STAT 230 - Probability

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Contents

1 Lecture 1

Chapter 1: Introduction to Probability

1.1 Definitions of Probability

We can define probability in three different ways:

1.1 Definition (Probability)

1. *Classical* definition:

$$P(\text{event}) = \frac{\#\text{ ways the event can occur}}{\#\text{ of all possible outcomes}}$$

this requires the outcomes to all be equally likely.
2. *Relative frequency* definition:

$$P(\text{event}) = \text{proportion of the time the event occurs in repeated experiments}$$

this requires the same conditions for each observation.
3. *Subjective probability* definition:

$$P(\text{event}) = \text{how certain we are that the event will occur}$$

However, all three of these definitions have serious limitations.

2 Lecture 2

Chapter 2: Mathematical Probability Models

2.1 Sample Spaces and Probability

We need a mathematical model to define probability.

2.1 Definition (Experiment, Trial, Outcome)

We define an *experiment* as a process that can be repeated with multiple possible results. We define a *trial* as a single repetition of an experiment. We define an *outcome* as the results on one trial of an experiment.

2.2 Definition (Set)

A set is a collection of well defined and distinct objects.

Remark 1. A set is an unordered list with no repetition.

2.3 Definition (Sample Space)

A *sample space* S is a set of distinct outcomes for an experiment or process, with the property that in a single trial, one and only one of these outcomes occur.

Sample spaces can be *discrete* or *non-discrete*.

2.4 Definition (Discrete)

Let S be a sample space. We say S is *discrete* if it consists of a finite or countably infinite set of simple events.

Example

Roll a fair 6-sided die repeatedly. Determine some possible sample spaces for this experiment.

Solution.

Sample space:

$$\begin{aligned} S_1 &= \{1,2,3,4,5,6\} \star \text{ easiest to work with} \\ S_2 &= \{\text{odd, even}\} \\ S_3 &= \{\text{prime, non-prime}\} \\ S_4 &= \{6, \text{not } 6\} \star \text{ outcomes don't have to be equally likely in a sample space} \\ S_5 &= \{\text{a number}\} \end{aligned}$$

$\star$ need not have equally likely outcomes

2.5 Definition (Event, Simple Event, Compound Event)

Let S be a discrete sample space. An *event* in a discrete sample space is a subset $A \subseteq S$. If the event is indivisible so it contains only one point, e.g. $A_1 = \{a_1\}$ we call it a *simple event*. An event A made up of two or more simple events such as $A = \{a_1, a_2\}$ is called a *compound event*.

Remark 2. The notation $A \subseteq S$ means $a \in A \implies a \in B$.

Example

Let A = a 5 is rolled. Let B = an odd # is rolled. Determine which of the events are simple events and compound events.

Solution.

$A = \{5\} \subseteq S$, $B = \{1,3,5\} \subseteq S$. Thus, A is a simple event and B is a compound event.

When the trial is conducted, the outcome determines which events occur.

If outcome is in the set, it occurs

5 rolled $\rightarrow$ A and B both occur

3 rolled $\rightarrow$ A does not occur, B occurs

2 rolled $\rightarrow$ neither events occur

2.6 Definition (Probability, Probability Distribution)

Let $S = \{a_1, \ldots\}$ be a discrete sample space S. Assign numbers (*probabilities*) $P(a_i)$ for $i = 1, \ldots$ to the a_i's such that the following two conditions hold:
(1) $0 \le P(a_i) \le 1$
(2) $\sum\limits_{\text{all } i} P(a_i) = 1$
The set of probabilities $\{P(a_i),\ i = 1, \ldots\}$ is called a *probability distribution* on S.

2.7 Definition (Probability of an Event)

The *probability* $P(A)$ *of an event* A is the sum of the probabilities for all the simple events that make up A or

$$P(A) = \sum_{a \in A} P(a)$$

If a sample space S has equally likely outcomes then,

$$P(\text{simple event}) = \frac{1}{|S|}$$

$$P(A) = \sum_{a \in A} P(a_i) = \frac{|A|}{|S|}$$

3 Lecture 3

Chapter 3: Probability and Counting Techniques

3.1 Addition and Multiplication Rules

We need a systematic way to count outcomes without listing them.

3.1 Counting Rules

There are two basic counting rules:
1. The *Addition* Rule: Suppose we can do job 1 in p ways and job 2 in q ways. Then we can do either job 1 OR job 2 (but not both), in $p+q$ ways.
2. The *Multiplication* Rule: Suppose we can do job 1 in p ways and job 2 in q ways. Then we can do both job 1 AND job 2 in $p \times q$ ways.

3.2 Counting Arrangements or Permutations

Sampling with replacement: it is possible to obtain the same result more than once. e.g. die rolls, coin flip, slot machine, password.

Sampling without replacement: once a result occurs, it cannot happen again. e.g. drawing cards, balls from an urn, eating candy of different colour.

3.2 Definition (Permutation)

A *permutation* is an ordered selection of k objects chosen from n objects.
If we select the objects above without replacement, we write

$$^{n}P_k = n(n-1)\cdots(n-k+1) = n^{(k)}$$

If we select the objects above with replacement, we write

$$n^k$$

$n^{(0)} = 1$

$n^{(n)} = n!$

$k > n \to 0$ not possible

Example

IP addresses: an ordered sequence of four numbers between 0 and 255. e.g. $192.168.1.1$, $129.97.95.107$, etc. Determine the total possible outcomes with and without replacement.

Solution.

Since order matters, we are immediately looking at a permutation.

With replacement: 256^4

Without replacement: $256^{(4)}$

4 Lecture 4

Always ask:

1. Can you get the same object twice?

- Yes $\rightarrow n^k$

- No $\rightarrow$ Step 2.

2. Does the order matter?

- Yes $\rightarrow n^{(k)}$

- No $\rightarrow$ today's lesson $\binom{n}{k}$

Examples

4 numbers between 0 and 255

- total possible: 256^4
- all odd numbers 128^4 so $P(\text{all odd}) = {}^1/_{16}$
- at least one odd number; work with the opposite: all even: 128^4, so $P(\text{at least one odd}) = 1 - {}^{128^4}/_{256^4}$

Example

5 people A, B, C, D 4 co-op jobs 1, 2, 3, 4

Find the probability that A gets a job.

Solution.

1. order matters $\rightarrow$ (permutation of some sort)

2. 1 + without replacement $\rightarrow n^{(k)}$

so total ways is $5^{(4)} = 120$

$A, \_, \_, \_$ or $\_, A, \_, \_$ or $\_, \_, A, \_$ or $\_, \_, \_, A$

$4^{(3)} = 96$

So probability they do is $\frac{96}{120} = 0.8$

Alternatively, # of ways for A to not get a job is $4^{(4)}$ or $4!$ (they are the same quantity). So probability they do is $1 - {}^{4!}/_{120} = 0.8$.

Intuitively this makes sense because each of the 5 is equally likely $({}^1/_5)$ to not get a job.

Find probability that B and C get adjacent jobs.

$BC, \_, \_$ or $\_, BC, \_$ or $\_, \_, BC$

$CB, \underbrace{\_, \_}_{3^{(2)}}$ or $\_, CB, \_$ or $\_, \_, CB$

So total ways is $6 \times 3^{(2)} = 36$, probability=$^{36}/_{120}$. Alternatively, treat BC as one unit with 2 ways it can look (BC or CB).

3.3 Counting Subsets or Combinations

4.1 Definition (Combination)

A *combination* is an unordered selection of k objects chosen from n objects.
If we select the objects above without replacement, we write

$$^nC_k = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

How many ways? If we did care, $n^{(k)}$. Then deliberately forgot the order.

e.g. select 3 digits $0 - 9$

$$\{8, 2, 1\}$$

if we care about the order, each set is counted $3! = 6$ times.

So, there are $^{10^{(3)}}/_{3!} = 120$ possible sets of 3 digits.

This quantity is called

$$\binom{n}{k} = \frac{n^{(k)}}{k!} = \frac{n!}{k!(n-k)!} = {}^nC_k$$

"n choose k", "binomial coefficient", "$\binom{n}{k}$ is the k^{th} element of the n^{th} row of Pascal's Δ"

Example

Lotto $6/49$, choose 6 winning # from 49. The order of the numbers does not matter.

$$\binom{49}{6} \text{ ways } \approx 13.9 \text{ million}$$

5 Lecture 5

5.1 Theorem (Properties of n choose k)

Let $n,\ k \in \mathbb{Z}$ be non-negative.
1. $n^{(k)} = \frac{n!}{(n-k)!} = n(n-1)^{(k-1)}$ for $k \geq 1$
2. $\binom{n}{k} = \frac{n^{(k)}}{k!} = \frac{n!}{k!(n-k)!}$
3. $\binom{n}{k} = \binom{n}{n-k}$ for all $k = 1, \ldots, n$
4. If we define $0! = 1$, then $\binom{n}{0} = \binom{n}{n} = 1$
5. Pascal's Identity: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$
6. Binomial Theorem: $(1+x)^n = \binom{n}{0} + \cdots + \binom{n}{n}x^n$

Example

Lotto $6/49$ $\binom{49}{6}$ possible sets of winning #'s. If your ticket contains all 6 winning #'s you win.

$$P(\text{win}) = \frac{\binom{6}{6}\binom{43}{0}}{\binom{49}{6}}$$

$$P(\text{match } 5) = \frac{\binom{6}{5}\binom{43}{1}}{\binom{49}{6}}$$

Example

Suppose you select 5 cards from 52 (13 cards of each 4 suits). Find the probability of 3 of one rank, 2 of another rank.

Total # of hands: $\binom{52}{5}$. # with 3 of one, 2 of another:

$$\underbrace{\text{rank of triple}}_{13} \times \binom{4}{3} \times \underbrace{\text{rank of pair}}_{12} \times \binom{4}{2}$$

3.4 Number of Arrangements When Symbols Are Repeated

Suppose we have 5 objects, 2 of which are alike:

D I A N A

If we arrange the objects in order, how many results can we get?

If we could tell them apart: $5!$ ways. But every possible arrangement has a matching one with A's flipped.

So, $5!/2! = 60$ ways (removing double counting)

In general, if we have n objects:

$$\left.\begin{array}{ccc} n_1 & \text{of} & \text{type } 1 \\ & \vdots & \\ n_k & \text{of} & \text{type } k \end{array}\right\} n_1 + \cdots + n_k = n$$

How many ways can the objects be arranged, where objects of the same type are identical?

$STATISTICS$: $n = 10$

$$\begin{array}{ll} S:3 & n_1 \\ T:3 & n_2 \\ A:1 & n_3 \\ I:2 & n_4 \\ C:1 & n_5 \end{array}$$

ways to place:

$$\begin{array}{cc} S & \binom{10}{3} \\ T & \binom{7}{3} \\ A & \binom{4}{1} \\ I & \binom{3}{2} \\ C & \binom{1}{1} \end{array}$$

So in total there are

$$\begin{aligned} \binom{10}{3}\binom{7}{3}\binom{4}{1}\binom{3}{2}\binom{1}{1} &= \frac{10!}{3!7!}\frac{7!}{3!4!}\frac{4!}{1!3!}\frac{3!}{2!1!}\frac{1!}{1!0!} \\ &= \frac{10!}{3!3!2!} \end{aligned}$$

In general,

$$\frac{n!}{n_1! \cdots n_k!}$$

6 Lecture 6

Example

7 Pokémon Go players, (2 M, 2 I, 3 V) are ranked $1-7$. Find the probability that 1 and 7 are on different teams.

Total # rankings: $\frac{7!}{2!2!3!} = 210$.

$M, \underbrace{\_,\_,\_,\_,\_}_{2I,3V}, M$: $\frac{3!}{2!3!} = 10$

$I, \underbrace{\_,\_,\_,\_,\_}_{2M,3V}, I$: $\frac{3!}{2!3!} = 10$

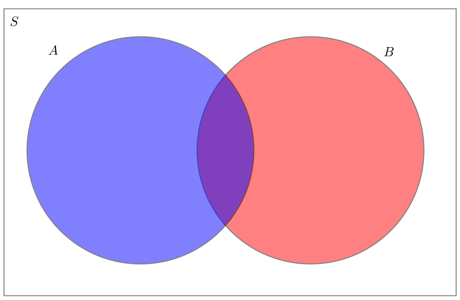
$V, \underbrace{\_,\_,\_,\_,\_}_{2M,2I,1V}, V$: $\frac{3!}{2!2!1!} = 50$

$210 - 50 = 160/210 < -$total

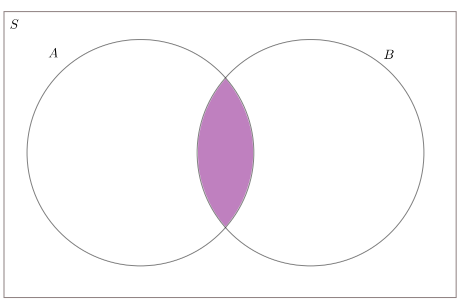
Chapter 4: Probability Rules and Conditional Probability

4.1 General Methods

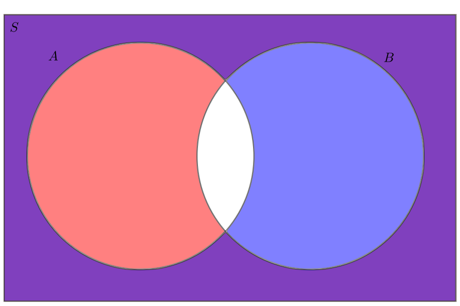
$A \cup B$



$A \cap B$



$\overline{A \cup B} = \overline{A} \cap \overline{B}$



6.1 Theorem (De Morgan's Laws)

(1) $\overline{A \cup B} = \bar{A} \cap \bar{B}$
(2) $\overline{A \cap B} = \bar{A} \cup \bar{B}$

4.2 Rules for Unions of Events

6.2 Rule 4a (Addition Law of Probability or the Sum Rule)

Let A and B be any events. Then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

6.3 Rule 4b (Probability of the Union of Three Events)

Let A, B and C be any events. Then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$$

Example

$P(J) = {}^{19}/_{22}$

$P(C) = {}^{7}/_{22}$

$P(\text{neither}) = {}^{2}/_{22}$

Find $P(JC)$.

