

# Geometric Invariance

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Created: March 2, 1999

## 1 Vector Fields

Let  $\mathbb{R}^n$  denote  $n$ -tuples of real numbers. A *vector field* is a function  $V : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . The  $k^{\text{th}}$  component of  $V$  is a function  $\xi_k : \mathbb{R}^n \rightarrow \mathbb{R}$ . The vector field can be written as an  $n$ -tuple

$$V = (\xi_1(x), \dots, \xi_n(x)), \quad x \in \mathbb{R}^n$$

or can be thought of as an  $n \times 1$  column vector (when used in matrix calculations). We also can write the vector field as a linear combination

$$V = \sum_{k=1}^n \xi_k(x) \frac{\partial}{\partial x_k}$$

where the symbols  $\partial/\partial x_k$  are place-keepers for the  $k^{\text{th}}$  component. In this form  $V$  can be thought of as a directional derivative operator  $V \bullet \nabla$  which can be applied to functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ .

Although the general notation uses indexed variables  $x = (x_1, \dots, x_n)$ , for small dimensions we might use different names. For example, if  $n = 2$ , then we might use  $(x, y)$  for the position vector; for  $n = 3$  we might use  $(x, y, z)$  for the position.

EXAMPLE. Let  $V = (xy, x^2)$  be a vector field on  $\mathbb{R}^2$ . In directional derivative form,

$$V = xy \frac{\partial}{\partial x} + x^2 \frac{\partial}{\partial y}.$$

Let  $f(x, y) = \sin(x + y^2)$ . The derivative of  $f$  in the direction  $V$  is

$$Vf = (V \bullet \nabla)f = xy \cos(x + y^2) + 2x^2y \cos(x + y^2).$$

Another notation we use for directional derivative is  $D_V f$ . ⊞

As a directional derivative operator,  $V$  measures an *infinitesimal* rate of change in the direction indicated. That is,  $V$  is a spatially varying tangent vector to certain curves. To determine the curves themselves, you need to solve the following system of ordinary differential equations:

$$\frac{dx}{dt} = V(x(t)), \quad t > 0; \quad x(0) = x_0$$

where  $x_0$  is the starting point on the curve. The curve represents the *global action* of the vector field.

EXAMPLE. Let  $V = (x, 0)$  with initial point  $(2, -1)$ . The system of equations is

$$\begin{aligned}\frac{dx}{dt} &= x, & x(0) &= 2 \\ \frac{dy}{dt} &= 0, & y(0) &= -1\end{aligned}$$

The second equation says  $y$  is constant along the curve, so  $y(t) \equiv -1$ . The first equation has solution  $x(t) = 2 \exp(t)$ . If you think of  $V$  as velocity of a particle initially at  $(2, -1)$ , then the particle flows along the straight line  $(x(t), y(t)) = (2 \exp(t), -1)$ , but has increasing speed.  $\bowtie$

EXAMPLE. This example shows that solving the system may not always be as easy as you would like. Let  $V = (-y, x)$  with initial point  $(x_0, y_0)$ . The system of equations is

$$\begin{aligned}\frac{dx}{dt} &= -y, & x(0) &= x_0 \\ \frac{dy}{dt} &= x, & y(0) &= y_0\end{aligned}$$

The equations are coupled. However, this is a linear problem and can be solved by methods learned in a standard ordinary differential equations course. In matrix form the system is

$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

A matrix system of the form  $x'(t) = Ax(t)$  where  $A$  is a constant  $n \times n$  matrix and  $x \in \mathbb{R}^n$ , with initial data  $x(0) = x_0$ , has solution  $x(t) = \exp(tA)x_0$  where the matrix  $\exp(tA)$  is formally obtained by taking the Taylor series for  $\exp(z)$  and replacing  $z$  by the matrix  $tA$ . For our example,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \exp \left( t \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

Note that the end result is a rotation of the initial data, so the particle trajectories are circles.  $\bowtie$

## 2 Prolongations

Let  $x, u \in \mathbb{R}$  and think of  $u = u(x)$  ( $u$  is a function of  $x$ ). If we make a change of variables  $g(x, y) = (\bar{x}, \bar{u})$  such that  $\bar{u} = \bar{u}(\bar{x})$ , then how is  $d\bar{u}/d\bar{x}$  related to  $du/dx$ ?

