Calculus 1 for Honours Mathematics

MATH 137

Fall 2018 ()

LaTeXer: *Cameron Roopnarine*

Instructor:

September 26, 2024

Contents

Contents 1

1 Sequences and Convergence 3
1.1 Absolute Values . . . 3
1.1.1 Inequalities Involving Absolute Values . . . 3
1.2 Sequences and Their Limits . . . 5
1.2.1 Introduction to Subsequences . . . 5
1.2.2 Recursively Defined Sequences . . . 5
1.2.3 Subsequences and Tails . . . 6
1.2.4 Limits of Sequences . . . 6
1.2.5 Divergence to Infinity . . . 9
1.2.6 Arithmetic For Limits . . . 9
1.3 Squeeze Theorem . . . 12
1.4 Monotone Convergence Theorem . . . 13

2 Limits and Continuity 17
2.1 Introduction to Function Limits . . . 17
2.2 Sequential Characterization of Limits . . . 18
2.3 Arithmetic Rules for limits of functions . . . 19
2.4 One-Sided Limits . . . 20
2.5 The Squeeze Theorem . . . 21
2.6 The Fundamental Trigonometric Limit . . . 22
2.7 Limits at infinity and Asymptotes . . . 22
2.7.1 Asymptotes and Limits at Infinity . . . 22
2.7.2 The Fundamental Log Limit . . . 22
2.7.3 Vertical Asymptotes and Infinite Limits . . . 22
2.8 Continuity . . . 22
2.8.1 Types of Discontinuities . . . 22
2.8.2 Continuity of Certain Functions . . . 22
2.8.3 Arithmetic Rules for Continuity . . . 22
2.8.4 Continuity On An Interval . . . 22
2.9 The Intermediate Value Theorem . . . 22
2.9.1 Approximating Solutions to Equations . . . 22
2.9.2 The Bisection Method . . . 22
2.10 The Extreme Value Theorem . . . 22

3 Derivatives 23
3.1 Instantaneous Velocity . . . 23
3.2 Definition of the Derivative . . . 24
3.2.1 The Tangent Line . . . 24
3.2.2 Differentiability versus Continuity . . . 25
3.3 The Derivative Function . . . 26

3.4 Derivatives of Elementary Functions 28
3.4.1 The Derivative of $\sin x$ and $\cos x$ 28
3.4.2 The Derivative of e^x 29
3.7 Arithmetic Rules for Differentiation 29
3.8 The Chain Rule 31
3.9 Derivatives of Other Trigonometric Functions 32
3.5 Tangent Lines and Linear Approximation 32
3.5.1 Error in Linear Approximation 33
3.5.2 Applications of Linear Approximation 34
3.6 Newton's Method 35
3.10 Derivatives of Inverse Functions 36
3.11 Derivatives of Inverse Trigonometric Functions 37
3.12 Implicit Differentiation 38
3.13 Local Extrema 40
3.13.1 The Local Extrema Theorem 40

4 The Mean Value Theorem **42**
4.1 The Mean Value Theorem 42
4.2 Applications of the Mean Value Theorem 42
4.2.1 Antiderivatives 42
4.2.2 Increasing Function Theorem 42
4.2.3 Functions with Bounded Derivatives 42
4.2.4 Comparing Functions Using Their Derivatives 42
4.3 L'Hopital's Rule 42
4.3.1 Interpreting the Second Derivative 42
4.3.2 Formal Definition of Concavity 42
4.3.3 Classifying Critical Points: The First and Second Derivative Tests 42
4.4 Curve Sketching 42

5 Taylor Polynomials and Taylor's Theorem **43**
5.1 Introduction to Taylor Polynomials and Approximation 43
5.2 Taylor's Theorem and Errors in Approximations 43
5.3 Big-O 43

Chapter 1

Sequences and Convergence

1.1 Absolute Values

What is an absolute value? We commonly think of it as an operation that removes negative signs.

EXAMPLE 1.1.1

$|-2| = 2$, $|-17| = 17$, $|3| = 3$, etc.

So, is $|-x| = x$ for all $x \in \mathbb{R}$? Not always! Let's give the definition to avoid ambiguity.

DEFINITION 1.1.2

Let $x \in \mathbb{R}$. The **absolute value** of x is denoted $|x|$, and is defined as follows:

$$|x| = \begin{cases} x, & x > 0, \\ 0, & x = 0, \\ -x, & x < 0. \end{cases}$$

This also tells us the distance from x to 0, or the magnitude (size of x).

EXAMPLE 1.1.3

How do we get the distance between two arbitrary numbers using absolute values? For example, what is the distance from 3 to 7? 4 units. Also, $|7-3| = 4 = |3-7|$.

So, the distance from a to b is $|b-a|$ for all $a, b \in \mathbb{R}$. Also, $|b-a| = |a-b|$, which makes sense since the distance from a to b should be the same as the distance from b to a.

1.1.1 Inequalities Involving Absolute Values

The main focus of this course is **approximation**. We will seek ways to approximate roots, curves, limits, etc., but if we make an approximation it will be useless unless we can talk about how close it is to the actual object! So, we will look for ways to determine the maximum size of the **error**. Before we do this, we will need to examine **inequalities**. Let's start with the triangle inequality.

THEOREM 1.1.4: Triangle Inequality

Let $x, y, z \in \mathbb{R}$. *Then*
$$|x-y| \leq |x-z| + |z-y|.$$

Proof: Since $|x-y| = |y-x|$, we can assume without loss of generality (WLOG) that $x \leq y$. Hence, we consider three cases.
Case 1 ($z < x$): Clearly, $|x-y| \leq |z-y|$, which means $|x-y| \leq |x-z| + |z-y|$.
Case 2 ($x \leq z \leq y$): In this case, $|x-y| = |x-z| + |z-y|$, which means $|x-y| = |x-z| + |z-y|$, as desired.
Case 3 ($y < z$): This time, $|x-y| \leq |x-z|$, so $|x-y| \leq |x-z| + |z-y|$.

We consider a useful variant of the triangle inequality.

THEOREM 1.1.5: Triangle Inequality II

Let $x, y \in \mathbb{R}$. *Then*
$$|x+y| \leq |x| + |y|.$$

Proof:

$$\begin{aligned} |x+y| &= |x-(-y)| \\ &\leq |x-0| + |0-(-y)| && \text{triangle inequality with } z = 0 \\ &= |x| + |y|. \end{aligned}$$

If we want to prove $|x| < \delta$, we just need to prove $x < \delta$ and $x > -\delta$, that is, $-\delta < x < \delta$. So, what do the inequalities of the form $|x-a| < \delta$ for $a, \delta \in \mathbb{R}$ look like? What set does this represent? Well, it's the set of all $x \in \mathbb{R}$ that are less than δ units away from a. So, starting at a, we move δ-units to the left and right, which means
$$|x-a| < \delta \iff -\delta < x-a < \delta \iff a-\delta < x < a+\delta.$$
So, it is the interval $(a-\delta, a+\delta)$, where we do not include the endpoints as the inequality is strict.

What about $|x-a| \leq \delta$? In this case,
$$|x-a| \leq \delta \iff -\delta \leq x-a \leq \delta \iff a-\delta \leq x \leq a+\delta.$$
So, it is the interval $[a-\delta, a+\delta]$.

What about $0 < |x-a| < \delta$? Now, the distance can't be zero which means $x \neq a$. So, it translates to $(a-\delta, a+\delta) \setminus \{a\}$ or $(a-\delta, a) \cup (a, a+\delta)$.

EXAMPLE 1.1.6

Find the corresponding sets for the inequalities.
(1) $|x-4| < 3$.
(2) $2 \leq |x-4| < 4$.
(3) $|x-1| + |x+2| \geq 4$.

Solution.
(1) $|x-4| < 3 \iff -3 < x-4 < 3 \iff 1 < x < 7$, so $(1, 7)$ is the corresponding interval.
(2) $2 \leq |x-4| < 4$ means $2 \leq |x-4|$ and $|x-4| < 4$, so
$$(2 \leq x-4) \vee (x-4) \leq -2 \iff (6 \leq x) \vee (x \leq 2)$$
and
$$-4 < x-4 < 4 \iff 0 < x < 8.$$

Putting these together, we get $0 < x \leq 2$ or $6 \leq x < 8$, so $(0, 2] \cup [6, 8)$ is the corresponding interval.

(3) We consider three cases.

(i) If $x > 1$, then both $x - 1 > 0$ and $x + 2 > 0$, then

$$x - 1 + x + 2 > 4 \iff 2x + 1 > 4 \iff 2x > 3 \iff x < 3/2.$$

(ii) If $-2 \leq x \leq 1$, then $|x - 1| = 1 - x$, but $|x + 2| = x + 2$, so we get

$$1 - x + x + 2 \geq 4 \iff 3 \geq 4,$$

which is not true for <u>any</u> x.

(iii) If $x < -2$, then $|x - \overline{1| = } 1 - x$ and $|x + 2| = -x - 2$, then

$$1 - x + (-x - 2) \geq 4 \iff -1 - 2x \geq 4 \iff -5 \geq 2x \iff -5/2 \geq x.$$

Putting it all together, we get $x \geq 3/2$ or $x \leq -5/2$, that is, $(-\infty, -5/2] \cup (3/2, \infty)$.

1.2 Sequences and Their Limits

1.2.1 Introduction to Subsequences

DEFINITION 1.2.1

An infinite sequence of numbers is a list of numbers in a definite order, e.g.,

$$a_1, a_2, a_3, a_4, \ldots, a_n,\ a_i \in \mathbb{R}.$$

<u>Notation</u>: $\{a_1, a_2, \ldots, a_n\}$ or $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$.

Sequences can be defined explicitly (in terms of n) or recursively (in terms of previous terms).

EXAMPLE 1.2.2: Explicit Sequences

- $\left\{\frac{1}{n+1}\right\}_{n=1}^{\infty}$: $1/2, 1/3, 1/4, 1/5, \ldots$.
- $\{\sqrt{n+2}\}_{n=2}^{\infty}$: $\sqrt{4}, \sqrt{5}, \sqrt{6}, \ldots$.
- $\{(-1)^n\}_{n=1}^{\infty}$: $-1, 1, -1, 1, \ldots$.

1.2.2 Recursively Defined Sequences

EXAMPLE 1.2.3: Recursive Sequences

- $a_1 = 1$, $a_{n+1} = \sqrt{1 + a_n}$, so $a_1 = 1$, $a_2 = \sqrt{2}$, $a_3 = \sqrt{1 + \sqrt{2}}$, and so on for $n \geq 1$.
- Fibonacci's sequence: $a_1 = 1$, $a_2 = 1$, $a_{n+2} = a_{n+1} + a_n$ for $n \geq 1$, i.e., $1, 1, 2, 3, 5, 8, 13, \ldots$.

We can plot sequences on a number line, or we could think of a sequence as a function $f\colon \mathbb{N} \to \mathbb{R}$, writing $f(n) = a_n$, e.g., for $a_n = 1/2$ we would write $f(n) = 1/2$.

<u>Why study sequences?</u>

- Lots of continuous processes can be modelled with discrete data, as we will see.
- We can use recursive sequences to approximate solutions to equations that can't be solved explicitly (Newton's Method).

- For another (ancient) application, see page 14 of the course notes about calculating square roots.

Our goal now will be to determine how to find the limit of a sequence, that is, find what the value of the terms of the sequence are approaching (if it exists).

We may want to build new sequences out of old ones or only discuss what happens to a sequence eventually, that is, after a certain index.

EXAMPLE 1.2.4

For $\{\frac{1}{n}\}_{n=1}^{\infty}$, if we consider only the odd terms, we get $1, 1/3, 1/5$, or the k^{th} term is

$$\frac{1}{2k-1}$$

for $k \in \mathbb{N}$. This is called a subsequence.

1.2.3 Subsequences and Tails

DEFINITION 1.2.5: Subsequence

If $\{a_n\}$ is a sequence and $n_1, n_2, \ldots$ is a sequence of natural numbers, where $n_1 < n_2 < n_3 < \cdots$, then the sequence

$$\{a_{n_1}, a_{n_2}, \ldots\} = \{a_{n_k}\}$$

is a **subsequence** of $\{a_n\}$.

