CO 331 - Coding Theory

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Contents

Chapter 1

Introduction and Fundamentals

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1.1 An Introduction to Coding Theory

$$\text{Data Source} \to \underset{\text{(digital data)}}{\text{Source Encoder}} \to \underset{\text{(encoding algorithm)}}{\text{Channel Encoder}} \underset{\substack{\uparrow \\ \text{Noise}}}{\xrightarrow{\text{Channel}}} \underset{\text{(decoding algorithm)}}{\text{Channel Decoder}} \to \text{Source Decoder} \to \text{Data}$$

EXAMPLE 1.1.1 (Repetition Code).

source message → codeword	# errors/codeword that can be detected	# errors/codeword that can be corrected	rate
$0 \to 0$ $1 \to 1$	0	0	1
$0 \to 00$ $1 \to 11$	1	0	$1/2$
$0 \to 000$ $1 \to 111$	2	1	$1/3$
$0 \to 00000$ $1 \to 11111$	4	2	$1/5$

Goal of Coding Theory

Design codes such that:

- High information rate
- High error-correcting capability
- Efficient encoding and decoding algorithms

Codes $\supset$ Block codes $\supset$ Linear codes $\supset$ Cyclic codes $\supset$ BCH Codes $\supset$ RS Codes

Codes not covered in this course:

- Flamming codes
- Golay codes

- Raptor codes
- LDPC codes
- Turbo codes

Requirements for this course:

- MATH 136
- Not required (but required to take the course): MATH 235
- Familiarity with: Groups, Fields, Ideals, Rings (these will be taught)
- Useful, if you have completed these you might be bored: PMATH 336, PMATH 334 [or the advanced equivalents]

The big picture

In its broadest sense, coding deals with the reliable, efficient, and secure transmissions of data over channels that are subject to inadvertent noise and malicious intrusion.

$$\text{Data source} \to \underset{\text{(data compression)}}{\text{Source encoder}} \to \underset{\text{(cryptography; co 487)}}{\text{Encryptory authentication}} \to \underset{\text{(error correction codes)}}{\text{Channel encoder}} \quad \underset{\substack{\Big\uparrow\\ \text{noise}\\ \text{(non-malicious)}}}{} \xrightarrow{\text{channel}} \underset{\substack{\Big\uparrow\\ \text{adversarial intrusion}\\ \text{(malicious)}}}{}$$

$$\to \text{Decriptory verification} \to \underset{\text{(data decompression)}}{\text{Source decoder}} \to \text{Data}$$

2020-01-08

1.2 Fundamental Concepts

$$\text{Message source} \to \underbrace{\underset{\text{(binary strings)}}{\text{Source encoder}} \to \underset{\text{(adds redundancy to message)}}{\text{Channel encoder}} \underset{\substack{\Big\uparrow\\ \text{noise}}}{\xrightarrow{\text{channel}}} \text{Channel decoder}}_{\text{this course}} \to \text{Source Decoder} \to$$

Message

DEFINITION 1.2.1. An **alphabet** A is a finite set of $|A| = q \geqslant 2$ symbols.

DEFINITION 1.2.2. A **word** is a finite sequence (**tuples** or **vectors**) of symbols from an alphabet A.

DEFINITION 1.2.3. The **length** of a word is the number of symbols in it.

DEFINITION 1.2.4. A **code** C over A is a finite set of words in A with $|C| \geqslant 2$.

DEFINITION 1.2.5. A **codeword** $\boldsymbol{c}$ is a word in code C.

DEFINITION 1.2.6. A **block code** is a code where all codewords have the same length. A block code C of length n containing M codewords over A is a subset $C \subseteq A^n$, with $|C| = M$. We refer to such a block code as an $[n, M]$-code over A.

EXAMPLE 1.2.7 (Block Code)**.** Let $A = \{0, 1\}$ and $C = \{00000,\ 11100,\ 00111,\ 10101\}$. C is a $[5, 4]$-code over $\{0, 1\}$.

Messages $\to$ Codewords
$00 \to 00000$
$10 \to 11100$
$01 \to 00111$
$11 \to 10101$

The encoding is a one-to-one map.

The channel encoder transmits only codewords, but what's received by the channel decoder might not be a codeword. For example, suppose the channel decoder receives $\boldsymbol{r} = 11001$. What should it do? In our above example, we can see that $\boldsymbol{r}$ is closest to 11100 and 10101 (only two bits are different), so it's possible that the codeword was one of those two. However, this may not be the case in practice.

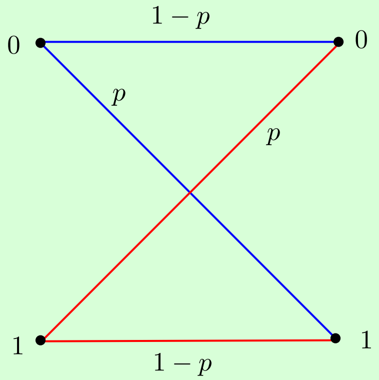
1.3 Assumptions About the Communications Channel

1) The channels only transmit symbols from A.

2) No symbols are deleted, added, or transposed.

3) Errors are random

EXAMPLE 1.3.1 (Binary Symmetric Channel, BSC)**.** Let $A = \{0, 1\}$, and p denote the symbol error probability. The encoding map is:



A similar encoding map can be drawn for $A = \{0, 1, 2\}$, with symbol error probability $p/2$.

Suppose that the symbols transmitted are $X_1, X_2, \ldots$, and the symbols received are $Y_1, Y_2, \ldots$. Then for all $i \geqslant 1$, $j \geqslant 1$, $k \leqslant q$, the probability that Y_i is received, given that X_i is transmitted is:

$$P(Y_i = a_j \mid X_i = a_k) = \begin{cases} 1-p, & \text{if } j = k \\ \frac{p}{q-1}, & \text{if } j \neq k \end{cases}$$

1.4 Notes about BSC

(i) If $p = 0$, the channel is perfect.

(ii) If $p = 1/2$, the channel is useless.

(iii) If $1/2 < p \leqslant 1$, then simply flip all bits that are received.

(iv) WLOG, we can assume $0 < p < 1/2$.

(v) Analogously, for a q-ary channel, we can assume that $0 < p < \frac{q-1}{q}$.

DEFINITION 1.4.1. If $\boldsymbol{x}, \boldsymbol{y} \in A^n$, the **Hamming distance** $d(\boldsymbol{x}, \boldsymbol{y})$ is the number of coordinate positions in which $\boldsymbol{x}$ and $\boldsymbol{y}$ differ.

EXAMPLE 1.4.2 (Hamming Distance)**.** Let $\boldsymbol{x} = 10111$ and $\boldsymbol{y} = 01010$. The Hamming distance of $\boldsymbol{x}$ and $\boldsymbol{y}$ is $d(\boldsymbol{x}, \boldsymbol{y}) = 4$ since $\boldsymbol{x}$ and $\boldsymbol{y}$ differ in the coordinate positions 1, 2, 3, and 5.

DEFINITION 1.4.3. Let C be an $[n, M]$-code. The **Hamming distance** d **of a code** C is

$$d(C) = \min\{d(\boldsymbol{x}, \boldsymbol{y}) : \boldsymbol{x}, \boldsymbol{y} \in C,\ \boldsymbol{x} \neq \boldsymbol{y}\}$$

THEOREM 1.4.4. *d is a **metric**. For all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in A^n$:*
(1) $d(\boldsymbol{x}, \boldsymbol{y}) \geqslant 0$
(2) $d(\boldsymbol{x}, \boldsymbol{y}) = 0$ if and only if $\boldsymbol{x} = \boldsymbol{y}$
(3) $d(\boldsymbol{x}, \boldsymbol{y}) = d(\boldsymbol{y}, \boldsymbol{x})$
(4) (Triangle inequality): $d(\boldsymbol{x}, \boldsymbol{z}) \leqslant d(\boldsymbol{x}, \boldsymbol{y}) + d(\boldsymbol{y}, \boldsymbol{z})$

Proof. (1)-(3) are trivially true.

(4) Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in A^n$. Suppose that $\boldsymbol{x}$ and $\boldsymbol{z}$ differ in exactly a positions; that is, $d(\boldsymbol{x}, \boldsymbol{z}) = a$. Out of the a positions in which $\boldsymbol{x}$ and $\boldsymbol{z}$ differ, there are b positions in which $\boldsymbol{y}$ is identical to $\boldsymbol{x}$, but not $\boldsymbol{z}$. Out of the a positions, there are $a - b$ positions in which $\boldsymbol{y}$ is identical to $\boldsymbol{z}$, but not $\boldsymbol{x}$. Lastly, in the $n - a$ positions where $\boldsymbol{x}$ is identical to $\boldsymbol{z}$, there are c positions in which $\boldsymbol{y}$ does not match either $\boldsymbol{x}$ or $\boldsymbol{z}$. We can see that $d(\boldsymbol{x}, \boldsymbol{y}) = b + c$ and $d(\boldsymbol{y}, \boldsymbol{z}) = a - b + c$. We get

$$d(\boldsymbol{x}, \boldsymbol{y}) + d(\boldsymbol{y}, \boldsymbol{z}) = (b + c) + (a - b + c) = a + 2c \geqslant a$$

Therefore d is a metric. □

DEFINITION 1.4.5. The **rate** (or **information rate**) of an $[n, M]$-code C over A, is

$$R = \frac{\log_q(M)}{n}$$

where $q = |A|$.
If the source messages are all k-tuples over A, then $M = q^k$, so we have

$$R = \frac{\log_q(q^k)}{n} = \frac{k}{n}$$

EXAMPLE 1.4.6 (Rate & Distance of Code)**.** Let $A = \{0, 1\}$ and $C = \{00000,\ 11100,\ 00111,\ 10101\}$ which is a $[2, 4]$-code over $\{0, 1\}$.

- Rate of code: $R = {}^{2}/_{5}$
- Distance of code: $d(C) = 2$, since the minimum distance are from the pair of codewords 00111 and 10101 which have Hamming distance of 2 as they differ in coordinate positions 1 and 4.

1.5 Decoding Strategy

Suppose we have an $[n, M]$-code C over A of distance d. We need to adopt a strategy for the channel decoder (henceforth called the decoder). When the decoder receives an n-tuple $\boldsymbol{r} \in A^n$ it must make some decision. This decision may be one of

(i) no errors hace occurred; accept $\boldsymbol{r}$ as a codeword.

(ii) errors have occurred; correct $\boldsymbol{r}$ to a codeword $\boldsymbol{c}$; e.g. $0 \to 0000,\ 1 \to 1111,\ r = 0001$ corrected to 0000.

(iii) errors have occurred; no correction is possible.

1.6 Nearest Neighbour Decoding

1.6.1 Incomplete Maximum Likelihood Decoding (IMLD)

Correct $\boldsymbol{r}$ to the unique codeword $\boldsymbol{c}$ for which $d(\boldsymbol{r}, \boldsymbol{c})$ is smallest. If $\boldsymbol{c}$ is not unique, reject $\boldsymbol{r}$.

1.6.2 Complete Maximum Likelihood Decoding (CMLD)

Same as IMLD, except ties are broken arbitrarily.

Question: Is IMLD a reasonable strategy?

THEOREM 1.6.1. *IMLD selects the codeword* $\mathbf{c}$ *that maximizes* $P(\boldsymbol{r} \mid \mathbf{c})$*; that is, it maximizes the probability* $\boldsymbol{r}$ *is received, given* $\mathbf{c}$ *was sent.*

We actually want to maximize $P(\boldsymbol{c} \mid \boldsymbol{r})$, but we will ignore that for now.

Proof. Suppose $\boldsymbol{c}_1, \boldsymbol{c}_2 \in C$ with $d(\boldsymbol{c}_1, \boldsymbol{r}) = d_1$ and $d(\boldsymbol{c}_2, \boldsymbol{r}) = d_2$. Suppose $d_1 > d_2$. Now,

$P(\boldsymbol{r} \mid \boldsymbol{c}_1) = (1-p)^{n-d_1}\left(\frac{p}{q-1}\right)^{d_1}$ and $P(\boldsymbol{r} \mid \boldsymbol{c}_2) = (1-p)^{n-d_2}\left(\frac{p}{q-1}\right)^{d_2}$.

Hence,

$$\begin{aligned}\frac{P(\boldsymbol{r} \mid \boldsymbol{c}_1)}{P(\boldsymbol{r} \mid \boldsymbol{c}_2)} &= (1-p)^{d_2-d_1}\left(\frac{p}{q-1}\right)^{d_1-d_2} \\ &= \left[\frac{p}{(1-p)(q-1)}\right]^{d_1-d_2}\end{aligned}$$

Recall that, for a q-ary channel, we can assume that $p < \frac{q-1}{q}$. Thus,

$$\begin{aligned}&\implies pq < q-1 \\ &\implies 0 < q-1-pq \\ &\implies p < q-1-pq+p \\ &\implies p < (1-p)(q-1) \\ &\implies \frac{p}{(1-p)(q-1)} < 1\end{aligned}$$

Since $d_1 > d_2$, we get $\frac{P(\boldsymbol{r}|\boldsymbol{c}_1)}{P(\boldsymbol{r}|\boldsymbol{c}_2)} < 1$, and so $P(\boldsymbol{r} \mid \boldsymbol{c}_1) < P(\boldsymbol{r} \mid \boldsymbol{c}_2)$. □

The ideal strategy is to correct $\boldsymbol{r}$ to $\boldsymbol{c} \in C$ such that $P(\boldsymbol{c} \mid \boldsymbol{r})$ is maximized. This is **Minimum Error Decoding (MED)**.

EXAMPLE 1.6.2 (IMLD != MED). Let $C = \{\underbrace{000}_{\boldsymbol{c}_1}, \underbrace{111}_{\boldsymbol{c}_2}\}$, $P(\boldsymbol{c}_1) = 0.1$, $P(\boldsymbol{c}_2) = 0.9$, $p = {}^1\!/_4$, and $\boldsymbol{r} = 100$.
IMLD $\boldsymbol{r}$ is decoded to to $\boldsymbol{c}_1 = 000$.
MED

$$\begin{aligned} P(\boldsymbol{c}_1 \mid \boldsymbol{r}) &= \frac{P(\boldsymbol{r} \mid \boldsymbol{c}_1)P(\boldsymbol{c}_1)}{P(\boldsymbol{r})} \\ &= \frac{p(1-p)^2(0.1)}{P(\boldsymbol{r})} \\ &= \frac{0.0140625}{P(\boldsymbol{r})} \end{aligned}$$

$$\begin{aligned} P(\boldsymbol{c}_2 \mid \boldsymbol{r}) &= \frac{P(\boldsymbol{r} \mid \boldsymbol{c}_2)P(\boldsymbol{c}_2)}{P(\boldsymbol{r})} \\ &= \frac{p^2(1-p)(0.9)}{P(\boldsymbol{r})} \\ &= \frac{0.0421875}{P(\boldsymbol{r})} \end{aligned}$$

Since $P(\boldsymbol{c}_1 \mid \boldsymbol{r}) < P(\boldsymbol{c}_2 \mid \boldsymbol{r})$, $\boldsymbol{r}$ is decoded to $\boldsymbol{c}_2 = 111$.

Notes:

(i) IMLD selects $\boldsymbol{c}$ such that $P(\boldsymbol{r} \mid \boldsymbol{c})$ is maximum.

(ii) MED selects $\boldsymbol{c}$ such that $P(\boldsymbol{c} \mid \boldsymbol{r})$ is maximum.

(iii) MED has a drawback that it requires knowledge of $P(\boldsymbol{c}_i)$ for each $i \in [1, M]$.

(iv) Suppose source messages are equally likely, so $P(\boldsymbol{c}_i) = \frac{1}{M}$ for each $i \in [1, M]$. Then,

$$P(\boldsymbol{r} \mid \boldsymbol{c}_i) = \frac{P(\boldsymbol{c}_i \mid \boldsymbol{r})P(\boldsymbol{r})}{P(\boldsymbol{c}_i)} = P(\boldsymbol{c}_i \mid \boldsymbol{r})\underbrace{MP(r)}_{\text{constant}}$$

So, maximizing $P(\boldsymbol{r} \mid \boldsymbol{c}_i)$ is the same as maximizing $P(\boldsymbol{c}_i \mid \boldsymbol{r})$. Thus, IMLD is the same as MED in this case.

In the remainder of the course, we will use IMLD/CMLD.