EXAMPLE. Let the change of variables be

$$\begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

where  $c = \cos(\theta)$  and  $s = \sin(\theta)$  for some  $\theta > 0$ . As long as  $du/dx$  is finite on the domain of  $u$ , then there is a small enough angle  $\theta$  so that  $\bar{u}$  is a function of  $\bar{x}$ .

We can determine the relationship between the derivatives with a little bit of calculus (using the chain rule). We have  $\bar{x} = cx - su$ , so taking derivatives with respect to  $\bar{x}$ , we get

$$1 = \frac{d\bar{x}}{d\bar{x}} = c \frac{dx}{d\bar{x}} - s \frac{du}{d\bar{x}} = c \frac{dx}{d\bar{x}} - \frac{du}{dx} \frac{dx}{d\bar{x}}$$

which implies that

$$\frac{dx}{d\bar{x}} = \frac{1}{c - su_x}$$

where  $u_x = du/dx$ . We also have  $\bar{u} = sx + cu$ , so taking derivatives again with respect to  $\bar{x}$  gives us

$$\frac{d\bar{u}}{d\bar{x}} = s \frac{dx}{d\bar{x}} + c \frac{du}{d\bar{x}} = s \frac{dx}{d\bar{x}} + c \frac{du}{dx} \frac{dx}{d\bar{x}} = \frac{s + cu_x}{c - su_x}.$$

The group action  $g(x, u) = (\bar{x}, \bar{u})$  and condition  $u = u(x)$  induce an action on the variables  $(x, u, u_x)$ , called the first-order *prolongation* of  $g$ . We denote it by  $\text{pr}^{(1)}g(x, u, u_x)$ . For our example,

$$\text{pr}^{(1)}g(x, u, u_x) = \left( cx - su, sx + cu, \frac{s + cu_x}{c - su_x} \right).$$

Note that when  $\theta = 0$ , the mapping is the identity. Here we are given the group action, but we can compute the infinitesimal action by taking the derivative with respect to  $\theta$  at  $\theta = 0$ . [In the last section we had  $V$  and found  $g$ . Now we do the reverse process.]

The vector field  $V$  corresponding to  $g$  is

$$V = \left( \frac{d\bar{x}}{d\theta} \Big|_{\theta=0} \right) \frac{\partial}{\partial x} + \left( \frac{d\bar{u}}{d\theta} \Big|_{\theta=0} \right) \frac{\partial}{\partial u} = -u \frac{\partial}{\partial x} + x \frac{\partial}{\partial u}.$$

The vector field corresponding to  $\text{pr}^{(1)}g$  is given by

$$\text{pr}^{(1)}V = V + \left( \frac{d\bar{u}_x}{d\theta} \Big|_{\theta=0} \right) \frac{\partial}{\partial u_x} = -u \frac{\partial}{\partial x} + x \frac{\partial}{\partial u} + (1 + u_x^2) \frac{\partial}{\partial u_x}.$$

I'll leave the derivative details up to you.

The same idea can be applied to determining the relationships between higher-order derivatives. In our example,

$$\bar{u}_{\bar{x}\bar{x}} = \frac{d}{d\bar{x}} \left( \frac{s + cu_x}{c - su_x} \right) = \frac{u_{xx}}{(c - su_x)^3}$$

and

$$\frac{d\bar{u}_{\bar{x}\bar{x}}}{d\theta} \Big|_{\theta=0} = 3u_x u_{xx}.$$

The second-order group prolongation is

$$\text{pr}^{(2)}g(x, u, u_x, u_{xx}) = \left( cx - su, sx + cu, \frac{s + cu_x}{c - su_x}, \frac{u_{xx}}{(c - su_x)^3} \right)$$

and the corresponding vector field is

$$\text{pr}^{(2)}V = \text{pr}^{(1)}V + 3u_x u_{xx} \frac{\partial}{\partial u_{xx}} = -u \frac{\partial}{\partial x} + x \frac{\partial}{\partial u} + (1 + u_x^2) \frac{\partial}{\partial u_x} + 3u_x u_{xx} \frac{\partial}{\partial u_{xx}}.$$

An important point to note is that the prolongation vectors have place keepers  $x$ ,  $u$ ,  $u_x$ , and  $u_{xx}$ . These are simply *names* for the components of the vector field and no analytic relationship between  $x$  and  $u$  is used.