Solution.

$$\frac{(19 - x) + x + (7 - x) + 2}{22} = 1 \implies x = 6$$

This relied on the fact that the regions were non-overlapping so we could add them up.

6.4 Definition (Mutually Exclusive)

Events A and B are *mutually exclusive* if

$$A \cap B = \emptyset \text{ (the empty set)}$$

6.5 Rule 5a (Probability of the Union of Two Mutually Exclusive Events)

Let A and B be mutually exclusive events. Then

$$P(A \cup B) = P(A) + P(B)$$

6.6 Rule 5c (Probability of the Union of n Mutually Exclusive Events)

Let $A_1, \ldots, A_n$ be mutually exclusive events. Then

$$P(A_1 \cup \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i)$$

6.7 Rule 6 (Probability of the Complement of an Event)

For any event A,

$$P(A) = 1 - P(\bar{A})$$

Proof. A and $\bar{A}$ are mutually exclusive. By rule 5a we have

$$A \cup \bar{A} = P(A) + P(\bar{A})$$

But since $P(A \cup \bar{A}) = P(S) = 1$,

$$1 = P(A) + P(\bar{A}) \implies P(A) = 1 - P(\bar{A})$$

□

7 Lecture 7

Roll two fair 12-sided die. What is the probability at least one of them is greater than 7.

$$\begin{aligned} 1 - P(\text{neither}) &= 1 - P(\overline{A \cap B}) \\ &= 1 - P(\bar{A} \cup \bar{B}) \\ &= 1 - \frac{7}{12}\frac{7}{12} \end{aligned}$$

In this example, we relied on the multiplication rule to find a probability on both events, but this requires the events to not influence each other.

<u>4.3 Intersections of Events and Independence</u>

7.1 Definition (Independent, Dependent)

A and B are *independent* if and only if

$$P(AB) = P(A)P(B)$$

If the events are not independent, we call the events *dependent*.

We can use this in two ways.

1. If we know both events are independent, we can calculate $P(AB)$.

2. Calculate/estimate all three probabilities and check whether independent. e.g. treatment vs recover, smoking vs cancer, income vs politics

Note for A, B, C to be independent, we need

$$\begin{aligned} &P(AB) = P(A)P(B) \\ &P(BC) = P(B)P(C) \\ &P(AC = P(A)P(C) \\ &P(ABC) = P(A)P(B)P(C) \end{aligned}$$

<u>Example</u>

Roll 2 fair 6-sided dice. Let $A =$ the first die is 3. Let $B =$ the total is 7. Are A and B independent?

$$\begin{aligned}
P(A) &= \frac{1}{6}\\
P(B) &= \frac{6}{36}\\
P(AB) &= \frac{1}{36}\\
P(AB) &= \frac{1}{36} = \frac{1}{6}\frac{1}{36} = P(A)P(B)
\end{aligned}$$

Now, let $C =$ the total is 8. Are A and C independent?

$$\begin{aligned}
P(C) &= \frac{5}{36}\\
P(AC) &= \frac{1}{36}\\
P(AC) &= \frac{1}{36} \neq \frac{1}{6}\frac{5}{36} = P(A)P(C)
\end{aligned}$$

Why? With 7 there is always a possible second roll, but with 8 it's not always possible (e.g. if the first die was a 1).

Independence vs. Mutual Exclusive

	ME	ID	Both
math	$AB = \emptyset$, $P(AB) = 0$	$P(AB) = P(A)P(B)$	$P(A)P(B) = 0$
logic	both can't happen	one doesn't affect the other	at least one is impossible

If events are dependent, we might want to quantify the effect of one on the other.

4.4 Conditional Probability

7.2 Definition (Conditional Probability)

The *conditional probability* of A, given B is

$$P(A|B) = \frac{P(AB)}{P(B)}$$

provided $P(B) > 0$.

Why? The classical definition of probability:

$$\frac{\#\text{ ways } A \text{ can occur}}{\#\text{ ways } B \text{ can occur}}$$

Since we need to restrict S to just be B,

$$P(A|C) = \frac{P(AC)}{P(C)} = \frac{\frac{1}{36}}{\frac{5}{36}} = \frac{1}{5} > P(A)$$

or $C : \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$

$$P(C|A) = \frac{P(CA)}{P(A)} = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6} > P(C)$$

or $A : \{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\}$

Dependence is a two way relationship. Both influence the other in the same direction.

8 Lecture 8*

4.5 Product Rules, Law of Total Probability and Bayes' Theorem

8.1 Rule 7 (Product Rules)

Let $A,\ B,\ C,\ D, \ldots$ be events in a sample space. Assume that $P(A) > 0$, $P(AB) > 0$, and $P(ABC) > 0$. Then

$$P(AB) = P(B|A)P(A)$$

$$P(ABC) = P(C|AB)P(AB)$$

$$P(ABCD) = P(D|ABC)P(ABC)$$

and so on.

8.2 Rule 8 (Law of Total Probability)

Let $A_1, \ldots, A_k$ be a partition of the sample space S into disjoint (mutually exclusive events), that is

$$A_1 \cup \cdots A_k = S$$

and

$$A_i \cap A_j = \emptyset \qquad i \neq j$$

Let B be an arbitrary event in S. Then

$$P(B) = \sum_{i=1}^{k} P(B|A_i)P(A_i)$$

8.3 Theorem (Bayes' Theorem)

Suppose A and B are events defined on a sample space S. Suppose also that $P(B) > 0$. Then

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\bar{A})P(\bar{A})}$$

9 Lecture 9

We might be interested in reversing the direction of a conditional probability.

- given a positive test, what is the probability that you have a disease?
- given an error in code, who wrote it?

Example

1. If you test for a disease, what is the probability that you have it?

$P(D) = 0.02$

$P(T|\bar{D}) = 0.05$ false positive

$P(\bar{D}|T) = 0.01$ false negative

We found $P(T) = 0.0688$, we want

$$\begin{aligned} P(D|T) &= \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|\bar{D})P(\bar{D})} \\ &= \frac{0.99 \times 0.02}{0.99 \times 0.02 + 0.05 \times 0.98} \\ &= 0.288 \end{aligned}$$

2. Given a line of code that has an error, what is the probability that A wrote it?

$P(A) = 0.5$

$P(B) = P(C) = 0.25$

$P(E|A) = 0.01$

$P(E|B) = 0.02$

$P(E|C) = 0.05$

We want

$$\begin{aligned} P(A|E) &= \frac{P(E|A)P(A)}{P(E|A)P(A) + P(E|B)P(B) + P(E|C)P(C)} \\ &= \frac{0.5 \times 0.01}{0.0225} \\ &= 0.222 < P(A) \end{aligned}$$

Similarly,

$$P(B|E) = 0.22 < P(B)$$
$$P(C|E) = 0.556 > P(C)$$

Note the three conditional probabilities sum to 1 which they must since exactly one of A, B, or C wrote the line.

3. Probability of a LoL player also playing Warcraft?

$$P(W|L) = \frac{P(L|W)P(W)}{0.0387} = \text{exercise}$$

4.6 Useful Series and Sums

9.1 Theorem (Geometric Series)

The geometric series $\sum_{n=0}^{\infty} ar^n$ converges if $|r| < 1$ and diverges otherwise. If $|r| < 1$, then

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots = \frac{a}{1-r}$$

9.2 Theorem (Binomial Theorem)

Let n be a positive integer, $x \in \mathbb{R}$.

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

9.3 Theorem (Exponential Series)

Let $x \in \mathbb{R}$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

9.4 Theorem (Hypergeometric Identity)

$$\binom{a+b}{n} = \sum_{x=0}^{n} \binom{a}{x}\binom{b}{n-x}$$

10 Lecture 10

MLIW # 3: Naïve Bayes' Classifier

In ML classification, we use evidence to decide what category something belongs to. That is,

$$P(\text{category}|\text{evidence}) = \frac{P(\text{category})P(\text{evidence})}{\sum\limits_{\text{cat } i} P(\text{cat } i)P(\text{evidence}|\text{cat } i)}$$

The NBC assumes that if there are multiple types of evidence, their probabilities are independent <u>conditional on the category</u>.

For example, Spam detection. Let A_1 = fail rdns checked (sender spoofed). Let A_2 = sends to over 100 people. Let A_3 = link text doesn't match actual URL. Then what is

$$P(\text{Spam}|A_1A_2A_3) = \frac{P(A_1A_2A_3|\text{Spam})}{P(A_1A_2A_3|\text{Spam})P(\text{Spam}) + P(A_1A_2A_3|\overline{\text{Spam}})P(\overline{\text{Spam}})}$$

<u>**Chapter 5: Random Variables**</u>

<u>5.1 Random Variables and Probability Functions</u>

10.1 Definition (Random Variable)

A *random variable* is a function that assigns a real number to each point in a sample space S.

We typically use X, Y, Z as random variables and x, y, z as the possible values the random variable can take on. There are two types of random variables based on the range.

10.2 Definition (Discrete Random Variables)

Discrete random variables take integer values or, more generally, values in a countable set.

10.3 Definition (Continuous Random Variables)

Continuous random variables take values in some interval of real numbers like $(0,1)$ or $(1,\infty)$ or $(-\infty,\infty)$.

We can define multiple random variables on the same sample space S. For example, roll a fair 6-sided die 3 times.

$$S = \{(x, y, z) | 1 \leq x, y, z \leq 6\}$$

- Let $X =$ sum on the three die. $range(X) = \{3, \ldots, 18\}$.
- Let $Y =$ product on the three die. $range(Y) = \{1, \ldots, 216\}$.
- Let $Z =$ number on the first die. $range(Z) = \{1, \ldots, 6\}$
- Let $\bar{X} =$ average. $range(\bar{X}) = \{1, {}^4\!/\!_3, \ldots, 6\}$
- Let $W =$ # of dice that are 1. $range(W) = \{0, 1, 2, 3\}$

Examples of continuous random variables include:

- $T =$ time until an event
- $P =$ positive in space of a particle
- $H =$ height of a random person.

For Chapter 5, we will focus on discrete random variables.

10.4 Definition (Probability Function, Probability Distribution)

Let X be a discrete random variable with $range(X) = A$. The *probability function* of X is the function

$$f(x) = P(X = x)$$

for all $x \in A$.
The set of pairs $\{(x, f(x)) : x \in A\}$ is called the *probability distribution* of X.

10.5 Theorem (Properties of Probability Functions)

All probability functions must have the two properties:
1. $0 \leq f(x) \leq 1$ for all $x \in A$
2. $\sum\limits_{\text{all } x \in A} f(x) = 1$

<u>Example</u>

Let $X =$ # of dice that are 1.

x	0	1	2	3
$f(x)$	$5^3/6^3$	$3{\times}5^2/6^2$	$3{\times}5/6^3$	$1^3/6^3$

11 Lecture 11

11.1 Definition (Cumulative Distribution Function)

The cumulative distribution function of X is the function usually denoted by $F(x)$

$$F(x) = P(X \leq x)$$

for all $x \in \mathbb{R}$.

<u>Example</u>

x	0	1	2	3
$f(x)$	$^{125}/_{216}$	$^{75}/_{216}$	$^{15}/_{216}$	$^{1}/_{216}$
$F(x)$	$^{125}/_{216}$	$^{200}/_{216}$	$^{215}/_{216}$	1

11.2 Theorem (Properties of Cumulative Distribution Functions)

All cumulative distribution functions must have the following properties:
1. $F(x)$ is a non-decreasing function of x for all $x \in \mathbb{R}$
2. $0 \leq F(x) \leq 1$ for all $x \in \mathbb{R}$
3. $\lim_{x \to -\infty} F(x) = 0$
4. $\lim_{x \to \infty} F(x) = 1$

11.3 Theorem

If X takes on integer values for x such that $x \in A$ and $(x-1) \in A$,

$$f(x) = F(x) - F(x-1)$$

Proof. $F(x) - F(x-1) = P(X \leq x) - P(X \leq x-1) = P(X = x) = f(x)$ □

5.2 Discrete Uniform Distribution

11.4 Definition (Discrete Uniform Distribution)

Suppose X takes values $a, a+1, \ldots, b$ with all values being equally likely. Then X has a Discrete Uniform distribution on the set $\{a, a+1, \ldots, b\}$ and we write

$$X \sim \mathrm{DU}[a, b]$$

Find the probability function, $f(x)$

we note there are $(b-a+1)$ values X can take so the probability at each of these values must be $\frac{1}{b-a+1}$ so that $\sum_{x=a}^{b} f(x) = 1$. Hence

$$f(x) = \begin{cases} \frac{1}{b-a+1}, \; x = a, \ldots, b \\ 0, \text{ otherwise} \end{cases}$$

5.3 Hypergeometric Distribution

11.5 Definition (Hypergeometric Distribution)

Suppose we have a collection of N objects which can be classified into two different types, a success (S) and a failure (F). Suppose there are r success and $N-r$ failures. Pick n objects at random without replacement. Let X be the number of successes obtained. Then X has a *Hypergeometric distribution* and we write

$$X \sim \mathrm{Hyp}(N, r, n)$$

Find the probability function, $f(x)$

There are $\binom{N}{n}$ points in the sample space S if we don't consider the order of selection. There are $\binom{r}{x}$ ways to choose the successes from the r available AND $\binom{N-r}{n-x}$ ways to choose the remaining $(n-x)$ objects from the

$(N-r)$ failures. Hence

$$f(x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$$

for $x = 0, \dots, \min(r, n)$.

