AMATH/PMATH 331 - Applied Real Analysis

Cameron Roopnarine

Last updated: November 28, 2020

Contents

Chapter 1

Real Limits, Continuity and Differentiation

1.1 Order Properties in R

THEOREM 1.1.1: Discreteness Property of Z

We state two equivalent definitions.

$$\forall k \in \mathbf{Z}\ \forall n \in \mathbf{Z}\ (k \leq n \iff k < n+1)$$

$$\forall n \in \mathbf{Z}\ \nexists k \in \mathbf{Z}\ (n < k < n+1)$$

Proof of: Theorem 1.1.1

Accepted axiomatically, without proof.

DEFINITION 1.1.2: Bounded above, Upper bound

A is **bounded above** (in **R**) when

$$\exists b \in \mathbf{R}\ \forall x \in A\ (x \leq b)$$

We say that b is an **upper bound** for A.

DEFINITION 1.1.3: Bounded below, Lower bound

A is **bounded below** (in **R**) when

$$\exists a \in \mathbf{R}\ \forall x \in A\ (a \leq x)$$

We say that a is a **lower bound** for A.

DEFINITION 1.1.4: Bounded

A is **bounded** when A is both bounded above and below.

DEFINITION 1.1.5: Supremum, Least upper bound, Maximum element

A has a **supremum** (or a **greatest lower bound**) when there exists an element $b \in \mathbf{R}$ such that b is an upper bound for A with $b \leq c$ for every upper bound $c \in \mathbf{R}$ for A. In this case, we say b is the **supremum** (or the **greatest lower bound**) of A and write $b = \sup\{A\}$. When $b = \sup\{A\} \in A$ we also say that b is the **maximum element** of A, and we write $b = \max\{A\}$.

DEFINITION 1.1.6: Infimium, Greatest lower bound, Minimum element

A has an **infimum** (or a **greatest lower bound**) when there exists an element $a \in \mathbf{R}$ such that a is a lower bound for A with $c \leq a$ for every lower bound c for A. In this case, we say a is the **infimum** (or the **greatest lower bound**) of A and write $a = \inf\{A\}$. When $a = \inf\{A\} \in A$ we also say that a is the **minimum element** of A, and we write $a = \min\{A\}$.

EXAMPLE 1.1.7

Let $A = \mathbf{R}_{>0} = (0, \infty) = \{x \in \mathbf{R} \mid x > 0\}$ and $B = [1, \sqrt{2}) = \{x \in \mathbf{R} \mid 1 \leq x < \sqrt{2}\}$.

- A is bounded below, but not above.
- -1 and 0 are both lower bounds for A.
- $\inf\{A\} = 0$
- A has no minimum element, and no maximum element.
- B is bounded both above and below.
- 0 and 1 are both lower bounds for B
- $\sqrt{2}$ and 3 are both upper bounds for B.
- $\inf\{B\} = 1$
- $\sup\{B\} = \sqrt{2}$
- B has a minimum element, namely $\min\{B\} = 1$, but has no maximum element.

THEOREM 1.1.8: The Supremum and Infemum Properties of R

(1) *Every non-empty subset of* $\mathbf{R}$ *which is bounded above in* $\mathbf{R}$ *has a supremum in* $\mathbf{R}$*.*
(2) *Every non-empty subset of* $\mathbf{R}$ *which is bounded below in* $\mathbf{R}$ *has an infimum in* $\mathbf{R}$*.*

Proof of: Theorem 1.1.8

Accepted axiomatically, without proof.

THEOREM 1.1.9: Approximation Property of Supremum and Infimum

Let $\emptyset \neq A \in \mathbf{R}$*.*
(1) $b = \sup\{A\} \implies \forall \varepsilon \in \mathbf{R}_{>0}\ \exists x \in A\ (b - \varepsilon < x \leq b)$
(2) $a = \inf\{A\} \implies \forall \varepsilon \in \mathbf{R}_{>0}\ \exists x \in A\ (a \leq x < a + \varepsilon)$

Proof of: Theorem 1.1.9

We prove (1). Let $b = \sup\{A\}$ and $\varepsilon > 0$. Suppose for a contradiction that there exists no element $x \in A$ with $b - \varepsilon < x$, or equivalently that for all $x \in A$ we have $b - \varepsilon \geq x$. Let $c = b - \varepsilon$. Note that c is an upper bound for A since $x \leq b - \varepsilon = c$ for all $x \in A$. Then, since $b = \sup\{A\}$ and c is an upper bound for A, we have $b \leq c$. However, since $\varepsilon > 0$ we have $b > b - \varepsilon = c$, contradiction. Therefore, there exists $x \in A$ with $b - \varepsilon < x$. Now, choose an element $x \in A$. Then, since $b = \sup\{A\}$, we know that b is an upper bound for A and hence $b \geq x$. Therefore, $b - \varepsilon < x \leq b$, as required.

THEOREM 1.1.10: Well-Ordering Properties of Z in R

(1) Every non-empty subset of **Z** *which is bounded above in* **R** *has a maximum element.*
(2) Every non-empty subset of **Z** *which is bounded below in* **R** *has a minimum element.*

Proof of: Theorem 1.1.10

We prove (1). Let A be a non-empty subset of **Z** which is bounded above. By Theorem 1.1.8 (1), A has a supremum in **R**. Let $n = \sup\{A\}$. We must show that $n \in A$. Suppose for a contradiction that $n \notin A$. By Theorem 1.1.9 (using $\varepsilon = 1$), we can choose $a \in A$ with $n - 1 < a \leq n$. Note that $a \neq n$ since $a \in A$ and $n \notin A$, so we have $a < n$. By Theorem 1.1.9 (using $\varepsilon = n - a$) we can choose $b \in A$ with $a < b \leq n$. Since $a < b$ we have $b - a > 0$. Since $n - 1 < a$ and $b \leq n$, we have $1 = n - (n - 1) > b - a$. However, we have $(b - a) \in \mathbf{Z}$ with $0 < b - a < 1$, which contradicts Theorem 1.1.1. Therefore, $n \in A$, and hence A has a maximum element.

