4. (10 marks) **Telephone numbers**
Let C be an $[n, M]$-code with distance d over an alphabet A of size q. Let $e = \lfloor \frac{d-1}{2} \rfloor$.

(a) Prove that $M \sum_{i=0}^{e} \binom{n}{i}(q-1)^i \leq q^n$.
(Hint: Consider the number of words in a sphere of radius e centred at a codeword)

Proof. Consider the number of words in a single sphere of radius e centred at a codeword $c \in C$, call it x. Every pair of spheres do not intersect since $d(C) = d \implies C$ is a e-error correcting code. There are up to $\binom{n}{e}$ (from $\binom{n}{0}$) ways to choose the components of c to differ by $(q-1)$ other values. Hence,

$$x = \sum_{i=0}^{e} \binom{n}{i}(q-1)^i$$

For the total number of spheres, M, we get

$$Mx = M\sum_{i=0}^{e} \binom{n}{i}(q-1)^i$$

which cannot possibly be more than the total number of words in the alphabet A of size q, q^n. Thus,

$$M\sum_{i=0}^{e} \binom{n}{i}(q-1)^i \leq q^n$$

□

(b) Suppose that there are 110 million telephones in a country. Is it possible to assign 10-digit decimal numbers to these telephones so that a single error in dialing can always be (automatically) corrected? (Explain)

Try the following parameters with the sphere packing bound:

- $M = 110000000$ (number of telephones/codewords)
- $q = 10$ (all decimal digits)
- $n = 10$ (length of each telephone number/length of words)
- $e = 1$ (errors to be corrected)

We get,

$$(110000000)\sum_{i=0}^{1} \binom{10}{i}(10-1)^i = 10010000000 \not\leq 10^{10}$$

The block code **does not** satisfy the *necessary* sphere packing bound condition, thus it is **not possible** to assign a 10-digit decimal numbers to these telephones so that a single error in dialing can always be (automatically) corrected.

(c) Suppose that there are 80 million telephones in a country. Is it possible to assign 10-digit decimal numbers to these telephones so that a single error in dialing can always be (automatically) corrected? (Explain)

Try the following parameters with the sphere packing bound:

- $M = 80000000$ (number of telephones/codewords)
- $q = 10$ (all decimal digits)
- $n = 10$ (length of each telephone number/length of words)
- $e = 1$ (errors to be corrected)

We get,

$$(80000000)\sum_{i=0}^{1}\binom{10}{i}(10-1)^i = 7280000000 \leq 10^{10}$$

The block code **does** satisfy the *necessary* sphere packing bound condition, but the bound is not *sufficient* to say whether there exists a code for this problem. Thus, the solution to this problem is **inconclusive**.