CO 250 - Introduction to Optimization

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1.1 Example (Manufacturing Tables and Chairs)

Process: raw materials → machine → labour → final products

Rules:

- Company has 30 workers and 40 machines available 40hrs/week.
- Manufacturing a table requires 2 machine-hours and 1 labour-hour.
- Manufacturing a chair requires 1 machine-hours and 3 labour-hours.
- Each manufacturer table yields \$10 of profit and each manufacturer chair yields \$15 of profit.

Goal: The company wants to prepare a weekly production plan which maximizes total profit.

Variables:

- $x_1 :=$ the number of tables manufactured per week
- $x_2 :=$ the number of chairs manufactured per week

The total profit per week can be modelled by $10x_1 + 15x_2$.

Constraints:

- Machine-hours used per week ≤ machine-hours available per week which can be modelled by $2x_1 + x_2 \leq 40 \times 40 = 1600$
- Labour-hours used per week ≤ labour-hours available per week which can be modelled by $x_1 + 3x_2 \leq 30 \times 40 = 1200$

We can then create a linear programming (LP) model.

$$\max 10x_1 + 15x_2$$

subject to

$$\begin{aligned} 2x_1 + x_2 &\leq 1600 \\ x_1 + 3x_2 &\leq 1200 \\ x_1, x_2 &\geq 0 \end{aligned}$$

1.2 Example (A General Production Planning Problem)

There are resources $I := \{1, \ldots, m\}$ and products $J := \{1, \ldots, n\}$. There are b_i units of resource i available per week $\forall i \in I$. One unit of product j yields c_j of profit for $\forall j \in J$. Manufacturing one unit of product j requires a_{ij} units of resource i. We want to maximize the total profit of this manufacturing process. $x_j :=$ amount of product j manufactured per week. (LP)

$$\max c_1x_1 + \cdots + c_nx_n = \sum_{j=1}^{n} c_jx_j$$

subject to

$$\begin{aligned} \sum_{j=1}^{n} a_{ij}x_{ij} &\le b_i & \forall i \in \{1, \dots, m\} \\ x_j &\ge 0 & \forall j \in \{1, \dots, n\} \end{aligned}$$

Remark 1. If $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $\boldsymbol{x} \le \boldsymbol{y}$, then $x_1 \le y_1, \dots x_n, \le y_n$.

Remark 2.

$$\boldsymbol{c} := \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \boldsymbol{x} := \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, A := \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \boldsymbol{b} := \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Given $A, \boldsymbol{b}, \boldsymbol{c}$ with $\boldsymbol{x} \in \mathbb{R}^n$ as the variable vector, we realize that $\boldsymbol{c}^\top \boldsymbol{x} = \sum_{j=1}^{n} c_jx_j$ is exactly the model that we wanted to maximize in 1.2 such that it satisfies $A\boldsymbol{x} \le \boldsymbol{b}$, with $\boldsymbol{x} \ge \mathbb{0}$.

1.3 Definition (Affine Function)

Let $f : \mathbb{R}^n \to \mathbb{R}$. f is an *affine function* if $f(x) = \boldsymbol{a}^\top \boldsymbol{x} + \beta$ for some $\boldsymbol{a} \in \mathbb{R}^n$, and $\beta \in \mathbb{R}$.

1.4 Definition (Linear Function)

Let $f : \mathbb{R}^n \to \mathbb{R}$. f is a *linear function* if f is an affine function such that $\beta = 0$.

Remark 3. Every linear function is affine, but the converse is not true.

1.5 Definition (Linear Constraint)

A *linear constraint* is one of

$$\begin{aligned} f(x) &\le \beta \\ f(x) &= \beta \\ f(x) &\ge \beta \end{aligned}$$

where f is a linear function and $\beta \in \mathbb{R}$

1.6 Definition (Linear Program)

A *linear program* (LP) is a problem of minimizing or maximizing an affine function subject to a finite number of constraints.

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Recall the family of LP problems:

$$\max \boldsymbol{c}^\top \boldsymbol{x}$$

subject to

$$\begin{aligned} A\boldsymbol{x} &\leq \boldsymbol{b} \\ \boldsymbol{x} &\geq \mathbb{0} \end{aligned}$$

An assignment of values to all variables such that every constraint is satisfied is called a feasible solution. A feasible region is the set of all feasible solutions. An optimal solution is a feasible solution which has the best possible objective value among all feasible solutions. Note that an optimization problem may have many optimal solutions, but it may have one optimal value.

2.1 Example (Refer to 1.1)

Suppose an entrepreneur offers at most 500 machine hours/week (rental) at $2.5/hour. Can we incorporate this new situation into our mathematical model? Can it still be a LP? Yes. $x_3 :=$ the number of machine hours rented from the business person per week.

$$\max 10x_1 + 15x_2 - 2.5x_3$$

subject to

$$\begin{aligned} 2x_1 + x_2 &\leq 1600 + x_3 \\ x_1 + 3x_2 &\leq 1200 \\ x_3 &\leq 500 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

2.2 Example (Constraints on Ratios, Percentages and Proportions)

Suppose we are required to manufacture at least 10 tables and 80 chairs per week. Also ${}^{\#\text{of tables manufactured}}/_{\text{week}} \big/ {}^{\#\text{of chairs manufactured}}/_{\text{week}} \geq 6$.

$$\left\{ \begin{aligned} x_1 &\geq 10 \\ x_2 &\geq 80 \\ {}^{x_2}/_{x_1} &\geq 6 \end{aligned} \right\} \iff \left\{ \begin{aligned} x_1 &\geq 10 \\ x_2 &\geq 80 \\ x_2 &\geq 6x_1 \end{aligned} \right\}$$

In general suppose f, g are affine functions

$$b_1 \leq {}^{f(x)}/_{g(x)} \leq b_2$$

provided that $g(x) > 0$ for every feasible solution x we can equivalently write

$$\begin{aligned} f(x) &\leq b_2 g(x) \\ f(x) &\geq b_1 g(x) \end{aligned}$$

2.3 Example (Multi-period, Multi-stage optimization problems)

Consider planning for multiple periods where in each period we want to decide how much to produce, how much to keep in stock (inventory) for the upcoming periods. Suppose we have just one period (WLOG),

- $d_t :=$ the demand for the end of period t in # of units (given)
- $s_t :=$ the # of units of products in stock at beginning of period t
- $p_t :=$ the # of units of products manufactured at period t

Key constraints

$$\begin{aligned} p_t + s_t = d_t + s_{t+1} & \qquad \forall t \in \{0, \dots, T\} \\ p_t, s_t, d_t \geq 0 & \qquad \forall t \in \{0, \dots, T\} \end{aligned}$$

Remark 4. Typically we have additional constraints on s_0 and s_{T+1}.

2.4 Definition (Integer Program)

An *integer program* (IP) is obtained from linear program by requiring a non-empty subset of variables to be integers.

