Calculus 1 for Honours Mathematics

MATH 137

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Chapter 1

Sequences and Convergence

1.1 Absolute Values

What is an absolute value? We commonly think of it as an operation that removes negative signs.

EXAMPLE 1.1.1

$|-2| = 2$, $|-17| = 17$, $|3| = 3$, etc.

So, is $|-x| = x$ for all $x \in \mathbb{R}$? Not always! Let's give the definition to avoid ambiguity.

DEFINITION 1.1.2

Let $x \in \mathbb{R}$. The **absolute value** of x is denoted $|x|$, and is defined as follows:

$$|x| = \begin{cases} x, & x > 0, \\ 0, & x = 0, \\ -x, & x < 0. \end{cases}$$

This also tells us the distance from x to 0, or the magnitude (size of x).

EXAMPLE 1.1.3

How do we get the distance between two arbitrary numbers using absolute values? For example, what is the distance from 3 to 7? 4 units. Also, $|7 - 3| = 4 = |3 - 7|$.

So, the distance from a to b is $|b - a|$ for all $a, b \in \mathbb{R}$. Also, $|b - a| = |a - b|$, which makes sense since the distance from a to b should be the same as the distance from b to a.

1.1.1 Inequalities Involving Absolute Values

The main focus of this course is **approximation**. We will seek ways to approximate roots, curves, limits, etc., but if we make an approximation it will be useless unless we can talk about how close it is to the actual object! So, we will look for ways to determine the maximum size of the **error**. Before we do this, we will need to examine **inequalities**. Let's start with the triangle inequality.

THEOREM 1.1.4: Triangle Inequality

Let $x, y, z \in \mathbb{R}$. Then

$$|x - y| \leq |x - z| + |z - y|.$$

Proof: Since $|x - y| = |y - x|$, we can assume without loss of generality (WLOG) that $x \leq y$. Hence, we consider three cases.

Case 1 ($z < x$): Clearly, $|x - y| \leq |z - y|$, which means $|x - y| \leq |x - z| + |z - y|$.

Case 2 ($x \leq z \leq y$): In this case, $|x - y| = |x - z| + |z - y|$, which means $|x - y| = |x - z| + |z - y|$, as desired.

Case 3 ($y < z$): This time, $|x - y| \leq |x - z|$, so $|x - y| \leq |x - z| + |z - y|$.

We consider a useful variant of the triangle inequality.

THEOREM 1.1.5: Triangle Inequality II

Let $x, y \in \mathbb{R}$. Then

$$|x + y| \leq |x| + |y|.$$

Proof:

$$\begin{aligned} |x + y| &= |x - (-y)| \\ &\leq |x - 0| + |0 - (-y)| && \text{triangle inequality with } z = 0 \\ &= |x| + |y|. \end{aligned}$$

If we want to prove $|x| < \delta$, we just need to prove $x < \delta$ and $x > -\delta$, that is, $-\delta < x < \delta$. So, what do the inequalities of the form $|x - a| < \delta$ for $a, \delta \in \mathbb{R}$ look like? What set does this represent? Well, it's the set of all $x \in \mathbb{R}$ that are less than δ units away from a. So, starting at a, we move δ-units to the left and right, which means

$$|x - a| < \delta \iff -\delta < x - a < \delta \iff a - \delta < x < a + \delta.$$

So, it is the interval $(a - \delta, a + \delta)$, where we do not include the endpoints as the inequality is strict.

What about $|x - a| \leq \delta$? In this case,

$$|x - a| \leq \delta \iff -\delta \leq x - a \leq \delta \iff a - \delta \leq x \leq a + \delta.$$

So, it is the interval $[a - \delta, a + \delta]$.

What about $0 < |x - a| < \delta$? Now, the distance can't be zero which means $x \neq a$. So, it translates to $(a - \delta, a + \delta) \setminus \{a\}$ or $(a - \delta, a) \cup (a, a + \delta)$.

EXAMPLE 1.1.6

Find the corresponding sets for the inequalities.

(1) $|x - 4| < 3$.
(2) $2 \leq |x - 4| < 4$.
(3) $|x - 1| + |x + 2| \geq 4$.

Solution.

(1) $|x - 4| < 3 \iff -3 < x - 4 < 3 \iff 1 < x < 7$, so $(1, 7)$ is the corresponding interval.
(2) $2 \leq |x - 4| < 4$ means $2 \leq |x - 4|$ and $|x - 4| < 4$, so

$$(2 \leq x - 4) \vee (x - 4) \leq -2 \iff (6 \leq x) \vee (x \leq 2)$$

and

$$-4 < x - 4 < 4 \iff 0 < x < 8.$$

Putting these together, we get $0 < x \leq 2$ or $6 \leq x < 8$, so $(0, 2] \cup [6, 8)$ is the corresponding interval.

(3) We consider three cases.

(i) If $x > 1$, then both $x - 1 > 0$ and $x + 2 > 0$, then

$$x - 1 + x + 2 > 4 \iff 2x + 1 > 4 \iff 2x > 3 \iff x < 3/2.$$

(ii) If $-2 \leq x \leq 1$, then $|x - 1| = 1 - x$, but $|x + 2| = x + 2$, so we get

$$1 - x + x + 2 \geq 4 \iff 3 \geq 4,$$

which is not true for any x.

(iii) If $x < -2$, then $|x - 1| = 1 - x$ and $|x + 2| = -x - 2$, then

$$1 - x + (-x - 2) \geq 4 \iff -1 - 2x \geq 4 \iff -5 \geq 2x \iff -5/2 \geq x.$$

Putting it all together, we get $x \geq 3/2$ or $x \leq -5/2$, that is, $(-\infty, -5/2] \cup (3/2, \infty)$.

1.2 Sequences and Their Limits

1.2.1 Introduction to Subsequences

DEFINITION 1.2.1

An infinite sequence of numbers is a list of numbers in a definite order, e.g.,

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ a_i \in \mathbb{R}.$$

Notation: $\{a_1, a_2, \ldots, a_n\}$ or $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$.

Sequences can be defined explicitly (in terms of n) or recursively (in terms of previous terms).

EXAMPLE 1.2.2: Explicit Sequences

- $\left\{\frac{1}{n+1}\right\}_{n=1}^{\infty}$: $1/2, 1/3, 1/4, 1/5, \ldots$.
- $\{\sqrt{n+2}\}_{n=2}^{\infty}$: $\sqrt{4}, \sqrt{5}, \sqrt{6}, \ldots$.
- $\{(-1)^n\}_{n=1}^{\infty}$: $-1, 1, -1, 1, \ldots$.

1.2.2 Recursively Defined Sequences

EXAMPLE 1.2.3: Recursive Sequences

- $a_1 = 1$, $a_{n+1} = \sqrt{1 + a_n}$, so $a_1 = 1$, $a_2 = \sqrt{2}$, $a_3 = \sqrt{1 + \sqrt{2}}$, and so on for $n \geq 1$.
- Fibonacci's sequence: $a_1 = 1$, $a_2 = 1$, $a_{n+2} = a_{n+1} + a_n$ for $n \geq 1$, i.e., $1, 1, 2, 3, 5, 8, 13, \ldots$.

We can plot sequences on a number line, or we could think of a sequence as a function $f\colon \mathbb{N} \to \mathbb{R}$, writing $f(n) = a_n$, e.g., for $a_n = 1/2$ we would write $f(n) = 1/2$.

Why study sequences?

- Lots of continuous processes can be modelled with discrete data, as we will see.
- We can use recursive sequences to approximate solutions to equations that can't be solved explicitly (Newton's Method).

- For another (ancient) application, see page 14 of the course notes about calculating square roots.

Our goal now will be to determine how to find the limit of a sequence, that is, find what the value of the terms of the sequence are approaching (if it exists).

We may want to build new sequences out of old ones or only discuss what happens to a sequence eventually, that is, after a certain index.

EXAMPLE 1.2.4

For $\{\frac{1}{n}\}_{n=1}^{\infty}$, if we consider only the odd terms, we get $1, 1/3, 1/5$, or the k^{th} term is

$$\frac{1}{2k-1}$$

for $k \in \mathbb{N}$. This is called a subsequence.

1.2.3 Subsequences and Tails

DEFINITION 1.2.5: Subsequence

If $\{a_n\}$ is a sequence and $n_1, n_2, \ldots$ is a sequence of natural numbers, where $n_1 < n_2 < n_3 < \cdots$, then the sequence

$$\{a_{n_1}, a_{n_2}, \ldots\} = \{a_{n_k}\}$$

is a **subsequence** of $\{a_n\}$.

One particular subsequence is $\{a_k, a_{k+1}, a_{k+2}\}$ for some $k \in \mathbb{N}$. This is called the tail of $\{a_n\}$ with cut-off k.

1.2.4 Limits of Sequences

We are going to see lots of different limits this term, but we will start with sequences.

EXAMPLE 1.2.6

$\{\frac{1}{n}\}$ seems like it converges to 0, or that 0 is the limit of the sequence. We saw this when we plotted the sequence. We will eventually want a formal definition, but let's start intuitively.

Given a sequence $\{a_n\}$, what does it mean to say that $\{a_n\}$ converges to L as n goes to infinity?

What about "as n gets larger, a_n gets closer to L?" Unfortunately, this isn't a good definition. For example, as n gets larger $\frac{1}{n}$ gets closer to 0, but it also gets closer to -1, -2, and so on. But, 0 is the limit! What makes it different? Well, the sequence gets infinitely close to 0, unlike the other numbers! Let's try to define this again: "the limit of $\{a_n\}$ is L if, as n gets infinitely large, a_n gets infinitely close to L." This is much better! But how can we formalize the idea of "infinitely close?"

DEFINITION 1.2.7: Formal Definition of the Limit of a Sequence I

Let $\{a_n\}$ be a sequence in $\mathbb{R}$. For $L \in \mathbb{R}$, we say that the sequence $\{a_n\}$ **converges** to L (or that the **limit** of $\{a_n\}$ is equal to L), and we write $a_n \to L$ (as $n \to \infty$), or we write $\lim\limits_{n\to\infty} a_k = L$, when

$$\forall \varepsilon \in \mathbb{R}_{>0} : \exists N \in \mathbb{R}_{>0} : \forall n \in \mathbb{N} : n \geq N \implies |a_n - L| < \varepsilon.$$

We say that the sequence $\{a_n\}$ **diverges to infinity** (in $\mathbb{R}$) when there exists $L \in \mathbb{R}$ such that $\{a_n\}$ converges to L. We say that the sequence **diverges** (in $\mathbb{R}$) when it does not converge (to any $L \in \mathbb{R}$).

EXAMPLE 1.2.8

Consider $a_n = \frac{1}{n^2}$. We'd guess that the limit is 0. Say $\varepsilon = \frac{1}{100}$, can we find a large enough $n \in N$ so that $\left|\frac{1}{n^2} - 0\right| < \frac{1}{100}$ if $n \geq N$? Well, we need

$$\left|\frac{1}{n^2} - 0\right| < \frac{1}{100} \implies \frac{1}{n^2} < \frac{1}{100} \implies n^2 > 100,$$

so $n > 10$. Let $N = 11$, then if $n \geq N$, we get $\left|\frac{1}{n^2} - 0\right| < \frac{1}{100}$. But wait! We aren't done yet! The definition says we need to prove it for any $\varepsilon > 0$, but we only proved it for $\varepsilon = \frac{1}{100}$. Let's adapt the proof to work for any $\varepsilon > 0$.

Proof that $\lim\limits_{n\to\infty} \frac{1}{n^2} = 0$. Let $\varepsilon > 0$ be given. Let $N > \frac{1}{\sqrt{\varepsilon}}$ for $N \in \mathbb{N}$. Then, if $n \geq N$, we get

$$\left|\frac{1}{n^2} - 0\right| = \frac{1}{n^2} \leq \frac{1}{N^2} < \frac{1}{(1/\sqrt{\varepsilon})^2} = \frac{1}{1/\varepsilon} = \varepsilon$$

as desired.

The point is: we have to give a method for choosing N that works for any $\varepsilon > 0$. Also, the logical order of the proof is important, so let's do some more examples.

EXAMPLE 1.2.9

Prove that $\lim\limits_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2}$.

Proof: Let $\varepsilon > 0$ be given. Let $N > \frac{1}{4}\left(\frac{3}{\varepsilon} - 6\right)$ for $N \in \mathbb{N}$. Then, if $n \geq N$, we get:

$$|a_n - L| = \left|\frac{n}{2n+3} - \frac{1}{2}\right| = \frac{3}{4n+6} \leq \frac{3}{4N+6} < \frac{3}{4\left(\frac{1}{4}(\frac{3}{\varepsilon} - 6)\right) + 6} = \varepsilon$$

as desired.
Aside: We want

$$\frac{3}{4n+6} < \varepsilon \iff \frac{3}{\varepsilon} < 4n+6 \iff \frac{3}{\varepsilon} - 6 < 4n \iff \frac{1}{4}\left(\frac{3}{\varepsilon} - 6\right) < n.$$

EXAMPLE 1.2.10

Prove that $\lim\limits_{n\to\infty} \frac{n^2}{3n^2+7n} = \frac{1}{3}$.

Proof: Let $\varepsilon > 0$ be given. Let $N > \frac{7}{9\varepsilon}$ for $N \in \mathbb{N}$. Then, if $n \geq N$, we get:

$$|a_n - L| = \left|\frac{n^2}{3n^2+7n} - \frac{1}{3}\right| = \frac{7n}{9n^2+21n} \leq \frac{7n}{9n^2} = \frac{7}{9n} \leq \frac{7}{9(\frac{7}{9\varepsilon})} = \varepsilon.$$

Aside: We want

$$\frac{7}{9n} < \varepsilon \iff \frac{7}{9\varepsilon} < n.$$

REMARK 1.2.11: Avoid Common Mistakes

- Don't choose ε! Let it be arbitrary.
- Never assume $|a_n - L| < \varepsilon$, make sure you only do work in an aside with that inequality since it is

what you are proving.

- In practice, unless you are asked to, do not use this formal definition. We will now try to develop better methods for finding limits.

Equivalent Definitions of the Limit

When proving $\lim_{n\to\infty} a_n = L$, we are given $\varepsilon > 0$, and we try to find $N \in \mathbb{N}$ so that if $n \geq N$, then $|a_n - L| < \varepsilon$. But, this is the same as saying $a_n \in (L-\varepsilon, L+\varepsilon)$. Also, the collection of $\{a_n\}$ for which $n \geq N$ is the tail of the sequence with cut-off N. So, here's another definition.

DEFINITION 1.2.12

$\lim_{n\to\infty} a_n = L$ if for any $\varepsilon > 0$, the interval $(L-\varepsilon, L+\varepsilon)$ contains a tail of the sequence $\{a_n\}$.

Let's push it further! Since the above is true for any $\varepsilon > 0$, if we pick any open interval (a, b) containing L, then we can find a small enough $\varepsilon > 0$ so that $(L-\varepsilon, L+\varepsilon) \subseteq (a, b)$. Therefore, any interval containing L also contains a tail of $\{a_n\}$. Let's collect all of these alternate (but equivalent) definitions together.

THEOREM 1.2.13

The following are equivalent:

(1) $\lim_{n\to\infty} a_n = L$.

(2) For any $\varepsilon > 0$, $(L-\varepsilon, L+\varepsilon)$ *contains a tail of* $\{a_n\}$.

(3) For any $\varepsilon > 0$, $(L-\varepsilon, L+\varepsilon)$ *contains all but finitely many terms of* $\{a_n\}$.

(4) Every interval (a, b) *containing* L *contains a tail of* $\{a_n\}$.