One particular subsequence is $\{a_k, a_{k+1}, a_{k+2}\}$ for some $k \in \mathbb{N}$. This is called the tail of $\{a_n\}$ with cut-off k.

1.2.4 Limits of Sequences

We are going to see lots of different limits this term, but we will start with sequences.

EXAMPLE 1.2.6

$\{\frac{1}{n}\}$ seems like it converges to 0, or that 0 is the limit of the sequence. We saw this when we plotted the sequence. We will eventually want a formal definition, but let's start intuitively.

Given a sequence $\{a_n\}$, what does it mean to say that $\{a_n\}$ converges to L as n goes to infinity?

What about "as n gets larger, a_n gets closer to L?" Unfortunately, this isn't a good definition. For example, as n gets larger $\frac{1}{n}$ gets closer to 0, but it also gets closer to -1, -2, and so on. But, 0 is the limit! What makes it different? Well, the sequence gets infinitely close to 0, unlike the other numbers! Let's try to define this again: "the limit of $\{a_n\}$ is L if, as n gets infinitely large, a_n gets infinitely close to L." This is much better! But how can we formalize the idea of "infinitely close?"

DEFINITION 1.2.7: Formal Definition of the Limit of a Sequence I

Let $\{a_n\}$ be a sequence in $\mathbb{R}$. For $L \in \mathbb{R}$, we say that the sequence $\{a_n\}$ **converges** to L (or that the **limit** of $\{a_n\}$ is equal to L), and we write $a_n \to L$ (as $n \to \infty$), or we write $\lim\limits_{n\to\infty} a_k = L$, when

$$\forall \varepsilon \in \mathbb{R}_{>0}\ \exists N \in \mathbb{N}\ \forall n \in \mathbb{N}\ \big(n \geq N \implies |a_n - L| < \varepsilon\big).$$

We say that the sequence $\{a_n\}$ **diverges to infinity** (in $\mathbb{R}$) when there exists $L \in \mathbb{R}$ such that $\{a_n\}$ converges to L. We say that the sequence **diverges** (in $\mathbb{R}$) when it does not converge (to any $L \in \mathbb{R}$).

EXAMPLE 1.2.8

Consider $a_n = \frac{1}{n^2}$. We'd guess that the limit is 0. Say $\varepsilon = \frac{1}{100}$, can we find a large enough $n \in N$ so that $\left|\frac{1}{n^2} - 0\right| < \frac{1}{100}$ if $n \geq N$? Well, we need

$$\left|\frac{1}{n^2} - 0\right| < \frac{1}{100} \implies \frac{1}{n^2} < \frac{1}{100} \implies n^2 > 100,$$

so $n > 10$. Let $N = 11$, then if $n \geq N$, we get $\left|\frac{1}{n^2} - 0\right| < \frac{1}{100}$. But wait! We aren't done yet! The definition says we need to prove it for any $\varepsilon > 0$, but we only proved it for $\varepsilon = \frac{1}{100}$. Let's adapt the proof to work for any $\varepsilon > 0$.

Proof that $\lim\limits_{n\to\infty} \frac{1}{n^2} = 0$. Let $\varepsilon > 0$ be given. Let $N > \frac{1}{\sqrt{\varepsilon}}$ for $N \in \mathbb{N}$. Then, if $n \geq N$, we get

$$\left|\frac{1}{n^2} - 0\right| = \frac{1}{n^2} \leq \frac{1}{N^2} < \frac{1}{(1/\sqrt{\varepsilon})^2} = \frac{1}{1/\varepsilon} = \varepsilon$$

as desired.

The point is: we have to give a method for choosing N that works for any $\varepsilon > 0$. Also, the logical order of the proof is important, so let's do some more examples.

EXAMPLE 1.2.9

Prove that $\lim\limits_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2}$.

Proof: Let $\varepsilon > 0$ be given. Let $N > \frac{1}{4}\left(\frac{3}{\varepsilon} - 6\right)$ for $N \in \mathbb{N}$. Then, if $n \geq N$, we get:

$$|a_n - L| = \left|\frac{n}{2n+3} - \frac{1}{2}\right| = \frac{3}{4n+6} \leq \frac{3}{4N+6} < \frac{3}{4\left(\frac{1}{4}(\frac{3}{\varepsilon} - 6)\right) + 6} = \varepsilon$$

as desired.
Aside: We want

$$\frac{3}{4n+6} < \varepsilon \iff \frac{3}{\varepsilon} < 4n+6 \iff \frac{3}{\varepsilon} - 6 < 4n \iff \frac{1}{4}\left(\frac{3}{\varepsilon} - 6\right) < n.$$

EXAMPLE 1.2.10

Prove that $\lim\limits_{n\to\infty} \frac{n^2}{3n^2+7n} = \frac{1}{3}$.

Proof: Let $\varepsilon > 0$ be given. Let $N > \frac{7}{9\varepsilon}$ for $N \in \mathbb{N}$. Then, if $n \geq N$, we get:

$$|a_n - L| = \left|\frac{n^2}{3n^2+7n} - \frac{1}{3}\right| = \frac{7n}{9n^2+21n} \leq \frac{7n}{9n^2} = \frac{7}{9n} \leq \frac{7}{9(\frac{7}{9\varepsilon})} = \varepsilon.$$

Aside: We want

$$\frac{7}{9n} < \varepsilon \iff \frac{7}{9\varepsilon} < n.$$

REMARK 1.2.11: Avoid Common Mistakes

- Don't choose ε! Let it be arbitrary.
- Never assume $|a_n - L| < \varepsilon$, make sure you only do work in an aside with that inequality since it is

what you are proving.
- In practice, unless you are asked to, do not use this formal definition. We will now try to develop better methods for finding limits.

Equivalent Definitions of the Limit

When proving $\lim_{n\to\infty} a_n = L$, we are given $\varepsilon > 0$, and we try to find $N \in \mathbb{N}$ so that if $n \geq N$, then $|a_n - L| < \varepsilon$. But, this is the same as saying $a_n \in (L-\varepsilon, L+\varepsilon)$. Also, the collection of $\{a_n\}$ for which $n \geq N$ is the tail of the sequence with cut-off N. So, here's another definition.

DEFINITION 1.2.12

$\lim_{n\to\infty} a_n = L$ if for any $\varepsilon > 0$, the interval $(L-\varepsilon, L+\varepsilon)$ contains a tail of the sequence $\{a_n\}$.

Let's push it further! Since the above is true for any $\varepsilon > 0$, if we pick any open interval (a, b) containing L, then we can find a small enough $\varepsilon > 0$ so that $(L-\varepsilon, L+\varepsilon) \subseteq (a, b)$. Therefore, any interval containing L also contains a tail of $\{a_n\}$. Let's collect all of these alternate (but equivalent) definitions together.

THEOREM 1.2.13

The following are equivalent:
(1) $\lim_{n\to\infty} a_n = L$.
(2) For any $\varepsilon > 0$, $(L-\varepsilon, L+\varepsilon)$ *contains a tail of* $\{a_n\}$.
(3) For any $\varepsilon > 0$, $(L-\varepsilon, L+\varepsilon)$ *contains all but finitely many terms of* $\{a_n\}$.
(4) Every interval (a, b) *containing* L *contains a tail of* $\{a_n\}$.
(5) Every interval (a, b) *containing* L *contains all but finitely many terms of* $\{a_n\}$.
Clearly, changing finitely many terms of $\{a_n\}$ *does not affect the convergence or the limit.*

EXAMPLE 1.2.14

Can a sequence have more than one limit? Consider $\{(-1)^n\} = -1, 1, -1, 1, \ldots$, it equals to both 1 and -1 infinitely often. Could both 1 and -1 be the limits? No! Let's prove -1 isn't a limit.

Proof: Consider the interval $(-2, 0)$. Clearly $-1 \in (-2, 0)$, but this interval does not contain any of the infinitely many 1's in the sequence. So, -1 is not a limit by (5) above. A similar argument can be used with the interval $(0, 2)$ to show 1 is also not a limit. So, does $\{(-1)^n\}$ have a limit at all? Let's prove it doesn't! Let $\varepsilon = 1/2$, and supposed for a contradiction that the sequence converges to $L \in \mathbb{R}$. That means the interval $(L - 1/2, L + 1/2)$ must contain all but finitely many terms of the sequence, that is, but 1 and -1 must lie in that interval. But the interval is only 1 unit long! So there is not $L \in \mathbb{R}$ for which both 1 and -1 lie inside $(L - 1/2, L + 1/2)$. So, $\{(-1)^n\}$ diverges.

A similar argument can be used to prove limits are unique.

THEOREM 1.2.15

Let $\{a_n\}$ *be a sequence in* $\mathbb{R}$. *If* $\{a_n\}$ *has a limit (finite or infinite), then the limit is unique.*

Proof: Suppose for a contradiction that L and M are both limits of $\{a_n\}$ and $L \neq M$ and WLOG that $L < M$. Consider two intervals:

$$(L-1, \tfrac{L+M}{2}) \ni L, \quad (\tfrac{L+M}{2}, M+1) \ni M.$$

This means, by definition, only finitely many terms of the sequence are not in the first interval and only finitely many terms are not in the second interval. But the sequence has infinitely many terms! So, at

least one term is in both intervals which is impossible. This is a contradiction, so $L = M$.
Note: This does not cover the cases where the limit is infinite.

REMARK 1.2.16: A Remark on Possible Limits

If $a_n \geq 0$ for all n, then $\{a_n\}$ can't converge to a negative number! If it did, say to $L < 0$, then the interval $(L-1, 0)$ would contain L but no terms of the sequence.

THEOREM 1.2.17

If $a_n \geq 0$ for all n and $\lim\limits_{n\to\infty} a_n = L$, then $L \geq 0$. More generally, if $\alpha \leq a_n \leq \beta$ for all n and $\lim\limits_{n\to\infty} a_n = L$, then $\alpha \leq L \leq \beta$.

- Q: If $a_n > 0$ for all n and $\lim\limits_{n\to\infty} a_n = L$ is $L > 0$?
- A: Not necessarily! Consider $a_n = \frac{1}{n} > 0$, but $L = 0$.

1.2.5 Divergence to Infinity

Consider $a_n = n$. It is clear that the sequence is getting larger without bound, so $\lim\limits_{n\to\infty} a_n$ does not exist. That is, $\{a_n\}$ diverges. But we can say more! Since it does not get infinitely large, we can make a definition to capture this.

DEFINITION 1.2.18

- We say that $\{a_n\}$ **diverges to** ∞, or that the limit of $\{a_n\}$ is equal to **infinity**, and we write $a_n \to \infty$ (as $n \to \infty$), or we write $\lim\limits_{n\to\infty} a_n = \infty$, when
$$\forall M \in \mathbb{R}_{>0}\, \exists N \in \mathbb{N}\, \forall n \in \mathbb{N}\, \big(n \geq N \implies a_n > M\big).$$
Equivalently, any interval of the form (M, ∞) contains a tail of $\{a_n\}$.
- We say that $\{a_n\}$ **diverges to** $-\infty$, or that the limit of $\{a_n\}$ is equal to **negative infinity**, and we write $a_n \to -\infty$ (as $n \to \infty$), or we write $\lim\limits_{n\to\infty} a_n = -\infty$, when
$$\forall M \in \mathbb{R}_{<0}\, \exists N \in \mathbb{N}\, \forall n \in \mathbb{N}\, \big(n \geq N \implies a_n < M\big).$$
Equivalently, any interval of the form $(-\infty, M)$ contains a tail of $\{a_n\}$.

EXAMPLE 1.2.19

Show $\lim\limits_{n\to\infty} (1-n) = -\infty$.

Proof: Let $M < 0$ be given, pick $N > 1 - M$ for $N \in \mathbb{N}$. Then, if $n \geq N$, we have
$$a_n = 1 - n \leq 1 - N < 1 - (1 - M) = M.$$
Aside: Want $1 - n < M \iff 1 - M < n$.

