

Euler Angle Formulas

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1 Introduction

Rotations about the coordinate axes are easy to define and work with. Rotation about the x -axis by angle θ is

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$

where $\theta > 0$ indicates a counterclockwise rotation in the plane $x = 0$. The observer is assumed to be positioned on the side of the plane with $x > 0$ and looking at the origin. Rotation about the y -axis by angle θ is

$$R_y(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

where $\theta > 0$ indicates a counterclockwise rotation in the plane $y = 0$. The observer is assumed to be positioned on the side of the plane with $y > 0$ and looking at the origin. Rotation about the z -axis by angle θ is

$$R_z(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $\theta > 0$ indicates a counterclockwise rotation in the plane $z = 0$. The observer is assumed to be positioned on the side of the plane with $z > 0$ and looking at the origin. Rotation by an angle θ about an arbitrary axis containing the origin and having unit length direction $\vec{U} = (U_x, U_y, U_z)$ is given by

$$R_{\vec{U}}(\theta) = I + (\sin \theta)S + (1 - \cos \theta)S^2$$

where I is the identity matrix,

$$S = \begin{bmatrix} 0 & -U_z & U_y \\ U_z & 0 & -U_x \\ -U_y & U_x & 0 \end{bmatrix}$$

and $\theta > 0$ indicates a counterclockwise rotation in the plane $\vec{U} \cdot (x, y, z) = 0$. The observer is assumed to be positioned on the side of the plane to which \vec{U} points and is looking at the origin.

2 Factoring Rotation Matrices

A common problem is to factor a rotation matrix as a product of rotations about the coordinate axes. The form of the factorization depends on the needs of the application and what ordering is specified. For example, one might want to factor a rotation as $R = R_x(\theta_x)R_y(\theta_y)R_z(\theta_z)$ for some angles θ_x , θ_y , and θ_z . The ordering is xyz . Five other possibilities are xzy , yxz , yzx , zxy , and zyx . One might also envision factorizations such as xyx —these are not discussed here. The following discussion uses the notation $c_a = \cos(\theta_a)$ and $s_a = \sin(\theta_a)$ for $a = x, y, z$.

2.1 Factor as $R_x R_y R_z$

Setting $R = [r_{ij}]$ for $0 \leq i \leq 2$ and $0 \leq j \leq 2$, formally multiplying $R_x(\theta_x)R_y(\theta_y)R_z(\theta_z)$, and equating yields

$$\begin{bmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} c_y c_z & -c_y s_z & s_y \\ c_z s_x s_y + c_x s_z & c_x c_z - s_x s_y s_z & -c_y s_x \\ -c_x c_z s_y + s_x s_z & c_z s_x + c_x s_y s_z & c_x c_y \end{bmatrix}$$

From this we have $s_y = r_{02}$, so $\theta_y = \text{asin}(r_{02})$. If $\theta_y \in (-\pi/2, \pi/2)$, then $c_y \neq 0$ and $c_y(s_x, c_x) = (-r_{12}, r_{22})$ in which case $\theta_x = \text{Tan}^{-1}(-r_{12}, r_{22})$. Similarly, $c_y(s_z, c_z) = (-r_{01}, r_{00})$ in which case $\theta_z = \text{Tan}^{-1}(-r_{01}, r_{00})$.

If $\theta_y = \pi/2$, then $s_y = 1$ and $c_y = 0$. In this case

$$\begin{bmatrix} r_{10} & r_{11} \\ r_{20} & r_{21} \end{bmatrix} = \begin{bmatrix} c_z s_x + c_x s_z & c_x c_z - s_x s_z \\ -c_x c_z + s_x s_z & c_z s_x + c_x s_z \end{bmatrix} = \begin{bmatrix} \sin(\theta_z + \theta_x) & \cos(\theta_z + \theta_x) \\ -\cos(\theta_z + \theta_x) & \sin(\theta_z + \theta_x) \end{bmatrix}.$$

Therefore, $\theta_z + \theta_x = \text{Tan}^{-1}(r_{10}, r_{11})$. There is one degree of freedom, so the factorization is not unique. One choice is $\theta_z = 0$ and $\theta_x = \text{Tan}^{-1}(r_{10}, r_{11})$. If $\theta_y = -\pi/2$, then $s_y = -1$ and $c_y = 0$. In this case

$$\begin{bmatrix} r_{10} & r_{11} \\ r_{20} & r_{21} \end{bmatrix} = \begin{bmatrix} -c_z s_x + c_x s_z & c_x c_z + s_x s_z \\ c_x c_z + s_x s_z & c_z s_x - c_x s_z \end{bmatrix} = \begin{bmatrix} \sin(\theta_z - \theta_x) & \cos(\theta_z - \theta_x) \\ \cos(\theta_z - \theta_x) & -\sin(\theta_z - \theta_x) \end{bmatrix}.$$

Therefore, $\theta_z - \theta_x = \text{Tan}^{-1}(r_{10}, r_{11})$. There is one degree of freedom, so the factorization is not unique. One choice is $\theta_z = 0$ and $\theta_x = -\text{Tan}^{-1}(r_{10}, r_{11})$.

