Polysolids and Boolean Operations

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This document is based solely on notes from Professors Hugh Maynard and Lucio Tavernini in the Computer Science Department at the University of Texas at San Antonio. The framework for polysolids was developed by them in the mid 1980s–they never published their work. The ideas are important for boolean operations in constructive planar geometry and in constructive solid geometry. Boolean operations on polysolids require no special–case handling as the framework is concise and powerful.

1 Polysolids

A *polysolid* is a generalization of the concept of polygonal regions (in 2D) and polyhedral regions (in 3D). The ideas apply in dimensions larger than three, but I do not discuss those extensions in this document. The definition is recursive. The intersection of a hyperplane with an n–dimensional polysolid is an $(n-1)$–dimensional polysolid within that hyperplane. The base case of the recursive definition is a singleton point, a $0D$ polysolid.

1.1 Polysolids in $\mathbb{R}^1$

Let $L = \{\ell_1, \ldots, \ell_n\} \subset \mathbb{R}^1$ be a finite set. The complement of L is

$$\mathbb{R}^1 \setminus L = \bigcup_{i=0}^{n} U_i$$

where $U_0 = (-\infty, \ell_1)$, $U_i = (\ell_i, \ell_{i+1})$ for $1 \le i \le n-1$, and $U_n = (\ell_n, +\infty)$. The *dual graph generated by* L is the graph $G_L = (V_L, E_L)$ where the graph vertices are $V_L = \{U_i\}_{i=0}^{n}$ and the graph edges are $E_L = \{(U_i, U_{i+1}) : 0 \le i \le n-1\}$. The vertices are the open intervals forming P and the edges are pairs of adjacent intervals. Note that G_L is a connected two–colorable graph. Define $c_L : V_L \to \{0,1\}$ to be the unique two–coloring of G_L such that $c_L(U_0) = 1$.

L defines two complementary *polysolids*, $P_0(L)$ and $P_1(L)$, defined by

$$P_\alpha(L) = \{U \in V_L : c_L(U) = \alpha\}$$

for $\alpha = 0, 1$. For example, if $L = \{a, b, c\}$, then $P_0(L) = (-\infty, a) \cup (b, c)$ and $P_1(L) = (a, b) \cup (c, +\infty)$. In this example both $P_0(L)$ and $P_1(L)$ are unbounded sets. If $L = \{a, b\}$, then $P_0(L) = (-\infty, a) \cup (b, +\infty)$ and $P_1(L) = (a, b)$. In this example $P_0(L)$ is unbounded and $P_1(L)$ is bounded. Generally, if L has an odd number of points, then both polysolids are unbounded. If L has an even number of points, then $P_0(L)$ is unbounded and $P_1(L)$ is bounded. In this case, $P_0(L)$ is said to be *cobounded* (its complement $\mathbb{R}^1 \setminus P_0(L)$ is a bounded set).

The set of 1–dimensional polysolids in $\mathbb{R}^1$ is defined by

$$\mathcal{P}^1(\mathbb{R}^1) = \{P_\alpha(L) : L \subset \mathbb{R}^1 \text{ is finite and } \alpha = 0, 1\}.$$

If $H \subset \mathbb{R}^n$ is a 1–dimensional affine manifold (fancy phrase for a line), then $\mathcal{P}^1(H)$ is defined to be the affine image of $\mathcal{P}^1(\mathbb{R}^1)$ under any affine transformation of $\mathbb{R}^1$ onto H. The set of all 1–dimensional polysolids in $\mathbb{R}^n$ is

$$\mathcal{P}^1(\mathbb{R}^n) = \bigcup\{\mathcal{P}^1(H) : H \text{ is a 1–dimensional affine manifold in } \mathbb{R}^n\}.$$

An important subset of $\mathcal{P}^1(\mathbb{R}^1)$ is the collection of pairs of bounded/cobounded polysolids, denoted $\mathcal{P}^1_b(\mathbb{R}^1)$. As noted earlier, these polysolids are generated by finite sets L with an even number of points. The set of bounded/cobounded polysolids corresponding to a 1–dimensional affine manifold H is denoted $\mathcal{P}^1_b(H)$ and the set of bounded/cobounded polysolids in $\mathbb{R}^n$ is denoted $\mathcal{P}^1_b(\mathbb{R}^n)$.

1.2 Polysolids in $\mathbb{R}^2$

Let L be a finite set of 1–dimensional polysolids in $\mathbb{R}^2$. If $e \in L$ is a polysolid, the set

$$\mathbb{R}^2 \setminus \bigcup_{e \in L} \text{Closure}(e)$$

is an open set and has a finite number of open components, $V_L = \{U_i\}_{i=0}^n$.

