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ECS 020 A03

Homework 8

Exercise 1: .

show that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$ for all $n \geqslant 1$

For $n = 1$:

$$\sum_{i=1}^{1} i^3 = 1 = \left(\frac{1(1+1)}{2}\right)^2$$

So the equation is hold when $n = 1$

If the equation is held for $n = k,\ k \geqslant 1$, then

$$\begin{aligned}
\sum_{i=1}^{k} i^3 &= \left(\frac{k(k+1)}{2}\right)^2 \\
\sum_{i=1}^{k+1} i^3 &= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\
&= \frac{(k+1)^2}{4}(k^2+4k+4) \\
&= \frac{(k+1)^2}{4}(k+2)^2 \\
&= \left(\frac{(k+2)(k+1)}{2}\right)^2
\end{aligned}$$

Therefore the statement holds when $n = (k+1)$.

Since both the basis and the inductive step have been performed, by mathematical induction, the statement holds for all $n \geqslant 1$, $n \in \mathbb{R}$

Exercise 2: .

Use mathematical induction to prove that:

$$1 * 2 * 3 + 2 * 3 * 4 + ... + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}, \text{for all } n \geqslant 1$$

let $f(n) = 1 * 2 * 3 + 2 * 3 * 4 + ... + n(n+1)(n+2)$

for $n = 1$

$$\begin{aligned}
f(1) &= 1 * 2 * 3 \\
&= 6 \\
&= \frac{1(1+1)(1+2)(1+3)}{4}
\end{aligned}$$

So the $f(n)$ holds when $n = 1$

If the $f(n)$ held when $n = k, k \geqslant 1$, then

$$\begin{aligned}
f(k+1) &= f(k) + (k+1)(k+2)(k+3) \\
&= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3) \\
&= (k+1)(k+2)(k+3)(\frac{k}{4} + 1) \\
&= \frac{(k+1)(k+2)(k+3)(k+4)}{4}
\end{aligned}$$

Therefore $f(n)$ holds when $n = (k+1)$.

Since both the basis and the inductive step have been performed, by mathematical induction, the $f(x)$ holds for all $n \geqslant 1$, $n \in \mathbb{N}$

Exercise 3: .

Prove that

$$1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} < 2 - \frac{1}{n}$$

Whenever n is a positive integer greater than 1

for $n = 2$

$$1 + \frac{1}{4} < 2 - \frac{1}{2}$$

so the inequality holds when $n = 2$

if the inequality holds when $n = k, k \geqslant 2$, then

$$1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{k^2} < 2 - \frac{1}{k}$$

so we can get that

$$1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

$$\begin{aligned}
&2 - \frac{1}{k+1} - \left(2 - \frac{1}{k} + \frac{1}{(k+1)^2}\right) \\
=&\frac{k(k+1)(2k+1) - ((2k-1)(k+1)^2 + k)}{k(k+1)^2} \\
=&\frac{2k^3 + 3k^2 + k - (2k^3 + 3k^2 - 2k + 1)}{k(k+1)^2} \\
=&\frac{2k-1}{k(k+1)^2}
\end{aligned}$$

Because $\frac{2k-1}{k(k+1)^2} > 0$ for all $k > 1$, therefore $2 - \frac{1}{k}$ is greater than $2 - \frac{1}{k} + \frac{1}{(k+1)^2}$ for all $k > 1$, thus $1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$

Therefore inequality holds when $n = (k+1)$.

Since both the basis and the inductive step have been performed, by mathematical induction, the $f(n)$ holds for all $n \geqslant 1$, $n \in \mathbb{N}$

Exercise 4: .

Use mathematical induction to show that $n^2 - 7n + 12$ is non negative if n is an integer greater than 3.

let $f(n) = n^2 - 7n + 12$

for $n = 4$ $f(4) = 4 * 4 - 7 * 4 + 12 = 0$

so $f(n)$ holds when $n = 4$

if $f(n)$ holds when $n = k, k > 4$,

$$\begin{aligned}
f(k+1) &= (k+1)^2 - 7(k+1) + 12 \\
&= k^2 + 2k + 1 - 7k - 7 + 12 \\
&= k^2 - 7k + 12 + 2k + 5 \\
&= f(k) + 2k + 5
\end{aligned}$$

because $f(k) > 0$ and $2k+5 > 0$, $f(k+1) > 0$

Since both the basis and the inductive step have been performed, by mathematical induction, the $f(n)$ holds for all $n > 3$, $n \in \mathbb{N}$

Exercise 5: .

Use mathematical induction to prove that a set with n elements has $n(n-1)/2$ sub sets containing exactly two elements whenever n is an integer greater than or equal to 2.

for $n = 2$, it only has one subset. And $2(2-1)/2 = 1$. So this statement holds when $n = 2$

for $n > 2$, if we add one more element, the new element can form a set with all of the element in the old set. So the total number of sub set will increase k.

$$\frac{k(k-1)}{2} + k = \frac{k(k+1)}{2}$$

Therefore inequality holds when $n = (k+1)$.

