U}_2$. Replacing this in the factored form leads to $(x/a)^2 + (y/b)^2 = 1$, as expected since originally $\vec{Y}$ was selected to be the coordinates representing the rotation and translation of the standard form ellipse with coordinates $\vec{X}$.

7. The bounding rectangle for the ellipse that has the same directions as the major and minor axes of the ellipse has center $\vec{K}$. The four corners are $\vec{K} \pm a\vec{U}_1 \pm b\vec{U}_2$.