Calculus 1 for Honours Mathematics

MATH 137

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Contents

Chapter 1

Sequences and Convergence

1.1 Absolute Values

What is an absolute value? We commonly think of it as an operation that removes negative signs.

EXAMPLE 1.1.1

$|-2| = 2$, $|-17| = 17$, $|3| = 3$, etc.

So, is $|-x| = x$ for all $x \in \mathbf{R}$? Not always! Let's give the definition to avoid ambiguity.

DEFINITION 1.1.2

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

Clearly, $|-x| = |x|$.

This also tells us the distance from x to 0, or the magnitude (size of x).

EXAMPLE 1.1.3

How do we get the distance between two arbitrary numbers using absolute values? For example, what is the distance from 3 to 7? 4 units. Also, $|7-3| = 4 = |3-7|$.

So, the distance from a to b is $|b-a|$ for all $a, b \in \mathbf{R}$. Also, $|b-a| = |a-b|$, which makes sense since the distance from a to b should be the same as the distance from b to a.

1.1.1 Inequalities Involving Absolute Values

The main focus of this course is **approximation**. We will seek ways to approximate roots, curves, limits, etc., but if we make an approximation it will be useless unless we can talk about how close it is to the actual object! So, we will look for ways to determine the maximum size of the **error**. Before we do this, we will need to examine **inequalities**. Let's start with the triangle inequality.

THEOREM 1.1.4: Triangle Inequality

For $x, y, z \in \mathbf{R}$, $|x-y| \leq |x-z| + |z-y|$.

Proof: Since $|x-y| = |y-x|$, we can assume without loss of generality (WLOG) that $x \leq y$. Hence, we consider three cases.
Case 1 ($z < x$): Clearly, $|x-y| \leq |z-y|$, which means $|x-y| \leq |x-z| + |z-y|$.
Case 2 ($x \leq z \leq y$): In this case, $|x-y| = |x-z| + |z-y|$, which means $|x-y| = |x-z| + |z-y|$, as desired.
Case 3 ($y < z$): This time, $|x-y| \leq |x-z|$, so $|x-y| \leq |x-z| + |z-y|$.

We consider a useful variant of the triangle inequality.

COROLLARY 1.1.5

For $x, y \in \mathbf{R}$, $|x+y| \leq |x| + |y|$.

Proof:

$$\begin{aligned} |x+y| &= |x-(-y)| \\ &\leq |x-0| + |0-(-y)| && \text{triangle inequality with } z = 0 \\ &= |x| + |y|. \end{aligned}$$

If we want to prove $|x| < \delta$, we just need to prove $x < \delta$ and $x > -\delta$, that is, $-\delta < x < \delta$. So, what do the inequalities of the form $|x-a| < \delta$ for $a, \delta \in \mathbf{R}$ look like? What set does this represent? Well, it's the set of all $x \in \mathbf{R}$ that are less than δ units away from a. So, starting at a, we move δ-units to the left and right, which means

$$|x-a| < \delta \iff -\delta < x-a < \delta \iff a-\delta < x < a+\delta.$$

So, it is the interval $(a-\delta, a+\delta)$, where we do not include the endpoints as the inequality is strict.

What about $|x-a| \leq \delta$? In this case,

$$|x-a| \leq \delta \iff -\delta \leq x-a \leq \delta \iff a-\delta \leq x \leq a+\delta.$$

So, it is the interval $[a-\delta, a+\delta]$.

What about $0 < |x-a| < \delta$? Now, the distance can't be zero which means $x \neq a$. So, it translates to $(a-\delta, a+\delta) \setminus \{a\}$ or $(a-\delta, a) \cup (a, a+\delta)$.

EXAMPLE 1.1.6

Find the corresponding sets for the inequalities.

(1) $|x-4| < 3$.
(2) $2 \leq |x-4| < 4$.
(3) $|x-1| + |x+2| \geq 4$.

Solution.

(1) $|x-4| < 3 \iff -3 < x-4 < 3 \iff 1 < x < 7$, so $(1,7)$ is the corresponding interval.
(2) $2 \leq |x-4| < 4$ means $2 \leq |x-4|$ and $|x-4| < 4$, so

$$(2 \leq x-4) \vee (x-4) \leq -2 \iff (6 \leq x) \vee (x \leq 2)$$

and

$$-4 < x-4 < 4 \iff 0 < x < 8.$$

Putting these together, we get $0 < x \leq 2$ or $6 \leq x < 8$, so $(0,2] \cup [6,8)$ is the corresponding interval.
(3) We consider three cases.
(i) If $x > 1$, then both $x-1 > 0$ and $x+2 > 0$, then

$$x-1+x+2 > 4 \iff 2x+1 > 4 \iff 2x > 3 \iff x < 3/2.$$

(ii) If $-2 \leq x \leq 1$, then $|x-1| = 1-x$, but $|x+2| = x+2$, so we get

$$1 - x + x + 2 \geq 4 \iff 3 \geq 4,$$

which is not true for any x.