Remark 5. If all variables are restricted to be integers $\to$ Pure IP, and if at least some variables may take real values $\to$ Mixed IP

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Recall, IP problems are obtained from LP problems by requiring a non-empty subset of variables to be integers. So, in IP problems we are allowed to have constraints $x_i \in \mathbb{Z}$, $x \in \mathbb{Z}^n$, $x_i \in \{0, 1\}$, x_i is an integer, $x_i \in \{0, 1\}^n$.

3.1 Example (SPIT - Smart People Institute of Technology)

SPIT has a campus near the North Pole. They have three buildings named A, B, C which need to be renovated to be served as one of a Library, Laboratory, or Gym (sometimes called functions). Each building must be assigned one activity, and each activity must be assigned one building. Renovation costs in millions of dollars are given:

	Library	Laboratory	Gym
A	10	60	20
B	60	70	50
C	20	60	40

Find an assignment of activities to buildings so that the total renovation cost is minimized.

Let us generalize to n buildings and n activities.

$$x_{ij} := \begin{cases} 1, \text{ if } i \text{ is assigned to activity } j \\ 0, \text{ otherwise} \end{cases} \qquad \forall i, j \in \{1, \ldots, n\}$$

$$c_{ij} := \text{renovation cost for assigning activity } j \text{ to building } i$$

(LP)

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad \forall j \in \{1, \ldots, n\} \tag{1}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall i \in \{1, \ldots, n\} \tag{2}$$

$$x_{ij} = \{0, 1\} \qquad \forall i, j \in \{1, \ldots, n\} \tag{3}$$

(1) $\implies$ every activity is assigned exactly one building
(2) $\implies$ every building is assigned exactly one activity

Remark 6. This example is frequently known as an *assignment problem*.

Suppose $c_{ij} \in \mathbb{R}$ and consider the inequality version:

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to

$$\sum_{i=1}^{n} x_{ij} \leq 1 \qquad \forall j \in \{1, \ldots, n\}$$

$$\sum_{j=1}^{n} x_{ij} \leq 1 \qquad \forall i \in \{1, \ldots, n\}$$

$$x_{ij} = \{0, 1\} \qquad \forall i, j \in \{1, \ldots, n\}$$

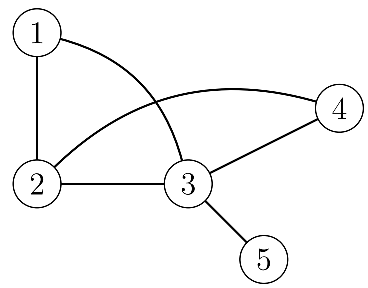
We can generalize this class optimization problem further.

3.2 Definition (Undirected Graph)

An *undirected graph* is a pair $G = (V, E)$, where V is a finite set of elements called *vertices*, and E is a set of pairs of distinct vertices called *edges*. All edges in an undirected graph are bidirectional.

3.3 Example (Undirected Graph)

Given $G :=$



we have

$$V = \{1, \dots, 5\}$$
$$E = \{12, 13, 23, 24, 35, 34\}$$

3.4 Definition (Matching)

Given a graph $G = (V, E)$, a *matching* M in G is a subset of edges in G such that no two edges in M share a common vertex.

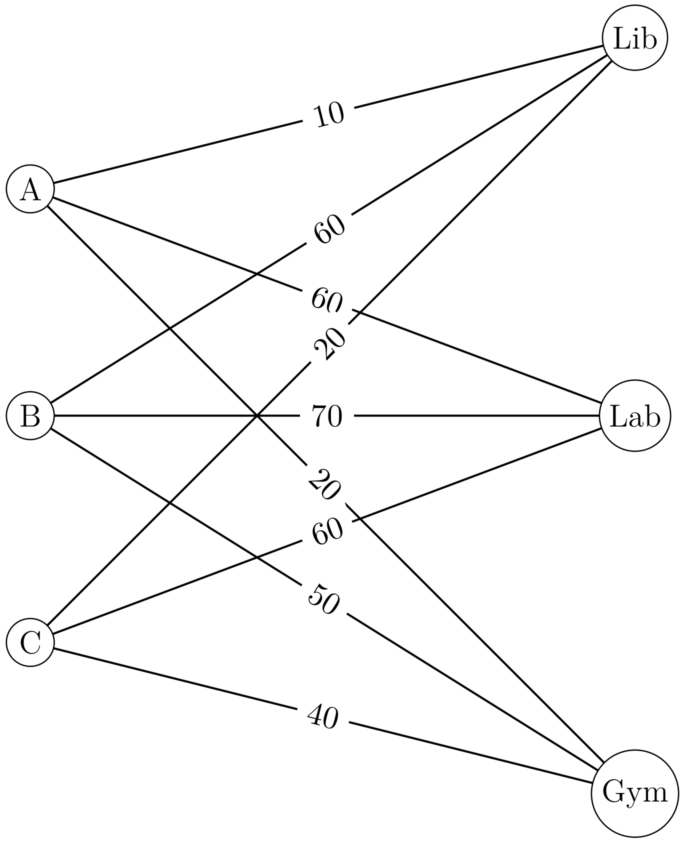
In the above example:

Matching	Not matching
$M := \{12\}$	$M := \{12, 25\}$
$M := \emptyset$	$M := \{67\}$
$M := \{12, 35\}$	

3.5 Definition (Perfect Matching)

Given a graph $G = (V, E)$, if every vertex V in G is an endpoint of an edge in M, we call the matching a *perfect matching*.

The assignment problem is a special case of a *minimum cost perfect matching problem* or weighted graphs (in this case every edge is given a weight/cost c_{ij})



Remark 7. In a perfect matching graph, there are n^2 edges, and $2n$ (an even number of) vertices.

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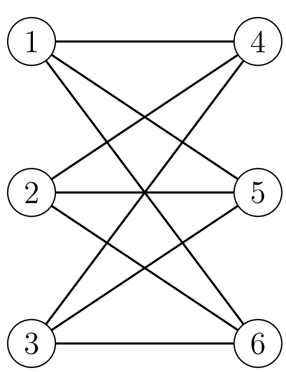
4.1 Example (Minimum-Cost Perfect Matching Problem)

Given an undirected graph $G = (V, E)$, and $w_e \in \mathbb{R}$, for every $e \in E$, we want to find a perfect matching in G with minimum total cost. The cost of matching M is $\sum_{e \in M} w_e$. For each $v \in V$, $\delta(v) :=$ the set of edges incident to v. $G :=$



$$\delta(1) = \{14, 15, 16\}$$
$$\delta(5) = \{15, 25, 35\}$$
$$x_e := \begin{cases} 1 \text{ , if } e \text{ is chosen in the matching} \\ 0 \text{ , otherwise} \end{cases}$$

(IP)

$$\min \sum_{e \in E} w_e x_e$$

subject to

$$\sum_{e \in E} x_e = 1 \qquad \forall v \in V$$
$$x_e \in \{0, 1\} \qquad \forall e \in E$$

4.2 Definition (Bipartite)

A graph $G = (V, E)$ is *bipartite* if there exists a partition V_1, V_2 of V ($V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$) such that

$$E \subseteq \{uv \mid u \in V_1, v \in V_2\}$$

Assignment problems are a special case of minimum cost perfect matching problems in bipartite graphs.

