Distance Between Point and Circle or Disk in 3D

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1 Point and Circle

A circle in 3D is represented by a center $\vec{C}$, a radius R, and a plane containing the circle, $\vec{N} \cdot (\vec{X} - \vec{C}) = 0$ where $\vec{N}$ is a unit length normal to the plane. If $\vec{U}$ and $\vec{V}$ are also unit length vectors so that $\vec{U}$, $\vec{V}$, and $\vec{N}$ form a right–handed orthonormal coordinate system (the matrix with these vectors as columns is orthonormal with determinant 1), then the circle is parameterized as

$$\vec{X} = \vec{C} + R(\cos(\theta)\vec{U} + \sin(\theta)\vec{V}) =: \vec{C} + R\vec{W}(\theta)$$

for angles $\theta \in [0, 2\pi)$. Note that $|\vec{X} - \vec{C}| = R$, so the $\vec{X}$ values are all equidistant from $\vec{C}$. Moreover, $\vec{N} \cdot (\vec{X} - \vec{C}) = 0$ since $\vec{U}$ and $\vec{V}$ are perpendicular to $\vec{N}$, so the $\vec{X}$ lie in the plane.

For each angle $\theta \in [0, 2\pi)$, the squared distance from a specified point $\vec{P}$ to the corresponding circle point is

$$F(\theta) = |\vec{C} + R\vec{W}(\theta) - \vec{P}|^2 = R^2 + |\vec{C} - \vec{P}|^2 + 2R(\vec{C} - \vec{P}) \cdot \vec{W}.$$

The problem is to minimize $F(\theta)$ by finding θ_0 such that $F(\theta_0) \leq F(\theta)$ for all $\theta \in [0, 2\pi)$. Since F is a periodic and differentiable function, the minimum must occur when $F'(\theta) = 0$. Also, note that $(\vec{C} - \vec{P}) \cdot \vec{W}$ should be negative and as large in magnitude as possible to reduce the right–hand side in the definition of F. The derivative is

$$F'(\theta) = 2R(\vec{C} - \vec{P}) \cdot \vec{W}'(\theta)$$

where $\vec{W} \cdot \vec{W}' = 0$ since $\vec{W} \cdot \vec{W} = 1$ for all θ. The vector $\vec{W}'$ is unit length vector since $\vec{W}'' = -\vec{W}$ and $0 = \vec{W} \cdot \vec{W}'$ implies $0 = \vec{W} \cdot \vec{W}'' + \vec{W}' \cdot \vec{W}' = -1 + \vec{W}' \cdot \vec{W}'$. Finally, $\vec{W}'$ is perpendicular to $\vec{N}$ since $\vec{N} \cdot \vec{W} = 0$ implies $0 = \vec{N} \cdot \vec{W}'$. All conditions imply that $\vec{W}$ is parallel to the projection of $\vec{P} - \vec{C}$ onto the plane and points in the same direction.

Let $\vec{Q}$ be the projection of $\vec{P}$ onto the plane. Then

$$\vec{Q} - \vec{C} = \vec{P} - \vec{C} - \left(\vec{N} \cdot (\vec{P} - \vec{C})\right) \vec{N}.$$

The vector $\vec{W}(\theta)$ must be the unitized projection $(\vec{Q} - \vec{C})/|\vec{Q} - \vec{C}|$. The closest point on the circle to $\vec{P}$ is

$$\vec{X} = \vec{C} + R\frac{\vec{Q} - \vec{C}}{|\vec{Q} - \vec{C}|}$$

assuming that $\vec{Q} \neq \vec{C}$. The distance from point to circle is then $|\vec{P} - \vec{X}|$.

If the projection of $\vec{P}$ is exactly the circle center $\vec{C}$, then all points on the circle are equidistant from $\vec{C}$. The distance from point to circle is the length of the hypotenuse of any triangle whose vertices are $\vec{C}$, $\vec{P}$, and any circle point. The lengths of the adjacent and opposite triangle sides are R and $|\vec{P} - \vec{C}|$, so the distance from point to circle is $\sqrt{R^2 + |\vec{P} - \vec{C}|^2}$.

The typical case where $\vec{P}$ does not project to circle center is shown in Figure 1.

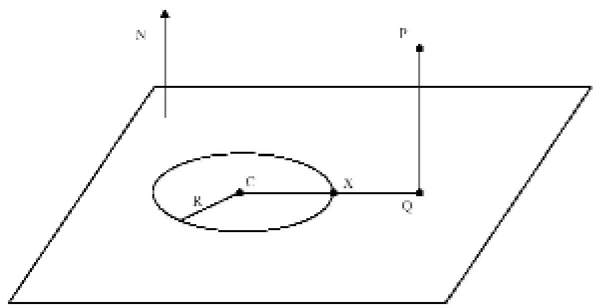

Figure 1. Typical case, closest point to circle.

The case when $\vec{P}$ does project to circle center is shown in Figure 2.

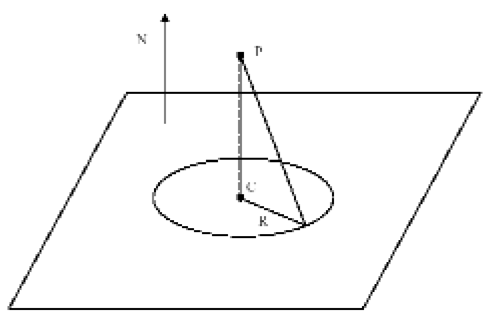

Figure 2. Typical case, closest point to circle.

2 Point and Disk

This requires a minor modification of the point and circle algorithm. The disk is the set of all points $\vec{X} = \vec{C} + \rho\vec{W}(\theta)$ where $0 \leq \rho \leq R$. If the projection of $\vec{P}$ is contained in the disk, then the projection is already the closest point to $\vec{P}$. If the projection is outside the disk, then the closest point to $\vec{P}$ is the closest point on the disk boundary, a circle.

Figure 3 shows the case when $\vec{P}$ projects inside the disk.

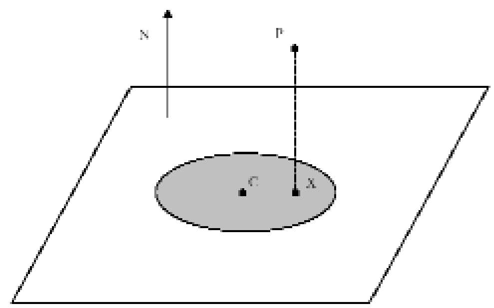

Figure 3. Closest point when $\vec{P}$ projects inside the disk.

Figure 4 shows the case when $\vec{P}$ projects outside the disk.

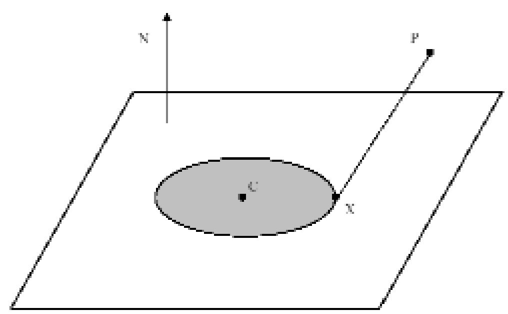

Figure 4. Closest point when $\vec{P}$ projects outside the disk.