The set L is said to be *decomposing in* $\mathbb{R}^2$ iff V_L is composed of open sets U that satisfy the following topological condition,

$$\text{Interior}(\text{Closure}(U)) = U.$$

An open set U satisfying this condition is said to be *regular*. The use of decomposing finite sets avoids "dangling" line segments. For example, the four 1–dimensional bounded polysolids $(x, 0)$, $(x, 1)$, $(0, y)$, and $(1, y)$ for $0 \le x \le 1$ and $0 \le y \le 1$ form a set L which is decomposing. The set $\cup_{e \in L}\text{Closure}(e)$ is the perimeter of the unit square. The two regions inside and outside the square are regular open sets. However if L additionally contains the polysolid (x, x) for $0 \le x \le 1/2$, then L is not decomposing. This diagonal edge is a dangling line segment for the polysolid. Figure 1 illustrates the two cases.

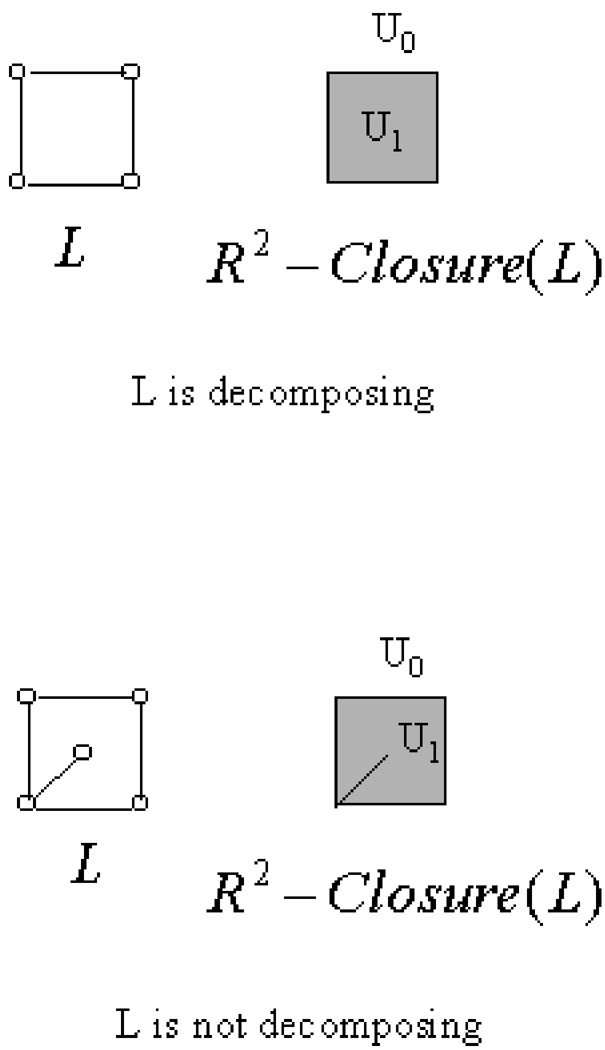

Figure 1. Decomposing and nondecomposing sets.

The *dual graph generated by* L is the graph $G_L = (V_L, E_L)$ where $(U_\alpha, U_\beta) \in E_L$ if and only if $U_\alpha \cup U_\beta$ is not a regular open set. Intuitively, U_α and U_β are separated by a 1–dimensional piecewise linear curve.

The set L is said to be *generating in* $\mathbb{R}^2$ if and only if it is decomposing and G_L is two–colorable. For example, the four polysolids mentioned earlier whose union of closures forms the unit square is generating. If L additionally contains the polysolid (x, x) for $0 \leq x \leq 1$, then L is decomposing, but not generating since the graph G_L is not two–colorable. Figure 2 illustrates this concept. The bottom example is not generating because L does not allow a two–coloring of the plane.