### 3 Invariants

Given a vector field  $V : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , an *invariant* is a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  such that the directional derivative satisfies  $Vf = 0$ . That is,  $f$  remains constant as you walk in the direction of  $V$ . Consequently the level sets of  $f$  are the solution curves to the system of differential equations  $dx/dt = V(x)$ .

EXAMPLE. Consider the vector field

$$V = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}.$$

An invariant is  $f(x, y) = x^2 + y^2$  since  $Vf = V \bullet \nabla f = (-y, x) \bullet (2x, 2y) \equiv 0$ . The level curves of  $f$  are circles, which is intuitive since as we saw before,  $V$  is the infinitesimal generator for rotation.  $\bowtie$

For most vector fields we may not be able to guess at the invariants  $f$ , so we need a constructive method. As indicated earlier, the level sets of an invariant  $f$  are solutions to a system of differential equations.

EXAMPLE. For the vector field in the last example, the level curves of invariants are given by

$$\begin{aligned} \frac{dx}{dt} &= -y, & x(0) &= x_0 \\ \frac{dy}{dt} &= x, & y(0) &= y_0 \end{aligned}$$

We saw earlier that this system has solution  $x = x_0 \cos(t) - y_0 \sin(t)$  and  $y = x_0 \sin(t) + y_0 \cos(t)$ . It can be shown that  $x^2 + y^2 = x_0^2 + y_0^2$ . Another way of determining the level curves (for a system of 2 equations) is to take the ratio of the two equations to get a single differential equation

$$\frac{dy}{dx} = \frac{x}{-y}.$$

This equation is separable:  $x dx + y dy = 0$ . Integrate to get  $\frac{x^2}{2} + \frac{y^2}{2} = c$  for constant  $c$ . Substituting in the initial data gives us  $x^2 + y^2 = x_0^2 + y_0^2$ .  $\bowtie$

We can also look for invariants with respect to more than one vector field. Let  $V_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$  for  $k = 1, \dots, m$  be vector fields. A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is an *invariant* for the vector fields if  $V_k f = 0$  for all  $k$ .

EXAMPLE. Consider the vector fields on  $\mathbb{R}^3$ ,

$$V_1 = 2 \frac{\partial}{\partial x} - 3 \frac{\partial}{\partial y}, \quad V_2 = 1 \frac{\partial}{\partial x} + 1 \frac{\partial}{\partial z}.$$

A function  $f(x, y, z)$  is an invariant if  $0 = V_1 f = (2, -3, 0) \bullet \nabla f$  and  $0 = V_2 f = (1, 0, 1) \bullet \nabla f$ . The gradient of  $f$  must be a vector which is orthogonal to both  $(2, -3, 0)$  and  $(1, 0, 1)$ . Consequently  $\nabla f$  is parallel to  $(-3, -2, 3) = (2, -3, 0) \times (1, 0, 1)$ . An invariant is  $f(x, y, z) = -3x - 2y + 3z$ .  $\bowtie$

#### 3.1 Independent Functions

One of the questions you should ask is: How many invariants are there for a vector field (or vector fields)? For example, an invariant for the rotation vector field  $V = -y \partial / \partial x + x \partial / \partial y$  is  $f(x, y) = x^2 + y^2$ . Another invariant is  $\sin(x^2 + y^2)$ . In fact for any differentiable function  $g : \mathbb{R} \rightarrow \mathbb{R}$ ,  $g(f(x, y))$  is an invariant. But in some sense these all *depend* on the quantity  $x^2 + y^2$ .

