Calculus 1 for Honours Mathematics

MATH 137

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Chapter 1

Sequences and Convergence

1.1 Absolute Values

What is an absolute value? We commonly think of it as an operation that removes negative signs.

EXAMPLE 1.1.1

$|-2| = 2$, $|-17| = 17$, $|3| = 3$, etc.

So, is $|-x| = x$ for all $x \in \mathbf{R}$? Not always! Let's give the definition to avoid ambiguity.

DEFINITION 1.1.2

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

Clearly, $|-x| = |x|$.

This also tells us the distance from x to 0, or the magnitude (size of x).

EXAMPLE 1.1.3

How do we get the distance between two arbitrary numbers using absolute values? For example, what is the distance from 3 to 7? 4 units. Also, $|7 - 3| = 4 = |3 - 7|$.

So, the distance from a to b is $|b - a|$ for all $a, b \in \mathbf{R}$. Also, $|b - a| = |a - b|$, which makes sense since the distance from a to b should be the same as the distance from b to a.

1.1.1 Inequalities Involving Absolute Values

The main focus of this course is **approximation**. We will seek ways to approximate roots, curves, limits, etc., but if we make an approximation it will be useless unless we can talk about how close it is to the actual object! So, we will look for ways to determine the maximum size of the **error**. Before we do this, we will need to examine **inequalities**. Let's start with the triangle inequality.

THEOREM 1.1.4: Triangle Inequality

For $x, y, z \in \mathbf{R}$, $|x - y| \leq |x - z| + |z - y|$.

Proof: Since $|x-y| = |y-x|$, we can assume without loss of generality (WLOG) that $x \leq y$. Hence, we consider three cases.
Case 1 ($z < x$): Clearly, $|x-y| \leq |z-y|$, which means $|x-y| \leq |x-z| + |z-y|$.
Case 2 ($x \leq z \leq y$): In this case, $|x-y| = |x-z| + |z-y|$, which means $|x-y| = |x-z| + |z-y|$, as desired.
Case 3 ($y < z$): This time, $|x-y| \leq |x-z|$, so $|x-y| \leq |x-z| + |z-y|$.

We consider a useful variant of the triangle inequality.

COROLLARY 1.1.5

For $x, y \in \mathbf{R}$, $|x+y| \leq |x| + |y|$.

Proof:

$$\begin{aligned} |x+y| &= |x-(-y)| \\ &\leq |x-0| + |0-(-y)| && \text{triangle inequality with } z = 0 \\ &= |x| + |y|. \end{aligned}$$

If we want to prove $|x| < \delta$, we just need to prove $x < \delta$ and $x > -\delta$, that is, $-\delta < x < \delta$. So, what do the inequalities of the form $|x-a| < \delta$ for $a, \delta \in \mathbf{R}$ look like? What set does this represent? Well, it's the set of all $x \in \mathbf{R}$ that are less than δ units away from a. So, starting at a, we move δ-units to the left and right, which means

$$|x-a| < \delta \iff -\delta < x-a < \delta \iff a-\delta < x < a+\delta.$$

So, it is the interval $(a-\delta, a+\delta)$, where we do not include the endpoints as the inequality is strict.

What about $|x-a| \leq \delta$? In this case,

$$|x-a| \leq \delta \iff -\delta \leq x-a \leq \delta \iff a-\delta \leq x \leq a+\delta.$$

So, it is the interval $[a-\delta, a+\delta]$.

What about $0 < |x-a| < \delta$? Now, the distance can't be zero which means $x \neq a$. So, it translates to $(a-\delta, a+\delta) \setminus \{a\}$ or $(a-\delta, a) \cup (a, a+\delta)$.

EXAMPLE 1.1.6

Find the corresponding sets for the inequalities.
(1) $|x-4| < 3$.
(2) $2 \leq |x-4| < 4$.
(3) $|x-1| + |x+2| \geq 4$.

Solution.
(1) $|x-4| < 3 \iff -3 < x-4 < 3 \iff 1 < x < 7$, so $(1, 7)$ is the corresponding interval.
(2) $2 \leq |x-4| < 4$ means $2 \leq |x-4|$ and $|x-4| < 4$, so

$$(2 \leq x-4) \vee (x-4) \leq -2 \iff (6 \leq x) \vee (x \leq 2)$$

and

$$-4 < x-4 < 4 \iff 0 < x < 8.$$

Putting these together, we get $0 < x \leq 2$ or $6 \leq x < 8$, so $(0, 2] \cup [6, 8)$ is the corresponding interval.
(3) We consider three cases.
(i) If $x > 1$, then both $x-1 > 0$ and $x+2 > 0$, then

$$x-1+x+2 > 4 \iff 2x+1 > 4 \iff 2x > 3 \iff x < 3/2.$$

(ii) If $-2 \leq x \leq 1$, then $|x-1| = 1-x$, but $|x+2| = x+2$, so we get

$$1 - x + x + 2 \geq 4 \iff 3 \geq 4,$$

which is not true for any x.

(iii) If $x < -2$, then $|x-1| = 1-x$ and $|x+2| = -x-2$, then

$$1 - x + (-x-2) \geq 4 \iff -1 - 2x \geq 4 \iff -5 \geq 2x \iff -5/2 \geq x.$$

Putting it all together, we get $x \geq 3/2$ or $x \leq -5/2$, that is, $(-\infty, -5/2] \cup (3/2, \infty)$.

1.2 Sequences and Their Limits

DEFINITION 1.2.1

An infinite sequence of numbers is a list of numbers in a definite order, e.g.,

$$a_1, a_2, a_3, a_4, \ldots, a_n,\ a_i \in \mathbf{R}.$$

Notation: $\{a_1, a_2, \ldots, a_n\}$ or $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$.

Sequences can be defined explicitly (in terms of n) or recursively (in terms of previous terms).

EXAMPLE 1.2.2: Explicit Sequences

- $\left\{\frac{1}{n+1}\right\}_{n=1}^{\infty}$: $1/2, 1/3, 1/4, 1/5, \ldots$.
- $\{\sqrt{n+2}\}_{n=2}^{\infty}$: $\sqrt{4}, \sqrt{5}, \sqrt{6}, \ldots$.
- $\{(-1)^n\}_{n=1}^{\infty}$: $-1, 1, -1, 1, \ldots$.

