CO 331 - Coding Theory

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Contents

1 2020-01-06

Data source $\to$ Source encoder $\to$ Channel encoder $\overset{\text{channel}}{\to}$ Channel decoder $\to$ Source Decoder $\to$ Data
(digital data) (encoding algorithm) (decoding algorithm)
noise

1.1 Example (Repetition Code)

source message $\to$ codeword	# errors/codeword that can be detected	# errors/codeword that can be corrected	rate
$0 \to 0$ $1 \to 1$	0	0	1
$0 \to 00$ $1 \to 11$	1	0	$1/2$
$0 \to 000$ $1 \to 111$	2	1	$1/3$
$0 \to 00000$ $1 \to 11111$	4	2	$1/5$

Goal of Coding Theory

Design codes so that:

- High information rate
- High error-correcting capability
- Efficient encoding/decoding algorithm

Codes $\subset$ Block codes $\subset$ Linear codes $\subset$ Cyclic codes $\subset$ BCH Codes $\subset$ RS Codes

Requirements for this course:

- MATH 136
- Not required (but required to take the course): MATH 235
- Familiarity with: Groups, Fields, Ideals, Rings (these will be taught)
- Useful, if you have completed these you might be bored: PMATH 336, PMATH 334 [or the advanced equivalents]

The big picture

In its broadest sense, coding deals with the reliable, efficient, and secure transmissions of data over channels that are subject to inadvertent noise and malicious intrusion.

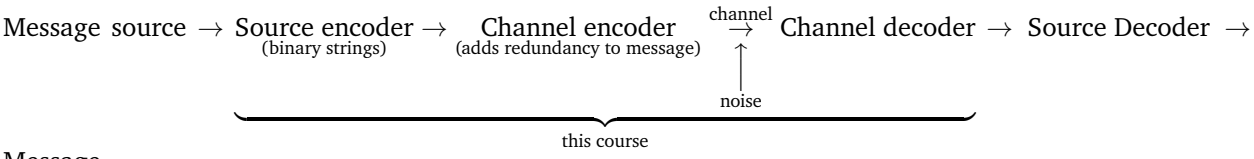
Data source $\to$ Source encoder $\to$ Encryptory authentication $\to$ Channel encoder
(data compression) (cryptography; co 487) (error correction codes)

channel
noise (non-malicious)
adversarial intrusion (malicious)

$\to$ Decriptory verification $\to$ Source decoder $\to$ Data
(data decompression)

2 2020-01-08

2.1 Chapter 1: Introduction and Fundamentals



Message

2.2 Definition (Alphabet)

An *alphabet* A is a finite set of $q \geq 2$ symbols.

Since we will be using the alphabet $A = \{0, 1\}$ very often, we make the following definition.

2.3 Definition (Binary Alphabet)

The alphabet $A = \{0, 1\}$ is a *binary alphabet*.

2.4 Definition (Word, Tuples, Vectors)

A *word* is a finite sequence of symbols from A (*tuples*, or *vectors*). We use the terms *vector* and *word* interchangeably for n-tuple.

2.5 Definition (Length)

The *length* of a word is the number of symbols in it.

2.6 Definition (Code)

A *code* C over A is a finite set of words over A. We define $|C| \geq 2$.

2.7 Definition (codeword)

A *codeword* is a word in C.

2.8 Definition (Block code)

A *block code* is a code where all codewords have the same length. A block code C of length n containing M codewords over A is a subset $C \subseteq A^n$, with $|C| = M$. We refer to such a block code as an $[n, M]$-code over A.

2.9 Example (Block code)

Message $\rightarrow$ Codeword
$00 \rightarrow 00000$
$10 \rightarrow 10110$
$01 \rightarrow 01011$
$11 \rightarrow 11101$

The alphabet is $A = \{0, 1\}$. We have a 5-tuple since the length of each word is $n = 5$. The code is $C = \{00000, 10110, 01011, 11101\}$. Each element in C is a codeword, thus there are a total of 4 codewords.

2.10 Example (Block code)

$A = \{0, 1\}$, $C = \{00000, 11100, 00111, 10101\}$ is a $[5, 4]$-code over $\{0, 1\}$.

Messages $\rightarrow$ codewords
$00 \rightarrow 00000$
$10 \rightarrow 11100$
$01 \rightarrow 00111$
$11 \rightarrow 10101$

This is an n-coding (1-1) map.

- The channel encoder transmits only codewords. But, what's received by the channel decoder might not be a codeword.

2.11 Example

Suppose the channel decoder receives $r = 11001$. What should it do?

2.12 Assumptions about the communications channel

1) Channels only transmit symbols from A
2) No symbols are deleted, added, or transposed
3) (Errors are "random")

2.13 Example (Binary symmetric channel, BSC)

$q = 2$ (0 or 1)

Suppose that the symbols transmitted are $X_1, X_2, X_3, \ldots$, and the symbols received are $Y_1, Y_2, Y_3, \ldots$. Then for all $i \geq 1$, and all $i \leq j$, $k \leq q$,

$$P_r(Y_i = a_j \mid X_i = a_k) = \begin{cases} 1 - p, \text{ if } j = k \\ \frac{p}{q-1}, \text{ if } j \neq k \end{cases}$$

Here, p is the symbol error probability.

2.14 Notes about BSC

(i) if $p = 0$, the channel is perfect

(ii) if $p = 1/2$, the channel is useless

(iii) if $1/2 < p \leq 1$, then simply flip all bits that aren't received

(iv) WLOG, we'll assume that $0 < p < 1/2$

(v) Analogously, for any q-ary channel, we can assume that $0 < p < \frac{q-1}{q}$

2.15 Definition (Hamming distance)

If $x, y \in A^n$, the *Hamming distance*, $d(x, y)$ is the # of coordinate positions in which x and y differ.

2.16 Example (Hamming distance)

The hamming distance of 10111 and 01010 is

$$d(10111, 01010) = 4$$

2.17 Definition (Hamming distance of a code)

Let C be an $[n, M]$-code. The *Hamming distance d of a code C* is

$$d(C) = \min\{d(x, y) : x, y \in C,\ x \neq y\}$$

2.18 Theorem

d is a *metric*. For all $x, y, z \in A^n$:
(i) $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$
(ii) $d(x, y) = d(y, x)$
(iii) ($\triangle$ inequality): $d(x, z) \leq d(x, y) + d(y, z)$

2.19 Definition (Rate)

The *rate* (or *information rate*) of an $[n, M]$-code C over A, is

$$R = \frac{\log_q(M)}{n}$$

where $q = |A|$.
If the source messages are all k-tuples over A, then

$$R = \frac{\log_q(q^k)}{n} = \frac{k}{n}$$

that is, there are q^k source messages.

2.20 Example (Rate)

$A = \{0, 1\}$, $C = \{00000,\ 11100,\ 00111,\ 10101\}$.
We have a $[2, 4]$-code over $\{0, 1\}$.
$R = \frac{2}{5}$, and $d(C) = 2$ since 00111 and 10101 differ by 2 in the first and fourth bit.