MATH 237 - Calculus 3

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Directional Derivative Recall
$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$
rate of change of f in the direction $x = a$.

Choose a direction $\hat{\boldsymbol{u}} = (u_1, u_2)$

$(x, y) = (a, b) + s(u_1, u_2)$ similar equation for tangent line

when $s = 0 \implies (x, y) = (a, b)$

Consider distance
$$||\boldsymbol{x} - \boldsymbol{a}|| = ||s(u_1, u_2)|| = |s|||(u_1, u_2)||$$
If we choose $||\boldsymbol{u}|| = 1$, then $|s|$ is the distance along this from some (x, y) to (a, b). So, $||\boldsymbol{u}|| = 1$ is a convention. If $||\boldsymbol{u}|| \neq 1$, normalize it
$$\boldsymbol{u} = \frac{1}{||\boldsymbol{u}||}\boldsymbol{u}$$

Rate of change of $\boldsymbol{u}$ is
$$\frac{\partial f}{\partial \boldsymbol{u}}(a,b) = \lim_{s\to 0} \frac{f(a+su_1, b+su_2)-f(a,b)}{s}$$

Other ways to write
$$\frac{\partial f}{\partial \boldsymbol{u}}(a,b) = D_{\mathbf{u}}f(\boldsymbol{a}) = \frac{d}{ds}f(\boldsymbol{a}+s\boldsymbol{u})\Big|_{s=0}$$
sub for $x = a + su_1, y = b + su_2$ then $f(x, y) = f(a + su_1, b + su_2) = f(\boldsymbol{a} + s\boldsymbol{u})$

2 Quiz

./figures/angle

2.1 Definition (Directional Derivative)

The *directional derivative* of $f(x, y)$ at a point (a, b) in the direction of a unit vector $\mathbf{u} = (u_1, u_2)$ defined by
$$D_{\mathbf{u}}f(a,b) = \frac{d}{ds}f(a+su_1, b+su_2)\Big|_{s=0}$$
provided the derivative exists.

2.2 Theorem

If $f(x, y)$ is differentiable at (a, b) and $\mathbf{u} = (u_1, u_2)$ is a unit vector, then
$$D_{\mathbf{u}}f(a,b) = \nabla f(a,b)\cdot\mathbf{u}$$

Remark 1. Be careful to check the condition of Theorem before applying it. If f is not differentiable at (a, b), then we must apply the definition of the directional derivative.

Remark 2. If we choose $\mathbf{u} = \mathbf{i} = (1, 0)$ or $\mathbf{u} = \mathbf{j} = (0, 1)$, then the directional derivative is equal to the partial derivatives f_x or f_y respectively.

2.3 Theorem

If $f(x, y)$ is differentiable at (a, b), and $\nabla f(a, b) \neq (0, 0)$, then the largest value of $D_{\mathbf{u}} f(a, b)$ is $||\nabla f(a, b)||$, and occurs when $\mathbf{u}$ is in the direction of $\nabla f(a, b)$.

2.4 Theorem

If $f(x, y) \in C^1$ in a neighborhood of (a, b) and $\nabla f(a, b) \neq (0, 0)$, then $\nabla f(a, b)$ is orthogonal to the level curve $f(x, y) = k$ through (a, b).

2.5 Theorem

If $f(x, y, z) \in C^1$ in a neighborhood of (a, b, c) and $\nabla f(a, b, c) \neq (0, 0, 0)$, then $\nabla f(a, b, c)$ is orthogonal to the level curve $f(x, y, z) = k$ through (a, b, c).