(5) Every interval (a, b) *containing* L *contains all but finitely many terms of* $\{a_n\}$.

Clearly, changing finitely many terms of $\{a_n\}$ *does not affect the convergence or the limit.*

EXAMPLE 1.2.14

Can a sequence have more than one limit? Consider $\{(-1)^n\} = -1, 1, -1, 1, \ldots$, it equals to both 1 and -1 infinitely often. Could both 1 and -1 be the limits? No! Let's prove -1 isn't a limit.

Proof: Consider the interval $(-2, 0)$. Clearly $-1 \in (-2, 0)$, but this interval does not contain any of the infinitely many 1's in the sequence. So, -1 is not a limit by (5) above. A similar argument can be used with the interval $(0, 2)$ to show 1 is also not a limit. So, does $\{(-1)^n\}$ have a limit at all? Let's prove it doesn't! Let $\varepsilon = 1/2$, and supposed for a contradiction that the sequence converges to $L \in \mathbb{R}$. That means the interval $(L - 1/2, L + 1/2)$ must contain all but finitely many terms of the sequence, that is, but 1 and -1 must lie in that interval. But the interval is only 1 unit long! So there is not $L \in \mathbb{R}$ for which both 1 and -1 lie inside $(L - 1/2, L + 1/2)$. So, $\{(-1)^n\}$ diverges.

A similar argument can be used to prove limits are unique.

THEOREM 1.2.15

Let $\{a_n\}$ *be a sequence in* $\mathbb{R}$. *If* $\{a_n\}$ *has a limit (finite or infinite), then the limit is unique.*

Proof: Suppose for a contradiction that L and M are both limits of $\{a_n\}$ and $L \neq M$ and WLOG that $L < M$. Consider two intervals:

$$(L-1, \tfrac{L+M}{2}) \ni L, \quad (\tfrac{L+M}{2}, M+1) \ni M.$$

This means, by definition, only finitely many terms of the sequence are not in the first interval and only finitely many terms are not in the second interval. But the sequence has infinitely many terms! So, at

least one term is in both intervals which is impossible. This is a contradiction, so $L = M$.
Note: This does not cover the cases where the limit is infinite.

REMARK 1.2.16: A Remark on Possible Limits

If $a_n \geq 0$ for all n, then $\{a_n\}$ can't converge to a negative number! If it did, say to $L < 0$, then the interval $(L-1, 0)$ would contain L but no terms of the sequence.

THEOREM 1.2.17

If $a_n \geq 0$ for all n and $\lim\limits_{n\to\infty} a_n = L$, then $L \geq 0$. More generally, if $\alpha \leq a_n \leq \beta$ for all n and $\lim\limits_{n\to\infty} a_n = L$, then $\alpha \leq L \leq \beta$.

- Q: If $a_n > 0$ for all n and $\lim\limits_{n\to\infty} a_n = L$ is $L > 0$?
- A: Not necessarily! Consider $a_n = \frac{1}{n} > 0$, but $L = 0$.

1.2.5 Divergence to Infinity

Consider $a_n = n$. It is clear that the sequence is getting larger without bound, so $\lim\limits_{n\to\infty} a_n$ does not exist. That is, $\{a_n\}$ diverges. But we can say more! Since it does not get infinitely large, we can make a definition to capture this.

DEFINITION 1.2.18

- We say that $\{a_n\}$ **diverges to** ∞, or that the limit of $\{a_n\}$ is equal to **infinity**, and we write $a_n \to \infty$ (as $n \to \infty$), or we write $\lim\limits_{n\to\infty} a_n = \infty$, when

$$\forall M \in \mathbb{R}_{>0}\, \exists N \in \mathbb{N}\, \forall n \in \mathbb{N}\, \big(n \geq N \implies a_n > M\big).$$

  Equivalently, any interval of the form (M, ∞) contains a tail of $\{a_n\}$.
- We say that $\{a_n\}$ **diverges to** $-\infty$, or that the limit of $\{a_n\}$ is equal to **negative infinity**, and we write $a_n \to -\infty$ (as $n \to \infty$), or we write $\lim\limits_{n\to\infty} a_n = -\infty$, when

$$\forall M \in \mathbb{R}_{<0}\, \exists N \in \mathbb{N}\, \forall n \in \mathbb{N}\, \big(n \geq N \implies a_n < M\big).$$

  Equivalently, any interval of the form $(-\infty, M)$ contains a tail of $\{a_n\}$.

EXAMPLE 1.2.19

Show $\lim\limits_{n\to\infty} (1-n) = -\infty$.

Proof: Let $M < 0$ be given, pick $N > 1 - M$ for $N \in \mathbb{N}$. Then, if $n \geq N$, we have

$$a_n = 1 - n \leq 1 - N < 1 - (1 - M) = M.$$

Aside: Want $1 - n < M \iff 1 - M < n$.

1.2.6 Arithmetic For Limits

If we can avoid using the definition to find a limit, we should. There are certain rules we can follow to compute lots of sequence limits. Let's see them now!

THEOREM 1.2.20

(1) $\alpha > 0 \implies \lim_{n\to\infty} n^\alpha = \infty$.

(2) $\alpha < 0 \implies \lim_{n\to\infty} n^\alpha = 0$.

THEOREM 1.2.21: Arithmetic Rules for Limits of Sequences

Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\mathbb{R}$ and let $c \in \mathbb{R}$. Suppose that $\{a_n\}$ and $\{b_n\}$ both converge with $a_n \to L$ and $b_n \to M$. Then

(1) *if $a_n = C$ for all n, then $C = L$,*

(2) $\lim_{n\to\infty} ca_n = cL$,

(3) $\lim_{n\to\infty} (a_n + b_n) = L + M$,

(4) $\lim_{n\to\infty} (a_n - b_n) = L - M$,

(5) $\lim_{n\to\infty} a_n b_n = LM$,

(6) *if $M \neq 0$,* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$,

(7) *for all $k \in \mathbb{N}$,* $\lim_{n\to\infty} a_{n+k} = L$.

Proof: Exercises, but let's prove (3) as an example. Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} a_n = L$, we can find $N_1 \in \mathbb{N}$ so that if $n \geq N_1$, we get $|a_n - L| < \varepsilon/2$. Also, since $\lim_{n\to\infty} b_n = M$, we can find $N_2 \in \mathbb{N}$ so that if $n \geq N_2$, we have $|b_n - M| < \varepsilon/2$. Now, let $N = \max(N_1, N_2)$. Then, if $n \geq N$ we get

$$|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_n - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

where we used the triangle inequality in the first inequality.

REMARK 1.2.22

To use any of the above properties, the limits need to exist!

EXAMPLE 1.2.23

(1) $\displaystyle \lim_{n\to\infty} \frac{3n+7}{n+2} = \lim_{n\to\infty} \frac{3+7/n}{1+2/n} = \frac{\lim_{n\to\infty} 3 + \lim_{n\to\infty} 7/n}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} 2/n} = \frac{3+0}{1+0} = 3.$

(2) $\displaystyle \lim_{n\to\infty} \frac{n^3+n^2+1}{2n^3+7n^2-1} = \lim_{n\to\infty} \frac{1+1/n+1/n^3}{2+7/n-1/n^3} = \frac{1+0+0}{2+0+0} = \frac{1}{2}.$

(3) $\displaystyle \lim_{n\to\infty} \frac{n+1}{n^2+1} = \lim_{n\to\infty} \frac{1/n+1/n^2}{1+1/n^2} = \frac{0+0}{1+0} = 0.$

REMARK 1.2.24

You don't need to write "arithmetic rules" every time, as we <u>always</u> use them! Just make sure you show your work!

EXAMPLE 1.2.25

What if in property (5), $M = 0$? Anything can happen!

- $\displaystyle \lim_{n\to\infty} \frac{1/n}{1/n} = 1$ even though $1/n \to 0$.
- $\displaystyle \lim_{n\to\infty} \frac{1/n}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{n} = \lim_{n\to\infty} n = \infty.$

- $\lim_{n\to\infty} \frac{1/n^2}{1/n} = \lim_{n\to\infty} \frac{1}{n} = 0.$

Hence, we will need to handle these on an individual basis.

However, there is one thing we can say.

THEOREM 1.2.26

If $\lim_{n\to\infty} b_n = 0$ *and* $\lim_{n\to\infty} \frac{a_n}{b_n}$ *exists, then* $\lim_{n\to\infty} a_n = 0$.

Proof: Suppose $\lim_{n\to\infty} b_n = 0$, and say $\lim_{n\to\infty} \frac{a_n}{b_n} = k \in \mathbb{R}$, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{a_n}{b_n} = k \cdot 0 = 0.$$

COROLLARY 1.2.27

If $\lim_{n\to\infty} b_n = 0$ *and* $\lim_{n\to\infty} a_n \neq 0$*, then* $\lim_{n\to\infty} \frac{a_n}{b_n}$ *does not exist.*

EXAMPLE 1.2.28

$$\lim_{n\to\infty} \frac{n^3+3n}{n^2+1} = \lim_{n\to\infty} \frac{1+3/n^2}{1/n+1/n^3}.$$

However, the numerator converges to 1, while the denominator converges to 0. Therefore, the limit does not exist.

We could also say

$$\lim_{n\to\infty} \frac{n^3+3n}{n^2+1} = \infty,$$

which means DNE <u>and</u> infinitely large!

Let's compute the limit of any ratios of powers of n.

PROPOSITION 1.2.29

$$\begin{aligned}\lim_{n\to\infty} \frac{b_0+b_1n+b_2n^2+\cdots+b_jn^j}{c_0+c_1n+c_2n^2+\cdots+c_kn^k} &= \lim_{n\to\infty} \frac{n^j}{n^k}\left[\frac{\frac{b_0}{n^j}+\frac{b_1}{n^{j-1}}+\cdots+b_j}{\frac{c_0}{n^k}+\frac{c_1}{n^{k-1}}+\cdots+c_k}\right] \\ &= \begin{cases} \frac{b_j}{c_k}, & j=k, \\ 0, & j<k, \\ \infty, & j>k \wedge b_j/c_k>0, \\ -\infty, & j>k \wedge b_k/c_k<0. \end{cases}\end{aligned}$$

EXAMPLE 1.2.30

- $$\lim_{n\to\infty} \frac{3n+2}{2n-1} = \frac{3}{2}.$$
- $$\lim_{n\to\infty} \frac{4n^2+5n}{n^3-1} = 0.$$

- $$\lim_{n\to\infty} \frac{7-n^4}{1+n^3} = -\infty.$$

REMARK 1.2.31

Still show work when writing solutions on a test though (e.g., dividing by highest power of n).

EXAMPLE 1.2.32

If we have something that “looks like” $\infty - \infty$, then multiply by the conjugate!

$$\begin{aligned}
\lim_{n\to\infty} \sqrt{n^2-4} - n &= \lim_{n\to\infty} \sqrt{n^2-4} - n\frac{\sqrt{n^2+4}+n}{\sqrt{n^2+4}+n} \\
&= \lim_{n\to\infty} \frac{n^2+4-n^2}{\sqrt{n^2+4}+n} \\
&= \lim_{n\to\infty} \frac{4}{\sqrt{n^2+4}+n} \\
&= \lim_{n\to\infty} \frac{4/n}{\sqrt{1+4/n^2}+1} \\
&= \frac{0}{2} \\
&= 0.
\end{aligned}$$

Recursive Sequence Limits

We will examine recursive sequences more closely in 1.4, but for now, if we know a recursive sequence converges, then we can use rule (7) to find the limit!

EXAMPLE 1.2.33

$a_1 = 2$, $a_{n+1} = \frac{5+a_n}{2}$. Suppose we know it has a limit, say $\lim\limits_{n\to\infty} a_n = L$. Then, using rule (7), we get:

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{5+a_n}{2} = \frac{5+L}{2}.$$

Therefore,

$$L = \frac{5+L}{2} \iff 2L = 5+L \iff L = 5.$$

1.3 Squeeze Theorem

THEOREM 1.3.1: Squeeze Theorem for Sequences of Real Numbers

Let $\langle a_n\rangle$, $\langle b_n\rangle$, and $\langle c_n\rangle$ be sequences in $\mathbb{R}$.
Let $\lim\limits_{n\to\infty} a_n = L = \lim\limits_{n\to\infty} c_n$.
Suppose that

$$\forall n \in \mathbb{N} : a_n \le b_n \le c_n$$

Then:

$$\lim_{n\to\infty} b_n = L.$$

Proof: Let $\varepsilon > 0$ be given. Since $a_n \to L$ and $c_n \to L$, we can find $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in (L-\varepsilon, L+\varepsilon)$, and $c_n \in (L-\varepsilon, L+\varepsilon)$. Then, for $n \geq N$,

$$L - \varepsilon \leq a_n \leq b_n \leq c_n \leq L + \varepsilon,$$

so $b_n \in (L-\varepsilon, L+\varepsilon)$, which means $\lim_{n\to\infty} b_n = L$.

REMARK 1.3.2

The Squeeze Theorem is great for dealing with $\sin / \cos$ and $(-1)^n$.

EXAMPLE 1.3.3

Compute the following limits.

(1) $\lim_{n\to\infty} \frac{(-1)^n}{n^2+1}$.

(2) $\lim_{n\to\infty} \frac{\cos(n^2+7)+7}{n}$.

Solution.

(1) Notice that $\frac{-1}{n^2+1} \leq \frac{(-1)^n}{n^2+1} \leq \frac{1}{n^2+1}$ and $\lim_{n\to\infty} \frac{-1}{n^2+1} = 0 = \lim_{n\to\infty} \frac{1}{n^2+1}$, so $\lim_{n\to\infty} \frac{(-1)^n}{n^2+1} = 0$ by the Squeeze Theorem.

(2) Notice that $\frac{6}{n} \leq \frac{\cos(n^2+7)+7}{n} \leq \frac{8}{n}$, and since $\lim_{n\to\infty} \frac{6}{n} = 0 = \lim_{n\to\infty} \frac{8}{n}$, we get $\lim_{n\to\infty} \frac{\cos(n^2+7)+7}{n} = 0$ by the Squeeze Theorem.

1.4 Monotone Convergence Theorem

First, we need to some terminology.

DEFINITION 1.4.1: Upper Bound of Set

Let $S \subseteq \mathbb{R}$.

An **upper bound for S (in $\mathbb{R}$)** is an element $\alpha \in \mathbb{R}$ such that

$$\forall x \in S : x \leq \alpha.$$

We say that S is **bounded above (in $\mathbb{R}$)** if and only if S admits an upper bound (in $\mathbb{R}$).

A **lower bound for S (in $\mathbb{R}$)** is an element $\beta \in \mathbb{R}$ such that

$$\forall x \in S : x \geq \beta.$$

We say that S is **bounded below (in $\mathbb{R}$)** if and only if S admits a lower bound (in $\mathbb{R}$).

We say that S **bounded (in $\mathbb{R}$)** if and only if it is bounded below and bounded above (in $\mathbb{R}$). In this case,

$$\exists M \in \mathbb{R} : \forall x \in S : |x| \leq M.$$

EXAMPLE 1.4.2

If $S = (-1, 1)$, then 7 is an upper bound and -12 is a lower bound, so S is bounded. Another example is $S \subseteq [-5, 5]$.

DEFINITION 1.4.3

Let $S \subseteq \mathbb{R}$. α is called the **least upper bound** of S if:
(i) α is an upper bound, and
(ii) α is the smallest, that is, if α' is another upper bound, then $\alpha' \geq \alpha$.
Denote this by $\alpha = \text{lub}(S)$ or $\alpha = \sup(S)$.
Similarly, β is the **greatest lower bound** if
(i) β is a lower bound, and
(ii) β is the largest, that is, if β' is another lower bound, then $\beta' \leq \beta$.
Denote this by $\beta = \text{glb}(S)$ or $\beta = \inf(S)$.