Pseudocode for the factorization is

```
thetaY = asin(r02);
if ( thetaY < PI/2 )
{
    if ( thetaY > -PI/2 )
```

```

{
    thetaX = atan2(-r12,r22);
    thetaZ = atan2(-r01,r00);
}
else
{
    // not a unique solution
    thetaX = -atan2(r10,r11);
    thetaZ = 0;
}
}
else
{
    // not a unique solution
    thetaX = atan2(r10,r11);
    thetaZ = 0;
}
}

```

2.2 Factor as $R_x R_z R_y$

Setting $R = [r_{ij}]$ for $0 \leq i \leq 2$ and $0 \leq j \leq 2$, formally multiplying $R_x(\theta_x)R_z(\theta_z)R_y(\theta_y)$, and equating yields

$$\begin{bmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} c_y c_z & -s_z & c_z s_y \\ s_x s_y + c_x c_y s_z & c_x c_z & -c_y s_x + c_x s_y s_z \\ -c_x s_y + c_y s_x s_z & c_z s_x & c_x c_y + s_x s_y s_z \end{bmatrix}$$

Analysis similar to the xyz case leads to the pseudocode

```

thetaZ = asin(-r01);
if ( thetaZ < PI/2 )
{
    if ( thetaZ > -PI/2 )
    {
        thetaX = atan2(r21,r11);
        thetaY = atan2(r02,r00);
    }
    else
    {
        // not a unique solution
        thetaX = -atan2(-r20,r22);
        thetaY = 0;
    }
}
else
{
    // not a unique solution
}

```

```

    thetaX = atan2(-r20,r22);
    thetaY = 0;
}

```

2.3 Factor as $R_y R_x R_z$

Setting $R = [r_{ij}]$ for $0 \leq i \leq 2$ and $0 \leq j \leq 2$, formally multiplying $R_y(\theta_y)R_x(\theta_x)R_z(\theta_z)$, and equating yields

$$\begin{bmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} c_y c_z + s_x s_y s_z & c_z s_x s_y - c_y s_z & c_x s_y \\ c_x s_z & c_x c_z & -s_x \\ -c_z s_y + c_y s_x s_z & c_y c_z s_x + s_y s_z & c_x c_y \end{bmatrix}$$

Analysis similar to the xyz case leads to the pseudocode

```

thetaX = asin(-r12);
if ( thetaX < PI/2 )
{
    if ( thetaX > -PI/2 )
    {
        thetaY = atan2(r02,r22);
        thetaZ = atan2(r10,r11);
    }
    else
    {
        // not a unique solution
        thetaY = -atan2(-r01,r00);
        thetaZ = 0;
    }
}
else
{
    // not a unique solution
    thetaY = atan2(-r01,r00);
    thetaZ = 0;
}

```

2.4 Factor as $R_y R_z R_x$

Setting $R = [r_{ij}]$ for $0 \leq i \leq 2$ and $0 \leq j \leq 2$, formally multiplying $R_y(\theta_y)R_z(\theta_z)R_x(\theta_x)$, and equating yields

$$\begin{bmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} c_y c_z & s_x s_y - c_x c_y s_z & c_x s_y + c_y s_x s_z \\ s_z & c_x c_z & -c_z s_x \\ -c_z s_y & c_y s_x + c_x s_y s_z & c_x c_y - s_x s_y s_z \end{bmatrix}$$

Analysis similar to the xyz case leads to the pseudocode

```

thetaZ = asin(r10);
if ( thetaZ < PI/2 )
{
    if ( thetaZ > -PI/2 )
    {
        thetaY = atan2(-r20,r00);
        thetaX = atan2(-r12,r11);
    }
    else
    {
        // not a unique solution
        thetaY = -atan2(r21,r22);
        thetaX = 0;
    }
}
else
{
    // not a unique solution
    thetaY = atan2(r21,r22);
    thetaX = 0;
}