1.2.6 Arithmetic For Limits

If we can avoid using the definition to find a limit, we should. There are certain rules we can follow to compute lots of sequence limits. Let's see them now!

THEOREM 1.2.20

(1) $\alpha > 0 \implies \lim_{n\to\infty} n^\alpha = \infty$.
(2) $\alpha < 0 \implies \lim_{n\to\infty} n^\alpha = 0$.

THEOREM 1.2.21: Arithmetic Rules for Limits of Sequences

Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\mathbb{R}$ and let $c \in \mathbb{R}$. Suppose that $\{a_n\}$ and $\{b_n\}$ both converge with $a_n \to L$ and $b_n \to M$. Then

(1) if $a_n = C$ for all n, then $C = L$,
(2) $\lim_{n\to\infty} ca_n = cL$,
(3) $\lim_{n\to\infty} (a_n + b_n) = L + M$,
(4) $\lim_{n\to\infty} (a_n - b_n) = L - M$,
(5) $\lim_{n\to\infty} a_n b_n = LM$,
(6) if $M \neq 0$, $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$,
(7) for all $k \in \mathbb{N}$, $\lim_{n\to\infty} a_{n+k} = L$.

Proof: Exercises, but let's prove (3) as an example. Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} a_n = L$, we can find $N_1 \in \mathbb{N}$ so that if $n \geq N_1$, we get $|a_n - L| < \varepsilon/2$. Also, since $\lim_{n\to\infty} b_n = M$, we can find $N_2 \in \mathbb{N}$ so that if $n \geq N_2$, we have $|b_n - M| < \varepsilon/2$. Now, let $N = \max(N_1, N_2)$. Then, if $n \geq N$ we get

$$|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_n - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

where we used the triangle inequality in the first inequality.

REMARK 1.2.22

To use any of the above properties, the limits need to exist!

EXAMPLE 1.2.23

(1) $\lim_{n\to\infty} \dfrac{3n+7}{n+2} = \lim_{n\to\infty} \dfrac{3+7/n}{1+2/n} = \dfrac{\lim_{n\to\infty} 3 + \lim_{n\to\infty} 7/n}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} 2/n} = \dfrac{3+0}{1+0} = 3.$
(2) $\lim_{n\to\infty} \dfrac{n^3+n^2+1}{2n^3+7n^2-1} = \lim_{n\to\infty} \dfrac{1+1/n+1/n^3}{2+7/n-1/n^3} = \dfrac{1+0+0}{2+0+0} = \dfrac{1}{2}.$
(3) $\lim_{n\to\infty} \dfrac{n+1}{n^2+1} = \lim_{n\to\infty} \dfrac{1/n+1/n^2}{1+1/n^2} = \dfrac{0+0}{1+0} = 0.$

REMARK 1.2.24

You don't need to write "arithmetic rules" every time, as we <u>always</u> use them! Just make sure you show your work!

EXAMPLE 1.2.25

What if in property (5), $M = 0$? Anything can happen!

- $\lim_{n\to\infty} \dfrac{1/n}{1/n} = 1$ even though $1/n \to 0$.
- $\lim_{n\to\infty} \dfrac{1/n}{1/n^2} = \lim_{n\to\infty} \dfrac{n^2}{n} = \lim_{n\to\infty} n = \infty.$

- $\lim\limits_{n\to\infty} \frac{1/n^2}{1/n} = \lim\limits_{n\to\infty} \frac{1}{n} = 0.$

Hence, we will need to handle these on an individual basis.

However, there is one thing we can say.

THEOREM 1.2.26

If $\lim\limits_{n\to\infty} b_n = 0$ *and* $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$ *exists, then* $\lim\limits_{n\to\infty} a_n = 0$.

Proof: Suppose $\lim\limits_{n\to\infty} b_n = 0$, and say $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = k \in \mathbb{R}$, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{a_n}{b_n} = k \cdot 0 = 0.$$

COROLLARY 1.2.27

If $\lim\limits_{n\to\infty} b_n = 0$ *and* $\lim\limits_{n\to\infty} a_n \neq 0$*, then* $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$ *does not exist.*

EXAMPLE 1.2.28

$$\lim_{n\to\infty} \frac{n^3+3n}{n^2+1} = \lim_{n\to\infty} \frac{1+3/n^2}{1/n+1/n^3}.$$

However, the numerator converges to 1, while the denominator converges to 0. Therefore, the limit does not exist.

We could also say

$$\lim_{n\to\infty} \frac{n^3+3n}{n^2+1} = \infty,$$

which means DNE <u>and</u> infinitely large!

Let's compute the limit of any ratios of powers of n.

PROPOSITION 1.2.29

$$\begin{aligned}\lim_{n\to\infty} \frac{b_0+b_1n+b_2n^2+\cdots+b_jn^j}{c_0+c_1n+c_2n^2+\cdots+c_kn^k} &= \lim_{n\to\infty} \frac{n^j}{n^k}\left[\frac{\frac{b_0}{n^j}+\frac{b_1}{n^{j-1}}+\cdots+b_j}{\frac{c_0}{n^k}+\frac{c_1}{n^{k-1}}+\cdots+c_k}\right] \\ &= \begin{cases} \frac{b_j}{c_k}, & j=k, \\ 0, & j<k, \\ \infty, & j>k \wedge b_j/c_k > 0, \\ -\infty, & j>k \wedge b_k/c_k < 0. \end{cases}\end{aligned}$$

EXAMPLE 1.2.30

- $$\lim_{n\to\infty} \frac{3n+2}{2n-1} = \frac{3}{2}.$$
- $$\lim_{n\to\infty} \frac{4n^2+5n}{n^3-1} = 0.$$

- $$\lim_{n\to\infty} \frac{7-n^4}{1+n^3} = -\infty.$$

REMARK 1.2.31

Still show work when writing solutions on a test though (e.g., dividing by highest power of n).

EXAMPLE 1.2.32

If we have something that "looks like" $\infty - \infty$, then multiply by the conjugate!

$$\begin{aligned}
\lim_{n\to\infty} \sqrt{n^2-4} - n &= \lim_{n\to\infty} \sqrt{n^2-4} - n\frac{\sqrt{n^2+4}+n}{\sqrt{n^2+4}+n} \\
&= \lim_{n\to\infty} \frac{n^2+4-n^2}{\sqrt{n^2+4}+n} \\
&= \lim_{n\to\infty} \frac{4}{\sqrt{n^2+4}+n} \\
&= \lim_{n\to\infty} \frac{4/n}{\sqrt{1+4/n^2}+1} \\
&= \frac{0}{2} \\
&= 0.
\end{aligned}$$

Recursive Sequence Limits

We will examine recursive sequences more closely in 1.4, but for now, if we know a recursive sequence converges, then we can use rule (7) to find the limit!

EXAMPLE 1.2.33

$a_1 = 2$, $a_{n+1} = \frac{5+a_n}{2}$. Suppose we know it has a limit, say $\lim\limits_{n\to\infty} a_n = L$. Then, using rule (7), we get:

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{5+a_n}{2} = \frac{5+L}{2}.$$

Therefore,

$$L = \frac{5+L}{2} \iff 2L = 5+L \iff L = 5.$$

1.3 Squeeze Theorem

THEOREM 1.3.1: Squeeze Theorem

If $a_n \le b_n \le c_n$ and $\lim\limits_{n\to\infty} a_n = L = \lim\limits_{n\to\infty} c_n$, then $\lim\limits_{n\to\infty} b_n = L$.

Proof: Let $\varepsilon > 0$ be given. Since $a_n \to L$ and $c_n \to L$, we can find $N \in \mathbb{N}$ such that if $n \ge N$, then $a_n \in (L-\varepsilon, L+\varepsilon)$, and $c_n \in (L-\varepsilon, L+\varepsilon)$. Then, for $n \ge N$,

$$L - \varepsilon \le a_n \le b_n \le c_n \le L + \varepsilon,$$

so $b_n \in (L-\varepsilon, L+\varepsilon)$, which means $\lim\limits_{n\to\infty} b_n = L$.

REMARK 1.3.2

The Squeeze Theorem is great for dealing with $\sin / \cos$ and $(-1)^n$.

EXAMPLE 1.3.3

Compute the following limits.

(1) $\lim\limits_{n\to\infty} \frac{(-1)^n}{n^2+1}$.

(2) $\lim\limits_{n\to\infty} \frac{\cos(n^2+7)+7}{n}$.

Solution.

(1) Notice that $\frac{-1}{n^2+1} \leq \frac{(-1)^n}{n^2+1} \leq \frac{1}{n^2+1}$ and $\lim\limits_{n\to\infty} \frac{-1}{n^2+1} = 0 = \lim\limits_{n\to\infty} \frac{1}{n^2+1}$, so $\lim\limits_{n\to\infty} \frac{(-1)^n}{n^2+1} = 0$ by the Squeeze Theorem.

(2) Notice that $\frac{6}{n} \leq \frac{\cos(n^2+7)+7}{n} \leq \frac{8}{n}$, and since $\lim\limits_{n\to\infty} \frac{6}{n} = 0 = \lim\limits_{n\to\infty} \frac{8}{n}$, we get $\lim\limits_{n\to\infty} \frac{\cos(n^2+7)+7}{n} = 0$ by the Squeeze Theorem.

1.4 Monotone Convergence Theorem

First, we need to some terminology.

DEFINITION 1.4.1

Let $S \subseteq \mathbb{R}$. We say that α is an **upper bound** of S if $x \leq \alpha$ for all $x \in S$. We call such a set **bounded above**.
Similarly, β is a **lower bound** if $x \leq \beta$ for all $x \in S$. In this case, S is **bounded below**.
We call S **bounded** if it is bounded both above and below. In this case, we could find $M \in \mathbb{R}$ such that $S \subseteq [-M, M]$.

EXAMPLE 1.4.2

If $S = (-1, 1)$, then 7 is an upper bound and -12 is a lower bound, so S is bounded. Another example is $S \subseteq [-5, 5]$.

DEFINITION 1.4.3

Let $S \subseteq \mathbb{R}$. α is called the **least upper bound** of S if:

(i) α is an upper bound, and

(ii) α is the smallest, that is, if α' is another upper bound, then $\alpha' \geq \alpha$.

Denote this by $\alpha = \text{lub}(S)$ or $\alpha = \sup(S)$.
Similarly, β is the **greatest lower bound** if

(i) β is a lower bound, and

(ii) β is the largest, that is, if β' is another lower bound, then $\beta' \leq \beta$.

Denote this by $\beta = \text{glb}(S)$ or $\beta = \inf(S)$.

EXAMPLE 1.4.4

If $S = (-1, 1)$, then $\inf(S) = -1$ and $\sup(S) = 1$.

REMARK 1.4.5

The $\inf(S)$ and $\sup(S)$ may or may not be in S. One of the properties (axioms) of $\mathbb{R}$ guarantees the existence of $\inf$ and $\sup$. If $S \subseteq \mathbb{R}$ is non-empty and bounded above (below), then S has $\sup$ ($\inf$).

DEFINITION 1.4.6

We say that a sequence $\{a_n\}$ is:

- **increasing** if $a_n < a_{n+1}$,
- **non-decreasing** if $a_n \leq a_{n+1}$,
- **decreasing** if $a_n > a_{n+1}$,
- **non-increasing** if $a_n \geq a_{n+1}$,
- **monotonic** if $\{a_n\}$ is either non-decreasing or non-increasing.

Now, we can state the theorem!

THEOREM 1.4.7: Monotone Convergence Theorem (MCT)

Let $\{a_n\}$ be a non-decreasing (non-increasing) sequence.

(1) If $\{a_n\}$ is bounded above (below), then $\{a_n\}$ converges to $L = lub(\{a_n\})$ ($L = glb(\{a_n\})$).

(2) If $\{a_n\}$ is not bounded above (below), then $\{a_n\}$ diverges to ∞ ($-\infty$).

Proof: We will prove the non-decreasing/bounded above case, the other case is similar. Suppose $\{a_n\}$ is non-decreasing.