Since both the basis and the inductive step have been performed, by mathematical induction, the statement holds for all $n \geqslant 2$, $n \in \mathbb{N}$

Exercise 6: .

Find the flaw with the following proof that $a^n = 1$ for all non negative integer n, when ever a is a non zero real number: Basis step: $a^0 = 1$ is true, by definition of a^0 Inductive step: assume that $a^j = 1$ for all non negative integers j with $j \leqslant k$. Then note that:

$$a^{k+1} = \frac{a^k \cdot a^k}{a^{k-1}}$$

because he prove a^{k+1} by using a^k and a^{k-1}, he need two basic case to let this induction available. However, he only give $a^0 = 1$. That is why this proof is wrong.

Exercise 7: .

Use mathematical induction to show that 21 divides $4^{n+1} + 5^{2n-1}$ whenever n is a positive integer.

for $n = 1$, we can get that $4^{n+1} + 5^{2n-1} = 4^2 + 5^1 = 21$

Therefore this statement holds for base statement.

if this statement holds for $n = k$, then we assume that $4^{n+1}\ mod\ 21 = i$, and $5^{2n-1}\ mod\ 21 = j$.

$$\begin{aligned}
&(4^{(n+1)+1} + 5^{2(n+1)-1})\ mod\ 21 \\
=&(4 * 4^{n+1} + 25 * 5^{2n-1})\ mod\ 21 \\
=&4\ mod\ 21 * i + 25\ mod\ 21 * j \\
=&4(i+j)\ mod\ 21 \\
=&0
\end{aligned}$$

thus this statement holds when $n = k+1$

Since both the basis and the inductive step have been performed, by mathematical induction, the statement holds for all positive integer.

Exercise 8: .

prove that $f_1^2 + f_2^2 + \ldots + f_n^2 = f_n f_{n+1}$, whenever n is a positive integer.

for $n = 1$, $f_1^2 = f_1 f_2 = 1 * 1 = 1$, so this statement holds when n = 1 if this statement holds when $n = k$, $k > 1$

$$\begin{aligned}
&f_1^2 + f_2^2 + \ldots + f_n^2 + f_{n+1}^2 \\
=&f_n f_{n+1} + f_{n+1}^2 \\
=&f_{n+1}(f_n + f_{n+1}) \\
=&f_{n+1} f_{n+2}
\end{aligned}$$

thus this statement holds when $n = k + 1$

Since both the basis and the inductive step have been performed, by mathematical induction, the statement holds for all positive integer.

Exercise 9: .

show that $f_0 - f_1 + f_2 - \ldots - f_{2n-1} + f_{2n} = f_{2n-1} - 1$

for $n = 1$

$$\begin{aligned}
LHS &= f_0 - f_1 + f_2 \\
&= 0 - 1 + 1 \\
&= 0 \\
RHS &= f_1 - 1 \\
&= 0
\end{aligned}$$

So this statement holds for $n = 1$

if this statement holds when $n = k$, $k > 1$

$$\begin{aligned}
&f_0 - f_1 + f_2 - \ldots - f_{2n-1} + f_{2n} - f_{2n+1} + f_{2n+2} \\
=&f_{2n-1} - 1 - f_{2n+1} + f_{2n+2} \\
=&f_{2n-1} + f_{2n}) - 1 \\
=&f_{2n+1} - 1
\end{aligned}$$

thus this statement holds when $n = k + 1$

Since both the basis and the inductive step have been performed, by mathematical induction, the statement holds for all positive integer.

Exercise 10: .

Use mathematical induction to prove that a set with n elements has $n(n-1)(n-2)/6$ subsets containing exactly three elements whenever n is an integer greater than or equal to 3.

for $x = 3$, the number of subset equals to $n(n-1)(n-2)/6 = 3 * 2 * 1/6 = 1$

if this statement holds when $n = k$, $k > 3$

if we add one more element, the new element can form a set with all of the subset with 2 elements in the old set.As I proved in exercise 5, we can know that a set with n elements has $n(n-1)/2$ sub sets containing exactly two elements whenever n is an integer greater than or equal to 2. So the total number of sub set will increase $k(k-1)/2$.

$$\begin{aligned}
&k(k-1)(k-2)/6 + k(k-1)/2 \\
=&k(k-1)((k-2)/6 + 1/2) \\
=&k(k-1)(k+1)/6
\end{aligned}$$

thus this statement holds when $n = k + 1$

Since both the basis and the inductive step have been performed, by mathematical induction, the statement holds for all positive integer,which is greater than or equal to 3.