Remark 8. A graph is *bipartite* $\iff$ it does not contain an odd cycle.

Given a situation where we have binary-valued variables

$$x_j := \begin{cases} 1, \text{ option } j \text{ is chosen} \\ 0, \text{ otherwise} \end{cases} \qquad \forall j \in \{1, \dots, n\}$$

We solve how to formulate in an IP in the following conditions:

- at most k options are chosen: $\sum_{j=1}^{n} x_j \leq k$
- at least k options are chosen: $\sum_{j=1}^{n} x_j \geq k$
- exactly k options are chosen: $\sum_{j=1}^{n} x_j = k$

We can also formulate many classes of the ”OR” type constraint in IP problems.

4.3 Example (Refer to 1.1)

(IP)

$$\max 10x_1 + 15x_2$$

subject to

$$\begin{aligned} 2x_1 + x_2 &\le 1600 \\ x_1 + 3x_2 &\le 1200 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Suppose C&O is required to produce at least 10 tables per week or at least 80 chairs per week, or possibly both. $x_1 \ge 10$ or $x_2 \ge 80$ or both. We introduce a new binary-valued variable $z \in \{0, 1\}$.

$$z := \begin{cases} 1, \text{ if } x_1 \ge 10 \\ 0, \text{ if } x_2 \ge 80 \end{cases}$$

$$\{(x_1 \ge 10 \text{ OR } x_2 \ge 80) \text{ AND } (x_1 \ge 0 \text{ OR } x_2 \ge 0)\} \iff \left\{ \begin{array}{r} x_1 \ge 10z \\ x_2 \ge 80(1-z) \\ z \in \{0,1\} \\ x_1, x_2 \ge 0 \end{array} \right\}$$

Remark 9. *Possibly both* means that you can choose either one of these conditions in the first OR above and it will be correct.

Now, suppose C&O has a new condition every week. We must manufacture either exactly 3 chairs for every table or exactly 8 chairs for every table. Show how to incorporate this in an IP formulation

$$\{x_2 = 3x_1 \text{ OR } x_2 = 8x_1\} \iff \{(x_2 \le 3x_1 \text{ AND } x_2 \ge 3x_1) \text{ OR } (x_2 \le 8x_1 \text{ AND } x_2 \ge 8x_1)\}$$

Introduce a new binary-valued variable $z \in \{0, 1\}$.

$$z := \begin{cases} 1, \text{ if } x_2 = 3x_1 \\ 0, \text{ if } x_2 = 8x_1 \end{cases}$$

Existing constraints:

$$\left\{ \begin{array}{r} 2x_1 + x_2 \le 1600 \\ x_1 + 3x_2 \le 1200 \\ x_1, x_2 \ge 0 \end{array} \right\} \begin{array}{l} \implies x_1 \in [0, 800] \\ \implies x_2 \in [0, 500] \end{array}$$

So,

$$\begin{aligned} x_2 &\le 3x_1 + 400(1-z) \\ x_2 &\ge 3x_1 - 2400(1-z) \\ x_2 &\le 8x_1 + 400z \\ x_2 &\ge 8x_1 - 6400z \\ z &\in \{0, 1\} \end{aligned}$$

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5.1 Definition (Non-linear Program)

A *non-linear program* has the form

$$\min f(x)$$

subject to

$$\begin{aligned} g_1(x) &\leq 0 \\ g_2(x) &\leq 0 \\ &\vdots \\ g_m(x) &\leq 0 \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g_i : \mathbb{R}^n \to \mathbb{R}$, $\forall i \in \{1, \dots, m\}$.

Every LP problem is a very special case of a NLP problem. IP problems can also be formulated as NLP problems.

5.2 Example (Formulating LP problems as NLP problems)

$$\begin{aligned} x_i \in \mathbb{Z} &\iff \sin(\pi x_i) = 0 \\ &\iff [\sin(\pi x_i)]^2 \leq 0 \end{aligned}$$

NLP problems have huge modelling power, as a result, one must understand the structure of the underlying problem and construct "good" NLP models that are amendable to analysis and solution techniques.

5.3 Example (Portfolio Optimization)

There are n stocks $1, \dots, n$ to invest in. We have a budget of B dollars. We have an expected return (for \$1 investment at the end of our planning horizon) of $\mu_1, \dots, \mu_n$. We are also given $V \in \mathbb{R}^{n \times n}$, a variance coefficient matrix so that if we invest in $x_1, \dots, x_n$ dollars in n stocks, $1, \dots, n$ respectively, then the expected risk of such an investment is given by $\boldsymbol{x}^\top V \boldsymbol{x}$.

$$\sum_{i=1}^{n} \sum_{j=1}^{n} V_{ij} x_i x_j$$

$x_j :=$ amount of investment in stock j in dollars.

Suppose we are also given a goal G (a dollar amount we want as the value of our portfolio at the end of the planning horizon).

Data

- Budget ($) → B
- Goal ($) → G
- Expected return → $\begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}$
- Variance-covariance matrix → $V \in \mathbb{R}^{n \times n}$

We want to minimize the risk of our portfolio while satisfying the budget and the goal constraints. (NLP)

$$\min \boldsymbol{x}^\top V \boldsymbol{x}$$

subject to

$$\begin{aligned} \sum_{j=1}^{n} x_j &\geq B \\ \sum_{j=1}^{n} \mu_j x_j &\leq G \\ \boldsymbol{x} &\geq \mathbb{0} \end{aligned}$$

There are many variants of such models and extensions. For example, instead of a goal G, we may given an upper bound on the risk, say $R \in \mathbb{R}_{>0}$. (NLP)

$$\max \sum_{j=1}^{n} \mu_j x_j$$

subject to

$$\begin{aligned} \sum_{j=1}^{n} x_j &\geq B \\ \boldsymbol{x}^\top V \boldsymbol{x} &\leq R \\ \boldsymbol{x} &\geq \mathbb{0} \end{aligned}$$

We can handle many more variants and extensions. Suppose investing in stock j below l_j dollars is not allowed. For diversity of our portfolio, we want to invest in at least 20 stocks, and for the sake of simplicity we want to invest in at most 150 stocks. We introduce a binary-valued variable z_j.

$$z_j := \begin{cases} 1, \text{ if we invest in stock } j \\ 0, \text{ otherwise} \end{cases} \qquad \forall j \in \{1, \dots, n\}$$

Add these constraints:

$$\begin{aligned} & l_j z_j \leq x_j \leq B z_j \qquad \forall j \in \{1, \dots, n\} \\ & 20 \leq \sum_{j=1}^{n} z_j \leq 150 \\ & z_j \in \{0, 1\} \end{aligned}$$

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6.1 Feasible and Infeasible Solutions

Consider an LP with variables $x_1, \dots, x_n$. Then the assignment of values to all variables such that all constraints are satisfied, gives a *feasible solution*.
An optimization problem is called *feasible* if it has at least one feasible solution, otherwise it is called *infeasible*.