(1) Suppose $\{a_n\}$ is bounded above and let $L = \text{lub}(\{a_n\})$. Let $\varepsilon > 0$ be given. Then, $L - \varepsilon < L$, which means that $L - \varepsilon$ is not an upper bound of $\{a_n\}$ (L is the least upper bound). So, there exists $N \in \mathbb{N}$ so that $L - \varepsilon < a_N$. Then, if $n \geq N$, we have $L - \varepsilon < a_N \leq a_n$ since the sequence is non-decreasing. Therefore, for $n \geq N$, $L - \varepsilon < a_n \leq L < L + \varepsilon$, so the tail of $\{a_n\}$ is in $(L - \varepsilon, L + \varepsilon)$, which means $\lim_{n\to\infty} a_n = L$.

(2) Suppose $\{a_n\}$ is not bounded above. Let $M \in \mathbb{R}$ be given. We can find $N \in \mathbb{N}$ so that $M < a_N$. Then, if $n \geq N$, we have $M < a_N < a_n$ ($\{a_n\}$ is non-decreasing). This shows $\lim_{n\to\infty} a_n = \infty$.

Introduction to Mathematical Induction

Before we can use the MCT, we need to develop one proof technique: Mathematical Induction (MATH 135 will explore it further). Induction is a proof technique that allows us to prove an infinite number of statements. Say we have statements $P_1, P_2, P_3, \ldots, P_n, \ldots$ for $n \in \mathbb{N}$. If we can:

(1) Prove P_1 is true (base case).

(2) Prove: if P_k is true for some k (inductive hypothesis), then P_{k+1} is true (inductive step).

Then, we can conclude that P_n is true for all $n \in \mathbb{N}$. Think of dominoes!

We will use the MCT and induction to find the limits of recursive sequences. To do this, we follow these steps:

(1) Prove the sequence is monotonic.

(2) Prove the sequence is bounded (above or below).

(3) Conclude the sequence converges by MCT.

(4) Find the limit using an earlier trick:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}.$$

Note that the order matters! We can't perform step 4 unless we know that the sequence converges.

EXAMPLE 1.4.8

Let $a_1 = 1$, $a_{n+1} = \frac{3+a_n}{2}$ for $n \geq 1$. Prove the sequence converges and find its limit.

Solution.

(1) Let's check a few terms: $a_1 = 1$, $a_2 = 2$, $a_3 = 5/2$, so it looks like the sequence is non-decreasing. Claim: $a_n \leq a_{n+1}$ for all $n \in N$.
- Base Case: Is $a_1 \leq a_2$? Yes, since $a_1 = 1 \leq 2 = a_2$.
- Inductive Hypothesis: Suppose $a_k \leq a_{k+1}$ for some $k \geq 1$.
- Inductive Step: Since $a_k \leq a_{k+1}$, $3 + a_k \leq 3 + a_{k+1}$, which means
$$\frac{3+a_k}{2} \leq \frac{3+a_{k+1}}{2},$$
that is, $a_{k+1} \leq a_{k+2}$.

Therefore, the sequence is non-decreasing by induction.

(2) What upper bound should we use? Don't try to guess the lub at this point, any upper bound will do!
Claim: $a_n \leq 5$ for all $n \in \mathbb{N}$.
- Base Case: $a_1 = 1 \leq 5$.
- Inductive Hypothesis: Suppose $a_k \leq 5$ for some $k \in \mathbb{N}$.
- Inductive Step: Since $a_k \leq 5$, $3 + a_k \leq 8$, so $\frac{3+a_k}{2} \leq 4$. Therefore, $a_{k+1} \leq 4 \leq 5$.

Therefore, $a_n \leq 5$ for all $n \in \mathbb{N}$ by induction, so the sequence is bounded above.

(3) Since $\{a_n\}$ is bounded above and non-decreasing, we know $\{a_n\}$ converges by MCT.

(4) Now, we know a limit exists, say $L = \lim\limits_{n\to\infty} a_n$. Then,
$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{3+a_n}{2} = \frac{3+L}{2}.$$
So,
$$L = \frac{3+L}{2} \iff 2L + 3L \iff L = 3.$$
Therefore, $\lim\limits_{n\to\infty} a_n = 3$.

EXAMPLE 1.4.9

Let $a_1 = 2$, $a_{n+1} = 7 + a_n$ for $n \geq 1$. Prove the sequence converges and find its limit.

Solution. Let's check a few terms: $a_1 = 2$, $a_2 = 3$, $a_3 = \sqrt{10}$, so it looks like the sequence is non-decreasing. Let's prove bounded above and non-decreasing in one step!

(1) Claim: $a_n \leq a_{n+1} \leq 9$ for all $n \in \mathbb{N}$.
- Base Case: $a_1 = 2 \leq 3 = a_2$ and $a_2 = 3 \leq 9$, so $a_1 \leq a_2 \leq 9$.
- Inductive Hypothesis: Assume $a_k \leq a_{k+1} \leq 9$ for some $k \in \mathbb{N}$.
- Inductive Step: Then,
$$\begin{aligned} & a_k \leq a_{k+1} \leq 9 \\ \implies & 7 + a_k \leq \sqrt{7 + a_{k+1}} \leq 4 \leq 9 \\ \implies & a_{k+1} \leq a_{k+2} \leq 9. \end{aligned}$$

So, by induction $a_n \leq a_{n+1} \leq 9$ for all $n \in \mathbb{N}$.

(2) The sequence converges by the MCT.

(3) Finally, we need to find the limit. Say $L = \lim_{n\to\infty} a_n$. Then,

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{7 + a_n} = \sqrt{7 + L};$$

see A2Q6 for the last equality. So,

$$L = \sqrt{7 + L} \implies L^2 = 7 + L \implies L^2 - L - 7 = 0 \implies L = \frac{1 \pm \sqrt{29}}{2}.$$

However, we know $L = \text{lub}(\{a_n\})$ and $a_1 = 2$. So, $L \neq \frac{1-\sqrt{29}}{2}$ since $\frac{1-\sqrt{29}}{2} < 2$, that is, it isn't even an upper bound. Hence,

$$L = \frac{1 + \sqrt{29}}{2}.$$

Chapter 2

Limits and Continuity

2.1 Introduction to Function Limits

Let's examine $\lim_{x \to a} f(x) = L$ for $a, L \in \mathbb{R}$. Intuitively, this means that $f(x)$ gets infinitely close to L as x gets infinitely close to a (but $x \neq a$). Let's translate this into a more precise definition.

DEFINITION 2.1.1: Limit of Real Function

Let $A \subseteq \mathbb{R}$ and let $f\colon A \to \mathbb{R}$. When $a \in \mathbb{R}$ is a limit point of A, and $L \in \mathbb{R}$, we say that the **limit** of $f(x)$ as x tends to a is equal to L, and we write $\lim_{x \to a} f(x) = L$ or we write $f(x) \to L$ as $x \to a$, when

$$\forall \varepsilon \in \mathbb{R}_{>0} : \exists \delta \in \mathbb{R}_{>0} : \forall x \in A : 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$

REMARK 2.1.2

(1) The limit is not affected by what happens at $x = a$.
(2) For the limit to exist, the function needs to approach L from <u>both sides</u>.

EXAMPLE 2.1.3

(1) Prove $\lim_{x \to 2} 5x + 1 = 11$.
(2) Prove $\lim_{x \to 5} x^2 = 25$.

Solution.

(1) Let $\varepsilon > 0$. Choose $\delta = \varepsilon/5$. If $0 < |x - 2| < \delta$, then

$$|f(x) - L| = |(5x + 1) - 11| = |5x - 10| = 5|x - 2| < 5\delta = \frac{5\varepsilon}{5} = \varepsilon.$$

(2) Let $\varepsilon > 0$. Choose $\delta = \min(1, \varepsilon/11)$. If $0 < |x - 5| < \delta$, then since $|x - 5| < \delta \leq 1$, we have $4 < x < 6$, so that $|x + 5| \leq |6 + 5| = 11$.

$$|x^2 - 25| = |(x - 5)(x + 5)| = |x - 5||x + 5| \leq 11|x - 5| < 11\delta \leq \frac{11\varepsilon}{11} = \varepsilon.$$

As before, it is tricky to work with the formal definition. We will strive to establish some better techniques!

REMARK 2.1.4: Some Comments

(1) For $\lim_{x\to a} f(x)$ to exist, f must be defined in an open interval (α, β), containing a (except possibly at $x = a$).
(2) $f(a)$ does not affect $\lim_{x\to a} f(x)$.
(3) If $f(x) = g(x)$, except possibly at $x = a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.

2.2 Sequential Characterization of Limits

We define $\overline{\mathbb{R}} = R \cup \{-\infty, \infty\}$.

THEOREM 2.2.1: The Sequential Characterization of Limits of Functions

Let $A \subseteq \mathbb{R}$ be open, let $f\colon A \to \mathbb{R}$, let $L \in \overline{\mathbb{R}}$, and let $a \in A$ be a limit point of A.

$$\lim_{x\to a} f(x) = L$$

if and only if

for all real sequences $\{x_n\}$ in $A \setminus \{a\}$ with $x_n \to a$ we have $f(x_n) \to L$.

- ($\implies$) Suppose $\lim_{x\to a} f(x) = L$. Let $\varepsilon > 0$. Since $\lim_{x\to a} f(x) = L$, we can choose $\delta > 0$ so that $0 < |x - a| < \delta \implies |f(x) - b| < \varepsilon$. Since $x_n \to a$, we can choose $N \in \mathbb{N}$ so that $n \geq N \implies |x_n - a| < \delta$. Then for $n \geq N$, we have $|x_n - a| < \delta$ and we have $x_n \neq a$ (since $\{x_k\}$ is in the set $A \setminus \{a\}$) and hence $|f(x_n) - L| < \varepsilon$. This shows that $f(x_n) \to L$.
- ($\impliedby$) Tricky exercise to think about.

Since we know sequences can only have one limit, we immediately get the following theorem.

THEOREM 2.2.2: Uniqueness of Limits

Let $A \subseteq \mathbb{R}$, let a be a limit point, and let $f\colon A \to \mathbb{R}$. For $L, M \in \overline{\mathbb{R}}$, if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = M$, then $L = M$. Similar results hold for limits $x \to a^{\pm}$ and $x \to \pm\infty$.

The sequential characterization can help us prove a limit does not exist.

Strategy [Showing Limits Do Not Exist]

1. Find a sequence $\{x_n\}$ with $x_n \to a$, $x_n \neq a$ for which $\lim_{n\to\infty} f(x_n)$ does not exist.
2. Find two sequences (x_n) and (y_n) with $x_n \to a$, $y_n \to a$, $x_n, y_n \neq a$ for which $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n) = M$

EXAMPLE 2.2.3

Prove that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

Solution. Let $x_n = 1/n$ and $y_n = -1/n$. Clearly, $x_n \to 0$, $y_n \to 0$, and $x_n, y_n \neq 0$. Since $x_n > 0$ and

$y_n < 0$ for all n, we have $|x_n|/x_n = 1$ and $|y_n|/y_n = -1$ for all n. Therefore,

$$\lim_{n\to\infty} \frac{|x_n|}{x_n} = 1 \neq -1 = \lim_{n\to\infty} \frac{|y_n|}{y_n}.$$

Therefore, $\lim\limits_{x\to 0} \frac{|x|}{x}$ does not exist.

2.3 Arithmetic Rules for limits of functions

THEOREM 2.3.1: Combination Theorem for Limits of Functions

Let f, g be real functions defined on an open subset $A \subseteq \mathbb{R}$, except possibly at a point $a \in A$. Let $\lim\limits_{x\to a} f(x) = L \in \mathbb{R}$ and $\lim\limits_{x\to a} g(x) = M \in \mathbb{R}$.

(1) $\forall x \in \mathbb{R} : f(x) = c \implies L = c$.

(2) ***Multiple Rule.***

$$\lim_{x\to a} cf(x) = cL.$$

(3) ***Sum Rule.***

$$\lim_{x\to a} f(x) + g(x) = L + M.$$

(4) ***Product Rule.***

$$\lim_{x\to a} f(x)g(x) = LM.$$

(5) ***Quotient Rule.***

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M} \text{ provided that } M \neq 0.$$

(6) ***Power Rule.***

$$\forall \alpha > 0 : \lim_{x\to a} (f(x))^\alpha = L^\alpha.$$

(7) If $M = 0$, and $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ exists, then $L = 0$.