2020-01-13

1.7 Error Correcting & Detecting Capabilities of a Code

- If C is used for error correction, the strategy is IMLD/CMLD.
- If C is used for error detection only, the strategy is to reject $\boldsymbol{r}$ if $\boldsymbol{r} \notin C$, otherwise accept $\boldsymbol{r}$.

DEFINITION 1.7.1. A code C is called an ***e*-error correcting code** if the decoder always makes the correct decision if at most e errors per codeword are introduced per transmission. We define ***e*-error detecting code** similarly.

EXAMPLE 1.7.2 (Error Detecting and Correcting Codes).

- $C = \{0000,\ 1111\}$ is a 1-error correcting code, but not a 2-error correcting code.
- $C = \{\underbrace{0\cdots 0}_{m},\ \underbrace{1\cdots 1}_{m}\}$ is a $\lfloor \frac{m-1}{2} \rfloor$-error correcting code.
- $C = \{0000,\ 1111\}$ is a 3-error detecting code.

THEOREM 1.7.3. *Suppose $d(C) = d$, then C is a $(d-1)$-error detecting code.*

Proof. Suppose $\boldsymbol{c} \in C$ is transmitted $\boldsymbol{r}$ is received. Let e denote the amount of errors that have occurred in transmission.

- If $e = 0$, then $\boldsymbol{r} = \boldsymbol{c} \in C$, and the decoder accepts $\boldsymbol{r}$.
- If $e \geq d$, then the decoder can make the wrong choice since $d(C) = d$.
- If $e \in [1, d-1]$, then $1 \leqslant d(\boldsymbol{r}, \boldsymbol{c}) \leqslant d - 1$. So, $\boldsymbol{r} \notin C$, hence the decoder rejects $\boldsymbol{r}$. Hence, C is a $(d-1)$-error detecting code.

□

THEOREM 1.7.4. *If $d(C) = d$, then C is not a d-error detecting code.*

Proof. Since $d(C) = d$, there exists codewords $\boldsymbol{c}_1, \boldsymbol{c}_2 \in C$ with $d(\boldsymbol{c}_1, \boldsymbol{c}_2) = d$. If $\boldsymbol{c}_1$ is sent and $\boldsymbol{r}$ is received with d errors, it is possible $\boldsymbol{r} = \boldsymbol{c}_2$ is received. In this case, the decoder accepts $\boldsymbol{c}_2$. Hence, C is not a d-error detecting code. □

THEOREM 1.7.5. *If $d(C) = d$, then C is a $\lfloor \frac{d-1}{2} \rfloor$-error correcting code.*

Proof. Suppose $\boldsymbol{c} \in C$ is transmitted, at most $\frac{d-1}{2}$ errors are introduced, and $\boldsymbol{r}$ is received. Let $\boldsymbol{z} \in C$ with $\boldsymbol{z} \neq \boldsymbol{c}$. By the triangle inequality, we have

$$\begin{aligned} d(\boldsymbol{c}, \boldsymbol{z}) \leqslant d(\boldsymbol{c}, \boldsymbol{r}) + d(\boldsymbol{r}, \boldsymbol{z}) \implies d(\boldsymbol{r}, \boldsymbol{z}) &\geqslant d(\boldsymbol{c}, \boldsymbol{z}) - d(\boldsymbol{c}, \boldsymbol{r}) \\ &\geqslant d - \frac{d-1}{2} \\ &= \frac{d+1}{2} \\ &> \frac{d-1}{2} \end{aligned}$$

So, $\boldsymbol{c}$ is the unique codeword closest to $\boldsymbol{r}$. Hence, IMLD/CMLD will decode $\boldsymbol{r}$ to $\boldsymbol{c}$. Thus, C is a $\lfloor \frac{d-1}{2} \rfloor$-error correcting code. □

THEOREM 1.7.6. *If $d(C) = d$, then C is not a $\left(\lfloor \frac{d-1}{2} \rfloor + 1\right)$- error correcting code.*

Proof. Exercise. □

Given q, n, M, d, does there exist an $[n, M]$-code over A with $|A| = q$ such that $d(C) = d$?

Let $C = \{\boldsymbol{c}_1, \dots, \boldsymbol{c}_M\}$ and $e = \lfloor \frac{d-1}{2} \rfloor$. For any codeword $\boldsymbol{c} \in C$, let $S_{\boldsymbol{c}}$ be the sphere of radius e centered at $\boldsymbol{c}$; that is,

$$S_{\boldsymbol{c}} = \{\boldsymbol{r} \in A^n : d(\boldsymbol{r}, \boldsymbol{c}) \leqslant e\}$$

We proved that if $\boldsymbol{c}_i, \boldsymbol{c}_j \in C$ with $i \neq j$, then $S_{\boldsymbol{c}_i} \cap S_{\boldsymbol{c}_j} = \varnothing$ for each $i \neq j$. This question can be viewed as a **sphere packing problem**: Can we place M spheres of radius e in A^n such that no two spheres overlap? This is a purely combinatorial problem.

Given $A = \{0, 1\}$, $n = 128$, $M = 2^{64}$, determine if an $[n, M]$-code C over A with $d(C) = d$ exists.

The answer to this problem is yes and we will see this in the following lectures.

Roadmap: We'll view $\{0, 1\}^n$ as a vector space of dimension n over $\mathbb{Z}_q$ where $|A| = q$. We will chose the code C to be an M-dimensional subspace of this vector space and we will choose special subspaces that satisfy the $d(C) = d$ requirement.

Chapter 2

Finite Fields

2020-01-15

2.1 Introduction

DEFINITION 2.1.1. A **field** F is a set of elements under two binary operations, which we denote by $+$ and $\cdot$ such that $+ : F \times F \to F$ and $\cdot : F \times F \to F$ where all the following axioms are satisfied:
V1 $a + (b + c) = (a + b) + c$
V2 $a + b = b + a$
V3 $\exists\, 0 \in F$ such that $a + 0 = a$
V4 $\exists\, (-a) \in F$ such that $a + (-a) = 0$
V5 $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
V6 $a \cdot b = b \cdot a$
V7 $\exists\, 1 \in F$ such that $a \cdot 1 = a$
V8 $\forall\, a \neq 0, \exists (a^{-1}) \in F$ such that $a \cdot (a^{-1}) = 1$
V9 $a \cdot (b + c) = a \cdot b + a \cdot c$

DEFINITION 2.1.2. A field F is **infinite** if $|F|$ is infinite.

DEFINITION 2.1.3. A field F is **finite** if $|F|$ is finite.

DEFINITION 2.1.4. The **order** of a field F, denoted $\operatorname{ord}(F)$ is $|F|$.

EXAMPLE 2.1.5 (Infinite and Finite Fields)**.**

- $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are infinite fields.
- $\mathbb{Z}$ is **not** a field since $3 \in \mathbb{Z}$, but $\left(\frac{1}{3}\right) \notin \mathbb{Z}$.

Question: For what $n \in \mathbb{Z}_{\geqslant 2}$ do there exists finite fields of order n? If a field of order n exists, how do we "construct" it?

Recall: Let $n \geqslant 2$. The integers modulo n, $\mathbb{Z}_n$ is the set of all equivalence classes $\mod n$.

$$\mathbb{Z}_n = \{[0], [1], [2], \dots, [n-1]\}$$

where $[a]+[b]=[a+b]$ and $[a][b]=[ab]$. More simply, $\mathbb{Z}_n=\{0,1,\ldots,n-1\}$ with addition and multiplication performed $\mod n$.

EXAMPLE 2.1.6 (Modulo). Let $\mathbb{Z}_9=\{0,1,\ldots,8\}$.

- $5+7=3$ (i.e. $5+7\equiv 3 \mod 9$)
- $5\cdot 7=8$ (i.e. $5+7\equiv 8 \mod 9$)

DEFINITION 2.1.7. A **commutative ring** satisfies field axioms V1-V9 except V8.

THEOREM 2.1.8. *$\mathbb{Z}_n$ is a commutative ring.*

THEOREM 2.1.9. *$\mathbb{Z}_n$ is a field if and only if n is prime.*

Proof. ($\Longleftarrow$) Suppose n is prime. Let $a\in\mathbb{Z}_n$, $a\neq 0$ (i.e. $1\leqslant a\leqslant n-1$). Since n is prime, $\gcd(a,n)=1$ so $\exists\, s,t\in\mathbb{Z}$ such that

$$as+nt=1$$

Reducing both sides $\mod n$ gives

$$as\equiv 1 \mod n$$

Define $a^{-1}=s$. Thus, V8 is satisfied and hence $\mathbb{Z}_n$ is a field of order n.

($\Longrightarrow$) Suppose for a contradiction that n is composite, say $n=ab$ where $2\leqslant a,b\leqslant n-1$. Suppose a^{-1} exists, and define $a^{-1}=s$. Then,

$$as\equiv 1 \mod n \implies abs\equiv b \mod n \implies ns\equiv b \mod n \implies 0\equiv b \mod n$$

So, $n\mid b$ which is impossible. Therefore, a^{-1} does not exist, and hence $\mathbb{Z}_n$ is not a field. □

Question: Do there exist finite fields of orders 4 and 6?

DEFINITION 2.1.10. The **characteristic** of a field, denoted $\operatorname{char}(F)$, is the smallest possible integer m such that

$$\underbrace{1+\cdots+1}_{m}=0$$

If no such m exists, then we define $\operatorname{char}(F)=0$

EXAMPLE 2.1.11 (Characteristic of Fields).

- $\operatorname{char}(\mathbb{Q})=0$
- $\operatorname{char}(\mathbb{R})=0$
- $\operatorname{char}(\mathbb{C})=0$
- $\operatorname{char}(\mathbb{Z}_p)=p$ where p is prime.

THEOREM 2.1.12. *If $\operatorname{char}(F)=0$, then F is infinite.*

Proof. Consider $1,1+1,\ldots,\underbrace{1+\cdots+1}_{a}\in F$. Suppose for a contradiction there exists distinct $a,b\in\mathbb{Z}$ such that

$$\underbrace{1+\cdots+1}_{a}=\underbrace{1+\cdots+1}_{b}$$

where $a > b$, then

$$\underbrace{1+\cdots+1}_{a} = \underbrace{1+\cdots+1}_{b} + \underbrace{1+\cdots+1}_{a-b} = \underbrace{1+\cdots+1}_{b}$$

Hence, $\underbrace{1+\cdots+1}_{a-b} = 0 \implies \mathrm{char}(F) = (a-b)$ which contradicts $\mathrm{char}(F) = 0$. Thus, F is infinite. □

THEOREM 2.1.13. *If F is a finite field, then $\mathrm{char}(F)$ is prime.*

Proof. Suppose for a contradiction that $\mathrm{char}(F) = m$ is composite, say $m = ab$ where $2 \leqslant a, b \leqslant m-1$. Now

$$(\underbrace{1+\cdots+1}_{a})(\underbrace{1+\cdots+1}_{b}) = \underbrace{1+\cdots+1}_{m} = 0$$

since $\mathrm{char}(F) = m$. Let $\underbrace{1+\cdots+1}_{a} = s$ and $\underbrace{1+\cdots+1}_{b} = t$, so $st = 0$ where $s \neq 0$. Since $\mathrm{char}(F) = m > a$, there exists $c \in F$ such that $cs = 1 \implies c = s^{-1}$. Therefore $s^{-1}st = 0$. Thus, $t = 0$ which is a contradiction to $\mathrm{char}(F) = m$. □

Roadmap: Let F be a finite field of order n. Then, $\mathrm{char}(F) = p$ where p is prime. Then, $\mathbb{Z}_p$ is a subfield of F. F is a vector space over $\mathbb{Z}_p$ of $\dim = k$. Then, order of F is p^k.

2020-01-17

DEFINITION 2.1.14. We say two fields F and S are **isomorphic** if they have the same binary operations and if there exists a bijection between them.

DEFINITION 2.1.15. Let F be a field. A subset $S \subseteq F$ is called a **subfield** of F if S is a field itself with respect to the same operations of F.

EXAMPLE 2.1.16 (Subfield). Let F be a finite field where $\mathrm{char}(F) = p$. Consider $E = \{0, 1, 1+1, \ldots, \underbrace{1+\cdots+1}_{p-1}\} \subseteq F$. We see that E is a field with the same field operations as F. Also, E has order p. If we label the elements of E in a natural way such that $\underbrace{1+\cdots+1}_{p-1} \longleftrightarrow p-1$, then

$$E = \{0, 1, 1+1, \ldots, \underbrace{1+\cdots+1}_{p-1}\} = \mathbb{Z}_p = \{0, 1, 2, \ldots, p-1\} \subseteq F$$

So E is isomorphic to $\mathbb{Z}_p$.

THEOREM 2.1.17. *If F is a finite field of characteristic p, then $\mathbb{Z}_p$ is a subfield of F.*

Proof. Exercise. □

DEFINITION 2.1.18. Let F be a finite field, and consider $\mathbb{Z}_p \subseteq F$.

- Each $v \in F$ is vector.
- Each $c \in \mathbb{Z}_p$ is a scalar.
- Addition in F is defined by vector addition.
- Multiplication in F by elements in $\mathbb{Z}_p$ is defined by scalar multiplication.

THEOREM 2.1.19. *If F is a finite field of characteristic p, then F is a vector space over $\mathbb{Z}_p$.*

Proof. Exercise. □

THEOREM 2.1.20. *If F is a finite field of characteristic p, then*

$$\mathrm{ord}(F) = p^k$$

for some $k \in \mathbb{Z}_{\geqslant 1}$.

Proof. Let k be the dimension of the vector space F over $\mathbb{Z}_p$. Let $\{\alpha_1, \ldots, \alpha_k\}$ be a basis for F. Then, every element in F can be written as

$$c_1\alpha_1 + \cdots + c_k\alpha_k$$

where $c_i \in \mathbb{Z}_p$. For each α_i, there are p possible choices for c_i, hence $\mathrm{ord}(F) = p^k$. □

EXAMPLE 2.1.21. There is no field of order 6.

Question: Is there a finite field of order 4, 8, 9?

2.2 Irreducible Polynomials

DEFINITION 2.2.1. Let F be a field. The **set of all polynomials in x over F** (polynomials with coefficients from F) is denoted $F[x]$. Addition and multiplication are both done in the usual way, with coefficient arithmetic in F.

EXAMPLE 2.2.2. In $\mathbb{Z}_{11}$, $(2 + 5x + 6x^2) + (3 + 9x + 5x^2) = 5 + 3x$.

THEOREM 2.2.3. *Let F be a field. $F[x]$ is a commutative ring.*

DEFINITION 2.2.4. Let F be a field and let $f \in F[x]$ with $\deg(f) \geqslant 1$. If $g, h \in F[x]$ with $f \mid (g - h)$, then we write

$$g \equiv h \mod f$$

or equivalently, we can write $g - h = \ell f$ for some $\ell \in F[x]$.

THEOREM 2.2.5. *Congruence is an equivalence relation.*

DEFINITION 2.2.6. For a given $f \in F[x]$, the **equivalence class containing $g \in F[x]$** is

$$[g] = \{h \in F[x] : h \equiv g \mod f\}$$

DEFINITION 2.2.7. For $g, h \in F[x]$, we define addition and multiplication as follows:

- Addition: $[g] + [h] = [g + h]$
- Multiplication: $[g][h] = [gh]$

THEOREM 2.2.8. *1. The set of all equivalence classes, denoted $F[x]/(f)$ where $f \in F[x]$ and $\deg(f) \geqslant 1$ is a commutative ring.*

2. The polynomials in $F[x]$ of degree less than degree of f are a system of distinct representatives of equivalence classes in $F[x]/(f)$.

Proof of 5:

Proof. Let $g \in F[x]$. By division algorithm for polynomials we can write $g = \ell f + r$ where $\deg(r) < \deg(f)$. So, $g - r = \ell f$. Hence, $g \equiv r \mod f$. Thus, $[g] = [r]$ and we have $\deg(r) < \deg(f)$. Also, if $r_1, r_2 \in F[x]$ with $r_1 \neq r_2$, and $\deg(r_1), \deg(r_2) < \deg(f)$, then

$$f \nmid (r_1 - r_2) \iff r_1 \not\equiv r_2 \mod f$$

Thus, $[r_1] \neq [r_2]$. □

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DEFINITION 2.2.9. Let F be a field, and $f \in F[x]$ of degree $n \geqslant 1$. f is **irreducible** over F if f cannot be written as $f = gh$, where $g, h \in F[x]$ and $\deg(g), \deg(n) \geqslant 1$.