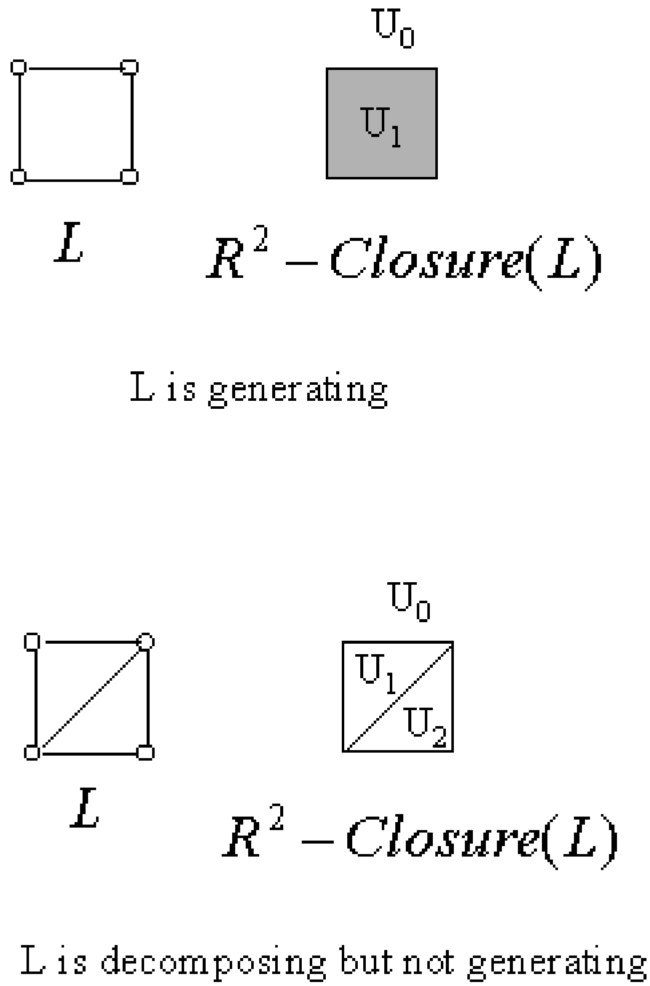

Figure 2. Generating and nongenerating sets.

If L is generating in $\mathbb{R}^2$, then L defines two polysolids $P_0(L)$ and $P_1(L)$ defined by

$$P_\alpha(L) = \{U \in V_L : c(U) = \alpha\}$$

where $c : V_L \to \{0, 1\}$ is a two–coloring of G_L. In this context the set L is referred to as the set of *polyfaces* of $P_\alpha(L)$.

The set of 2–dimensional polysolids in $\mathbb{R}^2$ is defined by

$$\mathcal{P}^2(\mathbb{R}^2) = \{P_\alpha(L) : L \in \mathcal{P}^1(\mathbb{R}^1) \text{ is finite and } \alpha = 0, 1\}.$$

If $H \subset \mathbb{R}^n$ is a 2–dimensional affine manifold (fancy name for a plane), then $\mathcal{P}^2(H)$ is defined to be the affine image of $\mathcal{P}^2(\mathbb{R}^2)$ under any affine transformation of $\mathbb{R}^2$ onto H. The set of all 2–dimensional polysolids in

$\mathbb{R}^n$ is

$$\mathcal{P}^2(\mathbb{R}^n) = \bigcup\{\mathcal{P}^2(H) : H \text{ is a 2–dimensional affine manifold in } \mathbb{R}^n\}.$$

An important subset of $\mathcal{P}^2(\mathbb{R}^2)$ is the collection of bounded/cobounded polysolids, denoted $\mathcal{P}^2_b(\mathbb{R}^2)$. These polysolids are generated by sets L of bounded 1–dimensional polysolids. The set of bounded/cobounded polysolids corresponding to a 2–dimensional affine manifold H is denoted $\mathcal{P}^2_b(H)$ and the set of 2–dimensional bounded/cobounded polysolids in $\mathbb{R}^n$ is denoted $\mathcal{P}^2_b(\mathbb{R}^n)$.

1.3 Polysolids in $\mathbb{R}^n$

Let L be a finite set of $(n-1)$–dimensional polysolids in $\mathbb{R}^n$. If $e \in L$ is a polysolid, denote $\bar{e}$ to be the topological closure of e. The set

$$\mathbb{R}^n \setminus \bigcup_{e \in L} \bar{e}$$

is an open set and has a finite number of open components, $V_L = \{U_i\}_{i=0}^m$.

The set L is said to be *decomposing in* $\mathbb{R}^n$ iff V_L is composed of regular open sets. The *dual graph generated by* L is the graph $G_L = (V_L, E_L)$ where $(U_\alpha, U_\beta) \in E_L$ if and only $U_\alpha \cup U_\beta$ is not a regular open set. The set L is said to be *generating in* $\mathbb{R}^n$ if and only if it is decomposing and G_L is two–colorable.

If L is generating in $\mathbb{R}^n$, then L defines two polysolids $P_0(L)$ and $P_1(L)$ defined by

$$P_\alpha(L) = \{U \in V_L : c(U) = \alpha\}$$

where $c : V_L \to \{0, 1\}$ is a two–coloring of G_L. In this context the set L is referred to as the set of *polyfaces* of $P_\alpha(L)$.