Example

Suppose we have 10 cards, with 7 that are money cards and 3 non-money cards. Let $X = \#$ of money cards in your hand. Then

$$X \sim Hyp(10, 7, 5)$$

$$f(x) = \frac{\binom{7}{x}\binom{3}{5-x}}{\binom{10}{5}}$$

for $x = 2, 3, 4, 5$ (any less and you run out of non-money cards).

5.4 Binomial Distribution

11.6 Definition (Bernoulli Trials)

Suppose an experiment has two distinct outcomes, call them a success (S) and a failure (F), and let their probabilities p for S and $1-p$ for F. Repeat the experiment n independent times. Then, the n individual experiments in the process are called *Bernoulli trials*.

11.7 Definition (Binomial Distribution)

Suppose an experiment follows the Bernoulli trials. Then X has a *Binomial distribution* and we write

$$X \sim \text{Bin}(n, p)$$

Find the probability function, $f(x)$

There are $\binom{n}{x}$ different arrangements of x S's and $(n-x)$ F's over n trials. Since the trials are independent, the probability of a success is p multiplied x times, and a failure is $(1-p)$ multiplied $(n-x)$ times. Thus we have

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

for $x = 0, \dots, n$.

12 Lecture 12

Example - Error Correcting Code

Suppose you send a 4-bit message over a noisy connection. Each bit is independently flipped with a probability $p = 0.1$. Find $P(\text{message is received correctly})$.

Solution.

Let $X = \#$ of bits that get flipped.

$$X \sim \text{Bin}(4, 0.1)$$

$$P(X = 0) = \binom{4}{0} 0.1^0 (1-0.1)^{4-0} = 0.6561$$

Example cont.

Now suppose we add 3 parity bits that allow the receiver to detect and correct up to 1 error in the message. Find $P(\text{message is received correctly})$.

Solution.

Let $Y = \#$ of bits flipped.

$$Y \sim Bin(7, 0.1)$$

$$P(Y=0) + P(Y=1) = \binom{7}{0}0.1^0 0.9^7 + \binom{7}{1}0.1^1 0.9^6 \approx 0.8503$$

Example (ION)

Suppose there are 100 people, 10 people with no bus pass, and 20 people are selected at random without replacement. Find the probability that there are two people with no bus pass.

Solution.

Let $X = \#$ people found with no bus pass.

$$X \sim \mathrm{Hyp}(100, 10, 20)$$

$$P(X=2) = \frac{\binom{10}{2}\binom{90}{18}}{\binom{100}{20}} \approx 0.3182$$

Hypergeometric Approximation with Binomial Distribution

If we did the selection with repetition, the Hypergeometric distribution would become a Binomial distribution.

If N is extremely large compared to n, then it doesn't make much difference if the sampling is with our without.

In that case, a $\mathrm{Hyp}(N, r, n)$ case can be approximated by $\mathrm{Bin}(n, {}^r\!/\!_N)$

Example

Suppose we have 100 cards, with 70 money cards and 30 non-money cards. Select 4 cards without replacement. Find a suitable approximation for this Hypergeometric distribution assuming $Z = \#$ of money cards selected.

Solution.

Let $Z = \#$ of money cards selected.

$$Z \sim \mathrm{Hyp}(100, 70, 5)$$

$$P(Z=4) = \frac{\binom{70}{4}\binom{30}{1}}{\binom{100}{5}} \approx 0.3654$$

With a Binomial approximation:

$$Z \sim \mathrm{Bin}(5, {}^{70}\!/\!_{100} = 0.7)$$

$$P(Z=4) = \binom{5}{4}0.7^4(1-0.7)^{5-4} = 0.36015$$

which is clearly a very good approximation.

13 Lecture 13

13.1 Negative Binomial Distribution (5.5)

Setup: Bernoulli Trials

- independent
- each trial is a success or fail (S or F)
- $P(\text{success}) = p = \text{constant}$

Suppose we want to get k S's. We do trials until we get k S's and let $X = \#$ of F's. We get

$$X \sim \text{NB}(k, p)$$

in a total of $k + X$ trials.

Binomial	Negative Binomial
know # of trials	unknown # of trials
unknown # of S's	known # of S's
$\binom{n}{x}p^x(1-p)^{n-x}$	$\binom{x+k-1}{k-1}p^k(1-p)^x$

13.2 Example

How many tails until we get the 10th head on a fair coin. $X \sim \text{NB}(10, \frac{1}{2})$

13.3 Example

If courses were independent with probability p of passing and you need 40 courses, then the number of failed courses would be $\text{NB}(40, p)$.

13.4 Range and Probability Function of the Negative Binomial Distribution

range $x \in \{0, 1, \dots\}$ (countably infinite)

$$\begin{aligned} f(x) = P(X = x) &= p(x \text{ F's before } k\text{th S}) \\ &= \binom{x+k-1}{x}p^k(1-p)^x \\ &= \binom{x+k-1}{k-1}p^k(1-p)^x \end{aligned}$$

In a picture:

$$\overbrace{\underbrace{\_\,\_\,\_\cdots\_}_{(k-1)\text{ S's, }x\text{ F's}}}^{x+(k-1)\text{ Trials}}\Big|\ \underbrace{\text{S}}_{k\text{th S}}$$

13.5 Example

Suppose a startup is looking for 5 investors. They ask investors repeatedly where each independently has a 20% chance of saying yes. Let $X = \text{total } \#$ of investors that they ask and note that X does not follow a negative binomial distribution. Find $f(x)$ and $f(10)$.

Let $Y = \#$ who say no before 5 say yes. $Y \sim \mathrm{NB}(5, 0.2)$, and $X = Y + 5$. So,

$$\begin{aligned} f(x) &= P(X = x) \\ &= P(Y + 5 = x) \\ &= P(Y = x - 5) \\ &= \binom{(x-5)+5-1}{5-1}(0.2)^5(0.8)^{x-5} \\ &= \binom{x-1}{4}(0.2)^5(0.8)^{x-5} \qquad \text{for } x = 5, \ldots \end{aligned}$$

$$f(10) = \binom{9}{4}(0.2)^5(0.8)^5$$

note that it's $\binom{9}{4}$ and not $\binom{10}{5}$ because the 10th investor must have said yes.

14 Lecture 14

14.1 Example

Suppose you send a bit string over a noisy connection with each bit independently having a probability 0.01 of being flipped. What is the probability that

(a) it takes 50 bits to get 5 errors?

(b) a 50 bit message has 5 errors?

(b) Let $Y = \#$ of errors in 50 bits. $Y \sim \mathrm{Bin}(50, 0.01)$.

Then, $P(Y = 5) = \binom{50}{5}(0.01)^5(0.99)^{45}$

(a) Let $X = \#$ of correct bits until 5 errors. $X \sim \mathrm{NB}(5, 0.01)$.

Then, $P(X = 45) = \binom{49}{4}(0.01)^5(0.99)^{45}$

14.2 Geometric Distribution (5.6)

The Geometric Distribution is just a special case of the Negative Binomial Distribution with $k = 1$. Let $X = \#$ of F's in Bernoulli trials before the first S. $X \sim \mathrm{Geo}(p)$

14.3 Range and Probability Function of the Geometric Distribution

range: $x \in \{0, 1, \ldots\}$

$$\begin{aligned} f(x) &= P(X = x) \\ &= P(\underbrace{\text{F, F, } \ldots}_{\text{all F's}}, \text{S}) \\ &= (1-p)^x p \end{aligned}$$

or sub $k = 1$ into the NB probability function.

Prove $\sum\limits_{\text{all } x} f(x) = 1$

Proof.

$$
\begin{aligned}
\sum_{x=0}^{\infty}(1-p)^x p &= \underbrace{p + p(1-p) + \dots}_{\text{(geo. series: } a=p,\, r=1-p)} \\
&= \frac{p}{1-(1-p)} \\
&= 1
\end{aligned}
$$

□

Find the cumulative distribution function.

$$
\begin{aligned}
F(x) &= P(X \le x) \\
&= 1 - P(X > x) \\
&= 1 - [f(x+1) + \dots] \\
&= \underbrace{1 - [p(1-p)^{x+1} + p(1-p)^{x+2} + \dots]}_{\text{(geo. series: } a=p(1-p)^{x-1},\, r=1-p)} \\
&= 1 - \frac{p(1-p)^{x+1}}{1-(1-p)} \\
&= 1 - (1-p)^{x+1} \text{ for } x = 0, 1, \dots
\end{aligned}
$$

if $x \in \mathbb{R}$, then

$$
F(x) = \begin{cases} 1-(1-p)^{\lfloor x \rfloor + 1}, \text{ if } x \ge 0 \\ 0, \text{ if } x < 0 \end{cases}
$$

	Discrete Uniform	Hypergeometric	Binomial	Negative Binomial	Geometric	Poisson
function range parameters	$\mathrm{DU}[a,b]$ $a, a+1, \dots, b$	$\mathrm{Hyp}(N,r,n)$ bad	$\mathrm{Bin}(n,p)$ $0,1,\dots,n$	$\mathrm{NB}(k,p)$ $0,1,\dots$	$\mathrm{Geo}(p)$ $0,1,\dots$	$\mathrm{Poi}(\mu)$ $0,1,\dots$ $\mu=np$, $\mu=\lambda t$
pf, $f(x)$	$\frac{1}{b-a+1}$	$\frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$	$\binom{n}{x}p^x(1-p)^{n-x}$	$\binom{x+k-1}{k-1}p^k(1-p)^x$	$p(1-p)^x$	$\frac{e^{-\mu}\mu^x}{x!}$
cumulative distribution function, $F(x)$	$\frac{x-a+1}{b-a+1}$				$1-(1-p)^{x+1}$	$e^{\mu}[1+\frac{\mu^1}{1!}+\cdots\frac{\mu^x}{x!}]$
how to tell	“equally likely” know min. & max.	know total # objects know # S’s know # trials without replacement selecting a subset	Bernoulli trials know # trials count # S’s	Bernoulli trials “until” “it take... to get” “before” know how many S’s we want	“until we get” “before the first”	Bin. with large amount of trials, small prob rate specified (#events/time) no pre-specified max. events occurring at any time (randomly) Poisson process & know time & count events doesn’t make sense to ask how often an event did **not** occur

Bernoulli trials:

- independent
- each outcome is a S or F
- P(success)=p=constant

15 Lecture 15

15.1 Example

Naomi invites 12 people to her party. If each independently comes with probability p. Let $X = \#$ of guests.

Binomial: $X \sim \mathrm{Bin}(12, p)$

15.2 Example

20 toys in a machine. Each time you grab one with a claw. Let $X = \#$ of tries to get one toy you want.

None.

15.3 Example

Trying to catch a pokemon, each time has a probability p of succeeding. Let $X = \#$ of failed attempts.

Geometric: $X \sim \mathrm{Geo}(p)$

15.4 Example

You have 5 classes randomly scheduled in a row. Let $X = \#$ of classes before your favourite.

range: $0, 1, 2, 3, 4$, and the probability is $1/5$ for each of the range.

Discrete Uniform: $X \sim \mathrm{DU}[1, 4]$

15.5 Poisson Distribution from Binomial (5.7)

Suppose we have a $X \sim \mathrm{Bin}(n, p)$ where n is very large and p is very small. Then, as $n \to \infty$ and $p \to 0$ such that np remains constant, the probability function of X approaches a limit.

Let $np = \mu$, so $p = \frac{\mu}{n}$. Then

$$\begin{aligned}
\lim_{n\to\infty} f(x) &= \lim_{n\to\infty} \binom{n}{x} p^x (1-p)^{n-x} \\
&= \lim_{n\to\infty} \frac{n(n-1)\cdots(n-x+1)}{x!} \frac{\mu^x}{n^x} \left(1-\frac{\mu}{n}\right)^n \left(1-\frac{\mu}{n}\right)^{-x} \\
&= \frac{\mu^x}{x!} \lim_{n\to\infty} \frac{n}{n}\frac{n-1}{n}\cdots\frac{n-x+1}{n} \left(1-\frac{\mu}{n}\right)^n \left(1-\frac{\mu}{n}\right)^{-x} \\
&= \frac{\mu^x}{x!} \lim_{n\to\infty} \left(1-\frac{\mu}{n}\right)^n \\
&= \frac{e^{-\mu}\mu^x}{x!}
\end{aligned}$$

We write: $X \sim \mathrm{Poi}(\mu)$, range: $0, 1, \ldots$

We can use the Poisson random variable as an approximation to the Binomial when n is large, and p is small. The only thing we need to do is $\mu = np$.