EXAMPLE 1.4.4

If $S = (-1, 1)$, then $\inf(S) = -1$ and $\sup(S) = 1$.

REMARK 1.4.5

The $\inf(S)$ and $\sup(S)$ may or may not be in S. One of the properties (axioms) of $\mathbb{R}$ guarantees the existence of $\inf$ and $\sup$. If $S \subseteq \mathbb{R}$ is non-empty and bounded above (below), then S has $\sup$ ($\inf$).

DEFINITION 1.4.6

We say that a sequence $\{a_n\}$ is:
- **increasing** if $a_n < a_{n+1}$,
- **non-decreasing** if $a_n \leq a_{n+1}$,
- **decreasing** if $a_n > a_{n+1}$,
- **non-increasing** if $a_n \geq a_{n+1}$,
- **monotonic** if $\{a_n\}$ is either non-decreasing or non-increasing.

Now, we can state the theorem!

THEOREM 1.4.7: Monotone Convergence Theorem (MCT)

Let $\{a_n\}$ be a non-decreasing (non-increasing) sequence.
(1) If $\{a_n\}$ is bounded above (below), then $\{a_n\}$ converges to $L = \text{lub}(\{a_n\})$ ($L = \text{glb}(\{a_n\})$).
(2) If $\{a_n\}$ is not bounded above (below), then $\{a_n\}$ diverges to ∞ ($-\infty$).

Proof: We will prove the non-decreasing/bounded above case, the other case is similar. Suppose $\{a_n\}$ is non-decreasing.

(1) Suppose $\{a_n\}$ is bounded above and let $L = \text{lub}(\{a_n\})$. Let $\varepsilon > 0$ be given. Then, $L - \varepsilon < L$, which means that $L - \varepsilon$ is <u>not</u> an upper bound of $\{a_n\}$ (L is the <u>least</u> upper bound). So, there exists $N \in \mathbb{N}$ so that $L - \varepsilon < a_N$. Then, if $n \geq N$, we have $L - \varepsilon < a_N \leq a_n$ since the sequence is non-decreasing. Therefore, for $n \geq N$, $L - \varepsilon < a_n \leq L < L + \varepsilon$, so the tail of $\{a_n\}$ is in $(L - \varepsilon, L + \varepsilon)$, which means $\lim\limits_{n\to\infty} a_n = L$.

(2) Suppose $\{a_n\}$ is not bounded above. Let $M \in \mathbb{R}$ be given. We can find $N \in \mathbb{N}$ so that $M < a_N$. Then, if $n \geq N$, we have $M < a_N < a_n$ ($\{a_n\}$ is non-decreasing). This shows $\lim\limits_{n\to\infty} a_n = \infty$.

Introduction to Mathematical Induction

Before we can use the MCT, we need to develop one proof technique: Mathematical Induction (MATH 135 will explore it further). Induction is a proof technique that allows us to prove an infinite number of statements. Say we have statements $P_1, P_2, P_3, \ldots, P_n, \ldots$ for $n \in \mathbb{N}$. If we can:

(1) Prove P_1 is true (base case).

(2) Prove: if P_k is true for some k (inductive hypothesis), then P_{k+1} is true (inductive step).

Then, we can conclude that P_n is true for all $n \in \mathbb{N}$. Think of dominoes!

We will use the MCT and induction to find the limits of recursive sequences. To do this, we follow these steps:

(1) Prove the sequence is monotonic.

(2) Prove the sequence is bounded (above or below).

(3) Conclude the sequence converges by MCT.

(4) Find the limit using an earlier trick:
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}.$$

Note that the order matters! We can't perform step 4 unless we know that the sequence converges.

EXAMPLE 1.4.8

Let $a_1 = 1$, $a_{n+1} = \frac{3+a_n}{2}$ for $n \geq 1$. Prove the sequence converges and find its limit.

Solution.

(1) Let's check a few terms: $a_1 = 1$, $a_2 = 2$, $a_3 = 5/2$, so it looks like the sequence is non-decreasing. Claim: $a_n \leq a_{n+1}$ for all $n \in N$.
- Base Case: Is $a_1 \leq a_2$? Yes, since $a_1 = 1 \leq 2 = a_2$.
- Inductive Hypothesis: Suppose $a_k \leq a_{k+1}$ for some $k \geq 1$.
- Inductive Step: Since $a_k \leq a_{k+1}$, $3 + a_k \leq 3 + a_{k+1}$, which means
$$\frac{3+a_k}{2} \leq \frac{3+a_{k+1}}{2},$$
that is, $a_{k+1} \leq a_{k+2}$.

Therefore, the sequence is non-decreasing by induction.

(2) What upper bound should we use? Don't try to guess the lub at this point, any upper bound will do!
Claim: $a_n \leq 5$ for all $n \in \mathbb{N}$.
- Base Case: $a_1 = 1 \leq 5$.
- Inductive Hypothesis: Suppose $a_k \leq 5$ for some $k \in \mathbb{N}$.
- Inductive Step: Since $a_k \leq 5$, $3 + a_k \leq 8$, so $\frac{3+a_k}{2} \leq 4$. Therefore, $a_{k+1} \leq 4 \leq 5$.

Therefore, $a_n \leq 5$ for all $n \in \mathbb{N}$ by induction, so the sequence is bounded above.

(3) Since $\{a_n\}$ is bounded above and non-decreasing, we know $\{a_n\}$ converges by MCT.

(4) Now, we know a limit exists, say $L = \lim\limits_{n\to\infty} a_n$. Then,
$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{3+a_n}{2} = \frac{3+L}{2}.$$
So,
$$L = \frac{3+L}{2} \iff 2L + 3L \iff L = 3.$$
Therefore, $\lim\limits_{n\to\infty} a_n = 3$.

EXAMPLE 1.4.9

Let $a_1 = 2$, $a_{n+1} = 7 + a_n$ for $n \geq 1$. Prove the sequence converges and find its limit.

Solution. Let's check a few terms: $a_1 = 2$, $a_2 = 3$, $a_3 = \sqrt{10}$, so it looks like the sequence is non-decreasing. Let's prove bounded above and non-decreasing in one step!

(1) Claim: $a_n \leq a_{n+1} \leq 9$ for all $n \in \mathbb{N}$.
   - Base Case: $a_1 = 2 \leq 3 = a_2$ and $a_2 = 3 \leq 9$, so $a_1 \leq a_2 \leq 9$.
   - Inductive Hypothesis: Assume $a_k \leq a_{k+1} \leq 9$ for some $k \in \mathbb{N}$.
   - Inductive Step: Then,

$$\begin{aligned} &a_k \leq a_{k+1} \leq 9 \\ &\implies 7 + a_k \leq \sqrt{7 + a_{k+1}} \leq 4 \leq 9 \\ &\implies a_{k+1} \leq a_{k+2} \leq 9. \end{aligned}$$

   So, by induction $a_n \leq a_{n+1} \leq 9$ for all $n \in \mathbb{N}$.

(2) The sequence converges by the MCT.

(3) Finally, we need to find the limit. Say $L = \lim\limits_{n\to\infty} a_n$. Then,

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{7 + a_n} = \sqrt{7 + L};$$

   see A2Q6 for the last equality. So,

$$L = \sqrt{7 + L} \implies L^2 = 7 + L \implies L^2 - L - 7 = 0 \implies L = \frac{1 \pm \sqrt{29}}{2}.$$

   However, we know $L = \operatorname{lub}(\{a_n\})$ and $a_1 = 2$. So, $L \neq \frac{1-\sqrt{29}}{2}$ since $\frac{1-\sqrt{29}}{2} < 2$, that is, it isn't even an upper bound. Hence,

$$L = \frac{1 + \sqrt{29}}{2}.$$

Chapter 2

Limits and Continuity

2.1 Introduction to Function Limits

Let's examine $\lim_{x \to a} f(x) = L$ for $a, L \in \mathbb{R}$. Intuitively, this means that $f(x)$ gets infinitely close to L as x gets infinitely close to a (but $x \neq a$). Let's translate this into a more precise definition.

DEFINITION 2.1.1: Limit of Real Function

Let (a, b) be an open real interval.
Let $c \in (a, b)$.
Let $f \colon (a, b) \setminus \{c\} \to \mathbb{R}$.
Let $L \in \mathbb{R}$.
We say that the **limit** of $f(x)$ as x tends to c is equal to L, and we write $\lim_{x \to c} f(x) = L$ or we write $f(x) \to L$ as $x \to c$, when

$$\forall \varepsilon \in \mathbb{R}_{>0} : \exists \delta \in \mathbb{R}_{>0} : \forall x \in A : 0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon.$$

REMARK 2.1.2

(1) The limit is not affected by what happens at $x = a$.
(2) For the limit to exist, the function needs to approach L from both sides.

EXAMPLE 2.1.3

(1) Prove $\lim_{x \to 2} 5x + 1 = 11$.
(2) Prove $\lim_{x \to 5} x^2 = 25$.

Solution.
(1) Let $\varepsilon > 0$. Choose $\delta = \varepsilon/5$. If $0 < |x - 2| < \delta$, then

$$|f(x) - L| = |(5x + 1) - 11| = |5x - 10| = 5|x - 2| < 5\delta = \frac{5\varepsilon}{5} = \varepsilon.$$

(2) Let $\varepsilon > 0$. Choose $\delta = \min(1, \varepsilon/11)$. If $0 < |x - 5| < \delta$, then since $|x - 5| < \delta \leq 1$, we have $4 < x < 6$, so that $|x + 5| \leq |6 + 5| = 11$.

$$|x^2 - 25| = |(x - 5)(x + 5)| = |x - 5||x + 5| \leq 11|x - 5| < 11\delta \leq \frac{11\varepsilon}{11} = \varepsilon.$$

As before, it is tricky to work with the formal definition. We will strive to establish some better techniques!

REMARK 2.1.4: Some Comments

(1) For $\lim\limits_{x\to a} f(x)$ to exist, f must be defined in an open interval (α, β), containing a (except possibly at $x = a$).
(2) $f(a)$ does not affect $\lim\limits_{x\to a} f(x)$.
(3) If $f(x) = g(x)$, except possibly at $x = a$, then $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x)$.

2.2 Sequential Characterization of Limits

We define $\overline{\mathbb{R}} = R \cup \{-\infty, \infty\}$.

THEOREM 2.2.1: The Sequential Characterization of Limits of Functions

Let $A \subseteq \mathbb{R}$ be open, let $f\colon A \to \mathbb{R}$, let $L \in \overline{\mathbb{R}}$, and let $a \in A$ be a limit point of A.

$$\lim_{x\to a} f(x) = L$$

if and only if

for all real sequences $\{x_n\}$ in $A \setminus \{a\}$ with $x_n \to a$ we have $f(x_n) \to L$.

- ($\implies$) Suppose $\lim\limits_{x\to a} f(x) = L$. Let $\varepsilon > 0$. Since $\lim\limits_{x\to a} f(x) = L$, we can choose $\delta > 0$ so that $0 < |x - a| < \delta \implies |f(x) - b| < \varepsilon$. Since $x_n \to a$, we can choose $N \in \mathbb{N}$ so that $n \geq N \implies |x_n - a| < \delta$. Then for $n \geq N$, we have $|x_n - a| < \delta$ and we have $x_n \neq a$ (since $\{x_k\}$ is in the set $A \setminus \{a\}$) and hence $|f(x_n) - L| < \varepsilon$. This shows that $f(x_n) \to L$.
- ($\impliedby$) Tricky exercise to think about.

Since we know sequences can only have one limit, we immediately get the following theorem.

THEOREM 2.2.2: Uniqueness of Limits

Let $A \subseteq \mathbb{R}$, let a be a limit point, and let $f\colon A \to \mathbb{R}$. For $L, M \in \overline{\mathbb{R}}$, if $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} f(x) = M$, then $L = M$. Similar results hold for limits $x \to a^{\pm}$ and $x \to \pm\infty$.

The sequential characterization can help us prove a limit does not exist.

Strategy [Showing Limits Do Not Exist]

1. Find a sequence $\{x_n\}$ with $x_n \to a$, $x_n \neq a$ for which $\lim\limits_{n\to\infty} f(x_n)$ does not exist.
2. Find two sequences (x_n) and (y_n) with $x_n \to a$, $y_n \to a$, $x_n, y_n \neq a$ for which $\lim\limits_{n\to\infty} f(x_n) \neq \lim\limits_{n\to\infty} f(y_n) = M$

EXAMPLE 2.2.3

Prove that $\lim\limits_{x\to 0} \dfrac{|x|}{x}$ does not exist.

Solution. Let $x_n = 1/n$ and $y_n = -1/n$. Clearly, $x_n \to 0$, $y_n \to 0$, and $x_n, y_n \neq 0$. Since $x_n > 0$ and

$y_n < 0$ for all n, we have $|x_n|/x_n = 1$ and $|y_n|/y_n = -1$ for all n. Therefore,

$$\lim_{n\to\infty} \frac{|x_n|}{x_n} = 1 \neq -1 = \lim_{n\to\infty} \frac{|y_n|}{y_n}.$$

Therefore, $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

2.3 Arithmetic Rules for limits of functions

THEOREM 2.3.1: Combination Theorem for Limits of Functions

Let f, g be real functions defined on an open subset $A \subseteq \mathbb{R}$, except possibly at a point $a \in A$. Let $\lim_{x\to a} f(x) = L \in \mathbb{R}$ and $\lim_{x\to a} g(x) = M \in \mathbb{R}$.

(1) $\forall x \in \mathbb{R} : f(x) = c \implies L = c$.

(2) ***Multiple Rule.***

$$\lim_{x\to a} cf(x) = cL.$$

(3) ***Sum Rule.***

$$\lim_{x\to a} f(x) + g(x) = L + M.$$

(4) ***Product Rule.***

$$\lim_{x\to a} f(x)g(x) = LM.$$

(5) ***Quotient Rule.***

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M} \text{ provided that } M \neq 0.$$

(6) ***Power Rule.***

$$\forall \alpha > 0 : \lim_{x\to a} (f(x))^{\alpha} = L^{\alpha}.$$

(7) If $M = 0$, and $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists, then $L = 0$.

THEOREM 2.3.2: Limits of Polynomials

If $p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_n x^n$ is any polynomial*, then*

$$\lim_{x\to a} p(x) = p(a).$$

Proof: Exercise.

Limits of Rational Functions

Consider $\frac{P(x)}{Q(x)}$, where P, Q are polynomials.

- Case 1: If $Q(a) \neq 0$, then

$$\lim_{x\to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}.$$

- Case 2: If $\lim_{x\to a} Q(x) = 0$ and $\lim_{x\to a} P(x) \neq 0$, then

$$\lim_{x\to a} \frac{P(x)}{Q(x)}$$

does not exist.

- Case 3: If $\lim_{x\to a} Q(x) = Q(a) = 0 = P(a) = \lim_{x\to a} P(x)$, then $(x-a)$ is a factor of both $P(x)$ and $Q(x)$, so we can write $P(x) = (x-a)P^*(x)$ and $Q(x) = (x-a)Q^*(x)$. Therefore,
$$\lim_{x\to a} \frac{P(x)}{Q(x)} = \lim_{x\to a} \frac{(x-a)P^*(x)}{(x-a)Q^*(x)} = \lim_{x\to a} \frac{P^*(x)}{Q^*(x)}$$
and return to step 1!