```

2.5 Factor as $R_z R_x R_y$

Setting $R = [r_{ij}]$ for $0 \leq i \leq 2$ and $0 \leq j \leq 2$, formally multiplying $R_z(\theta_z)R_x(\theta_x)R_y(\theta_y)$, and equating yields

$$\begin{bmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} c_y c_z - s_x s_y s_z & -c_x s_z & c_z s_y + c_y s_x s_z \\ c_z s_x s_y + c_y s_z & c_x c_z & -c_y c_z s_x + s_y s_z \\ -c_x s_y & s_x & c_x c_y \end{bmatrix}$$

Analysis similar to the xyz case leads to the pseudocode

```

thetaX = asin(r21);
if ( thetaX < PI/2 )
{
    if ( thetaX > -PI/2 )
    {
        thetaZ = atan2(-r01,r11);
        thetaY = atan2(-r20,r22);
    }
    else
    {
        // not a unique solution
        thetaZ = -atan2(r02,r00);
        thetaY = 0;
    }
}

```

```

}
else
{
    // not a unique solution
    thetaZ = atan2(r02,r00);
    thetaY = 0;
}

```

2.6 Factor as $R_z R_y R_x$

Setting $R = [r_{ij}]$ for $0 \leq i \leq 2$ and $0 \leq j \leq 2$, formally multiplying $R_z(\theta_z)R_y(\theta_y)R_x(\theta_x)$, and equating yields

$$\begin{bmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} c_y c_z & c_z s_x s_y - c_x s_z & c_x c_z s_y + s_x s_z \\ c_y s_z & c_x c_z + s_x s_y s_z & -c_z s_x + c_x s_y s_z \\ -s_y & c_y s_x & c_x c_y \end{bmatrix}$$

Analysis similar to the xyz case leads to the pseudocode

```

thetaY = asin(-r20);
if ( thetaY < PI/2 )
{
    if ( thetaY > -PI/2 )
    {
        thetaZ = atan2(r10,r00);
        thetaX = atan2(r21,r22);
    }
    else
    {
        // not a unique solution
        thetaZ = -atan2(-r01,r02);
        thetaX = 0;
    }
}
else
{
    // not a unique solution
    thetaZ = atan2(-r01,r02);
    thetaX = 0;
}

```

3 Factor Product of Two

Given a rotation R that is a product of two coordinate axis rotations, the problem is to factor it into three coordinate axis rotations using the ordering xyz . Derivations for the other orderings are similar. In

the subsections the matrices are $P_x = R_x(\phi_x)$, $P_y = R_y(\phi_y)$, and $P_z = R_z(\phi_z)$. Define $s_a = \sin(\phi_x)$, $s_b = \sin(\phi_y)$, $s_c = \sin(\phi_z)$, $c_a = \cos(\phi_x)$, $c_b = \cos(\phi_y)$, and $c_c = \cos(\phi_z)$.

3.1 Factor $P_x P_y$

This is a trivial case. The factorization is $R = R_x(\phi_x)R_y(\phi_y) = R_x(\theta_x)R_y(\theta_y)R_z(\theta_z)$. Therefore, $\theta_x = \phi_x$, $\theta_y = \phi_y$, and $\theta_z = 0$.

3.2 Factor $P_y P_x$

The factorization is $R = R_y(\phi_y)R_x(\phi_x) = R_x(\theta_x)R_y(\theta_y)R_z(\theta_z)$. Formal multiplication of the various terms leads to the equation

$$\begin{bmatrix} c_b & s_a s_b & c_a s_b \\ 0 & c_a & -s_a \\ -s_b & c_b s_a & c_a c_b \end{bmatrix} = \begin{bmatrix} c_y c_z & -c_y s_z & s_y \\ c_z s_x s_y + c_x s_z & c_x c_z - s_x s_y s_z & -c_y s_x \\ -c_x c_z s_y + s_x s_z & c_z s_x + c_x s_y s_z & c_x c_y \end{bmatrix}.$$

It is easy to see that $s_y = c_a s_b$ in which case $\theta_y = \text{asin}(\cos \theta_y \sin \theta_x)$. Adding the 10 and 21 terms yields

$$0 + c_b s_a = (c_z s_x s_y + c_x s_z) + (c_z s_x + c_x s_y s_z) = (1 + s_y)(c_z s_x + c_x s_z)$$

which leads to $\sin(\theta_x + \theta_z) = c_b s_a / (1 + c_a s_b)$. In the even that $c_a s_b = -1$, this leads to a special case in the coding that is easy to solve. Subtracting the 10 term from the 21 term yields

$$c_b s_a - 0 = (c_z s_x s_y + c_x s_z) - (c_z s_x + c_x s_y s_z) = (1 - s_y)(c_z s_x - c_x s_z)$$

which leads to $\sin(\theta_x - \theta_z) = c_b s_a / (1 - c_a s_b)$. In the event that $c_a s_b = 1$, this also leads to a special case in the coding that is easy to solve. The sine functions can be inverted and the two resulting equations for θ_x and θ_z can be solved. For the case $|c_a s_b| < 1$,