EXAMPLE 2.2.10 (Irreducible)**.**

- $x^2 + 1$ is irreducible over $\mathbb{R}$
- $x^2 + 1$ is reducible over $\mathbb{C}$ since $(x + i)(x - i) = x^1 + 1$
- $x^2 + 1$ is reducible over $\mathbb{Z}_2$ since $(x + 1)^2 = x^1 + 1$
- $x^2 + 1$ is irreducible over $\mathbb{Z}_3$

THEOREM 2.2.11. *Let F be a field and $f \in F[x]$ of degree $n \geqslant 1$. $F[x]/(f)$ is a field if and only if f is irreducible over F.*

Proof. Note that $F[x]/(f)$ is a commutative ring.

$(\Longleftarrow)$ Suppose $g \in F[x]/(f)$ where $g \neq 0$ and $\deg(g) < \deg(f)$. Then, $\gcd(g, f) = 1$ and so by EEA for polynomials, there exists $s, t \in F[x]$ such that

$$gs + ft = 1$$

Reducing both sides modulo f gives

$$gs \equiv 1 \mod f$$

So, $g^{-1} = s$. Hence $F[x]/(f)$ is a field.

$(\Longrightarrow)$ Exercise. □

We need an irreducible polynomial $f \in \mathbb{Z}_p[x]$ of degree n. Then, $\mathbb{Z}[x]/(f)$ is a finite field of order p^n.

THEOREM 2.2.12. *For any prime p and $n \in \mathbb{Z}_{\geqslant 2}$, there exists an irreducible polynomial of degree n over $\mathbb{Z}_p$.*

The proof is beyond the scope of this course.

THEOREM 2.2.13. *There exists a finite field of order q if and only if q is a prime power.*

EXAMPLE 2.2.14. Construct a finite field of order $2^2 = 4$.
Solution. Take $f(x) = x^2 + x + 1 \in \mathbb{Z}_2[x]$ which is irreducible over $\mathbb{Z}_2[x]$. Thus, the field is

$$\mathbb{Z}_2[x]/(x^2 + x + 1) = \{0, 1, x, x + 1\}$$

Examples of operations:

- $x + (x + 1) = 1$
- $x(x + 1) = x^2 + x = 1$
- $x^{-1} = x + 1$
- $1^{-1} = 1$
- $x^{-1} = x + 1$
- $(x + 1)^{-1} = x$

EXAMPLE 2.2.15. Construct a field of order $2^3 = 8$.
Solution. We need an irreducible polynomial of degree 3 over $\mathbb{Z}_2$. Take $f_1(x) = x^3 + x + 1$ which is irreducible over $\mathbb{Z}_2$. Then a field of order 8 is

$$F_1 = Z_2[x]/(x^3 + x + 1) = \left\{0, 1, x, x + 1, x^2, x^2 + 1, x^2 + x, x^2 + x + 1\right\}$$

Examples of operations:

- $x^2 + (x^2 + x + 1) = x + 1$
- $x^2(x^2 + x + 1) = x^4 + x^3 + x^2 = 1$
- $(x^2)^{-1} = x^2 + x + 1$
- $x^{-1} = x^2 + 1$

EXAMPLE 2.2.16. Construct a field of order $2^3 = 8$.
Solution. Take $f_2(x) = x^3 + x^2 + 1$. Then a field of order 8 is

$$F_2 = \mathbb{Z}_2[x]/(x^3 + x^2 + 1) = \left\{0, 1, x, x + 1, x^2, x^2 + 1, x^2 + x, x^2 + x + 1\right\}$$

Examples of operations:

- $x^{-1} = x^2 + x$

Note: F_1 and F_2 are two different fields of order $2^3 = 8$, but they are isomorphic. That is, there is a bijection $\alpha : F_1 \to F_2$ such that

$$\alpha(a + b) = \alpha(a) + \alpha(b)$$

$$\alpha(ab) = \alpha(a)\alpha(b)$$

for all $a, b \in F_1$.

THEOREM 2.2.17. *Any two finite fields of order q are isomorphic.*

Proof. Exericse. □

DEFINITION 2.2.18. We will denote the **Galois field of order** q by $GF(q)$.

We saw one representation of $GF(2^2)$ and two different representations of $GF(2^3)$.

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EXAMPLE 2.2.19. Construct $GF(2^4 = 16)$.
Solution. Take $f(x) = x^4 + x + 1 \in \mathbb{Z}_2[x]$.

- f has no roots in $\mathbb{Z}_2$ and hence no linear factors
- long division shows that $x^2 + x + 1 \nmid x^4 + x + 1$, so f has no irreducible quadratic factors
- f is irreducible over $\mathbb{Z}_2$.

Thus, $GF(16) = \mathbb{Z}_2[x]/(x^4 + x + 1)$.

2.3 Properties of Finite Fields

THEOREM 2.3.1 (Frosh's Dream). *Let* $\alpha, \beta \in GF(q)$ *where* $\operatorname{char}(GF(q)) = p$.

$$(\alpha + \beta)^p = \alpha^p + \beta^p$$

Proof.

$$(\alpha + \beta)^p = \alpha^p + \sum_{i=1}^{p-1} \binom{p}{i} \alpha^i \beta^{p-i} + \beta^p$$

Now,

$$\begin{aligned}\binom{p}{i} = \frac{p!}{i!(p-i)!} &= \frac{p(p-1)\cdots(p-i+1)(p-i)(p-i-1)\cdots(2)(1)}{[i(i-1)\cdots(2)(1)]\,[(p-i)(p-i-1)\cdots(2)(1)]} \\ &= p\left[\frac{(p-1)\cdots(p-i+2)}{i(i-1)\cdots(2)(1)}\right]\end{aligned}$$

If $1 \leqslant i \leqslant p-1$ then $p \mid$ numerator, but $p \nmid$ denominator. Thus,

$$p \mid \binom{p}{i} = p\lambda$$

where $\lambda \in \mathbb{N}$ with $\lambda \neq 0$ and $p \nmid \lambda$.

$$\begin{aligned}\sum_{i=1}^{p-1} \binom{p}{i} \alpha^i \beta^{p-i} &= \sum_{i=1}^{p-1} (p\lambda_i) \alpha^i \beta^{p-i} \\ &= \sum_{i=1}^{p-1} (\underbrace{1 + \cdots + 1}_{p}) \lambda_i \alpha^i \beta^{p-i} \\ &= 0\end{aligned}$$

Thus, $(\alpha + \beta)^p = \alpha^p + \beta^p$. □

COROLLARY 2.3.2.

$$(\alpha+\beta)^{p^m} = \alpha^{p^m} + \beta^{p^m}$$

for all $m \geqslant 1$.

Proof. Exercise. Hint: Induction on m. □

THEOREM 2.3.3. *Let $\alpha \in GF(q)$. Then*

$$\alpha^q = \alpha$$

Proof. If $\alpha = 0$, then $\alpha^q = 0 = \alpha$.

If $\alpha \neq 0$, let $\{\alpha_1, \ldots, \alpha_{q-1}\}$ be the non-zero elements in $GF(q)$. Consider

$$\{\alpha\alpha_1, \ldots, \alpha\alpha_{q-1}\}$$

Note that the elements in this list are pairwise distinct because if $\alpha\alpha_i = \alpha\alpha_j$ with $i \neq j$, then

$$\alpha^{-1}\alpha\alpha_i = \alpha^{-1}\alpha\alpha_j$$

which implies that $\alpha_i = \alpha_j$ which is a contradiction. Also $\alpha\alpha_i \neq 0$ for all $i \in [1, q-1]$. Hence, $\{\alpha_1, \ldots, \alpha_{q-1}\} = \{\alpha\alpha_1, \ldots, \alpha\alpha_{q-1}\}$. Therefore, $\alpha_1 \cdots \alpha_{q-1} = (\alpha\alpha_1) \cdots (\alpha\alpha_{q-1})$. Hence, $\alpha^{q-1} = 1$. Thus, $\alpha^q = \alpha$. □

DEFINITION 2.3.4. Let $GF(q)^* = GF(q)/\{0\}$.

DEFINITION 2.3.5. The **order of** $\alpha \in GF(q)^*$, denoted $\text{ord}(\alpha)$, is the smallest positive integer t such that $\alpha^t = 1$.

EXAMPLE 2.3.6. How many elements of order 1 are there in $GF(q)$?
Solution. $\alpha = 1$

EXAMPLE 2.3.7. Find $\text{ord}(x)$ in $GF(16) = \mathbb{Z}_2/(x^4+x+1)$.
Solution.

- $x^1 = x$
- $x^2 = x^2$
- $x^3 = x^3$
- $x^4 = x + 1$
- $x^5 = x^2 + x$
- $x^6 = x^3 + x^2$
- $x^7 = x^3 + x + 1$
- $x^8 = x^2 + 1$
- $x^9 = x^3 + x$
- $x^{10} = x^2 + x + 1$
- $x^{11} = x^3 + x^2 + x$
- $x^{12} = x^3 + x^2 + x + 1$
- $x^{13} = x^3 + x^2 + 1$
- $x^{14} = x^3 + 1$
- $x^15 \equiv 1 \mod x^4 + x + 1$

Since $\text{ord}(x) \neq 1, 3, 5$ $\text{ord}(x) \mid 15$, so we have $\text{ord}(x) = 15$.

LEMMA 2.3.8. *Let $\alpha \in GF(q)^*$, $\operatorname{ord}(\alpha) = t$ and $s \in \mathbb{Z}$.*

$$\alpha^s = 1 \iff t \mid s$$

Proof. Let $s \in \mathbb{Z}$. By the division algorithm for integers,

$$s = \ell t + r$$

where $0 \leqslant r \leqslant t - 1$. Then

$$\alpha^s = \alpha^{\ell t + r} = (\alpha^t)^\ell \alpha^r = \alpha^r$$

So,

$$\begin{aligned} \alpha^s = 1 &\iff a^r = 1 \\ &\iff r = 0 \qquad \text{since } 0 \leqslant 1 \leqslant t - 1 \\ &\iff t \mid s \end{aligned}$$

□

COROLLARY 2.3.9. *If $\alpha \in GF(q)^*$, then $\operatorname{ord}(\alpha) \mid (q - 1)$.*

Proof. We know $\alpha^{q-1} = 1$, so $\operatorname{ord}(\alpha) \mid (q - 1)$ by the previous Lemma. □

DEFINITION 2.3.10. An element $\alpha \in GF(q)$ is a **generator** of $GF(q)^*$ if

$$\{\alpha^i : i \geqslant 0\} = GF(q)^*$$

That is, α generates all the non-zero field elements. $\operatorname{ord}(\alpha) = q - 1$.

THEOREM 2.3.11. *If α is a generator of $GF(q)^*$, then*

$$\{\alpha^1, \dots, \alpha^{q-1}\} = GF(q)^*$$

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THEOREM 2.3.12. *If $GF(q)^*$ has order t, then*

$$\alpha^1, \dots, \alpha^{t-1}$$

are pairwise distinct.

Proof. Suppose for a contradiction that $\alpha^i = \alpha^j$ where $0 \leqslant i, j \leqslant t - 1$. WLOG suppose $j > i$, then $\alpha^{j-i} = 1$ which contradicts $\operatorname{ord}(\alpha) = t$ since $1 \leqslant j - i \leqslant t - 1$. □

2.4 † Existence of Generators

LEMMA 2.4.1. *Let $\alpha \in GF(q)^*$ with $\operatorname{ord}(\alpha) = t$. Then $\operatorname{ord}(\alpha^i) = t/\gcd(t,i)$.*

Proof. Let $d = \gcd(t,i)$. The order of a^i is the smallest positive integer s such that $\alpha^{is} = 1$. Now,

$$\alpha^{is} = 1 \iff t \mid is \iff \frac{t}{d} \mid \frac{i}{d}s \iff \frac{t}{d} \mid s$$

Since the smallest positive integer s satisfying $\frac{t}{d} \mid s$ is $s = \frac{t}{d}$, we have $\operatorname{ord}(\alpha^i) = \frac{t}{d}$. □

LEMMA 2.4.2. *Let $\alpha, \beta \in GF(q)^*$, with $\operatorname{ord}(\alpha) = m$ and $\operatorname{ord}(\beta) = n$. If $\gcd(m,n) = 1$ then $\operatorname{ord}(\alpha\beta) = mn$.*

Proof. Let $t = \operatorname{ord}(\alpha\beta)$. Now,

$$(\alpha\beta)^{mn} = \alpha^{mn}\beta^{mn} = 1,$$

so $t \mid mn$. Also,

$$1 = (\alpha\beta)^{tn} = \alpha^{tn}\beta^{tn} = \alpha^{tn},$$

so $m \mid tn$. And, since $\gcd(m,n) = 1$, we have $m \mid t$. Similarly,

$$1 = (\alpha\beta)^{tm} = \alpha^{tm}\beta^{tm} = \beta^{tm},$$

so $n \mid tm$. And, since $\gcd(m,n) = 1$, we have $n \mid t$. Hence, since $\gcd(m,n) = 1$, we have $mn \mid t$. Thus $t = mn$. □

THEOREM 2.4.3. *Every finite field $GF(q)$ has a generator.*

Proof. Let α be an element of highest order in $GF(q)^*$; say $\operatorname{ord}(\alpha) = t$. Suppose that $t < (q-1)$.

If the order of every element in $GF(q)^*$ were to divide t then the equation $y^t - 1 = 0$ would have $q-1$ roots in $GF(q)$, which is impossible since $(q-1) > t$. Hence there exists an element $\beta \in GF(q)^*$ whose order b does not divide t.

Now, let ℓ be a prime such that the highest power of ℓ which divides b (say ℓ^e) is greater than the highest power of ℓ which divides t (say ℓ^f) — such a prime ℓ must exist since b does not divide t.

Consider the field elements $\alpha' = \alpha^{\ell^f}$ and $\beta' = \beta^{b/\ell^e}$. We have

$$\operatorname{ord}(\alpha') = \frac{t}{\gcd(t,\ell^f)} = \frac{t}{\ell^f}$$

and

$$\operatorname{ord}(\beta') = \frac{b}{\gcd(b,\ell^e)} = \frac{b}{b/\ell^e} = \ell^e$$

Since $\gcd(t/\ell^f, \ell^e) = 1$, we have $\operatorname{ord}(\alpha'\beta') = (t/\ell^f)(\ell^e) = t\ell^{e-f} > t$. This contradicts the hypothesis that the highest order of any element in $GF(q)^*$ is t. Hence the hypothesis that $t < (q-1)$ is wrong, and so $t = q-1$. Thus α is a generator of $GF(q)^*$. □

Chapter 3

Linear Codes

3.1 Introduction

Let $F = GF(q)$. Let $V_n(F) = \underbrace{F \times \cdots \times F}_{n} = F^n$. Then, $V_n(F)$ is an n-dimensional vector space over F and we have $|V_n(F)| = q^n$.

DEFINITION 3.1.1. Let $F = GF(q)$. A **linear (n,k)-code** over F is an n-dimensional subspace of $V_n(F)$.

DEFINITION 3.1.2. A **subspace** of a vector space V over F is a subset $S \subseteq V$ such that
V1 $\mathbf{0} \in S \implies S \neq \varnothing$
V2 $\boldsymbol{v}_1 + \boldsymbol{v}_2 \in S$, $\forall \boldsymbol{v}_1, \boldsymbol{v}_2 \in S$
V3 $\lambda \boldsymbol{v} \in S$, $\forall \lambda \in F$ and $\boldsymbol{v} \in S$
Note that $S \subseteq V$ is also a vector space over F.

Let C be an (n,k)-code over F. Let $v_1, \ldots, v_k$ be an ordered basis for C.

(1) The codewords in C are precisely:

$$m_1 \boldsymbol{v}_1 + \cdots + m_k \boldsymbol{v}_k$$

where $m_i \in F$. So, $|C| = M = q^k$ since there are q choices for each m. The length of C is n and has dimension k.

(2) The rate of C is

$$R = \frac{\log_q(M)}{n} = \frac{k}{n}$$

DEFINITION 3.1.3. The **Hamming weight** of $\boldsymbol{v} \in V_n(F)$, denoted $w(\boldsymbol{v})$ is the number of non-zero coordinate positions in V.