EXAMPLE 1.2.3: Recursive Sequences

- $a_1 = 1$, $a_{n+1} = \sqrt{1 + a_n}$, so $a_1 = 1$, $a_2 = \sqrt{2}$, $a_3 = \sqrt{1+\sqrt{2}}$, and so on for $n \geq 1$.
- Fibonacci's sequence: $a_1 = 1$, $a_2 = 1$, $a_{n+2} = a_{n+1} + a_n$ for $n \geq 1$, i.e., $1, 1, 2, 3, 5, 8, 13, \ldots$.

We can plot sequences on a number line, or we could think of a sequence as a function $f\colon \mathbf{N} \to \mathbf{R}$, writing $f(n) = a_n$, e.g., for $a_n = 1/2$ we would write $f(n) = 1/2$.

Why study sequences?

- Lots of continuous processes can be modelled with discrete data, as we will see.
- We can use recursive sequences to approximate solutions to equations that can't be solved explicitly (Newton's Method).
- For another (ancient) application, see page 14 of the course notes about calculating square roots.

Our goal now will be to determine how to find the limit of a sequence, that is, find what the value of the terms of the sequence are approaching (if it exists).

1.2.1 Sequences and Tails

We may want to build new sequences out of old ones or only discuss what happens to a sequence eventually, that is, after a certain index.

EXAMPLE 1.2.4

For $\{\frac{1}{n}\}_{n=1}^{\infty}$, if we consider only the odd terms, we get $1, 1/3, 1/5$, or the k^{th} term is

$$\frac{1}{2k-1}$$

for $k \in \mathbf{N}$. This is called a subsequence.

DEFINITION 1.2.5: Subsequence

If $\{a_n\}$ is a sequence and $n_1, n_2, \ldots$ is a sequence of natural numbers, where $n_1 < n_2 < n_3 < \cdots$, then the sequence

$$\{a_{n_1}, a_{n_2}, \ldots\} = \{a_{n_k}\}$$

is a subsequence of $\{a_n\}$.

One particular subsequence is $\{a_k, a_{k+1}, a_{k+2}\}$ for some $k \in \mathbf{N}$. This is called the tail of $\{a_n\}$ with cut-off k.

Limits of Sequences

We are going to see lots of different limits this term, but we will start with sequences.

EXAMPLE 1.2.6

$\{\frac{1}{n}\}$ seems like it converges to 0, or that 0 is the limit of the sequence. We saw this when we plotted the sequence. We will eventually want a formal definition, but let's start intuitively.

Given a sequence $\{a_n\}$, what does it mean to say that $\{a_n\}$ converges to L as n goes to infinity?

What about "as n gets larger, a_n gets closer to L?" Unfortunately, this isn't a good definition. For example, as n gets larger $\frac{1}{n}$ gets closer to 0, but it also gets closer to -1, -2, and so on. But, 0 is the limit! What makes it different? Well, the sequence gets infinitely close to 0, unlike the other numbers! Let's try to define this again: "the limit of $\{a_n\}$ is L if, as n gets infinitely large, a_n gets infinitely close to L." This is much better! But how can we formalize the idea of "infinitely close?"

DEFINITION 1.2.7: Formal Definition of a Limit

$L \in R$ is the limit of $\{a_n\}$ if:
For all $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that if $n \geq N$, then

$$|a_n - L| < \varepsilon.$$

If such an L exists, we can say that $\{a_n\}$ converges and write $\lim\limits_{n\to\infty} a_n = L$ (or $a_n \to L$). If no such L exists, then we say that $\{a_n\}$ diverges.

Here, if $n \geq N$, then $|a_n - L| < \varepsilon$ (or $a_n \in (L-\varepsilon, L+\varepsilon)$). Let's take a look at some examples.

EXAMPLE 1.2.8

Consider $a_n = \frac{1}{n^2}$. We'd guess that the limit is 0. Say $\varepsilon = \frac{1}{100}$, can we find a large enough $n \in N$ so that $\left|\frac{1}{n^2} - 0\right| < \frac{1}{100}$ if $n \geq N$? Well, we need

$$\left|\frac{1}{n^2} - 0\right| < \frac{1}{100} \implies \frac{1}{n^2} < \frac{1}{100} \implies n^2 > 100,$$

so $n > 10$. Let $N = 11$, then if $n \geq N$, we get $\left|\frac{1}{n^2} - 0\right| < \frac{1}{100}$. But wait! We aren't done yet! The definition says we need to prove it for any $\varepsilon > 0$, but we only proved it for $\varepsilon = \frac{1}{100}$. Let's adapt the proof to work for any $\varepsilon > 0$.

Proof that $\lim\limits_{n\to\infty} \frac{1}{n^2} = 0$. Let $\varepsilon > 0$ be given. Let $N > \frac{1}{\sqrt{\varepsilon}}$ for $N \in \mathbf{N}$. Then, if $n \geq N$, we get

$$\left|\frac{1}{n^2} - 0\right| = \frac{1}{n^2} \leq \frac{1}{N^2} < \frac{1}{(1/\sqrt{\varepsilon})^2} = \frac{1}{1/\varepsilon} = \varepsilon$$

as desired.

The point is: we have to give a method for choosing N that works for any $\varepsilon > 0$. Also, the logical order of the proof is important, so let's do some more examples.

EXAMPLE 1.2.9

Prove that $\lim\limits_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2}$.

Proof: Let $\varepsilon > 0$ be given. Let $N > \frac{1}{4}\left(\frac{3}{\varepsilon} - 6\right)$ for $N \in \mathbf{N}$. Then, if $n \geq N$, we get:

$$|a_n - L| = \left|\frac{n}{2n+3} - \frac{1}{2}\right| = \frac{3}{4n+6} \leq \frac{3}{4N+6} < \frac{3}{4\left(\frac{1}{4}\left(\frac{3}{\varepsilon} - 6\right)\right) + 6} = \varepsilon$$

as desired.
Aside: We want

$$\frac{3}{4n+6} < \varepsilon \iff \frac{3}{\varepsilon} < 4n+6 \iff \frac{3}{\varepsilon} - 6 < 4n \iff \frac{1}{4}\left(\frac{3}{\varepsilon} - 6\right) < n.$$

EXAMPLE 1.2.10

Prove that $\lim\limits_{n\to\infty} \frac{n^2}{3n^2+7n} = \frac{1}{3}$.