Let  $f_k : \mathbb{R}^n \rightarrow \mathbb{R}$  for  $k = 1, \dots, m$  where  $m \leq n$  be  $m$  differentiable functions. These are said to be *functionally independent* at  $x \in \mathbb{R}^n$  if the  $m \times n$  matrix of first derivatives  $[\partial f_i / \partial x_j]$  has full rank  $m$ .

EXAMPLE. Let  $f_1(x, y) = x^2 + y^2$  and  $f_2(x, y) = \sin(x^2 + y^2)$ . The first derivative matrix is

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & 2y \\ 2x \cos(x^2 + y^2) & 2y \cos(x^2 + y^2) \end{bmatrix}.$$

The second row is just a scalar times the first row, so the matrix has rank 1 (not full rank), so the two functions are dependent.  $\boxtimes$

EXAMPLE. Let  $f_1(x, y) = x^2 + y^2$  and  $f_2(x, y) = x$ . The first derivative matrix is

$$\begin{bmatrix} 2x & 2y \\ 1 & 0 \end{bmatrix}.$$

For  $(x, y) \neq (0, 0)$ , the matrix can be row-reduced to the identity, so the matrix has full rank. Thus,  $f_1$  and  $f_2$  are functionally independent.  $\boxtimes$

EXAMPLE. Consider only the vector field on  $\mathbb{R}^3$ ,

$$V = 2 \frac{\partial}{\partial x} - 3 \frac{\partial}{\partial y} + 0 \frac{\partial}{\partial z}.$$

Two invariants are  $f_1(x, y, z) = -3x - 2y$  and  $f_2(x, y, z) = z$ . They are functionally independent since the second derivative matrix

$$\begin{bmatrix} -3 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has full rank (2).

### 3.2 Lie Algebras

Given vector fields  $V_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$  for  $k = 1, \dots, m$ , with  $m \leq n$ , how many functionally independent invariants are there? To answer this question we need a little background in *Lie algebras*.

Consider two vector fields

$$V(x) = \sum_{i=1}^n a_i(x) \frac{\partial}{\partial x_i} \quad \text{and} \quad W(x) = \sum_{j=1}^n b_j(x) \frac{\partial}{\partial x_j}.$$

Each vector field can be applied as a directional derivative to functions from  $\mathbb{R}^n \rightarrow \mathbb{R}$ . In particular, the compositions  $V(Wf)$  and  $W(Vf)$  are well-defined. Thus,

$$\begin{aligned} W(Vf) &= \sum_{j=1}^n b_j \frac{\partial}{\partial x_j} \left( \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i} \right) \\ &= \sum_{j=1}^n b_j \left[ \sum_{i=1}^n \left( a_i \frac{\partial^2 f}{\partial x_j \partial x_i} + \frac{\partial a_i}{\partial x_j} \frac{\partial f}{\partial x_i} \right) \right] \\ &= \sum_{i=1}^n \sum_{j=1}^n \left( a_i \frac{\partial^2 f}{\partial x_i \partial x_j} b_j + b_j \frac{\partial a_i}{\partial x_j} \frac{\partial f}{\partial x_i} \right) \end{aligned}$$

and

$$\begin{aligned}
V(Wf) &= \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} \left( \sum_{j=1}^n b_j \frac{\partial f}{\partial x_j} \right) \\
&= \sum_{i=1}^n a_i \left[ \sum_{j=1}^n \left( b_j \frac{\partial^2 f}{\partial x_i \partial x_j} + \frac{\partial b_j}{\partial x_i} \frac{\partial f}{\partial x_j} \right) \right] \\
&= \sum_{i=1}^n \sum_{j=1}^n \left( a_i \frac{\partial^2 f}{\partial x_i \partial x_j} b_j + a_j \frac{\partial b_i}{\partial x_j} \frac{\partial f}{\partial x_i} \right)
\end{aligned}$$

where we assume that  $f$  is sufficiently smooth to guarantee that its mixed partial derivatives are the same. The difference of the above quantities is

$$W(Vf) - V(Wf) = \sum_{i=1}^n \left( \sum_{j=1}^n \frac{\partial a_i}{\partial x_j} b_j - \frac{\partial b_i}{\partial x_j} a_j \right) \frac{\partial f}{\partial x_i} =: \sum_{i=1}^n c_i(x) \frac{\partial f}{\partial x_i}.$$

