Intersection of Cylinders

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1 Introduction

This document shows how to determine if two bounded cylinders intersect. The algorithm uses the method of separating axes, although the construction is more complicated than one encounters when separating convex polyhedra. The resulting algorithm is a fairly expensive one if you plan on using cylinders for bounding volumes in a real–time graphics engine. A better alternative to a cylinder is a *capsule*, the set of points a specified distance from a line segment. Two capsules intersect if and only if the distance between capsule line segments is smaller or equal to the sum of the capsule radii, a much cheaper test to perform.

2 Nonintersection of Convex Objects by Projection Methods

Consider the problem of determining whether or not two convex objects in 3D are intersecting. This *test intersection* geometric query is not concerned about constructing the intersection set, if it exists. The latter problem is denoted a *find intersections* geometric query. The methods described here involve projection of the objects onto linear subspaces and testing for intersection on the projected sets.

2.1 Separation by Projection onto a Line

A test for nonintersection of two convex objects is simply stated: If there exists a line for which the intervals of projection of the two objects onto that line do not intersect, then the objects are do not intersect. Such a line is called a *separating line* or, more commonly, a *separating axis*. The translation of a separating line is also a separating line, so it is sufficient to consider lines that contain the origin. Given a line containing the origin and with unit–length direction $\vec{D}$, the projection of a compact convex set C onto the line is the interval

$$I = [\lambda_{\min}(\vec{D}), \lambda_{\max}(\vec{D})] = [\min\{\vec{D}\cdot\vec{X} : \vec{X} \in C\}, \max\{\vec{D}\cdot\vec{X} : \vec{X} \in C\}].$$

Two compact convex sets C_0 and C_1 are separated if there exists a direction $\vec{D}$ such that the projection intervals I_0 and I_1 do not intersect, $I_0 \cap I_1 = \emptyset$. Specifically they do not intersect when

$$\lambda^{(0)}_{\min}(\vec{D}) > \lambda^{(1)}_{\max}(\vec{D}) \text{ or } \lambda^{(0)}_{\max}(\vec{D}) < \lambda^{(1)}_{\min}(\vec{D}).$$

The superscript corresponds to the index of the convex set. Although the comparisons are made where $\vec{D}$ is unit–length, the comparison results are invariant to changes in length of the vector. This follows from

$\lambda_{\min}(t\vec{D}) = t\lambda_{\min}(\vec{D})$ and $\lambda_{\max}(t\vec{D}) = t\lambda_{\max}(\vec{D})$ for $t > 0$. The Boolean value of the pair of comparisons is also invariant when $\vec{D}$ is replaced by the opposite direction $-\vec{D}$. This follows from $\lambda_{\min}(-\vec{D}) = -\lambda_{\max}(\vec{D})$ and $\lambda_{\max}(-\vec{D}) = -\lambda_{\min}(\vec{D})$. When $\vec{D}$ is not unit–length, the intervals obtained for the separating axis tests are not the projections of the object onto the line, rather they are scaled versions of the projection intervals. I make no distinction in this document between the scaled projection and regular projection. I will also use the terminology that the direction vector for a separating axis is called a *separating direction*, said direction not necessarily being unit–length.

For a pair of convex polyhedra, only a finite set of direction vectors needs to be considered for separation tests. That set includes the normal vectors to the faces of the polyhedra and vectors generated by a cross product of two edges, one from each polyhedron. I am aware of no general theory for constructing the smallest set of potential separating directions for other convex objects.

2.2 Separation by Projection onto a Plane

Another test for nonintersection of two convex objects is stated: If there exists a plane for which the regions of projection of the two objects onto that line do not intersect, then the objects are do not intersect. Such a plane is called a *separating plane*. This is not to be confused with a plane that is perpendicular to a separating line and for which the objects are on opposite sides of the plane. The translation of a separating plane is also a separating plane, so it is sufficient to consider planes that contain the origin. Given a plane containing the origin and with unit–length normal $\vec{N}$, the projection of a compact convex set C onto the line is the set of points

$$R = \{\vec{Y} : \vec{Y} = \vec{X} - (\vec{N} \cdot \vec{X})\vec{N} = (I - \vec{N}\vec{N}^{\mathrm{T}})\vec{X}, \ \ \vec{X} \in C\}$$

where I is the 3×3 identity matrix. The projection set is itself a compact convex set. Two compact convex sets C_0 and C_1 are separated if there exists a normal $\vec{N}$ such that the projection sets R_0 and R_1 do not intersect, $R_0 \cap R_1 = \emptyset$. The determination of this condition can involve one of many geometric methods, for example by showing that the distance between the two sets is positive. It might be possible to analyze the projections in native 2D and attempting to find a separating line in 2D, but such a construction should work as well in 3D.

3 Representation of a Cylinder

A cylinder has a center point $\vec{C}$, unit–length axis direction $\vec{W}$, radius r and height h. The end disks of the cylinder are located at $\vec{C} \pm (h/2)\vec{W}$. Let $\vec{U}$ and $\vec{V}$ be any unit–length vectors so that $\{\vec{U}, \vec{V}, \vec{W}\}$ is a right–handed set of orthonormal vectors. That is, the vectors are unit length, mutually orthogonal, and $\vec{W} = \vec{U} \times \vec{V}$. Points on the cylinder surface are parameterized by

$$\vec{X}(\theta, t) = \vec{C} + (r\cos\theta)\vec{U} + (r\sin\theta)\vec{V} + t\vec{W}, \ \ \theta \in [0, 2\pi), \ |t| \leq h/2.$$

The end disks are parameterized by

$$\vec{X}(\theta, \rho) = \vec{C} + (\rho\cos\theta)\vec{U} + (\rho\sin\theta)\vec{V} \pm (h/2)\vec{W}, \ \ \theta \in [0, 2\pi), \ \rho \in [0, r].$$

The projections of a cylinder onto a line or plane are determinely solely by the cylinder wall, not the end disks, so the second parameterization is not relevant for intersection testing.

The choice of $\vec{U}$ and $\vec{V}$ is arbitrary. Intersection queries between cylinders should be independent of this choice, but some of the algorithms are better handled if a choice is made. A quadratic equation that represents the cylinder wall is $(\vec{X}-\vec{C})^{\mathrm{T}}(I-\vec{W}\vec{W}^{\mathrm{T}})(\vec{X}-\vec{C}) = r^2$. The boundedness of the cylinder is specified by $|\vec{W}\cdot(\vec{X}-\vec{C})| \leq h/2$. This representation is dependent only on $\vec{C}$, $\vec{W}$, r, and h.