(iii) If $x < -2$, then $|x-1| = 1-x$ and $|x+2| = -x-2$, then

$$1 - x + (-x-2) \geq 4 \iff -1 - 2x \geq 4 \iff -5 \geq 2x \iff -5/2 \geq x.$$

Putting it all together, we get $x \geq 3/2$ or $x \leq -5/2$, that is, $(-\infty, -5/2] \cup (3/2, \infty)$.

1.2 Sequences and Their Limits

DEFINITION 1.2.1

An infinite sequence of numbers is a list of numbers in a definite order, e.g.,

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ a_i \in \mathbf{R}.$$

Notation: $\{a_1, a_2, \ldots, a_n\}$ or $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$.

Sequences can be defined explicitly (in terms of n) or recursively (in terms of previous terms).

EXAMPLE 1.2.2: Explicit Sequences

- $\left\{\frac{1}{n+1}\right\}_{n=1}^{\infty}$: $1/2, 1/3, 1/4, 1/5, \ldots$.
- $\{\sqrt{n+2}\}_{n=2}^{\infty}$: $\sqrt{4}, \sqrt{5}, \sqrt{6}, \ldots$.
- $\{(-1)^n\}_{n=1}^{\infty}$: $-1, 1, -1, 1, \ldots$.

EXAMPLE 1.2.3: Recursive Sequences

- $a_1 = 1$, $a_{n+1} = \sqrt{1 + a_n}$, so $a_1 = 1$, $a_2 = \sqrt{2}$, $a_3 = \sqrt{1+\sqrt{2}}$, and so on for $n \geq 1$.
- Fibonacci sequence: $a_1 = 1$, $a_2 = 1$, $a_{n+2} = a_{n+1} + a_n$ for $n \geq 1$, i.e., $1, 1, 2, 3, 5, 8, 13, \ldots$.

We can plot sequences on a number line or we could think of a sequence as a function $f\colon \mathbf{N} \to \mathbf{R}$, writing $f(n) = a_n$, e.g., for $a_n = 1/2$ we would write $f(n) = 1/2$.

Why study sequences?

- Lots of continuous processes can be modelled with discrete data, as we will see.
- We can use recursive sequences to approximate solutions to equations that can't be solved explicitly (Newton's Method).
- For another (ancient) application, see page 14 of the course notes about calculating square roots.

Our goal now will be to determine how to find the limit of a sequence, that is, find what the value of the terms of the sequence are approaching (if it exists).

1.2.1 Sequences and Tails

We may want to build new sequences out of old ones or only discuss what happens to a sequence eventually, that is, after a certain index.

EXAMPLE 1.2.4

For $\{\frac{1}{n}\}_{n=1}^{\infty}$, if we consider only the odd terms, we get $1, 1/3, 1/5$, or the k^{th} term is

$$\frac{1}{2k-1}$$

for $k \in \mathbf{N}$. This is called a subsequence.

DEFINITION 1.2.5: Subsequence

If $\{a_n\}$ is a sequence and $n_1, n_2, \ldots$ is a sequence of natural numbers, where $n_1 < n_2 < n_3 < \cdots$, then the sequence

$$\{a_{n_1}, a_{n_2}, \ldots\} = \{a_{n_k}\}$$

is a subseqeunce of $\{a_n\}$.

One particular subsequence is $\{a_k, a_{k+1}, a_{k+2}\}$ for some $k \in \mathbf{N}$. This is called the tail of $\{a_n\}$ with cut-off k.

Limits of Sequences

We are going to see lots of different limits this term, but we will start with sequences.

EXAMPLE 1.2.6

$\{\frac{1}{n}\}$ seems like it converges to 0, or that 0 is the limit of the sequence. We saw this when we plotted the sequence. We will eventually want a formal definition, but let's start intuitively.

Given a sequence $\{a_n\}$, what does it mean to say that $\{a_n\}$ converges to L as n goes to infinity?

What about "as n gets larger, a_n gets closer to L"? Unfortunately, this isn't a good definition. For example, as n gets larger $\frac{1}{n}$ gets closer to 0, but it also gets closer to -1, -2, and so on. But, 0 is the limit! What makes it different? Well, the sequence gets infinitely close to 0, unlike the other numbers! Let's try to define this again: "the limit of $\{a_n\}$ is L if, as n gets infinitely large, a_n gets infinitely close to L." This is much better! But how can we formalize the idea of "infinitely close"?

DEFINITION 1.2.7: Formal Definition of a Limit

$L \in R$ is the limit of $\{a_n\}$ if:
For all $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that if $n \geq N$, then

$$|a_n - L| < \varepsilon.$$

If such an L exists, we can say that $\{a_n\}$ converges and write $\lim_{n\to\infty} a_n = L$ (or $a_n \to L$). If no such L exists, then we say that $\{a_n\}$ diverges.

