Distance Between Ellipses in 2D

David Eberly
Magic Software, Inc.
http://www.magic-software.com

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This document shows how to compute the distance between two ellipses in two dimensions. The algorithm is based on constrained minimization and elimination theory. In this construction it is assumed that the ellipses are separated (no intersection, one not contained in the other).

An ellipse is represented parametrically by specifying a center $\vec{C}$, two unit–length and mutually orthogonal axes $\vec{\xi}$ and $\vec{\eta}$, and two lengths a and b. The four extreme points in the directions of the specified axes are $\vec{C} \pm a\vec{\xi}$ and $\vec{C} \pm b\vec{\eta}$. Introducing an angle $\theta \in [0, 2\pi)$, the parametric form is

$$\vec{X}(\theta) = \vec{C} + (a\cos\theta)\vec{\xi} + (b\sin\theta)\vec{\eta}.$$

Some vector algebra shows that

$$\cos\theta = \frac{\vec{\xi} \cdot (\vec{X} - \vec{C})}{a} \quad \text{and} \quad \sin\theta = \frac{\vec{\eta} \cdot (\vec{X} - \vec{C})}{b}.$$

It follows that

$$
\begin{aligned}
1 &= \cos^2\theta + \sin^2\theta \\
&= \left(\tfrac{1}{a}\vec{\xi} \cdot (\vec{X} - \vec{C})\right)^2 + \left(\tfrac{1}{b}\vec{\eta} \cdot (\vec{X} - \vec{C})\right)^2 \\
&= (\vec{X} - \vec{C})^T \left(\tfrac{\vec{\xi}\vec{\xi}^T}{a^2} + \tfrac{\vec{\eta}\vec{\eta}^T}{b^2}\right)(\vec{X} - \vec{C}) \\
&= (\vec{X} - \vec{C})^T R^T D^2 R(\vec{X} - \vec{C})
\end{aligned}
$$

where $R^T = [\vec{\xi} \,|\, \vec{\eta}]$, a rotation matrix, and $D = \text{Diag}(1/a, 1/b)$, a diagonal matrix whose diagonal entries are $1/a$ and $1/b$.

The two ellipses are represented algebraically by quadratic equations

$$Q_0(\vec{X}) = (\vec{X} - \vec{C}_0)^T R_0^T D_0^2 R_0(\vec{X} - \vec{C}_0) - 1 = 0 \tag{1}$$

and

$$Q_1(\vec{Y}) = (\vec{Y} - \vec{C}_1)^T R_1^T D_1^2 R_1(\vec{Y} - \vec{C}_1) - 1 = 0. \tag{2}$$

The problem is to compute $\vec{X}$ and $\vec{Y}$, a point on each ellipse, that minimize $|\vec{X} - \vec{Y}|^2$ subject to the constraints $Q_0(\vec{X}) = 0$ and $Q_1(\vec{Y}) = 0$. This may be solved by the method of Lagrange multipliers. However, it turns out that it is convenient to first make a change of variables. Define $\vec{U} = D_0 R_0(\vec{X} - \vec{C}_0)$ and $\vec{V} = D_1 R_1(\vec{Y} - \vec{C}_1)$. The constraints become $|\vec{U}| = 1$ and $|\vec{V}| = 1$. The goal now is to minimize $|\vec{X} - \vec{Y}|^2 = |A\vec{U} - B\vec{V} + \vec{C}|^2$ subject to $\vec{U}$ and $\vec{V}$ being unit length. The quantities on the right–hand side are $A = R_0^T D_0^{-1}$, $B = R_1^T D_1^{-1}$, and $\vec{\Delta} = \vec{C}_0 - \vec{C}_1$.