6.2 Example (Infeasible LP)

(LP)

$$\max x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5$$

subject to

$$\begin{array}{r} 1 \\ -2 \end{array} \underbrace{\begin{bmatrix} -3 & 2 & 7 & 1 & -7 \\ -2 & 1 & 2 & 0 & -4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} 6 \\ 4 \end{bmatrix}}_{\boldsymbol{b}}$$

$$\boldsymbol{x} \geq \mathbb{0}$$

Let $\boldsymbol{y} := (1, -2)^\top$ and consider the facts

$$\begin{aligned} & A\boldsymbol{x} = \boldsymbol{b} \\ & \implies \boldsymbol{y}^\top A\boldsymbol{x} = \boldsymbol{y}^\top \boldsymbol{b} \\ & \implies \underbrace{\begin{bmatrix} 1 & 0 & 3 & 1 & 1 \end{bmatrix}}_{\geq \mathbb{0}^\top} \underbrace{\boldsymbol{x}}_{\geq \mathbb{0}} = \underbrace{6 - 8}_{<0} = -2 \end{aligned}$$

Therefore, since $\nexists$ any solution to $A\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x} \geq 0$ the LP is infeasible.

6.3 Proposition (Infeasibility)

If $\exists \boldsymbol{y} \in \mathbb{R}^m$ such that
(1) $\boldsymbol{y}^\top A \geq \mathbb{0}^\top$
(2) $\boldsymbol{y}^\top \boldsymbol{b} < 0$
For every $\boldsymbol{c} \in \mathbb{R}^n$, the LP

$$\max\{\boldsymbol{c}^\top \boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b},\, \boldsymbol{x} \geq \mathbb{0}\}$$

is infeasible. In particular, we call a vector $\boldsymbol{y}$ a *certificate of infeasibility*.

Proof. Suppose there exists such a $\boldsymbol{y}$. Suppose for a contradiction that $\exists \bar{\boldsymbol{x}} \in \mathbb{R}^n$ (there is a feasible solution) such that

$$A\bar{\boldsymbol{x}} = \boldsymbol{b},\, \bar{\boldsymbol{x}} \geq \mathbb{0}$$

$$A\bar{\boldsymbol{x}} = \boldsymbol{b} \implies \underbrace{\boldsymbol{y}^\top A}_{\geq \mathbb{0}^\top} \underbrace{\bar{\boldsymbol{x}}}_{\geq \mathbb{0}} = \underbrace{\boldsymbol{y}^\top \boldsymbol{b}}_{\not< 0}$$

a contradiction to (2). □

An optimization problem is called unbounded if $\forall M \in \mathbb{R}$, there exists a feasible solution of the optimization problem with the objective value strictly better than M.

6.4 Example (Unbounded LP)

$$\max \begin{bmatrix} -1 & 3 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{x}$$

subject to

$$\begin{bmatrix} -1 & 3 & -1 & 1 & 0 \\ -2 & 4 & 1 & 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$\boldsymbol{x} \geq \mathbb{0}$$

Consider

$$\tilde{\boldsymbol{x}} := \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}}_{\boldsymbol{x}} + t \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}}_{\boldsymbol{d}}, \; t \geq 0$$

$$A\boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \bar{\boldsymbol{x}} \geq \mathbb{0}. \text{Therefore } \bar{\boldsymbol{x}} \text{ is a feasible solution.}$$

$$A\boldsymbol{d} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \boldsymbol{d} \geq \mathbb{0}.$$

$$A\tilde{\boldsymbol{x}} = A(\bar{\boldsymbol{x}} + t\boldsymbol{d}) = A\bar{\boldsymbol{x}} + t(A\boldsymbol{d}) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\tilde{\boldsymbol{x}} = \bar{\boldsymbol{x}} + t\boldsymbol{d}$$

Therefore, $\tilde{\boldsymbol{x}}$ is a feasible solution $\forall t \geq 0$.

Objective function value of $\tilde{\boldsymbol{x}}$:

$$\begin{bmatrix} -1 & 3 & 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right) = 1 + t(-1+2) = 1 + t \to +\infty \text{ as } t \to +\infty$$

Therefore the LP is unbounded.

6.5 Proposition (Unboundedness)

If $\exists \bar{\boldsymbol{x}} \in \mathbb{R}^n$ such that

$$A\bar{\boldsymbol{x}} = \boldsymbol{b}, \boldsymbol{x} \geq \mathbb{0}.$$

and $\exists \boldsymbol{d} \in \mathbb{R}^n$ such that
(1) $A\boldsymbol{d} = \mathbb{0}$
(2) $\boldsymbol{d} \geq \mathbb{0}$
(3) $\boldsymbol{c}^\top \boldsymbol{d} > 0$
For every $\boldsymbol{c} \in \mathbb{R}^n$, the LP

$$\max\{\boldsymbol{c}^\top \boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b},\, \boldsymbol{x} \geq \mathbb{0}\}$$

is unbounded. In particular, we call a pair of vectors $\bar{\boldsymbol{x}}$, $\boldsymbol{d}$ a *certificate of unboundedness.*

Proof. Suppose there exists such $\boldsymbol{d}$. Consider

$$\tilde{\boldsymbol{x}} = \bar{\boldsymbol{x}} + t\boldsymbol{d}, t \geq 0$$

Then,

$$A\tilde{\boldsymbol{x}} = \underbrace{A\bar{\boldsymbol{x}}}_{\boldsymbol{b}} + t\underbrace{(A\boldsymbol{d})}_{\mathbb{0}} = \boldsymbol{b}$$

Therefore $\tilde{\boldsymbol{x}}$ is a feasible solution of the LP, $t \geq 0$. The objective value of the function is

$$\boldsymbol{c}^\top \tilde{\boldsymbol{x}} = \boldsymbol{c}^\top \bar{\boldsymbol{x}} + t\underbrace{(\boldsymbol{c}^\top \boldsymbol{d})}_{>\mathbb{0}} \to +\infty \text{ as } t \to +\infty$$

Therefore, the LP is unbounded. □

Remark 10. If the LP is min, then flip the equality for (3).

6.6 Example (Optimal LP)

$$\max 10x_1 + 15x_2$$

subject to

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 1600 \\ x_1 + 3x_2 + x_4 &= 1200 \\ \boldsymbol{x} &\geq \mathbb{0} \end{aligned}$$

Consider $\bar{\boldsymbol{x}} := (720, 160, 0, 0)^\top$ and $\boldsymbol{y} := (3, 4)^\top$.

Note that $A\bar{\boldsymbol{x}} = \boldsymbol{b}$, with $\bar{\boldsymbol{x}} \geq \mathbb{0}$, so $\bar{\boldsymbol{x}}$ is a feasible solution.