4 Projection of a Cylinder onto a Line

Let the line be $s\vec{D}$ where $\vec{D}$ is a nonzero vector. The projection of a cylinder point onto the line is

$$\lambda(\theta,t) = \vec{D}\cdot\vec{X}(\theta,t) = \vec{D}\cdot\vec{C} + (r\cos\theta)\vec{D}\cdot\vec{U} + (r\sin\theta)\vec{D}\cdot\vec{V} + t\vec{D}\cdot\vec{W}.$$

The interval of projection has end points determined by the extreme values of this expression. The maximum value occurs when all three terms involving the parameters are made as large as possible. The t–term has a maximum of $(h/2)|\vec{D}\cdot\vec{W}|$. The θ–terms, not including the radius, can be viewed as a dot product $(\cos\theta,\sin\theta)\cdot(\vec{D}\cdot\vec{U},\vec{D}\cdot\vec{V})$. This is maximum when $(\cos\theta,\sin\theta)$ is in the same direction as $(\vec{D}\cdot\vec{U},\vec{D}\cdot\vec{V})$. Therefore,

$$(\cos\theta,\sin\theta) = \frac{(\vec{D}\cdot\vec{U},\vec{D}\cdot\vec{V})}{\sqrt{(\vec{D}\cdot\vec{U})^2+(\vec{D}\cdot\vec{V})^2}}.$$

and the maximum projection value is

$$\lambda_{\max} = \vec{D}\cdot\vec{C} + r\sqrt{|\vec{D}|^2-(\vec{D}\cdot\vec{W})^2} + (h/2)|\vec{D}\cdot\vec{W}|$$

where I have used the fact that $\vec{D} = (\vec{D}\cdot\vec{U})\vec{U} + (\vec{D}\cdot\vec{V})\vec{V} + (\vec{D}\cdot\vec{W})\vec{W}$, which implies $(\vec{D}\cdot\vec{U})^2 + (\vec{D}\cdot\vec{V})^2 + (\vec{D}\cdot\vec{W})^2 = |\vec{D}|^2$. The minimum projection value is similarly derived,

$$\lambda_{\min} = \vec{D}\cdot\vec{C} - r\sqrt{|\vec{D}|^2-(\vec{D}\cdot\vec{W})^2} - (h/2)|\vec{D}\cdot\vec{W}|.$$

5 Projection of a Cylinder onto a Plane

Let the plane be $\vec{N}\cdot\vec{X} = 0$ where $\vec{N}$ is a unit–length normal. The projection of a cylinder onto a plane has one of three geometric configurations

1. a disk when $\vec{W}$ is parallel to $\vec{N}$,
2. a rectangle when $\vec{W}$ is perpendicular to $\vec{N}$, or
3. a rectangle with hemielliptical caps.

The projection matrix is $P = I - \vec{N}\vec{N}^{\mathrm{T}}$. In the first case, the center of the disk is $P\vec{C}$ and the radius is r. In the second case, the rectangle has center $P\vec{C}$ and has unit–length axis directions $\vec{W}$ and $\vec{W}\times\vec{N}$. The four corners of the rectangle are $P\vec{C} \pm r\vec{W}\times\vec{N} \pm (h/2)\vec{W}$.

The third case is only slightly more complicated. The center point of the projection region is $P\vec{C}$. The axis of the projection region has non–unit–length direction $P\vec{W}$. An axis of the cylinder that is in the plane and perpendicular to $\vec{N}$ has direction $\vec{U} = (P\vec{W}) \times \vec{N}/|(P\vec{W}) \times \vec{N}|$. The four points on the cylinder that map to the four corners of the rectangular portion of the projection are $\vec{C} \pm r\vec{U} \pm (h/2)\vec{W}$. The four corners are $P\vec{C} \pm r\vec{U} \pm (h/2)P\vec{W}$.

Let $\vec{V} = \vec{W} \times \vec{U}$. The end circles of the cylinder are $\vec{X}(\theta) = \vec{C} \pm r((\cos\theta)\vec{U} + (\sin\theta)\vec{V}) \pm (h/2)\vec{W}$. Let $\vec{Y} = P(\vec{X} - \vec{C} \pm (h/2)\vec{W})$; then $\vec{Y} = r((\cos\theta)\vec{U} + (\sin\theta)P\vec{V})$. Therefore, $\vec{U} \cdot \vec{Y} = r\cos\theta$ and $P\vec{V} \cdot \vec{Y} = |P\vec{V}|^2 r \sin\theta$. Combining these yields

$$\begin{aligned}
1 &= \tfrac{1}{r^2}\left((\vec{U} \cdot \vec{Y})^2 + \tfrac{1}{|P\vec{V}|^4}(P\vec{V} \cdot \vec{Y})^2\right) \\
&= \tfrac{1}{r^2}\vec{Y}^{\mathrm{T}}\left(\vec{U}\vec{U}^{\mathrm{T}} + \tfrac{1}{|P\vec{V}|^2}\tfrac{P\vec{V}}{|P\vec{V}|}\tfrac{P\vec{V}}{|P\vec{V}|}^{\mathrm{T}}\right)\vec{Y} \\
&= (P(\vec{X} - \vec{C} \pm (h/2)\vec{W}))^{\mathrm{T}}\left(\tfrac{1}{r^2}\vec{U}\vec{U}^{\mathrm{T}} + \tfrac{1}{r^2|P\vec{V}|^2}\tfrac{P\vec{V}}{|P\vec{V}|}\tfrac{P\vec{V}}{|P\vec{V}|}^{\mathrm{T}}\right)(P(\vec{X} - \vec{C} \pm (h/2)\vec{W})).
\end{aligned}$$

This is the equation for two ellipses with centers at $P(\vec{C} \pm (h/2)\vec{W})$, axes $\vec{U}$ and $P\vec{V}/|P\vec{V}|$, and axis half–lengths r and $r|P\vec{V}|$.

6 Separating Line Tests for Two Cylinders

Given two cylinders with centers $\vec{C}_i$, axis directions $\vec{W}_i$, radii r_i, and heights h_i, for $i = 0, 1$, the cylinders are separated if there exists a nonzero direction $\vec{D}$ such that either

$$\vec{D} \cdot \vec{C}_0 - r_0\sqrt{|\vec{D}|^2 - (\vec{D} \cdot \vec{W}_0)^2} - (h_0/2)|\vec{D} \cdot \vec{W}_0| > \vec{D} \cdot \vec{C}_1 + r_1\sqrt{|\vec{D}|^2 - (\vec{D} \cdot \vec{W}_1)^2} + (h_1/2)|\vec{D} \cdot \vec{W}_1|$$

or

$$\vec{D} \cdot \vec{C}_0 + r_0\sqrt{|\vec{D}|^2 - (\vec{D} \cdot \vec{W}_0)^2} + (h_0/2)|\vec{D} \cdot \vec{W}_0| < \vec{D} \cdot \vec{C}_1 - r_1\sqrt{|\vec{D}|^2 - (\vec{D} \cdot \vec{W}_1)^2} - (h_1/2)|\vec{D} \cdot \vec{W}_1|.$$

Defining $\vec{\Delta} = \vec{C}_1 - \vec{C}_0$, these tests can be rewritten as a single expression, $f(\vec{D}) < 0$, where

$$f(\vec{D}) = r_0|P_0\vec{D}| + r_1|P_1\vec{D}| + (h_0/2)|\vec{D} \cdot \vec{W}_0| + (h_1/2)|\vec{D} \cdot \vec{W}_1| - |\vec{D} \cdot \vec{\Delta}|.$$

and where $P_i = I - \vec{W}_i\vec{W}_i^{\mathrm{T}}$ for $i = 0, 1$.