Introduce Lagrange multipliers s and t and define

$$F(\vec{U}, \vec{V}; s, t) = |A\vec{U} - B\vec{V} + \vec{\Delta}|^2 + s(|\vec{U}|^2 - 1) + t(|\vec{V}|^2 - 1).$$

The minimum occurs when $\vec{\nabla} F = \vec{0}$ where the gradient is with respect to six variables: the two components of $\vec{U}$, the two components of $\vec{V}$, and the two multipliers s and t. The derivative with respect to $\vec{U}$ leads to the equation

$$\frac{\partial F}{\partial \vec{U}} = A^T(A\vec{U} - B\vec{V} + \vec{\Delta}) + s\vec{U} = \vec{0}. \tag{3}$$

The derivative with respect to $\vec{V}$ leads to the equation

$$\frac{\partial F}{\partial \vec{V}} = B^T(A\vec{U} - B\vec{V} + \vec{\Delta}) - t\vec{V} = \vec{0}. \tag{4}$$

The derivatives with respect to s and t lead to equations that just reproduce the constraints,

$$\frac{\partial F}{\partial s} = |\vec{U}|^2 - 1 = 0 \text{ and } \frac{\partial F}{\partial t} = |\vec{V}|^2 - 1 = 0.$$

Multiplying by the inverses of the transposes in equations (3) and (4) leads to

$$sA^{-T}\vec{U} + (A\vec{U} - B\vec{V} + \vec{\Delta}) = \vec{0}$$
$$tB^{-T}\vec{V} - (A\vec{U} - B\vec{V} + \vec{\Delta}) = \vec{0}$$

The first equation implies that $A^{-T}\vec{U}$ and $A\vec{U} - B\vec{V} + \vec{\Delta}$ are parallel. The second equation implies that $B^{-T}\vec{V}$ and $A\vec{U} - B\vec{V} + \vec{\Delta}$ are parallel. Consequently, the three vectors $A^{-T}\vec{U}$, $B^{-T}\vec{V}$, and $A\vec{U} - B\vec{V} + \vec{\Delta}$ are all parallel.

Define $\vec{W} = (A^{-T}\vec{U})^{\perp}$ where $(x, y)^{\perp} = (y, -x)$. The fact that $A^{-T}\vec{U}$ and $A\vec{U} - B\vec{V} + \vec{\Delta}$ are parallel implies

$$\vec{W}^T(A\vec{U} - B\vec{V} + \vec{\Delta}) = 0.$$

The fact that $A^{-T}\vec{U}$ and $B^{-T}\vec{V}$ are parallel implies

$$\vec{W}^T(B^{-T}\vec{V}) = 0.$$

The two equations are rearranged in matrix form as

$$\begin{bmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{bmatrix} \vec{V} = \begin{bmatrix} \vec{W}^T B \\ \vec{W}^T B^{-T} \end{bmatrix} \vec{V} = \begin{bmatrix} \vec{W}^T(A\vec{U} + \vec{\Delta}) \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ 0 \end{bmatrix}$$

where the 2×2 coefficient matrix on the left–hand side is partitioned into two row vectors. Each entry of the coefficient matrix that multiplies $\vec{V}$ is a linear polynomial in the components of $\vec{U}$. The first row entry in the 2×1 column vector on the right–hand side is a quadratic polynomial in the components of $\vec{U}$.

The rotation matrices in the ellipses are

$$R_0 = \begin{bmatrix} c_0 & s_0 \\ -s_0 & c_0 \end{bmatrix} \text{ and } R_1 = \begin{bmatrix} c_1 & s_1 \\ -s_1 & c_1 \end{bmatrix}$$

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for some angles ϕ_0 and ϕ_1 with $c_i = \cos(\phi_i)$, $s_i = \sin(\phi_i)$ for $i = 0, 1$. The diagonal matrices are

$$D_0 = \text{Diag}(1/a_0, 1/b_0) \quad \text{and} \quad D_1 = \text{Diag}(1/a_1, 1/b_1).$$

Define $\vec{U} = (u_0, u_1)$, $\vec{V} = (v_0, v_1)$, and $\vec{\Delta} = (\Delta_0, \Delta_1)$. The components of the linear system are calculated to be

$$
\begin{aligned}
m_{00} &= \frac{a_1 \sin(\phi_0 - \phi_1)}{a_0} u_0 + \frac{a_1 \cos(\phi_0 - \phi_1)}{b_0} u_1 \\
m_{01} &= \frac{-b_1 \cos(\phi_0 - \phi_1)}{a_0} u_0 + \frac{b_1 \sin(\phi_0 - \phi_1)}{b_0} u_1 \\
m_{10} &= \frac{\sin(\phi_0 - \phi_1)}{a_1 a_0} u_0 + \frac{\cos(\phi_0 - \phi_1)}{a_1 b_0} u_1 \\
m_{11} &= \frac{-\cos(\phi_0 - \phi_1)}{b_1 a_0} u_0 + \frac{\sin(\phi_0 - \phi_1)}{b_1 b_0} u_1 \\
n &= \left(\frac{\Delta_0 s_0}{a_0} - \frac{\Delta_1 c_0}{a_0} \right) u_0 + \left(\frac{\Delta_0 c_0}{b_0} + \frac{\Delta_1 s_0}{b_0} \right) u_1 + \left(\frac{a_0}{b_0} - \frac{b_0}{a_0} \right) u_0 u_1.
\end{aligned}
$$