Also, $\boldsymbol{c}^\top \bar{\boldsymbol{x}} = 7200 + 2400 = 9600$. Every feasible solution satisfies

$$\begin{aligned} A\boldsymbol{x} &= \boldsymbol{b} \\ \implies \boldsymbol{y}^\top A\boldsymbol{x} &= \boldsymbol{y}^\top \boldsymbol{b} \end{aligned}$$

$$\boldsymbol{y}^\top A = \begin{bmatrix} 10 & 15 & 3 & 4 \end{bmatrix} \geq \begin{bmatrix} 10 & 15 & 0 & 0 \end{bmatrix} = \boldsymbol{c}^\top$$

$$\boldsymbol{y}^\top \boldsymbol{b} = 3 \times 1600 + 4 \times 1200 = 9600 = \boldsymbol{c}^\top \bar{\boldsymbol{x}}$$

Therefore $\bar{\boldsymbol{x}}$ is an optimal solution.

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7.1 Summary of outcomes

(LP)

$$\max\{\boldsymbol{c}^\top \boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b},\, \boldsymbol{x} \geq \mathbb{0}\}$$

- if $\exists \boldsymbol{y} \in \mathbb{R}^m$ such that
  (1) $\boldsymbol{y}^\top A \geq \mathbb{0}^\top$
  (2) $\boldsymbol{y}^\top \boldsymbol{b} < 0$
  then (LP) is infeasible.
- if $\exists (\bar{\boldsymbol{x}}, \boldsymbol{d}) \in \mathbb{R}^n$ such that $A\boldsymbol{x} = b$, $\boldsymbol{x} \geq \mathbb{0}$,
  (1) $A\boldsymbol{d} = \mathbb{0}$
  (2) $\boldsymbol{d} \geq \mathbb{0}$
  (3) $\boldsymbol{c}^\top \boldsymbol{d} > 0$
  then (LP) is unbounded.
- if $\exists (\bar{\boldsymbol{x}}, \boldsymbol{y})$ such that
  (1) $A\bar{\boldsymbol{x}} = \boldsymbol{b}$, $\boldsymbol{x} \geq 0$
  (2) $A^\top \boldsymbol{y} \geq \boldsymbol{c}$
  (3) $\boldsymbol{c}^\top \bar{\boldsymbol{x}} = \boldsymbol{y}^\top \boldsymbol{b}$
  then $\bar{\boldsymbol{x}}$ is an optimal solution of (LP).

7.2 Definition (Standard Equality Form)

An LP is said to be in *Standard Equality Form* (SEF) if it has the Form

$$\max \boldsymbol{c}^T\boldsymbol{x} + \bar{z}$$

subject to

$$\begin{aligned} A\boldsymbol{x} &= \boldsymbol{b} \\ \boldsymbol{x} &\geq \mathbb{0} \end{aligned}$$

where $\bar{z}$ is a constant. In other words, it satisfies all of the conditions:
(1) It is a maximization problem
(2) All constraints are equations (other than non-negativity constraints)
(3) Every variable has a non-negativity constraint

Every LP can be converted to SEF. A pair of LP problems LP1 and LP2 are equivalent if they both have the same status (infeasible, unbounded, or optimal) and certificate of such a status for one problem can easily be converted into a certificate of the same type for the other LP.

Given an arbitrary LP problem,

- if the objective function is a minimization problem, then $\min \boldsymbol{c}^\top\boldsymbol{x} \to -(\max -\boldsymbol{c}^\top\boldsymbol{x})$

  Remark 11. We often omit one negative sign from a TA on Piazza: "It's more just a convention of not putting - before max when doing this and it's understood that the optimal value of one is the negative of the optimal value of the other"

- if there are constraints $\alpha\boldsymbol{x} \leq \alpha$, introduce a new non-negative *slack variable* x_{n+1}, $x_{n+1} \geq 0$.

- if some x_j has no constraint on it, such variables are called *free variables* and we represent that free variable as a difference of two non-negative variables, $x_j = x_j^+ - x_j^-$, $x_j^+ \geq 0$, $x_j^- \geq 0$.

- if some $x_j < 0$ flip all signs correlating to x_j

7.3 Example (Converting an LP to SEF)

(P)

$$\max 100x_1 + 200x_2$$

subject to

$$\begin{array}{rcrcr} 11x_1 & + & 12x_2 & \leq & 2000 \\ 21x_1 & + & 22x_2 & \geq & 1000 \end{array}$$

$$x_1 \geq 0$$

Converting into SEF we get (P′):

$$\max 100x_1 + 200(x_2^+ - x_2^-)$$

subject to

$$\begin{array}{rcrcrcrcl} 11x_1 & + & 12(x_2^+ - x_2^-) & + & x_3 & & & = & 2000 \\ 21x_1 & + & 22(x_2^+ - x_2^-) & - & & & x_4 & = & 1000 \end{array}$$

$$x_1, x_2^+, x_2^-, x_3, x_4 \geq 0$$

(P) and (P′) are equivalent.

Let $(\bar{x_1}, \bar{x_2^+}, \bar{x_2}^-, \bar{x_3}, \bar{x_4})^\top$ be a feasible solution of (P′).

If

$$\hat{x_1} := \bar{x_1}$$
$$\hat{x_2} := \bar{x_2}^+ - \bar{x_2}^-$$

Then $(\hat{x_1}, \hat{x_2})^\top$ is a feasible solution of (P).

Let $(\bar{x_1}, \bar{x_2})^\top$ be a feasible solution of (P).

If

$$\bar{x_3} := 2000 - 11\bar{x_1} - 12\bar{x_2}$$
$$\bar{x_4} := 21\bar{x_1} + 22\bar{x_2} - 1000$$

and if $\bar{x_2} \geq 0$

$$\bar{x_2}^+ := \bar{x_2}$$
$$\bar{x_2}^- := 0$$

or $\bar{x_2} < 0$

$$\bar{x_2}^+ := 0$$
$$\bar{x_2}^- := -\bar{x_2}$$

then $(\bar{x_1}, \bar{x_2}^+, \bar{x_2}^-, \bar{x_3}, \bar{x_4})^\top$ is a feasible solution of (P′).

Remark 12. This example was a question from a past midterm and not covered in class. The class example was useless.

7.4 Definition (Basis)

Let $A \in \mathbb{R}^{m \times n}$, $B \subseteq \{1, \dots, n\}$ such that $|B| = m$. If

$$A_B := \left[\ a_i \ \middle|\ i \in B\ \right] \in \mathbb{R}^{m \times m}$$

where A_B is non-singular (i.e. IMT holds), then B is a basis of A. If B is a basis of A, then A_B is a basis for $\mathbb{R}^m$. We denote N as the set that does not have the elements of B.

7.5 Definition (Basic Solution)

A vector $\bar{\boldsymbol{x}}$ is a *basic solution* of $A\boldsymbol{x} = \boldsymbol{b}$ for a basis B if the following conditions hold:
(1) $A\bar{\boldsymbol{x}} = \boldsymbol{b}$
(2) $\bar{\boldsymbol{x}}_{\boldsymbol{N}} = 0$

7.6 Definition (Basic Feasible Solution)

If $\bar{\boldsymbol{x}}$ is a *basic solution* and $\bar{\boldsymbol{x}} \geq \mathbb{0}$, then $\bar{\boldsymbol{x}}$ is a *basic feasible solution* .