If $\vec{\Delta} = 0$, then $f \geq 0$. This is geometrically obvious since two cylinders with the same center already intersect. The remainder of the discussion assumes $\vec{\Delta} \neq \vec{0}$. If $\vec{D}$ is perpendicular to $\vec{\Delta}$, then $f(\vec{D}) \geq 0$. This shows that any line perpendicular to the line containing the two cylinder centers can never be a separating axis. This is also clear geometrically. The line of sight $\vec{C}_0 + s\vec{\Delta}$ intersects both cylinders at their centers. If you project the two cylinders onto the plane $\vec{\Delta} \cdot (\vec{X} - \vec{C}_0) = 0$, both regions of projection overlap. No matter which line you choose containing $\vec{C}_0$ in this plane, the line intersects both projection regions.

If $\vec{D}$ is a separating direction, then $f(\vec{D}) < 0$. Observe that $f(t\vec{D}) = |t|f(\vec{D})$, so $f(t\vec{D}) < 0$ for any t. This is consistent with the geometry of the problem. Any nonzero multiple of a separating direction must itself be a separating direction. This allows us to restrict our attention to the unit sphere, $|\vec{D}| = 1$. Function f is continuous on the unit sphere, a compact set, so f must attain its minimum at some point on the sphere. This is a minimization problem in two dimensions, but the spherical geometry complicates the analysis somewhat. Different restrictions on the set of potential separating directions can be made that yield minimization problems in a line or a plane rather than on a sphere.

The analysis of f involves computing its derivatives, $\vec{\nabla}(f)$, and determine its critical points. These are points for which $\vec{\nabla}(f)$ is zero or undefined. The latter category is easy to specify. The gradient is undefined when any of the terms inside the five absolute value signs is zero. Thus, $\vec{\nabla}(f)$ is undefined at $\vec{W}_0$, $\vec{W}_1$, at vectors that are perpendicular to $\vec{W}_0$, at vectors that are perpendicular to $\vec{W}_1$, and at vectors that are perpendicular to $\vec{\Delta}$. I already argued that $f \geq 0$ for vectors perpendicular to $\vec{\Delta}$, so we can ignore this case.

7 Tests at $\vec{W}_0$, $\vec{W}_1$, and $\vec{W}_0 \times \vec{W}_1$

The cylinder axis directions themselves can be tested first for separation. The test function values are

$$f(\vec{W}_0) = r_1|\vec{W}_0 \times \vec{W}_1| + (h_0/2) + (h_1/2)|\vec{W}_0 \cdot \vec{W}_1| - |\vec{W}_0 \cdot \vec{\Delta}|$$

and

$$f(\vec{W}_1) = r_0|\vec{W}_0 \times \vec{W}_1| + (h_0/2)|\vec{W}_0 \cdot \vec{W}_1| + (h_1/2) - |\vec{W}_1 \cdot \vec{\Delta}|.$$

If either function value is negative, the cylinders are separated. The square roots can be avoided. For example, the test $f(\vec{W}_0) < 0$ is equivalent to

$$r_1|\vec{W}_0 \times \vec{W}_1| < |\vec{W}_0 \cdot \vec{\Delta}| - h_0/2 - (h_1/2)|\vec{W}_0 \cdot \vec{W}_1| =: \rho.$$

The right–hand side is evaluated. If $\rho \leq 0$, then the inequality cannot be true since $\vec{W}_0 \times \vec{W}_1 \neq \vec{0}$ and the left–hand side is positive. Otherwise, $\rho > 0$ and it is now enough to test $r_1|\vec{W}_0 \times \vec{W}_1|^2 < \rho^2$. A similar construction applies to $f(\vec{W}_1) < 0$.

One last test that does not require many more operations and might lead to a quick no–intersection test is

$$f(\vec{W}_0 \times \vec{W}_1) = (r_0 + r_1)|\vec{W}_0 \times \vec{W}_1| - |\vec{W}_0 \times \vec{W}_1 \cdot \vec{\Delta}| < 0$$

or equivalently

$$(r_0 + r_1)^2|\vec{W}_0 \times \vec{W}_1|^2 < |\vec{W}_0 \times \vec{W}_1 \cdot \vec{\Delta}|^2,$$

assuming of course that $\vec{W}_0 \times \vec{W}_1 \neq \vec{0}$. This vector is actually one for which the gradient of f is undefined.

If $\vec{W}_0$ and $\vec{W}_1$ are parallel, then $\vec{W}_0 \times \vec{W}_1 = \vec{0}$ and $|\vec{W}_0 \cdot \vec{W}_1| = 1$. The identity

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$$

can be used to show that $|\vec{W}_0 \times \vec{W}_1|^2 = 1 - (\vec{W}_0 \cdot \vec{W}_1)^2$. The test function for $\vec{W}_0$ evaluates to

$$f(\vec{W}_0) = (h_0 + h_1)/2 - |\vec{W}_0 \cdot \vec{\Delta}|.$$

The two cylinders are separated if $(h_0 + h_1)/2 < |\vec{W}_0 \cdot \vec{\Delta}|$. If $f(\vec{W}_0) \geq 0$, the two cylinders are potentially separated by a direction that is perpendicular to $\vec{W}_0$. Geometrically it is enough to determine whether or not the circles of projection of the cylinders onto the plane $\vec{W}_0 \cdot \vec{X} = \vec{0}$ intersect. These circles are disjoint if and only if the length of the projection of $\vec{\Delta}$ onto that plane is larger than the sum of the radii of the circles. The projection of $\vec{\Delta}$ is $\vec{\Delta} - (\vec{W}_0 \cdot \vec{\Delta})\vec{W}_0$ and its squared length is

$$|\vec{\Delta} - (\vec{W}_0 \cdot \vec{\Delta})\vec{W}_0|^2 = |\vec{\Delta}|^2 - (\vec{W}_0 \cdot \vec{\Delta})^2.$$

The sum of the radii of the circles is the sum of the radii of the cylinders, $r_0 + r_1$, so the two cylinders are separated if $|\vec{\Delta}|^2 - (\vec{W}_0 \cdot \vec{\Delta})^2 > (r_0 + r_1)^2$.