DEFINITION 3.1.4. The **Hamming weight of an (n,k)-code** C is:

$$w(C) = \min\{w(\boldsymbol{c}) : \boldsymbol{c} \in C,\ \boldsymbol{c} \neq \mathbf{0}\}$$

THEOREM 3.1.5. *If C is a linear code, then $d(C) = w(C)$.*

Proof.

$$\begin{aligned} d(C) &= \min\{d(\boldsymbol{x},\boldsymbol{y}) : \boldsymbol{x},\boldsymbol{y} \in C,\ \boldsymbol{x} \neq \boldsymbol{y}\} \\ &= \min\{w(\boldsymbol{x}-\boldsymbol{y}) : \boldsymbol{x},\boldsymbol{y} \in C,\ \boldsymbol{x} \neq \boldsymbol{y}\} \qquad \text{by (A2Q1a)} \\ &= \min\{w(\boldsymbol{c}) : \boldsymbol{c} \in C,\ \boldsymbol{c} \neq \boldsymbol{0}\} \qquad \text{since } C \text{ is a vector space} \\ &= w(C) \end{aligned}$$

□

3.2 Generator Matrices and the Dual Code

Since $M = q^k$, there are q^k source messages. We'll assume that the source messages are elements of $V_k(F)$. Then, a natural encoding rule is, given $(m_1, \dots, m_k) \in V_k(F)$ we'll encode the message as

$$c = m_1\boldsymbol{v}_1 + \cdots + m_k\boldsymbol{v}_k$$

The encoding rule depends on the basis chosen for C.

If $m = (m_1, \dots, m_k)$, then the encoding rule can be written as follows:

$$\begin{aligned} C &= (m_1, \dots, m_k) \begin{bmatrix} -v_1- \\ -v_2- \\ \vdots \\ -v_k- \end{bmatrix}_{k\times n} \\ &= mG \end{aligned}$$

Note that v_i are row vectors in this course.

DEFINITION 3.2.1. Let C be an (n, k)-code. A **generator matrix** G for C is a $k \times n$ matrix whose rows form a basis for C.

Note: An encoding rule for C with respect to G is $C = mG$. Performing elementary row operations on G gives a different matrix for the same code C due to the order of the basis.

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EXAMPLE 3.2.2. Consider a $\underbrace{\text{binary}}_{F=GF(2)=\mathbb{Z}_2}$ $(\underbrace{5}_{n}, \underbrace{3}_{k})$-code C. Then $M = q^k = 2^3$ and $R = \frac{k}{n} = \frac{3}{5}$.

$C = \langle \underbrace{10010}_{v_1}, \underbrace{01011}_{v_2}, \underbrace{00101}_{v_3} \rangle$.

$$G = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]_{3\times 5}$$

$\operatorname{rank}(G) = 3$, thus G is a generator matrix for C.

M (source messages) $\to C$ (codewords)
$000 \to 00000$
$001 \to 00101$
$010 \to 01011$
$011 \to 01110$
$100 \to 10010$
$101 \to 10111$
$110 \to 11001$
$111 \to 11100$

$d(C) = 2$, $e = 0$.

Note: Any matrix equivalent to G is also a generator matrix for C, but yields a different encoding rule.

DEFINITION 3.2.3. Let $[\,I_k \mid A\,]_{k\times n}$ be a generator matrix for an (n,k)-code C. If an (n,k)-code has a generator matrix of this form, then C is **systematic**, and the generator matrix is in **standard form**.

EXAMPLE 3.2.4. $C = \langle 100011, 101010, 100110\rangle$ is a non-systematic $(6,3)$-code. Some generator matrices are:

$$G_1 = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$R_2 + R_1$:

$$G_2 = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$R_3 + R_1$:

$$G_3 = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

Clearly C is not systematic. However, if every codeword is permuted by moving the second bit to the fourth bit, we get C' that is linear and has the same length, dimension, and distance as C.

DEFINITION 3.2.5. Let C be an (n,k)-code. If π is a permutation on $\{1,\dots,n\}$. Then $\pi(C)$ (that is, apply π to each codeword) is an (n,k)-code which is said to be an **equivalent code** for C.

THEOREM 3.2.6. *(1) If C and C' are equivalent codes, then*

$$d(C) = d(C')$$

(2) Every linear code is equivalent to a systematic code.

Proof. Let C be an (n,k) code. Let G be a generator matrix for C in RREF. Them, one can permute the columns of G to get a matrix $G' = [\,I_k \mid A\,]$ in standard form. Then, G' is a generator matrix for a code C' that is equivalent to C. □

DEFINITION 3.2.7. Let $\boldsymbol{x}, \boldsymbol{y} \in V_n(F)$. The **inner product** of $\boldsymbol{x}$ and $\boldsymbol{y}$ is

$$\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^{n} x_i y_i \in F$$

THEOREM 3.2.8. *If $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in V_n(F)$ and $\lambda \in F$, then*
(1) $\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{y} \cdot \boldsymbol{x}$
(2) $\boldsymbol{x} \cdot (\boldsymbol{y} + \boldsymbol{z}) = \boldsymbol{x} \cdot \boldsymbol{y} + \boldsymbol{x} \cdot \boldsymbol{z}$
(3) $(\lambda x) \cdot \boldsymbol{y} = \lambda(\boldsymbol{x} \cdot \boldsymbol{y})$
(4) $\boldsymbol{x} \cdot \boldsymbol{x} = \boldsymbol{0}$ *does **not** imply* $\boldsymbol{x} = \boldsymbol{0}$

EXAMPLE 3.2.9. Consider $V_2(\mathbb{Z}_2)$. Then, $(1, 1) \cdot (1, 1) = 0$.

DEFINITION 3.2.10. Let C be an (n, k)-code over F. The **dual code** of C is

$$C^\perp = \{\boldsymbol{x} \in V_n(F) : \boldsymbol{x} \cdot \boldsymbol{c} = \boldsymbol{0}\, \forall \boldsymbol{c} \in C\}$$

THEOREM 3.2.11. *Let $\boldsymbol{x} \in V_n(F)$.*

$$\boldsymbol{x} \in C^\perp \iff \boldsymbol{v}_1 \cdot \boldsymbol{x} = \cdots = \boldsymbol{v}_k \cdot \boldsymbol{x} = 0$$

Proof. ($\implies$) If $\boldsymbol{x} \in C^\perp$, then $\boldsymbol{x} \cdot \boldsymbol{c} = 0$ for all $\boldsymbol{c} \in C$. In particular,

$$\boldsymbol{x} \cdot \boldsymbol{v}_1 = \cdots = \boldsymbol{x} \cdot \boldsymbol{v}_k = 0$$

($\impliedby$) Suppose $\boldsymbol{x} \cdot \boldsymbol{v}_1 = \cdots = \boldsymbol{x} \cdot \boldsymbol{v}_k = 0$. Let $\boldsymbol{c} \in C$. We can write

$$\boldsymbol{c} = \lambda_1 \boldsymbol{v}_1 + \cdots + \lambda_k \boldsymbol{v}_k$$

for all $\lambda_i \in F$. Then,

$$\boldsymbol{x} \cdot \boldsymbol{c} = \lambda_1(\boldsymbol{x} \cdot \boldsymbol{v}_1) + \cdots + \lambda_k(\boldsymbol{x} \cdot \boldsymbol{v}_k) = 0$$

Hence, $\boldsymbol{x} \in C^\perp$. □

THEOREM 3.2.12. *If C is an (n, k)-code over F, then $C^\perp$ is an $(n, n - k)$-code over F.*

Proof. Consider

$$G = \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}_{k \times n}$$

Then, $\boldsymbol{x} \in C^\perp$ if and only if $G\boldsymbol{x}^\top = \boldsymbol{0}$. So, $C^\perp$ is the nullspace of G. Hence, $C^\perp$ is an $(n - k)$-dimensional subspace of $V_n(F)$. □

DEFINITION 3.2.13. If $\boldsymbol{x}, \boldsymbol{y} \in V_n(F)$ and $\boldsymbol{x} \cdot \boldsymbol{y} = 0$, then x and y are **orthogonal**.

THEOREM 3.2.14. *If C is a linear code, then $(C^\perp)^\perp = C$.*

Proof. Let C be an (n,k)-code. Then $C^\perp$ is an $(n, n-k)$-code. So, $(C^\perp)^\perp$ is an (n,k)-code. But $C \subseteq (C^\perp)^\perp$ by definition of $C^\perp$. Suppose C is a code over $F = GF(q)$. Then $|C| = q^k$ and $|(C^\perp)^\perp| = q^k$. Thus, $C = (C^\perp)^\perp$. □

THEOREM 3.2.15. *Let C be an (n,k)-code with standard form $k \times n$ generator matrix. Then, a generator matrix for $C^\perp$ is*

$$H = \left[\, -A^\top \mid I_{n-k} \,\right]_{(n-k)\times n}$$

Proof. $\operatorname{rank}(H) = n-k$, so H is indeed a generator matrix for some $(n, n-k)$-code $\overline{C}$. Now,

$$\begin{aligned} GH^\top &= [\, I_k \mid A \,] \begin{bmatrix} -A \\ I_{n-k} \end{bmatrix} \\ &= -A + A \\ &= 0 \end{aligned}$$

Since $GH^\top = 0$, every row of H is orthogonal to every row of G, so every vector in the row space of H is orthogonal to every vector in the row space of G. Hence, $\overline{C} \subseteq C$. Since $\dim(\overline{C}) = \dim(C^\perp)$ we have $\overline{C} = C^\perp$. □

3.3 The Parity-Check Matrix

DEFINITION 3.3.1. A generator matrix for $C^\perp$ is called a **parity-check matrix** (PCM) for C.

EXAMPLE 3.3.2. Consider a $(5,2)$-code C over $\mathbb{Z}_3$ with generator matrix

$$G = \begin{bmatrix} 2 & 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \leftarrow c_1 \\ \leftarrow c_2 \end{matrix}$$

Find the length, dimension, order, number of codewords, codewords, distance, weight and errors that can be corrected for C.
Solution.

- Length: $n = 5$ ((n,k)-code)
- Dimension: $k = 2$ ((n,k)-code)
- Order: $q = 3$ ($\mathbb{Z}_3$)
- Number of codewords: $M = q^k = 3^2 = 9$
- Codewords: $C = \{00000, 20210, 10120, 11001, 22002, 01211, 12212, 21121, 02122\}$
- Distance: $d(C) = w(C) = 3$
- Error-correcting capability: $e = 1$

Find a generator matrix for $C^\perp$.
Solution.

$$\begin{bmatrix} 2 & 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{bmatrix}$$

So,

$$H = \left[\begin{array}{cc|ccc} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

is a generator matrix for $C^{\perp}$ which is a $(5,3)$-code over $\mathbb{Z}_3$. $M = 3^3 = 27$.

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THEOREM 3.3.3. *Let C be an (n,k)-code over F, and let H be a PCM for C. Then $d(C) \geqslant s$ if and only if every $(s-1)$ columns of H are linearly independent over F.*

Proof. Let $h_1, \dots, h_n$ be the columns of H.

$(\Longleftarrow)$ Suppose $d(C) \leqslant s-1$, so $w(C) \leqslant s-1$. Let $c \in C$, with $1 \leqslant w(c) \leqslant s-1$. WLOG, suppose $c_j = 0$ for all $s \leqslant j \leqslant n$. Since $c \in C$, we have $Hc^{\top} = 0$. Therefore, $c_1 h_1 + \cdots + c_{s-1} h_{s-1} = 0$. Since $w(C) \geqslant 1$, this is a non-trivial linear combination of $h_1, \dots, h_{s-1}$ that equal 0. So, $h_1, \dots, h_{s-1}$ are linearly dependent over F.

$(\Longrightarrow)$ Suppose there are $s-1$ columns of H that are linearly dependent over F, say $h_1, \dots, h_{s-1}$. So, we can write

$$c_1 h_1 + \cdots + c_{s-1} h_{s-1}$$

where $c_j \in F$ not all zero. Let $c = (c_1, \dots, c_{s-1}, \underbrace{0 \cdots 0}_{n-s+1}) \in V_n(F)$. Then, $Hc^{\top} = 0$. So, $c_i C$ and $1 \leqslant w(c) \leqslant s-1$, so $d(C) \leqslant s-1$. □

COROLLARY 3.3.4. *Let C be an (n,k)-code over F with PCM H. Then, $d(C)$ is the smallest number of columns of H that are linearly dependent over F.*

EXAMPLE 3.3.5. Recall, we found a PCM

$$H = \left[\begin{array}{cc|ccc} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

for a $(5,2)$-code C over $\mathbb{Z}_3$. Find $d(C)$.
Solution.

- No 0 column in $H \implies d(C) \geqslant 2$
- No two linearly dependent columns in H (since no repeated columns, and no column is two times another column $\implies d(C) \geqslant 2$)

$$\begin{bmatrix} 2 & 1 & 0 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

Therefore $d(C) \not\geqslant 4$, therefore $d(C) = 3$.

EXAMPLE 3.3.6. Let C be a binary code with PCM H.

- $d(C) = 1 \iff H$ has a 0 column.
- $d(C) = 2 \iff$ the columns of H are non-zero and two are the same.
- $d(C) = 3 \iff$ the columns of H are non-zero, distinct, and one column is the sum of two other (distinct) columns.

3.4 Hamming Codes and Perfect Codes

EXAMPLE 3.4.1. Construct a $(7, 4, 3)$-binary code C.
Solution. Consider a PCM for C:

$$H = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right]_{3\times 7}$$

This is a **Hamming Code** of order 3 over $GF(2)$.

DEFINITION 3.4.2. Let C be an $[n, M]$-code with distance d over an alphabet A of size q. Let $e = \left\lfloor \frac{d-1}{2} \right\rfloor$. The **sphere packing bound** or **Hamming bound** is:

$$M \sum_{i=0}^{e} \binom{n}{i} (q-1)^i \leqslant q^n$$

DEFINITION 3.4.3. Let C be an $[n, M]$-code over A of distance d. Then, C is perfect if

$$M \sum_{i=0}^{e} \binom{n}{i} (q-1)^i = q^n$$

Note: If C is perfect, then IMLD=CMLD.

2020-01-03

For fixed n, q, d, a perfect code maximizes

$$R = \frac{log_q(M)}{n}$$

EXAMPLE 3.4.4. $GF(q)^n$ is a trivial perfect code with $d = 1$.
$C = \{\underbrace{0\cdots 0}_{n}, \underbrace{1\cdots 1}_{n}\}$ over $\mathbb{Z}_2$ is a perfect code if n is odd.

Proof.

$$\begin{aligned} 2\left(\sum_{i=0}^{e}\binom{n}{i}\right) &= 2\left(\binom{n}{0} + \binom{n}{e}\right) \\ &= \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{e+1} + \cdots \binom{n}{n-1} + \binom{n}{n} \\ &= (1+1)^n \\ &= 2^n \end{aligned}$$

□

Exercise: Prove that every perfect code must have odd distance (without using the theorem below)

THEOREM 3.4.5 (Tietäväinen, 1973)**.** *The only perfect codes are:*
(1) $V_n(GF(q))$.
(2) The binary replication code of odd length.
(3) The $(23, 12, 7)$*-binary Golay code and all codes equivalent to it.*
(4) The $(11, 6, 5)$*-ternary Golay code and all codes equivalent to it. A generator matrix for this code is:*

$$G = \left[\begin{array}{c|ccccc} & 1 & 1 & 1 & 1 & 1 \\ & 0 & 1 & 2 & 2 & 1 \\ I_6 & 1 & 0 & 1 & 2 & 2 \\ & 2 & 1 & 0 & 1 & 2 \\ & 2 & 2 & 1 & 0 & 1 \\ & 1 & 2 & 2 & 1 & 0 \end{array}\right]_{6\times 11}$$

(5) The Hamming codes and all codes of the same $[n, M, d]$ *parameters as them with* $d = 3$.

EXAMPLE 3.4.6. A Hamming code of order $r = 3$ over $GF(3)$ is a $(13, 10, 3)$-code over $GF(3)$ with PCM:

$$H = \left[\begin{array}{ccc|ccc|ccc|ccc|c} 1 & 0 & 0 & 1 & 0 & 1 & 2 & 2 & 0 & 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 2 & 1 & 2 & 2 & 1 \end{array}\right]_{3\times 13}$$

Observations:

(i) For every non-zero vector $v \in V_r(GF(q))$, exactly one scalar multiple of v must be a column of a PCM (for the Hamming code of order r over $GF(q)$)

(ii) The dimension of the code is indeed k since $\operatorname{rank}(\text{PCM}) = r = n - k$ since $\lambda_i e_i$ are columns of the PCM.

(iii) The Hamming codes have distance 3.

