Intersection of Linear and Circular Components in 2D

David Eberly
Magic Software, Inc.
http://www.magic-software.com

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1 Introduction

A *line* in 2D is parameterized as $\vec{P}+t\vec{D}$ where $\vec{D}$ is a nonzero vector and where $t \in \mathbb{R}$. A *ray* is parameterized the same way except that $t \in [0, \infty)$. The point $\vec{P}$ is the origin of the ray. A *segment* is also parameterized the same way except that $t \in [0, 1]$. The points $\vec{P}$ and $\vec{P} + \vec{D}$ are the end points of the segment. A *linear component* is the general term for a line, a ray, or a segment.

A *circle* in 2D is represented by $|\vec{X} - \vec{C}|^2 = R^2$ where $\vec{C}$ is the center and $R > 0$ is the radius of the circle. The circle can be parameterized by $\vec{X}(\theta) = \vec{C} + R\vec{U}(\theta)$ where $\vec{U}(\theta) = (\cos\theta, \sin\theta)$ and where $\theta \in [0, 2\pi)$. An *arc* is parameterized the same way except that $\theta \in [\theta_0, \theta_1]$ with $\theta_0 \in [0, 2\pi)$, $\theta_1 \in [0, 4\pi)$, and $\theta_0 < \theta_1$. The larger interval for θ_1 allows for arcs that intersect the positive x–axis. It is also possible to represent an arc by center $\vec{C}$, radius R, and two end points $\vec{A}$ and $\vec{B}$ that correspond to angles θ_0 and θ_1, respectively.

2 Intersection of Two Linear Components

Given two lines $\vec{P}_0 + s\vec{D}_0$ and $\vec{P}_1 + t\vec{D}_1$ for $s, t \in \mathbb{R}$, they are either intersecting, nonintersecting and parallel, or the same line. To help determine which of these cases occurs, define the scalar–valued operation $\text{Cross}((x_0, y_0), (x_1, y_1)) = x_0y_1 - x_1y_0$. The name is related to the cross product in 3D given by $(x_0, y_0, 0) \times (x_1, y_1, 0) = (0, 0, \text{Cross}((x_0, y_0), (x_1, y_1)))$. The operation has the property that $\text{Cross}(\vec{U}, \vec{V}) = -\text{Cross}(\vec{V}, \vec{U})$.

A point of intersection, if any, can be found by solving the two equations in two unknowns implied by setting $\vec{P}_0 + s\vec{D}_0 = \vec{P}_1 + t\vec{D}_1$. Rearranging terms yields $s\vec{D}_0 - t\vec{D}_1 = \vec{P}_1 - \vec{P}_0$. Setting $\vec{\Delta} = \vec{P}_1 - \vec{P}_0$ and applying the cross operation yields

$$\begin{aligned}\text{Cross}(\vec{D}_0, \vec{D}_1)\, s &= \text{Cross}(\vec{\Delta}, \vec{D}_1) \\ \text{Cross}(\vec{D}_0, \vec{D}_1)\, t &= \text{Cross}(\vec{\Delta}, \vec{D}_0)\end{aligned}$$

If $\text{Cross}(\vec{D}_0, \vec{D}_1) \neq 0$, then the lines intersect in a single point determined by $s = \text{Cross}(\vec{\Delta}, \vec{D}_1)/\text{Cross}(\vec{D}_0, \vec{D}_1)$ or $t = \text{Cross}(\vec{\Delta}, \vec{D}_0)/\text{Cross}(\vec{D}_0, \vec{D}_1)$. If $\text{Cross}(\vec{D}_0, \vec{D}_1) = 0$, then the lines are either nonintersecting and parallel or the same line. If the cross operation of the direction vectors is zero, then the previous equations in s and t reduce to a single equation $\text{Cross}(\vec{\Delta}, \vec{D}_0) = 0$ since $\vec{D}_1$ is a scalar multiple of $\vec{D}_0$. The lines are the same if this equation is true; otherwise, the lines are nonintersecting and parallel.

If the two linear components are a line ($s \in \mathbb{R}$) and a ray ($t \geq 0$), the point of intersection (if it exists) is determined by solving for s and t as shown previously. However, it must be verified that $t \geq 0$. If $t < 0$, the first line intersects the line containing the ray, but not at a ray point. Similar tests on s and t must be applied when either linear component is a ray or a segment. Finally, if the lines containing the two linear components are the same line, the linear components intersect in a t–interval, possibly empty, bounded, semiinfinite, or infinite. Computing the interval of intersection is somewhat tedious, but not complicated.