Here, if $n \geq N$, then $|a_n - L| < \varepsilon$ (or $a_n \in (L-\varepsilon, L+\varepsilon)$). Let's take a look at some examples.

EXAMPLE 1.2.8

Consider $a_n = \frac{1}{n^2}$. We'd guess that the limit is 0. Say $\varepsilon = \frac{1}{100}$, can we find a large enough $n \in N$ so that $\left|\frac{1}{n^2} - 0\right| < \frac{1}{100}$ if $n \geq N$? Well, we need

$$\left|\frac{1}{n^2} - 0\right| < \frac{1}{100} \implies \frac{1}{n^2} < \frac{1}{100} \implies n^2 > 100,$$

so $n > 10$. Let $N = 11$, then if $n \geq N$, we get $\left|\frac{1}{n^2} - 0\right| < \frac{1}{100}$. But wait! We aren't done yet! The definition says we need to prove it for any $\varepsilon > 0$, but we only proved it for $\varepsilon = \frac{1}{100}$. Let's adapt the proof to work for any $\varepsilon > 0$.

Proof that $\lim_{n\to\infty} \frac{1}{n^2} = 0$. Let $\varepsilon > 0$ be given. Let $N > \frac{1}{\sqrt{\varepsilon}}$ for $N \in \mathbf{N}$. Then, if $n \geq N$, we get

$$\left|\frac{1}{n^2} - 0\right| = \frac{1}{n^2} \leq \frac{1}{N^2} < \frac{1}{(1/\sqrt{\varepsilon})^2} = \frac{1}{1/\varepsilon} = \varepsilon$$

as desired.

The point is: we have to give a method for choosing N that works for any $\varepsilon > 0$. Also, the logical order of the proof is important, so let's do some more examples.

EXAMPLE 1.2.9

Prove that $\lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2}$.

Proof: Let $\varepsilon > 0$ be given. Let $N > \frac{1}{4}(\frac{3}{\varepsilon} - 6)$ for $N \in \mathbf{N}$. Then, if $n \geq N$, we get:

$$|a_n - L| = \left|\frac{n}{2n+3} - \frac{1}{2}\right| = \frac{3}{4n+6} \leq \frac{3}{4N+6} < \frac{3}{4(\frac{1}{4}(\frac{3}{\varepsilon} - 6)) + 6} = \varepsilon$$

as desired.
Aside: We want

$$\frac{3}{4n+6} < \varepsilon \iff \frac{3}{\varepsilon} < 4n+6 \iff \frac{3}{\varepsilon} - 6 < 4n \iff \frac{1}{4}\left(\frac{3}{\varepsilon} - 6\right) < n.$$

EXAMPLE 1.2.10

Prove that $\lim_{n\to\infty} \frac{n^2}{3n^2+7n} = \frac{1}{3}$.

Proof: Let $\varepsilon > 0$ be given. Let $N > \frac{7}{9\varepsilon}$ for $N \in \mathbf{N}$. Then, if $n \geq N$, we get:

$$|a_n - L| = \left|\frac{n^2}{3n^2+7n} - \frac{1}{3}\right| = \frac{7n}{9n^2+21n} \leq \frac{7n}{9n^2} = \frac{7}{9n} \leq \frac{7}{9(\frac{7}{9\varepsilon})} = \varepsilon.$$

Aside: We want

$$\frac{7}{9n} < \varepsilon \iff \frac{7}{9\varepsilon} < n.$$

REMARK 1.2.11: Avoid Common Mistakes

- Don't choose ε! Let it be arbitrary.
- Never assume $|a_n - L| < \varepsilon$, make sure you only do work in an aside with that inequality since it is what you are proving.
- In practice, unless you are asked to, do not use this formal definition. We will now try to develop better methods for finding limits.

1.2.2 Equivalent Definitions of the Limit

When proving $\lim_{n\to\infty} a_n = L$, we are given $\varepsilon > 0$ and we try to find $N \in \mathbf{N}$ so that if $n \geq N$, then $|a_n - L| < \varepsilon$. But, this is the same as saying $a_n \in (L - \varepsilon, L + \varepsilon)$. Also, the collection of $\{a_n\}$ for which $n \geq N$ is the tail of the sequence with cut-off N. So, here's another definition.

DEFINITION 1.2.12

$\lim_{n\to\infty} a_n = L$ if for any $\varepsilon > 0$, the interval $(L-\varepsilon, L+\varepsilon)$ contains a tail of the sequence $\{a_n\}$.