This motivates the *Lie product* of two vector fields. Define the vector field

$$[V, W] = \sum_{i=1}^n c_i(x) \frac{\partial}{\partial x_i}$$

where  $c_i(x)$  was defined earlier. In terms of matrix operations, if  $DV$  and  $DW$  are the first derivative matrices of the coefficient functions of  $V$  and  $W$ , then  $[V, W] = (DV)W - (DW)V$ . Thus, the difference of the mixed directional derivatives is

$$W(Vf) - V(Wf) = [V, W]f.$$

NOTE: Given two directions  $V(x)$  and  $W(x)$ , the order of directional differentiation at  $x$  is irrelevant if and only if  $[V(x), W(x)]f(x) = 0$ .

Given a collection of vector fields  $V_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$  for  $k = 1, \dots, m$  with  $m \leq n$ , the *Lie algebra* of the vector fields is obtained by constructing the smallest vector space which contains all sums, scalar multiples, and Lie products of the  $V_k$ . We will call this vector space  $L(V_1, \dots, V_m)$ .

### 3.3 Independent Invariants

Now we get back to the question of how many functionally independent invariants are there for a collection of vector fields. First note that if  $f$  is an invariant, then by definition  $V_k f = 0$  for all  $1 \leq k \leq m$ . Intuitively if the  $V_k$  are linearly independent, then the actions of  $V_k$  should require you to move around on a  $k$ -dimensional surface. That gives you  $n - k$  degrees of freedom in which to “stack up” the level surfaces for invariants. So you might expect to find  $n - k$  functionally independent invariants, one corresponding to each degree of freedom.

However, the situation is slightly more complicated. Since  $V_k f = 0$  for all  $k$ , it is easily seen that  $[V_i, V_j]f = (DV_i)V_j f - (DV_j)V_i f = 0$ , so  $f$  is also an invariant for the vector field  $[V_i, V_j]$ . It is possible that the Lie product of two linearly independent vector fields is another vector field independent of the first two. In this case, you lose a degree of freedom for the invariants.

EXAMPLE. Let  $V = (x_1, x_2, x_3)$  and  $W = (x_1, x_2, 0)$ . The Lie product is

$$[V, W] = (DV)W - (DW)V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -x_3 \end{bmatrix}$$

The vectors  $V$ ,  $W$ , and  $[V, W]$  are linearly independent.

The last argument shows that the number of degrees of freedom is not  $n - k$ , but rather it should be  $n - \dim(L)$  where  $\dim(L)$  is the dimension of the Lie algebra.

EXAMPLE. Consider the example where  $x, u \in \mathbb{R}$ ,  $u = u(x)$ , and

$$V_1 = -u \frac{\partial}{\partial x} + x \frac{\partial}{\partial u}, \quad V_2 = 1 \frac{\partial}{\partial x}, \quad V_3 = 1 \frac{\partial}{\partial u}.$$

These correspond to rotation in  $(x, u)$ , translation in  $x$ , and translation in  $u$ . We want to find some low-order invariants of all three.

(1) Zero-order invariants: We want functions  $f$  such that  $V_k f = 0$  for  $k = 1, 2, 3$ . The Lie algebra generated by the vector fields is all of  $\mathbb{R}^2$ , so  $n - \dim(L) = 2 - 2 = 0$ , so there are no zero-order invariants.

(2) First-order invariants: We want functions  $f$  such that  $\text{pr}^{(1)} V_k f = 0$  for  $k = 1, 2, 3$ . The prolongations are

$$W_1 = \text{pr}^{(1)} V_1 = -u \frac{\partial}{\partial x} + x \frac{\partial}{\partial u} + (1 + u_x^2) \frac{\partial}{\partial u_x}, \quad W_2 = \text{pr}^{(1)} V_2 = V_2, \quad W_3 = \text{pr}^{(1)} V_3 = V_3.$$

Some computations will show you that

$$[W_1, W_2] = W_3, \quad [W_1, W_3] = -W_1, \quad [W_2, W_3] = 0.$$

So we get nothing new from the pairwise products. Since the  $W_k$  act on  $\mathbb{R}^3$  and are linearly independent,  $n - \dim(L) = 3 - 3 = 0$ , so there are no first-order invariants.