For the remainder of this document I assume that $\vec{W}_0$ and $\vec{W}_1$ are not parallel.

8 Tests at Vectors Perpendicular to $\vec{W}_0$ or $\vec{W}_1$

Considering the domain of f to be the unit sphere, the set of vectors perpendicular to $\vec{W}_0$ is a great circle on the sphere. The gradient of f is undefined on this great circle. Define $\vec{D}(\theta) = (\cos\theta)\vec{U}_0 + (\sin\theta)\vec{V}_0)$ and $F(\theta) = f(\vec{D}(\theta))$. If we can show that $F(\theta) < 0$ for some $\theta \in [0, 2\pi)$, then the corresponding direction is a separating line for the cylinders. However, F is a somewhat complicated function that does not lend itself to a simple analysis. Since $f(-\vec{D}) = f(\vec{D})$, we may restrict our attention to only half of the great circle. Rather than restricting f to a half circle, we can restrict it to a tangent line $\vec{D}(x) = x\vec{U}_0 + \vec{V}_0$ and define $F(x) = f(\vec{D}(x))$, so

$$
\begin{aligned}
F(x) &= r_0\sqrt{x^2+1} + r_1|(P_1\vec{U}_0)x + (P_1\vec{V}_0)| + (h_1/2)|(\vec{W}_1 \cdot \vec{U}_0)x + (\vec{W}_1 \cdot \vec{V}_0)| - |(\vec{\Delta} \cdot \vec{U}_0)x + (\vec{\Delta} \cdot \vec{V}_0)| \\
&= r_0\sqrt{x^2+1} + r_1|\vec{A}_0 x + \vec{B}_0| + (h_1/2)|a_1 x + b_1| - |a_2 x + b_2|.
\end{aligned}
$$

This function is more readily analyzed by breaking it up into four cases by replacing the last two absolute values with sign indicators,

$$
G(x) = r_0\sqrt{x^2+1} + r_1|\vec{A}_0 x + \vec{B}_0| + \sigma_1(h_1/2)(a_1 x + b_1) - \sigma_2(a_2 x + b_2)
$$

with $|\sigma_1| = |\sigma_2| = 1$. The minimum of G is calculated for each choice of (σ_1, σ_2) by computing $G'(x)$ and determining where it is zero or undefined. Any critical point x must first be tested to see if is consistent with the choice of signs. That is, a critical point must be tested to make sure $\sigma_1(a_1x+b_1) \geq 0$ and $\sigma_2(a_2x+b_2) \geq 0$. If so, then $G(x)$ is evaluated and compared to zero. The derivative is

$$
G'(x) = r_0\frac{x}{\sqrt{x^2+1}} + r_1\vec{A}_0 \cdot \frac{\vec{A}_0 x + \vec{B}_0}{|\vec{A}_0 x + \vec{B}_0|} + (\sigma_1 h_1/2)a_1 - \sigma_2 b_2.
$$

The derivative is undefined when $|\vec{A}_0 x + \vec{B}_0|$, but this case is actually generated when the original direction is parallel to $\vec{W}_0 \times \vec{W}_1$, discussed earlier. To algebraically solve $G'(x) = 0$, a few squaring operations can be applied. Note that $G'(x) = 0$ is of the form

$$
L_0\sqrt{Q_0} + L_1\sqrt{Q_1} = c\sqrt{Q_0 Q_1}
$$

where L_i are linear in x, Q_i are quadratic in x, and c is a constant. Squaring and rearranging terms yields

$$
2L_0L_1\sqrt{Q_0Q_1} = c^2Q_0Q_1 - L_0^2Q_0 - L_1^2Q_1.
$$

Squaring again and rearranging terms yields

$$
4L_0^2L_1^2Q_0Q_1 - (c^2Q_0Q_1 - L_0^2Q_0 - L_1^2Q_1)^2 = 0.
$$

The left–hand side is a polynomial in x of degree 8. The roots can be compute by numerical methods, tested for validity as shown earlier, and then G can be tested for negativity.

Yet one more alternative is to notice that attempting to locate a separating direction that is perpendicular to $\vec{W}_0$ is equivalent to projecting the two cylinders onto the plane $\vec{W}_0 \cdot \vec{X} = 0$ and determining if the projections are disjoint. The first cylinder projects to a disk. The second cylinder projects to a disk, a rectangle, or a rectangle with hemielliptical caps depending on the relationship of $\vec{W}_1$ to $\vec{W}_0$. Separation can be determined by showing that (1) the projection of $\vec{C}_0$ is not inside the projection of the second cylinder and (2) the distance from $\vec{C}_0$ to the projection of the second cylinder is larger than r_0. If the second projection is a disk, the distance is just the length of the projection of $\vec{\Delta}$. If the second projection is a rectangle, then the problem amounts to computing the distance between a point and a rectangle in the plane. This test is an inexpensive one. If the second projection is a rectangle with hemielliptical caps, then the problem amounts to computing the minimum of the distances between a point and a rectangle and two ellipses, then comparing it to r_0. Calculating the distance between a point and an ellipse in the plane requires finding roots of a polynomial of degree 4. This alternative trades off, in worst case, finding the roots to a polynomial of degree 8 for finding the roots of two polynomials of degree 4.

9 Tests for Directions at which $\vec{\nabla}(f) = \vec{0}$

The symmetry $f(-\vec{D}) = f(\vec{D})$ implies that we only need to analyze f on a hemisphere; the other hemisphere values are determined automatically. Since $f \geq 0$ on the great circle of vectors that are perpendicular to $\vec{\Delta}$, we can restrict our attention to the hemisphere whose pole is $\vec{W} = \vec{\Delta}/|\vec{\Delta}|$. Rather than project onto the hemisphere, we can project onto the tangent plane at the pole. The mapping is $\vec{D} = x\vec{U} + y\vec{V} + \vec{W}$ where $\vec{U}$, $\vec{V}$, and $\vec{W}$ form a right–handed orthonormal set. Defining the rotation matrix $R = [\vec{U}|\vec{V}|\vec{W}]$ and $\vec{\xi} = (x, y, 1)$, the function f reduces to

$$F(x, y) = r_0|P_0R\vec{\xi}| + r_1|P_1R\vec{\xi}| + (h_0/2)|\vec{W}_0 \cdot R\vec{\xi}| + (h_1/2)|\vec{W}_1 \cdot R\vec{\xi}| - |\vec{\Delta}|$$

for $(x, y) \in \mathbb{R}^2$. To determine if $F(x, y) < 0$ for some (x, y), it is enough to show that the minimum of F is negative. The point at which the minimum is attained occurs when the gradient of F is zero or undefined. $\vec{\nabla}(F)$ is undefined at points for which any of the first four absolute value terms is zero. In terms of points $\vec{D}$ on the unit sphere, the first term is zero at $\vec{W}_0$, the second term is zero at $\vec{W}_1$, the third term is zero at any vector perpendicular to $\vec{W}_0$, and the fourth term is zero at any vector perpendicular to $\vec{W}_1$. After all such points have been tested only to find that $F \geq 0$, the next phase of the separation test is to compute solutions to $\vec{\nabla}(F) = \vec{0}$ and test if any of those force $F < 0$.