7.7 Example (Bases of A)

$$A := \begin{bmatrix} 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & 2 & -1 & 2 \end{bmatrix}, \boldsymbol{b} := \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Bases of A: $\{1, 2\}$, $\{2, 3\}$, $\{1, 4\}$.
Not a bases of A: $\emptyset$, $\{1\}$, $\{1, 2, 3\}$, $\{3, 4\}$.
To find the basic solution determined by $B := \{1, 4\}$, solve

$$\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

and we get $\bar{\boldsymbol{x}} = (-3, 0, 0, -5, 0)^\top$.

8 2019-10-01

Let $A \in \mathbb{R}^{m \times n}, \boldsymbol{b} \in \mathbb{R}^m, \boldsymbol{c} \in \mathbb{R}^n$. Consider (P)

$$\max \boldsymbol{c}^\top \boldsymbol{x}$$

subject to

$$A\boldsymbol{x} = \boldsymbol{b}$$

$$\boldsymbol{x} \geq \mathbb{0}$$

Suppose we are given $\tilde{\boldsymbol{x}} \in \mathbb{R}^n$ such that, $A\tilde{\boldsymbol{x}} = \boldsymbol{b}$, $\tilde{\boldsymbol{x}} \geq \mathbb{0}$ and $\boldsymbol{y} \in \mathbb{R}^m$ such that $A^\top \boldsymbol{y} \geq \boldsymbol{c}$, $\boldsymbol{y}^\top \boldsymbol{b} = \boldsymbol{c}^\top \bar{\boldsymbol{x}}$ with objective function value $= \boldsymbol{c}^\top \bar{\boldsymbol{x}}$.

Computing $\boldsymbol{c}^\top \bar{\boldsymbol{x}}$ we get

$$\begin{aligned} \boldsymbol{c}^\top \bar{\boldsymbol{x}} &= \boldsymbol{y}^\top \boldsymbol{b} \\ &= \boldsymbol{y}^\top (A\tilde{\boldsymbol{x}}) \\ &= \underbrace{(\boldsymbol{y}^\top A)}_{\geq \boldsymbol{c}^\top} \underbrace{\tilde{\boldsymbol{x}}}_{\geq \mathbb{0}} \\ &\geq \boldsymbol{c}^\top \tilde{\boldsymbol{x}} \end{aligned}$$

Since $\bar{\boldsymbol{x}}$ achieves the objective value of $\boldsymbol{c}^\top \bar{\boldsymbol{x}}$ and for every feasible solution the objective value is at most $\boldsymbol{c}^\top \bar{\boldsymbol{x}}$, $\bar{\boldsymbol{x}}$ is an optimal solution of (P).

8.1 Definition (Canonical form)

Consider the following LP in SEF: (P)

$$\max \boldsymbol{c}^\top \boldsymbol{x} + \bar{z}$$

subject to

$$\begin{aligned} A\boldsymbol{x} &= \boldsymbol{b} \\ \boldsymbol{x} &\geq \mathbb{0} \end{aligned}$$

We say (P) is in *canonical form* for a basis B of A if
(C1) A_B is an identity matrix
(C2) $\boldsymbol{c_B} = \mathbb{0}$

Now,

$$\begin{aligned} A\boldsymbol{x} &= \sum_{j=1}^{n} \boldsymbol{a_j} x_j \\ &= \sum_{j \in B}^{n} \boldsymbol{a_j} x_j + \sum_{j \in N}^{n} \boldsymbol{a_j} x_j \\ &= A_B \boldsymbol{x_B} + A_N \boldsymbol{x_N} \end{aligned}$$

Since B is a basis of A, A_B is non-singular,

$$\begin{aligned} & A\boldsymbol{x} = \boldsymbol{b} \\ & \iff A_B^{-1} A\boldsymbol{x} = A_B^{-1}\boldsymbol{b} \\ & \iff A_B^{-1}(A_B\boldsymbol{x_B} + A_N\boldsymbol{x_N}) = A_B^{-1}\boldsymbol{b} \\ & \iff (\underbrace{A_B^{-1}A_B}_{I}\boldsymbol{x_B}) + (A_B^{-1}A_N\boldsymbol{x_N}) = A_B^{-1}\boldsymbol{b} \\ & \iff \boldsymbol{x_B} = A_B^{-1}\boldsymbol{b} - (A_B^{-1}A_N\boldsymbol{x_N}) \end{aligned}$$

Consider (C2). For any $\boldsymbol{y} := (y_1, \ldots, y_m)^\top$ the equation

$$\boldsymbol{y}^\top A\boldsymbol{x} = \boldsymbol{y}^\top \boldsymbol{b}$$

can be written as

$$0 = \boldsymbol{y}^\top \boldsymbol{b} - \boldsymbol{y}^\top A\boldsymbol{x}$$

Since this equation holds for every feasible solution, we can add this constraint to the objective function of (Q). The objective function is now

$$\max \boldsymbol{c}^\top \boldsymbol{x} + \bar{z} + \boldsymbol{y}^\top \boldsymbol{b} - \boldsymbol{y}^\top A\boldsymbol{x} \implies \max(\boldsymbol{c}^\top - \boldsymbol{y}^\top A)\boldsymbol{x} + \boldsymbol{y}^\top \boldsymbol{b} + \bar{z}$$

Let $\bar{\boldsymbol{c}}^\top := \boldsymbol{c}^\top - \boldsymbol{y}^\top A$. For (C2) to be satisfied we need $\bar{\boldsymbol{c}}_{\boldsymbol{B}} = \mathbb{0}$, so we need to choose $\boldsymbol{y}$ accordingly, such as

$$\bar{\boldsymbol{c}}_{\boldsymbol{B}}^\top = \boldsymbol{c}_{\boldsymbol{B}}^\top - \boldsymbol{y}^\top A_B = \mathbb{0}^\top$$

equivalently,

$$\boldsymbol{y}^\top A_B = \boldsymbol{c}_{\boldsymbol{B}}^\top \implies \boldsymbol{y}^\top = \boldsymbol{c}_{\boldsymbol{B}}^\top A_B^{-1}$$

We have shown the following:

8.2 Proposition (Canonical Form)

$$\max(\boldsymbol{c}^\top - \boldsymbol{y}^\top A)\boldsymbol{x} + \boldsymbol{y}^\top \boldsymbol{b} + \bar{z}$$

subject to

$$\begin{aligned} A_B^{-1} A\boldsymbol{x} &= A_B^{-1}\boldsymbol{b} \\ \boldsymbol{x} &\geq \mathbb{0} \end{aligned}$$

where $\boldsymbol{y}^\top = \boldsymbol{c}_{\boldsymbol{B}} A_B^{-1}$, is an equivalent LP in canonical form for the basis B of A.