EXAMPLE 2.3.3

(1) $\lim_{x\to 2} \dfrac{x^2-4}{x^2+3x+1} = \dfrac{0}{11} = 0.$

(2) $\lim_{x\to 2} \dfrac{x^2-4}{x^2-x-2} = \lim_{x\to 2} \dfrac{(x-2)(x+2)}{(x-2)(x+1)} = \lim_{x\to 2} \dfrac{x+2}{x+1} = \dfrac{4}{3}.$

2.4 One-Sided Limits

We may want to examine the behaviour of a function at a point but only from one side, instead of both sides at the same time. Let's see how to do that, and what the behaviour means for the overall limit.

DEFINITION 2.4.1: One-Sided Limits

Let $A = (a, b)$ be an open real interval, let $f\colon A \to \mathbb{R}$, and let $L \in \mathbb{R}$.

- *Limit from Right*.
$$\lim_{x\to a^+} f(x) = L \iff \forall \varepsilon \in \mathbb{R}_{>0} : \exists \delta \in \mathbb{R}_{>0} : \forall x \in A : a < x < a + \delta \implies |f(x) - L| < \varepsilon.$$

- *Limit from Left*.
$$\lim_{x\to b^-} f(x) = L \iff \forall \varepsilon \in \mathbb{R}_{>0} : \exists \delta \in \mathbb{R}_{>0} : \forall x \in A : b - \delta < x < b \implies |f(x) - L| < \varepsilon.$$

EXAMPLE 2.4.2

If $f(x) = \dfrac{|x|}{x} = \begin{cases} 1, & x > 0, \\ -1, & x < 0, \end{cases}$ then $\lim_{x\to 0^+} f(x) = 1$ and $\lim_{x\to 0^-} f(x) = -1$

EXAMPLE 2.4.3

If $f(x) = \begin{cases} -100, & x \le 0, \\ 0, & 0 < x \le 1, \\ x^2+1, & x > 1, \end{cases}$ then $\lim_{x\to 1^+} f(x) = 1^2 + 1 = 2$, $\lim_{x\to 1^-} f(x) = 0$, $\lim_{x\to 0^+} f(x) = 0$ and $\lim_{x\to 0^-} f(x) = -100$.

THEOREM 2.4.4: One-sided versus Two-sided Limits

Let A be a function defined on an open real interval, let $f\colon A \to \mathbb{R}$, and let $a \in A$.
$$\lim_{x\to a} f(x) \text{ exists and equals } L$$
if and only if both one-sided limits exist and
$$\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x).$$

REMARK 2.4.5

All arithmetic rules and sequential characterization hold for one-sided limits as well.

2.5 The Squeeze Theorem

There is an analogue of the Squeeze Theorem for Sequences for functions!

THEOREM 2.5.1: Squeeze Theorem

Let a be a point on an open real interval A, and let $f, g, h\colon A \to R$. If

$$\forall x \neq a \in A : g(x) \leq f(x) \leq h(x)$$
$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} f(x) = L.$$

EXAMPLE 2.5.2

Find the following limits.

(1) $\lim\limits_{x\to 0} x^2\cos(e^x+7)$.

(2) $\lim\limits_{x\to 0} \frac{\sin x}{x}$.

Solution.

(1) We know that $-1 \leq \cos(e^x+7) \leq 1$, so $-x^2 \leq \cos(e^x+7) \leq x^2$. Also, $\lim\limits_{x\to 0} -x^2 = 0 = \lim\limits_{x\to 0} x^2$, so by the Squeeze Theorem, $\lim\limits_{x\to 0} x^2\cos(e^x+7) = 0$

(2) If $0 < x < \pi/2$, then $\sin x \leq x \leq \tan x$, so $|\sin x| \leq |x| \leq |\tan x|$ if $-\pi/2 < x < \pi/2$. So,

$$1 \leq \frac{|x|}{|\sin x|} \leq \frac{|\tan x|}{|\sin x|} = \frac{1}{|\cos x|} \text{ if } -\frac{\pi}{2} < x < \frac{\pi}{2}, x \neq 0.$$

Therefore,

$$1 \geq \left|\frac{\sin x}{x}\right| \geq |\cos x|,$$

but $\dfrac{\sin x}{x} \geq 0$ and $\cos x > 0$ on $(-\pi/2, \pi/2)$, so

$$1 \geq \frac{\sin x}{x} \geq \cos x.$$

Also, $\lim\limits_{x\to 0} 1 = 1 = \lim\limits_{x\to 0} \cos x$, so by the Squeeze Theorem, $\lim\limits_{x\to 0} \frac{\sin x}{x} = 0$.

2.6 The Fundamental Trigonometric Limit

We have already seen that $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$. The proof relied on a geometric argument that $x \leq \tan x$ for $x \in (0, \pi/2)$. Let's look at another argument that uses areas! Proof that $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$:

$$\text{Area of small triangle} = \frac{1}{2}\sin(x)\cos(x).$$

$$\text{Area of pie piece} = \frac{x}{2\pi}\pi = \frac{x}{2}.$$
$$\text{Area of large triangle} = \frac{1}{2}\tan x.$$

So,
$$\frac{1}{2}\sin(x)\cos(x) \le \frac{x}{2} \le \frac{\tan x}{2} \implies \cos x \le \frac{x}{\sin x} \le \frac{1}{\cos x}.$$

So,
$$\frac{1}{\cos x} \ge \frac{\sin x}{x} \ge \cos x \text{ for } x \in (0, \pi/2).$$
$$\lim_{x\to 0} = 1 = \lim_{x\to 0}\cos x,$$

so by the Squeeze Theorem,
$$\lim_{x\to 0^+}\frac{\sin x}{x} = 1.$$

Similar arguments can show $\lim_{x\to 0^-}\frac{\sin x}{x} = 1$, so $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

Now that we have this limit, we can solve similar limits!

EXAMPLE 2.6.1

(1) $\lim_{x\to 0}\dfrac{\sin(5x)}{\sin(2x)} = \lim_{x\to 0}\dfrac{\sin(5x)}{5x}\dfrac{2x}{\sin(2x)}\dfrac{5x}{2x} = (1)(1)(5/2) = 5/2$, noting that $\lim_{x\to 0}\frac{\sin(ax)}{ax} = 1$ for $a \in \mathbb{R}$.

(2) $\lim_{x\to 0}\dfrac{\tan(3x)}{\sin(x)} = \lim_{x\to 0}\dfrac{x}{\sin x}\dfrac{\sin(3x)}{3x}\dfrac{1}{\cos(3x)}\dfrac{3x}{x} = (1)(1)(1)(3) = 3.$

EXERCISE 2.6.2

Let $a, b \in \mathbb{R}\setminus\{0\}$. Prove that

(i) $\lim_{x\to 0}\dfrac{\sin(ax)}{\sin(bx)} = \dfrac{a}{b}$,

(ii) $\lim_{x\to 0}\dfrac{\tan(ax)}{\tan(bx)} = \dfrac{a}{b}$, and

(iii) $\lim_{x\to 0}\dfrac{\tan(ax)}{\sin(bx)} = \dfrac{a}{b}$.

2.7 Limits at infinity and Asymptotes

We want to extend the concept of limit in two ways:

(1) Limits at infinity ($x \to \pm\infty$) $\to$ horizontal asymptotes,

(2) Infinite limits ($f(x) \to \pm\infty$) $\to$ vertical asymptotes.

Recall: When we say a limit $= \infty$, we mean it does not exist and gets infinitely large.

2.7.1 Asymptotes and Limits at Infinity

Let's mimic the definition of sequence limit to define a limit as $x \pm \infty$.

DEFINITION 2.7.1: Limit at Infinity

Let $f : \mathbb{R} \to \mathbb{R}$ be a real function.
Let $L \in \mathbb{R}$.

$$\lim_{x\to\infty} f(x) = L \iff \forall\varepsilon \in \mathbb{R}_{>0} : \exists N \in \mathbb{R} : \forall x > N \implies |f(x) - L| < \varepsilon.$$
$$\lim_{x\to-\infty} f(x) = L \iff \forall\varepsilon \in \mathbb{R}_{>0} : \exists N \in \mathbb{R} : \forall x < N \implies |f(x) - L| < \varepsilon.$$

EXAMPLE 2.7.2

$\lim\limits_{x\to\infty} e^{-x} = 0$ we can see that e^{-x} approaches 0 as x gets large.

We can see that $\lim\limits_{x\to\infty} f(x) = L$ means the graph of $f(x)$ approaches the line $y = L$ as x gets large. We have a name for such lines.

DEFINITION 2.7.3: Horizontal Asymptote

The horizontal line $y = L$ is a **horizontal asymptote** of the graph of a real function f if and only if either of the following limits exist:
$\lim\limits_{x\to\infty} f(x) = L_1$
$\lim\limits_{x\to-\infty} f(x) = L_2$

This will be useful when we explore curve sketching later in the course. We can also define what it means for $f(x)$ to diverge to $\pm\infty$ as $x \to \pm\infty$.

DEFINITION 2.7.4

$$\lim_{x\to\infty} f(x) = \infty \iff \forall M \in \mathbb{R}_{>0} : \exists N \in \mathbb{R} : \forall x \in A : x > N \implies f(x) > M.$$

Similarly, we can define $\lim\limits_{x\to\infty} f(x) = -\infty$ and $\lim\limits_{x\to-\infty} f(x) = \pm\infty$.

The Squeeze Theorem also still holds in these cases!

THEOREM 2.7.5

If $g(x) \leq f(x) \leq h(x)$ for all $x \geq N$ for some $N \in \mathbb{R}$, and if $\lim\limits_{x\to\infty} g(x) = L = \lim\limits_{x\to\infty} h(x)$, then $\lim\limits_{x\to\infty} f(x) = L$ as well. We can also let $x \to -\infty$ here also!

Let's do some examples!

EXAMPLE 2.7.6

(1) $\lim\limits_{x\to\infty} \dfrac{2x^2 - 3x + 7}{x^2 - 4x + 5} = \lim\limits_{x\to\infty} \dfrac{x^2(2 - 3/x + 7/x^2)}{x^2(1 - 4/x + 5/x^2)} = \dfrac{2}{1} = 2.$

(2) $\lim\limits_{x\to-\infty} \dfrac{x^2 + 2x + 1}{x - 7} = \lim\limits_{x\to-\infty} \dfrac{x + 2 + 1/x}{1 - 7/x} = -\infty$. In general, for $f(x) = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$,

$$\lim_{x\to\pm\infty} f(x) = \begin{cases} \frac{a_n}{b_m}, & n = m, \\ 0, & m > n, \\ \text{DNE}, & m < n. \end{cases}$$

(3) $\lim\limits_{x\to\infty} \dfrac{\sin(3x^2 + 7)}{x}$. Note that

$$-1 \leq \sin(3x^2 + 7) \leq 1 \implies -\frac{1}{x} \leq \frac{\sin(3x^2 + 7)}{x} \leq \frac{1}{x} \text{ for } x > 0.$$

Taking the limit of both sides as $x \to \infty$ yields $\lim_{x\to\infty} \frac{\sin(3x^2+7)}{x} = 0$ by the Squeeze Theorem.

EXERCISE 2.7.7

$$\lim_{x\to-\infty} \frac{\cos(3x+2)+2}{x^3+1}.$$

2.7.2 The Fundamental Log Limit

Our goal here is to use the Squeeze Theorem to prove that $\lim_{x\to\infty} \frac{\ln x}{x} = 0$. First, if we look at the graphs of $y = x$ and $y = \ln x$, we see that $\ln x < x$ for all $x > 0$. So,

$$\frac{\ln x}{x} \leq 1 \text{ for } x > 0.$$

Since $x \to \infty$, we may assume that $x \geq 1$. Then $\ln x \geq 0$, so we get

$$\frac{\ln x}{x} \geq 0.$$

For the upper bound, there's a trick!

$$0 \leq \frac{\ln x}{x} = \frac{\ln(\sqrt{x}^2)}{\sqrt{x}\sqrt{x}} = \frac{2}{\sqrt{x}} \frac{\ln(\sqrt{x})}{\sqrt{x}} \leq \frac{2}{\sqrt{x}} \text{ since } \frac{\ln(\sqrt{x})}{\sqrt{x}} \leq 1.$$

So,

$$0 \leq \frac{\ln x}{x} \leq \frac{2}{\sqrt{x}}$$

and applying the Squeeze Theorem yields the result. This tells us that x grows <u>much faster</u> than $\ln x$. What about other powers of x? Let's see!

EXAMPLE 2.7.8

$$\lim_{x\to\infty} \frac{\ln x}{x^{1/50}} = \lim_{x\to\infty} \frac{50\ln(x^{1/50})}{x^{1/50}} = (50)(0) = 0.$$

In fact,

$$\lim_{x\to\infty} \frac{\ln x}{x^p} = 0 \text{ for any } p > 0.$$

EXAMPLE 2.7.9

$$\lim_{x\to\infty} \frac{\ln(x^p)}{x} = \lim_{x\to\infty} \frac{p\ln x}{x} = (p)(0) = 0.$$

$$\lim_{x\to\infty} \frac{\ln(x^{100})}{\sqrt{x}} = \lim_{x\to\infty} \frac{100\ln x}{\sqrt{x}} = (100)(0) = 0.$$

What about exponential functions?

EXAMPLE 2.7.10

Let $p \in \mathbb{R}_{>0}$. Let $u = e^x$ so that $x = \ln u$ and

$$\lim_{x\to\infty} \frac{x^p}{e^x} = \lim_{u\to\infty} \frac{(\ln u)^p}{u} = \lim_{u\to\infty} \left(\frac{\ln u}{u^{1/p}}\right)^p = 0^p = 0.$$

We can also get results when $x \to 0^+$.

EXAMPLE 2.7.11

Let $u = 1/x$ or $x = 1/u$, so $x \to 0^+ \implies u \to \infty$ and

$$\lim_{x\to 0^+} x^p \ln x = \lim_{u\to\infty} \frac{\ln(1/u)}{u^p} = \lim_{u\to\infty} \frac{-\ln u}{u^p} = 0.$$

This shows that $x^p \to 0$ faster than $\ln x \to -\infty$. To summarize, $\ln x$ grows an order of magnitude slower than x^p, and x^p grows an order of magnitude slower than p^x. For $p > 0$, as $x \to \infty$, we can write

$$(\ln x)^p \ll x^p \ll p^x \ll x^x,$$

where $\ll$ is the **much less than** symbol.

2.7.3 Vertical Asymptotes and Infinite Limits

If we examine a function near a point, one or both sided limits could go to $\pm\infty$.

DEFINITION 2.7.12

$$\lim_{x\to a^+} f(x) = \infty \iff \forall M \in \mathbb{R}_{>0} : \exists \delta \in \mathbb{R}_{>0} : \forall x \in A : a < x < a + \delta \implies f(x) > M.$$
$$\lim_{x\to b^-} f(x) = \infty \iff \forall M \in \mathbb{R}_{>0} : \exists \delta \in \mathbb{R}_{>0} : \forall x \in A : b - \delta < x < b \implies f(x) > M.$$

Finally, we say $\lim_{x\to a} f(x) = \infty$ if $\lim_{x\to a^+} f(x) = \infty = \lim_{x\to a^-} f(x)$. If $\lim_{x\to a^\pm} = \pm\infty$, then we say the line $x = a$ is a **vertical asymptote** of f.

REMARK 2.7.13

Reminder: Saying $= \infty$ means the limit does not exist and gets infinitely large.