2.6 Definition (2nd degree Taylor polynomial)

The *second degree Taylor polynomial* $P_{2,(a,b)}$ of $f(x, y)$ at (a, b) is given by

$$P_{2,(a,b)}(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) +$$
$$\frac{1}{2}[f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(a, b)(y - b)^2]$$

2.7 Theorem

If $f''(x)$ exists on $[a, x]$, then there exists a number c between a and x such that

$$f(x) = f(a) + f'(a)(x - a) + R_{1,a}(x)$$

where

$$R_{1,a}(x) = \frac{1}{2} f''(c)(x - a)^2$$

2.8 Theorem (Taylor's Theorem

If $f(x, y) \in C^2$ in some neighborhood $N(a, b)$ of (a, b), then for all $(x, y) \in N(a, b)$ there exists a point (c, d) on the line segment joining (a, b) and (x, y) such that

$$f(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + R_{1,(a,b)}(x, y)$$

where

$$R_{1,(a,b)}(x, y) = \frac{1}{2}[f_{xx}(c, d)(x - a)^2 + 2f_{xy}(c, d)(x - a)(y - b) + f_{yy}(c, d)(y - b)^2]$$

Remark 3. Like the one variable case, Taylor's Theorem for $f(x, y)$ is an existence theorem. That is, it only tells us that the point (c, d) exists, but not how to find it.

Remark 4. The most important thing about the error term $R_{1,(a,b)(x,y)}$ is not its explicit form, but rather its dependence on the magnitude of the displacement $||(x, y)(a, b)||$. We state the result as a Corollary.

2.9 Corollary

If $f(x,y) \in C^2$ in some closed neighborhood $N(a,b)$ of (a,b), then there exists a positive constant M such that

$$R_{1,(a,b)}(x,y) \leq M||(x,y)-(a,b)||^2$$

for all $(x,y) \in N(a,b)$.

2.10 Taylor's Theorem of order k

If $f(x,y) \in C^{k+1}$ at each point on the line segment joining (a,b) and (x,y), then there exists a point (c,d) on the line segment between (a,b) and (x,y) such that

$$f(x,y) = P_{k,(a,b)}(x,y) + R_{k,(a,b)}(x,y)$$

where

$$R_{k,(a,b)}(x,y) = \frac{1}{(k+1)!}|(x-a)D_1 + (y-b)D_2|^{k+1} f(c,d)$$

2.11 Corollary

If $f(x,y) \in C^k$ in some neighborhood of (a,b) then

$$\lim_{(x,y)\to(a,b)} \frac{|f(x,y) - P_{k,(a,b)}(x,y)|}{||(x,y)-(a,b)||^k} = 0$$

2.12 Corollary

If $f(x,y) \in C^{k+1}$ in some closed neighborhood $N(a,b)$ of (a,b), then there exists a constant $M > 0$ such that

$$|f(x,y) - P_{k,(a,b)}(x,y)| \leq M||(x,y)-(a,b)||^{k+1}$$

for all $(x,y) \in N(a,b)$.

3 2019-10-23

3.1 Definition (Local Maximum and Minimum)

A point (a,b) is a *local maximum point* for $f(x,y)$ if

$$f(x,y) \leq f(a,b)$$

for all (x,y) in some neighborhood of (a,b).
A point (a,b) is a *local minimum point* for $f(x,y)$ if

$$f(x,y) \geq f(a,b)$$

for all (x,y) in some neighborhood of (a,b).

3.2 Theorem

Let $f(x, y)$ have continuous partials. If (a, b) is a local maximum or minimum point of f, then

$$\nabla f(a, b) = 0$$

or at least one of f_x, f_y does not exist at (a, b).

Proof. Let (a, b) be a local maximum or minimum point of f. Fix $x = a$, consider $f(a, y) = z$ (cross section), it has a local maximum/minimum point at $y = b \implies \frac{\partial f}{\partial y}(a, b) = 0$ (or DNE) when $y = b$.

Similarly, $\frac{\partial f}{\partial x}(a, b) = 0$ (or DNE). □

3.3 Definition (Critical Point)

A point (a, b) in the domain of $f(x, y)$ is called a *critical point* of f if $\frac{\partial f}{\partial x}(a, b) = 0$ or $\frac{\partial f}{\partial x}(a, b)$ does not exist, and $\frac{\partial f}{\partial y}(a, b) = 0$ or $\frac{\partial f}{\partial y}(a, b)$ does not exist.