If $\vec{D}$ is a separating direction, then $f(\vec{D}) < 0$. Observe that $f(t\vec{D}) = |t|f(\vec{D})$, so $f(t\vec{D}) < 0$ for any t. This is consistent with the geometry of the problem. Any nonzero multiple of a separating direction must itself be a separating direction. This allows us to restrict our attention to the unit sphere, $|\vec{D}| = 1$. Function f is continuous on the unit sphere, a compact set, so f must attain its minimum at some point on the sphere. This is a minimization problem in two dimensions, but the spherical geometry complicates the analysis somewhat. A different restriction on the set of potential separating directions can be made that yields a two–dimensional minimization in the plane rather than a two–dimensional minimization on a sphere.

First, some notation. The function $f(\vec{D})$ can be written as

$$f(\vec{D}) = r_0|A_0^{\mathrm{T}}\vec{D}| + r_1|A_1^{\mathrm{T}}\vec{D}| + (h_0/2)|\vec{D} \cdot \vec{W}_0| + (h_1/2)|\vec{D} \cdot \vec{W}_1| - |\vec{D} \cdot \vec{\Delta}|$$

where the matrices $A_i = [\vec{U}_i|\vec{V}_i]$ are 3×2. Observe that $A_i^{\mathrm{T}}A_i = I_2$, the 2×2 identity, and $A_iA_i^{\mathrm{T}} = I_3 - \vec{W}_i\vec{W}_i^{\mathrm{T}}$ where I_3 is the 3×3 identity matrix.

The symmetry $f(-\vec{D}) = f(\vec{D})$ implies that we only need to analyze f on a hemisphere; the other hemisphere values are determined automatically. The complicating factor in directly analyzing f turns out to be the presence of the absolute value terms $|\vec{D} \cdot \vec{W}_0|$, $|\vec{D} \cdot \vec{W}_1|$, and $|\vec{D} \cdot \vec{\Delta}|$. Instead I will look at functions where the absolute values are removed. To illustrate, consider

$$g_0(\vec{D}) = r_0|A_0^{\mathrm{T}}\vec{D}| + r_1|A_1^{\mathrm{T}}\vec{D}| - \vec{D} \cdot \vec{\phi}$$

where $\vec{\phi} = \vec{\Delta} - (h_0/2)\vec{W}_0 - (h_1/2)\vec{W}_1$. If the analysis of g_0 produces a direction $\vec{D}$ for which $g_0(\vec{D}) < 0$ *and* if $\vec{D} \cdot \vec{W}_0 \geq 0$, $\vec{D} \cdot \vec{W}_1 \geq 0$, and $\vec{D} \cdot \vec{\Delta} \geq 0$, then $f(\vec{D}) < 0$ and we have a separating direction. However, the inequality constraints might not be satisfied, even when $g_0(\vec{D}) < 0$, in which case $\vec{D}$ is rejected as a candidate for separation. The *companion* function is

$$g_1(\vec{D}) = r_0|A_0^{\mathrm{T}}\vec{D}| + r_1|A_1^{\mathrm{T}}\vec{D}| + \vec{D} \cdot \vec{\phi}.$$

If the analysis of g_1 produces a direction $\vec{D}$ for which $g_1(\vec{D}) < 0$ *and* if $\vec{D} \cdot \vec{W}_0 \leq 0$, $\vec{D} \cdot \vec{W}_1 \leq 0$, and $\vec{D} \cdot \vec{\Delta} \leq 0$, then $f(\vec{D}) < 0$ and we have a separating direction. However, the inequality constraints might not be satisfied, even when $g_1(\vec{D}) < 0$, in which case $\vec{D}$ is rejected as a candidate for separation. There are four such pairs of functions to consider, exhausting all eight sign possibilities on the three absolute value terms.

Let us now analyze $g_0(\vec{D})$. If $\vec{\phi} = \vec{0}$, then clearly $g_0(\vec{D}) \geq 0$ for all directions, so no separation can occur. For the remainder of the argument, assume $\vec{\phi} \neq \vec{0}$. Any direction $\vec{D}$ for which $\vec{D} \cdot \vec{\phi} \leq 0$ cannot be a separating direction. This allows us to restrict our attention to a hemisphere of directions whose pole is $\vec{W} = \vec{\phi}/|\vec{\phi}|$. Moreover, we can avoid working on the hemisphere by projecting those points radially outward onto the tangent plane at the pole. That is, we need only analyze g_0 for directions $\vec{D} = x\vec{U} + y\vec{V} + \vec{W}$ where $\{\vec{U}, \vec{V}, \vec{W}\}$ forms a right–handed orthonormal set of vectors. Defining the rotation matrix $R = [\vec{U}|\vec{V}|\vec{W}]$ whose columns are the indicated vectors, the restriction of g_0 to the plane is $F(x, y) = g_0(\vec{D}) = g_0(R\vec{\xi})$ where $\vec{\xi} = (x, y, 1)$, so

$$F(x, y) = r_0|A_0^{\mathrm{T}}R\vec{\xi}| + r_1|A_1^{\mathrm{T}}R\vec{\xi}| - |\vec{\phi}|. \tag{1}$$

In order to determine if $F(x, y) < 0$ for some (x, y), I will determine the minimum of F and test if it is negative. The minimum must occur at critical points, those points where $\vec{\nabla}F$ is zero or undefined. Any critical points that do not satisfy the inequality constraints for g_0 are rejected since F can be viewed as the restriction of g_0 to a convex subset of the plane defined by the inequality constraints. We only need to compute the minimum of F on this convex subset, so critical points outside that convex set are irrelevant. Analysis of the corresponding $F(x, y)$ for the companion function g_1 uses the projection $\vec{D} = x\vec{U} + y\vec{V} - \vec{W}$.