THEOREM 3.4.7. *Hamming codes are perfect.*

Proof. Recall that Hamming codes have $e = 1$ and $n = \frac{q^r-1}{q-1}$ with $r = n - k$.

$$\begin{aligned} M\sum_{i=0}^{e}\binom{n}{i}(q-1)^i &= q^{n-r}(1 + n(q-1)) \\ &= q^{n-r}\left(1 + \frac{q^r - 1}{q-1}(q-1)\right) \\ &= q^n \end{aligned}$$

□

DEFINITION 3.4.8. Suppose $\boldsymbol{c} \in C$ is transmitted. Suppose $\boldsymbol{r} \in V_n(F)$ is received. Then, the **error vector** is $\boldsymbol{e} = \boldsymbol{r} - \boldsymbol{c}$.

EXAMPLE 3.4.9 (Error Vector)**.** Over $\mathbb{Z}_3$, if $\boldsymbol{c} = (120212)$ is sent, and $\boldsymbol{r} = (122102)$ is received, then the error vector is $\boldsymbol{e} = (002220)$.

3.5 Decoding Single-Error Correcting Codes

Let H be a PCM for an (n, k)-code C over $GF(q)$ with $d \geqslant 3$.

$$\begin{aligned} H\boldsymbol{r}^\top &= H(\boldsymbol{c}+\boldsymbol{e})^\top \\ &= H\boldsymbol{c}^\top + H\boldsymbol{e}^\top \\ &= H\boldsymbol{e}^\top \qquad \text{since } c^\top \text{ is in nullspace of } H \end{aligned}$$

DEFINITION 3.5.1. Let H be a parity-check matrix for an (n, k)-code. The **syndrome** $\boldsymbol{s}$ of $\boldsymbol{r}$ is defined to be $\boldsymbol{s} = H\boldsymbol{r}^\top$.

Notes:

(1) $\boldsymbol{r}$ and $\boldsymbol{e}$ have the same syndrome

(2) If $\boldsymbol{e} = \boldsymbol{0}$, then $H\boldsymbol{e}^\top = \boldsymbol{0}$.

(3) If $w(\boldsymbol{e}) = 1$, say $\boldsymbol{e} = (0, \ldots, 0, \alpha, 0, \ldots, 0)$ where α is in the i^{th} position with $\alpha \neq 0$, then $H\boldsymbol{e}^\top = \alpha h_i$ where h_i is the i^{th} column of H.

(4) The converse of (2) and (3) are false.

```
Algorithm 1: Decoding Algorithm for Single-Error Correcting Codes
  Input  : $H$, $r$
  Output: Decoded vector
  Compute syndrome: $\boldsymbol{s} = H\boldsymbol{r}^\top$
  If $w(\boldsymbol{s}) = 0$, then accept $\boldsymbol{r}$; STOP.
  Compare $\boldsymbol{r}$ with the columns of $H$. If $\boldsymbol{s} = \alpha h_i$ with $\alpha \neq 0$, then correct $\boldsymbol{r}$ to $\boldsymbol{c} = \boldsymbol{r} - \boldsymbol{e}$; STOP.
  Reject, (not needed if $H$ is a Hamming code).
```

Claim: If $w(\boldsymbol{e}) \leqslant 1$, then the decoding algorithm always makes the correct decision.

Note: If H is a Hamming code and $w(\boldsymbol{e}) \geqslant 2$, then this decoding algorithm will always make the wrong decision.

EXAMPLE 3.5.2 (Single-Error Decoding)**.** Consider the $(7, 4, 3)$-binary Hamming code with PCM

$$H = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}_{3\times 7}$$

Decode $\boldsymbol{r} = (0111110)$.
Solution.
1. Compute $\boldsymbol{s} = H\boldsymbol{r}^\top = (011)^\top$.
2. $\boldsymbol{s}$ is the 6th column of H, so $\boldsymbol{e} = (0000010)$.
3. $\boldsymbol{r} \to 0111100$. Verify that $H\boldsymbol{c}^\top = \boldsymbol{0}$.

General Decoding Problem for Binary Linear Codes

Instance: An $(n-k) \times n$ matrix H over $GF(2)$ with $\operatorname{rank}(H) = n - k$. $\boldsymbol{r} \in V_n(GF(2))$.

Find: A vector $\boldsymbol{e} \in V_n(GF(2))$ of minimum weight with $H\boldsymbol{r}^\top = H\boldsymbol{e}^\top$.

Fact: This problem is NP-hard.

- P = problems solvable in "polynomial time"; that is, efficiently.
- NP = a "certain" class of problems including problems of strong practical interest which we do not know how to solve efficiently
- NP-hard = If any single problem in this class of problems can be solved efficiently, then so can all problems in NP, in which case P=NP.

2020-02-07

3.6 Decoding Linear Codes

Let C be an (n, k)-code over $F = GF(q)$ with PCM H.

DEFINITION 3.6.1. We write $\boldsymbol{x} \equiv \boldsymbol{y} \mod C$, where $\boldsymbol{x}, \boldsymbol{y} \in V_n(F)$ if $\boldsymbol{x} - \boldsymbol{y} \in C$.

Notes:

(1) $\equiv \mod C$ is an equivalence relation.

(2) The set of equivalence classes partitions $V_n(F)$.

(3) The equivalence classes containing $\boldsymbol{x} \in V_n(F)$ is called a **coset** of $V_n(F)$. This class is:

$$\begin{aligned}\{\boldsymbol{y} \in V_n(F) : \boldsymbol{y} \equiv \boldsymbol{x} \mod C\} &= \{\boldsymbol{x} + \boldsymbol{c} : \boldsymbol{c} \in C\} \\ &= C + \boldsymbol{x}\end{aligned}$$

We call $C + \boldsymbol{x}$ the coset of C represented by $\boldsymbol{x}$.

EXAMPLE 3.6.2 (Cosets)**.** Consider a $(5, 2)$-binary code with generator matrix

$$G = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}_{2\times 5}$$

with $d(C) = 3$. Find all cosets of C.
Solution. The cosets of C are:
(1) $C + 00000 = \{00000, 10111, 01110, 11001\} = \{\mathbf{0}, R_1, R_2, R_1 + R_2\} = C + 10111 = C + 01110 = C + 11001$
(2) $C + 10000 = \{10000, 00111, 11110, 01001\}$
(3) $C + 01000 = \{01000, 11111, 00000, 10001\}$
(4) $C + 00100 = \{00100, 10011, 01010, 11101\}$
(5) $C + 00010 = \{00010, 10101, 01100, 11011\}$
(6) $C + 00001 = \{00001, 10110, 01111, 11000\}$
(7) $C + 00011 = \{00011, 10100, 01101, 11010\}$
(8) $C + 11100 = \{11100, 01011, 10010, 00101\}$
In total, there are 8 cosets.

Notes:

(1) $C + \mathbf{0} = C$

(2) If $\boldsymbol{y} \in C + \boldsymbol{x}$, then $C + \boldsymbol{y} = C + \boldsymbol{x}$ by definition of equivalence.

(3) The number of cosets is ${}^{q^n}/_{q^k} = q^{n-k}$.

Recall: If $\boldsymbol{x} \in V_n(F)$, then it's syndrome is

$$\boldsymbol{s} = H\boldsymbol{r}^\top \in V_{n-k}(F)$$

THEOREM 3.6.3. *Let $\boldsymbol{x}, \boldsymbol{y} \in V_n(F)$. Then $\boldsymbol{x} \equiv \boldsymbol{y} \mod C$ if and only if $H\boldsymbol{x}^\top = H\boldsymbol{y}^\top$.*

Proof.

$$\begin{aligned} \boldsymbol{x} \equiv \boldsymbol{y} \mod C &\iff \boldsymbol{x} - \boldsymbol{y} \in C \\ &\iff H(\boldsymbol{x} - \boldsymbol{y})^\top = \boldsymbol{0} \\ &\iff H\boldsymbol{x}^\top = \boldsymbol{y}^\top \end{aligned}$$

□

So, cosets are characterized by their syndromes.

Decoding

- $\boldsymbol{c} \in C$ is sent.
- $\boldsymbol{r} \in V_n(F)$ is received.
- $\boldsymbol{e} = \boldsymbol{r} - \boldsymbol{c} \in V_n(F)$
- $H\boldsymbol{r}^\top = H\boldsymbol{e}^\top$.

So, $\boldsymbol{r}$ and $\boldsymbol{e}$ belong to the same coset of C.

CMLD

Given $\boldsymbol{r}$, find a vector $\boldsymbol{e}$ of smallest weight in $C + \boldsymbol{r}$ or equivalently, find a vector $\boldsymbol{e}$ of smallest weight with the same syndrome as $\boldsymbol{r}$. Then, decode $\boldsymbol{r}$ to $\boldsymbol{c} = \boldsymbol{r} - \boldsymbol{e}$.

IMLD

Find the unique vector $\boldsymbol{e}$ of smallest weight having the same syndrome as $\boldsymbol{r}$. If no such $\boldsymbol{e}$ exists, then reject $\boldsymbol{r}$. Otherwise, decode $\boldsymbol{r}$ to $\boldsymbol{c} = \boldsymbol{r} - \boldsymbol{e}$.

3.7 Syndrome Decoding Algorithm

Given a PCM H for an (n, k)-code C over $F = GF(q)$.

DEFINITION 3.7.1. A vector of smallest weight is a coset of C is distinguished and called a **coset leader** (of that coset).

Algorithm 2: Syndrome Decoding Algorithm

```
Input  : Table of cosets and $\boldsymbol{r}$.
Output: Decoded vector
while do
    Compute $\boldsymbol{s} = H\boldsymbol{r}^\top$
    Look up the coset leader corresponding to $\boldsymbol{s}$, say $\boldsymbol{\ell}$.
    Decode $\boldsymbol{r}$ to $\boldsymbol{c} = \boldsymbol{r} - \boldsymbol{\ell}$.
end
```

EXAMPLE 3.7.2 (Syndrome Decoding).

$$G = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}_{2\times 5}$$

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}_{3\times 5}$$

There are $q^{n-k} = 2^{5-2} = 2^3 = 8$ cosets in total.
Coset Leaders → Syndromes:

- $00000 \to 000$
- $10000 \to 111$
- $01000 \to 110$
- $00100 \to 100$
- $00010 \to 010$
- $00001 \to 001$
- $00011 \to 011$
- $10010 \to 101$

Suppose $\boldsymbol{r} = 10111$. Compute $\boldsymbol{s} = H\boldsymbol{r}^\top = (000)^\top$. The closest leader is $\boldsymbol{\ell} = 00000$, so $\boldsymbol{c} = \boldsymbol{r} - \boldsymbol{e} = 10111$.

Chapter 4

Some Special Linear Codes

2020-02-07

DEFINITION 4.0.1. A linear code C is **self-orthogonal** if $C \subseteq C^{\perp}$.

DEFINITION 4.0.2. A linear code C is **self-dual** if $C = C^{\perp}$.

For a binary (n, k)-code C, the syndrome table has size $2^{n-k} \times n$ which is exponentially large.

Goal: Design decoding algorithm which require very little space.

EXAMPLE 4.0.3. Use only the PCM H which is $(n-k) \times n$ bits.

4.1 The Binary Golay Code C23 (1949)

Let

$$\hat{B} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}_{12 \times 11}$$

Then, $\hat{G} = \begin{bmatrix} I_{12} \mid \hat{B} \end{bmatrix}_{12 \times 23}$ is a generator matrix for a $(23, 12)$-binary code called C_{23}.

Note: In $\hat{B}$,

- R_1 in only contains 1's.
- R_3 to R_{12} are left cyclic shifts of R_2.

THEOREM 4.1.1. *Facts:*

1. $d(C_{23}) = 7$.
2. C_{23} *is perfect.*

Proof. We know that $e = 3$, so $2^{12}\left[\binom{23}{0} + \binom{23}{1} + \binom{23}{2} + \binom{23}{3}\right] = 2^{23}$. □

4.2 The Extended Golay Code C24

Let

$$B = \left[\begin{array}{c|c} 0 & \hat{B} \\ \mathbf{1} & \end{array}\right]_{12\times 12}$$

where $\mathbf{1}$ is the column vector $(\underbrace{1,\ldots,1}_{\text{11 times}})^\top$.

Then, $G = [\ I_{12} \mid B\ ]_{12\times 24}$ is a generator matrix for a $(24, 12)$-binary code called C_{24}.

Notes:

(i) C_{24} is a $(24, 12, 8)$-binary code ($e = 3$)

(ii) $GG^\top = \mathbf{0}$

(iii) $C_{24} \subseteq C_{24}^\perp$, C_{24} is a self-orthogonal code.

(iv) $\dim(C_{24}) = 12$ and $d(C^\perp) = 12$, so $C_{24} = C_{24}^\perp$ (C_{24} is a self-dual code)

(v) B is symmetric

(vi) PCM for C_{24} is $H = \left[-B^\top \mid I_{12}\right] = [B \mid I_{12}]$

(vii) Since $C_{24} = C_{24}^\perp$, H is also a GM for C_{24}.

(viii) G is also a PCM for $C_{24}^\perp$.

4.2.1 Decoding Algorithm for C24

Compute a syndrome of $\boldsymbol{r}$. Find a vector $\boldsymbol{e}$ with $w(\boldsymbol{e}) \leqslant 3$, that has the same syndrome as $\boldsymbol{r}$. If no such $\boldsymbol{e}$ exists, then reject $\boldsymbol{r}$, otherwise decode $\boldsymbol{r}$ to $\boldsymbol{c} = \boldsymbol{r} - \boldsymbol{e}$.

Let $\boldsymbol{r} = (\boldsymbol{x}, \boldsymbol{y})$ and $\boldsymbol{e} = (\boldsymbol{e}_1, \boldsymbol{e}_2)$. There are five (not mutually exclusive) cases to consider. In the event that $w(\boldsymbol{e}) \leqslant 3$,

(A)) $w(\boldsymbol{e}_1) = 0$, $w(\boldsymbol{e}_2) = 0$

(B)) $1 \leqslant w(\boldsymbol{e}_1) \leqslant 3$, $w(\boldsymbol{e}_2) = 0$

(C)) $w(\boldsymbol{e}_1) = 1$ or 2, $w(\boldsymbol{e}_2) = 1$

(D)) $w(\boldsymbol{e}_1) = 0$, $1 \leqslant w(\boldsymbol{e}_2) \leqslant 3$

(E)) $w(\boldsymbol{e}_1) = 1$, $w(\boldsymbol{e}_2) = 1$ or 2

THEOREM 4.2.1. *Let C be an (n, k, d)-code over $GF(q)$. Let $\boldsymbol{x} = V_n(GF(q))$ with $w(\boldsymbol{x}) \leqslant \lfloor \frac{d-1}{2} \rfloor$. Then $\boldsymbol{x}$ is the unique vector of minimum weight in the coset of C containing $\boldsymbol{x}$ (so, it must be a coset leader).*

Proof. Suppose for a contradiction that $\boldsymbol{y}$ is a vector in the same coset of C as $\boldsymbol{x}$ with $\boldsymbol{y} \neq \boldsymbol{x}$ and

$$w(\boldsymbol{y}) \leqslant w(\boldsymbol{x}) \leqslant \left\lfloor \frac{d-1}{2} \right\rfloor$$