3.4 Examples

Consider $f(x, y) = \sqrt{x^2 + y^2}$ which is a cone (upper half). $(0, 0)$ is a local minimum point.

$$f(x, y) = \sqrt{x^2 + y^2} > 0 = f(0, 0)$$

However, $f_x(0, 0)$ and $f_y(0, 0)$ does not exist.

Consider $g(x, y) = x^2 - y^2$ which is a hyperbolic paraboloid (saddle surface)

$$\begin{aligned} g_x &= 2x \\ g_y &= 2y \end{aligned}$$

So, $(0, 0)$ is the only critical point of g, but

$$\begin{aligned} g(x, 0) &> g(0, 0) \\ h(0, y) &< h(0, 0) \end{aligned}$$

for all $x, y \in \mathbb{R}$, so $(0, 0)$ is neither a local maximum or minimum point. We classify it as a *saddle point*.

To summarize, all critical points are either local maxima, minima or saddle points.

3.5 Example (Finding Critical Points)

Find all critical points of $f(x, y) = xy(1 - x^2 - y^2)$.

$$\begin{aligned} f_x &= y(1 - x^2 - y^2) + xy(-2x) && (1) \\ &= y[(1 - x^2 - y^2) + x(-2x)] && (2) \\ &= y(1 - x^2 - y^2 - 2x^2) && (3) \\ &= y(1 - y^2 - 3x^2) && (4) \\ & && (5) \end{aligned}$$

Similarly, we get

$$f_y = x(1 - 3y^2 - x^2) = 0 \tag{6}$$

Note that (1) and (6) are both non-linear systems. (6) yields roots $y = 0$ and $y = \pm\sqrt{1 - 3x^2}$ for roots, split into two cases.

Case 1 $y = 0$

Substituting into (6) we get $x(1 - x^2) = 0$, giving $x = -1, 0, 1$. Thus, the corresponding critical points are: $(-1, 0), (0, 0), (1, 0)$.

Case 2 $y = \pm\sqrt{1 - 3x^2}$

Substituting into (6) we get $x(8x^2 - 2)$, giving $x = 0, \frac{1}{2}, -\frac{1}{2}$. To find the corresponding y values, plug the x values into $y = \pm\sqrt{1 - 3x^2}$. Thus, the corresponding critical points are: $(0, 0), \underbrace{(\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, \frac{1}{2})}_{\text{+ sqrt}}, \underbrace{(\frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2})}_{\text{- sqrt}},$.

We need an analogy to the 2nd derivative test for $y = f(x)$.

$f\prime\prime > 0 \to$ local minimum, $f\prime\prime < 0 \to$ local maximum

Consider the Taylor Series for $f(x, y)$ about (a, b) such that $\nabla f(a, b) = (0, 0)$

$$f(x, y) - f(a, b) \approx \frac{1}{2}\left[f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(y - b) + f_{yy}(a, b)(y - b)^2\right] + \underbrace{\cdots}_{\text{H.O.T}} \tag{7}$$

If x is close to a and y is close to b, then the higher order terms can be neglected. So,

$$f(x, y) - f(a, b) \approx \frac{1}{2}\left[f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(y - b) + f_{yy}(a, b)(y - b)^2\right] \tag{8}$$

3.6 Theorem (Second Partials Derivatives Test)

Suppose $f(x, y) \in C^2$ in some neighborhood of (a, b) and that

$$\nabla f(a, b) = 0$$

(1) If $f(x, y) - f(a, b) > 0$ (positive definite) for all (x, y) near (a, b), $(x, y) \neq (0, 0) \neq (a, b)$ then (a, b) is a local minimum point of f.
(2) If $f(x, y) - f(a, b) < 0$ (negative definite) for all (x, y) near (a, b), $(x, y) \neq (0, 0) \neq (a, b)$ then (a, b) is a local maximum point of f.
(3) If $f(x, y) - f(a, b) < 0$ for some (x, y) near (a, b) and $f(x, y) - f(a, b) > 0$ for some other (x, y) near (a, b), then (a, b) is a saddle point (indefinite) of f.