Proof: Let $\varepsilon > 0$ be given. Let $N > \frac{7}{9\varepsilon}$ for $N \in \mathbf{N}$. Then, if $n \geq N$, we get:

$$|a_n - L| = \left|\frac{n^2}{3n^2+7n} - \frac{1}{3}\right| = \frac{7n}{9n^2+21n} \leq \frac{7n}{9n^2} = \frac{7}{9n} \leq \frac{7}{9\left(\frac{7}{9\varepsilon}\right)} = \varepsilon.$$

Aside: We want

$$\frac{7}{9n} < \varepsilon \iff \frac{7}{9\varepsilon} < n.$$

REMARK 1.2.11: Avoid Common Mistakes

- Don't choose ε! Let it be arbitrary.
- Never assume $|a_n - L| < \varepsilon$, make sure you only do work in an aside with that inequality since it is what you are proving.
- In practice, unless you are asked to, do not use this formal definition. We will now try to develop better methods for finding limits.

1.2.2 Equivalent Definitions of the Limit

When proving $\lim\limits_{n\to\infty} a_n = L$, we are given $\varepsilon > 0$, and we try to find $N \in \mathbf{N}$ so that if $n \geq N$, then $|a_n - L| < \varepsilon$. But, this is the same as saying $a_n \in (L - \varepsilon, L + \varepsilon)$. Also, the collection of $\{a_n\}$ for which $n \geq N$ is the tail of the sequence with cut-off N. So, here's another definition.

DEFINITION 1.2.12

$\lim_{n\to\infty} a_n = L$ if for any $\varepsilon > 0$, the interval $(L-\varepsilon, L+\varepsilon)$ contains a tail of the sequence $\{a_n\}$.

Let's push it further! Since the above is true for any $\varepsilon > 0$, if we pick any open interval (a, b) containing L, then we can find a small enough $\varepsilon > 0$ so that $(L-\varepsilon, L+\varepsilon) \subseteq (a, b)$. Therefore, any interval containing L also contains a tail of $\{a_n\}$. Let's collect all of these alternate (but equivalent) definitions together.

THEOREM 1.2.13

The following are equivalent:

(1) $\lim_{n\to\infty} a_n = L$.
(2) For any $\varepsilon > 0$, $(L-\varepsilon, L+\varepsilon)$ *contains a tail of* $\{a_n\}$.
(3) For any $\varepsilon > 0$, $(L-\varepsilon, L+\varepsilon)$ *contains all but finitely many terms of* $\{a_n\}$.
(4) Every interval (a, b) *containing* L *contains a tail of* $\{a_n\}$.
(5) Every interval (a, b) *containing* L *contains all but finitely many terms of* $\{a_n\}$.

Clearly, changing finitely many terms of $\{a_n\}$ *does not affect the convergence or the limit.*

EXAMPLE 1.2.14

Can a sequence have more than one limit? Consider $\{(-1)^n\} = -1, 1, -1, 1, \ldots$, it equals to both 1 and -1 infinitely often. Could both 1 and -1 be the limits? No! Let's prove -1 isn't a limit.

Proof: Consider the interval $(-2, 0)$. Clearly $-1 \in (-2, 0)$, but this interval does not contain any of the infinitely many 1's in the sequence. So, -1 is not a limit by (5) above. A similar argument can be used with the interval $(0, 2)$ to show 1 is also not a limit. So, does $\{(-1)^n\}$ have a limit at all? Let's prove it doesn't! Let $\varepsilon = 1/2$, and supposed for a contradiction that the sequence converges to $L \in \mathbf{R}$. That means the interval $(L - 1/2, L + 1/2)$ must contain all but finitely many terms of the sequence, that is, but 1 and -1 must lie in that interval. But the interval is only 1 unit long! So there is not $L \in \mathbf{R}$ for which both 1 and -1 lie inside $(L - 1/2, L + 1/2)$. So, $\{(-1)^n\}$ diverges.

A similar argument can be used to prove limits are unique.

THEOREM 1.2.15

Let $\{a_n\}$ *be a sequence. If it has a limit* L*, then the limit is unique.*

Proof: Suppose for a contradiction that L and M are both limits of $\{a_n\}$ and $L \neq M$ and WLOG that $L < M$. Consider two intervals:

$$(L-1, \tfrac{L+M}{2}) \ni L, \quad (\tfrac{L+M}{2}, M+1) \ni M.$$

This means, by definition, only finitely many terms of the sequence are not in the first interval and only finitely many terms are not in the second interval. But the sequence has infinitely many terms! So, at least one term is in both intervals which is impossible. This is a contradiction, so $L = M$.

REMARK 1.2.16: A Remark on Possible Limits

If $a_n \geq 0$ for all n, then $\{a_n\}$ can't converge to a negative number! If it did, say to $L < 0$, then the interval $(L-1, 0)$ would contain L but no terms of the sequence.

THEOREM 1.2.17

If $a_n \geq 0$ for all n and $\lim_{n\to\infty} a_n = L$, then $L \geq 0$. More generally, if $\alpha \leq a_n \leq \beta$ for all n and $\lim_{n\to\infty} a_n = L$, then $\alpha \leq L \leq \beta$.

- Q: If $a_n > 0$ for all n and $\lim_{n\to\infty} a_n = L$ is $L > 0$?
- A: Not necessarily! Consider $a_n = \frac{1}{n} > 0$, but $L = 0$.

1.2.3 Divergence to Infinity

Consider $a_n = n$. It is clear that the sequence is getting larger without bound, so $\lim_{n\to\infty} a_n$ does not exist. That is, $\{a_n\}$ diverges. But we can say more! Since it does not get infinitely large, we can make a definition to capture this.

DEFINITION 1.2.18

$\lim_{n\to\infty} a_n = \infty$ if for all $M > 0$, we can find $N \in \mathbf{N}$ so that if $n \geq N$, then $a_n > M$.
Equivalently, any interval of the form (M, ∞) contains a tail of $\{a_n\}$.

It does look strange to write "$= \infty$" but with the above definition we know it means "does not exist but gets infinitely large."
Similarly, if $\lim_{n\to\infty} a_n = -\infty$ for all $M < 0$, there is $N \in \mathbf{N}$ so that if $n \geq N$, then $a_n < M$.

EXAMPLE 1.2.19

Show $\lim_{n\to\infty} (1-n) = -\infty$.

Proof: Let $M < 0$ be given, pick $N > 1 - M$ for $N \in \mathbf{N}$. Then, if $n \geq N$, we have

$$a_n = 1 - n \leq 1 - N < 1 - (1 - M) = M.$$

Aside: Want $1 - n < M \iff 1 - M < n$.