Then, $\boldsymbol{y} - \boldsymbol{x} \neq 0$, $\boldsymbol{x} \equiv \boldsymbol{y} \mod C$, and so $\boldsymbol{x} - \boldsymbol{y} \in C$. Now,

$$\begin{aligned} w(\boldsymbol{x} - \boldsymbol{y}) = w(\boldsymbol{x} + (-\boldsymbol{y})) &\leqslant w(\boldsymbol{x}) + w(\boldsymbol{y}) \\ &= w(\boldsymbol{x}) + w(\boldsymbol{y}) \\ &\leqslant \left\lfloor \frac{d-1}{2} \right\rfloor + \left\lfloor \frac{d-1}{2} \right\rfloor \\ &\leqslant d - 1 \end{aligned}$$

contradicting $d(C) = d$. □

2020-02-12

```
Algorithm 3: Decoding Algorithm for C24
  Input: $\boldsymbol{r} = (\boldsymbol{x}, \boldsymbol{y})$ is recieved.
  (1) Compute the syndrome $\boldsymbol{s}_1 = [\, I_{12} \mid B \,]\, \boldsymbol{r}^\top$. If $\boldsymbol{s}_1 = \boldsymbol{0}$, then accept $\boldsymbol{r}$ and STOP.
  (2) If $w(\boldsymbol{s}_1) \leqslant 3$, then correct $\boldsymbol{x}$ in the positions corresponding to the 1's in $\boldsymbol{s}_1$ and STOP.
  (3) Compare $\boldsymbol{s}_1$ to the columns (or rows) of $B$. If any column, say column $i$, differs in $1$ position from
      $\boldsymbol{s}_1$ (say position $j$) or 2 positions (say positions $j$ and $k$), then correct $\boldsymbol{r}$ as follows and STOP:
        • Correct $\boldsymbol{x}$ in positions $j$ and $k$.
        • Correct $\boldsymbol{y}$ in position $i$.
  (4) Compute the syndrome $\boldsymbol{s}_2 = [\, B \mid I_{12} \,]\, \boldsymbol{r}^\top$.
  (5) If $w(\boldsymbol{s}_2) \leqslant 3$, then correct $\boldsymbol{y}$ in the positions corresponding to the 1's in $\boldsymbol{s}_2$ and STOP.
  (6) Compare $\boldsymbol{s}_2$ to the columns (or rows) of $B$. If any column, say column $i$, differs in $1$ position from
      $\boldsymbol{s}_2$ (say position $j$) or 2 positions (say positions $j$ and $k$), then correct $\boldsymbol{r}$ as follows and STOP:
        • Correct $\boldsymbol{y}$ in positions $j$ and $k$.
        • Correct $\boldsymbol{x}$ in position $i$.
  (7) Reject $\boldsymbol{r}$.
```

EXAMPLE 4.2.2 (Decoding Algorithm for C24).
(i) Decode $\boldsymbol{r} = (1000\ 1000\ 0000\ 1001\ 0001\ 1101)$.
Solution. Compute $\boldsymbol{s}_1 = [\, I_{12} \mid B \,]\, \boldsymbol{r}^\top = (0100\ 1000\ 0000)^\top$. Since $w(\boldsymbol{s}_1) \leqslant 3$, we set $\boldsymbol{e} = (\boldsymbol{s}_1^\top, 0)$ and decode $\boldsymbol{r}$ to

$$\boldsymbol{c} = \boldsymbol{r} - \boldsymbol{e} = (1100\ 0000\ 0000\ 1001\ 0001\ 1101)$$

(ii) Decode $\boldsymbol{r} = (1000\ 0010\ 0000\ 1000\ 1101\ 0010)$.
Solution. Compute $\boldsymbol{s}_1 = [\, I_{12} \mid B \,]\, \boldsymbol{r}^\top = (1011\ \ 1110\ \ 1011)^\top$. Note that $w(\boldsymbol{s}_1) > 3$. Comparing $\boldsymbol{s}_1$ with the rows of B, we see that $\boldsymbol{s}_1$ differs in poisitions 6 and 7 from row 4 of B. Hence we set $\boldsymbol{e} = (0000\ 0110\ 0000\ 0001\ 0000\ 0000)$ and decode $\boldsymbol{r}$ to

$$\boldsymbol{c} = \boldsymbol{r} - \boldsymbol{e} = (1000\ 0100\ 0000\ 1001\ 1101\ 0010)$$

NOTE: In both examples we should check out answers by verifying that $H\boldsymbol{c}^\top = \boldsymbol{0}$ (i.e., $\boldsymbol{c}$ is indeed a codeword).

Note:

(1) If $w(\boldsymbol{e}) \leqslant 3$, then the algorithm makes the correct decision.

(2) No storage is needed:

$$\boldsymbol{s}_1 = [\, I_{12} \mid B \,]\, \boldsymbol{r}^\top = [\, I_{12} \mid B \,] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \boldsymbol{x} + B\boldsymbol{y}$$

where B is a left cyclic shift of the first row.

(3) The algorithm is very simple and efficient for hardware.

4.2.2 Reliability of C24

- $p =$ symbol error probability
- $C = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_M\}$
- $w_i =$ probability that the decoding algorithm makes an incorrect decision if $\boldsymbol{c}_i$ is sent.
- $P_C = \frac{1}{M}\sum\limits_{i=1}^{M} w_i$ error probability of C.
- $1 - P_C$ = reliability of C (correct decision)

p	$(1-p)^{12}$	$1-P_{C_{24}}$	$1-P_T$	$1-P_H$
0.1	0.28243	0.785738	0.71121	0.549043
0.01	0.886385	0.999909	0.99643	0.99037
0.001	0.988066	≈ 1	0.999964	0.999896
Rate	1	$1/2 = 0.5$	$1/3 = 0.3\overline{3}$	$11/15 = 0.7\overline{3}$

(1) If no source is used, then the reliability for 12-bit messages is

$$(1-p)^{12}$$

(2) C_{24}

$$1 - P_{C_{24}} = \left[(1-p)^{24} + \binom{24}{1}p(1-p)^{23} + \binom{24}{2}p^2(1-p)^{22} + \binom{24}{3}p^3(1-p)^{21}\right]$$

(3) Triplication Code T

$$1 - P_T = \left[(1-p)^3 + 3p(1-p)^2\right]^{12}$$

(4) $(15, 11)$-binary Hamming Code

$$1 - P_H = (1-p)^{15} + 15p(1-p)^{14}$$

Chapter 5

Cyclic Codes

2020-02-14 ♡

5.1 Introduction

DEFINITION 5.1.1. A subspace S of $V_n(F)$ is a **cyclic subspace** if $(a_0, a_1, \ldots, a_{n-1}) \in S \implies (a_{n-1}, a_0, \ldots, a_{n-2}) \in S$.

DEFINITION 5.1.2. A **cyclic code** is a cyclic subspace of $V_n(F)$.

5.2 Rings and Ideals

Let $R = F[x]/(x^n - 1)$. We write

$$\underbrace{(a_0, a_1, \ldots, a_{n-1})}_{\in V_n(F)} \longleftrightarrow \underbrace{a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}}_{\in R}$$

That is, there is an isomorphism between $V_n(F)$ and R.

- Addition is preserved: $a + b \longleftrightarrow a(x) + b(x)$
- Scalar multiplication is preserved: $\lambda a \longleftrightarrow \lambda a(x)$

Why choose $x^n - 1$?

Let $a = (a_0, \ldots, a_{n-1}) \in V_n(F)$. Let $a(x)$ be its associative polynomial in R. Then,

$$\begin{aligned} xa(x) &= a_0 x + a_1 x^2 + \cdots + a_{n-2}x^{n-1} + a_{n-1}x^n \\ &= a_{n-1} + a_0 x + \cdots + a_{n-2}x^{n-1} \mod x^n - 1 \\ &\longleftrightarrow (a_{n-1}, a_0, \ldots, a_{n-2}) \end{aligned}$$

So, multiplying a polynomial in R by x corresponds to a right cyclic shift of the associated vector.

We'll define $\cdot : V_n(F) \times V_n(F) \to V_n(F)$ by

$$a \cdot b \longleftrightarrow a(x)b(x) \mod x^n - 1$$

DEFINITION 5.2.1. Let R be a commutative finite ring. Then, the non-empty subset I of R is an **ideal** of R if
(1) For all $a, b \in I$, $a + b \in I$
(2) For all $a \in I$, $b \in R$, $ab \in I$
$\{0\}$ and R are defined to be **trivial** ideals of R.

THEOREM 5.2.2. *Let $S \subseteq V_n(F)$ be non-empty. Let I be the associated polynomials. Then S is a cyclic subspace of $V_n(F)$ if and only if I is an ideal of $R = F[x]/(x^n - 1)$.*

Proof. ($\Longrightarrow$) Suppose S is a cyclic subspace of $V_n(F)$. Since S is closed under addition, so is I. Let $a(x) \in I$, $b(x) = b_0 + \cdots + b_{n-1}x^{n-1} \in R$. Then $xa(x) \in I$ since S is a cyclic subspace. So, $x^i a(x) \in I$ for each $i \in [0, n-1]$. Also, $b_i x^i a(x) \in I$ since S is closed under scalar multiplication. Finally, $a(x)b(x) = a(x)(b_0 + \cdots + b_{n-1}x^{n-1})$ which is in I since I is closed under addition. Thus, I is an ideal.

($\Longleftarrow$) Suppose I is an ideal of R. Since I is closed under addition, so is S. Since I is closed under multiplication by constant polynomials, S is closed under scalar multiplication. Since I is closed under multiplication by x, S is closed under (right) cyclic shifts. Thus, S is a cyclic subspace. □

DEFINITION 5.2.3. Let $g(x) \in R$. Then $\langle g(x) \rangle = \{g(x)a(x) : a(x) \in R\}$ is an ideal of R called the **ideal generated by $g(x)$**. If I is an ideal of R, then I is a **principal** ideal if there exists a $g(x) \in I$ such that $I = \langle g(x) \rangle$. R is called the **principal ideal ring** if every ideal ring of R is principal.

THEOREM 5.2.4. *$R = F[x]/(x^n - 1)$ is a principal ideal ring.*

Proof. Let I be an ideal of R.

Suppose $I = \{0\}$, then $I = \langle 0 \rangle$ is principal.

Suppose $I \neq 0$. Let $g(x)$ be a polynomial of smallest degree in I. Let $a(x) \in I$. Long division gives

$$a(x) = \ell(x)g(x) + r(x)$$

where $\ell, r \in F[x]$ and $\deg(r) < \deg(g)$, but $\ell(x)g(x) \in I$ since I is closed under multiplication by R and $a(x) = \ell(x)g(x) \in I$. Therefore, $r(x) \in I$. Since $\deg(r) < \deg(g)$, we must have $r(x) = 0$ (since we define $\deg(0) = -\infty$). Hence, $a(x) = \ell(x)g(x)$. Therefore, $I = \langle g(x) \rangle$. Thus, R is a principal ideal ring. □

2020-02-24

5.3 Ideals and Cyclic Subspaces

DEFINITION 5.3.1. A **monic polynomial** $g(x)$ is a single-variable polynomial in which the non-zero coefficient of the highest degree of x is 1. That is,

$$g(x) = c_0 + \cdots + c_{\ell-1}x^{\ell-1} + x^{\ell}$$

for some constants c_i where $i \in [\ell - 1, 1]$.

If $I \neq \{0\}$, then we took $g(x) = a$ non-zero polynomial of smallest degree in I. Note, we can take $g(x)$ to be monic. If $g(x)$ is not monic, say

$$g(x) = c_0 + \cdots + c_\ell x^\ell$$

where $c_\ell \neq 0, 1$, then

$$c_\ell^{-1} g(x) = c_\ell^{-1} g_0 + \cdots x^\ell$$

is monic and is also in I. We'll call this process **making $\boldsymbol{g(x)}$ monic**.

DEFINITION 5.3.2. Let I be an ideal in $R = F[x]/(x^n - 1)$.
The **generator polynomial of $\boldsymbol{I}$** is:
(1) $x^n - 1$ since $x^n - 1 \equiv 0 \mod x^n - 1$ when $I = \{0\}$.
(2) **the** monic polynomial of least degree in I when $I \neq \{0\}$.

THEOREM 5.3.3. *Let I be a non-zero ideal in $R = F[x]/(x^n - 1)$.*
*(1) There is a **unique** monic polynomial g(x) of smallest degree in I.*
(2) $g(x) \mid (x^n - 1)$

Proof. (1) Suppose there exists two monic polynomials $g(x)$ and $h(x)$ of the same smallest degree in I. Then, $g(x) - h(x) \in I$ and $\deg(g - h) < \deg(g)$. Hence, we must have $g - h = 0$, so $g = h$.

(2) We can write

$$x^n - 1 = \ell(x)g(x) + r(x)$$

where $\ell, r \in F[x]$ and $\deg(r) < \deg(g)$. Then,

$$0 \equiv \ell(x)g(x) + r(x) \mod x^n - 1 \iff r(x) \equiv -\ell(x)g(x) \mod x^n - 1$$

Since $\langle g(x) \rangle = I$, we must have $r(x) \in I$. Hence, $\deg(r) < \deg(g)$ so we must have $r(x) = 0$. Thus,

$$g(x) \mid (x^n - 1)$$

□

THEOREM 5.3.4. *Let $h(x)$ be a monic divisor of $x^n - 1$ in $F[x]$. Then, **the** generator polynomial of $\langle h(x) \rangle$ is $h(x)$.*

Proof. If $h(x) = x^n - 1$, then $I = \{0\}$ and by definition, its generator polynomial is $x^n - 1$.

If $\deg(h) < n$, then $I \neq \{0\}$. Let $g(x)$ be **the** monic polynomial of smallest degree in I. Since g is a generator of I, we can write

$$g(x) \equiv a(x)h(x) \mod x^n - 1 \implies g(x) = a(x)h(x) + \ell(x)(x^n - 1)$$

for some $\ell \in F[x]$. Since $h \mid (x^n - 1)$, and $h \mid ah$, we have $h \mid g$. So, $\deg(h) \leqslant \deg(g)$ since g is a monic polynomial of smallest degree in I, we must have $\deg(g) \leqslant \deg(h)$, so $\deg(g) = \deg(h)$. Since g and h are both monic, we have $g = h$. □

COROLLARY 5.3.5. *There is a 1-1 correspondence between monic divisors of $x^n - 1$ in $F[x]$ and ideals in R. There is a 1-1 correspondence between monic divisors of $x^n - 1$ in $F[x]$ and cyclic subspaces of $V_n(F)$.*

EXAMPLE 5.3.6. Find all cyclic subspaces of $V_3(\mathbb{Z}_2)$.

Solution. The complete factorization of $x^3 - 1$ over $\mathbb{Z}_2$ is

$$x^3 - 1 = (1+x)(1+x+x^2)$$

Monic divisor of $x^3 - 1$	$\langle g_i(x) \rangle$	$\dim\langle g_i(x) \rangle$
$g_1(x) = 1$	$\{000, 001, \ldots, 111\}$	3
$g_2(x) = 1 + x$	$\{000, 110, 001, 101\}$	2
$g_3(x) = 1 + x + x^2$	$\{000, 111\}$	1
$g_4(x) = 1 + x^3$	$\{0\}$	0

2020-02-26

Midterm review session.

2020-02-28

$$\begin{array}{ccc}
V_n(F) & \longleftrightarrow & R = F[x]/(x^n - 1) \\
a = (a_0, a_1 \ldots, a_{n-1}) \in V_n(F) & \longleftrightarrow & a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} \in R \\
C : \text{ cyclic subspace, with } \dim(C) = k & \longleftrightarrow & I : \text{ ideal in } R \\
 & & g(x) \text{ with } \deg(g) = n - k \\
\text{GM for } C : G \text{ in terms of } g(x) & & \\
\text{Encoding: } mG & \longleftrightarrow & m(x)g(x) \\
C^{\perp} & \longleftrightarrow & h^*(x) \\
\text{PCM for } C : H & \longleftrightarrow & s(x) \equiv r(x) \mod g(x)
\end{array}$$

To find $h^*(x)$, we need $h(x) = (x^n - 1)/(g(x))$ where $\deg(h) = k$. Then, we find the reciprocal polynomial $h_R(x)$, and we make it monic to obtain $h^*(x)$.

Note: We do not know the distance of C, but we can use a BCH code and specifically select $g(x)$ to give a lower bound on $d(C)$.

LEMMA 5.3.7. *Let $g(x)$ be a monic divisor with $\deg(g) = n - k$ of $x^n - 1$ in $F[x]$. In fact,*

$$\langle g(x) \rangle = \{g(x)\overline{a}(x) : \deg(\overline{a}) < k\}$$

Proof. Let $h(x) = g(x)a(x) \mod x^n - 1$ where $\deg(a) < n$. So,

$$h(x) - g(x) = \ell(x)(x^n - 1)$$

for some $\ell \in F[x]$. Therefore, $g \mid h$. So, $h(x) = g(x)\overline{a}(x)$, for some $\overline{a} \in F[x]$ with $\deg(\overline{a}) \leqslant k - 1$. □

THEOREM 5.3.8. *Let $g(x)$ be a monic divisor of $x^n - 1$ with $\deg(g) = n - k$ of $x^n - 1$ in $F[x]$. Then, the cyclic code C generated by $g(x)$ has dimension k.*

Proof. We'll show that

$$B = \{g(x), xg(x), \ldots, x^{k-1}g(x)\}$$

is a basis of C.

