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ECS 020 A04

Homework 5

Exercise 1: .

Find this values:

$$\lfloor 1.4 \rfloor = 1 \qquad \lceil 1.4 \rceil = 2$$
$$\lfloor -0.4 \rfloor = -1 \qquad \lceil -0.4 \rceil = 0$$
$$\lfloor 4.99 \rfloor = 5 \qquad \lceil -4.99 \rceil = -4$$
$$\left\lfloor \frac{1}{3} + \left\lceil \frac{1}{3} \right\rceil \right\rfloor = 1 \qquad \left\lceil \left\lfloor \frac{1}{3} \right\rfloor + \left\lceil \frac{1}{3} \right\rceil + \frac{1}{3} \right\rceil = 2$$

Exercise 2: (proof)

a) Show that the following statement is true:

"If x is a real number such that $x^2 + 1 = 0$, then $x^4 = -1$".

p: x is a real number such that $x^2 + 1 = 0$

q: $x^4 = -1$

for $x^2 + 1 = 0$, $\triangle = b^2 - 2ac = -2$

$\because \triangle < 0$, The solution of the equation cannot be real.

$\therefore p$ is always false.

$\therefore p \to q$ is always true.

b) Constructive proof:

" If x and y are real numbers such that $x < y$, show that there exists a real number z. with $x < z < y$"

$$let \;\; z = x + \frac{x+y}{2}$$
$$\therefore x < z, \;\; y > z$$
$$\forall x, y \in \mathbb{R}, \exists z \; (x < z < y)$$

Exercise 3: .

Let x be a real number. Show that

$$\lfloor 4x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{4} \right\rfloor + \left\lfloor x + \frac{2}{4} \right\rfloor + \left\lfloor x + \frac{3}{4} \right\rfloor$$

assume that $x = a + \varepsilon,\ a \in \mathbb{Z}, 0 \leqslant \varepsilon < 1$

if $0 \leqslant \varepsilon < \frac{1}{4}$:

$$\begin{aligned} LHS &= \lfloor 4 \cdot (a + \varepsilon) \rfloor = 4a \\ RHS &= \lfloor x + \varepsilon \rfloor + \left\lfloor x + \varepsilon + \frac{1}{4} \right\rfloor + \left\lfloor x + \varepsilon + \frac{2}{4} \right\rfloor + \left\lfloor x + \varepsilon + \frac{3}{4} \right\rfloor \\ &= 4a \\ LHS &= RHS \end{aligned}$$

if $\frac{1}{4} \leqslant \varepsilon < \frac{2}{4}$:

$$\begin{aligned}
LHS &= \lfloor 4 \cdot (a+\varepsilon) \rfloor = 4a+1 \\
RHS &= \lfloor x+\varepsilon \rfloor + \left\lfloor x+\varepsilon+\frac{1}{4} \right\rfloor + \left\lfloor x+\varepsilon+\frac{2}{4} \right\rfloor + \left\lfloor x+\varepsilon+\frac{3}{4} \right\rfloor \\
&= a+a+a+(a+1) \\
&= 4a+1 \\
LHS &= RHS
\end{aligned}$$

if $\frac{2}{4} \leqslant \varepsilon < \frac{3}{4}$:

$$\begin{aligned}
LHS &= \lfloor 4 \cdot (a+\varepsilon) \rfloor = 4a+2 \\
RHS &= \lfloor x+\varepsilon \rfloor + \left\lfloor x+\varepsilon+\frac{1}{4} \right\rfloor + \left\lfloor x+\varepsilon+\frac{2}{4} \right\rfloor + \left\lfloor x+\varepsilon+\frac{3}{4} \right\rfloor \\
&= a+a+(a+1)+(a+1) \\
&= 4a+2 \\
LHS &= RHS
\end{aligned}$$

if $\frac{3}{4} \leqslant \varepsilon < 1$:

$$\begin{aligned}
LHS &= \lfloor 4 \cdot (a+\varepsilon) \rfloor = 4a+3 \\
RHS &= \lfloor x+\varepsilon \rfloor + \left\lfloor x+\varepsilon+\frac{1}{4} \right\rfloor + \left\lfloor x+\varepsilon+\frac{2}{4} \right\rfloor + \left\lfloor x+\varepsilon+\frac{3}{4} \right\rfloor \\
&= a+(a+1)+(a+1)+(a+1) \\
&= 4a+3 \\
LHS &= RHS
\end{aligned}$$

Therefore, for any $x \in \mathbb{R}$, $\lfloor 4x \rfloor = \lfloor x \rfloor + \left\lfloor x+\frac{1}{4} \right\rfloor + \left\lfloor x+\frac{2}{4} \right\rfloor + \left\lfloor x+\frac{3}{4} \right\rfloor$

Exercise 4: .