The canonical form is useful because it:

- allows us to simply read a basic solution
- gives us easy ways to move in the feasible region to improve the current basic feasible solution
- gives us a way to obtain optimality certificates if $\boldsymbol{c}_{\boldsymbol{N}}^\top - \boldsymbol{c}_{\boldsymbol{B}}^\top A_B^{-1} A_N \leq \mathbb{0}^\top$

8.3 Example (Canonical Form)

(P)

$$\max \begin{bmatrix} 0 & 0 & -4 & 1 & 0 \end{bmatrix} \boldsymbol{x}$$

subject to

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & -3 & 2 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$$
$$\boldsymbol{x} \geq \mathbb{0}$$

Note that $B := \{1, 2, 5\}$ is a basis of A, so the basic solution corresponding to the basis is $\bar{\boldsymbol{x}} := (4, 2, 0, 0, 6)^\top$.

$c_3 = -4$, increasing the value of x_3 from 0 will decrease the objective value by -4 units
$c_4 = 1$, we want to increase the value of x_4, so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} - x_4 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \geq 0$$

Let t denote the maximum value we can assign to x_4 and stay feasible.
So, $t = \min\{4/1, \_, 6/2\} = 3$

9 2019-10-03

Remark 13. The following lecture will not be 1-1 since the explanations in class were useless.

9.1 Example (Continuation of 8.3)

So, the new basic feasible solution is $\bar{\boldsymbol{x}} := (1, 5, 0, 3, 0)^\top$ determined by the basis $B := \{1, 2, 5\} \cup \{4\} \ \{5\} = \{1, 2, 4\}$. Note that we exclude $\{5\}$ since the index of which t achieved the minimum was at $6/2$, i.e. index 5 (row x_5). The canonical form determined by the new basis is

$$\max \begin{bmatrix} 0 & 0 & -5/2 & 0 & -1/2 \end{bmatrix} \boldsymbol{x} + 3$$

subject to

$$\begin{bmatrix} 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 1 & 1/2 & 0 & 1/2 \\ 0 & 0 & -3/2 & 1 & 1/2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$$

Remark 14. $\bar{\boldsymbol{x}}$ is the optimal solution with optimal value 3.

Remark 15. How did we arrive to this LP? Using the formulae in Proposition 8.2. If you didn't want to calculate A_B^{-1}, then follow the below instructions.

9.2 Example (Canonical form without computing the inverse)

Remark 16. The following was not taught in class or the textbook. This method can be confusing and not intuitive.

Write

$$A := \left[\begin{array}{ccccc|c} 1 & 0 & -1 & 1 & 0 & 4 \\ 0 & 1 & 2 & -1 & 0 & 2 \\ 0 & 0 & -3 & 2 & 1 & 6 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 1/2 & 0 & -1/2 & 1 \\ 0 & 1 & 1/2 & 0 & 1/2 & 5 \\ 0 & 0 & -3/2 & 1 & 1/2 & 3 \end{array}\right]$$

and row reduce A to make fourth column get a leading one as seen above. The row-reduced matrix and the augment are your new constraints.
The objective function is tricky, we want a 0 in the fourth column of our $\boldsymbol{c}^\top$. Also, we denote x_1, x_2, x_4 as the rows of the matrix respectively. Using x_4 (which is our row-reduced A), we get

$$-1 \left(\begin{bmatrix} 0 & 0 & -3/2 & 1 & 1/2 \end{bmatrix} \boldsymbol{x} - 3 \right) + \begin{bmatrix} 0 & 0 & -4 & 1 & 0 \end{bmatrix} \boldsymbol{x}$$

The -3 right after the first matrix was the row of $\boldsymbol{b}$.

9.3 Simplex Algorithm

Algorithm 1: Simplex Algorithm

Input : $A \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{c} \in \mathbb{R}^n$

1 Compute the canonical form for B, let $\bar{\boldsymbol{x}}$ be the basic feasible solution.
2 If $\boldsymbol{c_N} \leq 0$, then stop ($\bar{\boldsymbol{x}}$ is optimal).
3 Choose $k \in N$ such that $c_k > 0$.
4 If $A_k \leq \mathbb{0}$, then stop (the LP is unbounded).
5 Let r be the index which attains $t = \min\{{}^{b_i}/_{A_{ik}} \mid A_{ik} > 0\}$.
6 Let $l \in B$ be the r^{th} basis element.
7 Set $B := B \cup \{k\}\ \{l\}$.
8 Go to step 1.

9.4 Bland's Rule

In step 3, among all $k \in N$, with $c_k > 0$ and in step 5, $r \in B$, choose the smallest index for both k and r.

10 2019-10-08

10.1 Convergence of Simplex Algorithm

In each iteration, we choose $k \in N$ such that $c_k > 0$. Then, we compute $t = \min\{{}^{b_i}/_{A_{ik}} \mid A_{ik} > 0\}$. Then, throughout the rest of the Simplex iterations, we never see the same basis again. There are at most $\binom{n}{m}$ bases of A. Therefore, if $t > 0$ in each iteration, the Simplex Algorithm will terminate in at most $\binom{n}{m}$ iterations. The only way the algorithm will not terminate is when $t = 0$ for all iterations (after some # of iterations). If our choices for k and l are deterministic and consistent in this case if we repeat a basis we call it a *cycle*.

10.2 Theorem

The Simplex Algorithm starting from a basic feasible solution and Bland's Rule terminates.

10.3 Implementation of the Simplex Algorithm in "Big Data"

In a given iteration of the Simplex Algorithm, what information do we need to execute the algorithm?

We have the original data (A, $\boldsymbol{b}$, $\boldsymbol{c}$) and we have the current B, $\bar{\boldsymbol{x}}$, $\bar{\boldsymbol{v}}$.

Pick any $k \in N$ such that $\bar{c_k} \geq 0$. $\bar{c_k} = c_k - \bar{\boldsymbol{y}}^\top A\boldsymbol{x}$ (where $\bar{\boldsymbol{y}}^\top = c_B A_B^{-1}$).

Then to compute t, we need $t = \min\{{}^{b_i}/_{A_{ik}} \mid A_{ik} > 0\}$.

So, we need $\bar{A_k}$: $\bar{A_k} = A_B^{-1} A_k$ and note that $\bar{\boldsymbol{x_N}} = \mathbb{0}$, $\bar{\boldsymbol{x_B}} = \bar{\boldsymbol{b}}$

We solve linear systems $A_B^\top \boldsymbol{y} = \boldsymbol{c_B}$ and $A_B \boldsymbol{d_B} = A_k$.

In implementations, we typically express A_B or A_B^{-1} as a product of elementary matrices.

In practice, good implementations of the Simplex Algorithm terminates after $2m$ to $n/2$ iterations. Each iteration is very fast.

It is an open problem whether there exists a variant of Simplex Algorithm which is guaranteed to terminate in at most pn^q iterations for LP problems in SEF with n variables, where p, q are constants.

2019-10-10

Midterm 1 was written on this day, as a result no classes were held.

11 2019-10-22

Given any LP problem, we know how to convert it into an equivalent LP problem in SEF:

(P)

$$\max z := \boldsymbol{c}^\top \boldsymbol{x}$$

subject to

$$\begin{aligned} A\boldsymbol{x} &= \boldsymbol{b} \\ \boldsymbol{x} &\geq 0 \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}, \operatorname{rank}(A) = m$.

Given an LP in SEF, with a given basic feasible solution, we know how to solve it.