We first show B is linearly independent. Suppose

$$\lambda_0 g(x) + \lambda_1 x g(x) + \cdots + \lambda_{k-1} x^{k-1} g(x) = 0$$

where $\lambda_i \in F$ for each $i \in [0, k-1]$. The coefficient x^{n-1} in the LHS is λ_{k-1}. The coefficient of x^{n-1} in the RHS is 0. Hence, $\lambda_{k-1} = 0$. Similarly,

$$\lambda_0 = \lambda_1 = \cdots = \lambda_{k-2} = 0$$

Thus, B is linearly independent.

We now show B spans C. Let $h(x) \in \langle g(x) \rangle$. By Lemma, we can write

$$h(x) = \underbrace{g(x)}_{\deg = n-k} \underbrace{a(x)}_{\deg = k-1}$$

for some $a \in F[x]$ where $\deg(a) \leqslant k-1$. Let

$$a(x) = \sum_{i=0}^{k-1} a_i$$

where $a_i \in F$ for each $i \in [0, k-1]$. Then,

$$h(x) = g(x)a(x) = \sum_{i=0}^{k-1} a_i(x^i g(x))$$

Thus, $\dim(C) = k$. □

5.4 Generator Matrices and Parity-Check Matrices

Therefore, a generator matrix for C is:

$$G = \begin{bmatrix} g(x) \\ xg(x) \\ \vdots \\ x^{k-1}g(x) \end{bmatrix}_{k \times n} = \begin{bmatrix} g(x) & 0 & \cdots & 0 & 0 \\ 0 & xg(x) & 0 & \cdots & 0 \\ & & \vdots & & \\ 0 & \cdots & 0 & 0 & x^{k-1}g(x) \end{bmatrix}_{k \times n}$$

Note: G is a non-systematic generator matrix for C.

Encoding

$$\begin{aligned} c &= mG \\ &= (m_0, \cdots, m_{k-1}) \begin{bmatrix} g(x) \\ xg(x) \\ \vdots \\ x^{k-1}g(x) \end{bmatrix} \\ &= m_0 g(x) + m_{k-1}x^{k-1}g(x) \\ &= g(x)(m_0 + \cdots + m_{k-1}x^{k-1}) \\ &\implies c(x) = m(x)g(x) \end{aligned}$$

EXAMPLE 5.4.1. Construct a cyclic $(7,4)$-code over $\mathbb{Z}_2$.
Solution. We need a monic divisor of degree 3 of $x^7 - 1$ in $\mathbb{Z}_2[x]$. Using Table 3 on page 157:

$$(x^7 - 1) = (1+x)(1+x+x^3)(1+x^2+x^3)$$

Let's take $g(x) = 1 + x + x^3$. Then, $\langle g(x) \rangle$ is a $(7,4)$-cyclic code over $\mathbb{Z}_2$. A generator matrix for C is:

$$G = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}_{4\times 7}$$

Encode $m = (1011)$.
Solution.

$$c = mG = (1111111)$$
$$\implies c(x) = m(x)g(x) = (1 + x + x^3)(1 + x + x^3) = (1 + x + \cdots + x^6) = c$$

2020-03-02

Let C be an (n,k)-cyclic cover over F with generator polynomial $g(x)$. Let

$$g(x) = \underbrace{g_0}_{\neq 0} + g_1 x + \cdots + \underbrace{g_{n-k}x^{n-k}}_{=1} \underbrace{g_{n-k+1}x^{n-k+1} + \cdots + g_{n-1}x^{n-1}}_{=0}$$

Let

$$h(x) = (x^n - 1)/(g(x)) = h_0 + h_1 x + \cdots + h_{k-1}x^{k-1} + h_k x^k + \cdots + h_{n-1}x^{n+0}$$

Let $a(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$. We know that

$$a(x) = g(x)h(x) \mod x^n - 1 \qquad (\star)$$

Note: $a(x) = 0$. Equating coefficients of x^i for each $i \in [0, n-1]$ of $(\star)$:

$$a_i = 0 = g_0 h_i + g_1 h_{i-1} + \cdots + g_i h_0 + g_{i+1} h_{n-1} + g_{i+1} h_{n-2} + \cdots + g_{n-1} h_{i-1}$$

Let $g = (g_0, \ldots, g_{n-1})$, $\overline{h} = (h_{n-1}, \ldots, h_0)$. Then, g is orthogonal to $\overline{h}$ and all the cyclic shifts of $\overline{h}$. Every cyclic shift of g is orthogonal to every click shift of $\overline{h}$.

Recall: A generator matrix for C is:

$$G = \begin{bmatrix} g_0 & g_1 & \cdots & g_{n-k} & 0 & 0 & \cdots & 0 \\ 0 & g_0 & \cdots & g_{n-k-1} & g_{n-k} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & g_0 & g_1 & g_2 & \cdots & g_{n-k} \end{bmatrix}_{k\times n}$$

Consider

$$H = \begin{bmatrix} h_k & h_{k-1} & \cdots & h_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & h_k & \cdots & h_1 & h_0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & h_k & h_{k-1} & h_{k-2} & \cdots & h_1 & h_0 \end{bmatrix}_{(n-k)\times n}$$

We have observed $GH^\top = 0$. Let C' be the code spanned by the rows of H. Then, $C' \subseteq C^\perp$. But, $\operatorname{rank}(H) = n - k$ (since $h_k = 1$). So, $\dim(C') = n - k$, hence we have $C' = C^\perp$. Thus, H is a PCM for C.

DEFINITION 5.4.2. Let $h(x) = h(x) = h_0 + h_1 x + \cdots h_k x^k$ be a degree k polynomial. The **reciprocal of h** is

$$h_R(x) = h_k x^0 + \cdots + h_1 x^{k-1} + h_0 x^k$$

Note:

- $h_R(x) = x^k h\left(\frac{1}{x}\right)$
- If $h_0 \neq 0$, then $h^*(x) = h_0^{-1} h_R(x)$.

THEOREM 5.4.3. *If C is an (n,k)-cyclic code, then $C^\perp$ is an $(n,n-k)$ cyclic code.*

Proof.

$$\begin{aligned} & g(x)h(x) = x^n - 1 \\ \implies & g\left(\frac{1}{x}\right) h\left(\frac{1}{x}\right) = \left(\frac{1}{x^n} - 1\right) \\ \implies & x^{n-k} g\left(\frac{1}{x}\right) \left(x^k h\left(\frac{1}{x}\right)\right) = (1 - x^n) \\ \implies & g_R(x) h_R(x) = -(x^n - 1) \\ \implies & h_R(x) \mid (x^n - 1) \end{aligned}$$

So, $h_R(x)$ is a degree k divisor of $x^n - 1$. Hence, the matrix H is a generator matrix for the cyclic code generated by $h^*(x)$. Thus, $C^\perp$ is cyclic with generator polynomial $h^*(x)$. □

5.5 Syndromes and Simple Decoding Procedures

$s = H\boldsymbol{r}^\top$. Let's find a more convenient PCM for C.

(i) Find a generator matrix for C of the form $[\ R \mid I_k\ ]_{k\times n}$ is (essentially systematic). For each $i \in [0, k-1]$, long division gives:

$$x^{n-k+i} = \underbrace{\ell_i(x)g(x)}_{\deg = n-k} + \underbrace{r_i(x)}_{\deg \leqslant n-k-1}$$

Then, $-r_i(x) + x^{n-k+i} = \ell_i(x)g(x) \in C$. Let

$$G = \begin{bmatrix} -r_0(x) + x^{n-k} \\ -r_1(x) + x^{n-k+1} \\ \vdots \\ -r_{k-1}(x) + x^{n-1} \end{bmatrix} = [\ R \mid I_k\ ]_{k\times n}$$

G has $\operatorname{rank} = k$, so G is a GM for C.

(ii) Construct a PCM for C.

This is $H = \left[\ I_{n-k} \mid -R^\top\ \right]_{(n-k)\times n}$. Then, $H\boldsymbol{r}^\top = r(x) \mod g(x)$.

2020-03-04

Recall: Let C be an (n,k)-cyclic code over $GF(q)$ with generator polynomial $g(x)$. One generator matrix for C is:

$$\begin{bmatrix} g(x) \\ xg(x) \\ \vdots \\ x^{k-1}g(x) \end{bmatrix}_{k\times n}$$

One PCM for C is:

$$H = \begin{bmatrix} h^*(x) \\ xh^*(x) \\ \vdots \\ x^{n-k-1}h^*(x) \end{bmatrix}_{(n-k)\times n}$$

Another generator matrix for C is:

$$G = [\ R \mid I_k\ ] = \left[\begin{array}{c|c} -r_0(x) & \\ -r_1(x) & \\ \vdots & I_k \\ -r_{k-2}(x) & \\ -r_{k-1}(x) & \end{array}\right]$$

where $x^{n-k+i} = \ell_i(x)g(x) + r_i(x) \implies -r_i(x) + x^{n-k+i} = \ell_i(x)g(x)$ for each $i \in [0, k-1]$. Then, another PCM for C is: $H = \left[\ I_{n-k} \mid -R^\top\ \right]_{(n-k)\times n}$. So,

$$H^\top = \begin{bmatrix} I_{n-k} \\ -R \end{bmatrix}_{n\times(n-k)} = \begin{bmatrix} x^0 \mod g(x) \\ x \mod g(x) \\ \vdots \\ x^{n-k-1} \mod g(x) \\ x^{n-k} \mod g(x) \\ x^{n-k+1} \mod g(x) \\ x^{n-1} \mod g(x) \end{bmatrix}$$

Hence, if $r = (r_0, r_1, \ldots, r_{n-1}) \in V_{n-1}(F)$, then

$$\begin{aligned} \boldsymbol{s} &= H\boldsymbol{r}^\top \\ &= (r_0x^0 \mod g(x)) + \cdots + (r_{n-1}x^{n-1} \mod g(x)) \\ &= (r_0x^0 + r_1x + \cdots + r_{n-1}x^{n-1}) \mod g(x) \\ &= r(x) \mod g(x) \end{aligned}$$

THEOREM 5.5.1. *Let C be a cyclic code with generator polynomial $g(x)$, and $\boldsymbol{r} \in V_n(F)$. Then, the syndrome of $\boldsymbol{r}$ with respect to the previous PCM is:*

$$s(x) = r(x) \mod g(x)$$

EXAMPLE 5.5.2. $g(x) = 1 + x + x^2 + x^3 + x^6$ is the generator polynomial for a $(15, 9)$-binary cyclic code. Check $g(x) \mid (x^15 - 1)$ over $GF(2)$. Compute the syndrome of $\boldsymbol{r} = (1110\ 1110\ 1100\ 000)$.
Solution. Long division of $(x^9+x^8+x^6+x^5+x^4+x^2+x+1)/(x^6+x^3+x^2+x+1)$ gives x^5+x^4+x+1 as the remainder. Thus,

$$s(x) = 1 + x + x^4 + x^5 \implies \boldsymbol{s} = (110011)$$

REMARK 5.5.3. Given the syndrome $\boldsymbol{s}$ of $\boldsymbol{r}$, the syndromes of cyclic shifts of $\boldsymbol{r}$ can be easily computed.

THEOREM 5.5.4. *Let $\boldsymbol{r} \in V_n(F)$, and $s \equiv r(x) \mod g(x) = s_0 + xs_1 + \cdots + s_{n-k-1}x^{n-k-1}$. Then the syndrome of $xr(x)$ is:*
(i) $xs(x)$, *if* $s_{n-k-1} = 0$
(ii) $xs(x) - s_{n-k-1}g(x)$, *if* $s_{n-k-1} \neq 0$

Proof. We have

$$r(x) = \ell(x)g(x) + s(x)$$

Multiply by x,

$$xr(x) = x\ell(x)g(x) + xs(x)$$

Case 1 If $s_{n-k-1} = 0$, then $\deg(s) \leqslant n-k-2$, so $\deg(xs(x)) \leqslant n-k-1$. So, $xs(x)$ is **the** remainder upon dividing $xr(x)$ by $g(x)$. So, $xs(x)$ is **the** syndrome of $r(x)$.

Case 2 If $s_{n-k-1} \neq 0$, then $\deg(s) = n-k-1$. Then

$$xr(x) = x\ell(x)g(x) + xs(x) + s_{n-k-1}g(x) - s_{n-k-1}g(x)$$

$$\implies xr(x) = (x\ell(x) + s_{n-k-1})g(x) + (xs(x) - s_{n-k-1}g(x))$$

Now,

$$xs(x) - s_{n-k-1} = (s_0 + \cdots + s_{n-k-1}x^{n-k}) - (\cdots + s_{n-k-1}x^{n-k}) = xr(x)$$

So, $xs(x) - s_{n-k-1}g(x)$ is **the** syndrome of $xr(x)$. □

2020-03-06

5.6 Burst Error Correcting

“Cyclic codes are good for (cyclic) burst error correcting.”

Suppose we have a $C : (n, k, d)$ code, with $e = \lfloor \frac{d-1}{2} \rfloor = 5$. In practice, errors typically happen in bursts (not spread out). We expect typically one burst per codeword, or bursts to carry through two codewords.

DEFINITION 5.6.1. Let $\boldsymbol{e} \in V_n(F)$. The **cyclic burst length of $\boldsymbol{e}$** is the length of the smallest cyclic block that contain all the non-zero entries of $\boldsymbol{e}$.

EXAMPLE 5.6.2. $\boldsymbol{e} = \mathbf{011}000001$ has cyclic burst length 4.

DEFINITION 5.6.3. We say $\boldsymbol{e}$ is a **cyclic burst error of length $\boldsymbol{t}$** if its cyclic burst length is t.

DEFINITION 5.6.4. A linear code C is a ***t*-cyclic burst error correcting code** if every cyclic burst error of length at most t lies in a unique coset of C. The largest such t is called the **cyclic burst error capability of $\boldsymbol{C}$**.

EXAMPLE 5.6.5. $g(x) = 1 + x + x^2 + x^3 + x^6$ generates a $(15, 9)$-binary cyclic code C that is a 3-cyclic burst error correcting code.

$d(C) \leqslant 5$, so $e \leqslant 2$. We verify this by checking that each cyclic burst of length $\leqslant 3$ has a unique syndrome.

Cyclic burst errors	Syndromes
0	000000
x^0	100000
x^1	010000
x^2	001000
x^3	000100
$\vdots$	
x^6	111100 ($x^6 + g(x)$)
x^7	011110
x^8	001111
x^9	111011
	(0001111+1111001)
$\vdots$	
x^{14}	111001
$1 + x$	110000
$x(1 + x)$	011000
$\vdots$	
$x^{14}(1 + x)$	011001
$1 + x + x^2$	111100
$x(1 + x + x^2)$	011100
$\vdots$	011100
$x^{14}(1 + x + x^2)$	001001
$1 + x^2$	101000
$x(1 + x^2)$	010100
$\vdots$	
$x^{14}(1 + x^2)$	101001

The number of cyclic bursts of length $\leqslant 3$ is 61. The number of syndromes is 64.

EXAMPLE 5.6.6. $g(x) = 1 + x^4 + x^6 + x^7 + x^8$ generates a $(15, 7)$-binary cyclic code that is 4-cyclic burst error correcting. Distance $\leqslant 5$ so $e \leqslant 2$.

Question: How to construct codes with high cyclic burst error correcting capability?

(1) Use a computer search

(2) RS Codes

(3) Interleaving

THEOREM 5.6.7. *Let C be an (n, k, d)-code over $GF(q)$. Let t be its cyclic burst error correcting capability.*

$$\left\lfloor \frac{d-1}{2} \right\rfloor \leqslant t \leqslant n - k$$

Proof. Every cyclic burst of length $\leqslant t$ has weight $\leqslant t$. Since every vector of weight $\leqslant \lfloor \frac{d-1}{2} \rfloor$ has a unique syndrome, we have $\lfloor \frac{d-1}{2} \rfloor \leqslant t$.

The number of cyclic burst errors where all the non-zero entries lie in the first t coordinate positions is q^t. Each of them has a unique coset and the total number of cosets is q^{n-k}. Thus,

$$q^t \leqslant q^{n-k} \implies t \leqslant n - k$$

□

Exercise: Prove that $t \leqslant \frac{n-k}{2}$.

5.7 Decoding Cyclic Burst Errors

Let C be a t-cyclic burst error correcting code generated by $g(x)$ which is a degree-k monic divisor of $x^n - 1$ over $GF(q)$.

Recall: A PCM for C is:

$$H = \left[\ I_{n-k} \mid -R^\top\ \right]$$

whose columns are $x^0 \mod g(x), \ldots, x^{n-1} \mod g(x)$. The syndrome of $r(x)$ is $s(x) \equiv r(x) \mod g(x)$.

