MATH 239 - Introduction to Combinatorics

Cameron Roopnarine

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Contents

1 Math 239 Tutorial 2 (Shayla)

Problem 1. Use the negative binomial theorem and substitutions to give a formula for the coefficient of x^n in $(1-3x)^{-1} + 2(1-2x)^{-2}$.

$$(1-x)^{-k} = \sum_{n\geq 0} \binom{n+k-1}{k-1} x^n$$

Solution.

Hint: Start with $[x^n](1-3x)^{-1}$

$x \to 3x$, $k \to 1$

$$(1-3x)^{-1} = \sum_{n\geq 0} \binom{n+1-1}{1-1}(3x)^n = \sum_{n\geq 0} \binom{n}{0} 3^n x^n = \sum_{n\geq 0} 3^n x^n$$

$$(1-2x)^{-2} = \sum_{n\geq 0} \binom{n+2-1}{2-1}(2x)^n = \sum_{n\geq 0} \binom{n+1}{1} 2^n x^n = \sum_{n\geq 0} (n+1)2^{n+1}$$

$$[x^n](\text{expr.}) = 3^n + (n+1)2^{n+1}$$

Problem 2. Let $F(x) = x + x^2 + \cdots$ and let $G(x) = 1 + 3x + 2x^2$. Compute the coefficient of x^n in $G(F(x))$.

Solution.

$$G(F(x)) = 1 + 3(x + x^2 + \cdots) + 2(x + x^2 + \cdots)^2$$

$$\begin{aligned}
(x + x^2 + \cdots)^2 &= \left(\frac{1}{1-x} - 1\right)^2 \\
&= \left(\frac{x}{1-x}\right)^2 \\
&= x^2(1-x)^{-2} \\
&= x^2 \sum_{n\geq 0} \binom{n+2-1}{2-1} x^n \\
&= x^2 \sum_{n\geq 0} (n+1)x^n
\end{aligned}$$

$$[x^{n-2}] \sum_{n\geq 0} \binom{n+1}{1} x^n \qquad [x^n] \sum_{n\geq 0} (n+1)x^{n+2}$$

$$[x^{n-2}] \sum_{n\geq 0} \binom{n+1}{1} x^n = n-1 \text{ where } m = n-2 \geq 0$$

$$[x^n]2(x + x^2 + \cdots)^2 = \begin{cases} 2(n-1),\ n \geq 2 \\ 0,\ n = 0, 1 \end{cases}$$

$$[x^n]1 = \begin{cases} 0,\ n > 0 \\ 1,\ n = 0 \end{cases}$$

$$[x^n]3(x + x^2 + \cdots) = \begin{cases} 3,\ n \geq 1 \\ 0,\ n = 0 \end{cases}$$

Thus,

$$= \begin{cases} 2n+1,\ n \geq 2 \\ 3,\ n = 1 \\ 1,\ n = 0 \end{cases}$$

Problem 3. Show that if $F(x) = a_0 + a_1 x + a_2 x^2 + \cdots$. Then

$$F(x)(1-x) = a_0 + (a_1 - a_0)x + (a_2 - a_1)x^2 + \cdots$$

and

$$F(x)(1-x)^{-1} = \sum_{n=0}^{\infty} c_n x^n,$$

where $c_n = a_0 + a_1 + \cdots + a_n$.

Solution.

Note that $F(x)(1-x) = F(x) - xF(x)$

We know that $(1-x)^{-1} = 1 + x + x^2 + \cdots$, thus

$$\begin{aligned} F(x)(1-x)^{-1} &= F(x) + xF(x) + x^2F(x) + \cdots \\ &= \sum_{i \geq 0} x^i F(x) \\ &= \sum_{i \geq 0} [x^{n-i}] F(x) \\ &= \sum_{i \geq 0}^{n} a_{n-i} \text{ for } 0 \leq k = n - i \leq n \\ &= \sum_{k=0}^{n} a_k \end{aligned}$$

where $[x^{n-i}]F(x) = a_{n-i} = \begin{cases} a_{n-i},\ i \leq n \\ 0,\ i > n \end{cases}$

Problem 4. Show that for $k \geq 1$ and $n \geq 1$, we have

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{n-i+k-1}{k-1} = 0,$$

where we interpret $\binom{j}{i} = 0$ when $j < i$. Hint: Look at $1 = (1-x)^k(1-x)^{-k}$) and compute the coefficient of x^n in both sides.

