Homework 6

Exercise 1: .

Prove or disprove each of these statements about the floor and ceiling functions.

a) $\lceil\lfloor x\rfloor\rceil = \lfloor x\rfloor$, for all real number x.

$$\begin{aligned} &\text{let } x = n+\varepsilon, n\in\mathbb{Z}, \varepsilon\in\mathbb{R}, 0\leqslant\varepsilon<1 \\ &LHS = \lceil\lfloor x\rfloor\rceil = \lceil n\rceil = n \\ &RHS = \lfloor x\rfloor = n \\ \therefore &LHS = RHS \end{aligned}$$

So this statement is true

b) $\lfloor xy\rfloor = \lfloor x\rfloor\lfloor y\rfloor$, for all real number x and y.

let $x = n_x+\varepsilon_x, n_x\in\mathbb{Z}, \varepsilon_x\in\mathbb{R}, 0\leqslant\varepsilon<1$, let $y = n_y+\varepsilon_y, n_y\in\mathbb{Z}, \varepsilon_y\in\mathbb{R}, 0\leqslant\varepsilon<1$

$$\begin{aligned} LHS &= \lfloor n_x n_y + n_x\varepsilon_y + n_y\varepsilon_x + \varepsilon_x\varepsilon_y\rfloor \\ &= \lfloor n_x n_y + n_x\varepsilon_y + n_y\varepsilon_x\rfloor \\ RHS &= n_x n_y \\ \therefore LHS &\neq RHS \end{aligned}$$

So this statement is false

c) $\left\lfloor\sqrt{\lceil x\rceil}\right\rfloor = \lfloor\sqrt{x}\rfloor$, for all real number x.

let $x = 0.5$, then:

$$\begin{aligned} LHS &= \left\lfloor\sqrt{\lceil x\rceil}\right\rfloor = 1 \\ RHS &= \lfloor\sqrt{x}\rfloor = 0 \\ LHS &\neq RHS \end{aligned}$$

So this statement is false

Exercise 2: .

Show that x^3 is $O(x^4)$, but that x^4 is not $O(x^3)$.

$$\begin{aligned} \text{if } x^3 &\leqslant cx^4 \\ 1 &\leqslant cx \end{aligned}$$

This statement is true for fall $x>0$ and $c>0$, so $x^3 = O(x^4)$

$$\begin{aligned} \text{if } x^4 &\leqslant cx^3 \\ x &\leqslant c \end{aligned}$$

we cannot find an x_0 which make $\forall x > x_0$, such that $0 < x < c$, So $x^4 \neq O(x^3)$

Exercise 3: .

a) Show that $2x - 9$ is $\Theta(x)$

From the definition of big Theta:

$$c_1 x \leqslant 2x - 9 < c_2 x$$

For all $x \geqslant x_0$:

$$c_1 \leqslant 2 - \frac{9}{x} \leqslant c_2$$

The right-hand inequality can be made to hold for any value if $x \geqslant 3$ by choosing $c_2 \geqslant 1$

The left-hand inequality can be made to hold for any value if $x \geqslant 1$ by choosing $c_1 \leqslant -7$

Thus, by choosing $c_1 = -7$, $c_2 = 1$, and $x_0 = 3$, we can verify that $2x - 9 = \Theta(x)$

b) Show that $3x^2 + x - 5$ is $\Theta(x^2)$

From the definition of big Theta:

$$c_1 x^2 \leqslant 3x^2 + x - 5 < c_2 x^2$$

For all $x \geqslant x_0$:

$$c_1 \leqslant 3 + \frac{1}{x} - \frac{5}{x^2} \leqslant c_2$$

The right-hand inequality can be made to hold for any value if $x \geqslant 5$ by choosing $c_2 \geqslant 4$

The left-hand inequality can be made to hold for any value if $x \geqslant 5$ by choosing $c_1 \leqslant 3$