The set of n–dimensional polysolids in $\mathbb{R}^2$ is defined by

$$\mathcal{P}^n(\mathbb{R}^n) = \{P_\alpha(L) : L \in \mathcal{P}^{n-1}(\mathbb{R}^{n-1}) \text{ is finite and } \alpha = 0, 1\}.$$

If $H \subset \mathbb{R}^n$ is a k–dimensional affine manifold, then $\mathcal{P}^k(H)$ is defined to be the affine image of $\mathcal{P}^k(\mathbb{R}^k)$ under any affine transformation of $\mathbb{R}^k$ onto H. The set of all k–dimensional polysolids in $\mathbb{R}^n$ is

$$\mathcal{P}^k(\mathbb{R}^n) = \bigcup\{\mathcal{P}^k(H) : H \text{ is a } k\text{–dimensional affine manifold in } \mathbb{R}^n\}.$$

An important subset of $\mathcal{P}^n(\mathbb{R}^n)$ is the collection of bounded/cobounded polysolids, denoted $\mathcal{P}^n_b(\mathbb{R}^n)$. These polysolids are generated by sets L of bounded $(n-1)$–dimensional polysolids. The set of bounded/cobounded polysolids corresponding to a k–dimensional affine manifold H is denoted $\mathcal{P}^k_b(H)$ and the set of k–dimensional bounded/cobounded polysolids in $\mathbb{R}^n$ is denoted $\mathcal{P}^k_b(\mathbb{R}^n)$.

For general dimension n, a polysolid $\pi \in \mathcal{P}^n(\mathbb{R}^n)$ is described by its polyface set $L = f(\pi)$ and the color of the polysolid $c(\pi)$. The implementation of boolean operations on polysolids needs to determined both polyfaces and color.

2 Boolean Operations on Polysolids

Given polysolids in $\mathcal{P}^n(H)$, boolean operations are defined on these using the topological nature of the polysolid definitions. Let $\pi = P_\alpha(L)$, $\pi_1 = P_{\alpha_1}(L_1)$, and $\pi_2 = P_{\alpha_2}(L_2)$ denote polysolids where L, L_1, and L_2 are generating sets. Boolean operations are defined by:

NEGATION:	$\neg\pi = P_{1-\alpha}(L)$
INTERSECTION:	$\pi_1 \wedge \pi_2 = \pi_1 \cap \pi_2$
UNION:	$\pi_1 \vee \pi_2 = \text{Interior}(\text{Closure}(\pi_1 \cup \pi_2))$
DIFFERENCE:	$\pi_1 \neg \pi_2 = \pi_1 \cap \text{Complement}(\text{Closure}(\pi_2))$
EXCLUSIVE OR:	$\pi_1 \oplus \pi_2 = (\pi_1 \neg \pi_2) \cup (\pi_2 \neg \pi_1)$.

At the highest level, the algorithm for computing a boolean function of two polysolids $\pi = B(\pi_1, \pi_2)$ is as follows.

1. Determine the color α of the result of the function. This step is trivial in the implementation.
2. Determine which subelements of L_1 and L_2 can be used to define the generating set L for the result of the function. This step requires the following.
   (a) NORMALIZATION. Decompose the polyfaces L_1 and L_2 into components which are non–intersecting. This is accomplished by a *segmentation* of each polyface of one polysolid relative to the other polysolid. The segmentation is recursive through dimension. The essential work is done in segmenting a line (containing an edge of one polysolid) relative to the edges of a 2–dimensional polysolid.
   (b) ACCEPTANCE. Determine which of the normalized polyfaces to keep for the specified boolean operation. This is accomplished by maintaining *tags* on the segmented polyfaces relative to a polysolid according to the following relationships.

      o The polyface is *outside* the polysolid.

      i The polyface is *inside* the polysolid.

      \+ The polyface is a *positive boundary* of the polysolid. That is, the polyface lies on the boundary of the polysolid with the interior of the polysolid to the positive side of the hyperplane in which the polyface lives.

      \- The polyface is a *negative boundary* of the polysolid. That is, the polyface lies on the boundary of the polysolid with the interior of the polysolid to the negative side of the hyperplane in which the polyface lives.

The tags form the Klein–4 group whose binary operation is defined in the table below.

$\cdot$	o	i	-	+
o	o	i	-	+
i	i	o	+	-
-	-	+	o	i
+	+	-	i	o

The algorithm involves building four lists of polyfaces, one list per tag. The lists are merged according to the boolean operation, the final list yielding the resulting polysolid.

To briefly illustrate, consider two polysolids, π_1 a square and the π_2 an s–shaped object. They are shown normalized and superimposed in Figure 3 with the various tags on the edges.

