Distance Between Two Circles in 3D

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A circle in 3D is represented by a center $\vec{C}$, a radius R, and a plane containing the circle, $\vec{N}\cdot(\vec{X}-\vec{C})=0$ where $\vec{N}$ is a unit length normal to the plane. If $\vec{U}$ and $\vec{V}$ are also unit length vectors so that $\vec{U}$, $\vec{V}$, and $\vec{N}$ form a right–handed orthonormal coordinate system (the matrix with these vectors as columns is orthonormal with determinant 1), then the circle is parameterized as

$$\vec{X}=\vec{C}+R(\cos(\theta)\vec{U}+\sin(\theta)\vec{V}) =: \vec{C}+R\vec{W}(\theta)$$

for angles $\theta\in[0,2\pi)$. Note that $|\vec{X}-\vec{C}|=R$, so the $\vec{X}$ values are all equidistant from $\vec{C}$. Moreover, $\vec{N}\cdot(\vec{X}-\vec{C})=0$ since $\vec{U}$ and $\vec{V}$ are perpendicular to $\vec{N}$, so the $\vec{X}$ lie in the plane.

Let the two circles be $\vec{C}_0+R_0\vec{W}_0(\theta)$ for $\theta\in[0,2\pi)$ and $\vec{C}_1+R_1\vec{W}_1(\phi)$ for $\phi\in[0,2\pi)$. The squared distance between any two points on the circles is

$$\begin{aligned} F(\theta,\phi) &= |\vec{C}_1-\vec{C}_0+R_1\vec{W}_1-R_0\vec{W}_0|^2 \\ &= |\vec{D}|^2+R_0^2+R_1^2+2R_1\vec{D}\cdot\vec{W}_1-2R_0R_1\vec{W}_0\cdot\vec{W}_1-2R_0\vec{D}\cdot\vec{W}_0 \end{aligned}$$

where $\vec{D}=\vec{C}_1-\vec{C}_0$. Since F is doubly–periodic and continuously differentiable, its global minimum must occur when $\nabla F=(0,0)$. The partial derivatives are

$$\frac{\partial F}{\partial\theta}=-2R_0\vec{D}\cdot\vec{W}_0'-2R_0R_1\vec{W}_0'\cdot\vec{W}_1$$

and

$$\frac{\partial F}{\partial\phi}=2R_1\vec{D}\cdot\vec{W}_1'-2R_0R_1\vec{W}_0\cdot\vec{W}_1'.$$

Define $c_0=\cos(\theta)$, $s_0=\sin(\theta)$, $c_1=\cos(\phi)$, and $s_1=\sin(\phi)$. Then $\vec{W}_0=c_0\vec{U}_0+s_0\vec{V}_0$, $\vec{W}_1=c_1\vec{U}_1+s_1\vec{V}_1$, $\vec{W}_0'=-s_0\vec{U}_0+c_0\vec{V}_0$, and $\vec{W}_1'=-s_1\vec{U}_1+c_1\vec{V}_1$. Setting the partial derivatives equal to zero leads to

$$\begin{aligned} 0 &= s_0(a_0+a_1c_1+a_2s_1)+c_0(a_3+a_4c_1+a_5s_1) \\ 0 &= s_1(b_0+b_1c_0+b_2s_0)+c_1(b_3+b_4c_0+b_5s_0) \end{aligned}$$

where

$$\begin{aligned} &a_0=-\vec{D}\cdot\vec{U}_0,\quad a_1=-R_1\vec{U}_0\cdot\vec{U}_1,\quad a_2=-R_1\vec{U}_0\cdot\vec{V}_1,\quad a_3=\vec{D}\cdot\vec{V}_0,\quad a_4=R_1\vec{U}_1\cdot\vec{V}_0,\quad a_5=R_1\vec{V}_0\cdot\vec{V}_1, \\ &b_0=-\vec{D}\cdot\vec{U}_1,\quad b_1=R_0\vec{U}_0\cdot\vec{U}_1,\quad b_2=R_0\vec{U}_1\cdot\vec{V}_0,\quad b_3=\vec{D}\cdot\vec{V}_1,\quad b_4=-R_0\vec{U}_0\cdot\vec{V}_1,\quad b_5=-R_0\vec{V}_0\cdot\vec{V}_1. \end{aligned}$$