Solution.

$[x^n]1 = 0$

$[x^n](1-x)^k(1-x)^{-k}$; coefficient of x^i in $(1-x)^k$ and x^{n-i} for $(1-x)^{-k}$; add them up.

$$\begin{aligned} &= \sum_{i=0}^{n} [x^i](1-x)^{-k}[x^{n-i}](1-x)^{-k} \\ &= \sum_{i=0}^{n} (-1)^i \binom{k}{i} \binom{n-i+k-1}{k-1} \text{ by Bin. \& NB .thm} \\ &= \sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{n-i+k-1}{k-1} \end{aligned}$$

2 Math 239 Tutorial 3 (Shayla)

Problem 1. Consider the set of non-negative integers $\mathbb{N}_0$, but with a non-standard weight function

$$w(a) = \begin{cases} \frac{3}{2}a + 1, \text{ if } a \text{ is even,} \\ 2(a+1), \text{ if } a \text{ is odd.} \end{cases}$$

Find the generating series for $\mathbb{N}_0$ with respect to this weight function and express it as a simplified rational expression.

Solution.

$\mathbb{N}_0 = \mathbb{N}_{\text{even}} \cup \mathbb{N}_{\text{odd}}$

$$\begin{aligned} \Phi_{\mathbb{N}_0}(x) &= \Phi_{\mathbb{N}_{\text{even}}}(x) + \Phi_{\mathbb{N}_{\text{odd}}}(x) \qquad \text{Sum Lemma} \\ &= \sum_{a \text{ even}} x^{3/2a+1} + \sum_{a \text{ odd}} x^{2(a+1)} \\ &= x \sum_{a \text{ even}} x^{3/2a} + x^2 \sum_{a \text{ odd}} x^{2a} \\ &= x \sum_{i \geq 0} (x^{3/2})^{2i} + x^2 \sum_{i \geq 0} (x^2)^{2i+1} \\ &= x \sum_{i \geq 0} (x^3)^i + x^4 \sum_{i \geq 0} (x^4)^i \\ &= \frac{x}{1-x^3} + \frac{x^4}{1-x^4} \end{aligned}$$

Problem 2. Let m, n be positive integers and α, β positive real numbers. Find the generating series for the cartesian product

$$\{1, \ldots, m\} \times \{1, \ldots, n\}$$

with respect to the weight function

$$w(a, b) = \alpha a + \beta b$$

and express it as a simplified rational expression.

Solution.

<u>Partial Geometric series</u>

$$\sum_{i=0}^{k} x^i = \frac{1-x^{k+1}}{1-x}$$

$$\begin{aligned} \Phi_{A \times B}(x) &= \sum_{(a,b) \in A \times B} x^{\alpha a + \beta B} \\ &= \sum_{a \in A} x^{\alpha a} + \sum_{b \in B} x^{\beta b} \\ &= \sum_{i=1}^{m} (x^{\alpha})^i + \sum_{j=1}^{n} (x^{\beta})^j \\ &= x^{\alpha} \sum_{i=0}^{m-1} (x^{\alpha})^i + x^{\beta} \sum_{j=0}^{n} (x^{\beta})^j \\ &= x^{\alpha+\beta} \frac{(1-x^{\alpha m})(1-x^{\beta n})}{(1-x^{\alpha})(1-x^{\beta})} \end{aligned}$$

Problem 3. Let a, b, n, k be positive integers with $a \le b$ and $k \le n$. How many compositions of n with k parts are there in which all parts are elements of $\{a, \ldots, b\}$? Expressing the result as a finite sum $\sum_{i=0}^{k} s_i$ is sufficient.

Rough.

Start with a small example $a = 2,\ b = 4,\ n = 9,\ k = 3$.

Let $C = \{$compositions with 3 parts where each part is 2, 3, or 4$\}$. We want $[x^9]\Phi_C(x)$.