1.2.4 Arithmetic For Limits

If we can avoid using the definition to find a limit, we should. There are certain rules we can follow to compute lots of sequence limits. Let's see them now!

THEOREM 1.2.20: Arithmetic Rules for Limits

Let $\{a_n\}$ and $\{b_n\}$ be sequences. Say $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$. Then:

(1) $\forall c \in \mathbf{R},\ a_n = C \forall n \implies C = L$.
(2) $\forall c \in \mathbf{R},\ \lim_{n\to\infty} ca_n = cL$.
(3) $\lim_{n\to\infty} (a_n + b_n) = L + M$.
(4) $\lim_{n\to\infty} a_n b_n = LM$.
(5) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ if $M \neq 0$.
(6) $a_n \geq 0 \forall n \wedge \alpha > 0 \implies \lim_{n\to\infty} a_n^\alpha = L^\alpha$.
(7) $\forall k \in \mathbf{N},\ \lim_{n\to\infty} a_{n+k} = L$.
(8) $\alpha > 0 \implies \lim_{n\to\infty} n^\alpha = \infty$.

(9) $\alpha < 0 \implies \lim_{n\to\infty} n^\alpha = 0.$

Proof: Exercises, but let's prove (3) as an example. Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} a_n = L$, we can find $N_1 \in \mathbf{N}$ so that if $n \geq N_1$, we get $|a_n - L| < \varepsilon/2$. Also, since $\lim_{n\to\infty} b_n = M$, we can find $N_2 \in \mathbf{N}$ so that if $n \geq N_2$, we have $|b_n - M| < \varepsilon/2$. Now, let $N = \max N_1, N_2$. Then, if $n \geq N$ we get

$$|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_n - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

where we used the triangle inequality in the first inequality.

REMARK 1.2.21

To use any of the above properties, the limits need to exist!

EXAMPLE 1.2.22

(1) $\lim_{n\to\infty} \frac{3n+7}{n+2} = \lim_{n\to\infty} \frac{3+7/n}{1+2/n} = \frac{\lim_{n\to\infty} 3 + \lim_{n\to\infty} 7/n}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} 2/n} = \frac{3+0}{1+0} = 3.$

(2) $\lim_{n\to\infty} \frac{n^3+n^2+1}{2n^3+7n^2-1} = \lim_{n\to\infty} \frac{1+1/n+1/n^3}{2+7/n-1/n^3} = \frac{1+0+0}{2+0+0} = \frac{1}{2}.$

(3) $\lim_{n\to\infty} \frac{n+1}{n^2+1} = \lim_{n\to\infty} \frac{1/n+1/n^2}{1+1/n^2} = \frac{0+0}{1+0} = 0.$

REMARK 1.2.23

You don't need to write "arithmetic rules" every time, as we <u>always</u> use them! Just make sure you show your work!

EXAMPLE 1.2.24

What if in property (5), $M = 0$? Anything can happen!

- $\lim_{n\to\infty} \frac{1/n}{1/n} = 1$ even though $1/n \to 0$.
- $\lim_{n\to\infty} \frac{1/n}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{n} = \lim_{n\to\infty} n = \infty.$
- $\lim_{n\to\infty} \frac{1/n^2}{1/n} = \lim_{n\to\infty} \frac{1}{n} = 0.$

Hence, we will need to handle these on an individual basis.

However, there is one thing we can say.

THEOREM 1.2.25

If $\lim_{n\to\infty} b_n = 0$ *and* $\lim_{n\to\infty} \frac{a_n}{b_n}$ *exists, then* $\lim_{n\to\infty} a_n = 0$.

Proof: Suppose $\lim_{n\to\infty} b_n = 0$, and say $\lim_{n\to\infty} \frac{a_n}{b_n} = k \in \mathbf{R}$, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{a_n}{b_n} = k \cdot 0 = 0.$$

COROLLARY 1.2.26

If $\lim_{n\to\infty} b_n = 0$ *and* $\lim_{n\to\infty} a_n \neq 0$*, then* $\lim_{n\to\infty} \frac{a_n}{b_n}$ *does not exist.*

EXAMPLE 1.2.27

$$\lim_{n\to\infty} \frac{n^3+3n}{n^2+1} = \lim_{n\to\infty} \frac{1+3/n^2}{1/n+1/n^3}.$$

However, the numerator converges to 1, while the denominator converges to 0. Therefore, the limit does not exist.
We could also say

$$\lim_{n\to\infty} \frac{n^3+3n}{n^2+1} = \infty,$$

which means DNE and infinitely large!

Let's compute the limit of any ratios of powers of n.

PROPOSITION 1.2.28

$$\begin{aligned}\lim_{n\to\infty} \frac{b_0+b_1n+b_2n^2+\cdots+b_jn^j}{c_0+c_1n+c_2n^2+\cdots+c_kn^k} &= \lim_{n\to\infty} \frac{n^j}{n^k}\left[\frac{\frac{b_0}{n^j}+\frac{b_1}{n^{j-1}}+\cdots+b_j}{\frac{c_0}{n^k}+\frac{c_1}{n^{k-1}}+\cdots+c_k}\right] \\ &= \begin{cases} \frac{b_j}{c_k}, & j=k, \\ 0, & j<k, \\ \infty, & j>k \wedge b_j/c_k>0, \\ -\infty, & j>k \wedge b_k/c_k<0. \end{cases}\end{aligned}$$

EXAMPLE 1.2.29

- $$\lim_{n\to\infty} \frac{3n+2}{2n-1} = \frac{3}{2}.$$
- $$\lim_{n\to\infty} \frac{4n^2+5n}{n^3-1} = 0.$$
- $$\lim_{n\to\infty} \frac{7-n^4}{1+n^3} = -\infty.$$

REMARK 1.2.30

Still show work when writing solutions on a test though (e.g., dividing by highest power of n).