15.6 Example

Tim Hortons roll up the rim says 1 in 6 cups win a prize. Suppose you have 80 cups. Find the probability that you get 10 or fewer winners.

Let $X = \#$ of winning cups. $X \sim \text{Bin}(80, 1/6)$ We want

$$\begin{aligned} F(10) &= P(X \leq 10) \\ &= \sum_{x=0}^{10} f(x) \\ &= \binom{80}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^{80} + \cdots + \binom{80}{10}\left(\frac{1}{6}\right)^{10}\left(\frac{5}{6}\right)^{70} \\ &= 0.2002 \text{ (tedious)} \end{aligned}$$

Try a Poisson approximation. $Y \sim \text{Poi}(\mu = np = \frac{80}{6} \approx 13.33)$. Then,

$P(Y \leq 10) = e^{-13.33}\left[1 + \frac{13.33}{1!} + \cdots + \frac{13.33^{10}}{10!}\right] = 0.224$

Not a good approximation since p was too large.

15.7 Poisson Distribution from Poisson Process (5.8)

The Poisson Process: Suppose events occur randomly in time or space according to three conditions:

(1) Independence: the number of events in one period cannot affect another non-overlapping period

(2) Individuality: events occur one at a time (cannot have two at the exact same time)

(3) Homogeneity or Uniformity: events occur at a constant rate

16 Lecture 16

16.1 Example

Request coming in from a web server at a rate of 100 requests per minute. $\lambda = 100, t = \frac{1}{60}$ The # of requests per second would be

$$\text{Poi}\left(\mu = \frac{100}{60} = \frac{5}{3}\right)$$

16.2 Combining Other Models with the Poisson Process (5.9)

Problems may involve many different random variables!

16.3 Example (Continued)

We say that a second is quiet if it has no requests.

(a) Find probability that a second is quiet

(b) In a minute (60 non-overlapping seconds), find the probability of 10 quiet seconds

(c) Find the probability of having to wait 30 non-overlapping seconds to get 2 quiet seconds

(d) Given (c), find the probability of 1 quiet second in the first 15 seconds

(a) Let $X = \#$ requests in a second. $X \sim \text{Poi}(5/3)$.

We want $P(X = 0) = \frac{e^{-\frac{5}{3}}\left(\frac{5}{3}\right)^0}{0!} = 0.189$

(b) Let $Y = \#$ quiet seconds out of 60. $Y \sim \text{Bin}(60, 0.189)$.

We want $P(Y = 10) = \binom{60}{10}(0.189)^{10}(0.811)^{50} = 0.124$

(c) Let $Z = \#$ non-quiet seconds before getting 2 quiet seconds. $Z \sim \text{NB}(2, 0.189)$.

We want $P(Z = 28) = \binom{29}{1}(0.189)^2(0.811)^{28} = 0.003$

(d) $D_x = \#$ of quiet seconds out of 15. $D_x \sim \text{Bin}(15, 0.189)$.

$$P(D_x = 1) = \binom{15}{1}(0.189)^1(0.811)^{14}$$

We get,

$P(1 \text{ quiet second in the first 15 seconds} \mid \text{wait 30 to get 2 quiet}) =$

$$\begin{aligned}
&= \frac{P(1 \text{ quiet second in the first 15 seconds AND wait 30 to get 2 quiet})}{P(\text{wait 30 to get 2 quiet})} && (1)\\
&= \frac{P(1 \text{ quiet second in the first 15 seconds AND wait an additional 15 to get 1 additional quiet})}{P(C)} && (2)\\
&= \frac{P(1 \text{ quiet second in the first 15 seconds})P(\text{wait an additional 15 to get 1 additional quiet})}{P(C)} && (3)\\
&= \frac{\binom{15}{1}(0.189)^1(0.811)^{14} \times (0.811)^{14}(0.189)}{\binom{29}{28}(0.189)^2(0.811)^{28}} && (4)\\
&= \frac{\binom{15}{1}}{\binom{29}{28}} && (5)\\
&= \frac{15}{29} && (6)
\end{aligned}$$

In (3) we used the independence of non-overlapping time intervals and constant probability of events.

16.4 Summarizing Data on Random Variables (7.1)

Let $X = \#$ of kids in a family.

Value	Frequency
1	3
2	10
3	1
4	1

16.4.1 Definition (Median)

The *median* of a sample is a value such that half the results are below it and half above it, when the results are arranged in numerical order.

16.4.2 Definition (Mode)

The *mode* of the sample is the value which occurs most often. There is no guarantee there will be only a single mode.

Mean: average $\rightarrow \frac{1\times3+2\times10+3\times1+4\times1}{15}$

Median: 2

Mode: 2

17 Lecture 17*

- If you're interested, see if you can determine how small the probability of a cache hit would have to be (in our example) in order for it not to be worth it to use a cache. Post the answer in the follow-up if you get it.
- In Roulette, a game where there are 38 sections that can be chosen with equal probability, and you can bet on lots of different outcomes. It turns out that no matter what betting strategy you use or how you split up your money, the expected payoff from any $1 bet is always 0.94737, so you essentially lose about 5.3 cents every time you play! (Over Reading Week, I encourage you to imagine a betting strategy and verify this fact - but I do not encourage actually gambling!)
- Of course, different betting strategies will have different amounts of risk, even if the expected value is the same. This is the idea of Variance, which we'll start talking about on Monday after Reading Week. :)

Have a fantastic Reading Week! I recommend setting realistic goals for yourself (including both some dedicated time to relax and dedicated time to catch up / get ahead on school work) and have both a productive and fun week!

17.1 Expectation of a Random Variable (7.2)

Imagine we know the theoretical probability of each # of kids in a family.

x	1	2	3	4	5
$f(x)$	0.43	0.4	0.12	0.04	0.01

Now we replace the observed proportion in the sample mean with $f(x)$

$$\sum_{\text{all } x} xf(x) = (1)(0.43) + (2)(0.4) + (3)(0.12) + (4)(0.04) + (5)(0.01) = 1.8$$

which is the theoretical mean.

Why do we have sample mean > theoretical mean?

- urban vs rural population
- income level
- sampled max family size but theoretical includes growing families
- selection bias (if you randomly select people rather than families, people with lots of siblings will be over-represented

17.1.1 Definition (Expected Value)

Let X be a discrete random variable and probability function $f(x)$. The *expected value* (also called the mean or the expectation) of X is given by

$$\mu = E[X] = \sum_{\text{all } x} xf(x)$$

Remark 3. μ will be within the range but not necessarily equal to a possible value of x.

We might be interested in the expected value of some function of X, $g(X)$.

17.2 Example

Tax credit of $\$1000$ plus $\$250$ per kid. Find the average cost.

x	1	2	3	4	5
$g(x)$	1250	1500	1750	2000	2250

Average cost = weighted average of $g(x)$ values $= (1250)(0.43) + \cdots + (2250)(0.01) = 1450$

17.2.1 Theorem

Let X be a discrete random variable and probability function $f(x)$. The expected value of a some function $g(X)$ of X is given by

$$E[g(X)] = \sum_{\text{all } x} g(x)f(x)$$

Note that $g(x) = 1000 + 250x$ from last example.

$$E[g(X)] = 1000 + 250E[X] = 1450$$

What if tax credit $= \frac{2000}{x}$

$$E[g(X)] = (2000)(0.43) + (1000)(0.40) + \cdots + (400)(0.01) = 1364$$

But $\frac{2000}{E[X]} = \frac{2000}{1.8} = 1111.11$. Therefore

$$E[g(X)] \neq g(E[X])$$

unless g is a linear function. That is, if $g(X) = aX + b$, then $E[g(X)] = aE[X] + b$

17.3 Example

A web server has a cache. Takes 10ms to check, 20% of the requests are found (cache hit) and immediately shown. If it's not found (cache miss), it takes $\underbrace{50}_{\text{to server}} + \underbrace{70}_{\text{lookup}} + \underbrace{50}_{\text{to client}}$ additional milliseconds to get info and display. Is it worth it? Let $X = \#$ of milliseconds to display the information.

x	10	10+50+70+50=180
$f(x)$	0.2	0.8

$$E[X] = (10)(0.2) + (180)(0.8) = 146\text{ms}$$

Time no cache $= 50 + 70 + 50 = 170$ms.

Since 146ms < 170ms, it's worth it!

17.4 Example

Roulette: each of 38 numbers is equally likely

(1) If you bet 1 dollar on number $7 \to$ pays $35:1$

OR

(2) If you bet 50 cents on red $\to$ pays $1:1$ and 50 cents on first 12 numbers $\to$ pays $2:1$

(1)

x	0	36
$f(x)$	$37/38$	$1/38$

(2)

y	0	1	1.50	2.50
$f(y)$	$\underbrace{14/38}_{\text{neither}}$	$\underbrace{12/38}_{\text{red}}$	$\underbrace{6/38}_{\text{black}}$	$\underbrace{6/38}_{\text{both red}}$

$$E[X] = 0(\frac{37}{38}) + 36(\frac{1}{38}) = 0.94737$$

$$E[Y] = 0(\frac{14}{38}) + 1(\frac{12}{38}) + 1.5(\frac{6}{38}) + 2.5(\frac{6}{38}) = 0.94737$$

18 Lecture 18

18.1 Means and Variances of Distributions (7.4)

The mean $E[X]$ tells us where the distribution is on average. We also need a way to describe how spread out a distribution is. Variance could be $E[X - \mu] = 0$.

What about $E[|X - \mu|]$

- need cases to evaluate
- non-differentiable at point $X - \mu$
- linear penalty for being away from the mean

Instead we use $E[(X - \mu)^2]$

18.1.1 Definition (Variance)

The *variance* of a random variable X, denoted by $Var(X)$ or by σ^2, is

$$\sigma^2 = Var(X) = E[(X - \mu)^2] = \sum_{\text{all } x} (X - \mu)^2 f(x)$$

18.2 Example

$X = \#$ on fair 6-sided die

$E[X] = 3.5$

$E[(x - 3.5)^2]$

$E[X]^2 - 3.5^2$

x	1	2	3	4	5	6
x^2	1	4	9	16	25	36

Alternate form (calculation form)

$$\begin{aligned}
Var(X) &= E[(X - E[X])^2] \\
&= E[X^2 - 2XE[X] + E[X]^2] \\
&= E[X^2] - 2E[X]E[X] + E[X]^2 \\
&= E[X^2] - 2(E[X])^2 + E[X]^2 \\
&= E[X^2] - E[X]^2 \\
&= \sum_{\text{all } x} x^2 f(x) - \left(\sum_{\text{all} x} x f(x)\right)^2
\end{aligned}$$

18.3 Example (Roulette)

$X = 0$ or 36 (dollars)

x	0	36
$f(x)$	$37/38$	$1/38$

$$E[X] = 0.94737$$

$$Var(X) = E[X^2] - 0.94737^2 = 36^2(\frac{1}{36}) - 0.94737^2 = 33.207 \text{ dollars}^2$$

To interpret the variance better, we often take the square root to get the same units of the original variable.

18.3.1 Definition (Standard Deviation)

The *standard deviation* of a random variable X is

$$\sigma = SD(X) = \sqrt{Var(X)}$$

$$SD(X) = \sqrt{33.207} = 5.76$$

What if we bet $\$1$ on red. Y=winnings

y	0	2
$f(y)$	$20/38$	$18/38$

$$E[Y] = 0.94737$$

$$Var(Y) = E[Y^2] - 0.94737^2 = 0.97723$$

$$SD(Y) = 0.9986$$

18.4 Linear Transformations

If $Y = aX + b$, and we know $E[X]$ and $Var(X)$, what can we say about $E[Y]$ and $Var(Y)$.

$$E[Y] = aE[X] + b$$

$$\begin{aligned}
Var(Y) &= E[(Y - E[Y])^2] \\
&= E[(aX + b) - (aE[X] + b)^2] \\
&= E[a^2X^2 - 2XE[X] + E[X]^2] \\
&= a^2E[(X - E[X])^2]
\end{aligned}$$

$$Var(Y) = a^2 Var(X)$$
$$SD(Y) = |a| SD(X)$$

19 Lecture 19*

19.1 Example

Suppose X has probability function:

x	0	1	2	3	4
y	1	3	5	7	9
$f(x)$	0.1	0.1	0.1	0.5	0.2

Let $Y = 2X + 1$.