EXAMPLE 2.7.14

(1) $\lim_{x\to 1^-} \dfrac{x^2+1}{x-1}$. We know it is $\pm\infty$ since it is of the form $\#/0$, but is it positive or negative? If $x \to 1^-$, then $x \to 1$ and $x < 1$ so $x^2 + 1 > 0$, $x - 1 < 0$, which means the whole function is negative. Therefore, the limit is $-\infty$.

(2) $\lim_{x\to 3^+} \dfrac{(x+1)(x-7)}{(x-3)(x-1)} = -\infty$. We can do a quick check "$\frac{(4)(-4)}{0^+(2)}$" is negative.

EXAMPLE 2.7.15

Find all vertical/horizontal asymptotes for $f(x) = \frac{x-3}{x+1}$.

Solution. Since $\lim_{x\to\pm\infty} \frac{x-3}{x+1} = 1$, f has a horizontal asymptote at $y = 1$. Also, $\lim_{x\to -1^+} \frac{x-3}{x+1} = -\infty$, so $x = -1$ is a vertical asymptote.

2.8 Continuity

DEFINITION 2.8.1: Continuity at a Point #1

f is continuous at $x = a$ if and only if the limit $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x) = f(a)$.

Otherwise, we say f is discontinuous at $x = a$ or that $x = a$ is a point of discontinuity for f.

Intuitively, a function is continuous at $x = a$ if its behaviour at $x = a$ is determined by its behaviour near $x = a$. We can also define continuity in terms of $\varepsilon - \delta$'s.

DEFINITION 2.8.2: Continuity at a Point #2

$$\forall \varepsilon \in \mathbb{R}_{>0} : \exists \delta \in \mathbb{R}_{>0} : |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon.$$

THEOREM 2.8.3: The Sequential Characterization of Continuity

Let $A \subseteq \mathbb{R}$, let $a \in A$, and let $f\colon A \to \mathbb{R}$. Then f is continuous at a if and only if for every sequence (x_n) in A with $x_n \to a$, we have $f(x_n) \to f(a)$.

REMARK 2.8.4: Useful Observation

When we look at $\lim_{x\to a} f(x)$ and assume $x \neq a$, we can write $x = a + h$ for some $h \in \mathbb{R} \setminus \{0\}$. Then $x \to a \iff h \to 0$. So we can say that f is continuous at $x = a$ if $\lim_{h\to 0} f(a + h) = f(a)$.

EXAMPLE 2.8.5

- Is $f(x) = \frac{x+1}{x-7}$ continuous at $x = 1$? Well,
$$\lim_{x\to 1} \frac{x+1}{x-7} = \frac{2}{-6} = \frac{-1}{3}$$
and $f(1) = 2/6 = -1/3$, so yes.
- Is $f(x) = |x|$ continuous at $x = 0$? Well,
$$\lim_{x\to 0^+} x = \lim_{x\to 0^+} x = 0,$$
$$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0,$$
so $\lim_{x\to 0} |x| = 0 = |0| = f(0)$, so yes.
- Is $f(x) = \frac{1}{x}$ continuous at $x = 0$? Well,
$$\lim_{x\to 0} \frac{1}{x}$$
does not exist, so no.

2.8.1 Continuity of Certain Functions

Let's look at some functions that we know are continuous.

- **Polynomials**. We already know that if P is a polynomial, then $\lim_{x\to a} P(x) = P(a)$, so polynomials are continuous at all $a \in \mathbb{R}$.

- $\sin x$. First, let's show that $\lim_{x\to 0}\sin x = \sin 0 = 0$. For $0 < x < \pi/2$, $0 < \sin x < x$. Since $\lim_{x\to 0^+} 0 = 0 = \lim_{x\to 0^+} x$, we have $\lim_{x\to 0}\sin x = 0$ by the Squeeze Theorem. Next, we know $\sin(-x) = -\sin x$, and if $x \to 0^-$, then $-x \to 0^+$, so

$$\lim_{x\to 0^-} = \lim_{x\to 0^-} -\sin(-x) = \lim_{-x\to 0^+} -\sin(-x) = (-1)(0) = 0.$$

  So we get $\lim_{x\to 0}\sin x = 0$.

- $\cos x$. $\lim_{x\to 0}\cos x = \lim_{x\to 0}\sqrt{1-\sin^2 x}$ for $x \in (-\pi/2, \pi/2) = \sqrt{1-0} = 1 = \cos 0$.

Therefore, both $\sin x$ and $\cos x$ are continuous at $x = 0$. Let $a \in \mathbb{R}$ be given. Let's prove that $\lim_{x\to a}\sin x = \sin a$.

$$\begin{aligned}\lim_{x\to a}\sin x &= \lim_{h\to 0}\sin(a+h)\\ &= \lim_{h\to 0}\sin(a)\cos(h) + \sin(h)\cos(a)\\ &= \sin(a)(1) + (0)\cos(a)\\ &= \sin(a).\end{aligned}$$

EXERCISE 2.8.6

Show that $\lim_{x\to a}\cos x = \cos a$.

- e^x. This one is surprisingly hard to prove! We would need more info about e^x, like Power/Taylor series from MATH 138, but we can do it with the following.

  Fact: e^x is continuous at $x = 0$, i.e., $\lim_{x\to 0} e^x = 1$.

  Claim: For all $a \in \mathbb{R}$, $\lim_{x\to a} e^x = e^a$.

  Proof: We know $\lim_{x\to 0} e^x = e^0 = 1$, so let $a \neq 0$ and

$$\lim_{x\to a} e^x = \lim_{h\to 0} e^{a+h} = \lim_{h\to 0} e^a e^h = (e^a)(1) = e^a.$$

- $\ln x$. To prove $\ln x$ is continuous on its domain, let's use a more general theorem.

  THEOREM 2.8.7

  If $f(x)$ is invertible, $f(a) = b$ and f is continuous at $x = a$, then f^{-1} is continuous at $x = b$.

  Proof Idea: To get the graph of $f^{-1}(x)$, we reflect the graph of $f(x)$ over the line $y = x$. So, if $f(x)$ is continuous, reflecting it won't create any discontinuities! So, we can conclude that $\ln x$ is continuous since it is the inverse of e^x.

2.8.2 Arithmetic Rules for Continuity

THEOREM 2.8.8: Operations on Continuous Functions

Let $A \subseteq \mathbb{R}$, let $f, g\colon A \to \mathbb{R}$, let $a \in A$, and let $c \in \mathbb{R}$. Suppose that f and g are continuous at a. Then the functions cf, $f+g$, $f-g$, and fg are all continuous at a, and f/g is continuous at a provided that $g(a) \neq 0$.

Proof: Easy consequences of the corresponding limit rules.

EXAMPLE 2.8.9

Consider $f(x) = \frac{x^2+x-2}{x^2-4x+3} = \frac{(x-1)(x+2)}{(x-1)(x-3)}$. All component functions are continuous, so the only possible discontinuities are at $x = 1$ and $x = 3$.
$x = 1$: $\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{(x-1)(x+2)}{(x-1)(x-3)} = \lim_{x\to 1} \frac{x+2}{x-3} = \frac{-3}{2}$ exists, but $f(1)$ does not exist, so f is not continuous at $x = 1$.
$x = 3$: $\lim_{x\to 3^+} = \infty$, so f is not continuous at $x = 3$. Therefore, f is continuous on $(-\infty, 1)\cup(1, 3)\cup(3, \infty)$.
If we defined $f(1) = -3/2$, then f would be continuous at $x = 1$.

THEOREM 2.8.10: Composition of Continuous Functions

Let $A, B \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, let $g : B \to \mathbb{R}$, and let $h = g \circ f : C \to \mathbb{R}$ where $C = A \cap f^{-1}(B)$.
(1) If f is continuous at $a \in C$ and g is continuous at $f(a)$, then h is continuous at a.
(2) If f is continuous (on A) and g is continuous (on B), then h is continuous (on C).

Proof: Note that (2) follows immediately from (1), so it suffices to prove (1). Suppose f is continuous at $a \in A$ and g is continuous at $b = f(a) \in B$. Let (x_n) be a sequence in C with $x_n \to a$. Since f is continuous at a, we have $f(x_n) \to f(a) = b$ by the Sequential Characterization of Continuity. Since $(f(x_n))$ is a sequence in B with $f(x_n) \to b$ and since g is continuous at b, we have $g(f(x_n)) \to g(b)$ by the Sequential Characterization of Continuity. Thus, we have $h(x_n) = g(f(x_n)) \to g(b) = g(f(a)) = h(a)$. We have shown that for every sequence (x_n) in C with $x_n \to a$ we have $h(x_n) \to h(a)$. Thus, h is continuous at a by the Sequential Characterization of Continuity.

EXAMPLE 2.8.11

$\cos(e^{x^2})$ is continuous at each $a \in \mathbb{R}$ since x^2, e^x, and $\cos x$ are continuous by the Composition of Continuous Functions.

2.8.3 Continuity On An Interval

We should make it clear what we mean by 'continuous on an interval.' We will need to treat open and closed intervals separately.

DEFINITION 2.8.12: Continuity on an Interval (Open)

Let f be a real function defined on an open interval (a, b). f is **continuous on** (a, b) if and only if it is continuous at every point of (a, b).

What about closed intervals? The problem is that at the endpoints, f may not be defined outside!

EXAMPLE 2.8.13

$f(x) = \sqrt{x}$, the domain is $[0, \infty)$. Technically, $\lim_{x\to 0} \sqrt{x}$ does not exist since $\lim_{x\to 0^-} \sqrt{x}$ is not defined. But we would still like to say $\sqrt{x}$ is continuous at $x = 0$. Just ignore $x < 0$.

DEFINITION 2.8.14: Continuity on an Interval (Closed)

Let f be a real function defined on a closed interval $[a, b]$. f is **continuous on** $[a, b]$ if and only if it is:
(i) f is continuous on (a, b),
(ii) $\lim_{x\to a^+} f(x)$ exists and $\lim_{x\to a^+} f(x) = f(a)$, and
(iii) $\lim_{x\to b^-} f(x)$ exists and $\lim_{x\to b^-} f(x) = f(b)$.

In other words, we only consider continuity (and limits) as we approach from inside the interval in question. So, we can say that $\sqrt{x}$ is continuous on $[0, \infty)$.

2.8.4 Types of Discontinuities

Now that we know what it means for a function to be continuous, let's look at the various ways it can be discontinuous.

For $f(x)$ to be continuous at $x = a$, we need $\lim_{x\to a} f(x) = f(a)$. We classify four kinds of discontinuities.

(I) If $\lim_{x\to a} f(x)$ exists, but $\lim_{x\to a} f(x) \neq f(a)$, then we say that f has a **removable discontinuity**.

EXAMPLE 2.8.15

$f(x) = \begin{cases} x, & x \neq 1, \\ 3, & x = 1. \end{cases}$ $\lim_{x\to 1} f(x) = 1 \neq 3 = f(1)$.

REMARK 2.8.16

Called "removable" because we could re-define $f(x)$ at $x = a$ to equal the limit and "remove" the discontinuity. These are the least serious kinds of discontinuity.

(II) $\lim_{x\to a} f(x)$ does not exist, but both $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ exist (so are finite, but don't agree). Then we say that $f(x)$ has a **(finite) jump discontinuity**.

EXAMPLE 2.8.17

$f(x) = \begin{cases} x, & x \leq 0, \\ 3, & x > 0. \end{cases}$ $\lim_{x\to 0^+} f(x) = 3$, but $\lim_{x\to 0^-} f(x) = 0$, so $\lim_{x\to 0} f(x)$ does not exist. Therefore, $f(x)$ has a jump discontinuity at $x = 3$.

(III) If one or both of $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-}$ is $\pm\infty$, then we say that f has a **infinite discontinuity** at $x = a$.

EXAMPLE 2.8.18

$f(x) = \frac{1}{x}$. $\lim_{x\to 0^+} f(x) = \infty$, $\lim_{x\to 0^-} f(x) = -\infty$. So f has an infinite discontinuity at $x = 0$.

(IV) If $\lim_{x\to a} f(x)$ does not exist, but f is bounded near $x = a$ and is oscillating infinitely often near $x = a$, then f has an oscillatory discontinuity at $x = a$.

EXAMPLE 2.8.19

$f(x) = \sin(1/x)$. $\lim_{x\to 0} f(x)$ does not exist.

REMARK 2.8.20

Note that for types II, III, and IV, there is no easy way to get rid of the discontinuity by simply re-defining $f(a)$. So, they are **essential singularities** or **essential discontinuities**.

2.9 The Intermediate Value Theorem

One important tool we can use if we know a function is continuous is:

THEOREM 2.9.1: Intermediate Value Theorem (IVT)

Let $A = [a, b] \subseteq \mathbb{R}$ be a real interval, $f\colon A \to \mathbb{R}$ be continuous on A, and let $\alpha \in \mathbb{R}$ lie between $f(a)$ and $f(b)$. That is, either $f(a) < \alpha < f(b)$ or $f(b) < \alpha < f(a)$. Then $\exists c \in (a, b)$ such that $f(c) = \alpha$.

The proof is beyond the scope of the course, but it is intuitively clear! If f is above α at one point and below at another, then somewhere in between $f(x) = \alpha$, as long as f is "nice" (i.e., continuous).

EXAMPLE 2.9.2

Prove that $f(x) = x^5 - 2x^3 - 2$ has a root between 0 and 2.

Solution. Note that f is a polynomial, so it is continuous on $[0, 2]$. Also, $f(0) = -2 < 0$, $f(2) = 14 > 0$, so by the IVT, there exists $c \in (0, 2)$ such that $f(c) = 0$.

EXAMPLE 2.9.3

Prove that there exists a point $c \in (0, 1)$ such that $\cos(c) = c$.

Solution. Let's look at the function $f(x) = \cos x - x$ and prove it equals zero for some $c \in (0, 1)$. First, f is continuous since both $\cos x$ and x are. Also, $f(0) = \cos(0) - 0 = 1 > 0$, $f(1) = \cos(1) - 1 < 0$ since $\cos(1) < 1$. Therefore, by the IVT, there exists a point $c \in (0, 1)$ so that $f(c) = 0 \implies \cos(c) - c = 0 \implies \cos(c) = c$.

REMARK 2.9.4

The issue with the IVT is that it doesn't give us any indication of what c is! It also doesn't say that c is unique! However, we can use the IVT to estimate solutions.

2.9.1 Approximating Solutions to Equations

Let's start with polynomials!

- If $P(x)$ is a polynomial of degree 1, how can we solve $P(x) = 0$? Easy! $ax + b = 0 \implies x = -b/a$.
- Degree 2? Quadratic Formula!
- Degree 3 or 4? There are also formulas for these.
- Degree 5 or higher? No formula exists! But we can use the IVT to approximate solutions!

EXAMPLE 2.9.5

Recall we showed $P(x) = x^5 - 2x^3 - 2$ has a root in $(0, 2)$. Can we narrow it down further? Well, $P(1) = 1^5 - 2(1)^3 - 2 = -3 < 0$, so $P(2) > 0$, $P(1) < 0$, and so there is a root somewhere between $x = 1$ and $x = 2$.
Check the new midpoint! $P(3/2) = -37/32 < 0$, so there is a root between $x = 3/2$ and $x = 2$.
New midpoint is $7/4$, $P(7/4) - 3.694 > 0$, so the root is between $x = 3/2$ and $x = 7/4$. We could keep going or use a computer!
The method is great because each additional step cuts the potential error in half! Also, since $1/2^4 = 1/16 < 1/10$, every four iterations give us another decimal place of accuracy. $1/2^{10} < 1/1000$, so every 10 iterations gives 3 decimal places of accuracy.