EXAMPLE 1.2.31

If we have something that "looks like" $\infty - \infty$, then multiply by the conjugate!

$$\begin{aligned}
\lim_{n\to\infty} \sqrt{n^2-4} - n &= \lim_{n\to\infty} \sqrt{n^2-4} - n\frac{\sqrt{n^2+4}+n}{\sqrt{n^2+4}+n} \\
&= \lim_{n\to\infty} \frac{n^2+4-n^2}{\sqrt{n^2+4}+n} \\
&= \lim_{n\to\infty} \frac{4}{\sqrt{n^2+4}+n} \\
&= \lim_{n\to\infty} \frac{4/n}{\sqrt{1+4/n^2}+1} \\
&= \frac{0}{2} \\
&= 0.
\end{aligned}$$

Recursive Sequence Limits

We will examine recursive sequences more closely in 1.4, but for now, if we know a recursive sequence converges, then we can use rule (7) to find the limit!

EXAMPLE 1.2.32

$a_1 = 2$, $a_{n+1} = \frac{5+a_n}{2}$. Suppose we know it has a limit, say $\lim\limits_{n\to\infty} a_n = L$. Then, using rule (7), we get:

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{5+a_n}{2} = \frac{5+L}{2}.$$

Therefore,

$$L = \frac{5+L}{2} \iff 2L = 5+L \iff L = 5.$$

THEOREM 1.2.33: Squeeze Theorem

If $a_n \le b_n \le c_n$ and $\lim\limits_{n\to\infty} a_n = L = \lim\limits_{n\to\infty} c_n$, then $\lim\limits_{n\to\infty} b_n = L$.

Proof: Let $\varepsilon > 0$ be given. Since $a_n \to L$ and $c_n \to L$, we can find $N \in \mathbf{N}$ such that if $n \ge N$, then $a_n \in (L-\varepsilon, L+\varepsilon)$, and $c_n \in (L-\varepsilon, L+\varepsilon)$. Then, for $n \ge N$,

$$L - \varepsilon \le a_n \le b_n \le c_n \le L + \varepsilon,$$

so $b_n \in (L-\varepsilon, L+\varepsilon)$, which means $\lim\limits_{n\to\infty} b_n = L$.

REMARK 1.2.34

The Squeeze Theorem is great for dealing with sin / cos and $(-1)^n$.

EXAMPLE 1.2.35

Compute the following limits.

(1) $\lim\limits_{n\to\infty} \frac{(-1)^n}{n^2+1}$.

(2) $\lim_{n\to\infty} \frac{\cos(n^2+7)+7}{n}$.

Solution.

(1) Notice that $\frac{-1}{n^2+1} \leq \frac{(-1)^n}{n^2+1} \leq \frac{1}{n^2+1}$ and $\lim_{n\to\infty} \frac{-1}{n^2+1} = 0 = \lim_{n\to\infty} \frac{1}{n^2+1}$, so $\lim_{n\to\infty} \frac{(-1)^n}{n^2+1} = 0$ by the Squeeze Theorem.

(2) Notice that $\frac{6}{n} \leq \frac{\cos(n^2+7)+7}{n} \leq \frac{8}{n}$, and since $\lim_{n\to\infty} \frac{6}{n} = 0 = \lim_{n\to\infty} \frac{8}{n}$, we get $\lim_{n\to\infty} \frac{\cos(n^2+7)+7}{n} = 0$ by the Squeeze Theorem.

1.3 Monotone Convergence Theorem

First, we need to some terminology.

DEFINITION 1.3.1

Let $S \subseteq \mathbf{R}$. We say that α is an **upper bound** of S if $x \leq \alpha$ for all $x \in S$. We call such a set **bounded above**.
Similarly, β is a **lower bound** if $x \leq \beta$ for all $x \in S$. In this case, S is **bounded below**.
We call S **bounded** if it is bounded both above and below. In this case, we could find $M \in \mathbf{R}$ such that $S \subseteq [-M, M]$.

EXAMPLE 1.3.2

If $S = (-1, 1)$, then 7 is an upper bound and -12 is a lower bound, so S is bounded. Another example is $S \subseteq [-5, 5]$.

DEFINITION 1.3.3

Let $S \subseteq \mathbf{R}$. α is called the **least upper bound** of S if:
(i) α is an upper bound, and
(ii) α is the smallest, that is, if α' is another upper bound, then $\alpha' \geq \alpha$.
Denote this by $\alpha = \text{lub}(S)$ or $\alpha = \sup(S)$.
Similarly, β is the **greatest lower bound** if
(i) β is a lower bound, and
(ii) β is the largest, that is, if β' is another lower bound, then $\beta' \leq \beta$.
Denote this by $\beta = \text{glb}(S)$ or $\beta = \inf(S)$.

EXAMPLE 1.3.4

If $S = (-1, 1)$, then $\inf(S) = -1$ and $\sup(S) = 1$.

REMARK 1.3.5

The $\inf(S)$ and $\sup(S)$ may or may not be in S. One of the properties (axioms) of $\mathbf{R}$ guarantees the existence of inf and sup. If $S \subseteq \mathbf{R}$ is non-empty and bounded above (below), then S has sup (inf).

DEFINITION 1.3.6

We say that a sequence $\{a_n\}$ is:
- **increasing** if $a_n < a_{n+1}$,
- **non-decreasing** if $a_n \leq a_{n+1}$,
- **decreasing** if $a_n > a_{n+1}$,

- **non-increasing** if $a_n \geq a_{n+1}$,
- **monotonic** if $\{a_n\}$ is either non-decreasing or non-increasing.

Now, we can state the theorem!

THEOREM 1.3.7: Monotone Convergence Theorem (MCT)

Let $\{a_n\}$ be a non-decreasing (non-increasing) sequence.
(1) If $\{a_n\}$ is bounded above (below), then $\{a_n\}$ converges to $L = lub(\{a_n\})$ ($L = glb(\{a_n\})$).
(2) If $\{a_n\}$ is not bounded above (below), then $\{a_n\}$ diverges to ∞ ($-\infty$).

Proof: We will prove the non-decreasing/bounded above case, the other case is similar. Suppose $\{a_n\}$ is non-decreasing.

(1) Suppose $\{a_n\}$ is bounded above and let $L = \text{lub}(\{a_n\})$. Let $\varepsilon > 0$ be given. Then, $L - \varepsilon < L$, which means that $L - \varepsilon$ is not an upper bound of $\{a_n\}$ (L is the least upper bound). So, there exists $N \in \mathbf{N}$ so that $L - \varepsilon < a_N$. Then, if $n \geq N$, we have $L - \varepsilon < a_N \leq a_n$ since the sequence is non-decreasing. Therefore, for $n \geq N$, $L - \varepsilon < a_n \leq L < L + \varepsilon$, so the tail of $\{a_n\}$ is in $(L - \varepsilon, L + \varepsilon)$, which means $\lim_{n\to\infty} a_n = L$.