3 Intersection of a Linear and a Circular Component

Consider first a parameterized line $\vec{X}(t) = \vec{P} + t\vec{D}$ and a circle $|\vec{X} - \vec{C}|^2 = R^2$. Substitute the line equation into the circle equation, define $\vec{\Delta} = \vec{P} - \vec{C}$, and obtain the quadratic equation in t:

$$|\vec{D}|^2 t^2 + 2\vec{D} \cdot \vec{\Delta} t + |\vec{\Delta}|^2 - R^2 = 0.$$

The formal roots of the equation are

$$t = \frac{-\vec{D} \cdot \vec{\Delta} \pm \sqrt{(\vec{D} \cdot \vec{\Delta})^2 - |\vec{D}|^2(|\vec{\Delta}|^2 - R^2)}}{|\vec{D}|^2}.$$

Define $\delta = (\vec{D} \cdot \vec{\Delta})^2 - |\vec{D}|^2(|\vec{\Delta}|^2 - R^2)$. If $\delta < 0$, the line does not intersect the circle. If $\delta = 0$, the line is tangent to the circle (one point of intersetion). If $\delta > 0$, the line intersects the circle in two points.

If the linear component is a ray, and if $\bar{t}$ is a real–valued root of the quadratic equation, then the corresponding point of intersection between line and circle is a point of intersection between ray and circle if $\bar{t} \geq 0$. Similarly, if the linear component is a segment, the line–circle point of intersection is also one for the segment and circle if $\bar{t} \in [0, 1]$.

If the circular component is an arc, the points of intersection between the linear component and circle must be tested to see if they are on the arc. Let the arc have end points $\vec{A}$ and $\vec{B}$ where the arc is that portion of the circle obtained by traversing the circle counterclockwise from $\vec{A}$ to $\vec{B}$. Notice that the line containing $\vec{A}$ and $\vec{B}$ separates the arc from the remainder of the circle. If $\vec{P}$ is a point on the circle, it is on the arc if and only if it is on the same side of that line as the arc. The algebraic condition for the circle point $\vec{P}$ to be on the arc is

$$\text{Cross}(\vec{P} - \vec{A}, \vec{B} - \vec{A}) \geq 0.$$

4 Intersection of Circular Components

Let the two circles be represented by $|\vec{X} - \vec{C}_i|^2 = R^2$ for $i = 0, 1$. The points of intersection, if any, are determined by the following construction. Define $\vec{U} = \vec{C}_1 - \vec{C}_0 = (u_0, u_1)$. Define $\vec{V} = (u_1, -u_0)$. Note that $|\vec{U}|^2 = |\vec{V}|^2 = |\vec{C}_1 - \vec{C}_0|^2$ and $\vec{U} \cdot \vec{V} = 0$. The intersection points can be written in the form $\vec{X} = \vec{C}_0 + s\vec{U} + t\vec{V} = \vec{C}_1 + (s-1)\vec{U} + t\vec{V}$. Substituting the first of these into the first circle equation yields

$$(s^2 + t^2)|\vec{U}|^2 = R_0^2.$$

Substituting the second of these into the second circle equation yields

$$((s-1)^2 + t^2)|\vec{U}|^2 = R_1^2.$$

Subtracting and solving for s yields

$$s = \frac{1}{2}\left(\frac{R_0^2 - R_1^2}{|\vec{U}|^2} + 1\right).$$

Replacing this in the first equation and solving for t^2 yields

$$t^2 = \frac{R_0^2}{|\vec{U}|^2} - s^2.$$

In order for there to be solutoins, the right–hand side of the t^2 equation must be nonnegative. Some algebraic manipulation of these equations leads to the condition for existence of solutions:

$$(|\vec{U}|^2 - (R_0 + R_1)^2)(|\vec{U}|^2 - (R_0 - R_1)^2) \leq 0.$$

This in turn can be reduced to

$$|R_0 - R_1| \leq |\vec{U}| \leq |R_0 + R_1|.$$

If $|\vec{U}| = |R_0 + R_1|$, then each circle is outside the other circle, but just tangent. The point of intersection is $\vec{C}_0 + (R_0/(R_0 + R_1))\vec{U}$. If $|\vec{U}| = |R_0 - R_1|$, then the circles are nested and just tangent. The circles are the same if $|\vec{U}| = 0$ and $R_0 = R_1$, otherwise the point of intersection is $\vec{C}_0 + (R_0/(R_0 - R_1))\vec{U}$. If $|R_0 - R_1| < |\vec{U}| < |R_0 + R_1|$, then the two circles intersect in two points. The s and t values from the previous equations can be used to compute the intersection points as $\vec{C}_0 + s\vec{U} + t\vec{V}$.

If either or both circular components are arcs, the circle–circle points of intersection must be tested if they are on the arc (or arcs) using the circular–point–on–arc test described in the last section.