The solution to the system for $\vec{V}$ is

$$
\begin{aligned}
v_0 &= m_{11} n / (m_{00} m_{11} - m_{01} m_{10}) \\
v_1 &= -m_{10} n / (m_{00} m_{11} - m_{01} m_{10}).
\end{aligned}
\tag{5}
$$

Since $\vec{V}$ is unit length,

$$1 = \frac{(m_{11} n)^2 + (-m_{10} n)^2}{(m_{00} m_{11} - m_{01} m_{10})^2}$$

or

$$P(u_0, u_1) = (m_{11}^2 + m_{10}^2) n^2 - (m_{00} m_{11} - m_{01} m_{10})^2 = 0.$$

The polynomial P is of degree 6 when both ellipses are not circles. If the first ellipse is a circle, then the coefficient of $u_0 u_1$ in n is zero, in which case P is of degree 4.

Suppose P has degree 6; then

$$P(u_0, u_1) = \sum_{k=0}^{6} p_k(u_0) u_1^k = \left(p_0 + p_2 u_1^2 + p_4 u_1^4 + p_6 u_1^6 \right) + u_1 \left(p_1 + p_3 u_1^2 + p_5 u_1^4 \right) \tag{6}$$

where $p_k(u_0)$ are polynomials in u_0 whose degree is at most $6 - k$. We also know that $\vec{U}$ is unit length, so

$$u_1^2 = 1 - u_0^2. \tag{7}$$

replacing this in equation (6) leads to

$$
\begin{aligned}
P(u_0, u_1) &= \left(p_0 + p_2(1 - u_0^2) + p_4(1 - u_0^2)^2 + p_6(1 - u_0^2)^3 \right) + u_1 \left(p_1 + p_3(1 - u_0^2) + p_5(1 - u_0^2)^2 \right) \\
&= \alpha(u_0) + u_1 \beta(u_0)
\end{aligned}
$$

where $\alpha(u_0)$ is a polynomial of degree 6 and $\beta(u_0)$ is a polynomial of degree 5. Setting this to zero and solving yields

$$u_1 = -\frac{\alpha(u_0)}{\beta(u_0)}.$$

Squaring, using equation (7), and rearranging terms leads to

$$S(u_0) = \alpha^2(u_0) + (u_0^2 - 1)\beta^2(u_0) = 0. \tag{8}$$

The polynomial S has degree 12.

The following procedure creates pairs $\vec{U}$ and $\vec{V}$, each potentially yielding the minimum distance between ellipses.

1. Solve equation (8) for its real–valued roots. For each root u_0 do:

2. Solve equation (7) for $u_1 = \pm\sqrt{1 - u_0^2}$. For each of the two pairs $\vec{U} = (u_0, u_1)$ do:

3. Compute $\vec{V}$ from equation (5) by evaluating m_{ij} and n at $\vec{U}$ and do:

4. Evaluate the squared distance $|A\vec{U} - B\vec{V} + \vec{\Delta}|^2$.

5. For all squared distances obtained by the above steps, the minimum value is the squared distance between the ellipses.

In the case of a circle and an ellipse that is not a circle, P has degree 4, leading to a polynomial S of degree 8. It is better to formulate the problem by translating the circle to the origin and translating the ellipse center by the same amount. The distance between ellipse and circle is now obtained by computing distance from the ellipse to the origin, a calculation that requires solving a degree 4 polynomial equation, then subtracting the circle radius from that distance.

Another observation is that you can set up the distance calculator as a numerical minimizer of a function of two variables. Just use the parametric representations for the ellipses. The two variables are the angles ϕ_0 and ϕ_1. Try your favorite minimizer on the squared distance between ellipses points, a function of the two angles.