(2) Suppose $\{a_n\}$ is not bounded above. Let $M \in \mathbf{R}$ be given. We can find $N \in \mathbf{N}$ so that $M < a_N$. Then, if $n \geq N$, we have $M < a_N < a_n$ ($\{a_n\}$ is non-decreasing). This shows $\lim_{n\to\infty} a_n = \infty$.

Introduction to Mathematical Induction

Before we can use the MCT, we need to develop one proof technique: Mathematical Induction (MATH 135 will explore it further). Induction is a proof technique that allows us to prove an infinite number of statements. Say we have statements $P_1, P_2, P_3, \ldots, P_n, \ldots$ for $n \in \mathbf{N}$. If we can:

(1) Prove P_1 is true (base case).

(2) Prove: if P_k is true for some k (inductive hypothesis), then P_{k+1} is true (inductive step).

Then, we can conclude that P_n is true for all $n \in \mathbf{N}$. Think of dominoes!

We will use the MCT and induction to find the limits of recursive sequences. To do this, we follow these steps:

(1) Prove the sequence is monotonic.

(2) Prove the sequence is bounded (above or below).

(3) Conclude the sequence converges by MCT.

(4) Find the limit using an earlier trick:
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}.$$

Note that the order matters! We can't perform step 4 unless we know that the sequence converges.

EXAMPLE 1.3.8

Let $a_1 = 1$, $a_{n+1} = \frac{3+a_n}{2}$ for $n \geq 1$. Prove the sequence converges and find its limit.

Solution.

(1) Let's check a few terms: $a_1 = 1$, $a_2 = 2$, $a_3 = 5/2$, so it looks like the sequence is non-decreasing. Claim: $a_n \leq a_{n+1}$ for all $n \in N$.
   - Base Case: Is $a_1 \leq a_2$? Yes, since $a_1 = 1 \leq 2 = a_2$.
   - Inductive Hypothesis: Suppose $a_k \leq a_{k+1}$ for some $k \geq 1$.

- Inductive Step: Since $a_k \leq a_{k+1}$, $3 + a_k \leq 3 + a_{k+1}$, which means

$$\frac{3+a_k}{2} \leq \frac{3+a_{k+1}}{2},$$

that is, $a_{k+1} \leq a_{k+2}$.

Therefore, the sequence is non-decreasing by induction.

(2) What upper bound should we use? Don't try to guess the lub at this point, any upper bound will do!

Claim: $a_n \leq 5$ for all $n \in \mathbf{N}$.

- Base Case: $a_1 = 1 \leq 5$.
- Inductive Hypothesis: Suppose $a_k \leq 5$ for some $k \in \mathbf{N}$.
- Inductive Step: Since $a_k \leq 5$, $3 + a_k \leq 8$, so $\frac{3+a_k}{2} \leq 4$. Therefore, $a_{k+1} \leq 4 \leq 5$.

Therefore, $a_n \leq 5$ for all $n \in \mathbf{N}$ by induction, so the sequence is bounded above.

(3) Since $\{a_n\}$ is bounded above and non-decreasing, we know $\{a_n\}$ converges by MCT.

(4) Now, we know a limit exists, say $L = \lim\limits_{n\to\infty} a_n$. Then,

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{3+a_n}{2} = \frac{3+L}{2}.$$

So,

$$L = \frac{3+L}{2} \iff 2L + 3L \iff L = 3.$$

Therefore, $\lim\limits_{n\to\infty} a_n = 3$.

EXAMPLE 1.3.9

Let $a_1 = 2$, $a_{n+1} = 7 + a_n$ for $n \geq 1$. Prove the sequence converges and find its limit.

Solution. Let's check a few terms: $a_1 = 2$, $a_2 = 3$, $a_3 = \sqrt{10}$, so it looks like the sequence is non-decreasing. Let's prove bounded above and non-decreasing in one step!

(1) Claim: $a_n \leq a_{n+1} \leq 9$ for all $n \in \mathbf{N}$.

- Base Case: $a_1 = 2 \leq 3 = a_2$ and $a_2 = 3 \leq 9$, so $a_1 \leq a_2 \leq 9$.
- Inductive Hypothesis: Assume $a_k \leq a_{k+1} \leq 9$ for some $k \in \mathbf{N}$.
- Inductive Step: Then,

$$\begin{aligned} &a_k \leq a_{k+1} \leq 9 \\ &\implies 7 + a_k \leq \sqrt{7 + a_{k+1}} \leq 4 \leq 9 \\ &\implies a_{k+1} \leq a_{k+2} \leq 9. \end{aligned}$$

So, by induction $a_n \leq a_{n+1} \leq 9$ for all $n \in \mathbf{N}$.

(2) The sequence converges by the MCT.

(3) Finally, we need to find the limit. Say $L = \lim\limits_{n\to\infty} a_n$. Then,

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{7 + a_n} = \sqrt{7 + L};$$

see A2Q6 for the last equality. So,

$$L = \sqrt{7+L} \implies L^2 = 7 + L \implies L^2 - L - 7 = 0 \implies L = \frac{1 \pm \sqrt{29}}{2}.$$

However, we know $L = \mathrm{lub}(\{a_n\})$ and $a_1 = 2$. So, $L \neq \frac{1-\sqrt{29}}{2}$ since $\frac{1-\sqrt{29}}{2} < 2$, that is, it isn't even an upper bound. Hence,

$$L = \frac{1+\sqrt{29}}{2}.$$

1.4 The Derivative Function

DEFINITION 1.4.1

We say that f is **differentiable** on an interval I if $f'(a)$ exists for each $a \in I$. In this case, we define the **derivative function** as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \ x \in I.$$

Alternative (Leibniz) notation:

$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(f),$$

where "$\frac{\mathrm{d}}{\mathrm{d}x}$" is called a **differential operator**.

If $y = f(x)$, write $\frac{\mathrm{d}y}{\mathrm{d}x}$. For $f'(a)$, write $\frac{\mathrm{d}f}{\mathrm{d}x}\Big|_{x=a}$.

Let's look at some examples!

EXAMPLE 1.4.2

For $f(x) = 7$, find $f'(x)$ for $x \in \mathbf{R}$.

Solution.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{7-7}{h} = 0.$$

Therefore, $f'(x) = 0$ for all $x \in \mathbf{R}$.