Idea: Suppose $\boldsymbol{e}$ is a cyclic burst of length $\leqslant t$.

Compute $\boldsymbol{s} = H\boldsymbol{r}^\top \equiv r(x) \mod g(x)$.

Suppose $\boldsymbol{e} = \boxed{\text{x o} \cdots \text{o x x x}}$. We multiply x^3 by $\boldsymbol{e}$, so we get $\boxed{\text{x x x x o} \cdots \text{o}}$.

$\boldsymbol{s} = H\boldsymbol{r}^\top = H\boldsymbol{e}^\top$.

$\boldsymbol{s}_1 = H(x\boldsymbol{r})^\top = H(x\boldsymbol{e})^\top$

$\boldsymbol{s}_2 = H(x^2\boldsymbol{r})^\top = H(x^2\boldsymbol{e})^\top$

$\boldsymbol{s}_3 = H(x^3\boldsymbol{r})^\top = H(x^3\boldsymbol{e})^\top$

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Recall: Let C be an (n,k) code with generator polynomial $g(x)$. Suppose C is a t-c.b.e.c.c. So, $t \leqslant n-k$.

$$H = \left[\ I_{n-k} \mid -R^\top\right]$$

is a PCM for C; $s(x) = r(x) \mod g(x)$.

Idea: Suppose e is a cyclic burst of length at most t. Compute shifts of e, say $e_i = x^i e$ has all its non-zero entries in the first $(n-k)$ positions. Then,

$$s_i(x) = e_i(x) \mod g_i(x)$$

and we can recognize such an $s_i(x)$ since it is a non-cyclic burst of length at most t. Them, $e = x^{n-i}e_i$. How do we compute $s_i(x)$? Recall, $\boldsymbol{r} = \boldsymbol{c} + \boldsymbol{e}$. So, $x^i\boldsymbol{r} = x^i\boldsymbol{c} + x^i\boldsymbol{e}$, so $x^i\boldsymbol{r}$ and $x^i\boldsymbol{e}$ have the same syndrome.

5.8 Error Trapping Decoding (For Cyclic Burst Errors)

Let $r(x)$ = received polynomial. Let $s_i(x)$ = syndrome of $x^i r(x)$ for each $i \in [0, n-1]$. So, $s_0 = r(x) \mod g(x)$.

Algorithm 4:

```
Input :
Output:
for ( $i = 0;\ i < n - 1;\ i++$ ) {
    Compute $s_i(x)$. If $s_i(x)$ is a non-cyclic burst of length at most $t$, then
    (1) Let $e_i(x) = (s_i(x), 0)$
    (2) Let $e(x) = x^{n-i} e_i(x)$
    (3) Decode $r(x)$ to $r(x) - e(x)$.
}
```

EXAMPLE 5.8.1. $g(x) = 1 + x + x^2 + x^3 + x^6$ is the generator polynomial for $(15, 9)$-binary cyclic code with c.b.e.c.c 3. Decode $r = (1110\ 1110\ 1100\ 000)$.

Solution. Compute $s_0(x) = r(x) \mod g(x) = x^5 + x^4 + x + 1$.

i	$s_i(x)$
0	110011
1	100101
2	101110
3	110111
4	110111
5	100111
6	101111
7	101011
8	101001
9	101000

$$\Longrightarrow\ e_9 = (101000\ 0000000000)$$

$$\Longrightarrow\ \boldsymbol{e} = x^6 e_9 = (000000\ 101000\ 000)$$

$$\Longrightarrow\ \boldsymbol{c} = \boldsymbol{r} - \boldsymbol{e} = (1110\ 1100\ 0100\ 999)$$

Check: $H\boldsymbol{c}^\top = \mathbf{0} \leftarrow$ bad OR $g(x) \mid c(x)$.

5.9 Interleaving

Goal: Improve the c.b.e.c.c of a code.

Suppose C is an (n, k)-code with c.b.e.c.c t.

Suppose the following codewords are transmitted:

$$\begin{gathered} v_1 = (v_{11}, v_{12}, \ldots, v_{1n}) \in C \\ v_2 = (v_{21}, v_{22}, \ldots, v_{2n}) \in C \\ \vdots \\ v_s = (v_{s1}, v_{s2}, \ldots, v_{sn}) \in C \end{gathered}$$

Suppose $v_1, \ldots, v_s$ are transmitted in that order. If a cyclic burst error of length at most t occurs in any codeword, that error can be corrected.

Instead, we transmit: the columns in order:

$$[v_{11}, v_{21}, \ldots, v_{s1}, \ldots, v_{1n}, v_{2n}, \ldots, v_{sn}]$$

Now, if a cyclic burst error of length at most st occurs in this (fat) codeword, this means that each original codeword suffered a cyclic error burst of length at most t.

THEOREM 5.9.1. *Suppose C is an (n, k)-cyclic code with generator polynomial $g(x)$ and cyclic burst error correcting capability t. C^*, the code obtained by interleaving C to a depth s is an (ns, ks)-cyclic code with generator polynomial $g^*(x) = g(x^s)$.*

2020-03-11

5.10 Minimal Polynomials

Recall that if $F = GF(p^m)$ is a finite field of characteristic p, then $\mathbb{Z}_p$ is a subfield of F, and we can view F as an m-dimensional vector space over $\mathbb{Z}_p$. More generally, for any prime power q, $GF(q)$ is a subfield of $GF(q^m)$, and we can view $GF(q^m)$ as an m-dimensional vector space over $GF(q)$.

EXAMPLE 5.10.1. $GF(2^{16})$ is:

- a 16-dimensional vector space over $GF(2)$,
- an 8-dimensional vector space over $GF(2^2)$,
- a 4-dimensional vector space over $GF(2^4)$,
- a 2-dimensional vector space over $GF(2^8)$, and
- a 1-dimensional vector space over $GF(2^{16})$.

We call $GF(q^m)$ the **extension field**, and $GF(q)$ the **subfield**. Informally, $GF(q^m)$ is the “big field”, and $GF(q)$ is the “small field”.

Here is the main definition in this section:

DEFINITION 5.10.2. Let $\alpha \in GF(q^m)$. **The minimal polynomial of $\boldsymbol{\alpha}$ over $\boldsymbol{GF(q)}$**, denoted $m_\alpha(x)$, is the monic polynomial of smallest degree in $GF(q)[x]$ that has α as a root; that is, $m_\alpha(\alpha) = 0$.

REMARK 5.10.3.

(1) If $m_\alpha(x) \in GF(q)[x]$ is a non-zero polynomial with $m_\alpha(\alpha)$ and c is the leading coefficient of $m_\alpha(x)$, then $m'_\alpha(x) = c^{-1}m_\alpha(x)$ is a monic polynomial in $GF(q)[x]$ with $m'_\alpha(\alpha) = 0$.

(2) More generally, multiplying a polynomial by a non-zero constant does not change the roots of the polynomial.

(3) We have $m_0(x) = x$.

(4) If $\alpha \neq 0$, let t be the order of α (recall that $t \mid (q^m - 1)$). Then, α is a root of $x^t - 1 \in GF(q)[x]$. It follows that there does indeed exist a monic polynomial of smallest degree in $GF(q)[x]$ having α as a root.

EXAMPLE 5.10.4. We found the minimal polynomial of elements in $GF(2^2) = \mathbb{Z}_2[x]/(x^2 + x + 1)$ over $GF(2)$ by trial and error:

- $m_0(y) = y$.
- $m_1(y) = y + 1$.
- $m_x(y) = y^2 + y + 1$.
- $m_{x+1}(y) = y^2 + y + 1$.

THEOREM 5.10.5. *Let $\alpha \in GF(q^m)$.*

(i) The minimal polynomial, $m_\alpha(x)$ of α over $GF(q)$ is unique.

(ii) $m_\alpha(x)$ is irreducible over $GF(q)$.

(iii) $\deg(m_\alpha) \leqslant m$.

(iv) If $f(x) \in GF(q)[x]$, then, $f(\alpha) = 0$ if and only if $m_\alpha(x) \mid f(x)$.

Proof.

(i) Suppose there are two monic polynomials, $m_1(x)$ and $m_2(x)$, of (the same) smallest degree in $GF(q)[x]$ that have α as a root. Consider $r(x) = m_1(x) - m_2(x)$. Then,

$$r(\alpha) = m_1(\alpha) - m_2(\alpha) = 0 - 0 = 0$$

But, $\deg(r) < \deg(m_1)$, and so we conclude that $r(x) = 0$. Hence, $m_1(x) = m_2(x)$.

(ii) Suppose that m_α is reducible over $GF(q)$. Then, we can write

$$m_\alpha(x) = s(x)t(x)$$

for some $s, t \in GF(q)[x]$ with $\deg(s), \deg(t) < \deg(m_\alpha)$. Then,

$$m_\alpha(\alpha) = 0 = s(\alpha)t(\alpha),$$

and hence either of $s(\alpha) = 0$ or $t(\alpha) = 0$. In either case, we have a contradiction of the minimality of $\deg(m_\alpha)$. We conclude that m_α is irreducible over $GF(q)$.

(iii) Recall that $GF(q^m)$ can be viewed as an m-dimensional vector space over $GF(q)$. Thus, the $m+1$ field elements $1, \alpha, \alpha^2, \dots, \alpha^m$ are linearly dependent over $GF(q)$. Thus, we can write

$$a_0 + a_1\alpha + \cdots + a_m\alpha^m = 0,$$

where $a_0, a_1, \dots, a_m \in GF(q)$, and not all are 0. Hence, α is a root of the non-zero polynomial

$$a_0 + a_1x + \cdots + a_mx^m \in GF(q)[x]$$

having degree $\leqslant m$. It follows that $\deg(m_\alpha) \leqslant m$. □

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We proved $(i) - (iii)$ last class. We now prove (iv).

Proof. Let $f \in GF(q)[x]$. Using the division algorithm for polynomials, we can write

$$f(x) = \ell(x)m_\alpha(x) + r(x)$$

where $\ell, r \in GF(q)[x]$ and $\deg(r) < \deg(m_\alpha)$. Now,

$$f(\alpha) = \ell(\alpha)m_\alpha(\alpha) + r(\alpha) = r(\alpha)$$

Hence,

$$f(\alpha) = 0 \iff r(\alpha) = 0 \iff r(x) = 0 \text{ (since } \deg(r) < \deg(m_\alpha)\text{)} \iff m_\alpha(x) \mid f(x).$$

□

THEOREM 5.10.6. *Let $\alpha \in GF(q^m)$. Then, $\alpha \in GF(q)$ if and only if $\alpha^q = \alpha$.*

Proof. Since $\alpha^q = \alpha$ for all $\alpha \in GF(q)$, the elements of $GF(q)$ are roots of the polynomial $X^q - X$. Since this polynomial has degree q, it can't have any other roots in $GF(q^m)$. Thus, $\alpha \in GF(q)$ if and only if $\alpha^q = \alpha$. □

DEFINITION 5.10.7. Let $\alpha \in GF(q^m)$. Let t be the smallest positive integer such that $\alpha^{q^t} = \alpha$ (note that $t \leqslant m$). Then, **the set of conjugates of α with respect to $GF(q)$** is

$$C(\alpha) = \{\alpha, \alpha^q, \alpha^{q^2}, \ldots, \alpha^{q^{t-1}}\}$$

Note that the elements of $C(\alpha)$ are distinct.

THEOREM 5.10.8. *Let $\alpha \in GF(q^m)$. Then the minimal polynomial of α over $GF(q)$ is*

$$\begin{aligned} m_\alpha(x) &= \prod_{\beta \in C(\alpha)} (x - \beta) \\ &= (x-\alpha)(x-\alpha^q)(x-\alpha^{q^2})\cdots(x-\alpha^{q^{t-1}}). \end{aligned}$$

Proof.

(i) Clearly, $m_\alpha(x)$ is monic.

(ii) Clearly, $m_\alpha(\alpha) = 0$.

(iii) † Let $m_\alpha(x) = \sum_{i=0}^{t} m_i x^i$. The coefficients m_i are in $GF(q^m)$. We need to prove that $m_\alpha(x) \in GF(q)$. Now,

$$\begin{aligned} m_\alpha(x)^q &= \prod_{\beta \in C(\alpha)} (x-\beta)^q \\ &= \prod_{\beta \in C(\alpha)} (x^q - \beta^q) \\ &= \prod_{\beta \in C(\alpha)} (x^q - \beta), \qquad \text{since } C(\alpha) = \{B^q : \beta \in C(\alpha)\} \qquad (1) \\ &= m_\alpha(x^q) \\ &= \sum_{i=0}^{t} m_i x^{iq}. \end{aligned}$$

Also,

$$\begin{aligned} m_\alpha(x)^q &= \left(\sum_{i=0}^{t} m_i x^i\right)^q \\ &= \sum_{i=0}^{t} m_i^q x^{iq} \end{aligned} \tag{2}$$

Comparing coefficients of x^{iq} in (1) and (2) gives $m_i = m_i^q$ for all $i \in [0, t]$. Hence, $m_i \in GF(q)$. Thus, $m_\alpha(x) \in GF(q)[x]$.

(iv) † Let $f \in GF(q)[x]$ with $f(x) \neq 0$, and assume $f(\alpha) = 0$. Let $f(x) = \sum_{i=0}^{d} f_i x^i$. Then,

$$f(\alpha^q) = \sum_{i=0}^{d} f_i \alpha^{iq} = \left(\sum_{i=0}^{d} f_i \alpha_i\right)^q = f(\alpha)^q = 0.$$

Hence, the elements of $C(\alpha)$ are the roots of $f(x)$. Since the roots of $m_\alpha(x)$ are precisely the elements of $C(\alpha)$, we conclude that $m_\alpha(x)$ is the monic polynomial of smallest degree in $GF(q)[x]$ that has α as a root.

□

EXAMPLE 5.10.9. Consider $GF(2^4) = \mathbb{Z}_2[x]/(x^4+x+1)$. Find the minimal polynomial of $\beta = x^2+x^3$ over $\mathbb{Z}_2$. (In this example, we have $q = 2$ and $m = 4$)
Solution. When doing computations by hand, it will help to have a generator α of $GF(2^4)^*$, and a table of powers of α. It turns out that $\alpha = x$ is a generator as the following table shows.

$$\begin{array}{cccc}
\alpha^0 = 1 & \alpha^4 = 1+\alpha & \alpha^8 = 1+\alpha^2 & \alpha^{12} = 1+\alpha+\alpha^2+\alpha^3 \\
\alpha^1 = \alpha & \alpha^5 = \alpha+\alpha^2 & \alpha^9 = \alpha+\alpha^3 & \alpha^{13} = 1+\alpha^2+\alpha^3 \\
\alpha^2 = \alpha^2 & \alpha^6 = \alpha^2+\alpha^3 & \alpha^{10} = 1+\alpha+\alpha^2 & \alpha^{14} = 1+\alpha^3 \\
\alpha^3 = \alpha^3 & \alpha^7 = 1+\alpha+\alpha^3 & \alpha^{11} = \alpha+\alpha^2+\alpha^3 & \alpha^{15} = 1
\end{array}$$

Now, $\beta = \alpha^6$. Hence, $C(\beta) = C(\alpha^6) = \{\alpha^6, \alpha^{12}, \alpha^9 = \alpha^{24}, \alpha^3 = \alpha^{18}\}$. Therefore,

$$\begin{aligned}
m_\beta(y) &= (y-\alpha^6)(y-\alpha^{12})(y-\alpha^9)(y-\alpha^3) \\
&= [(y-\alpha^6)(y-\alpha^{12})][(y-\alpha^9)(y-\alpha^3)] \\
&= [y^2+(\alpha^6+\alpha^{12})y+\alpha^3][y^2+(\alpha^9+\alpha^3)y+\alpha^{12}] \\
&= [y^2+\alpha^4 y+\alpha^3][y^2+\alpha y+\alpha^{12}] \\
&= y^4+(\alpha+\alpha^4)y^3+(\alpha^{12}+\alpha^3+\alpha^5)y^2+(\alpha^{16}+\alpha^4)y+1 \\
&= y^4+y^3+y^2+y+1 \in \mathbb{Z}_2
\end{aligned}$$

Note that the coefficients of $m_\beta(y)$ are indeed in $GF(2)$.
Note also that we simplified terms such as $\alpha^3+\alpha^6$ to α^2 by using the table powers of α.