Thus, by choosing $c_1 = 3$, $c_2 = 4$, and $x_0 = 5$, we can verify that $3x^2 + x - 5 = \Theta(x^2)$

c) Show that $\left\lfloor x + \frac{2}{3} \right\rfloor$ is $\Theta(x)$

From the definition of big Theta:

$$c_1 x \leqslant \left\lfloor x + \frac{2}{3} \right\rfloor < c_2 x$$

The right-hand inequality can be made to hold for any value if $x \geqslant 2$ by choosing $c_2 \geqslant 3$

The left-hand inequality can be made to hold for any value if $x \geqslant 0$ by choosing $c_1 \leqslant 1$

Thus, by choosing $c_1 = 1$, $c_2 = 3$, and $x_0 = 2$, we can verify that $\left\lfloor x + \frac{2}{3} \right\rfloor = \Theta(x)$

d) Show that $log_{10}(x)$ is $\Theta(log_2(x))$

From the definition of big Theta:

$$c_1 log_2(x) \leqslant log_{10}(x) < c_2 log_2(x)$$

For all $x \geqslant x_0$:

$$c_1 \leqslant \frac{1}{log_2 10} \leqslant c_2$$

The right-hand inequality can be made to hold for any value if $x \geqslant 0$ by choosing $c_2 \geqslant \frac{1}{log_2 10}$

The left-hand inequality can be made to hold for any value if $x \geqslant 5$ by choosing $c_1 \leqslant \frac{1}{log_2 10}$

Thus, by choosing $c_1 = \frac{1}{log_2 10}$, $c_2 = \frac{1}{log_2 10}$, and $x_0 = 0$, we can verify that $\Theta(log_2(x))$

Exercise 4: .

Describe an algorithm that uses only assignment statements that replaces the quadruplet(w,x,y,z) with (x,y,z,w). What is the minimum number of assignment statements needed?

the minimum number of assignments is 5.

Algorithm 1 replaces the quadruplet

procedure REPLACE(w, x, y, z) ▷ A is a sequence with n elements
    $temp \leftarrow w$
    $w \leftarrow x$
    $x \leftarrow y$
    $y \leftarrow z$
    $z \leftarrow temp$
end procedure

Exercise 5: .

Devise an algorithm for finding both the largest and the smallest integers in a finite sequence of integers.What is the complexity of your algorithm?

The complexity of the algorithm is $O(n)$.

Algorithm 2 Find Max and Min

procedure GETMAXANDMIN(A, n) ▷ A is a sequence with n elements
    $max \leftarrow -\infty$ ▷ initialize max as negative infinity
    $min \leftarrow \infty$ ▷ initialize min as positive infinity
    for $i \leftarrow 0, n-1$ **do** ▷ pick the *ith* element in A
        if $min > A(i)$ **then**
            $min \leftarrow A(i)$ ▷ if $A(i) < min$, let A(i) become the new minimum
        end if
        if $max < A(i)$ **then**
            $max \leftarrow A(i)$ ▷ if $A(i) > max$, let A(i) become the new maximum
        end if
    end for
    return (min, max)
end procedure

Exercise 6: Extra Credit

We call a positive integer perfect if it equals the sum of its positive divisors other than itself.

a) Show that 6 and 28 are perfect

$$\begin{aligned}
&6 = 1 * 2 * 3 \\
\therefore\ &\text{sum} = 1 + 2 + 3 = 6 \\
\therefore\ &6 \text{ is a perfect number} \\
&28 = 1 * 2 * 2 * 7 \\
\therefore\ &\text{sum} = 1 + 2 + 7 + 4 + 14 = 28 \\
\therefore\ &28 \text{ is a perfect number}
\end{aligned}$$

b) Show that $2^{p-1}(2^p - 1)$ is a perfect number when $2p - 1$ is prime.

$$\begin{aligned}
\text{sum} &= 2^{p-1} + (1 + 2^p - 1)\sum_{n=0}^{p-2} 2^n \\
&= 2^{p-1} + 2^p\ (2^{p-2} - 1) \\
&= 2^{p-1} + 2^{p-1}\ (2^{p-1} - 2) \\
&= 2^{p-1}(2^p - 1)
\end{aligned}$$

so $2^{p-1}(2^p - 1)$ is a perfect number