EXAMPLE 1.4.3

Find the equation of the tangent line to $f(x) = x^2 + 3x + 2$ at $x = 2$.

Solution. First, what is the y-coordinate?

$$f(2) = 2^2 + 3(2) + 2 = 12,$$

so the point is $(2, 12)$. Next,

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 + 3(x+h) + 2 - x^2 - 3x - 2}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 3x + 3h - x^2 - 3x}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2 + 3h}{h} \\ &= \lim_{h \to 0} (2x + h + 3) \\ &= 2x + 3. \end{aligned}$$

At $x = 2$, $\frac{\mathrm{d}f}{\mathrm{d}x}\Big|_{x=2} = 2(2) + 3 = 7$. Therefore, $y = 7x - 2$. What about at $x = 3$?

$$f(3) = 3^2 + 3(3) + 2 = 20$$

and $f'(3) = 2(3) + 3 = 9$, so $y - 20 = 9(x - 3) \implies y = 9x - 7$.

REMARK 1.4.4

- Much faster than computing $f'(a)$ each time!
- We will soon learn ways to find $f'(x)$ much faster, but if asked to use the definition, then you must use the formula

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

EXAMPLE 1.4.5

Using the definition, find $f'(x)$ where

(1) $f(x) = x$;
(2) $f(x) = x^2$;
(3) $f(x) = x^3$;
(4) $f(x) = \sqrt{x}$.

Solution.

(1)
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{x+h-x}{h} \\ &= \lim_{h\to 0} \frac{h}{h} \\ &= 1. \end{aligned}$$

(2)
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2-x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h} \\ &= \lim_{h\to 0} \frac{2xh+h^2}{h} \\ &= \lim_{h\to 0} (2x+h) \\ &= 2x. \end{aligned}$$

(3)
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^3-x^3}{h} \\ &= \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} \\ &= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h} \\ &= \lim_{h\to 0} (3x^2+3xh+h^2) \\ &= 3x^2. \end{aligned}$$

(4) $$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\ &= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}}. \end{aligned}$$

1.4.1 Higher-Order Derivatives

DEFINITION 1.4.6

If f is differentiable with derivative f' and f' is also differentiable, then we call $\frac{\mathrm{d}}{\mathrm{d}x}(f')$ the **second derivative** of f, denoted $f''(x)$ or $f^{(2)}(x)$, or $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$.
In general, $f^{(n+1)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}(f^{(n)}(x))$, where $f^{(n)}$ is the n^{th} derivative.

EXERCISE 1.4.7

Prove the following with the limit definition, where $f(x) = x^4$.

- $f'(x) = 4x^3$.
- $f''(x) = 12x^2$.
- $f'''(x) = 24x$.
- $f^{(4)} = 24$.
- $f^{(5)} = 0$.

Note that using the limit definition is very inefficient (not to mention awful and ugly). So, let's develop some rules to help us calculate derivatives more quickly!

1.4.2 Derivatives of Elementary Functions

Now that we know the definition of the derivative, let's work on finding derivatives of elementary functions to speed up the process.

- **Constants**: If $f(x) = c$ where $c \in \mathbf{R}$, then $f'(x) = 0$.
- **Lines**: If $f(x) = mx + b$ where $m, b \in \mathbf{R}$, then $f'(x) = m$.
- **Quadratics**: If $f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbf{R}$ and $a \neq 0$, then $f'(x) = 2ax + b$.
- **Sine and Cosine**: First, we need to prove a different claim.

Claim: $\lim\limits_{x\to 0} \dfrac{\cos(x)-1}{x} = 0$.

$$\begin{aligned}
\lim_{x\to 0} \frac{\cos(x)-1}{x} \cdot \frac{\cos(x)+1}{\cos(x)+1} &= \lim_{x\to 0} \frac{\cos^2(x)-1}{x(\cos(x)+1)} \\
&= \lim_{x\to 0} \frac{-\sin^2(x)}{x(\cos(x+1))} \\
&= \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \frac{-\sin(x)}{\cos(x)+1} \\
&= 1 \cdot 0 \\
&= 0,
\end{aligned}$$

using the fundamental trigonometry limit. Now, we can compute $(\sin(x))'$.

$$\begin{aligned}
(\sin(x))' &= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(h)}{h}\cos(x) + \left(\frac{\cos(h)-1}{h}\right)\sin(x) \\
&= 1\cdot\cos(x) + 0\cdot\sin(x) \\
&= \cos(x).
\end{aligned}$$

EXERCISE 1.4.8

Show that $(\cos(x))' = -\sin(x)$.

- e^x: First, what is the number e? There are lots of ways to define it, for example: $\lim\limits_{x\to\infty}(1+\frac{1}{x})^x = e$ or $\sum_{n=0}^{\infty}\frac{1}{n!} = e$. But for us, we will define e to be the unique number $a \in \mathbf{R}$ such that the tangent line to a^x has slope 1 at $x = 0$. That is,

$$\lim_{h\to 0}\frac{e^h - e^0}{h} = 1 \implies \lim_{h\to 0}\frac{e^h-1}{h} = 1.$$

So, we get $(e^x)' = \lim\limits_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim\limits_{h\to 0} e^x(\frac{e^h-1}{h}) = e^x$. So, $(e^x)' = e^x$.

1.4.3 Arithmetic Rules for Differentiation

Now that we know how to find the derivatives of certain basic functions, let us look at some rules that tell us how to differentiate combinations of functions.

THEOREM 1.4.9: Arithmetic Rules for Differentiation

Suppose f and g are differentiable at $x = a$.

(1) ***Constant Multiple Rule****. Let $h(x) = cf(x)$. Then h is differentiable at $x = a$ and*

$$h'(a) = cf'(a).$$

(2) ***Sum Rule****. Let $h(x) = f(x) + g(x)$. Then h is differentiable at $x = a$ and*

$$h'(a) = f'(a) + g'(a).$$

(3) ***Product Rule****. Let $h(x) = f(x)g(x)$. Then h is differentiable at $x = a$ and*

$$h'(a) = f'(a)g(a) + f(a)g'(a).$$

(4) ***Reciprocal Rule****. Let $h(x) = \frac{1}{g(x)}$. If $g(a) \neq 0$, then h is differentiable at $x = a$ and*

$$h'(a) = -\frac{g'(a)}{[g(a)]^2}.$$

(5) ***Quotient Rule****: Let $h(x) = \frac{f(x)}{g(x)}$. If $g(a) \neq 0$, then h is differentiable at $x = a$ and*

$$h'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}.$$

Proof:

(1) Easy exercise.