$E[X] = 2.6$

$E[X^2] = 6.2$

$E[Y] = 6.2$

$E[Y^2] = 94.2$

$Var(X) = 8.2 - 2.6^2 = 1.44$

$SD(X) = 1.2$

$Var(Y) = 44.2 - 6.2^{=}5.76$

$SD(Y) = 2.4$

Now we can verify,

$$\begin{aligned} E[Y] &= E[2X+1] \\ &= 2E[X] + 1 \\ &= 2(2.6) + 1 \\ &= 6.2 \end{aligned}$$

$$Var(Y) = 2^2 Var(X) = 4(1.44) = 5.76$$
$$SD(Y) = |2| SD(X) = 2(1.2) = 2.4$$

Let $X \sim \mathrm{Bin}(n,p)$. Find $E[X]$.

$$E[X] = \sum_{\text{all } x} x f(x) \tag{1}$$
$$= \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} \tag{2}$$
$$= \sum_{x=1}^{n} x \binom{n}{x} p^x (1-p)^{n-x} \tag{3}$$
$$= \sum_{x=1}^{n} x \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \tag{4}$$
$$= \sum_{x=1}^{n} x \frac{n!}{x(x-1)!(n-x)!} p^x (1-p)^{n-x} \tag{5}$$
$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} p^x (1-p)^{n-x} \tag{6}$$
$$= \sum_{x=1}^{n} \frac{n(n-1)!}{(x-1)![(n-1)-(x-1)]!} p p^{x-1} (1-p)^{(n-1)-(x-1)} \tag{7}$$
$$= np(1-p)^{n-1} \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} (1-p)^{-(x-1)} \tag{8}$$
$$= np(1-p)^{n-1} \sum_{x=1}^{n} \binom{n-1}{x-1} \left(\frac{p}{1-p}\right)^{x-1} \tag{9}$$

From (2) to (3) we used to fact that when $x = 0$ the value of the expression is 0. Provided that $x \neq 0$, we can expand $x!$ as $x(x-1)!$ as seen from (4) to (5). Let $y = x - 1$, we get

$$E[X] = np(1-p)^{n-1} \sum_{x=1}^{n} \binom{n-1}{y} \left(\frac{p}{1-p}\right)^{y} \tag{10}$$
$$= np(1-p)^{n-1} \left(1 + \frac{p}{1-p}\right)^{n-1} \tag{11}$$
$$= np(1-p)^{n-1} \frac{(1-p+p)^{n-1}}{(1-p)^{n-1}} \tag{12}$$
$$= np \tag{13}$$

From (10) to (11) we used the Binomial Theorem.

Let $X \sim \mathrm{Poi}(\mu)$. Find $E[X]$.

$$E[X] = \sum_{\text{all } x} x f(x) \tag{1}$$
$$= \sum_{x=0}^{\infty} x \frac{e^{-\mu}\mu^x}{x!} \tag{2}$$
$$= \sum_{x=1}^{\infty} x \frac{e^{-\mu}\mu^x}{x(x-1)!} \tag{3}$$
$$= \sum_{x=1}^{\infty} \mu \frac{e^{-\mu}\mu^{x-1}}{(x-1)!} \tag{4}$$
$$\tag{5}$$

Let $y = x - 1$, we get

$$\begin{aligned} E[X] &= \mu e^{-\mu} \sum_{x=1}^{\infty} \frac{\mu^y}{y!} && (6) \\ &= \mu e^{-\mu} e^{\mu} && (7) \\ &= \mu && (8) \end{aligned}$$

From (6) to (7) we used the fact that $e^x = \sum\limits_{y=0}^{\infty} \frac{x^y}{y!}$.

Similarly,

$$X \sim \text{DU}[a, b],\ E[X] = \frac{a+b}{2}$$
$$X \sim \text{Hyp}(N, r, n),\ E[X] = \frac{nr}{N}$$
$$X \sim \text{NB}(k, p),\ E[X] = \frac{k(1-p)}{p}$$
$$X \sim \text{Geo}(p),\ E[X] = \frac{1-p}{p}$$

Let $X \sim \text{Poi}(\mu)$. Find $Var(X)$.

Since there's $x!$ in the denominator of $f(x)$, let's find $E[X(X-1)]$.

$$\begin{aligned} E[X(X-1)] &= \sum_{x=0}^{\infty} x(x-1) \frac{\mu^x e^{-\mu}}{x!} \\ &= \sum_{x=2}^{\infty} x(x-1) \frac{\mu^x e^{-\mu}}{x(x-1)(x-2)!} \\ &= \mu^2 e^{-\mu} \sum_{x=2}^{\infty} \frac{\mu^{x-2}}{(x-2)!} \end{aligned}$$

Let $y = x - 2$, we get

$$\begin{aligned} E[X(X-1)] &= \mu^2 e^{-\mu} \sum_{x=2}^{\infty} \frac{\mu^y}{y!} \\ &= \mu^2 \end{aligned}$$

$$\begin{aligned} Var(X) &= E[X(X-1)] + E[X] - E[X]^2 \\ &= \mu^2 + \mu - \mu^2 \\ &= \mu \end{aligned}$$

20 Lecture 20*

20.1 Example

Suppose the amount of data you use on your phone (in units of 100MB) has a Poisson distribution with mean 7 per month. You pay 15 per month plus 3 per 100MB. Find the standard deviation of random month's phone bill.

Let $X = \#$ of units of data used. $X \sim \text{Poi}(7)$. Let $Y = 15 + 3X \rightarrow E[Y] = 15 + 3(7) = 36$.

$SD(Y) = 3SD(X) = 3\sqrt{7} = 7.94$.

21 Lecture 21

A *continuous random variable* X maps points in a continuous sample space to real numbers such that the range is uncountably infinite.

Examples of continuous random variables

Let X be the number the point spots at.

(1) temperature of a day

(2) length of time until a bus arrives

(3) height of a random person

(4) average height of 10 people

$F(x) = P(X \leq x)$

$F(a) = P(X \leq a)$

21.1 Example

For $x < 0$, no chance of the point stopping at a number < 0.

For $x > 4$, $F(x) = 1$ since the point is certain to stop at a number below 4.

$P(0 < x \leq 1) = \frac{1}{4} = F(1)$

$$F(x) = \begin{cases} 0, \ x < 0 \\ \frac{x}{4}, \ 0 \leq x \leq 4 \\ 1, \ x > 4 \end{cases}$$

Properties of $F(x)$

(1) For all x, $P(X = x) = 0$. So,

$$\begin{aligned} P(a < x \leq b) &= P(a \leq x \leq b) \\ &= P(a < x < b) \\ &= P(a \leq x < b) \end{aligned}$$

Remark 4. End points don't matter.

(2)

$$\begin{aligned} \lim_{\varepsilon \to 0} F(x) - F(x - \varepsilon) &= \lim_{\varepsilon \to 0} P(x - \varepsilon < X \leq x) \\ &= P(X = x) \\ &= 0 \end{aligned}$$

Thus $\lim\limits_{\varepsilon \to 0} F(x - \varepsilon) = F(x)$, so $F(x)$ is continuous.

(3) $F(x)$ is non-decreasing.

(4) $\lim\limits_{x \to +\infty} F(x) = 1$, $\lim\limits_{x \to -\infty} F(x) = 0$

(5) $0 \leq F(x) \leq 1$

21.1.1 Definition (Probability Density Function)

The *probability density function* (p.d.f) $f(x)$ for a continuous random variable X is the derivative

$$f(x) = \frac{d}{dx}F(x)$$

where $F(x)$ is the cumulative distribution function for X.

Remark 5. $f(x)$ is not a probability. It can be > 1 relative likelihood that X takes a value near X.

PROPERTIES OF $f(x)$

(1)

$$\begin{aligned} P(a \leq X \leq b) &= F(b) - F(a) \\ &= \int_a^b f(x)dx \end{aligned}$$

(2)

$$\begin{aligned} \int_{-\infty}^{+\infty} f(x)dx &= F(+\infty) - F(-\infty) \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

(3) $f(x) \geq 0$ (since $F(x)$ is non-decreasing, it's derivative is non-negative)

(4)

$$F(x) = \int_{-\infty}^{x} f(u)du$$

21.2 Example

Suppose a continuous random variable X is on the range $[0, 1]$ has the cumulative distribution function $F(x) = x^2$.

WHAT IS THE PROBABILITY DENSITY FUNCTION?

$f(x) = \frac{d}{dx}F(x) = 2x$.

WHAT IS $P(X = 0.25)$?

$P(X = 0.25) = 0$

WHAT IS $P(X \leq 0.25)$?

(1) $P(X \leq 0.25) = F(0.25) = (0.25)^2 = 0.0625$

(2)

$$P(X \leq 0.25) = \int_0^{0.25} f(x)dx = \int_0^{0.25} 2xdx = 0.625$$

Expectation:

$$E[X] = \int_{-\infty}^{+\infty} x f(x) dx = \int_{x \in \text{range}} x f(x) dx$$

Variance:

$$Var(X) = E[X^2] - E[X]^2 = \int_{-\infty}^{+\infty} x^2 f(x) dx - \int_{-\infty}^{+\infty} x f(x) dx$$

21.2.1 Definition (Percentiles)

The p^{th} *percentile* of a distribution x_p such that $F(x_p) = p$.

22 Lecture 22

22.1 Example

$F(x) = x^2$ for $0 < x < 1$.

Find the mean, median, and mode

Mean:

$$E[X] = \int_0^1 x2x dx = \int_0^1 2x^2 dx = \left[\frac{2x^3}{3}\right]_0^1 = \frac{2}{3}$$

Median: $x_{0.5}$ satisfies $F(x_{0.5}) = 0.5 \implies (x_{0.5})^2 = 0.5 \implies x_{0.5} = \sqrt{0.5} = 0.707$

Mode: 1 (x value that maximizes $f(x)$)

22.2 Continuous Uniform Distribution (8.2)

A *continuous* random variable takes real values between a and b with $a < b$ such that any interval of fixed size is equally likely.

Notation

$X \sim U(a, b)$

Remark 6. Can include or exclude endpoints, doesn't matter.

Find $f(x)$

$f(x) = c$, (since it can't depend on x). We need

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\int_a^b c\, dx = 1$$

$[cx]_a^b = 1 \implies c(b-a) = 1 \implies c = \frac{1}{b-a}$

So,

$$f(x) = \begin{cases} \frac{1}{b-a}, \ a \leq x \leq b \\ 0, \ \text{otherwise} \end{cases}$$

FIND $F(x)$

$$F(x) = \int_{-\infty}^{x} f(u)du = \int_{a}^{x} \frac{1}{b-a}du = \left[\frac{u}{b-a}\right]_a^x = \frac{x-a}{b-a}$$

$$F(x) = \begin{cases} \frac{x-a}{b-a}, \ a < x < b \\ 0, \ x < a \\ 1, \ x > b \end{cases}$$

FIND THE MEAN, MEDIAN AND MODE

Mean:

$$E[X] = \int_{a}^{b} x\frac{1}{b-a}dx = \left[\left(\frac{x^2}{2}\right)\left(\frac{1}{b-a}\right)\right]_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{b+a}{2}$$

Median: is also $\frac{a+b}{2}$

Mode: no unique mode

Similarly,

$$Var(x) = \frac{(b-a)^2}{12}$$

SPECIAL CASE

$U \sim U(0,1)$ (i.e. $a=0, \ b=0$)

$$f(u) = \begin{cases} 1, \ 0 < u < 1 \\ 0, \ \text{otherwise} \end{cases}$$

$$F(u) = \begin{cases} u, \ 0 < u < 1 \\ 0, \ u < 0 \\ 1, \ u > 1 \end{cases}$$

$U(0,1)$ random variables are easy to generate.

22.3 Change of Variables

Suppose you know the distribution of X and you want the distribution of $Y = g(X)$.

1. Write the cumulative distribution function of Y in terms of the cumulative distribution function of X
2. Sub in what we know about X, then differentiate to get the pdf
3. Determine the range of Y

22.4 Example (Change of Variable)

Let $X \sim U(0,4)$, $F_X(x) = \frac{x}{4}$, $f_X(x) = \frac{1}{4} \ x \in (0,4)$

Let $Y = \frac{1}{X}$

1.

$$\begin{aligned} F_Y(y) &= P\left(Y \le y\right) \\ &= P\left(\frac{1}{X} \le y\right) \\ &= P\left(X > \frac{1}{y}\right) \\ &= 1 - F_X\left(\frac{1}{y}\right) \end{aligned}$$

2.

$$\begin{aligned} F_Y(y) &= 1 - F_X\left(\frac{1}{y}\right) \\ &= 1 - \frac{\frac{1}{y}}{4} \\ &= 1 - \frac{1}{4y} \end{aligned}$$

$f_Y(y) = \frac{d}{dx}F_Y(y) = \frac{1}{4y^2}$

OR differentiate $F_Y(y)$ before substituting in the information about X. You need the chain rule!

$$\frac{d}{dy}\left[1 - F_X\left(\frac{1}{y}\right)\right] = -f_X\left(\frac{1}{y}\right)\left(-\frac{1}{y^2}\right) = \frac{1}{4y^2}$$

3. $y \in (\frac{1}{4}, \infty)$

23 Lecture 23

23.1 Example

Let $Y \sim U(0,1) \implies f_Y(y) = \frac{1}{1-0} = 1,\ F_Y(y) = \frac{y-0}{1-0} = y$

$X = 2\sqrt[3]{Y}$.

Find $f_X(x)$

1.

$$\begin{aligned} F_X(x) &= P(X \le x) \\ &= P\left(2\sqrt[3]{Y} \le x\right) \\ &= P\left(Y \le \frac{x^3}{8}\right) \\ &= F_Y\left(\frac{x^3}{8}\right) \end{aligned}$$

2.

$$\begin{aligned} F_X(x) &= F_Y\left(\frac{x^3}{8}\right) \\ &= \frac{x^3}{8} \end{aligned}$$

$f_X(x) = \frac{d}{dx}F_X(x) = \frac{3}{8}x^2$

3. $x \in (0,2)$

23.2 Exponential Distribution (8.3)

Suppose we have a Poisson Process with rate λ. Let $X =$ time until the next event occurs. X has an exponential distribution.