Let's push it further! Since the above is true for any $\varepsilon > 0$, if we pick any open interval (a, b) containing L, then we can find a small enough $\varepsilon > 0$ so that $(L-\varepsilon, L+\varepsilon) \subseteq (a, b)$. Therefore, any interval containing L also contains a tail of $\{a_n\}$. Let's collect all of these alternate (but equivalent) definitions together.

THEOREM 1.2.13

The following are equivalent:

(1) $\lim_{n\to\infty} a_n = L$.

(2) For any $\varepsilon > 0$, $(L-\varepsilon, L+\varepsilon)$ *contains a tail of* $\{a_n\}$.

(3) For any $\varepsilon > 0$, $(L-\varepsilon, L+\varepsilon)$ *contains all but finitely many terms of* $\{a_n\}$.

(4) Every interval (a, b) *containing* L *contains a tail of* $\{a_n\}$.

(5) Every interval (a, b) *containing* L *contains all but finitely many terms of* $\{a_n\}$.

Clearly, changing finitely many terms of $\{a_n\}$ *does not affect the convergence or the limit.*

EXAMPLE 1.2.14

Can a sequence have more than one limit? Consider $\{(-1)^n\} = -1, 1, -1, 1, \ldots$, it equals to both 1 and -1 infinitely often. Could both 1 and -1 be the limits? No! Let's prove -1 isn't a limit.

Proof: Consider the interval $(-2, 0)$. Clearly $-1 \in (-2, 0)$, but this interval does not contain any of the infinitely many 1's in the sequence. So, -1 is not a limit by (5) above. A similar argument can be used with the interval $(0, 2)$ to show 1 is also not a limit. So, does $\{(-1)^n\}$ have a limit at all? Let's prove it doesn't't! Let $\varepsilon = 1/2$, and supposed for a contradiction that the sequence converges to $L \in \mathbf{R}$. That means the interval $(L-1/2, L+1/2)$ must contain all but finitely many terms of the sequence, that is, but 1 and -1 must lie in that interval. But the interval is only 1 unit long! So there is not $L \in \mathbf{R}$ for which both 1 and -1 lie inside $(L-1/2, L+1/2)$. So, $\{(-1)^n\}$ diverges.

A similar argument can be used to prove limits are unique.

THEOREM 1.2.15

Let $\{a_n\}$ *be a sequence. If it has a limit* L, *then the limit is unique.*

Proof: Suppose for a contradiction that L and M are both limits of $\{a_n\}$ and $L \neq M$ and WLOG that $L < M$. Consider two intervals:

$$(L-1, \tfrac{L+M}{2}) \ni L, \quad (\tfrac{L+M}{2}, M+1) \ni M.$$

This means, by definition, only finitely many terms of the sequence are not in the first interval and only finitely many terms are not in the second interval. But the sequence has infinitely many terms! So, at least one term is in both intervals which is impossible. This is a contradiction, so $L = M$.

REMARK 1.2.16: A Remark on Possible Limits

If $a_n \geq 0$ for all n, then $\{a_n\}$ can't converge to a negative number! If it did, say to $L < 0$, then the interval $(L-1, 0)$ would contain L but no terms of the sequence.

THEOREM 1.2.17

If $a_n \geq 0$ for all n and $\lim_{n\to\infty} a_n = L$, then $L \geq 0$. More generally, if $\alpha \leq a_n \leq \beta$ for all n and $\lim_{n\to\infty} a_n = L$, then $\alpha \leq L \leq \beta$.

- Q: If $a_n > 0$ for all n and $\lim_{n\to\infty} a_n = L$ is $L > 0$?
- A: Not necessarily! Consider $a_n = \frac{1}{n} > 0$, but $L = 0$.

1.2.3 Divergence to Infinity

Consider $a_n = n$. It is clear that the sequence is getting larger without bound, so $\lim_{n\to\infty} a_n$ does not exist. That is, $\{a_n\}$ diverges. But we can say more! Since it does not get infinitely large, we can make a definition to capture this.

DEFINITION 1.2.18

$\lim_{n\to\infty} a_n = \infty$ if for all $M > 0$, we can find $N \in \mathbf{N}$ so that if $n \geq N$, then $a_n > M$.
Equivalently, any interval of the form (M, ∞) contains a tail of $\{a_n\}$.

It does look strange to write "$= \infty$" but wit hthe above definition we know it means "does not exist but gets infinitely large."
Similarly, if $\lim_{n\to\infty} a_n = -\infty$ for all $M < 0$, there is $N \in \mathbf{N}$ so that if $n \geq N$, then $a_n < M$.

EXAMPLE 1.2.19

Show $\lim_{n\to\infty} (1-n) = -\infty$.

Proof: Let $M < 0$ be given, pick $N > 1 - M$ for $N \in \mathbf{N}$. Then, if $n \geq N$, we have

$$a_n = 1 - n \leq 1 - N < 1 - (1 - M) = M.$$

<u>Aside</u>: Want $1 - n < M \iff 1 - M < n$.