THEOREM 2.3.2: Limits of Polynomials

If $p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_n x^n$ is any polynomial, *then*

$$\lim_{x\to a} p(x) = p(a).$$

Proof: Exercise.

Limits of Rational Functions

Consider $\frac{P(x)}{Q(x)}$, where P, Q are polynomials.

- Case 1: If $Q(a) \neq 0$, then
$$\lim_{x\to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}.$$
- Case 2: If $\lim\limits_{x\to a} Q(x) = 0$ and $\lim\limits_{x\to a} P(x) \neq 0$, then
$$\lim_{x\to a} \frac{P(x)}{Q(x)}$$

does not exist.

- Case 3: If $\lim_{x \to a} Q(x) = Q(a) = 0 = P(a) = \lim_{x \to a} P(x)$, then $(x-a)$ is a factor of both $P(x)$ and $Q(x)$, so we can write $P(x) = (x-a)P^*(x)$ and $Q(x) = (x-a)Q^*(x)$. Therefore,
$$\lim_{x \to a} \frac{P(x)}{Q(x)} = \lim_{x \to a} \frac{(x-a)P^*(x)}{(x-a)Q^*(x)} = \lim_{x \to a} \frac{P^*(x)}{Q^*(x)}$$
and return to step 1!

EXAMPLE 2.3.3

(1) $\lim_{x \to 2} \frac{x^2-4}{x^2+3x+1} = \frac{0}{11} = 0.$

(2) $\lim_{x \to 2} \frac{x^2-4}{x^2-x-2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)(x+1)} = \lim_{x \to 2} \frac{x+2}{x+1} = \frac{4}{3}.$

2.4 One-Sided Limits

We may want to examine the behaviour of a function at a point but only from one side, instead of both sides at the same time. Let's see how to do that, and what the behaviour means for the overall limit.

DEFINITION 2.4.1: One-Sided Limits

Let $A = (a, b)$ be an open real interval, let $f\colon A \to \mathbb{R}$, and let $L \in \mathbb{R}$.

- *Limit from Right.*
$$\lim_{x \to a^+} f(x) = L \iff \forall \varepsilon \in \mathbb{R}_{>0} : \exists \delta \in \mathbb{R}_{>0} : \forall x \in A : a < x < a + \delta \implies |f(x) - L| < \varepsilon.$$
- *Limit from Left.*
$$\lim_{x \to b^-} f(x) = L \iff \forall \varepsilon \in \mathbb{R}_{>0} : \exists \delta \in \mathbb{R}_{>0} : \forall x \in A : b - \delta < x < b \implies |f(x) - L| < \varepsilon.$$

EXAMPLE 2.4.2

If $f(x) = \frac{|x|}{x} = \begin{cases} 1, & x > 0, \\ -1, & x < 0, \end{cases}$ then $\lim_{x \to 0^+} f(x) = 1$ and $\lim_{x \to 0^-} f(x) = -1$

EXAMPLE 2.4.3

If $f(x) = \begin{cases} -100, & x \le 0, \\ 0, & 0 < x \le 1, \\ x^2+1, & x > 1, \end{cases}$ then $\lim_{x \to 1^+} f(x) = 1^2 + 1 = 2$, $\lim_{x \to 1^-} f(x) = 0$, $\lim_{x \to 0^+} f(x) = 0$ and $\lim_{x \to 0^-} f(x) = -100$.

THEOREM 2.4.4: One-sided versus Two-sided Limits

Let A be a function defined on an open real interval, let $f\colon A \to \mathbb{R}$, and let $a \in A$.
$$\lim_{x \to a} f(x) \text{ exists and equals } L$$
if and only if both one-sided limits exist and
$$\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x).$$

REMARK 2.4.5

All arithmetic rules and sequential characterization hold for one-sided limits as well.

2.5 The Squeeze Theorem

There is an analogue of the Squeeze Theorem for Sequences for functions!

THEOREM 2.5.1: Squeeze Theorem

Let a be a point on an open real interval A, and let $f, g, h\colon A \to R$. If

$$\forall x \neq a \in A : g(x) \leq f(x) \leq h(x)$$
$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} f(x) = L.$$

EXAMPLE 2.5.2

Find the following limits.

(1) $\lim_{x\to 0} x^2 \cos(e^x + 7)$.

(2) $\lim_{x\to 0} \frac{\sin x}{x}$.

Solution.

(1) We know that $-1 \leq \cos(e^x + 7) \leq 1$, so $-x^2 \leq \cos(e^x + 7) \leq x^2$. Also, $\lim_{x\to 0} -x^2 = 0 = \lim_{x\to 0} x^2$, so by the Squeeze Theorem, $\lim_{x\to 0} x^2 \cos(e^x + 7) = 0$

(2) If $0 < x < \pi/2$, then $\sin x \leq x \leq \tan x$, so $|\sin x| \leq |x| \leq |\tan x|$ if $-\pi/2 < x < \pi/2$. So,

$$1 \leq \frac{|x|}{|\sin x|} \leq \frac{|\tan x|}{|\sin x|} = \frac{1}{|\cos x|} \text{ if } -\frac{\pi}{2} < x < \frac{\pi}{2}, x \neq 0.$$

Therefore,

$$1 \geq \left|\frac{\sin x}{x}\right| \geq |\cos x|,$$

but $\dfrac{\sin x}{x} \geq 0$ and $\cos x > 0$ on $(-\pi/2, \pi/2)$, so

$$1 \geq \frac{\sin x}{x} \geq \cos x.$$

Also, $\lim_{x\to 0} 1 = 1 = \lim_{x\to 0} \cos x$, so by the Squeeze Theorem, $\lim_{x\to 0} \frac{\sin x}{x} = 0$.

2.6 The Fundamental Trigonometric Limit

2.7 Limits at infinity and Asymptotes

2.7.1 Asymptotes and Limits at Infinity

2.7.2 The Fundamental Log Limit

2.7.3 Vertical Asymptotes and Infinite Limits

2.8 Continuity

2.8.1 Types of Discontinuities

2.8.2 Continuity of Certain Functions

2.8.3 Arithmetic Rules for Continuity

2.8.4 Continuity On An Interval

2.9 The Intermediate Value Theorem

2.9.1 Approximating Solutions to Equations

2.9.2 The Bisection Method

2.10 The Extreme Value Theorem

Chapter 3

Derivatives

3.1 Instantaneous Velocity

Suppose you are driving down a highway. Every 30 minutes you record your distance:

Time (min)	0	30	60	90	120	150	180
Distance (km)	0	55	100	130	200	250	300

- What was your average speed in these three hours?

$$\text{Average speed} = \frac{\text{distance}}{\text{time}} = \frac{300 \text{ km}}{1.5 \text{ h}} = 100 \text{ km/h}.$$

- First 1.5 hours?

$$\frac{130}{1.5} \approx 86.6 \text{ km/h}.$$

- Last 1.5 hours?

$$\frac{300 - 130}{1.5} \approx 113 \text{ km/h}.$$

In general, the formula for the **average velocity**, V_{ave} from $t = t_0$ to $t = t_1$ is

$$V_{\text{ave}} = \frac{s(t_1) - s(t_0)}{t_1 - t_0},$$

where $s(t)$ is the distancea at time t. To get the instantaneous velocity, we need to use limits! The instantaneous velocity at $t = t_0$ is

$$\lim_{t \to t_0} \frac{s(t) - s(t_0)}{t - t_0}$$

or

$$\lim_{h \to 0} \frac{s(t_0 + h) - s(t_0)}{h}.$$

EXAMPLE 3.1.1

Find the instantaneous velocity for $s(t) = t^2 + 3t$ at $t = 1$, $t = 2$, and $t_0 \in \mathbb{R}$.

Solution.

- $$\begin{aligned}\lim_{h\to 0}\frac{s(1+h)-s(1)}{h} &= \lim_{h\to 0}\frac{(1+h)^2+3(1+h)-(1^2+3(1))}{h}\\ &= \lim_{h\to 0}\frac{5h+h^2}{h}\\ &= \lim_{h\to 0}(5+h)\\ &= 5.\end{aligned}$$
- $$\begin{aligned}\lim_{h\to 0}\frac{(2+h)^2+3(2+h)-(2^3+3(2))}{h} &= \lim_{h\to 0}\frac{7h+h^2}{h}\\ &= 7.\end{aligned}$$
- $$\begin{aligned}\lim_{h\to 0}\frac{(t_0+h)^2+3(t_0+h)-(t_0^2+3t_0)}{h} &= \lim_{h\to 0}(2t_0+3+h)\\ &= 2t_0+3.\end{aligned}$$

The instantaneous velocity is a special case of a derivative!

3.2 Definition of the Derivative

We can perform the same analysis that we did on $s(t)$ in the previous section on any function!

DEFINITION 3.2.1

The **average rate of change of** $f(x)$ from $x = a$ to $x = b$ is

$$f_{\text{ave}} = \frac{f(b)-f(a)}{b-a}.$$

DEFINITION 3.2.2

The **instantaneous rate of change of** $f(x)$ at $x = a$, or the derivative of $f(x)$ at $x = a$, denoted $f'(a)$ is defined as

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}.$$

If this limit exists, we say that f is **differentiable** at $x = a$.

3.2.1 The Tangent Line

DEFINITION 3.2.3

The **tangent line** to the graph of f at $x = a$ is the line passing through $(a, f(a))$ with slope $m = f'(a)$. It follows that the equation of the tangent line is

$$y = f(a) + f'(a)(x-a).$$

EXAMPLE 3.2.4

Find the equation of the tangent line to $f(x) = x^2 + x + 1$ at $x = 3$.

Solution. First, we should compute $f'(3)$:

$$\begin{aligned} f'(3) &= \lim_{h\to 0}\frac{f(3+h)-f(3)}{h} \\ &= \lim_{h\to 0}\frac{(3+h)^2+(3+h)+1-(3^2+3+1)}{h} \\ &= \lim_{h\to 0}\frac{9+6h+h^2+3+h+1-9-3-1}{h} \\ &= \lim_{h\to 0}\frac{7h+h^2}{h} \\ &= \lim_{h\to 0}(7+h) \\ &= 7. \end{aligned}$$

So, $f'(3) = 7$. The point on the graph is $(3, f(3)) = (3, 13)$. So, the tangent line is

$$y = 13 + 7(x-3) = 13 + 7x - 21 = 7x - 8.$$

REMARK 3.2.5

Can't define the derivative as the slope of the tangent line! Without knowing what the derivative is first, we can't even define the tangent line!

3.2.2 Differentiability versus Continuity

- Q: Does continuity imply differentiability?
- A: No! Consider $f(x) = |x|$ at $x = 0$. Clearly,

  $$\lim_{x\to 0}|x| = 0 = |0|,$$

  so f is continuous at $x = 0$, but

  $$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{|h|}{h}$$

  does not exist. Therefore, f is not differentiable at $x = 0$. Therefore, continuity <u>does not</u> imply differentiability.
- Q: Does differentiability imply continuity?
- A: Yes!

THEOREM 3.2.6: Differentiability Implies Continuity

Let $A \subseteq \mathbb{R}$ be open, let $f\colon A \to \mathbb{R}$ and let $a \in A$. If f is differentiable at a, then f is continuous at a.

Proof: We have

$$f(x) - f(a) = \frac{f(x)-f(a)}{x-a}(x-a) \to f'(a)\cdot 0 = 0 \text{ as } x \to a$$

and so

$$f(x) = \big(f(x)-f(a)\big) + f(a) \to 0 + f(a) = f(a) \text{ as } x \to a.$$

This proves that f is continuous at a.

3.3 The Derivative Function

DEFINITION 3.3.1: The Derivative Function

We say that f is **differentiable** on an interval I if $f'(a)$ exists for each $a \in I$. In this case, we define the **derivative function** as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \ x \in I.$$

Alternative (Leibniz) notation:

$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(f),$$

where "$\frac{\mathrm{d}}{\mathrm{d}x}$" is called a **differential operator**.