Find range, $F(x)$, and $f(x)$

$x \in (0, \infty)$

$$\begin{aligned} F(x) &= P(X \leq x) \\ &= P(\text{time to next event} \leq x) \\ &= P(\text{number of events in } (0,x) \geq 1 \\ &= P(Y \geq 1) \qquad Y \sim \text{Poi}(\lambda x) \\ &= 1 - P(Y = 0) \\ &= 1 - \frac{(e^{-\lambda x})(\lambda x)^0}{0!} \\ &= \begin{cases} 1 - e^{-\lambda x},\ x > 0 \\ 0,\ x \leq 0 \end{cases} \end{aligned}$$

Alternate forms: $\theta = \frac{1}{\lambda}$, so

$$F(x) = 1 - e^{-\frac{x}{\theta}}$$

$$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}$$

We say $X \sim Exp(\theta)$.

24 Lecture 24

Find the mean and variance

$$E[X] = \int_0^\infty x \frac{1}{\theta} e^{-\frac{x}{\theta}}\, dx \text{ IBP}$$

Trick: **Gamma Function**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha - 1} e^{-x} dx$$

where $\alpha > 0$

Properties of the Gamma Function

(1) if $\alpha > 1$, then $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$

(2) if α is an integer ≥ 1,

$$\Gamma(1) = 1$$

$$\Gamma(2) = 1\Gamma(1) = 1$$

$$\Gamma(3) = 2\Gamma(2) = 2$$

$$\Gamma(4) = 3\Gamma(3) = 6$$

In general,

$$\Gamma(\alpha) = (\alpha - 1)!$$

So, back to our example:

$$
\begin{aligned}
E[X] &= \int_0^\infty x\frac{1}{\theta}e^{-\frac{x}{\theta}}\,dx \qquad y = \frac{x}{\theta} \implies x = (\theta y) \wedge \theta dy = dx \\
&= \int_0^\infty (\theta y)\frac{1}{\theta}e^{-y}\theta dy \\
&= \theta \int_0^\infty y^{2-1}e^{-y}dy \qquad \Gamma(2) = (2-1)! = 1 \\
&= \theta
\end{aligned}
$$

$$E[X] = \theta = \frac{1}{\lambda}$$

Why? If λ is higher, events happen more often, which means shorter wait time.

To find $Var(X)$,

$$
\begin{aligned}
E[X]^2 &= \int_0^\infty x^2\frac{1}{\theta}e^{-\frac{x}{\theta}}\,dx \qquad y = \frac{x}{\theta} \implies x^2 = (\theta y)^2 \wedge \theta dy = dx \\
&= \int_0^\infty (\theta y)^2\frac{1}{\theta}e^{-y}\theta dy \\
&= \theta^2 \int_0^\infty y^{3-1}e^{-y}dy \qquad \Gamma(3) = (3-1)! = 2 \\
&= 2\theta^2
\end{aligned}
$$

So $Var(X) = 2\theta^2 - \theta^2 = \theta^2$, $SD(X) = \theta = E[X]$

24.1 Memoryless Property

24.2 Example

Suppose busses follow a Poisson process with average 5 per hour.

(a) Find the probability that you wait > 15 mins.

Let $X =$ time until next bus. $X \sim Exp(12)$

$P(X > 15) = 1 - F(X \leq 15) = 1 - \left(1 - e^{-\frac{15}{12}}\right) = e^{-\frac{15}{12}} \approx 0.2865$

(b) If you have been waiting 6 minutes already, what is the probability that you wait another > 15 more minutes.

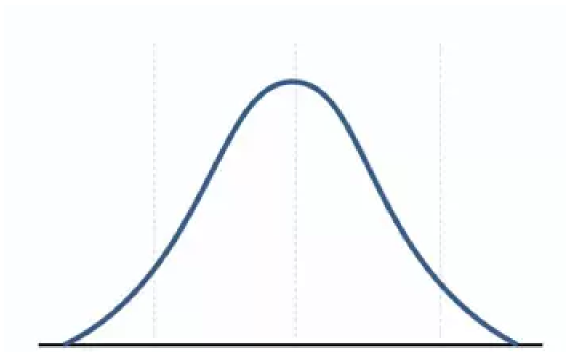
$$
\begin{aligned}
P(X > 21 \mid X > 6) &= \frac{P(X > 21 \text{ AND } X > 6)}{P(X > 6)} \\
&= \frac{P(X > 21)}{P(X > 6)} \\
&= \frac{1 - F(21)}{1 - F(6)} \\
&= \frac{1 - (1 - e^{-\frac{21}{12}})}{1 - (1 - e^{-\frac{6}{12}})} \\
&= e^{-\frac{15}{12}} \approx 0.2865
\end{aligned}
$$

The memoryless property says the past is irrelevant in the future distribution. In general, if $s, t > 0$:

$$P(X > t + s \mid X > s) = P(X > t)$$

24.3 Normal Distribution (8.5)

Many natural phenomena tend to follow a shape like this:



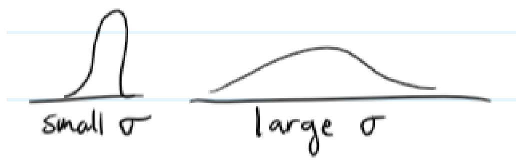
- amount of precipitation
- heights/weights of large populations
- measurement errors
- grades in courses

A normal rv X with parameters μ and σ^2 has pdf

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$$

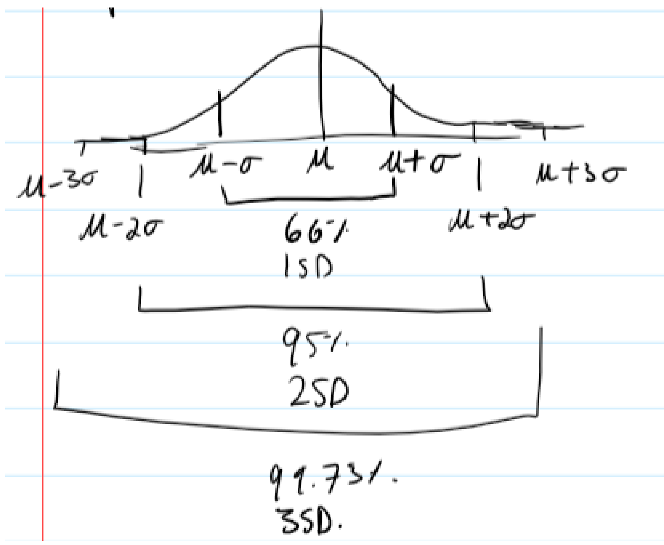
for $x \in \mathbb{R}$

- symmetric around μ
- both tails go to zero quickly
- $\frac{1}{\sqrt{2\pi}\sigma}$ makes it integrate to 1.



We can show that $E[X] = \mu$ and $Var(X) = \sigma^2$

24.4 Empirical rule



FIND $f(x)$

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-(u-\mu)^2/(2\sigma^2)} du$$

- not analytically integrable
- look it up or numerically evaluate

Standard Normal rv (special case with $\mu = 0,\ \sigma^2 = 1$)

$Z \sim N(0, 1)$

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

$F(z)$ still has no closed form.

25 Lecture 25

FIND $P(Z \leq 2.31)$

$$P(Z \leq \underbrace{2}_{\text{row}} . \underbrace{31}_{\text{col}}) = 0.98956$$

FIND $P(Z \leq -0.63)$

$$P(Z \leq -0.63) = P(Z > 0.63) = 1 - P(Z \leq 0.63) = 1 - 0.73565 = 0.26435$$

25.1 Example

A voltage of $+2$ means 1 and -2 means 0. The connection adds a $N(0,1)$ distribution amount of noise. If they receive a voltage of > 0.5, they interpret as 1, < 0.5 as 0.

FIND P(ERROR) IF A 1 WAS SENT Let $R =$ received $= 2 + Z$. $Z \sim N(0,1)$

$$\begin{aligned} P(\text{error}) &= P(R < 0.5) \\ &= P(2 + Z < 0.5) \\ &= P(Z < -1.5) \\ &= P(Z > 1.5) \\ &= 1 - P(Z \leq 1.5) \\ &= 1 - 0.93319 \\ &= 0.06681 \end{aligned}$$

Similarly P(error) if 0 was sent: $R = -2 + Z$,

$$\begin{aligned} P(\text{error}) &= P(R > 0.5) \\ &= P(-2 + Z > 0.5) \\ &= P(Z > 2.5) \\ &= 1 - P(Z \leq 2.5) \\ &= 1 - 0.99379 \\ &= 0.06621 \end{aligned}$$

FIND PERCENTILES OF $N(0,1)$

Suppose we want c such that $P(Z < c) = 0.85$

- look in body of table for ≈ 0.85 and read off row and column: c is between 1.03 and 1.04
- use reverse table, look up row and column: 1.0364

TRANSFORMING A NORMAL RV

Suppose $X \sim (\mu, \sigma^2)$, $\mu, \sigma^2 < \infty$.

Claim: if

$$Z = \frac{X - \mu}{\sigma}$$

then $Z \sim N(0,1)$

Proof.

1.

$$\begin{aligned} F_Z(z) &= P(Z \leq z) \\ &= P\left(\frac{X - \mu}{\sigma} \leq z\right) \\ &= P\left(X \leq z\sigma + \mu\right) \\ &= F_X(\sigma z + \mu) \end{aligned}$$

2. Differentiate

$$\begin{aligned} f_Z(z) &= \frac{d}{dz}F_Z(z) \\ &= \frac{d}{dz}F_X(\sigma z + \mu) \\ &= f_X(\sigma z + \mu)\sigma \qquad \text{CHAIN RULE} \\ &= \left(\frac{1}{\sqrt{2\pi}\sigma}e^{-((\sigma z+\mu)-\mu)^2/(2\sigma^2)}\right)\sigma \\ &= \frac{1}{\sqrt{2\pi}}e^{-z^2/2} \end{aligned}$$

3. range of Z is $\mathbb{R}$, so $Z \sim N(0,1)$

25.2 Example

MCAT scores are normal with mean 25.3 and standard deviation 6.5.

A score of 41 is how good?

Find $P(X > 41)$ where $X \sim N(25.3, 6.5^2)$

$$P\left(\frac{X-25.3}{6.5} > \frac{41-25.3}{6.5}\right) = P(Z > 2.42) = 1 - 0.99202 = 0.00798$$

25.3 Example

You want 98% of the population to use a ride by height. X = height $\sim N(69, 2.4^2)$. That is, find h such that $P(X < h) = 0.98$, so

$$P\left(\frac{X-69}{2.4} < \frac{h-69}{2.4}\right) = 0.98 \implies P\left(Z < \frac{h-69}{2.4}\right) = 0.98$$

Set $F(\frac{h-69}{2.4}) = 0.98$, and solve for h. You can also take F^{-1} on each side.

$$2.0537 = \frac{h-69}{2.4} \implies h = (2.0537)(2.4) + 69 = 73.93 \text{ inches}$$

In general,

$$x_p = \sigma z_p + \mu$$

26 Lecture 26

26.1 Example

If $Z \sim N(0,1)$, find d such that $P(|Z| < d) = 0.9$.

$$\begin{aligned} P(|Z| < d) &= P(-d < Z < d) \\ &= P(Z < d) - P(Z > -d) \\ &= P(Z \le d) - [1 - P(Z \le d)] \\ &= 2P(Z \le d) - 1 \end{aligned}$$

$$
\begin{aligned}
& 2P(Z \le d) - 1 = 0.90 \\
\implies & P(Z \le d) = \frac{0.90 + 1}{2} \\
\implies & F(d) = 0.95 \\
\implies & F^{-1}(F(d)) = F^{-1}(0.95) \\
\implies & d = 1.6449
\end{aligned}
$$

26.2 Basic Terminology and Techniques (9.1)

We have models for a single RV (both discrete or cts) but we often care about two or more RV's at the same time (and their relationship) Examples:

- two stock returns
- heights and weights
- number of cards of a rank vs number of a suit
- treatment vs recovery time
- all machine learning classification and regression

In this course, we focus on all discrete random variables

26.3 Definition (Joint Probability Function)

Let $X_1, \ldots, X_n$ be n discrete random variables. We define the *joint probability function* $f(x_1, \ldots, x_n)$ of $(X_1, \ldots, X_n)$ as

$$
\begin{aligned}
f(x_1, \ldots, x_n) &= P(X_1 = x_1 \text{ and } \cdots \text{ and } X_n = x_n) \\
&= P(X_1 = x_1, \ldots, X_n = x_n)
\end{aligned}
$$

26.4 Theorem

- $\sum\limits_{\text{all } (x_1, \ldots, x_n)} f(x_1, \ldots, x_n) = 1$
- $f(x_1, \ldots, x_n) \ge 0$ for all $(x_1, \ldots, x_n)$

26.5 Example

Suppose we flip a coin 3 times. Let $X = \#$ heads. Let

$$
Y = \begin{cases} 1, & \text{if first flip is a H} \\ 0, & \text{otherwise} \end{cases}
$$

Find $f(x, y)$.