THEOREM 1.1.11: Floor and Ceiling Properties of Z in R

(1) $\forall x \in \mathbf{R}\ \exists! n \in \mathbf{Z}\ (x - 1 < n \leq x)$.
(2) $\forall x \in \mathbf{R}\ \exists! m \in \mathbf{Z}\ (x \leq m < x + 1)$.

Proof of: Theorem 1.1.11

We prove (1).
Uniqueness. Let $x \in \mathbf{R}$, suppose $n, m \in \mathbf{Z}$ with $x - 1 < n \leq x$ and $x - 1 < m \leq x$. Since $x - 1 < n$ we have $x < n + 1$. Since $m \leq x$ and $x < n + 1$, we have $m < n + 1$, hence $m \leq n$ by Theorem 1.1.1. Similarly, $n \leq m$. Since $n \leq m$ and $m \leq n$, we have $n = m$ as required.
Existence. Let $x \in \mathbf{R}$. First, let us consider the case that $x \geq 0$. Let $A = \{k \in \mathbf{Z} \mid k \leq x\}$. Note that $A \neq \emptyset$ (because $0 \in A$), and A is bounded above by x. By Theorem 1.1.10, A has a maximum element. Let $n = \max\{A\}$. Since $n \in A$, we have $n \in \mathbf{Z}$ and $n \leq x$. Also, note that $x - 1 < n$ since $x - 1 \geq n \implies x \geq n + 1 \implies n + 1 \in A \implies n \neq \max\{A\}$. Thus, for $n = \max\{A\}$, we have $n \in \mathbf{Z}$ with $x - 1 < n \leq x$ as required.
Next, consider the case that $x < 0$. If $x \in \mathbf{Z}$, we can take $n = x$. Suppose that $x \notin \mathbf{Z}$. We have $-x > 0$ so, by the previous paragraph, we can choose $m \in \mathbf{Z}$ with $-x - 1 < -m < x + 1$. Thus, we can take $n = -m - 1$ to get $x - 1 < n < x$.

DEFINITION 1.1.12: Floor, Floor function

Let $x \in \mathbf{R}$. The **floor** of x, denoted by $\lfloor x \rfloor$, is the unique $n \in \mathbf{Z}$ with $x - 1 < n \leq x$. The function $f : \mathbf{R} \to \mathbf{Z}$ given by $f(x) = \lfloor x \rfloor$ is called the **floor function**.

DEFINITION 1.1.13: Ceiling, Ceiling function

Let $x \in \mathbf{R}$. The **ceiling** of x, denoted by $\lceil x \rceil$, is the unique $n \in \mathbf{Z}$ with $x \leq n < x + 1$. The function $f : \mathbf{R} \to \mathbf{Z}$ given by $f(x) = \lceil x \rceil$ is called the **ceiling function**.

THEOREM 1.1.14: Archimedean Properties of Z in R

(1) $\forall x \in \mathbf{R}\ \exists n \in \mathbf{Z}\ (n > x)$.
(2) $\forall x \in \mathbf{R}\ \exists m \in \mathbf{Z}\ (m < x)$.

Proof of: Theorem 1.1.14

Let $x \in \mathbf{R}$. Let $n = \lfloor x \rfloor + 1$ and $m = \lfloor x \rfloor - 1$. Since $x - 1 < \lfloor x \rfloor$, we have $x < \lfloor x \rfloor + 1 = n$ and since $\lfloor x \rfloor \leq x$, we have $m = \lfloor x \rfloor - 1 \leq x - 1 < x$.

THEOREM 1.1.15: Density of Q in R

$$\forall a \in \mathbf{R}\ \forall b \in \mathbf{R}\ \exists q \in \mathbf{Q}(a < b \implies a < q < b)$$

Proof of

Let $a, b \in \mathbf{R}$ with $a < b$. By Theorem 1.1.14, we can choose $n \in \mathbf{Z}$ with $n > \frac{1}{b-a} > 0$. Then, $n(b-a) > 1$ and so $nb > na + 1$. Let $k = \lfloor na + 1 \rfloor$. Then we have $na < k \leq na + 1 < nb$ hence $a < \frac{k}{n} < b$. Thus, we can take $q = \frac{k}{n}$ to get $a < q < b$.

1.2 Limit of Sequences in R

DEFINITION 1.2.1: Sequence, Term

For $p \in \mathbf{Z}$, let $Z_{\geq p} = \{k \in \mathbf{Z} \mid k \geq p\}$. A **sequence** in a set A is a function of the form $x : \mathbf{Z}_{\geq p} \to A$ for some $p \in \mathbf{Z}$. Given a sequence $x : \mathbf{Z}_{\geq p} \to A$, the k^{th} **term** of the sequence is the element $x_k = x(k) \in A$, and we denote the sequence x by

$$(x_k)_{k \geq p} = (x_p, x_{p+1}, \ldots)$$

Note that the range of the sequence $(x_k)_{k \geq p}$ is the set $\{x_k\}_{k \geq p} = \{x_k \mid k \geq p\}$.