If $y = f(x)$, write $\frac{\mathrm{d}y}{\mathrm{d}x}$. For $f'(a)$, write $\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=a}$.

Let's look at some examples!

EXAMPLE 3.3.2

For $f(x) = 7$, find $f'(x)$ for $x \in \mathbb{R}$.

Solution.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{7-7}{h} = 0.$$

Therefore, $f'(x) = 0$ for all $x \in \mathbb{R}$.

EXAMPLE 3.3.3

Find the equation of the tangent line to $f(x) = x^2 + 3x + 2$ at $x = 2$.

Solution. The tangent line passes through $(a, f(a)) = (2, f(2)) = (2, 12)$ since $f(2) = 2^2 + 3(2) + 2 = 12$. Next,

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^2 + 3(x+h) + 2 - x^2 - 3x - 2}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 3x + 3h - x^2 - 3x}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2 + 3h}{h} \\
&= \lim_{h \to 0} (2x + h + 3) \\
&= 2x + 3,
\end{aligned}$$

which gives $f'(2) = 2(2) = 3 = 7$. Therefore, the tangent line to f at $x = 2$ is

$$y = f(2) + f'(2)(x-2) = 12 + 7(x-2) = 12 + 7x - 14 = 7x - 2.$$

REMARK 3.3.4

- Much faster than computing $f'(a)$ each time!
- We will soon learn ways to find $f'(x)$ <u>much faster</u>, but if asked to use the <u>definition</u>, then you

must use the formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

EXAMPLE 3.3.5

Using the definition, find $f'(x)$ where

(1) $f(x) = x$;
(2) $f(x) = x^2$;
(3) $f(x) = x^3$;
(4) $f(x) = \sqrt{x}$.

Solution.

(1)
$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{x + h - x}{h} \\ &= \lim_{h \to 0} \frac{h}{h} \\ &= 1. \end{aligned}$$

(2)
$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \to 0} (2x + h) \\ &= 2x. \end{aligned}$$

(3)
$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h \to 0} (3x^2 + 3xh + h^2) \\ &= 3x^2. \end{aligned}$$

$$\begin{aligned}
\text{(4)}\quad f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\\
&= \lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}\\
&= \lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})}\\
&= \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}\\
&= \frac{1}{2\sqrt{x}}.
\end{aligned}$$

Higher-Order Derivatives

DEFINITION 3.3.6

If f is differentiable with derivative f' and f' is also differentiable, then we call $\frac{\mathrm{d}}{\mathrm{d}x}(f')$ the **second derivative** of f, denoted $f''(x)$ or $f^{(2)}(x)$, or $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$.
In general, $f^{(n+1)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}(f^{(n)}(x))$, where $f^{(n)}$ is the n^{th} derivative.

EXERCISE 3.3.7

Prove the following with the limit definition, where $f(x) = x^4$.

- $f'(x) = 4x^3$.
- $f''(x) = 12x^2$.
- $f'''(x) = 24x$.
- $f^{(4)} = 24$.
- $f^{(5)} = 0$.

Note that using the limit definition is very inefficient (not to mention awful and ugly). So, let's develop some rules to help us calculate derivatives more quickly!

3.4 Derivatives of Elementary Functions

Now that we know the definition of the derivative, let's work on finding derivatives of elementary functions to speed up the process.

- **Constants**: If $f(x) = c$ where $c \in \mathbb{R}$, then $f'(x) = 0$.
- **Lines**: If $f(x) = mx + b$ where $m, b \in \mathbb{R}$, then $f'(x) = m$.
- **Quadratics**: If $f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$ and $a \neq 0$, then $f'(x) = 2ax + b$.

3.4.1 The Derivative of $\sin x$ and $\cos x$

First, we need to prove a different claim:

$$\lim_{x\to 0}\frac{\cos x - 1}{x} = 0.$$

$$\begin{aligned}\lim_{x\to 0}\frac{\cos x-1}{x}\cdot\frac{\cos x+1}{\cos x+1} &= \lim_{x\to 0}\frac{\cos^2 x-1}{x(\cos x+1)}\\ &= \lim_{x\to 0}\frac{-\sin^2 x}{x(\cos(x+1))}\\ &= \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{-\sin x}{\cos x+1}\\ &= 1\cdot 0\\ &= 0,\end{aligned}$$

using the fundamental trigonometry limit. Now, we can compute $(\sin x)'$.

$$\begin{aligned}(\sin x)' &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\ &= \lim_{h\to 0}\frac{\sin x\cos(h)+\cos x\sin(h)-\sin x}{h}\\ &= \lim_{h\to 0}\frac{\sin(h)}{h}\cos x+\left(\frac{\cos(h)-1}{h}\right)\sin x\\ &= 1\cdot\cos x+0\cdot\sin x\\ &= \cos x.\end{aligned}$$

EXERCISE 3.4.1

Show that $(\cos x)' = -\sin x$.

3.4.2 The Derivative of e^x

First, what is the number e? There are lots of ways to define it, for example: $\lim\limits_{x\to\infty}(1+\frac{1}{x})^x = e$ or $\sum_{n=0}^{\infty}\frac{1}{n!} = e$. But for us, we will define e to be the unique number $a \in \mathbb{R}$ such that the tangent line to a^x has slope 1 at $x = 0$. That is,

$$\lim_{h\to 0}\frac{e^h-e^0}{h} = 1 \implies \lim_{h\to 0}\frac{e^h-1}{h} = 1.$$

So, we get $(e^x)' = \lim\limits_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim\limits_{h\to 0} e^x(\frac{e^h-1}{h}) = e^x$. So, $(e^x)' = e^x$.

3.7 Arithmetic Rules for Differentiation

Now that we know how to find the derivatives of certain basic functions, let us look at some rules that tell us how to differentiate combinations of functions.

THEOREM 3.7.1: Arithmetic Rules for Differentiation

Suppose f and g are differentiable at $x = a$.

*(1) **Constant Multiple Rule**. Let $h(x) = cf(x)$. Then h is differentiable at $x = a$ and*

$$h'(a) = cf'(a).$$

*(2) **Sum Rule**. Let $h(x) = f(x) + g(x)$. Then h is differentiable at $x = a$ and*

$$h'(a) = f'(a) + g'(a).$$

*(3) **Product Rule**. Let $h(x) = f(x)g(x)$. Then h is differentiable at $x = a$ and*

$$h'(a) = f'(a)g(a) + f(a)g'(a).$$

(4) ***Reciprocal Rule***. *Let* $h(x) = \frac{1}{g(x)}$. *If* $g(a) \neq 0$, *then* h *is differentiable at* $x = a$ *and*

$$h'(a) = -\frac{g'(a)}{[g(a)]^2}.$$

(5) ***Quotient Rule***: *Let* $h(x) = \frac{f(x)}{g(x)}$. *If* $g(a) \neq 0$, *then* h *is differentiable at* $x = a$ *and*

$$h'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}.$$

Proof:

(1) Easy exercise.

(2) Easy exercise.

(3)
$$\begin{aligned}(fg)'(a) &= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h} \\ &= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a+h)g(a) + f(a+h)g(a) - f(a)g(a)}{h} \\ &= \lim_{h\to 0} f(a+h)\frac{g(a+h) - g(a)}{h} + g(a)\frac{f(a+h) - f(a)}{h} \\ &= f(a)g'(a) + g(a)f'(a).\end{aligned}$$

(4)
$$\begin{aligned}\left(\frac{1}{f}\right)'(a) &= \lim_{h\to 0} \frac{\frac{1}{f(a+h)} - \frac{1}{f(a)}}{h} \\ &= \lim_{h\to 0} \frac{f(a) - f(a+h)}{hf(a+h)f(a)} \\ &= \lim_{h\to 0} \frac{-(f(a+h) - f(a))}{h} \frac{1}{f(a+h)f(a)} \\ &= \frac{-f'(a)}{[f(a)]^2}.\end{aligned}$$

(5) We can combine the product and reciprocal rules!
$$\begin{aligned}\left(\frac{f}{g}\right)'(a) &= \left(f\frac{1}{g}\right)'(a) \\ &= f'(a)\frac{1}{g(a)} + f(a)\left(\frac{1}{g}\right)'(a) \\ &= \frac{f'(a)}{g(a)} - \frac{f(a)g'(a)}{[g(a)]^2} \\ &= \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}.\end{aligned}$$

THEOREM 3.7.2: The Power Rule for Differentiation

Assume that $\alpha \in \mathbb{R}$, $\alpha \neq 0$, *and* $f(x) = x^\alpha$. *Then* f *is differentiable and*

$$f'(a) = \alpha x^{\alpha-1},$$

where $x^{\alpha-1}$ *is defined.*

In general, the proof is difficult. If $\alpha \in \mathbb{N}$, then it is a simple application of the Binomial Theorem. For $\alpha \in \mathbf{Q}$, it is possible with more tools (chain rule and inverse function theorem). But for general $\alpha \in \mathbb{R}$, we would need more tools, and it outside the scope of this course. So, we omit the proof. Let's look at some examples!

EXAMPLE 3.7.3

(1) $f(x) = x^2 \sin x$.

$$f'(x) = (x^2)' \sin x + x^2 (\sin x)' = 2x \sin x + x^2 \cos x.$$

(2) $f(x) = \frac{x^4-1}{x-7}$.

$$f'(x) = \frac{(x-7)(x^4+1)' - (x^4+1)(x-7)'}{(x-7)^2} = \frac{(x-7)(4x^3) - (x^4+1)(1)}{(x-7)^2}.$$

(3) $f(x) = \sec x = \frac{1}{\cos x}$.

$$f'(x) = \frac{-(\cos x)'}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x}\frac{1}{\cos x} = \tan x \sec x.$$

(4) $f(x) = e^x \cos x$.

$$f'(x) = e^x \cos x - e^x \sin x.$$

(5) $f(x) = 3x^5$.

$$f'(x) = 15x^4,\ f''(x) = 60x^3,\ f^{(3)}(x) = 180x^2,\ f^{(4)}(x) = 360x,\ f^{(5)}(x) = 360,\ f^{(\geq 6)}(x) = 0.$$

3.8 The Chain Rule

THEOREM 3.8.1: Chain Rule

Let $A, B \subseteq \mathbb{R}$ be open, let $f\colon A \to \mathbb{R}$, let $g\colon B \to \mathbb{R}$, and let $h = g \circ f\colon C \to \mathbb{R}$, where $C = A \cap f^{-1}(B)$. Let $a \in C$ and let $b = f(a) \in B$. Suppose that f is differentiable at a and g is differentiable at b. Then h is differentiable at a with

$$h'(a) = g'(f(a))f'(a).$$

In Leibniz notation, if $z = g(y)$ and $y = f(x)$, then

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

The proof is quite involved, for a geometric argument see the course notes.

COROLLARY 3.8.2: Generalized Power Rule

If $g(x) = f(x)^\alpha$ for $\alpha \in \mathbb{R} \setminus \{0\}$, then

$$g'(x) = \alpha f(x)^{\alpha-1} f'(x).$$

EXAMPLE 3.8.3

Find $f'(x)$.

(1) $f(x) = (3x^2 + 2x + 7)^{19}$.
(2) $f(x) = \sin(e^x + x^e)$.
(3) $f(x) = e^{\sin(x^2)}$.

Solution.

(1) $f'(x) = 38(3x+1)(3x^2+2x+7)^{18}$.
(2) $f'(x) = \cos(e^x + x^e)(e^x + exe^{e-1})$.

(3) $f'(x) = e^{\sin(x^2)}(\sin(x^2))' = e^{\sin(x^2)}\cos(x^2)(x^2)' = e^{\sin(x^2)}\cos(x^2)(2x)$.

Also, with the chain rule and the derivative of e^x, we can get the derivative of a^x for $a > 0$.

$$a^x = e^{x\ln(a)} \implies (a^x)' = (e^{x\ln(a)})' = e^{x\ln(a)}(x\ln(a))' = a^x\ln(a).$$

EXAMPLE 3.8.4

$f(x) = 2^{3x} + 5^{\cos x}$. $f'(x) = 2^{3x}\ln(2)(3) + 5^{\cos x}\ln(5)(-\sin x)$.