11.1 Finding a Feasible Solution To LPs (Two Phase Method)

Given an LP in SEF with $\operatorname{rank}(A) = m$, how do we find a feasible solution or prove that none exists.

We will construct an *auxiliary LP problem.*

We can always make sure $b \geq 0$. (If any $b_i < 0$, multiply both sides of that equation by (-1)) Introduce artificial variables $x_{n+1}, x_{n+2}, \cdots, x_{n+m}$

(P_{aux})

$$\min w := x_{n+1} + x_{n+2} + \cdots + x_{n+m}$$

subject to

$$\left[\, A \mid I \,\right] \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+m} \end{bmatrix}}_{\boldsymbol{x}} = \boldsymbol{b}$$

$$\boldsymbol{x} \geq \mathbb{0}$$

For every feasible solution of (P_{aux}), $w \geq 0$.

Therefore, (P_{aux}) is not unbounded.

If the optimal value of (P_{aux}) is zero, let $\begin{bmatrix} \hat{x_1} \\ \vdots \\ \hat{x_{n+m}} \end{bmatrix}$ be the corresponding basic feasible solution.

Then, $\begin{bmatrix} \hat{x_1} \\ \vdots \\ \hat{x_n} \end{bmatrix}$ is a basic feasible solution of (P).

It is basic since $\{A_j : \hat{x_j} > 0\}$ is linearly independent.

If $|\{j : \hat{x_j} > 0\}| = m$, this index set is a basis of A which determines $\begin{bmatrix} \hat{x_1} \\ \vdots \\ \hat{x_n} \end{bmatrix}$.

Otherwise, $|\{j : \hat{x_j} > 0\}| \leq m - 1$, we can extend this index set to be a basis of A, since $\text{rank}(A) = m$.

If the optimal value of (P_{aux}) is positive, then (P) is infeasible.

11.2 Two Phase Method

Algorithm 2: Two Phase Method

Input : $A, \boldsymbol{b}, \boldsymbol{c}$ data for LP in SEF such that full row rank and $\boldsymbol{b} \geq \mathbb{0}$.

1. Construct (P_{aux}) put into SEF, $B := \{n+1, n+2, \dots, n+m\}$
2. Put (P_{aux}) into the canonical form determined by B.
3. Solve (P_{aux}) starting with basis B by Simplex Method.
4. If the optimal value of (P_{aux}) is zero, then we have a basic feasible solution of (P). Solve (P) using Simplex Method.
5. If the optimal objective value of (P_{aux}) is not zero, then (P) is infeasible (a certificate of infeasibility is given by the last $\bar{\boldsymbol{y}}$ computed).

11.3 Example

(P)

$$\max z := \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \boldsymbol{x}$$

subject to

$$\begin{bmatrix} 1 & -2 & -3 \\ -1 & 1 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \tag{4}$$

$$\boldsymbol{x} \geq \mathbb{0} \tag{5}$$

Since $b_1 < 0$ we write

$$\begin{bmatrix} -1 & 2 & 3 \\ -1 & 1 & 1 \end{bmatrix} x = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Introduce artificial variables: x_4, x_5

(P_{aux})

$$\min w := \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \end{bmatrix} \boldsymbol{x}$$

subject to

$$\begin{bmatrix} -1 & 2 & 3 & 1 & 0 \\ -1 & 1 & 1 & 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \tag{6}$$

$$\boldsymbol{x} := \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \geq \mathbb{0} \tag{7}$$

Apply Simplex Method starting with the basis $B := \{4, 5\}$

$$\max -w = \begin{bmatrix} 0 & 0 & 0 & -1 & -1 \end{bmatrix} \boldsymbol{x}$$

subject to

$$\begin{bmatrix} 1 & 0 & 1 & 1 & -2 \\ 0 & 1 & 2 & 1 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{8}$$

$$\boldsymbol{x} \geq \mathbb{0} \tag{9}$$

Optimal basic feasible solution of (P_{aux}) is

$$\begin{bmatrix} \hat{x_1} \\ \hat{x_2} \\ \hat{x_3} \\ \hat{x_4} \\ \hat{x_5} \end{bmatrix} := \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} \hat{x_1} \\ \hat{x_2} \\ \hat{x_3} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \text{ is a basic feasible solution of (P).}$$

11.4 Example

(P)

$$\max z := \begin{bmatrix} 3 & 2 & 4 \end{bmatrix} \boldsymbol{x}$$

subject to

$$\underbrace{\begin{bmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}}_{A} \boldsymbol{x} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$\boldsymbol{x} \geq \mathbb{0}$$

(P_{aux})

$$\max w := \begin{bmatrix} 0 & 0 & 0 & -1 & -1 \end{bmatrix} \boldsymbol{x}$$

subject to

$$\underbrace{\begin{bmatrix} 5 & 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{bmatrix}}_{\tilde{A}} \boldsymbol{x} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$\boldsymbol{x} := (x_1, x_2, x_3, x_4, x_5)^\top \geq \mathbb{0}$$

Turn (P_{aux}) into canonical form for $B := \{4, 5\}$

$$\max w := \begin{bmatrix} 4 & 2 & 3 & 0 & 0 \end{bmatrix} \boldsymbol{x}$$

subject to

$$\underbrace{\begin{bmatrix} 5 & 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{bmatrix}}_{\tilde{A}} \boldsymbol{x} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$\boldsymbol{x} \geq \mathbb{0}$$

Starting with the basis $B = \{4, 5\}$, solve (P_{aux}) by Simplex Method

$$\max -w = \begin{bmatrix} -11 & -1 & 0 & -3 & 0 \end{bmatrix} \boldsymbol{x} - 3$$

subject to

$$\begin{bmatrix} 5 & 1 & 1 & 1 & 0 \\ -11 & -1 & 0 & -2 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\boldsymbol{x} \geq \mathbb{0}$$

The optimal value of (P_{aux}) is not zero. Therefore, (P) is infeasible.

$\bar{\boldsymbol{y}}$ is the unique solution of

$$\boldsymbol{y}^\top = \tilde{\boldsymbol{c}}_{\boldsymbol{B}}^\top \tilde{A}_B^{-1} \iff \boldsymbol{y}^\top = \underbrace{\begin{bmatrix} 0 & -1 \end{bmatrix}}_{\text{SEF of } (P_{aux})} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1}$$

$\bar{\boldsymbol{y}} = (2, -1)^\top$ is a certificate of infeasibility of (P).

$\bar{\boldsymbol{y}}^\top A = \begin{bmatrix} 11 & 1 & 0 \end{bmatrix} \geq \mathbb{0}^\top$, $\bar{\boldsymbol{y}}^\top \boldsymbol{b} = -3$ So, $\bar{\boldsymbol{y}}$ optimality of (P_{aux})

11.5 Theorem (Fundamental Theorem of LP (SEF))

Every LP problem in SEF, with a full row rank coefficient matrix, is either infeasible, or unbounded or has optimal solution(s). Moreover

(1) if the LP is feasible, it has a basic feasible solution

(2) if the LP has an optimal solution, then it has a basic feasible solution that is optimal.