REMARK 2.9.6

We can use this method on functions that aren't polynomials too! It is explored in the next section.

2.9.2 The Bisection Method

DEFINITION 2.9.7: Bisection Method

Let f be a real function such that:
f is continuous over a closed interval $[a, b]$
$f(a)$ and $f(b)$ are of opposite sign.

The **bisection method** is an iterative technique for finding an approximation to at least one solution to the equation $f(x) = 0$ to any desired accuracy.

So, we assume that $f(a)f(b) < 0$ and that $a < b$.
We evaluate $c = \dfrac{a+b}{2}$, thereby bisecting $[a, b]$.
We evaluate $f(c)$.
If $f(c) = 0$, then we have a solution to $f(x) = 0$.
Otherwise, $f(c)$ is of opposite sign to either $f(a)$ or $f(b)$.
If $f(c)$ is of opposite sign to $f(a)$, then there exists a solution to $f(x) = 0$ in $[a, c]$.
If $f(c)$ is of opposite sign to $f(b)$, then there exists a solution to $f(x) = 0$ in $[c, b]$.
In either case, a closed interval has been constructed of half the length of $[a, b]$.

This process can be repeated until the interval of interest is arbitrarily small, enabling the solution to be known to whatever accuracy is required.

REMARK 2.9.8

The bisection method is good, but later we will see Newton's Method which is more efficient.

2.10 The Extreme Value Theorem

It turns out that continuity on a closed interval is different from continuity on an open interval: we can say more about a function on a closed interval! But first, we need some definitions.

DEFINITION 2.10.1

Suppose $f : I \to \mathbb{R}$, where I is an interval.

- c is a **global maximum** for f on I if and only if
$$\exists c \in I : \forall x \in I : f(x) \leq f(c).$$
- c is a **global minimum** for f on I if and only if
$$\exists c \in I : \forall x \in I : f(x) \geq f(c).$$
- c is a **global extremum** for f on I if it is either a global maximum or a global minimum.

REMARK 2.10.2

Global max/mins are also called absolute max/mins.

- If f is defined on an interval I, does f achieve both its global max and global min?

- No! Consider $f(x) = x$ on $(0, 1)$. f has neither a global max nor a global min!

  The max/min look like they should be at $x = 1$ and $x = 0$, but these aren't in the interval!

EXAMPLE 2.10.3

$f(x) = x^2$ on $(-1, 1)$. f has a global min at $x = 0$, but no global max again! Okay, but let's include the endpoints! Is that enough? No, unfortunately.

EXAMPLE 2.10.4

$f(x) = \frac{1}{x}$ on $[-1, 1]$. No global max/min again! f goes to $\pm\infty$ as $x \to 0^{\pm}$.

So what conditions do we need to guarantee that f has a global max/min? It turns out that we need the interval to be <u>closed</u> and for f to be <u>continuous</u>.

THEOREM 2.10.5: Extreme Value Theorem for a Real Function (EVT)

Let f be a real function which is continuous in a closed real interval $[a, b]$.
Then:

$$\exists c_1, c_2 \in [a, b] : \forall x \in [a, b] : f(c_1) \leq f(x) \leq f(c_2).$$

The issue we face now is how to actually <u>find</u> the global extrema. The EVT doesn't tell us how! Also, as we saw in the $f(x) = x^2$ example, they aren't always at the endpoints.

We will return to this when we have more tools, in a few weeks.

Chapter 3

Derivatives

3.1 Instantaneous Velocity

Suppose you are driving down a highway. Every 30 minutes you record your distance:

Time (min)	0	30	60	90	120	150	180
Distance (km)	0	55	100	130	200	250	300

- What was your average speed in these three hours?

$$\text{Average speed} = \frac{\text{distance}}{\text{time}} = \frac{300\text{ km}}{1.5\text{ h}} = 100\text{ km/h}.$$

- First 1.5 hours?

$$\frac{130}{1.5} \approx 86.6\text{ km/h}.$$

- Last 1.5 hours?

$$\frac{300-130}{1.5} \approx 113\text{ km/h}.$$

In general, the formula for the **average velocity**, V_{ave} from $t = t_0$ to $t = t_1$ is

$$V_{\text{ave}} = \frac{s(t_1) - s(t_0)}{t_1 - t_0},$$

where $s(t)$ is the distance at time t. To get the instantaneous velocity, we need to use limits! The instantaneous velocity at $t = t_0$ is

$$\lim_{t \to t_0} \frac{s(t) - s(t_0)}{t - t_0}$$

or

$$\lim_{h \to 0} \frac{s(t_0 + h) - s(t_0)}{h}.$$

EXAMPLE 3.1.1

Find the instantaneous velocity for $s(t) = t^2 + 3t$ at $t = 1$, $t = 2$, and $t_0 \in \mathbb{R}$.

Solution.

- $$\begin{aligned}\lim_{h\to 0}\frac{s(1+h)-s(1)}{h} &= \lim_{h\to 0}\frac{(1+h)^2+3(1+h)-(1^2+3(1))}{h}\\ &= \lim_{h\to 0}\frac{5h+h^2}{h}\\ &= \lim_{h\to 0}(5+h)\\ &= 5.\end{aligned}$$
- $$\begin{aligned}\lim_{h\to 0}\frac{(2+h)^2+3(2+h)-(2^3+3(2))}{h} &= \lim_{h\to 0}\frac{7h+h^2}{h}\\ &= 7.\end{aligned}$$
- $$\begin{aligned}\lim_{h\to 0}\frac{(t_0+h)^2+3(t_0+h)-(t_0^2+3t_0)}{h} &= \lim_{h\to 0}(2t_0+3+h)\\ &= 2t_0+3.\end{aligned}$$

The instantaneous velocity is a special case of a derivative!

3.2 Definition of the Derivative

We can perform the same analysis that we did on $s(t)$ in the previous section on any function!

DEFINITION 3.2.1

The **average rate of change of** $f(x)$ from $x = a$ to $x = b$ is

$$f_{\text{ave}} = \frac{f(b)-f(a)}{b-a}.$$

DEFINITION 3.2.2

The **instantaneous rate of change of** $f(x)$ at $x = a$, or the derivative of $f(x)$ at $x = a$, denoted $f'(a)$ is defined as

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}.$$

If this limit exists, we say that f is **differentiable** at $x = a$.

3.2.1 The Tangent Line

DEFINITION 3.2.3

The **tangent line** to the graph of f at $x = a$ is the line passing through $(a, f(a))$ with slope $m = f'(a)$. It follows that the equation of the tangent line is

$$y = f(a) + f'(a)(x-a).$$

EXAMPLE 3.2.4

Find the equation of the tangent line to $f(x) = x^2 + x + 1$ at $x = 3$.

Solution. First, we should compute $f'(3)$:

$$\begin{aligned} f'(3) &= \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h\to 0} \frac{(3+h)^2 + (3+h) + 1 - (3^2 + 3 + 1)}{h} \\ &= \lim_{h\to 0} \frac{9 + 6h + h^2 + 3 + h + 1 - 9 - 3 - 1}{h} \\ &= \lim_{h\to 0} \frac{7h + h^2}{h} \\ &= \lim_{h\to 0} (7 + h) \\ &= 7. \end{aligned}$$

So, $f'(3) = 7$. The point on the graph is $(3, f(3)) = (3, 13)$. So, the tangent line is

$$y = 13 + 7(x - 3) = 13 + 7x - 21 = 7x - 8.$$

REMARK 3.2.5

Can't define the derivative as the slope of the tangent line! Without knowing what the derivative is first, we can't even define the tangent line!

3.2.2 Differentiability versus Continuity

- Q: Does continuity imply differentiability?
- A: No! Consider $f(x) = |x|$ at $x = 0$. Clearly,

  $$\lim_{x\to 0} |x| = 0 = |0|,$$

  so f is continuous at $x = 0$, but

  $$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

  does not exist. Therefore, f is not differentiable at $x = 0$. Therefore, continuity does not imply differentiability.
- Q: Does differentiability imply continuity?
- A: Yes!

THEOREM 3.2.6: Differentiability Implies Continuity

Let $A \subseteq \mathbb{R}$ be open, let $f\colon A \to \mathbb{R}$ and let $a \in A$. If f is differentiable at a, then f is continuous at a.

Proof: We have

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a}(x - a) \to f'(a) \cdot 0 = 0 \text{ as } x \to a$$

and so

$$f(x) = \big(f(x) - f(a)\big) + f(a) \to 0 + f(a) = f(a) \text{ as } x \to a.$$

This proves that f is continuous at a.

3.3 The Derivative Function

DEFINITION 3.3.1: The Derivative Function

We say that f is **differentiable** on an interval I if $f'(a)$ exists for each $a \in I$. In this case, we define the **derivative function** as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \ x \in I.$$

Alternative (Leibniz) notation:

$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(f),$$

where "$\frac{\mathrm{d}}{\mathrm{d}x}$" is called a **differential operator**.
If $y = f(x)$, write $\frac{\mathrm{d}y}{\mathrm{d}x}$. For $f'(a)$, write $\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=a}$.

Let's look at some examples!

EXAMPLE 3.3.2

For $f(x) = 7$, find $f'(x)$ for $x \in \mathbb{R}$.

Solution.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{7-7}{h} = 0.$$

Therefore, $f'(x) = 0$ for all $x \in \mathbb{R}$.

EXAMPLE 3.3.3

Find the equation of the tangent line to $f(x) = x^2 + 3x + 2$ at $x = 2$.

Solution. The tangent line passes through $(a, f(a)) = (2, f(2)) = (2, 12)$ since $f(2) = 2^2 + 3(2) + 2 = 12$. Next,

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 + 3(x+h) + 2 - x^2 - 3x - 2}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 3x + 3h - x^2 - 3x}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2 + 3h}{h} \\ &= \lim_{h \to 0} (2x + h + 3) \\ &= 2x + 3, \end{aligned}$$

which gives $f'(2) = 2(2) = 3 = 7$. Therefore, the tangent line to f at $x = 2$ is

$$y = f(2) + f'(2)(x-2) = 12 + 7(x-2) = 12 + 7x - 14 = 7x - 2.$$

REMARK 3.3.4

- Much faster than computing $f'(a)$ each time!
- We will soon learn ways to find $f'(x)$ <u>much faster</u>, but if asked to use the <u>definition</u>, then you

must use the formula

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

EXAMPLE 3.3.5

Using the definition, find $f'(x)$ where
(1) $f(x) = x$;
(2) $f(x) = x^2$;
(3) $f(x) = x^3$;
(4) $f(x) = \sqrt{x}$.

Solution.

(1) $$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{x+h-x}{h} \\ &= \lim_{h\to 0} \frac{h}{h} \\ &= 1. \end{aligned}$$

(2) $$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h\to 0} (2x+h) \\ &= 2x. \end{aligned}$$

(3) $$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h\to 0} (3x^2 + 3xh + h^2) \\ &= 3x^2. \end{aligned}$$

(4) $$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\ &= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}}. \end{aligned}$$

Higher-Order Derivatives

DEFINITION 3.3.6

If f is differentiable with derivative f' and f' is also differentiable, then we call $\frac{\mathrm{d}}{\mathrm{d}x}(f')$ the **second derivative** of f, denoted $f''(x)$ or $f^{(2)}(x)$, or $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$.
In general, $f^{(n+1)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}(f^{(n)}(x))$, where $f^{(n)}$ is the n^{th} derivative.

EXERCISE 3.3.7

Prove the following with the limit definition, where $f(x) = x^4$.

- $f'(x) = 4x^3$.
- $f''(x) = 12x^2$.
- $f^{(3)}(x) = 24x$.
- $f^{(4)} = 24$.
- $f^{(5)} = 0$.

Note that using the limit definition is very inefficient (not to mention awful and ugly). So, let's develop some rules to help us calculate derivatives more quickly!

3.4 Derivatives of Elementary Functions

Now that we know the definition of the derivative, let's work on finding derivatives of elementary functions to speed up the process.

- **Constants**: If $f(x) = c$ where $c \in \mathbb{R}$, then $f'(x) = 0$.
- **Lines**: If $f(x) = mx + b$ where $m, b \in \mathbb{R}$, then $f'(x) = m$.
- **Quadratics**: If $f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$ and $a \neq 0$, then $f'(x) = 2ax + b$.

3.4.1 The Derivative of $\sin x$ and $\cos x$

First, we need to prove a different claim:

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = 0.$$

$$
\begin{aligned}
\lim_{x\to 0}\frac{\cos x-1}{x}\cdot\frac{\cos x+1}{\cos x+1} &= \lim_{x\to 0}\frac{\cos^2 x-1}{x(\cos x+1)}\\
&= \lim_{x\to 0}\frac{-\sin^2 x}{x(\cos(x+1))}\\
&= \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{-\sin x}{\cos x+1}\\
&= 1\cdot 0\\
&= 0,
\end{aligned}
$$

using the fundamental trigonometry limit. Now, we can compute $(\sin x)'$.

$$
\begin{aligned}
(\sin x)' &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x\cos(h)+\cos x\sin(h)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin(h)}{h}\cos x+\left(\frac{\cos(h)-1}{h}\right)\sin x\\
&= 1\cdot\cos x+0\cdot\sin x\\
&= \cos x.
\end{aligned}
$$

EXERCISE 3.4.1

Show that $(\cos x)' = -\sin x$.

3.4.2 The Derivative of e^x

First, what is the number e? There are lots of ways to define it, for example: $\lim\limits_{x\to\infty}(1+\frac{1}{x})^x = e$ or $\sum_{n=0}^{\infty}\frac{1}{n!} = e$. But for us, we will define e to be the unique number $a \in \mathbb{R}$ such that the tangent line to a^x has slope 1 at $x = 0$. That is,

$$\lim_{h\to 0}\frac{e^h-e^0}{h} = 1 \implies \lim_{h\to 0}\frac{e^h-1}{h} = 1.$$

So, we get $(e^x)' = \lim\limits_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim\limits_{h\to 0} e^x(\frac{e^h-1}{h}) = e^x$. So, $(e^x)' = e^x$.

3.7 Arithmetic Rules for Differentiation

Now that we know how to find the derivatives of certain basic functions, let us look at some rules that tell us how to differentiate combinations of functions.

THEOREM 3.7.1: Arithmetic Rules for Differentiation

Suppose f and g are differentiable at $x = a$.

(1) ***Constant Multiple Rule***. *Let $h(x) = cf(x)$. Then h is differentiable at $x = a$ and*

$$h'(a) = cf'(a).$$

(2) ***Sum Rule***. *Let $h(x) = f(x) + g(x)$. Then h is differentiable at $x = a$ and*

$$h'(a) = f'(a) + g'(a).$$

(3) ***Product Rule***. *Let $h(x) = f(x)g(x)$. Then h is differentiable at $x = a$ and*

$$h'(a) = f'(a)g(a) + f(a)g'(a).$$

(4) ***Reciprocal Rule***. *Let $h(x) = \frac{1}{g(x)}$. If $g(a) \neq 0$, then h is differentiable at $x = a$ and*

$$h'(a) = -\frac{g'(a)}{[g(a)]^2}.$$

(5) ***Quotient Rule***: *Let $h(x) = \frac{f(x)}{g(x)}$. If $g(a) \neq 0$, then h is differentiable at $x = a$ and*

$$h'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}.$$

Proof:

(1) Easy exercise.

(2) Easy exercise.