(3) Second-order invariants: We want functions  $f$  such that  $\text{pr}^{(2)} V_k f = 0$  for  $k = 1, 2, 3$ . The prolongations are

$$W_1 = \text{pr}^{(2)} V_1 = -u \frac{\partial}{\partial x} + x \frac{\partial}{\partial u} + (1 + u_x^2) \frac{\partial}{\partial u_x} + (3u_x u_{xx}) \frac{\partial}{\partial u_{xx}}, \quad W_2 = \text{pr}^{(2)} V_2 = V_2, \quad W_3 = \text{pr}^{(2)} V_3 = V_3.$$

Some computations will show you that

$$[W_1, W_2] = W_3, \quad [W_1, W_3] = -W_1, \quad [W_2, W_3] = 0.$$

So we get nothing new from the pairwise products. Since the  $W_k$  act on  $\mathbb{R}^4$  and are linearly independent,  $n - \dim(L) = 4 - 3 = 1$ , so there is exactly one functionally independent second-order invariant.

To construct an invariant  $f = f(x, u, u_x, u_{xx})$ , we need  $W_k f = 0$  for  $k = 1, 2, 3$ . Note that  $W_2 f = 0$  implies  $f_x = 0$  and  $W_3 f = 0$  implies  $f_u = 0$ . Thus,  $f = f(u_x, u_{xx})$  only. To avoid confusion we will rename  $u_x, u_{xx}$  to  $v$  and  $w$ . Then  $f = f(v, w)$  and  $W_1 f = 0$  implies  $0 = (1 + v^2)f_v + (3vw)f_w = (1 + v^2, 3vw) \bullet \nabla f$ . The level curves of  $f$  must be solutions to the system of equations

$$\begin{aligned} \frac{dv}{dt} &= 1 + v^2 \\ \frac{dw}{dt} &= 3vw \end{aligned}$$

Taking the ratio, we get  $dv/dw = (1 + v^2)/(3vw)$ . Separating variables leads to

$$\frac{3v}{1 + v^2} dv = \frac{1}{w} dw$$

and integrating gives us

$$\frac{3}{2} \ln(1 + v^2) = \ln(w) + \ln(c)$$

for positive constant  $c$ . We can solve for  $c$  and replace  $v$  and  $w$  to get

$$f(u_x, u_{xx}) = \frac{u_{xx}}{(1 + u_x^2)^{3/2}} = c$$

so  $f$  is the invariant. Notice that this is the curvature of the graph of  $u(x)$ . If you translate or rotate the graph of  $u$ , the curvature is preserved.

Higher-order invariants can be constructed similarly. For a function  $u(x, y)$ , the most important vector fields on  $\mathbb{R}^3$  for us to consider are

$1 \frac{\partial}{\partial x}$	translation in $x$
$1 \frac{\partial}{\partial y}$	translation in $y$
$-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$	rotation in $(x, y)$
$x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$	uniform magnification in $(x, y)$
$(a + bu) \frac{\partial}{\partial u}$	affine transformation of $u$

The corresponding global actions obtained by solving the system  $d(\bar{x}, \bar{y}, \bar{u})/dt = V(\bar{x}, \bar{y}, \bar{u})$  with initial data  $(\bar{x}(0), \bar{y}(0), \bar{u}(0)) = (x, y, u)$  are respectively

$$\begin{aligned} (\bar{x}, \bar{y}, \bar{u}) &= (x + t, y, u) \\ (\bar{x}, \bar{y}, \bar{u}) &= (x, y + t, u) \\ (\bar{x}, \bar{y}, \bar{u}) &= (x \cos(t) - y \sin(t), x \sin(t) + y \cos(t), u) \\ (\bar{x}, \bar{y}, \bar{u}) &= (x \exp(t), y \exp(t), u) \\ (\bar{x}, \bar{y}, \bar{u}) &= (x, y, u \exp(bt) + at) \end{aligned}$$

The ideas of invariants also apply when you have more than one dependent variable.