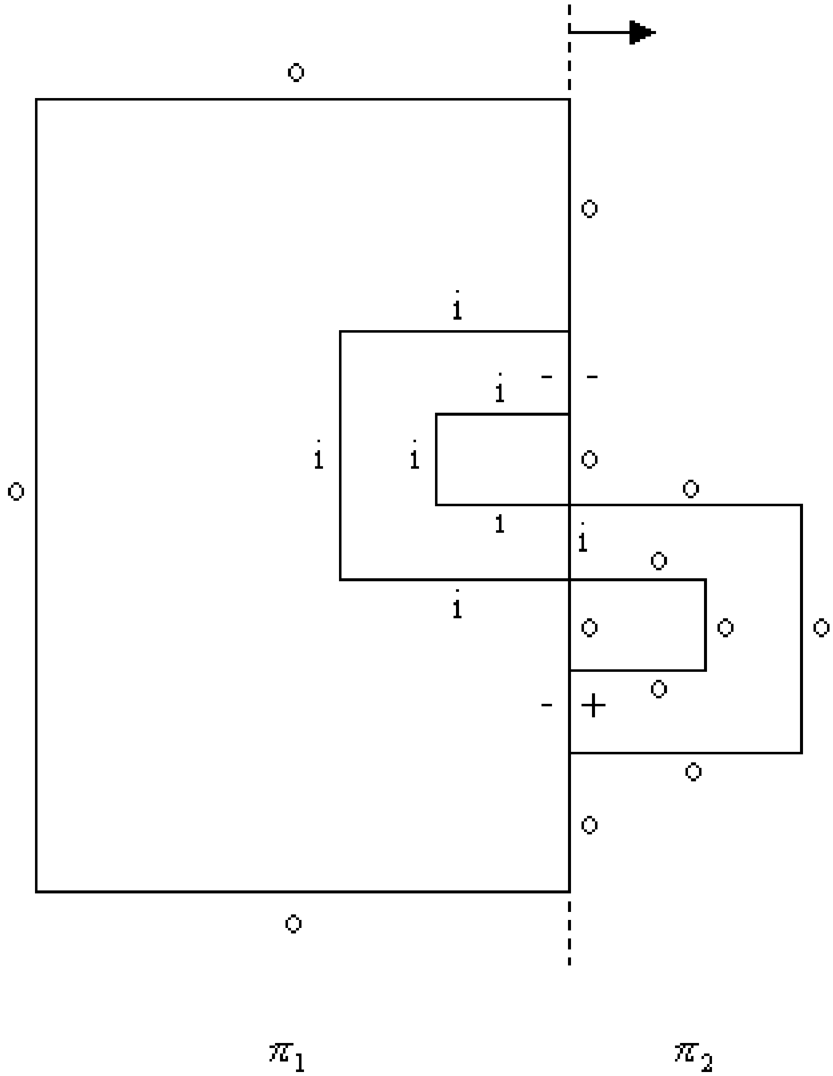

Figure 3. Segmentation of two bounded polysolids.

The union, intersection, difference, and exclusive or are shown in the next four figures.

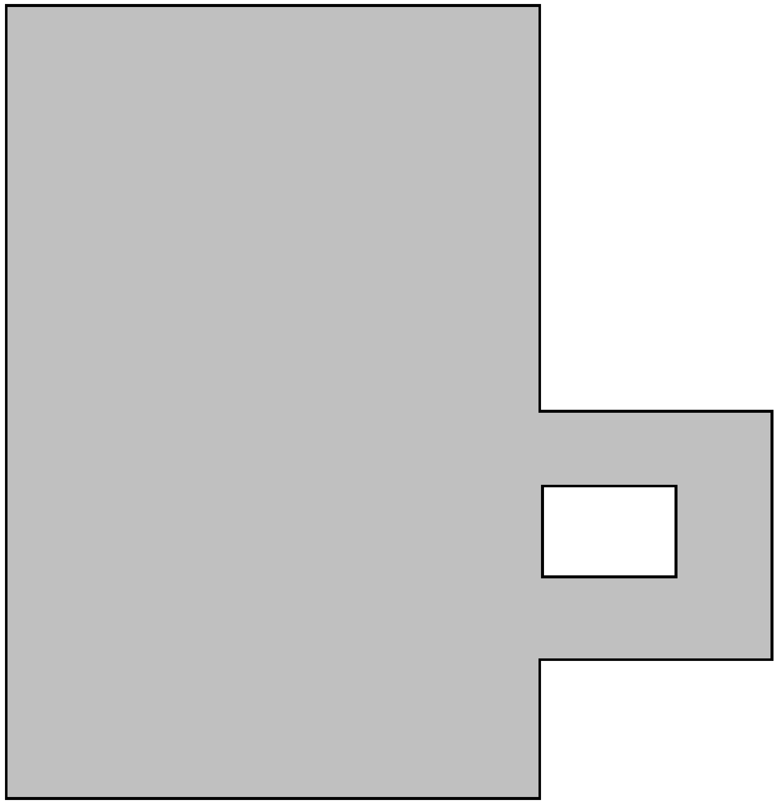

Figure 4. Union of the two polysolids.