(3)
$$\begin{aligned}(fg)'(a) &= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h} \\ &= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a+h)g(a) + f(a+h)g(a) - f(a)g(a)}{h} \\ &= \lim_{h\to 0} f(a+h)\frac{g(a+h) - g(a)}{h} + g(a)\frac{f(a+h) - f(a)}{h} \\ &= f(a)g'(a) + g(a)f'(a).\end{aligned}$$

(4)
$$\begin{aligned}\left(\frac{1}{f}\right)'(a) &= \lim_{h\to 0} \frac{\frac{1}{f(a+h)} - \frac{1}{f(a)}}{h} \\ &= \lim_{h\to 0} \frac{f(a) - f(a+h)}{hf(a+h)f(a)} \\ &= \lim_{h\to 0} \frac{-(f(a+h) - f(a))}{h}\frac{1}{f(a+h)f(a)} \\ &= \frac{-f'(a)}{[f(a)]^2}.\end{aligned}$$

(5) We can combine the product and reciprocal rules!
$$\begin{aligned}\left(\frac{f}{g}\right)'(a) &= \left(f\frac{1}{g}\right)'(a) \\ &= f'(a)\frac{1}{g(a)} + f(a)\left(\frac{1}{g}\right)'(a) \\ &= \frac{f'(a)}{g(a)} - \frac{f(a)g'(a)}{[g(a)]^2} \\ &= \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}.\end{aligned}$$

THEOREM 3.7.2: The Power Rule for Differentiation

Assume that $\alpha \in \mathbb{R}$, $\alpha \neq 0$, and $f(x) = x^\alpha$. Then f is differentiable and

$$f'(a) = \alpha x^{\alpha - 1},$$

where $x^{\alpha-1}$ is defined.

In general, the proof is difficult. If $\alpha \in \mathbb{N}$, then it is a simple application of the Binomial Theorem. For $\alpha \in \mathbf{Q}$, it is possible with more tools (chain rule and inverse function theorem). But for general $\alpha \in \mathbb{R}$, we would need more tools, and it outside the scope of this course. So, we omit the proof. Let's look at some examples!

EXAMPLE 3.7.3

(1) $f(x) = x^2 \sin x$.

$$f'(x) = (x^2)' \sin x + x^2 (\sin x)' = 2x \sin x + x^2 \cos x.$$

(2) $f(x) = \frac{x^4-1}{x-7}$.

$$f'(x) = \frac{(x-7)(x^4+1)' - (x^4+1)(x-7)'}{(x-7)^2} = \frac{(x-7)(4x^3) - (x^4+1)(1)}{(x-7)^2}.$$

(3) $f(x) = \sec x = \frac{1}{\cos x}$.

$$f'(x) = \frac{-(\cos x)'}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x}\frac{1}{\cos x} = \tan x \sec x.$$

(4) $f(x) = e^x \cos x$.

$$f'(x) = e^x \cos x - e^x \sin x.$$

(5) $f(x) = 3x^5$.

$$f'(x) = 15x^4,\ f''(x) = 60x^3,\ f^{(3)}(x) = 180x^2,\ f^{(4)}(x) = 360x,\ f^{(5)}(x) = 360,\ f^{(\geq 6)}(x) = 0.$$

3.8 The Chain Rule

THEOREM 3.8.1: Chain Rule

Let $A, B \subseteq \mathbb{R}$ be open, let $f\colon A \to \mathbb{R}$, let $g\colon B \to \mathbb{R}$, and let $h = g \circ f\colon C \to \mathbb{R}$, where $C = A \cap f^{-1}(B)$. Let $a \in C$ and let $b = f(a) \in B$. Suppose that f is differentiable at a and g is differentiable at b. Then h is differentiable at a with

$$h'(a) = g'(f(a))f'(a).$$

In Leibniz notation, if $z = g(y)$ and $y = f(x)$, then

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

The proof is quite involved, for a geometric argument see the course notes.

COROLLARY 3.8.2: Generalized Power Rule

If $g(x) = f(x)^\alpha$ for $\alpha \in \mathbb{R} \setminus \{0\}$, then

$$g'(x) = \alpha f(x)^{\alpha-1} f'(x).$$

EXAMPLE 3.8.3

Find $f'(x)$.

(1) $f(x) = (3x^2 + 2x + 7)^{19}$.
(2) $f(x) = \sin(e^x + x^e)$.
(3) $f(x) = e^{\sin(x^2)}$.

Solution.

(1) $f'(x) = 38(3x+1)(3x^2+2x+7)^{18}$.
(2) $f'(x) = \cos(e^x + x^e)(e^x + exe^{e-1})$.

(3) $f'(x) = e^{\sin(x^2)}(\sin(x^2))' = e^{\sin(x^2)}\cos(x^2)(x^2)' = e^{\sin(x^2)}\cos(x^2)(2x)$.

Also, with the chain rule and the derivative of e^x, we can get the derivative of a^x for $a > 0$.

$$a^x = e^{x\ln(a)} \implies (a^x)' = (e^{x\ln(a)})' = e^{x\ln(a)}(x\ln(a))' = a^x\ln(a).$$

EXAMPLE 3.8.4

$f(x) = 2^{3x} + 5^{\cos x}$. $f'(x) = 2^{3x}\ln(2)(3) + 5^{\cos x}\ln(5)(-\sin x)$.

3.9 Derivatives of Other Trigonometric Functions

So far, we've seen:

$$\begin{aligned}(\sin x)' &= \cos x\\ (\cos x)' &= -\sin x\\ (\sec x)' &= \sec x\tan x.\end{aligned}$$

EXAMPLE 3.9.1

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x(\sin x)' - \sin x(\cos x)'}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$
$$= \frac{1}{\cos^2 x}$$
$$= \sec^2 x.$$

EXERCISE 3.9.2

Prove that $(\cot x)' = -\csc^2 x$ and $(\csc x)' = -\csc x\cot x$.

Recap:

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x\tan x$
$\csc x$	$-\csc x\cot x$

3.5 Tangent Lines and Linear Approximation

The main idea of this section is: general functions are hard to understand, while lines are easy to understand. So, let's develop a way to approximate a function with a line!

More precisely, for a differentiable function f, we want to find a linear function $h(x)$ so that $f(a) = h(a)$, $f'(a) = h'(a)$, and if x is close to a, then $f(x)$ is close to $h(x)$. How do we find $h(x)$? Well, if f is differentiable, then

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = f'(a).$$

So, if x is close to a, then
$$\frac{f(x)-f(a)}{x-a} \approx f'(a).$$
Solving for $f(x)$, we get
$$f(x) \approx f(a) + f'(a)(x-a).$$
Hence, let's define
$$l(x) = f(a) + f'(a)(x-a).$$
This is a good choice for $h(x)$. Note that $l(x)$ is the tangent line to $f(x)$ at $(a, f(a))$, which leads us to the following definition.

DEFINITION 3.5.1: Linearization, Tangent Line

When $f\colon A \to \mathbb{R}$ is differentiable at $x = a$ with derivative $f'(a)$, the function
$$l(x) = f(a) + f'(a)(x-a)$$
is called the **linearization (linear approximation)** of f at $x = a$. Note that the graph $y = l(x)$ of the linearization is the line through the point $(a, f(a))$ with slope $f'(a)$. This line is called the **tangent line** to the graph $y = f(x)$ at the point $(a, f(a))$.

EXAMPLE 3.5.2

For $f(x) = \sqrt{x}$, find the linearization at $x = 4$. Use this to approximate $\sqrt{3.98}$ and $\sqrt{4.05}$.

Solution. $f(4) = 2$, $f'(x) = \frac{1}{2\sqrt{x}}$, so $f'(4) = \frac{1}{4}$. Hence, the linearization of f at $x = 4$ is
$$l(x) = 2 + \frac{1}{4}(x-4) = \frac{x}{4} + 1.$$
Then, $\sqrt{3.98} \approx l(3.98) = 1 + 3.98/4 = 1.995$ and $\sqrt{4.05} \approx l(4.05) = 1 + 4.05/4 = 2.0125$. These values are fairly close to the "exact" values: $\sqrt{3.98} = 1.994993734326\ldots$ and $\sqrt{4.05} = 2.0124611797498106\ldots$.

Q: From the graph of $f(x)$, how can we tell if these are over- or under-estimates? They are overestimates since the line is above the graph (TODO image).

REMARK 3.5.3

Note that this is only a good approximation <u>nearby</u> $x = 4$. If we try to approximate $\sqrt{9}$, we get $\sqrt{9} = l(9) = 1 + 9/4 = 3.25$. The exact value is $\sqrt{9} = 3$ (obviously).

3.5.1 Error in Linear Approximation

Without an upper bound on the error, an approximation is useless! Note that
$$|\text{error}| = |f(x) - l(x)|,$$
i.e., the distance from $f(x)$ to $l(x)$.

- Q: What factors affect the size of the error?
- A: First, the farther we go from $x = a$, the larger the error gets! Also, how <u>curved</u> the graph is also affects it. Of course, if we don't fully understand $f(x)$, we can't calculate the error exactly, but we can approximate it! How do we quantify "more curved?" Well, we can say the slopes of the tangent lines are changing faster on the more curved graph.

  Hence, the rate of change of $f'(x)$ is measured by $f''(x)$, so $|f''(x)|$ being larger means a larger error.

THEOREM 3.5.4: The Error in Linear Approximation

Assume f is such that $|f''(x)| \leq M$ for each x in an interval I containing a point a. Then

$$\left|f(x) - l(x)\right| \leq \frac{M}{2}(x-a)^2$$

for each $x \in I$.

This is a special case of Taylor's Inequality which we will discuss later.

EXAMPLE 3.5.5

Find an upper bound on the error using $l(x)$ at $x = 4$ to approximate $\sqrt{x}$ on $[1, 6]$.

Solution. We know that $f'(x) = \frac{1}{2\sqrt{x}}$, so $f''(x) = -\frac{1}{4x^{3/2}}$. So, if $x \in [1, 6]$, we have

$$|f''(x)| = \left|-\frac{1}{4x^{3/2}}\right| \leq \frac{1}{4} = M.$$

Hence,

$$|\text{error}| = |l(x) - f(x)| \leq \frac{M}{2}(x-4)^2 \leq \frac{1}{8}(1-4)^2 = \frac{9}{8},$$

where we note that the maximum of $|x - 4|$ is 3, so we let $x = 1$ in the final inequality.

3.5.2 Applications of Linear Approximation

We will explore one application: estimating change. (Qualitative analysis is another that we will discuss later).

Suppose we are looking at $f(x)$ near $x = a$. We want to know how much it could change if we move to a point x_1 near $x = a$. That is, we want to know $\Delta y = f(x_1) - f(a)$ if we change the input by $\Delta x = x_1 - a$. Then, using $f(x) \approx l(x)$, we get

$$\Delta y = f(x_1) - f(a) \approx l(x_1) - f(a) = f'(a)(x_1 - a) = f'(a)\Delta x.$$

So, $\Delta y = f'(a)\Delta x$.

EXAMPLE 3.5.6

Suppose you are inflating a giant spherical balloon and it currently has a radius of 20cm. You exhale once and it goes up to 20.01m. Then, the change in volume would be

$$\Delta V = V'(20)\Delta r,$$

where $V(r) = \frac{4}{3}\pi r^3$. So, $V'(r) = 4\pi r^2$ and $V'(20) = 1600\pi$. Therefore,

$$\Delta V = 1600\pi(0.01) = 16\pi,$$

so the volume would increase by approximately 16πm^3.

REMARK 3.5.7

For a qualitative analysis, we will explore it more when we discuss Taylor polynomials.

3.6 Newton's Method

We have a method for finding zeros of a function already: The Bisection Algorithm. Another way is using Newton's Method, which converges much faster but has its own issues as we will see!

Idea: to solve $f(x) = 0$, start with an initial guess, call it x_1. To get the next x-value, find the intersection of the tangent line $l(x)$ at $x = x_1$ and the x-axis. The numbers $x_1, x_2, x_3, \dots$ converge to a root (hopefully)! Let's find a formula for $x_2, x_3, \dots$.

Given x_1, the tangent line is

$$l(x) = f(x_1) + f'(x_1)(x - x_1).$$

Find the intersection with the x-axis:

$$0 = f(x_1) + f'(x_1)(x - x_1) \implies x = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Repeating this, we get the **Newton's Iterative Procedure**:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

EXAMPLE 3.6.1

Find the positive root of $3x^4 + 15x^3 - 125x - 1500 = 0$ with error at most 10^{-5}. Use $x_1 = 4$.

Solution. We can check that $f(4) < 0$ and $f(5) > 0$, so there is a root between $x = 4$ and $x = 5$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n^4 + 15x_n^3 - 125x_n - 1500}{12x_n^3 + 45x_n^2 - 125}.$$

$$\begin{aligned}
x_1 &= 4, \\
x_2 &= 4 - \frac{3(4)^4 + 15(4)^3 - 125(4) - 1500}{12(4)^3 + 45(4)^2 - 125} \approx 4.19956 \\
x_3 &\approx 4.187268 \\
x_4 &\approx 4.1872187 \\
x_5 &\approx 4.1872187
\end{aligned}$$

To 5 decimal places, this is 4.18722. Check that $f(4.187621) < 0$ and $f(4.18723) > 0$, so IVT says there is a root between!

Some Problems with Newton's Method

This method only works on differentiable functions, but more importantly it only works if x_1 is chosen "close enough" to a root! What is "close enough?" It depends! Sometimes any x_1 works, sometimes most don't.

EXAMPLE 3.6.2

Consider $f(x) = x^3 - 3x + 1$, pick $x_1 = 1$. Then

$$x_2 = x_1 - \frac{x_1^3 - 3x_1 + 1}{3x_1^2 - 3} = 1 - \frac{1 - 3 + 1}{0}?$$

Actually, at $x = 1$, f has a horizontal tangent that never intersects the x-axis, so we can't find x_2. Also, if we pick $x_1 = 2$, we will find a different root than if we pick $x_1 = -2$. So, pick a good starting point! A bad choice could make Newton's method diverge.

3.10 Derivatives of Inverse Functions

Suppose we want to find the derivative of an inverse function, how could we proceed? Let's start with the tangent line to $f(x)$ at $x = a$ and assume f is invertible.

$$l(x) = f(a) + f'(a)(x - a).$$

What would the tangent line to $f^{-1}(x)$ be at $x = f(a)$? $(l)^{-1}(x)$.

EXERCISE 3.10.1

If $L_a^f(x) = f(a) + f'(a)(x - a)$, show that

$$(L_a^f(x))^{-1} = a + \frac{1}{f'(a)}(x - f(a)).$$

So, if $f(a) = b$, then $a = f^{-1}(b)$, and the tangent line to $f^{-1}(x)$ at $x = b$ is

$$L_b^{f^{-1}}(x) = f^{-1}(b) + \frac{1}{f'(a)}(x - b) = f^{-1}(b) + \frac{1}{f'(f^{-1}(b))}(x - b).$$

But

$$L_b^{f^{-1}}(x) = f^{-1}(b) + (f^{-1})'(b)(x - b) \implies (f^{-1})' = \frac{1}{f'(f^{-1}(b))}.$$

This leads us to the following theorem.

THEOREM 3.10.2: Inverse Function Theorem (IFT)

Let I be an interval in $\mathbb{R}$, let $f\colon I \to \mathbb{R}$, and let a be a point in I which is not an endpoint. If f is bijective and continuous, and f is differentiable at a with $f'(a) \neq 0$, then its inverse f^{-1} is differentiable at $b = f(a)$ with

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}.$$

Moreover, L_a^f is invertible and $(L_a^f)^{-1} = L_{f(a)}^{f^{-1}} = L_b^{f^{-1}}$.

EXAMPLE 3.10.3

Let $f(x) = x^3$ so that $f^{-1}(x) = x^{1/3}$. Find $(f^{-1})'(3)$.