3.9 Derivatives of Other Trigonometric Functions

So far, we've seen:

$$(\sin x)' = \cos x$$
$$(\cos x)' = -\sin x$$
$$(\sec x)' = \sec x\tan x.$$

EXAMPLE 3.9.1

$$\begin{aligned}(\tan x)' &= \left(\frac{\sin x}{\cos x}\right)' &&= \frac{\cos x(\sin x)' - \sin x(\cos x)'}{\cos^2 x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x}\\ &= \sec^2 x.\end{aligned}$$

EXERCISE 3.9.2

Prove that $(\cot x)' = -\csc^2 x$ and $(\csc x)' = -\csc x\cot x$.

Recap:

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x\tan x$
$\csc x$	$-\csc x\cot x$

3.5 Tangent Lines and Linear Approximation

The main idea of this section is: general functions are hard to understand, while lines are easy to understand. So, let's develop a way to approximate a function with a line!

More precisely, for a differentiable function f, we want to find a linear function $h(x)$ so that $f(a) = h(a)$, $f'(a) = h'(a)$, and if x is close to a, then $f(x)$ is close to $h(x)$. How do we find $h(x)$? Well, if f is differentiable, then

$$\lim_{x\to a}\frac{f(x) - f(a)}{x - a} = f'(a).$$

So, if x is close to a, then
$$\frac{f(x)-f(a)}{x-a} \approx f'(a).$$
Solving for $f(x)$, we get
$$f(x) \approx f(a) + f'(a)(x-a).$$
Hence, let's define
$$l(x) = f(a) + f'(a)(x-a).$$
This is a good choice for $h(x)$. Note that $l(x)$ is the tangent line to $f(x)$ at $(a, f(a))$, which leads us to the following definition.

DEFINITION 3.5.1: Linearization, Tangent Line

When $f\colon A \to \mathbb{R}$ is differentiable at $x = a$ with derivative $f'(a)$, the function
$$l(x) = f(a) + f'(a)(x-a)$$
is called the **linearization (linear approximation)** of f at $x = a$. Note that the graph $y = l(x)$ of the linearization is the line through the point $(a, f(a))$ with slope $f'(a)$. This line is called the **tangent line** to the graph $y = f(x)$ at the point $(a, f(a))$.

EXAMPLE 3.5.2

For $f(x) = \sqrt{x}$, find the linearization at $x = 4$. Use this to approximate $\sqrt{3.98}$ and $\sqrt{4.05}$.

Solution. $f(4) = 2$, $f'(x) = \frac{1}{2\sqrt{x}}$, so $f'(4) = \frac{1}{4}$. Hence, the linearization of f at $x = 4$ is
$$l(x) = 2 + \frac{1}{4}(x-4) = \frac{x}{4} + 1.$$
Then, $\sqrt{3.98} \approx l(3.98) = 1 + 3.98/4 = 1.995$ and $\sqrt{4.05} \approx l(4.05) = 1 + 4.05/4 = 2.0125$. These values are fairly close to the "exact" values: $\sqrt{3.98} = 1.994993734326\ldots$ and $\sqrt{4.05} = 2.0124611797498106\ldots$.

Q: From the graph of $f(x)$, how can we tell if these are over- or under-estimates? They are overestimates since the line is above the graph (TODO image).

REMARK 3.5.3

Note that this is only a good approximation nearby $x = 4$. If we try to approximate $\sqrt{9}$, we get $\sqrt{9} = l(9) = 1 + 9/4 = 3.25$. The exact value is $\sqrt{9} = 3$ (obviously).

3.5.1 Error in Linear Approximation

Without an upper bound on the error, an approximation is useless! Note that
$$|\text{error}| = |f(x) - l(x)|,$$
i.e., the distance from $f(x)$ to $l(x)$.

- Q: What factors affect the size of the error?
- A: First, the farther we go from $x = a$, the larger the error gets! Also, how curved the graph is also affects it. Of course, if we don't fully understand $f(x)$, we can't calculate the error exactly, but we can approximate it! How do we quantify "more curved?" Well, we can say the slopes of the tangent lines are changing faster on the more curved graph.

  Hence, the rate of change of $f'(x)$ is measured by $f''(x)$, so $|f''(x)|$ being larger means a larger error.

THEOREM 3.5.4: The Error in Linear Approximation

Assume f is such that $|f''(x)| \leq M$ for each x in an interval I containing a point a. Then

$$\left|f(x) - l(x)\right| \leq \frac{M}{2}(x-a)^2$$

for each $x \in I$.

This is a special case of Taylor's Inequality which we will discuss later.

EXAMPLE 3.5.5

Find an upper bound on the error using $l(x)$ at $x = 4$ to approximate $\sqrt{x}$ on $[1, 6]$.

Solution. We know that $f'(x) = \frac{1}{2\sqrt{x}}$, so $f''(x) = -\frac{1}{4x^{3/2}}$. So, if $x \in [1, 6]$, we have

$$|f''(x)| = \left|-\frac{1}{4x^{3/2}}\right| \leq \frac{1}{4} = M.$$

Hence,

$$|\text{error}| = |l(x) - f(x)| \leq \frac{M}{2}(x-4)^2 \leq \frac{1}{8}(1-4)^2 = \frac{9}{8},$$

where we note that the maximum of $|x - 4|$ is 3, so we let $x = 1$ in the final inequality.

3.5.2 Applications of Linear Approximation

We will explore one application: estimating change. (Qualitative analysis is another that we will discuss later).

Suppose we are looking at $f(x)$ near $x = a$. We want to know how much it could change if we move to a point x_1 near $x = a$. That is, we want to know $\Delta y = f(x_1) - f(a)$ if we change the input by $\Delta x = x_1 - a$. Then, using $f(x) \approx l(x)$, we get

$$\Delta y = f(x_1) - f(a) \approx l(x_1) - f(a) = f'(a)(x_1 - a) = f'(a)\Delta x.$$

So, $\Delta y = f'(a)\Delta x$.

EXAMPLE 3.5.6

Suppose you are inflating a giant spherical balloon and it currently has a radius of 20cm. You exhale once and it goes up to 20.01m. Then, the change in volume would be

$$\Delta V = V'(20)\Delta r,$$

where $V(r) = \frac{4}{3}\pi r^3$. So, $V'(r) = 4\pi r^2$ and $V'(20) = 1600\pi$. Therefore,

$$\Delta V = 1600\pi(0.01) = 16\pi,$$

so the volume would increase by approximately 16πm^3.

REMARK 3.5.7

For a qualitative analysis, we will explore it more when we discuss Taylor polynomials.

3.6 Newton's Method

We have a method for finding zeros of a function already: The Bisection Algorithm. Another way is using Newton's Method, which converges much faster but has its own issues as we will see!

Idea: to solve $f(x) = 0$, start with an initial guess, call it x_1. To get the next x-value, find the intersection of the tangent line $l(x)$ at $x = x_1$ and the x-axis. The numbers $x_1, x_2, x_3, \ldots$ converge to a root (hopefully)! Let's find a formula for $x_2, x_3, \ldots$.

Given x_1, the tangent line is

$$l(x) = f(x_1) + f'(x_1)(x - x_1).$$

Find the intersection with the x-axis:

$$0 = f(x_1) + f'(x_1)(x - x_1) \implies x = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Repeating this, we get the **Newton's Iterative Procedure**:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

EXAMPLE 3.6.1

Find the positive root of $3x^4 + 15x^3 - 125x - 1500 = 0$ with error at most 10^{-5}. Use $x_1 = 4$.

Solution. We can check that $f(4) < 0$ and $f(5) > 0$, so there is a root between $x = 4$ and $x = 5$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n^4 + 15x_n^3 - 125x_n - 1500}{12x_n^3 + 45x_n^2 - 125}.$$

$$\begin{aligned} x_1 &= 4, \\ x_2 &= 4 - \frac{3(4)^4 + 15(4)^3 - 125(4) - 1500}{12(4)^3 + 45(4)^2 - 125} \approx 4.19956 \\ x_3 &\approx 4.187268 \\ x_4 &\approx 4.1872187 \\ x_5 &\approx 4.1872187 \end{aligned}$$

To 5 decimal places, this is 4.18722. Check that $f(4.187621) < 0$ and $f(4.18723) > 0$, so IVT says there is a root between!

Some Problems with Newton's Method

This method only works on differentiable functions, but more importantly it only works if x_1 is chosen "close enough" to a root! What is "close enough?" It depends! Sometimes any x_1 works, sometimes most don't.

EXAMPLE 3.6.2

Consider $f(x) = x^3 - 3x + 1$, pick $x_1 = 1$. Then

$$x_2 = x_1 - \frac{x_1^3 - 3x_1 + 1}{3x_1^2 - 3} = 1 - \frac{1 - 3 + 1}{0}?$$

Actually, at $x = 1$, f has a horizontal tangent that never intersects the x-axis, so we can't find x_2. Also, if we pick $x_1 = 2$, we will find a different root than if we pick $x_1 = -2$. So, pick a good starting point! A bad choice could make Newton's method diverge.

3.10 Derivatives of Inverse Functions

Suppose we want to find the derivative of an inverse function, how could we proceed? Let's start with the tangent line to $f(x)$ at $x = a$ and assume f is invertible.

$$l(x) = f(a) + f'(a)(x - a).$$

What would the tangent line to $f^{-1}(x)$ be at $x = f(a)$? $(l)^{-1}(x)$.

EXERCISE 3.10.1

If $L_a^f(x) = f(a) + f'(a)(x - a)$, show that

$$(L_a^f(x))^{-1} = a + \frac{1}{f'(a)}(x - f(a)).$$

So, if $f(a) = b$, then $a = f^{-1}(b)$, and the tangent line to $f^{-1}(x)$ at $x = b$ is

$$L_b^{f^{-1}}(x) = f^{-1}(b) + \frac{1}{f'(a)}(x - b) = f^{-1}(b) + \frac{1}{f'(f^{-1}(b))}(x - b).$$

But

$$L_b^{f^{-1}}(x) = f^{-1}(b) + (f^{-1})'(b)(x - b) \implies (f^{-1})' = \frac{1}{f'(f^{-1}(b))}.$$

This leads us to the following theorem.

THEOREM 3.10.2: Inverse Function Theorem (IFT)

Let I be an interval in $\mathbb{R}$, let $f\colon I \to \mathbb{R}$, and let a be a point in I which is not an endpoint. If f is bijective and continuous, and f is differentiable at a with $f'(a) \neq 0$, then its inverse f^{-1} is differentiable at $b = f(a)$ with

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}.$$

Moreover, L_a^f is invertible and $(L_a^f)^{-1} = L_{f(a)}^{f^{-1}} = L_b^{f^{-1}}$.

EXAMPLE 3.10.3

Let $f(x) = x^3$ so that $f^{-1}(x) = x^{1/3}$. Find $(f^{-1})'(3)$.

Solution 1. Direct computation yields

$$(f^{-1})'(x) = \frac{1}{x}x^{-2/3} \implies (f^{-1})'(3) = \frac{1}{3}3^{2/3} = \frac{1}{3(3^{2/3})}.$$

Solution 2. Use the IFT:

$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))}.$$

Note that $f'(x) = 3x^2$ and $f^{-1}(3) = 3^{1/3}$, so

$$(f^{-1})' = \frac{1}{f'(f^{-3}(3))} = \frac{1}{3(3^{1/3})^2} = \frac{1}{3(3^{2/3})}.$$

This example is somewhat silly since we could compute $(f^{-1})'$ directly. An important application of the IFT is that it allows us to find derivatives of inverse functions if we don't know them already!

EXAMPLE 3.10.4

Find $(\ln x)'$.