Let $P = \{$parts of value 2, 3, or 4$\}$. $C = P \times P \times P$ (3 parts). Thus, $\Phi_C(x) = (\Phi_P(x))^3$.

$$(\Phi_P(x))^3 = \left(\sum_{i=2}^{4} x^i\right)^3$$

We want $[x^9](x^2 + x^3 + x^4)^3$.

Solution.

So, in general we have

$$\begin{aligned}\Phi_C(x) &= (\Phi_P(x))^k \\ &= \left(\sum_{i=a}^{b} x^i\right)^k\end{aligned}$$

We want $[x^n]\left(\sum_{i=a}^{b} x^i\right)^k$.

$$\begin{aligned}[x^n]\left(\sum_{i=a}^{b} x^i\right)^k &= [x^n]\left(x^a + \cdots + x^b\right)^k \\ &= [x^n]\, x^{ak}(1 + \cdots + x^{b-a})^k \\ &= \left[x^{n-ak}\right]\left(\frac{1 - x^{b-a+1}}{1-x}\right)^k \\ &= \left[x^{n-ak}\right]\left(1 + (-x^{b-a+1})\right)^k (1-x)^{-k} \\ &= \left[x^{n-ak}\right]\sum_{i\ge 0}\binom{k}{i}(-1)^i x^{i(b-a+1)} \sum_{j\ge 0}\binom{j+k-1}{k-1}x^j \\ &= \left[x^{n-ak}\right]\sum_{i\ge 0}\sum_{j\ge 0}\binom{k}{i}\binom{j+k-1}{k-1}(-1)^i x^{i(b-a+1)+j}\end{aligned}$$

$i(b-a+1) + j = n - ak$

$$\sum_{i=0}^{\lfloor\frac{n-ak}{b-a+1}\rfloor}\binom{k}{i}\binom{n-ak-i(b-a+1)+k-1}{k-1}(-1)^i$$

3 Math 239 Tutorial 4

Problem 1. Let S denote the set of strings of the form $\{1\}^*\{0\}^*\{1\}^*\{0\}^*$. Find the generating function for $\Phi_S(x)$, where the weight of a string is given by its length.

Solution.

$\{1\}^* \left(\{0\}\{0\}^*\{1\}\{1\}^*\right) \{0\}^* \cup \{1\}^*\{0\}^*$

Problem 2. Let $S = \{00, 111\}^*$. Find a formula for $\Phi_S(x)$.

Solution.

$$\begin{aligned}\Phi_S(x) &= \Phi_{\{00,111\}^*} \\ &= \frac{1}{1 - \Phi_{\{00,111\}}} \\ &= \frac{1}{1 - (x^2 + x^3)}\end{aligned}$$

Problem 3. Let S be $\{00, 111\}^*$ and let S_n denote the set of strings of length n in S. Give a combinatorial proof that $|S_n| = |S_{n-2}| + |S_{n-3}|$ for $n \geqslant 3$.

Look at $n = 0, ..., 6$.

$n = 2 : |S_2| = 1 \to 00$

$n = 3 : |S_3| = 1 \to 111$

$n = 4 : |S_4| = 1 \to 0000$

$n = 5 : |S_5| = 2 \to 00111, 11100$

$n = 6 : |S_6| = 2 \to 111111, 000000$

Solution.

Let $s \in S_n$, $n \geqslant 3$. What could s start with?

Case 1: $00t$, $t \in S_{n-2}$

Case 2: $111t$, $t \in S_{n-3}$

$f : S_n \to S_{n-2} \cup S_{n-3}$, $\forall s \in S_n$, $f(s) = t$ where

$$s = \begin{cases} 00t, \text{ if } s \text{ starts with } 00 \\ 111t, \text{ if } s \text{ starts with } 111 \end{cases}$$

$g : S_{n-2} \cup S_{n-3} \to S_n$

$$g(t) = \begin{cases} 00t, \ t \in S_{n-2} \\ 111t, \ t \in S_{n-3} \end{cases}$$

Explain how $f(g(t)) = t$ and $g(f(s)) = s$.

Problem 4. Explain why $(\{1\}^*\{0\}^*)$ is ambiguous.

Solution.