$$\begin{aligned} \theta_x &= \frac{1}{2} \left[\text{asin} \left(\frac{c_b s_a}{1 + c_b s_a} \right) + \text{asin} \left(\frac{c_b s_a}{1 - c_b s_a} \right) \right] \\ \theta_y &= \text{asin}(c_a s_b) \\ \theta_z &= \frac{1}{2} \left[\text{asin} \left(\frac{c_b s_a}{1 + c_b s_a} \right) - \text{asin} \left(\frac{c_b s_a}{1 - c_b s_a} \right) \right] \end{aligned}$$

3.3 Factor $P_x P_z$

This is a trivial case. The factorization is $R = R_x(\phi_x)R_z(\phi_z) = R_x(\theta_x)R_y(\theta_y)R_z(\theta_z)$. Therefore, $\theta_x = \phi_x$, $\theta_y = 0$, and $\theta_z = \phi_z$.

3.4 Factor $P_z P_x$

A construction similar to the case $P_y P_x$ leads to

$$\begin{aligned}\theta_x &= \frac{1}{2} \left[\text{asin} \left(\frac{c_a c_c}{1+s_a s_c} \right) + \text{asin} \left(\frac{c_a c_c}{1-s_a s_c} \right) \right] \\ \theta_y &= \text{asin}(s_a s_c) \\ \theta_z &= \frac{1}{2} \left[\text{asin} \left(\frac{c_a c_c}{1+s_a s_c} \right) - \text{asin} \left(\frac{c_a c_c}{1-s_a s_c} \right) \right]\end{aligned}$$

3.5 Factor $P_y P_z$

This is a trivial case. The factorization is $R = R_y(\phi_y)R_z(\phi_z) = R_x(\theta_x)R_y(\theta_y)R_z(\theta_z)$. Therefore, $\theta_x = 0$, $\theta_y = \phi_y$, and $\theta_z = \phi_z$.

3.6 Factor $P_z P_y$

A construction similar to the case $P_y P_x$ leads to

$$\begin{aligned}\theta_x &= \frac{1}{2} \left[\text{asin} \left(\frac{c_b s_c}{1+s_b c_c} \right) - \text{asin} \left(\frac{c_b s_c}{1-s_b c_c} \right) \right] \\ \theta_y &= \text{asin}(s_b c_c) \\ \theta_z &= \frac{1}{2} \left[\text{asin} \left(\frac{c_b s_c}{1+s_b c_c} \right) + \text{asin} \left(\frac{c_b s_c}{1-s_b c_c} \right) \right]\end{aligned}$$

4 Product of Coordinate Rotation with zyx Factorization

Given $R = R_z(\theta_z)R_y(\theta_y)R_x(\theta_x)$ and coordinate axis rotation P , factor $PR = R_z(\psi_z)R_y(\psi_y)R_x(\psi_x)$.

4.1 $P = P_z$

Trivial case. $P = R_z(\phi_z)$ and $PR = R_z(\theta_z + \phi_z)R_y(\theta_y)R_x(\theta_x)$. Thus, $\psi_z = \theta_z + \phi_z$, $\psi_y = \theta_y$, and $\psi_x = \theta_x$.

4.2 $P = P_y$

Set $P = R_y(\phi_y) = P_y$. Then

$$\begin{aligned}
 PR &= P_y R_z R_y R_x \\
 &= (P_y R_z) R_y R_x \\
 &= (A_z A_x A_y) R_y R_x \\
 &= A_z (A_x B_y) R_x \\
 &= A_z (C_z C_y C_x) R_x \\
 &= D_z C_y D_x
 \end{aligned}$$

This requires factorization $P_y R_z = A_z A_x A_y$, $B_y = A_y R_y$, factorization $A_x B_y = C_z C_y C_x$, $D_z = A_z C_z$, and $D_x = C_x R_x$.

4.3 $P = P_x$

Set $P = R_x(\phi_y) = P_x$. Then

$$\begin{aligned}
 PR &= P_x R_z R_y R_x \\
 &= (P_x R_z) R_y R_x \\
 &= (A_z A_x A_y) R_y R_x \\
 &= A_z (A_x B_y) R_x \\
 &= A_z (C_z C_y C_x) R_x \\
 &= D_z C_y D_x
 \end{aligned}$$

This requires factorization $P_x R_z = A_z A_x A_y$, $B_y = A_y R_y$, factorization $A_x B_y = C_z C_y C_x$, $D_z = A_z C_z$, and $D_x = C_x R_x$.