10 Analysis of $F(x, y)$

Using $\partial R\vec{\xi}/\partial x = \vec{U}$ and $\partial R\vec{\xi}/\partial y = \vec{V}$, the partial derivatives of F are

$$\frac{\partial F}{\partial x} = \vec{U} \cdot \left(r_0 A_0 \frac{A_0^{\mathrm{T}}R\vec{\xi}}{|A_0^{\mathrm{T}}R\vec{\xi}|} + r_1 A_1 \frac{A_1^{\mathrm{T}}R\vec{\xi}}{|A_1^{\mathrm{T}}R\vec{\xi}|} \right) \quad \text{and} \quad \frac{\partial F}{\partial y} = \vec{V} \cdot \left(r_0 A_0 \frac{A_0^{\mathrm{T}}R\vec{\xi}}{|A_0^{\mathrm{T}}R\vec{\xi}|} + r_1 A_1 \frac{A_1^{\mathrm{T}}R\vec{\xi}}{|A_1^{\mathrm{T}}R\vec{\xi}|} \right).$$

If we define $A = [\vec{U}|\vec{V}]$, the equation $\vec{\nabla}F(x, y) = (0, 0)$ can be summarized by

$$A^{\mathrm{T}} \left(r_0 A_0 \frac{A_0^{\mathrm{T}}R\vec{\xi}}{|A_0^{\mathrm{T}}R\vec{\xi}|} + r_1 A_1 \frac{A_1^{\mathrm{T}}R\vec{\xi}}{|A_1^{\mathrm{T}}R\vec{\xi}|} \right) = \vec{0}.$$

Define the unit–length vectors $\vec{\eta}_i = A_i^{\mathrm{T}} R\vec{\xi}/|A_i^{\mathrm{T}} R\vec{\xi}|$ for $i = 0, 1$. Define the 2×2 matrices $B_i = A^{\mathrm{T}} A_i$. The system of equations to be solved is

$$r_0 B_0 \vec{\eta}_0 + r_1 B_1 \vec{\eta}_1 = \vec{0}, \ \ |\vec{\eta}_0|^2 = 1, \ \text{ and } \ |\vec{\eta}_1|^2 = 1. \tag{2}$$

Given any solution $\vec{\eta}_0$ and $\vec{\eta}_1$ to these equations, it must be that $\vec{\eta}_i$ and $A_i^{\mathrm{T}} R\vec{\xi}$ point in the same direction. That is,

$$\vec{\eta}_0^{\perp} \cdot A_0^{\mathrm{T}} R\vec{\xi} = 0, \ \ \vec{\eta}_1^{\perp} \cdot A_1^{\mathrm{T}} R\vec{\xi} = 0, \ \ \vec{\eta}_0 \cdot A_0^{\mathrm{T}} R\vec{\xi} > 0, \ \text{ and } \ \vec{\eta}_1 \cdot A_1^{\mathrm{T}} R\vec{\xi} > 0, \tag{3}$$

where $(a, b)^{\perp} = (b, -a)$. Each pair (x, y) that satisfies these conditions is a critical point for $F(x, y)$ with $\vec{\nabla} F(x, y) = (0, 0)$. The critical point can then be tested to see if $F(x, y) < 0$, in which case the cylinders are separated.

The outline of the algorithm for the analysis of $g0(\vec{D}(x, y)) = F(x, y)$ is

- Using the notation $R_i = [\vec{U}_i|\vec{V}_i|\vec{W}_i]$ for $i = 0, 1$, the various dot products of vectors required in the algorithm need to be computed. The eighteen values are represented abstractly as $G_0 = R^{\mathrm{T}} R_0 = [g_{ij}^{(0)}]$ and $G_1 = R^{\mathrm{T}} R_1 = [g_{ij}^{(1)}]$.
- Solve $r_0 B_0 \vec{\eta}_0 + r_1 B_1 \vec{\eta}_1 = \vec{0}$, $|\vec{\eta}_0|^2 = 1$, and $|\vec{\eta}_1|^2 = 1$ for $\vec{\eta}_0$ and $\vec{\eta}_1$. Note that there are multiple solution pairs, the obvious one being $(-\vec{\eta}_0, -\vec{\eta}_1)$ whenever $(\vec{\eta}_0, \vec{\eta}_1)$ is. This negated pair leads to the same system of equations to extract (x, y) in step 4, so it can be ignored.
- For each solution pair $(\vec{\eta}_0, \vec{\eta}_1)$, solve $\vec{\eta}_0^{\perp} \cdot A_0^{\mathrm{T}} R\vec{\xi} = 0$ and $\vec{\eta}_1^{\perp} \cdot A_1^{\mathrm{T}} R\vec{\xi} = 0$ for $\vec{\xi}$. This set of equations can also have multiple solutions.
- For each solution $\vec{\xi}$, verify that $\vec{W}_0 \cdot R\vec{\xi} \geq 0$, $\vec{W}_1 \cdot R\vec{\xi}$, $\vec{\eta}_0 \cdot A_0^{\mathrm{T}} R\vec{\xi} > 0$, and $\vec{\eta}_1 \cdot A_1^{\mathrm{T}} R\vec{\xi} > 0$.
- For each pair (x, y) from a valid $\vec{\xi}$ in the last step, test if $F(x, y) < 0$. If so, then $\vec{D} = R\vec{\xi}$ is a separating direction for the cylinders and the algorithm terminates.

The algorithm for the analysis of $g_1(x\vec{U} + y\vec{V} - \vec{W})$ is identical in the first three steps. The only difference in steps 4 and 5 is that $\vec{\xi} = (x, y, 1)$ for g_0 and $\vec{\xi} = (x, y, -1)$ for g_1.

11 Solving for $\vec{\eta}_i$

Note that

$$B_i = \begin{bmatrix} \vec{U} \cdot \vec{U}_i & \vec{U} \cdot \vec{V}_i \\ \vec{V} \cdot \vec{U}_i & \vec{V} \cdot \vec{V}_i \end{bmatrix},$$

so

$$\det(B_i) = (\vec{U} \cdot \vec{U}_i)(\vec{V} \cdot \vec{V}_i) - (\vec{U} \cdot \vec{V}_i)(\vec{V} \cdot \vec{U}_i) = (\vec{U} \times \vec{V}) \cdot (\vec{U}_i \times \vec{V}_i) = \vec{W} \cdot \vec{W}_i.$$

If $\det(B_0) = 0$ and $\det(B_1) = 0$, then $\vec{W}$ must be perpendicular to both $\vec{W}_0$ and $\vec{W}_1$. Since $\vec{W} = \vec{\phi}/|\vec{\phi}|$, $\vec{\phi}$ is perpendicular to both $\vec{W}_0$ and $\vec{W}_1$. Observe that $\vec{\phi} = (\vec{C}_1 - (h_1/2)\vec{W}_1) - (\vec{C}_0 + (h_0/2)\vec{W}_0)$, a difference of two cylinder end points, one from each cylinder. The line segment connecting the two end points is therefore perpendicular to each cylinder. Draw yourself a picture to see that intersection/separation is determined solely by testing the direction $\vec{D} = \vec{W}$. Note that this direction does satisfy the inequality constraints since $\vec{W} \cdot \vec{W}_0 = 0$, $\vec{W} \cdot \vec{W}_1 = 0$, and $\vec{W} \cdot \vec{\Delta} = \vec{W} \cdot \vec{\phi} = |\vec{\phi}| > 0$. The two cylinders are separated if and only if $|\vec{\phi}|^2 > (r_0 + r_1)^2$.