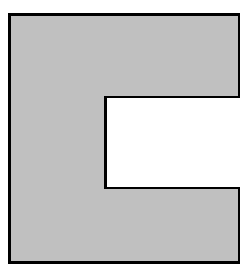

Figure 5. Intersection of the two polysolids.

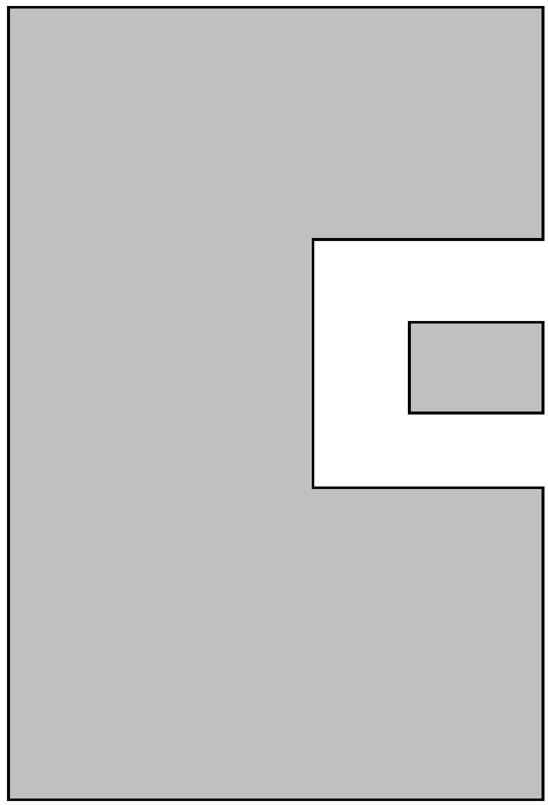

Figure 6. Difference of the two polysolids.

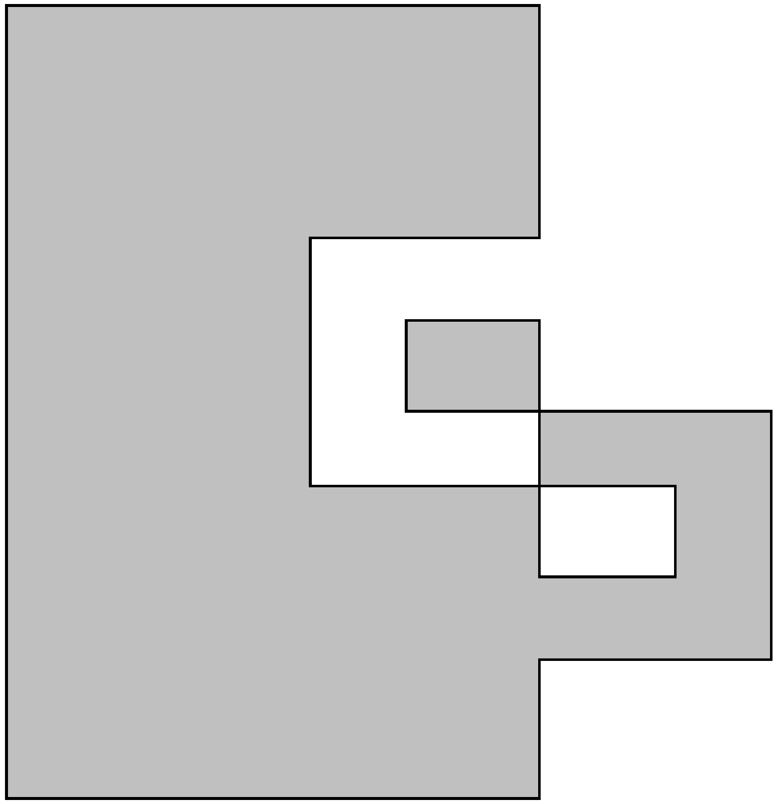

Figure 7. Exclusive Or of the two polysolids.

In the notes from Maynard and Tavernini, the line shown is drawn as an oriented line with normal pointing to the right. That oriented line was used for segmentation for both polysolids. In my implementation, the line normal is chosen to point to the side containing the bounded part of the segmenting polysolid. As such, my tags differ from theirs on a subedge which has the bounded parts of the two polysolids to opposite sides of that subedge There is one such subedge in Figure 3. The Maynard algorithm will mark the subedge with a $+$ and a $-$. My algorithm marks it with a $+$ and a $+$. When comparing two subedges for acceptance into a union, the Maynard algorithm sees that the subedge tags multiply to i and is rejected. In my algorithm, the subedge tags multiply to o, but the edge is still rejected because I compare the two subedges assuming that the end points are ordered. That is, the line has a specific direction vector determined from its normal vector and the subedges are constructed in the segmentation using the line direction to order the end points. While the subedges have the same tags, they have opposite directions. In the Maynard algorithm, the subedges have different tags and the same direction.