$y \backslash x$	0	1	2	3
0	$1/8$	$2/8$	$1/8$	$0/8$
1	$0/8$	$1/8$	$2/8$	$1/8$

$f(x, y)$ can be represented in a table or as a function of x and y (not usually a histogram).

Now suppose we are only interested in one of the random variables. e.g. suppose we are only want to find out about X.

$$
P(X = x) = f(0, 0) + f(0, 1) = \frac{1}{8} + 0 = \frac{1}{8}
$$

26.6 Definition (Marginal Probability Function)

Let X and Y be two discrete random variables. We define the *marginal probability function* of X as

$$f_X(x) = \sum_{\text{all } y} f(x,y)$$

and the *marginal probability function* of Y as

$$f_Y(y) = \sum_{\text{all } x} f(x,y)$$

26.7 Definition (Independent Random Variables)

$X_1, \dots, X_n$ are *independent random variables* if and only if

$$f(x_1, \dots, x_n) = f_1(x_1) \cdots f_n(x_n)$$

for all $(x_1, \dots, x_n)$

From example: Are X and Y independent? No. $f(0,0) = \frac{1}{8} \neq f_X(0)f_Y(0) = \frac{1}{8} \cdot \frac{1}{2}$ Shortcut: any 0 in your table $\rightarrow$ dependent.

27 Lecture 27

27.1 Thought Question

For a full-time UW Math Faculty student, let $X =$ number of courses taking and $Y = 1$ if in co-op, or 0 if in regular. The joint pf is given by (this is real data)

$y \backslash x$	3	4	5	6	$f_Y(y)$
0	0.09	0.17	0.22	0.01	
1	0.05	0.10	0.32	0.04	0.51
$f_X(x)$			0.54		1

Are X and Y independent?

a) Yes, b) No, c) Not enough information

Correct answer is b): No. $f(5,1) = 0.32 \neq f_X(5)f_Y(1) = (0.54)(0.51) = 0.2754$

27.2 Example

Imagine you have a card game with a total of 12 cards. Classified in three different categories: 5 cards (money), 4 cards (action), 3 cards (useless). Draw a hand of them, in this case 3 without replacement, and let $X = \#$ of useless, $Y = \#$ action.

Find the joint pf and range

$y \backslash x$	0	1	2	3	$f_Y(y)$
0	$^{10}/_{220}$	$^{30}/_{220}$	$^{15}/_{220}$	$^{1}/_{220}$	$^{56}/_{220}$
1	$^{40}/_{220}$	$^{60}/_{220}$	$^{12}/_{220}$	0	$^{112}/_{220}$
2	$^{30}/_{220}$	$^{18}/_{220}$	0	0	$^{48}/_{220}$
3	$^{4}/_{220}$	0	0	0	$^{4}/_{220}$
$f_X(x)$	$^{84}/_{220}$	$^{108}/_{220}$	$^{27}/_{220}$	$^{1}/_{220}$	1

Range: $x \in \{0,1,2,3\}$, $y \in \{0,1,2,3\}$ such that $x+y \leq 3$

$f(0,0)$ (no useless, no action)$=P$(all money)

$$\frac{\binom{5}{3}}{\binom{12}{3}} = \frac{10}{220}$$

$f(1,1)$ (1 useless, 1 action)$=P$(one of each type)

$$\frac{\binom{3}{1}\binom{4}{1}\binom{5}{1}}{\binom{12}{3}} = \frac{60}{220}$$

$$f(x,y) = \frac{\binom{3}{x}\binom{4}{y}\binom{5}{3-x-y}}{\binom{12}{3}}$$

Find marginal pfs (sum), $X \sim \text{Hyp}(12,3,3)$. $Y \sim \text{Hyp}(12,4,3)$. Check that the marginal pfs match.

Are they independent? No (don't have a cartesian product)

Recall: conditional probability:

$$P(A \mid B) = \frac{P(AB)}{P(B)}$$

27.3 Definition (Conditional Probability Function)

The *conditional probability function* of X given $Y=y$ is

$$f(x \mid y) = P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)} = \frac{f(x,y)}{f_Y(y)}$$

provided $f_Y(y) > 0$.
Similarly, the *conditional probability function* of Y given $X=x$ is

$$f(y \mid x) = \frac{f(x,y)}{f_X(x)}$$

provided $f_X(x) > 0$.

27.4 Example

What is the probability that someone taking 4 courses is a co-op student?

In other words, $P(Y=1 \mid X=4) = \frac{0.1}{0.27} = 0.37$

For 6 courses, $\frac{0.04}{0.05} = 0.80$.

27.5 Example

If you have 1 action card, find the pf of the number of useless cards.

i.e. the pf of $X \mid Y=1$

x	0	1	2
$f(x \mid 1)$	$40/112$	$60/112$	$12/112$

27.6 Functions of 2 or more random variables

Suppose U is some function of X and Y, e.g. $U = X - Y$. To find the pf of U.

$y\backslash x$	3	4	5	6
0	3	4	5	6
1	2	3	4	5

1. determine the possible values of U for each pair (x, y)

so the range is $u \in \{2, 3, 4, 5, 6\}$

2. $f(u)$ is the sum of $f(x, y)$ for all combos that map to u.

$$f(u) = \sum_{(x,y) \text{ s.t. } x-y=u} f(x, y)$$

u	2	3	4	5	6
$f(u)$	0.05	0.19	0.49	0.24	0.01

$u = 2 \to (x = 3, y = 1) = 0.05$

$u = 3 \to (x = 4, y = 1) + (x = 3, y = 0) = 0.1 + 0.09 = 0.19$

Using the earlier table:

$y\backslash x$	3	4	5	6
0	0.09	0.17	0.22	0.07
1	0.05	0.1	0.32	0.04

28 Lecture 28

28.1 Thought Question

Suppose X = # apple products and Y = # Microsoft products (given at least one of each) have a joint pf:

$y\backslash x$	1	2	3
1	0.30	0.17	0.20
2	0.17	0.10	0.06

Find $P(X + Y = 4)$

a) 0.10, b) 0.20, c) 0.30, d) 0.40, e) none

Correct answer is c): $P(X + Y = 4) = (3, 1) + (2, 2) = 0.20 + 0.10 = 0.30$

28.2 Sums of rvs

Suppose $T = X + Y$, and X, Y are non-negative.

The range of T is $0, 1, \ldots, \max(X) + \max(Y)$ pf of T is

$$\begin{aligned} f_T(t) &= \sum \sum_{x+y=t} f(x, y) \\ &= f(0, t) + f(1, t-1) + f(2, t-2) + \cdots + f(t, 0) \\ &= \sum_{x=0}^{t} f(x, t-x) \end{aligned}$$

If X and Y are independent, then

$$f_T(t) = \sum_{x=0}^{t} f_X(x) f_Y(y)(t-x)$$

This can be used to prove:

- sum of two independent Poission is a Poission rv
- sum of k independent $\mathrm{Geo}(P)$ is $\mathrm{NB}(k, p)$

28.3 Multinomial Distribution (9.2)

An extension of Binomial, where each independent trial can have k possible outcomes.

The probability of type i is p_i which is constant.

$$p_1 + p_2 + \cdots + p_k = 1$$

We do n trials and let $X_i = \#$ of outcome i's that occur.

$$X_1 + X_2 + \cdots + X_k = n$$

where n is the total number of trials.

Then we say $X_1, \ldots, X_k \sim \mathrm{Mult}(n, p_1, p_2, \ldots p_k)$.

Remark 7. X_k can be written as $n - \sum_{i=1}^{k-1} x_i$ and p_k can be written as $1 - \sum_{i=1}^{k-1} p_i$

28.4 Example

Roll a fair 6-sided die 10 times. $X_1 = \#$ 1's $X_2 = \#$ composites (4,6) $X_3 = \#$ primes (2,3,5)

Find range: $X_i \in \{0, \ldots, n\}$ $n = 10$ in this case. So,

$$X_1 + X_2 + X_3 = 10$$

Find joint pf: $f(x_1, x_2, x_3) = P(X_1 1's, X_2 C's, X_3 P's)$. So,

$$\underbrace{\frac{10!}{x_1! x_2! x_3!}}_{\text{arrangements}} \underbrace{\left(\frac{1}{6}\right)^{x_1} \left(\frac{2}{6}\right)^{x_2} \left(\frac{3}{6}\right)^{x_3}}_{\text{outcomes}}$$

In general,

$$f(x_1, \ldots, x_k) = \frac{n!}{x_1! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k}$$

for $x_1 + \cdots + x_k = n$ OR

$$f(x_1, \ldots, x_{k-1}) = \frac{n!}{x_1! \cdots x_{k-1}!} p_1^{x_1} \cdots p_{k-1}^{x_{k-1}}$$

for $x_1 + \cdots + x_{k-1} \leq n$

Find marginal pf of x_1.

$$\begin{aligned}
f_1(x_1) &= \sum_{x_2}^{x_3} f(x_1, x_2, x_3) \\
&= \sum_{x_2=0}^{10-x_1} f(x_1, x_2) \\
&= \sum_{x_2=0}^{10-x_1} \frac{10!}{x_1! x_2! (10 - x_1 - x_2)!} \left(\frac{1}{6}\right)^{x_1} \left(\frac{1}{3}\right)^{x_2} \left(\frac{1}{2}\right)^{(10-x_1-x_2)} \\
&= \frac{10!}{x_1!(10-x_1)!} \left(\frac{1}{6}\right)^{x_1} \sum_{x_2=0}^{10-x_1} \frac{(10-x_1)!}{x_2!(10-x_1-x_2)!} \left(\frac{1}{3}\right)^{x_2} \left(\frac{1}{2}\right)^{(10-x_1-x_2)} \\
&= \binom{10}{x_1} \left(\frac{1}{6}\right)^{x_1} \sum_{x_2=0}^{10-x_1} \binom{10-x_1}{x_2} \left(\frac{1}{3}\right)^{x_2} \left(\frac{1}{2}\right)^{(10-x_1-x_2)} \\
&= \binom{10}{x_1} \left(\frac{1}{6}\right)^{x_1} \left(\frac{1}{3} + \frac{1}{2}\right)^{10-x_1}
\end{aligned}$$

$$f(x_1) = \binom{10}{x_1} \left(\frac{1}{6}\right)^{x_1} \left(\frac{5}{6}\right)^{10-x_1} \quad \sim \text{Bin}(10, 1/6)$$

In general:

$$X_i \sim \text{Bin}(n, p_i)$$

29 Lecture 30

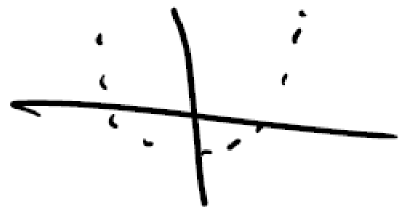
- Steve $\neq$ Diana
- Midterm # M3-3116
- Covariance
- Correlation

If X and Y are independent:

$$\begin{aligned}
E(XY) &= \sum_x \sum_y xyf(x,y) && \text{by defn} \\
&= \sum_x \sum_y xyf_X(x)f_Y(y) \\
&= \sum_x xf_X(x) \sum_y yf_Y(y) && X, Y \text{ indep.} \\
&= E[X]E[Y]
\end{aligned}$$

$\implies Cov(X,Y) = E[XY] - E[X]E[Y]$. Thus, X, Y independent $\implies Cov(X,Y) = 0$, but $Cov(X,Y) = 0$ does not mean X, Y are independent.

Uncorrelated variables could still be dependent. If, for instance there is a non-linear relationship.



~ Cov ≠ 0
clearly a relationship.

Other points about covariance:

- if $Cov > 0$, then $X \uparrow \iff Y \uparrow$ OR $X \downarrow \iff Y \downarrow$.
- if $Cov < 0$, then $X \downarrow \iff Y \uparrow$ OR $X \uparrow \iff Y \downarrow$.
- the magnitude of $Cov(X, Y)$ can't be interpreted. We need to rescale to a restricted range to interpret size.

Correlation

29.1 Definition (Correlation Coefficient)

The *correlation coefficient* ρ_{xy} of X and Y is:

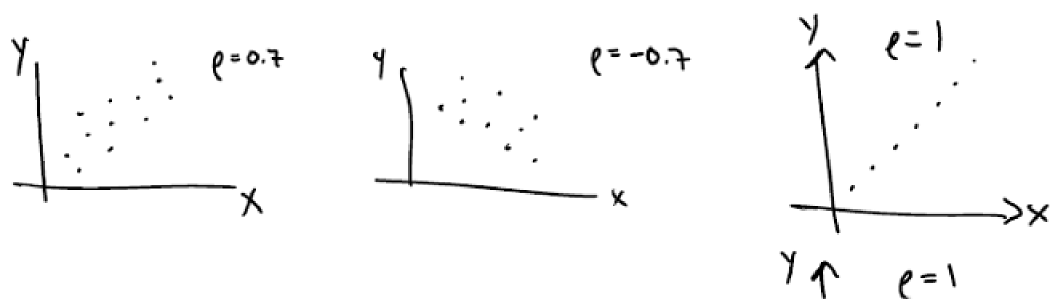
$$Corr(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

$$\rho = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

Notes

- sign of $Corr$ = sign of Cov for any given X, Y.
- $-1 \leq \rho_{xy} \leq 1$ (see course notes)
- only equal to ± 1 if $Y = aX + b$

We interpret the magnitude of the correlation as the strength of the linear relationship.