Let f be a bijection from a set A to a set B. Let S and T be two subsets of A.

a) Show that $f(S \cup T) = f(S) \cup f(T)$

let $x \in S \cup T$:

$$\begin{aligned}
&\because x \in S \text{ or } x \in T \\
&\therefore f(x) \in f(S) \text{ or } f(x) \in f(T) \\
&\therefore f(x) \in f(S) \cup f(T) \\
&\therefore f(S \cup T) \subseteq f(S) \cup f(T)
\end{aligned}$$

let $f(x) \in f(S) \cup f(T)$

$$\begin{aligned}
&\because \text{f is bijection} \\
&\therefore x \in S \text{ or } x \in T \\
&\therefore x \in S \cup T \\
&\therefore f(x) \in f(S \cup T) \\
&\therefore f(S) \cup f(T) \subseteq f(S \cup T)
\end{aligned}$$

Because $f(S) \cup f(T) \subseteq f(S \cup T)$ and $f(S \cup T) \subseteq f(S) \cup f(T)$, We can conclude that $f(S) \cup f(T) = f(S \cup T)$

b)Show that $f(S \cap T) \subseteq f(S) \cap f(T)$

let $x \in S \cap T$:

$$\begin{aligned}
&\because x \in S \cap T \\
&\therefore (x \in S \wedge x \notin T) \vee (x \in T \wedge x \notin S) \\
&\therefore (f(x) \in f(S) \wedge f(x) \notin f(T)) \vee (f(x) \in f(T) \wedge f(x) \notin f(S)) \\
&\therefore f(x) \in f(S) \cap f(T) \\
&\therefore f(S \cap T) \subseteq f(x) \in f(S) \cap f(T)
\end{aligned}$$

c) Show that $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$

let $x \in S \cup T$:

$$\begin{aligned}
&\because x \in S \text{ or } x \in T \\
&\therefore f^{-1}(x) \in f^{-1}(S) \text{ or } f^{-1}(x) \in f^{-1}(T) \\
&\therefore f^{-1}(x) \in f^{-1}(S) \cup f^{-1}(T) \\
&\therefore f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T)
\end{aligned}$$

let $f^{-1}(x) \in f^{-1}(S) \cup f^{-1}(T)$

$$\begin{aligned}
&\because \text{f is bijection} \\
&\therefore x \in S \text{ or } x \in T \\
&\therefore x \in S \cup T \\
&\therefore f^{-1}(x) \in f^{-1}(S \cup T) \\
&\therefore f^{-1}(S) \cup f^{-1}(T) \subseteq f^{-1}(S \cup T)
\end{aligned}$$

Because $f^{-1}(S) \cup f^{-1}(T) \subseteq f^{-1}(S \cup T)$ and $f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T)$, We can conclude that $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$

Exercise 5: Extra credit

Let us consider a generalization of exercise 1. Let x be a real number, and N an integergreater or equal to 3. Show that:

$$\lfloor Nx \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{N} \right\rfloor + \left\lfloor x + \frac{2}{N} \right\rfloor + ... + \left\lfloor x + \frac{N-1}{N} \right\rfloor$$

My own method:

The formula can be rewritten as $\lfloor Nx \rfloor = \sum\limits_{i=0}^{N-1} \left\lfloor x + \frac{i}{N} \right\rfloor$

Assume that $x = a + \varepsilon,\ a \in \mathbb{Z}, 0 \leqslant \varepsilon < 1$

Let's choose $n \in \mathbb{R}$, $0 \leqslant n < N$ such that $\frac{n}{N} \leqslant \varepsilon < \frac{n+1}{N}$

if $i \geqslant N - n$, $\left\lfloor x + \frac{i}{N} \right\rfloor = a + 1$, then if $i < N - n$, $\left\lfloor x + \frac{i}{N} \right\rfloor = a$

$\therefore RHS = n \cdot (a + 1) + (N - n) \cdot (a) = Na + n$

$LHS = \lfloor N \cdot (a + \varepsilon) \rfloor = Na + n$

$\therefore LHS = RHS$

Follow the hints:

$$\text{let } f(x) = \lfloor Nx \rfloor - \sum_{i=0}^{N-1} \left\lfloor x + \frac{i}{N} \right\rfloor$$

$$\begin{aligned} \text{f}(x + \frac{1}{N}) &= \lfloor Nx + 1 \rfloor - \sum_{i=0}^{N-1} \left\lfloor x + \frac{i+1}{N} \right\rfloor \\ &= \lfloor Nx + 1 \rfloor + (f(x) - \lfloor Nx \rfloor + \lfloor x \rfloor - \lfloor x + 1 \rfloor) \\ &= f(x) \end{aligned}$$

Therefore, the f(x) is a periodic function, whose period is $1/N$

$\forall x (0 \leqslant < 1/N), f(x) = 0 - 0 = 0$

$\because$ f(x) is a periodic function.

$\therefore f(x) = 0$ for all real number x.

$\therefore \forall x \in \mathbb{R}$, $\lfloor Nx \rfloor = \sum_{i=0}^{N-1} \left\lfloor x + \frac{i}{N} \right\rfloor$