3 Boolean Operations on Polysolids in $\mathbb{R}^2$

3.1 Normalization

Each polysolid is segmented against the edges of the other polysolid. If P and Q are the two polysolids, segmenting Q against P is given by the following pseudocode.

```
for each edge E of Q do
{
    L = directed line containing E;
    S = segmentation of L by P;

    if S is not empty then
    {
        // L intersects P
        S = S intersected with E;
        if S is not empty then
        {
            compute tagged edge lists T[o,i,+,i] from S;
        }
    }

    if S is empty then
    {
        // E did not intersect P, E must be outside
        add E to list T[o];
    }
}
```

The tagged edge lists are used in the acceptance phase (see next subsection).

The key operation is the segmentation of a line L by a polysolid P. This is done by iterating over all edges of P and determining if and edge E amd L (1) intersect at an interior point of E, (2) intersect at an end point of E, or (3) do not intersect. In case (1) the tag on the intersection is i. In case (2) the tag on the intersection is $+$ if E is on the positive side of the line (the side to which the normal vector points) or i if E is on the negative side of the line. If E is contained in L, then no tagging is necessary (edges of P parallel to L need not be processed).

Figure 8 shows a line and a polysolid consisting of three components, the last of which has a hole. The normal vector for the line is drawn and the positive side of the line is that side pointed to by the normal. The tagging of the intersection points is described for the four points labeled a, b, c, and d. The shading is used to help visual how the polysolid is built from the edges and which parts are above or below the line.

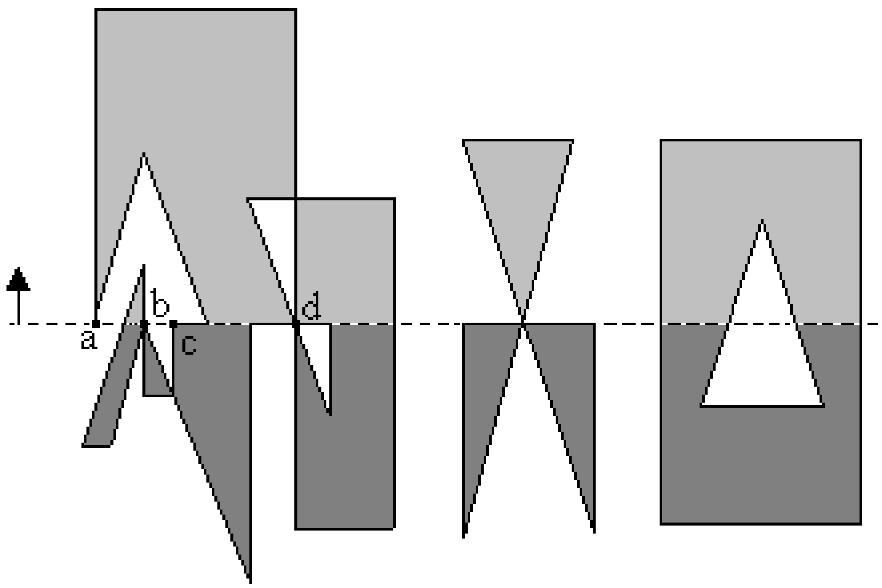

Figure 8. Polysolid segmenting a line.

At point a there are two edges intersecting the line. The common intersection point has tag $+$ from the first edge and tag $+$ from the second edge. The final tag on the intersection point is the Klein–4 product of the tags, $o = + \cdot +$. At point b there are three edges intersecting the line. The tags are $-$, $-$, and i. The final tag is the product $i = - \cdot - \cdot i$. At point c there are two edges, but one edge is contained in the line and can be ignored. The final tag is $-$, the tag generated from the other edge. At point d there are two edges intersecting the line, both with a i tag. The final tag is the product $o = i \cdot i$. Figure 9 shows the four situations.