Important: correlation does not imply causation!

if $\rho = 0.95$, then

$$\begin{cases} X \text{ causes } Y \text{ OR} \\ Y \text{ causes } X \text{ OR} \\ X, Y \text{ are caused by } Z \\ X, Y \text{ are correlated by chance} \end{cases}$$

Example

(from past lecture)

Suppose X = # apple products and Y = # Microsoft products (given at least one of each) have a joint pf:

$y\backslash x$	1	2	3
1	0.30	0.17	0.20
2	0.17	0.10	0.06

$Cov(X, Y) = -0.0407$

$Var(X) = E(X^2) - 1.79^2 = 0.6859$

$Var(Y) = E(Y^2) - 1.33^2 = 0.2211$

$Corr(X, Y) = \frac{-0.0407}{\sqrt{0.6959}\sqrt{0.2211}} = -0.1045$ ... a weak negative correlation.

Example

Roll a fair 6-sided die 10 times. X_1 = # 1's X_2 = # even composites (4,6)

$Cov(X_1, X_2) = 5 - {}^{10}/_6 \times {}^{10}/_3 = -0.556$

$Var(X_1) = 10\,({}^1/_6)\,({}^5/_6) = 1.389\ (npq)$

$Var(X_2) = 10\,({}^1/_3)\,({}^2/_3) = 2.222\ (npq)$

$Corr(X, Y) = \frac{-0.556}{\sqrt{1.389}\sqrt{2.222}} = -0.316$

Next class: Linear Combinations of Random Variables

30 Lecture 31

Suppose two variables X and Y have non-zero covariance. What can we say?

a) X and Y are independent. b) X and Y are not independent. c) we cannot tell if they are independent.

Same question, but for zero covariance.

Today

- Linear Combinations of RVs (9.5 & 9.6), which connects nicely to CLT
- a couple of examples

Friday

- Indicator Variables

Rules of Linear Combinations

$P = \alpha X + (1-\alpha)Y \rightarrow$ two stocks

$S = 0.05A + 0.3M + 0.15Q + 0.5F$

Means

1. $E(aX + bY) = aE(X) + bE(Y)$

2. $E\left(\sum\limits_{i=1}^{n} a_i X_i\right) = \sum\limits_{i=1}^{n} a_i E(X_i) = a_1 E(X_1) + a_2 E(X_2) + \cdots + a_n E(X_n)$

3. $E\left(\sum\limits_{i=1}^{n} \frac{X_i}{n}\right) = \sum\limits_{i=1}^{n} \frac{\mu}{n} = \frac{n}{n}\mu = \mu$

X_i's all have mean μ

$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \implies E(\bar{X}) = \mu$

Variances

1. $Var(aX + bY) = a^2Var(X) + b^2Var(Y) + 2abCov(X, Y)$

2. $Var(\bar{X}) = Var\left(\sum_{i=1}^{n} \frac{X_i}{n}\right) = \sum_{i=1}^{n} \left(\frac{1}{n}\right)^2 Var(X_i) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$

where X_i's are independent. Thus, if we have independent RV's $X_1, \ldots, X_n$ all with μ, σ^2, then $E(\bar{X}) = \mu$, $Var(\bar{X}) = \frac{\sigma^2}{n}$

$\boxed{\sigma/\sqrt{n} \to \text{ std error of the mean}}$

Note:

$$\begin{aligned} Cov(X, X) &= E[XX] - E[X]E[X] \\ &= E[X^2] - (E[X])^2 \\ &= Var(X) \end{aligned}$$

$\implies Corr(X, X) = 1$

Covariances

$Cov(aX + bY, cZ + dW) = acCov(X, Z) + adCov(X, W) + bcCov(Y, Z) + bdCov(Y, W)$

Normal RVs

Claim: If $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, 2, \ldots, n$ are random variables, then

$$\sum_{i=1}^{n} a_i X_i \sim N\left(\sum_{i=1}^{n} a_i\mu_i, \sum_{i=1}^{n} a_i^2\sigma_i^2\right)$$

$$\boxed{X_i \sim N(\mu, \sigma^2) \implies \bar{X} \sim N(\mu, \sigma^2/n)}$$

Example

Weight of a cat $C \sim N(4.1, 1.6^2)$, weight of a dog $D \sim (9.4, 3.6^2)$. Find the probability that a cat weighs more than a dog.

Solution.

$P(C > D) \implies P(C - D > 0) \to C - D \sim N(4.1 - 9.4, 1.6^2 + (-3.6)^2)$

$$\begin{aligned} &= P\left(\frac{C - D - (-5.3)}{\sqrt{15.52}} > \frac{0 - (-5.3)}{\sqrt{15.52}}\right) \\ &= P(Z > 1.35) \\ &= 1 - 0.91149 \\ &= 0.08851 \end{aligned}$$

Example

Heights of cats are $N(24, 1.5^2)$. Find probability a cat has height within 1cm of average.

Solution.

$$
\begin{aligned}
P(23 < X < 25) &= P\left(\frac{23-24}{1.5} < \frac{X-24}{1.5} < \frac{25-24}{1.5}\right) \\
&= P(-0.67 < Z < 0.67) \\
&= 2(0.74857) - 1 \\
&= 0.49714
\end{aligned}
$$

Example

Find the probability the average height of 5 cats is within 1cm of average.

Solution.

$\bar{X} = \sum_{i=1}^{5} X_i \sim N(24, {}^{1.5^2}/_{5})$

$$
\begin{aligned}
P\left(\left|\bar{X} - 24\right| < 1\right) &= P(23 < \bar{X} < 25) \\
&= P\left(\frac{23-24}{1.5/\sqrt{5}} < Z < \frac{25-25}{1.5/\sqrt{5}}\right) \\
&= P(-1.49 < Z < 1.49) \\
&= 0.86378
\end{aligned}
$$

31 Lecture 32

Today

1. Quiz #3 - Nov. 29$^{\text{th}}$, 7-8pm [Sec. 9.2-9.7 except 9.3 (Markov Chains)]
2. Cat exercise
3. Indicator variables

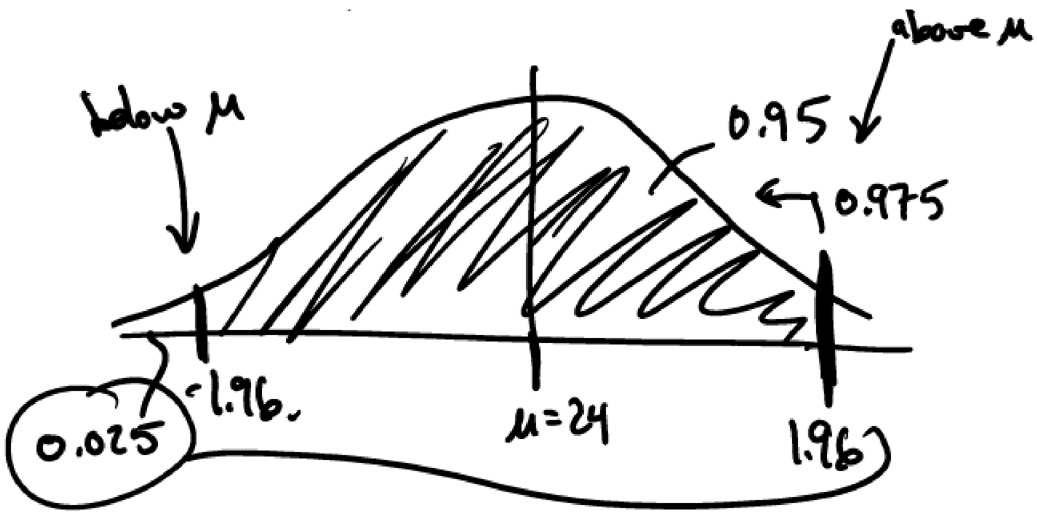
Cat Exercise

How many cats would you need to have for a 0.95 probability that the average height is within 1cm of the true average?

Solution. Let X be the height of the cat.

$$X \sim N(24, 1.5^2)$$

(finding the middle 0.95, tails are 0.025 each; in the table we look for 0.975)



$$P\left(\left|\bar{X}-24\right|<1\right)=0.95$$

$$P(0.975)=1.96$$

$$Z=\pm 1.96 \implies Z=\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \implies 1.96=\frac{25-24}{1.5/\sqrt{n}}$$

$n=8.64=9$

31.1 Indicator Variables (9.7)

A tool you can use to evaluate more complicated distributions.

31.2 Definition (Indicator Variable)

An *indicator variable* (Bernoulli variables)

$$I_A=\begin{cases}1, \text{ if } A \text{ occurs}\\ 0, \text{ if } A \text{ does not occur}\end{cases}$$

$E(I_A)=1P(A)+0(1-P(A))=P(A)$

$E(I_A^2)=1^2P(A)+0^2(1-P(A))=P(A)$

$Var(I_A)=E(I_A^2)-(E(I_A))^2=P(A)-(P(A))^2=P(A)(1-P(A))$

$$I_B=\begin{cases}1, \text{ if } B \text{ occurs}\\ 0, \text{ if } B \text{ does not occur}\end{cases}$$

$Cov(I_A,I_B)$

$I_B \backslash I_A$	0	1
0	0	0
1	0	1

$$I_AI_B=\begin{cases}\left.\begin{matrix}0\\0\\0\end{matrix}\right\}\text{otherwise}\\ 1, \text{ if } A \text{ and } B \text{ occur}\end{cases}$$

$E(A) = P(A)$

$E(B) = P(B)$

$E(I_A I_B) = P(AB)$

$$Cov(I_A, I_B) = P(AB) - P(A)P(B)$$

Remark 8. If A and B are independent, I_A and I_B will be uncorrelated.

1. Let $X \sim \text{Bin}(n, p)$ use indicator variables to find μ and σ^2

Let

$$X_i = \begin{cases} 1, & \text{if trial } i \text{ is a success} \\ 0, & \text{if trial } j \text{ is a failure} \end{cases}$$

then $X = X_1 + X_2 + \cdots + X_n$.

$E(X_i) = p$

$E(X) = E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i) = np = \mu$

$Var(X) = \sum_{i=1}^{n} Var(X_i) = np(1-p) = \sigma^2$

2. $X \sim \text{Hyp}(N, r, n)$; reminder: ($N$ trials, r S's, n selections) Let

$$X_i = \begin{cases} 1, & \text{if object } i \text{ is a success (S)} \\ 0, & \text{if object } j \text{ is a failure (F)} \end{cases}$$

$E(X_i) = P(\text{select object is an S}) = {}^r/_N$ [from $E(I_A) = P(A)$]

$Var(X_i) = {}^r/_N(1 - {}^r/_N)$ [from $Var(I_A) = P(A) - (1 - P(A))$]

$$\begin{aligned} Cov(X_i, X_j) &= P(\text{objects } i \text{ and } j \text{ are S's}) - \left(\frac{r}{N}\right)\left(\frac{r}{N}\right) \\ &= \frac{\binom{r}{2}}{\binom{N}{2}} - \left(\frac{r}{N}\right)^2 \\ &= \frac{r(r-1)}{N(N-1)} - \frac{r^2}{N^2} \\ &= -\frac{r(N-r)}{N^2(N-1)} < 0 \end{aligned}$$

$X = \sum_{i=1}^{n} X_i \implies E(X) = \sum_{i=1}^{n} {}^r/_N = {}^{nr}/_N$

$Var(X) = \sum_{i=1}^{n} Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j)$

where $Var(X)$ comes from properties in 9.5, the first term has n terms, the second term has $\binom{n}{2}$ terms.

$$= \frac{nr}{N}\left(1 - \frac{r}{N}\right) + 2\left(\frac{n(n-1)}{2}\right)\left(-\frac{r(N-r)}{N^2(N-1)}\right) = \frac{nr}{N}\left(1 - \frac{r}{N}\right)\left(\frac{N-n}{N-1}\right)$$

where $\frac{nr}{N}\left(1 - \frac{r}{N}\right)$ is $\text{Bin}(n, {}^r/_N)$ and $\left(\frac{N-n}{N-1}\right)$ reduces variance because sampling without replacement.

<u>Example</u>

N messages come to a server which randomly gives one message to each intended recipient. Find the mean and variance of the # of correctly delivered messages.

Solution.

Let

$$X_i = \begin{cases} 1, \text{ if msg } i \text{ is correct} \\ 0, \text{ otherwise} \end{cases}$$

then $X = \sum_{i=1}^{N} X_i$

$$\left.\begin{aligned} E(X_i) = P(\text{msg } i \text{ is correct}) = \frac{1}{N} \\ Var(X_i) = \frac{1}{N}\left(1 - \frac{1}{N}\right) \end{aligned}\right\} \text{properties of indicator variables}$$

$$\begin{aligned} E(X_iX_j) &= P(i \text{ correct})P(j \text{ correct}|i \text{ correct}) \\ &= \frac{1}{N}\frac{1}{N-1} \\ &= \frac{1}{N(N-1)} \end{aligned}$$

$Cov(X_i, X_j) = \frac{1}{N(N-1)} - \left(\frac{1}{N}\right)^2 = \frac{1}{N^2(N-1)} > 0$

$E(X) = \sum_{i=1}^{N} E(X_i) = \sum_{i=1}^{N} \frac{1}{N} = N\frac{1}{N} = 1$

$Var(X) = 1 \rightarrow$ for Diana