In matrix form we have

$$\begin{bmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{bmatrix} \begin{bmatrix} s_0 \\ c_0 \end{bmatrix} = \begin{bmatrix} a_0 + a_1 c_1 + a_2 s_1 & a_3 + a_4 c_1 + a_5 s_1 \\ b_2 s_1 + b_5 c_1 & b_1 s_1 + b_4 c_1 \end{bmatrix} \begin{bmatrix} s_0 \\ c_0 \end{bmatrix} = \begin{bmatrix} 0 \\ -(b_0 s_1 + b_3 c_1) \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda \end{bmatrix}$$

Let M denote the 2×2 matrix on the right–hand side of the equation. Multiplying by the adjoint of M yields

$$\det(M) \begin{bmatrix} s_0 \\ c_0 \end{bmatrix} = \begin{bmatrix} m_{11} & -m_{01} \\ -m_{10} & m_{00} \end{bmatrix} \begin{bmatrix} 0 \\ \lambda \end{bmatrix} = \begin{bmatrix} -m_{01}\lambda \\ m_{00}\lambda \end{bmatrix}. \tag{1}$$

Summing the squares of the vector components and using $s_0^2 + c_0^2 = 1$ yields

$$(m_{00}m_{11} - m_{01}m_{10})^2 = \lambda^2 \left(m_{00}^2 + m_{01}^2\right).$$

The above equation can be reduced to a polynomial of degree 8 whose roots $c_1 \in [-1, 1]$ are the candidates to provide the global minimum of F. Formally computing the determinant and using $s_1^2 = 1 - c_1^2$ leads to

$$m_{00}m_{11} - m_{01}m_{10} = p_0(c_1) + s_1 p_1(c_1)$$

where $p_0(z) = \sum_{i=0}^{2} p_{0i} z^i$ and $p_1(z) = \sum_{i=0}^{1} p_{1i} z$. The coefficients are

$$\begin{aligned} p_{00} &= a_2 b_1 - a_5 b_2, \\ p_{01} &= a_0 b_4 - a_3 b_5, \\ p_{02} &= a_5 b_2 - a_2 b_1 + a_1 b_4 - a_4 b_5, \\ p_{10} &= a_0 b_1 - a_3 b_2, \\ p_{11} &= a_1 b_1 - a_5 b_5 + a_2 b_4 - a_4 b_2. \end{aligned}$$

Similarly,

$$m_{00}^2 + m_{01}^2 = q_0(c_1) + s_1 q_1(c_1)$$

where $q_0(z) = \sum_{i=0}^{2} q_{0i} z^i$ and $q_1(z) = \sum_{i=0}^{1} q_{1i} z$. The coefficients are

$$\begin{aligned} q_{00} &= a_0^2 + a_2^2 + a_3^2 + a_5^2, \\ q_{01} &= 2(a_0 a_1 + a_3 a_4), \\ q_{02} &= a_1^2 - a_2^2 + a_4^2 - a_5^2, \\ q_{10} &= 2(a_0 a_2 + a_3 a_5), \\ q_{11} &= 2(a_1 a_2 + a_4 a_5). \end{aligned}$$

Finally,

$$\lambda^2 = r_0(c_1) + s_1 r_1(c_1)$$

where $r_0(z) = \sum_{i=0}^{2} r_{0i} z^i$ and $r_1(z) = \sum_{i=0}^{1} r_{1i} z$. The coefficients are

$$\begin{aligned} r_{00} &= b_0^2, \\ r_{01} &= 0, \\ r_{02} &= b_3^2 - b_0^2, \\ r_{10} &= 0, \\ r_{11} &= 2b_0 b_3. \end{aligned}$$