If $\det(B_0) \neq 0$ and $\det(B_1) = 0$, then the columns of B_1 are linearly dependent. Moreover, one of them must be nonzero. If not, then $0 = (\vec{U} \cdot \vec{U}_1)^2 + (\vec{V} \cdot \vec{U}_1)^2 = 1 - (\vec{W} \cdot \vec{U}_1)^2$ which implies $|\vec{W} \cdot \vec{U}_1| = 1$ and $\vec{U}_1$ is either $\vec{W}$ or $-\vec{W}$. Similarly $\vec{V}_1$ is either $\vec{W}$ or $-\vec{W}$. This cannot happen since $\vec{U}_1$ and $\vec{V}_1$ are orthogonal. Let $\vec{\psi}$ be a nonzero column of B_1. The vector $\vec{\zeta} = \vec{\psi}^\perp$ satisfies the condition $B_1^{\mathrm{T}} \vec{\zeta} = \vec{0}$; therefore,

$$0 = \vec{\zeta}^{\mathrm{T}} (r_0 B_0 \vec{\eta}_0 + r_1 B_1 \vec{\eta}_1) = r_0 (B_0^{\mathrm{T}} \vec{\zeta}) \cdot \vec{\eta}_0.$$

If $B_0^{\mathrm{T}} \vec{\zeta} = (a, b)$, then $\vec{\eta}_0 = \pm(b, -a)/\sqrt{a^2 + b^2}$. The vector $\vec{\eta}_1$ is determined by $|\vec{\eta}_1| = 1$ and the linear equation

$$r_1 (B_1^{\mathrm{T}} \vec{\psi}) \cdot \vec{\eta}_1 = -r_0 (B_0^{\mathrm{T}} \vec{\psi}) \cdot \vec{\eta}_0.$$

The $\vec{\eta}_1$ are therefore points of intersection, if any, between a circle and a line. The normalization of $\vec{\eta}_0$ can be avoided by defining $\vec{P}_0 = |B_0^{\mathrm{T}} \vec{\zeta}| \vec{\eta}_0$ and $\vec{P}_1 = |B_0^{\mathrm{T}} \vec{\zeta}| \vec{\eta}_1$. In this case $\vec{P}_0 = (B_0^{\mathrm{T}} \vec{\zeta})^\perp$ and $r_1 (B_1^{\mathrm{T}} \vec{\psi}) \cdot \vec{P}_1 = -r_0 (B_0^{\mathrm{T}} \vec{\psi}) \cdot \vec{P}_0$. The extraction of (x, y) discussed later in fact does not require the normalizations. The intersection of line and circle does require solving a quadratic equation, so a square root has to be calculated (or the quadratic must be solved iteratively to avoid the cost of the square root). A similar construction applies if $\det(B_0) = 0$ and $\det(B_1) \neq 0$.

If $\det(B_0) \neq 0$ and $\det(B_1) \neq 0$, then B_0 is invertible and

$$\vec{\eta}_0 = -(r_1/r_0) B_0^{-1} B_1 \vec{\eta}_1$$

with $|\vec{\eta}_0| = 1$ and $|\vec{\eta}_1| = 1$. The extraction of (x, y) discussed later does not require unit length quantities for $\vec{\eta}_0$ and $\vec{\eta}_1$, so the three equations can be rewritten to avoid some divisions and normalizations. Rewrite the displayed equation as

$$r_0 \det(B_0) \vec{\eta}_0 = -r_1 \mathrm{Adj}(B_0) B_1 \vec{\eta}_1.$$

Define $\vec{P}_0 = r_0 \det(B_0) \vec{\eta}_0$, $\vec{P}_1 = r_1 \vec{\eta}_1$, and $C = \mathrm{Adj}(B_0) B_1$. The equations are now $\vec{P}_0 = -C \vec{P}_1$, $|\vec{P}_0|^2 = r_0^2 \det(B_0)^2$, and $|\vec{P}_1|^2 = r_1^2$.

The quadratic equations for $\vec{P}_1$ are $r_0^2 \det(B_0)^2 = \vec{P}_1^{\mathrm{T}} C^{\mathrm{T}} C \vec{P}_1$ and $|\vec{P}_1|^2 = r_1^2$. Factor $C^{\mathrm{T}} C = Q E Q^{\mathrm{T}}$ where $E = \mathrm{Diag}(e_0, e_1)$ are eigenvalues and the columns of Q are eigenvectors. Let $\vec{\psi} = Q^{\mathrm{T}} \vec{P}_1$. The equations become $|\vec{\psi}|^2 = r_1^2$ and $r_0^2 \det(B_0)^2 = \vec{\psi}^{\mathrm{T}} E \vec{\psi}$. If $\vec{\psi} = (a, b)$, then $a^2 + b^2 = r_1^2$ and $e_0 a^2 + e_1 b^2 = r_0^2 \det(B_0)^2$. These are two linear equations in the two unknowns a^2 and b^2. The formal solution is: $a^2 = (e_1 r_1^2 - r_0^2 \det(B_0)^2)/(e_1 - e_0)$ and $b^2 = (r_1^2 - e_0 r_0^2 \det(B_0)^2)/(e_1 - e_0)$. Assuming both right–hand sides are nonnegative, you have four solutions (a, b), $(-a, b)$, $(a, -b)$, and $(-a, -b)$, as expected (intersection of ellipse and circle). Only (a, b) and $(-a, b)$ need to be considered, the others generate no new information in the extraction of (x, y). Given a solution for $\vec{\psi}$, the corresponding nonnormalized vectors for extraction are $\vec{P}_1 = Q \vec{\psi}$ and $\vec{P}_0 = -C \vec{P}_1$.

12 Solving for (x, y)

The first two equations in (3) can be written as two systems of equations in the unknowns x and y as

$$C \begin{bmatrix} x \\ y \end{bmatrix} = \vec{d}$$

where $\vec{\eta}_0 = (a_0, b_0)$, $\vec{\eta}_1 = (a_1, b_1)$, and

$$C = \begin{bmatrix} b_0 g_{00}^{(0)} - a_0 g_{01}^{(0)} & b_0 g_{10}^{(0)} - a_0 g_{11}^{(0)} \\ b_1 g_{00}^{(1)} - a_1 g_{01}^{(1)} & b_1 g_{10}^{(1)} - a_1 g_{11}^{(1)} \end{bmatrix}, \quad \vec{D} = \begin{bmatrix} a_0 g_{21}^{(0)} - b_0 g_{20}^{(0)} \\ a_1 g_{21}^{(1)} - b_1 g_{20}^{(1)} \end{bmatrix}.$$

If C is invertible, then a unique solution is obtained for (x, y).