Solution. Let $f(x) = e^x$, so that $f^{-1}(x) = \ln x$ for $x > 0$. So,

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

Therefore,

$$(\ln x)' = \frac{1}{x}.$$

REMARK 3.10.5

We can prove IFT by using the chain rule: Suppose f and f^{-1} are differentiable, we get

$$f(f^{-1}(x)) = x.$$

Differentiate both sides with chain rule:

$$f'(f^{-1}(x))(f^{-1})'(x) = 1 \implies (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

3.11 Derivatives of Inverse Trigonometric Functions

Let's use the IFT (or just the chain rule) to find $(\arcsin x)'$. We know $\sin(\arcsin x) = x$ for $x \in [-1, 1]$. Differentiating, we get

$$(\cos(\arcsin x))(\arcsin x)' = 1 \implies (\arcsin x)' = \frac{1}{\cos(\arcsin x)}.$$

Can we simplify $\cos(\arcsin x)$? Yes! Let $\theta = \arcsin x$, then $\sin\theta = x$. Visualizing a triangle, we get the hypotenuse as 1, height as x so that the base $\sqrt{1-x^2}$. Hence, $\cos\theta = \sqrt{1-x^2}$. Therefore,

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}.$$

- Q: Wait a minute, how do we know $\arcsin x$ is differentiable?
- A: IFT says so! Since $\sin x$ is differentiable for $x \in (-1, 1)$, $\arcsin x$ is too.

EXERCISE 3.11.1

Prove that

- $(\arccos x)' = \dfrac{-1}{\sqrt{1-x^2}}$, and
- $(\arctan x) = \dfrac{1}{1+x^2}$.

EXAMPLE 3.11.2

Find $f'(x)$, where

1. $f(x) = \arctan(e^{\sin x})$,
2. $f(x) = \arcsin x + \arccos x$, and
3. $f(x) = \ln(\arctan x)$.

Solution.

1. $f'(x) = \dfrac{1}{1+(e^{\sin x})^2}(e^{\sin x})'$
$$= \frac{1}{1+e^{2\sin x}} e^{\sin x}(\sin x)'$$
$$= \frac{e^{\sin x}\cos x}{1+e^{2\sin x}}.$$
2. $f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0.$
3. $f'(x) = \frac{1}{\arctan x}\frac{1}{1+x^2} = \frac{1}{(\arctan x)(1+x^2)}.$

3.12 Implicit Differentiation

So far, we have examined derivatives of explicitly-defined functions (e.g., $y = f(x)$), but what about implicitly-defined functions?

EXAMPLE 3.12.1

If $x^2 + y^2 = 1$, then this isn't even a function (as it does not pass the vertical line test). But, if we divide up the curve into positive and negative parts on the y-axis then it can be a function. Then, we could find the derivative of each piece! The good news is that it doesn't matter if we break it up first! We can differentiate both sides of an implicit equation using the chain rule and solve for y. We do need to assume that the equation defines an implicit function though, more on this later.

EXAMPLE 3.12.2

Find y' if $3x^3y^3 + x^2y + 13x = 12$.

Solution. We let $y = y(x)$, take the derivative with respect to x on both sides, and then solve for $y'(x)$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}[3x^3y(x)^3 + x^2y(x) + 13x] &= \frac{\mathrm{d}}{\mathrm{d}x}[12] \\
\frac{\mathrm{d}}{\mathrm{d}x}[3x^3y(x)^3] + \frac{\mathrm{d}}{\mathrm{d}x}[x^2y(x)] + \frac{\mathrm{d}}{\mathrm{d}x}[13x] &= 0 \\
3\big(3x^2y(x)^3 + 3x^3(3)y(x)^2y'(x)\big) + \big(2xy(x) + x^2y'(x)\big) + 13 &= 0 \\
9x^2y(x)^3 + 9x^3y(x)^2y'(x) + 2xy(x) + x^2y'(x) + 13 &= 0 \\
9x^3y(x)^2y'(x) + x^2y'(x) &= -13 - 9x^2y(x)^3 - 2xy(x) \\
y'(x)\big(9x^3 + x^2\big) &= -13 - 9x^2y(x)^3 - 2xy(x) \\
y'(x) &= \frac{-13 - 9x^2y(x)^3 - 2xy(x)}{9x^3 + x^2}.
\end{aligned}$$

Therefore,

$$y' = \frac{-13 - 9x^2y^3 - 2xy}{9x^3 + x^2}.$$

REMARK 3.12.3

We can't always find the derivative of both sides of an equation unless we have a function!

EXAMPLE 3.12.4

If x^2+y^2+1, we can show that $y' = -\frac{x}{y}$, but for which $(x, y) \in \mathbb{R}^2$ is this valid for? None! $x^2+y^2+1 \neq 0$ for <u>any</u> $(x, y) \in \mathbb{R}^2$, so we differentiated nothing! Another example is if $2x = x$, we would differentiate to get $2 = 1$ (nonsense). The issue is $2x = x$ is only true if $x = 0$, so we can't compute the derivative as we can't take a limit! So be careful, use this power wisely!

Logarithmic Differentiation

We can use implicit differentiation to find the derivative of functions of the form

$$y = (f(x))^{g(x)}, \ f(x) > 0$$

by taking the "$\ln$" of both sides.

EXAMPLE 3.12.5

Let $y = (\ln x)^{\sin x}$ for $x > 1$. Find y'.

Solution. Let $y = y(x)$ so that $y(x) = (\ln x)^{\sin x}$. Taking the logarithm (and then the derivative with respect to x) on both sides gives

$$\begin{aligned}
\ln y(x) &= (\sin x)\ln(\ln x) \\
\frac{\mathrm{d}}{\mathrm{d}x}[\ln y(x)] &= \frac{\mathrm{d}}{\mathrm{d}x}[(\sin x)\ln(\ln x)] \\
\frac{y'(x)}{y(x)} &= (\cos x)\ln(\ln x) + \sin x \frac{1}{\ln x}\frac{1}{x} \\
\implies y'(x) &= y(x)\left[(\cos x)\ln(\ln x) + \frac{\sin x}{x\ln x}\right] \\
\implies y'(x) &= (\ln x)^{\sin x}\left[(\cos x)\ln(\ln x) + \frac{\sin x}{x\ln x}\right].
\end{aligned}$$

EXAMPLE 3.12.6

Let $y = x^{\arctan x}$. Find y'.

Solution. Let $y = y(x)$ so that $y(x) = x^{\arctan x}$. Taking the logarithm (and then the derivative with respect to x) on both sides gives

$$\begin{aligned}
\ln y(x) &= \arctan(x)\ln x \\
\frac{\mathrm{d}}{\mathrm{d}x}[\ln y(x)] &= \frac{\mathrm{d}}{\mathrm{d}x}[\arctan(x)\ln x] \\
\frac{y'(x)}{y(x)} &= \frac{1}{1+x^2}\ln x + \arctan(x)\frac{1}{x} \\
\implies y'(x) &= y(x)\left[\frac{\ln x}{1+x^2} + \frac{\arctan x}{x}\right] \\
\implies y'(x) &= x^{\arctan x}\left[\frac{\ln x}{1+x^2} + \frac{\arctan x}{x}\right].
\end{aligned}$$

3.13 Local Extrema

DEFINITION 3.13.1: Local Maximum, Local Minimum

Let $A \subseteq \mathbb{R}$ be open, let $f\colon A \to \mathbb{R}$, and let $a \in A$. Then f has a **local maximum** at a if and only if

$$\forall x \in A : f(x) \leq f(a).$$

Similarly, we say f has a **local minimum** at a if and only if

$$\forall x \in A : f(x) \geq f(a).$$

We also present an equivalent definition.

DEFINITION 3.13.2: Local Maximum, Local Minimum

Let $A \subseteq \mathbb{R}$ be open, let $f\colon A \to \mathbb{R}$, and let $a \in A$. Then f has a **local maximum** at a if and only if

$$\exists \delta > 0 : \forall x \in A : |x - a| \leq \delta \implies f(x) \leq f(a).$$

Similarly, we say f has a **local minimum** at a if and only if

$$\exists \delta > 0 : \forall x \in A : |x - a| \leq \delta \implies f(x) \geq f(a).$$

REMARK 3.13.3

Local maximum/minimum means max/min nearby a point (i.e., in a small neighbourhood). Global max/min means max/min over the entire interval in question. So, global max/mins that occur inside the interval are also local max/mins.

How do we find local extrema? We will use the following theorem.

3.13.1 The Local Extrema Theorem

THEOREM 3.13.4: Fermat's Theorem/Local Extrema Theorem

Let $A \subseteq \mathbb{R}$ be open, let $f\colon A \to \mathbb{R}$, and let $a \in A$. Suppose that f is differentiable at a and that f has a local maximum or minimum value at a. Then $f'(a) = 0$.

Proof: We suppose that f has a local maximum value at a (the case that f has a local minimum value at a is similar). Choose $\delta > 0$ so that $|x - a| \leq \delta \implies f(x) \leq f(a)$. For $x \in A$ with $a < x < a + \delta$, since $x > a$ and $f(x) \geq a$ we have $\frac{f(x)-f(a)}{x-a} \geq 0$, and so

$$f'(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \geq 0$$

by the Comparison Theorem. Similarly, for $x \in A$ with $a - \delta \leq x < a$, since $x < a$ and $f(x) \geq f(a)$ we have $\frac{f(x)-f(a)}{x-a} \leq 0$, and so

$$f'(a) = \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} \leq 0.$$

- Q: Is the converse true?
- A: No! $f(x) = x^3$ has a critical point at $x = 0$, but 0 is neither a local max nor a local min.
- Q: If c is a local max/min, then is $f'(c) = 0$?

- A: No! $f(x) = |x|$ has a local min at $x = 0$, but $f'(0)$ does not exist.

Finding Global Extrema

We just saw that if we want to find a local extrema, we should look at points where $f' = 0$ or f' does not exist. Let's give a name to points like this.

DEFINITION 3.13.5: Critical Point

A point c in the domain of a function f is called a **critical point** for f if either $f'(c) = 0$ or $f'(c)$ does not exist.

Now, the EVT guarantees a continuous function has a global max/min on a closed interval. Either these are at the endpoints or they are inside, and therefore local max/mins, and hence critical points!

So here is the algorithm for finding the global max/min of a continuous function $f(x)$ on $[a, b]$.

(i) Find all critical points of f in $[a, b]$.

(ii) Evaluate $f(a)$, $f(b)$, and $f(c)$, where c are all the critical points.

(iii) The largest value tells you where the global maximum is, and the smallest tells you what the global minimum is.

EXAMPLE 3.13.6

Find the global maximum and minimum for $f(x) = x^3 - 3x + 2$ on $[-3, 3]$.

Solution. $f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1) = 0$ if $x = \pm 1$. These critical points are both inside $[-3, 3]$. Now, we check $f(-3) = -16$, $f(-1) = 4$, $f(1) = 0$, $f(3) = 20$. Therefore, the global maximum is at $(3, 20)$ and the global minimum is at $(-3, -16)$.

EXAMPLE 3.13.7

Find the global maximum and minimum for $f(x) = 1/x$ on $[3, 7]$.

Solution. $f'(x) = -1/x^2$ and $f'(x)$ does not exist if $x = 0$. However, 0 is not a critical point of f since $0 \notin [3, 7]$. So, f has no critical points. Now, $f(3) = 1/3$ and $f(7) = 1/7$, so the global maximum is at $(3, 1/3)$ and the global minimum is at $(7, 1/7)$.

We will re-visit this when we discuss curve sketching.

Chapter 4

The Mean Value Theorem

4.1 The Mean Value Theorem

4.2 Applications of the Mean Value Theorem

4.2.1 Antiderivatives

4.2.2 Increasing Function Theorem

4.2.3 Functions with Bounded Derivatives

4.2.4 Comparing Functions Using Their Derivatives

4.3 L'Hopital's Rule

4.3.1 Interpreting the Second Derivative

4.3.2 Formal Definition of Concavity

4.3.3 Classifying Critical Points: The First and Second Derivative Tests

4.4 Curve Sketching

Chapter 5

Taylor Polynomials and Taylor's Theorem

5.1 Introduction to Taylor Polynomials and Approximation

5.2 Taylor's Theorem and Errors in Approximations

5.3 Big-O