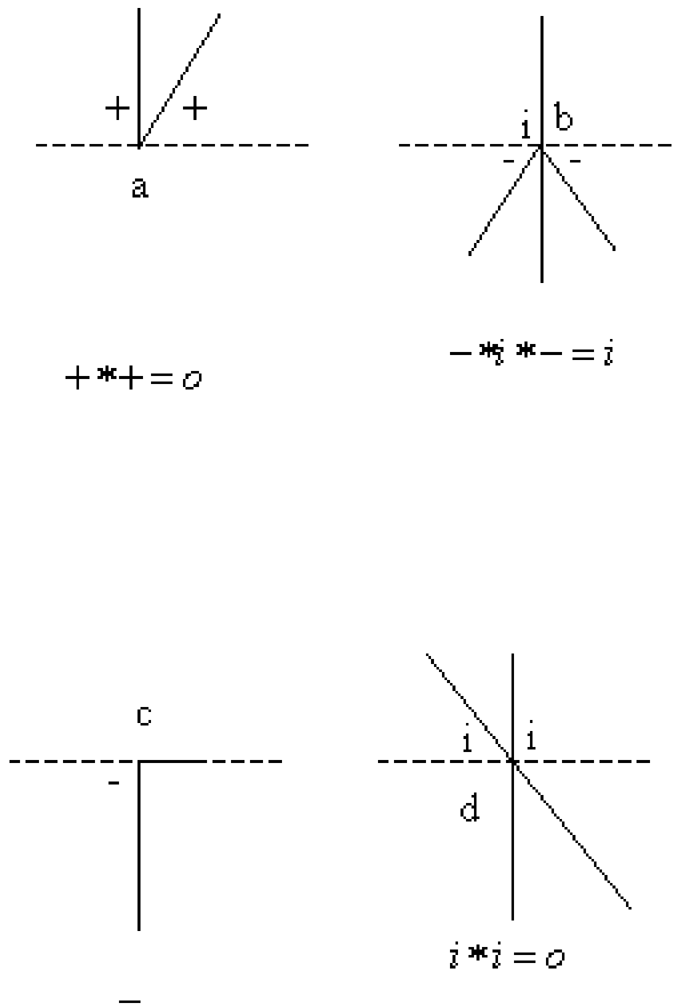

Figure 9. Computation of point tags.

For the entire line there are 16 points of intersection (counting only the two end points for each edge contained in the line). The intervals are tagged starting with the left–most half–infinite interval having tag *o*. Traversing from left to right, the tag on the next interval is the tag of the previous interval times the tag of the point separating the two intervals. Figure 10 shows the point and interval tags.

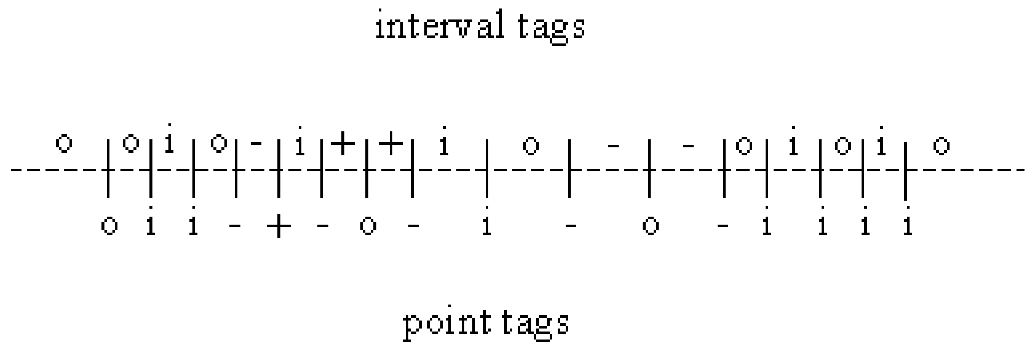

Figure 10. Point and interval tags.

This shows how the Klein–4 group multiplications allow the tagging of higher dimensional structures from the lower dimensional ones. This idea carries over into polysolids in higher dimensions.

3.2 Acceptance

After normalization we have four lists of tagged edges corresponding to the o, i, $+$, and $-$ tags for each of the two polysolids. These lists are merged as described below to obtain the various boolean combinations of the polysolids. It is instructive to apply these rules to the polysolids in Figure 1 to obtain the boolean results in Figure 2.

- Union. All o tagged edges are in the union. Pairs of edges having the same direction and both $+$ tags or both $-$ tags are in the union.
- Intersection. All i tagged edges are in the intersection. All $+$ tags are in the intersection (duplicates between the two $+$ lists must be avoided).
- Difference. Let the polysolids be labeled P and Q. The difference is $P \setminus Q$. This can be thought of as the intersection of P and $\neg Q$ where the negation indicates to change the color of the polysolid. When comparing P against Q, the tags on P are computed based on the bounded portion of Q. To compare against $\neg Q$ requires the tags on P to be negated (in a sense). Thus, all o tagged edges of P and all i tagged edges of Q are in the difference. When merging, the directions of the i tagged edges of Q must be reversed. Also, all $+$ tagged edges are kept in an intersection, but because we are comparing against $\neg Q$, the $-$ tagged edges of P and the $+$ tagged edges of Q are in the difference (duplicates between these two lists must be avoided).
- Exclusive Or. For polysolids P and Q, this is simply computed as the union of the differences $P \setminus Q$ and $Q \setminus P$.