Low–Degree Polynomial Roots

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The roots of polynomials of degrees 2, 3, or 4 can be found using algebraic methods. The implementation in `MgcPolynomial.cpp` computes only the real–valued roots. The formulas are taken from the CRC Handbook of Mathematics. The cubic roots are given in complex form, but I have done a bit more algebra to show what the real roots are. I have also assumed that the leading coefficients of the polynomials are 1.

1 Quadratic Roots

The roots to $x^2 + ax + b = 0$ are

$$x = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

and are real when $a^2 - 4b \geq 0$.

2 Cubic Roots

The roots to $y^3 + py^2 + qy + r = 0$ are obtained by first eliminating the squared term via $y = x - p/3$, $a = (3q - p^2)/3$, and $b = (2p^3 - 9pq + 27r)/27$, to obtain $x^3 + ax + b = 0$. For real coefficients p, q, and r, define $Q = b^2/4 + a^3/27$.

- If $Q > 0$, there is exactly one real root
$$x_1 = \left(-\frac{b}{2} + \sqrt{Q}\right)^{1/3} + \left(-\frac{b}{2} - \sqrt{Q}\right)^{1/3}.$$
- If $Q = 0$, there will be three real roots, at least two are equal
$$x_1 = x_2 = \left(\frac{b}{2}\right)^{1/3},\ x_3 = -2\left(\frac{b}{2}\right)^{1/3}.$$
- If $Q < 0$, there are three distinct real roots. Let $-b/2 + i\sqrt{-Q} = \rho\exp(i\theta)$ be the polar representation of the number; then
$$x_1 = 2\rho^{1/3}\cos(\theta/3),\ x_2 = -\rho^{1/3}(\cos(\theta/3) + \sqrt{3}\sin(\theta/3)),\ x_3 = -\rho^{1/3}(\cos(\theta/3) - \sqrt{3}\sin(\theta/3)).$$

The roots of the original polynomial are then $y_k = x_k - p/3$ for all valid k.

3 Quartic Roots

The roots to $x^4 + ax^3 + bx^2 + cx + d = 0$ are obtained by first finding a real root y to the resolvent cubic equation

$$y^3 - by^2 + (ac - 4d)y + (-a^2 d + 4bd - c^2) = 0.$$

Define $R = \sqrt{a^2/4 - b + y}$. If $R \neq 0$, define

$$D = \sqrt{\frac{3a^2}{4} - R^2 - 2b + \frac{4ab - 8c - a^3}{4R}}$$

and

$$E = \sqrt{\frac{3a^2}{4} - R^2 - 2b - \frac{4ab - 8c - a^3}{4R}}.$$

If $R = 0$, define

$$D = \sqrt{\frac{3a^2}{4} - 2b + 2\sqrt{y^2 - 4d}}$$

and

$$E = \sqrt{\frac{3a^2}{4} - 2b - 2\sqrt{y^2 - 4d}}.$$

The four roots of the polynomial are

$$x = -\frac{a}{4} + \frac{R}{2} \pm \frac{D}{2}, -\frac{a}{4} - \frac{R}{2} \pm \frac{E}{2}.$$

In the code, if any of the arguments for the square roots are negative, then the roots are not real. (Proof of this? What if the imaginary parts of R, D, or E cancel? This can happen only when $y = a^2/4$ and $4ab - 8c - a^3 = 0$.)

4 Real Parts of Polynomial Roots

Let $P_n(z) = \sum_{k=0}^{n} a_k^{(n)} z^k$ be a polynomial of degree n with complex coefficients. Let the roots be $z_j^{(n)}$, $j = 1, \ldots, n$. Define the $n-1$ degree polynomial

$$P_{n-1}(z) = [a_n^{(n)} \bar{a}_{n-1}^{(n)} + a_{n-1}^{(n)} \bar{a}_n^{(n)} - a_n^{(n)} \bar{a}_n^{(n)} z] P_n(z) + \left(a_n^{(n)}\right)^2 z \sum_{k=0}^{n} (-1)^{n-k} \bar{a}_k^{(n)} z^k = \sum_{k=0}^{n-1} a_k^{(n-1)} z^k$$

where $\bar{c}$ denotes the complex conjugate of c. Let the roots be $z_j^{(n-1)}$, $j = 1, \ldots, n-1$. Let $\mathrm{Re}(c)$ denote the real part of c. The Routh–Hurwitz criterion states:

$$\mathrm{Re}(z_j^{(n)}) < 0,\ 1 \le j \le n \text{ if and only if } \mathrm{Re}(a_{n-1}^{(n)}/a_n^{(n)}) > 0 \text{ and } \mathrm{Re}(z_j^{(n-1)}) < 0,\ 1 \le j \le n-1.$$

This gives a recursive way of deciding if all the real parts of a polynomial are negative.

I use this criterion for applications where $P_n(z)$ is the characteristic polynomial for a real symmetric $n \times n$ matrix M. Necessarily the roots are all real, so the criterion allows us to decide if the polynomial has all negative or all positive real roots. Thus, M is negative definite if all roots of $P_n(z)$ are negative, and M is positive definite if all roots of $P_n(z)$ are positive.

For polynomials with all real coefficients we have the recurrence relations

$$\begin{aligned} a_0^{(n-1)} &= 2a_0^{(n)} a_{n-1}^{(n)} a_n^{(n)} \\ a_k^{(n-1)} &= a_n^{(n)} [2a_{n-1}^{(n)} a_k^{(n)} - a_n^{(n)} a_{k-1}^{(n)} (1 + (-1)^{n-k})], \quad 1 \le k \le n-1 \end{aligned}$$

The code is a direct implementation of these relations, except that the coefficients of the polynomials are adjusted so that the leading coefficient is 1. Note that all roots of $P(z)$ have positive real parts if and only if all roots of $P(-z)$ have negative real parts.

For quadratic polynomials $P(z) = z^2 + az + b$, all roots have negative real part if and only if $a > 0$ and $b > 0$. For cubic polynomials $P(z) = z^3 + az^2 + bz + c$, all roots have negative real part if and only if $a > 0$, $ab - c > 0$, and $c > 0$. For quartic polynomials $P(z) = z^4 + az^3 + bz^2 + cz + d$, all roots have negative real part if and only if $a > 0$, $ab - c > 0$, $d > 0$, and $c(ab - c) > a^2 d$.