Combining these yields

$$0 = \left[(p_0^2 - r_0 q_0) + (1 - c_1^2)(p_1^2 - r_1 q_1)\right] + s_1 \left[2p_0 p_1 - r_0 q_1 - r_1 q_0\right] = g_0(c_1) + s_1 g_1(c_1) \tag{2}$$

where $g_0(z) = \sum_{i=0}^{4} g_{0i} z^i$ and $g_1(z) = \sum_{i=0}^{3} g_{1i} z^i$. The coefficients are

$$\begin{aligned} g_{00} &= p_{00}^2 + p_{10}^2 - q_{00} r_{00} \\ g_{01} &= 2(p_{00} p_{01} + p_{10} p_{11}) - q_{01} r_{00} - q_{10} r_{11} \\ g_{02} &= p_{01}^2 + 2p_{00} p_{02} + p_{11}^2 - p_{10}^2 - q_{02} r_{00} - q_{00} r_{02} - q_{11} r_{11} \\ g_{03} &= 2(p_{01} p_{02} - p_{10} p_{11}) - q_{01} r_{02} + q_{10} r_{11} \\ g_{04} &= p_{02}^2 - p_{11}^2 - q_{02} r_{02} + q_{11} r_{11} \\ g_{10} &= 2p_{00} p_{10} - q_{10} r_{00} \\ g_{11} &= 2(p_{01} p_{10} + p_{00} p_{11}) - q_{11} r_{00} - q_{00} r_{11} \\ g_{12} &= 2(p_{02} p_{10} + p_{01} p_{11}) - q_{10} r_{02} - q_{01} r_{11} \\ g_{13} &= 2p_{02} p_{11} - q_{11} r_{02} - q_{02} r_{11} \end{aligned}$$

We can eliminate the s_1 term by solving $g_0 = -s_1 g_1$ and squaring to obtain

$$0 = g_0^2 - (1 - c_1^2) g_1^2 = h(c_1)$$

where $h(z) = \sum_{i=0}^{8} h_i z^i$. The coefficients are

$$\begin{aligned} h_0 &= g_{00}^2 - g_{10}^2, \\ h_1 &= 2(g_{00} g_{01} - g_{10} g_{11}), \\ h_2 &= g_{01}^2 + g_{10}^2 - g_{11}^2 + 2(g_{00} g_{02} - g_{10} g_{12}), \\ h_3 &= 2(g_{01} g_{02} + g_{00} g_{03} + g_{10} g_{11} - g_{11} g_{12} - g_{10} g_{13}), \\ h_4 &= g_{02}^2 + g_{11}^2 - g_{12}^2 + 2(g_{01} g_{03} + g_{00} g_{04} + g_{10} g_{12} - g_{11} g_{13}), \\ h_5 &= 2(g_{02} g_{03} + g_{01} g_{04} + g_{11} g_{12} + g_{10} g_{13} - g_{12} g_{13}), \\ h_6 &= g_{03}^2 + g_{12}^2 - g_{13}^2 + 2(g_{02} g_{04} + g_{11} g_{13}), \\ h_7 &= 2(g_{03} g_{04} + g_{12} g_{13}), \\ h_8 &= g_{04}^2 + g_{13}^2. \end{aligned}$$

To find the minimum squared distance, compute all the real–valued roots of $h(c_1) = 0$. For each c_1, compute $s_1 = \pm\sqrt{1 - c_1^2}$ and choose either (or both) of these that satisfy equation (2). For each pair (c_1, s_1), solve for (c_0, s_0) in equation (1). The main numerical issue to deal with is how close to zero is $\det(M)$. (TO DO: Show that this case only occurs when circles are parallel and $\vec{D}$ is normal to both planes?)