If C is not invertible, the problem is slightly more complicated. There are no solutions if $\mathrm{Adj}(C)\vec{d} \neq \vec{0}$. Otherwise, the system only has one independent equation. Since $\vec{\eta}_0 \neq \vec{0}$ and since $A_0^{\mathrm{T}} R$ has full rank (equal to 2), the 3×1 vector $R^{\mathrm{T}} A_0 \vec{\eta}_0^{\perp}$ cannot be the zero vector. In fact, $\vec{\eta}_0^{\perp}$ is unit length which implies $A_0 \vec{\eta}^{\perp}$ is unit length. Finally, since R is a rotation matrix, $R^{\mathrm{T}} A_0 \vec{\eta}_0^{\perp}$ is a unit length vector. The same argument shows that $R^{\mathrm{T}} A_1 \vec{\eta}_1^{\perp}$ is a unit length vector. Both of these conditions and the fact that the system has infinitely many solutions implies that $c_{00}^2 + c_{01}^2 \neq 0$ and $c_{10}^2 + c_{11}^2 \neq 0$.

If $c_{01} \neq 0$, then $y = (d_0 - c_{00} x)/c_{01}$. Replacing this in $A_0^{\mathrm{T}} R \vec{\xi}$ yields

$$A_0^{\mathrm{T}} R \vec{\xi} = \frac{(g_{00}^{(0)} g_{11}^{(0)} - g_{01}^{(0)} g_{10}^{(0)})x + (g_{11}^{(0)} g_{20}^{(0)} - g_{10}^{(0)} g_{21}^{(0)})}{a_0 g_{11}^{(0)} - b_0 g_{10}^{(0)}} =: (\alpha_0 x + \beta_0)\vec{\eta}_0.$$

The numerator of α_0 is $\det(B_0)$. If $c_{01} = 0$ instead, then $c_{00} \neq 0$ and a similar expression is obtained for $A_0^{\mathrm{T}} R \vec{\xi}$ in terms of y, namely $\alpha_0' y + \beta_0'$ where the numerator of α_0' is also $\det(B_0)$. Similarly, if $c_{11} \neq 0$, then $y = (d_1 - c_{10} x)/c_{11}$ and

$$A_1^{\mathrm{T}} R \vec{\xi} = \frac{(g_{00}^{(1)} g_{11}^{(1)} - g_{01}^{(1)} g_{10}^{(1)})x + (g_{11}^{(1)} g_{20}^{(1)} - g_{10}^{(1)} g_{21}^{(1)})}{a_1 g_{11}^{(1)} - b_1 g_{10}^{(1)}} =: (\alpha_1 x + \beta_1)\vec{\eta}_1.$$

The numerator of α_1 is $\det(B_1)$. If $c_{11} = 0$ instead, then $c_{10} \neq 0$ and a similar expression is obtained for $A_1^{\mathrm{T}} R \vec{\xi}$ in terms of y, namely $\alpha_1' y + \beta_1'$ where the numerator of α_1' is also $\det(B_1)$.

In the case $c_{01} \neq 0$ and $c_{11} \neq 0$, then $F(x, y)$ reduces to

$$F(x, y) = r_0 |\alpha_0 x + \beta_0| + r_1 |\alpha_1 x + \beta_1| - |\vec{\phi}|.$$

If $\alpha_0 \neq 0$ and $\alpha_1 \neq 0$, then the minimum of F is attained at either $x = -\beta_0/\alpha_0$ or $x = -\beta_1/\alpha_1$. Notice that the first x forces $A_0^{\mathrm{T}} R \vec{\xi} = \vec{0}$, in which case the corresponding direction must have been $\vec{D} = \vec{W}_0$. The second x forces $A_1^{\mathrm{T}} R \vec{\xi} = \vec{0}$, in which case the corresponding direction must have been $\vec{D} = \vec{W}_1$. Both of these these directions were tested earlier, so this case can be ignored. If $\alpha_0 \neq 0$ and $\alpha_1 = 0$, then the minimum of F is attained at $x = -\beta_0/\alpha_0$. The corresponding direction must have been $\vec{D} = \vec{W}_0$, again handled earlier. The same argument applies to $\alpha_0 = 0$ and $\alpha_1 \neq 0$. The final case is $\alpha_0 = \alpha_1 = 0$, in which case

$\det(B_0) = \det(B_1) = 0$, yet another case that was handled earlier. Therefore, these cases can be ignored in an implementation. A similar argument applies when $c_{00} \neq 0$ and $c_{10} \neq 0$ and F reduces to

$$F(x, y) = r_0|\alpha'_0 y + \beta'_0| + r_1|\alpha'_1 y + \beta'_1| - |\vec{\phi}|.$$

All possibilities can be ignored in an implementation since they are handled by other separation tests. Finally, if there is a mixture of x and y terms,

$$F(x, y) = r_0|\alpha_0 x + \beta_0| + r_1|\alpha'_1 y + \beta'_1| - |\vec{\phi}|$$

or

$$F(x, y) = r_0|\alpha'_0 y + \beta'_0| + r_1|\alpha_1 x + \beta_1| - |\vec{\phi}|,$$

then the minimization is applied in each dimension separately, but just as before, other separation tests cover these cases. The conclusion is that an implementation does not have to do anything when C is not invertible.

13 Fast Method to Test $F(x, y) < 0$

The two square roots, $|A_i^{\mathrm{T}} R\vec{\xi}|$, in equation (1) can be avoided. The test $F(x, y) < 0$ is equivalent to

$$r_0|A_0^{\mathrm{T}} R\vec{\xi}| + r_1|A_1^{\mathrm{T}} R\vec{\xi}| < |\vec{\phi}|.$$

The inequality can be squared and rearranged to yield the test

$$2r_0 r_1|A_0^{\mathrm{T}} R\vec{\xi}||A_1^{\mathrm{T}} R\vec{\xi}| < |\vec{\phi}|^2 - r_0^2|A_0^{\mathrm{T}} R\vec{\xi}|^2 - r_1^2|A_1^{\mathrm{T}} R\vec{\xi}|^2 =: \rho.$$

If $\rho \leq 0$, then $F(x, y) \geq 0$ is forced and no more work needs to be done. If $\rho > 0$, then squaring one more time yields the test

$$4r_0^2 r_1^2|A_0^{\mathrm{T}} R\vec{\xi}|^2|A_1^{\mathrm{T}} R\vec{\xi}|^2 < \rho^2.$$