(2) Easy exercise.

(3)
$$\begin{aligned}(fg)'(a) &= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h} \\ &= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a+h)g(a) + f(a+h)g(a) - f(a)g(a)}{h} \\ &= \lim_{h\to 0} f(a+h)\frac{g(a+h) - g(a)}{h} + g(a)\frac{f(a+h) - f(a)}{h} \\ &= f(a)g'(a) + g(a)f'(a).\end{aligned}$$

(4)
$$\begin{aligned}\left(\frac{1}{f}\right)'(a) &= \lim_{h\to 0} \frac{\frac{1}{f(a+h)} - \frac{1}{f(a)}}{h} \\ &= \lim_{h\to 0} \frac{f(a) - f(a+h)}{hf(a+h)f(a)} \\ &= \lim_{h\to 0} \frac{-(f(a+h) - f(a))}{h}\frac{1}{f(a+h)f(a)} \\ &= \frac{-f'(a)}{[f(a)]^2}.\end{aligned}$$

(5) We can combine the product and reciprocal rules!
$$\begin{aligned}\left(\frac{f}{g}\right)'(a) &= \left(f\frac{1}{g}\right)'(a) \\ &= f'(a)\frac{1}{g(a)} + f(a)\left(\frac{1}{g}\right)'(a) \\ &= \frac{f'(a)}{g(a)} - \frac{f(a)g'(a)}{[g(a)]^2} \\ &= \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}.\end{aligned}$$

THEOREM 1.4.10: The Power Rule for Differentiation

Assume that $\alpha \in \mathbf{R}$, $\alpha \neq 0$, and $f(x) = x^\alpha$. Then f is differentiable and

$$f'(a) = \alpha x^{\alpha-1},$$

where $x^{\alpha-1}$ is defined.

In general, the proof is difficult. If $\alpha \in \mathbf{N}$, then it is a simple application of the Binomial Theorem. For $\alpha \in \mathbf{Q}$, it is possible with more tools (chain rule and inverse function theorem). But for general $\alpha \in \mathbf{R}$, we would need more tools, and it outside the scope of this course. So, we omit the proof. Let's look at some examples!

EXAMPLE 1.4.11

(1) $f(x) = x^2 \sin(x)$.

$$f'(x) = (x^2)' \sin(x) + x^2(\sin(x))' = 2x \sin(x) + x^2 \cos(x).$$

(2) $f(x) = \frac{x^4-1}{x-7}$.

$$f'(x) = \frac{(x-7)(x^4+1)' - (x^4+1)(x-7)'}{(x-7)^2} = \frac{(x-7)(4x^3) - (x^4+1)(1)}{(x-7)^2}.$$

(3) $f(x) = \sec(x) = \frac{1}{\cos(x)}$.

$$f'(x) = \frac{-(\cos(x))'}{\cos^2(x)} = \frac{\sin(x)}{\cos^2(x)} = \frac{\sin(x)}{\cos(x)} \frac{1}{\cos(x)} = \tan(x)\sec(x).$$

(4) $f(x) = e^x \cos(x)$.

$$f'(x) = e^x \cos(x) - e^x \sin(x).$$

(5) $f(x) = 3x^5$.

$$f'(x) = 15x^4,\ f''(x) = 60x^3,\ f^{(3)}(x) = 180x^2,\ f^{(4)}(x) = 360x,\ f^{(5)}(x) = 360,\ f^{(\geq 6)}(x) = 0.$$

1.4.4 The Chain Rule

THEOREM 1.4.12

Assume that $y = f(x)$ is differentiable at $x = a$ and $z = g(y)$ is differentiable at $y = f(a)$. Then $h(x) = g \circ f(x) = g(f(x))$ is differentiable at $x = a$ and

$$h'(a) = g'(f(a))f'(a).$$

In Leibniz notation, if $z = g(y)$ and $y = f(x)$, then

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

The proof is quite involved, for a geometric argument see the course notes.

COROLLARY 1.4.13: Generalized Power Rule

If $g(x) = f(x)^\alpha$ for $\alpha \in \mathbf{R} \setminus \{0\}$, then

$$g'(x) = \alpha f(x)^{\alpha-1} f'(x).$$

EXAMPLE 1.4.14

Find $f'(x)$.

(1) $f(x) = (3x^2 + 2x + 7)^{19}$.

(2) $f(x) = \sin(e^x + x^e)$.

(3) $f(x) = e^{\sin(x^2)}$.

Solution.

(1) $f'(x) = 38(3x+1)(3x^2 + 2x + 7)^{18}$.

(2) $f'(x) = \cos(e^x + x^e)(e^x + exe^{e-1})$.

(3) $f'(x) = e^{\sin(x^2)}(\sin(x^2))' = e^{\sin(x^2)}\cos(x^2)(x^2)' = e^{\sin(x^2)}\cos(x^2)(2x)$.

Also, with the chain rule and the derivative of e^x, we can get the derivative of a^x for $a > 0$.

$$a^x = e^{x\ln(a)} \implies (a^x)' = (e^{x\ln(a)})' = e^{x\ln(a)}(x\ln(a))' = a^x\ln(a).$$

EXAMPLE 1.4.15

$f(x) = 2^{3x} + 5^{\cos(x)}$. $f'(x) = 2^{3x}\ln(2)(3) + 5^{\cos(x)}\ln(5)(-\sin(x))$.

1.4.5 Derivatives of Other Trig. Functions

So far, we've seen:

$$(\sin(x))' = \cos(x)$$
$$(\cos(x))' = -\sin(x)$$
$$(\sec(x))' = \sec(x)\tan(x).$$

EXAMPLE 1.4.16

$$\begin{aligned}(\tan(x))' &= \left(\frac{\sin(x)}{\cos(x)}\right)' &= \frac{\cos(x)(\sin(x))' - \sin(x)(\cos(x))'}{\cos^2(x)}\\ &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}\\ &= \frac{1}{\cos^2(x)}\\ &= \sec^2(x).\end{aligned}$$

EXERCISE 1.4.17

Prove that $(\cot(x))' = -\csc^2(x)$ and $(\csc(x))' = -\csc(x)\cot(x)$.

Recap:

$f(x)$	$f'(x)$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\cot(x)$	$-\csc^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\csc(x)\cot(x)$