Ambiguous means there are multiple ways to create a string. So, taking ε works since,

$$\begin{aligned}\left(\{1\}^0\{0\}^0\right)^x &= (\varepsilon)^x \\ &= \varepsilon\end{aligned}$$

4 Math 239 Tutorial 5

Problem 1. Let S denote the set of binary strings not containing the string 101 as a substring. Find an unambiguous expression for S, and use it to give a rational expression for $\Phi_S(x)$, weighted by length.

Solution.

$\{0\}^*(\{1\}\{1\}^*\{0\}\{0\}^*)^*\{1\}^*$

$S = \{0\}^*(\{1\}\{1\}^*\{00\}\{0\}^*)^*\{1\}^*\{\varepsilon, 10\}$

$T = \{$binary strings containing exactly one copy of 101 as a suffix$\}$

$$\{\varepsilon\} \cup S\{0,1\} = S \cup T$$
$$S\{101\} = T \cup T\{01\}$$
$$1 + \Phi_S(x)2x = \Phi_S(x) + \Phi_T(x)$$
$$\Phi_S(x)x^3 = \Phi_T(x) + \Phi_T(x)x^2$$
$$\implies 1 + \Phi_S(x)2x - \Phi_S(x) = \Phi_T(x)$$

substituting,

$$\Phi_S(x)x^3 = 1 + \Phi_S(x)2x - \Phi_S(x) + x^2 + \Phi_S(x)2x^3 - \Phi_S(x)x^2$$
$$\implies \Phi_S(x)x^3 = 1 + \Phi_S(x)2x - \Phi_S(x) + \Phi_S(x)2x^3 - \Phi_S(x)x^2 + x^2$$
$$\implies \Phi_S(x)x^3 - \Phi_S(x)2x + \Phi_S(x) - \Phi_S(x)2x^3 + \Phi_S(x)x^2 = 1 + x^2$$
$$\implies -\Phi_S(x)x^3 - \Phi_S(x)2x + \Phi_S(x) + \Phi_S(x)x^2 = 1 + x^2$$
$$\implies \Phi_S(x)(-x^3 - 2x + 1 + x^2) = 1 + x^2$$
$$\implies \Phi_S(x) = \frac{1 + x^2}{-x^3 + x^2 - 2x + 1}$$

Problem 2. Let S be the set of binary strings with an odd number of blocks. Find an unambiguous recursive decomposition for S, and use it to find a rational expression for $\Phi_S(x)$, weighted by length.

$X = \{$binary strings with odd number of blocks, beginning with $1\}$

$Y = \{$binary strings with odd number of blocks, beginning with $0\}$

$S \cup T = T\{0,1\} \cup S\{0,1\} \cup \{\varepsilon\}$

$S = X \cup Y$

$X = \{1\}\{1\}^*(\{0\}\{0\}^*X \cup \{\varepsilon\}) \to \Phi_X(x) = \frac{x}{1-x}\left(\frac{x}{1-x}\Phi_X(x) + 1\right)$

$Y = \{0\}\{0\}^*(\{1\}\{1\}^*Y \cup \{\varepsilon\})$

$$\Phi_X(x) = \frac{x^2}{(1-x)^2}\Phi_X(x) + \frac{x}{1-x}$$

$$\begin{aligned}\implies \Phi_X(x) &= \frac{\frac{x}{1-x}}{1 - \frac{x^2}{(1-x)^2}} \\ &= \frac{x(1-x)}{(1-x)^2 - x^2} \\ &= \frac{x - x^2}{1 - 2x}\end{aligned}$$

$$\Phi_S(x) = \frac{2x - 2x^2}{1 - 2x}$$

Problem 3. Let k and ℓ be non negative, and S be the set of binary strings in which no block of zeros has length greater than k and no blocks of ones has length greater than ℓ. Find an unambiguous recursive decomposition for S, and use it to find a rational expression for $\Phi_S(x)$, weighted by length.

$T = \{0, 00, \ldots, 0^k\}$, $U = \{1, 11, \ldots, 1^\ell\}$

$(T \cup \{\varepsilon\})(UT)^*(U \cup \{\varepsilon\})$

$$\Phi_T(x) = \frac{x(1 - x^k)}{1 - x}$$

$$\Phi_U(x) = \frac{x(1 - x^\ell)}{1 - x}$$