Solution 1. Direct computation yields

$$(f^{-1})'(x) = \frac{1}{x}x^{-2/3} \implies (f^{-1})'(3) = \frac{1}{3}3^{2/3} = \frac{1}{3(3^{2/3})}.$$

Solution 2. Use the IFT:

$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))}.$$

Note that $f'(x) = 3x^2$ and $f^{-1}(3) = 3^{1/3}$, so

$$(f^{-1})' = \frac{1}{f'(f^{-3}(3))} = \frac{1}{3(3^{1/3})^2} = \frac{1}{3(3^{2/3})}.$$

This example is somewhat silly since we could compute $(f^{-1})'$ directly. An important application of the IFT is that it allows us to find derivatives of inverse functions if we don't know them already!

EXAMPLE 3.10.4

Find $(\ln x)'$.

Solution. Let $f(x) = e^x$, so that $f^{-1}(x) = \ln x$ for $x > 0$. So,

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

Therefore,

$$(\ln x)' = \frac{1}{x}.$$

REMARK 3.10.5

We can prove IFT by using the chain rule: Suppose f and f^{-1} are differentiable, we get

$$f(f^{-1}(x)) = x.$$

Differentiate both sides with chain rule:

$$f'(f^{-1}(x))(f^{-1})'(x) = 1 \implies (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

3.11 Derivatives of Inverse Trigonometric Functions

Let's use the IFT (or just the chain rule) to find $(\arcsin x)'$. We know $\sin(\arcsin x) = x$ for $x \in [-1, 1]$. Differentiating, we get

$$(\cos(\arcsin x))(\arcsin x)' = 1 \implies (\arcsin x)' = \frac{1}{\cos(\arcsin x)}.$$

Can we simplify $\cos(\arcsin x)$? Yes! Let $\theta = \arcsin x$, then $\sin\theta = x$. Visualizing a triangle, we get the hypotenuse as 1, height as x so that the base $\sqrt{1-x^2}$. Hence, $\cos\theta = \sqrt{1-x^2}$. Therefore,

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}.$$

- Q: Wait a minute, how do we know $\arcsin x$ is differentiable?
- A: IFT says so! Since $\sin x$ is differentiable for $x \in (-1, 1)$, $\arcsin x$ is too.

EXERCISE 3.11.1

Prove that

- $(\arccos x)' = \dfrac{-1}{\sqrt{1-x^2}}$, and
- $(\arctan x) = \dfrac{1}{1+x^2}$.

EXAMPLE 3.11.2

Find $f'(x)$, where

1. $f(x) = \arctan(e^{\sin x})$,
2. $f(x) = \arcsin x + \arccos x$, and
3. $f(x) = \ln(\arctan x)$.

Solution.

1. $f'(x) = \dfrac{1}{1+(e^{\sin x})^2}(e^{\sin x})'$
$= \dfrac{1}{1+e^{2\sin x}} e^{\sin x}(\sin x)'$
$= \dfrac{e^{\sin x}\cos x}{1+e^{2\sin x}}.$
2. $f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0.$
3. $f'(x) = \frac{1}{\arctan x}\frac{1}{1+x^2} = \frac{1}{(\arctan x)(1+x^2)}.$

3.12 Implicit Differentiation

So far, we have examined derivatives of explicitly-defined functions (e.g., $y = f(x)$), but what about implicitly-defined functions?

EXAMPLE 3.12.1

If $x^2 + y^2 = 1$, then this isn't even a function (as it does not pass the vertical line test). But, if we divide up the curve into positive and negative parts on the y-axis then it can be a function. Then, we could find the derivative of each piece! The good news is that it doesn't matter if we break it up first! We can differentiate both sides of an implicit equation using the chain rule and solve for y. We do need to assume that the equation defines an implicit function though, more on this later.

EXAMPLE 3.12.2

Find y' if $3x^3y^3 + x^2y + 13x = 12$.

Solution. We let $y = y(x)$, take the derivative with respect to x on both sides, and then solve for $y'(x)$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}[3x^3y(x)^3 + x^2y(x) + 13x] &= \frac{\mathrm{d}}{\mathrm{d}x}[12] \\
\frac{\mathrm{d}}{\mathrm{d}x}[3x^3y(x)^3] + \frac{\mathrm{d}}{\mathrm{d}x}[x^2y(x)] + \frac{\mathrm{d}}{\mathrm{d}x}[13x] &= 0 \\
3\big(3x^2y(x)^3 + 3x^3(3)y(x)^2y'(x)\big) + \big(2xy(x) + x^2y'(x)\big) + 13 &= 0 \\
9x^2y(x)^3 + 9x^3y(x)^2y'(x) + 2xy(x) + x^2y'(x) + 13 &= 0 \\
9x^3y(x)^2y'(x) + x^2y'(x) &= -13 - 9x^2y(x)^3 - 2xy(x) \\
y'(x)\big(9x^3 + x^2\big) &= -13 - 9x^2y(x)^3 - 2xy(x) \\
y'(x) &= \frac{-13 - 9x^2y(x)^3 - 2xy(x)}{9x^3 + x^2}.
\end{aligned}$$

Therefore,

$$y' = \frac{-13 - 9x^2y^3 - 2xy}{9x^3 + x^2}.$$

REMARK 3.12.3

We can't always find the derivative of both sides of an equation unless we have a function!

EXAMPLE 3.12.4

If x^2+y^2+1, we can show that $y' = -\frac{x}{y}$, but for which $(x,y) \in \mathbb{R}^2$ is this valid for? None! $x^2+y^2+1 \neq 0$ for any $(x,y) \in \mathbb{R}^2$, so we differentiated nothing! Another example is if $2x = x$, we would differentiate to get $2 = 1$ (nonsense). The issue is $2x = x$ is only true if $x = 0$, so we can't compute the derivative as we can't take a limit! So be careful, use this power wisely!

Logarithmic Differentiation

We can use implicit differentiation to find the derivative of functions of the form

$$y = (f(x))^{g(x)},\ f(x) > 0$$

by taking the "$\ln$" of both sides.

EXAMPLE 3.12.5

Let $y = (\ln x)^{\sin x}$ for $x > 1$. Find y'.

Solution. Let $y = y(x)$ so that $y(x) = (\ln x)^{\sin x}$. Taking the logarithm (and then the derivative with respect to x) on both sides gives

$$\begin{aligned}
\ln y(x) &= (\sin x)\ln(\ln x) \\
\frac{\mathrm{d}}{\mathrm{d}x}[\ln y(x)] &= \frac{\mathrm{d}}{\mathrm{d}x}[(\sin x)\ln(\ln x)] \\
\frac{y'(x)}{y(x)} &= (\cos x)\ln(\ln x) + \sin x \frac{1}{\ln x}\frac{1}{x} \\
\implies y'(x) &= y(x)\left[(\cos x)\ln(\ln x) + \frac{\sin x}{x \ln x}\right] \\
\implies y'(x) &= (\ln x)^{\sin x}\left[(\cos x)\ln(\ln x) + \frac{\sin x}{x \ln x}\right].
\end{aligned}$$

EXAMPLE 3.12.6

Let $y = x^{\arctan x}$. Find y'.

Solution. Let $y = y(x)$ so that $y(x) = x^{\arctan x}$. Taking the logarithm (and then the derivative with respect to x) on both sides gives

$$\begin{aligned}
\ln y(x) &= \arctan(x)\ln x \\
\frac{\mathrm{d}}{\mathrm{d}x}[\ln y(x)] &= \frac{\mathrm{d}}{\mathrm{d}x}[\arctan(x)\ln x] \\
\frac{y'(x)}{y(x)} &= \frac{1}{1+x^2}\ln x + \arctan(x)\frac{1}{x} \\
\implies y'(x) &= y(x)\left[\frac{\ln x}{1+x^2} + \frac{\arctan x}{x}\right] \\
\implies y'(x) &= x^{\arctan x}\left[\frac{\ln x}{1+x^2} + \frac{\arctan x}{x}\right].
\end{aligned}$$

3.13 Local Extrema

DEFINITION 3.13.1: Local Maximum, Local Minimum

Let $A \subseteq \mathbb{R}$ be open, let $f\colon A \to \mathbb{R}$, and let $a \in A$. Then f has a **local maximum** at a if and only if

$$\forall x \in A : f(x) \leq f(a).$$

Similarly, we say f has a **local minimum** at a if and only if

$$\forall x \in A : f(x) \geq f(a).$$

We also present an equivalent definition.

DEFINITION 3.13.2: Local Maximum, Local Minimum

Let $A \subseteq \mathbb{R}$ be open, let $f\colon A \to \mathbb{R}$, and let $a \in A$. Then f has a **local maximum** at a if and only if

$$\exists \delta > 0 : \forall x \in A : |x-a| \leq \delta \implies f(x) \leq f(a).$$

Similarly, we say f has a **local minimum** at a if and only if

$$\exists \delta > 0 : \forall x \in A : |x-a| \leq \delta \implies f(x) \geq f(a).$$

REMARK 3.13.3

Local maximum/minimum means max/min nearby a point (i.e., in a small neighbourhood). Global max/min means max/min over the entire interval in question. So, global max/mins that occur inside the interval are also local max/mins.

How do we find local extrema? We will use the following theorem.

3.13.1 The Local Extrema Theorem

THEOREM 3.13.4: Fermat's Theorem/Local Extrema Theorem

Let $A \subseteq \mathbb{R}$ be open, let $f\colon A \to \mathbb{R}$, and let $a \in A$. Suppose that f is differentiable at a and that f has a local maximum or minimum value at a. Then $f'(a) = 0$.

Proof: We suppose that f has a local maximum value at a (the case that f has a local minimum value at a is similar). Choose $\delta > 0$ so that $|x-a| \leq \delta \implies f(x) \leq f(a)$. For $x \in A$ with $a < x < a + \delta$, since $x > a$ and $f(x) \geq a$ we have $\frac{f(x)-f(a)}{x-a} \geq 0$, and so

$$f'(a) = \lim_{x \to a^+} \frac{f(x)-f(a)}{x-a} \geq 0$$

by the Comparison Theorem. Similarly, for $x \in A$ with $a - \delta \leq x < a$, since $x < a$ and $f(x) \geq f(a)$ we have $\frac{f(x)-f(a)}{x-a} \leq 0$, and so

$$f'(a) = \lim_{x \to a^-} \frac{f(x)-f(a)}{x-a} \leq 0.$$

- Q: Is the converse true?
- A: No! $f(x) = x^3$ has a critical point at $x = 0$, but 0 is neither a local max nor a local min.
- Q: If c is a local max/min, then is $f'(c) = 0$?

- A: No! $f(x) = |x|$ has a local min at $x = 0$, but $f'(0)$ does not exist.

Finding Global Extrema

We just saw that if we want to find a local extrema, we should look at points where $f' = 0$ or f' does not exist. Let's give a name to points like this.

DEFINITION 3.13.5: Critical Point

A point c in the domain of a function f is called a **critical point** for f if either $f'(c) = 0$ or $f'(c)$ does not exist.

Now, the EVT guarantees a continuous function has a global max/min on a closed interval. Either these are at the endpoints or they are inside, and therefore local max/mins, and hence critical points!

So here is the algorithm for finding the global max/min of a continuous function $f(x)$ on $[a, b]$.

(i) Find all critical points of f in $[a, b]$.

(ii) Evaluate $f(a)$, $f(b)$, and $f(c)$, where c are all the critical points.

(iii) The largest value tells you where the global maximum is, and the smallest tells you what the global minimum is.

EXAMPLE 3.13.6

Find the global maximum and minimum for $f(x) = x^3 - 3x + 2$ on $[-3, 3]$.

Solution. $f'(x) = 3x^2 - 3 = 3(x-1)(x+1) = 0$ if $x = \pm 1$. These critical points are both inside $[-3, 3]$. Now, we check $f(-3) = -16$, $f(-1) = 4$, $f(1) = 0$, $f(3) = 20$. Therefore, the global maximum is at $(3, 20)$ and the global minimum is at $(-3, -16)$.

EXAMPLE 3.13.7

Find the global maximum and minimum for $f(x) = 1/x$ on $[3, 7]$.

Solution. $f'(x) = -1/x^2$ and $f'(x)$ does not exist if $x = 0$. However, 0 is not a critical point of f since $0 \notin [3, 7]$. So, f has no critical points. Now, $f(3) = 1/3$ and $f(7) = 1/7$, so the global maximum is at $(3, 1/3)$ and the global minimum is at $(7, 1/7)$.

REMARK 3.13.8

If we considered $f(x) = 1/x$ on its entire domain $\{x \in \mathbb{R} : x \neq 0\}$, then $x = 0$ is still not a critical point as it's not in the domain of the function.

We will re-visit this when we discuss curve sketching.

Chapter 4

The Mean Value Theorem

As we will see, the Mean Value Theorem (MVT) has lots of applications! But first we should prove it! Let's start with:

THEOREM 4.0.1: Rolle's Theorem

Let f be a real function which is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b).Then:

$$\exists c \in (a, b) : f'(c) = 0.$$

Proof: We consider 3 cases:

- Case 1: $f(x) = 0$ for all $x \in [a, b]$. Then $f'(x) = 0$ for all $x \in (a, b)$, so there are lots of choices for $c \in (a, b)$ where $f'(c) = 0$.
- Case 2: There exists a point $x_0 \in (a, b)$ such that $f(x_0) > 0$. By EVT, f attains its global max on $[a, b]$ and since $f(x_0) > 0$, while $f(a) = f(b) = 0$, we can see that the global max will occur at $c \in (a, b)$. This means c is a critical point of f, and since $f'(c)$ exists, it must be the case that $f'(c) = 0$.
- Case 3: There exists a point $x_0 \in (a, b)$ such that $f(x_0) < 0$. The proof is left as an exercise (similar to case 2, but use minimum).

4.1 The Mean Value Theorem

THEOREM 4.1.1: The Mean Value Theorem (MVT)

Let f be a real function which is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b).Then:

$$\exists c \in (a, b) : f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof: Define

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Since f is continuous on $[a, b]$, so is h. Also,

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

so $h'(x)$ exists for $x \in (a,b)$. Lastly, $h(a) = h(b) = 0$. Rolle's theorem says there exists $c \in (a,b)$ such that $h'(c) = 0$. So,

$$0 = f'(c) - \frac{f(b)-f(a)}{b-a} \implies f'(c) = \frac{f(b)-f(a)}{b-a}.$$

EXAMPLE 4.1.2

Let $f(x) = x^2 + 2x + 1$ and $x \in [1,2]$. Find the c's that satisfy the MVT.

Solution. Note that

$$\frac{f(2)-f(1)}{2-1} = \frac{(4+4+1)-(1+2+1)}{1} = 5.$$

Need $f'(c) = 5 \implies f'(c) = 2c+2 \implies 2c+2 = 5 \implies c = 3/2$.

REMARK 4.1.3

Why do we need continuity at the endpoints? Consider $f(x) = x$ where $f(a) = f(b) = 0$, but $f'(x) \neq 0$ for any $x \in (a,b)$.

EXAMPLE 4.1.4

Can MVT be applied to $f(x)$?

(1) $f(x) = |x|$ where $x \in [-1,1]$. No! $f'(x)$ DNE at $x = 0$.
(2) $f(x) = \frac{x+1}{x+3}$ where $x \in [-4,0]$. No! $f(x)$ is not continuous at $x = -3$.
(3) $f(x) = e^{-x}$ where $x \in [-1,1]$. Yes!
(4) $f(x) = \sec x$ where $x \in [0,\pi]$. No! $\sec x$ is not continuous at $x = \pi/2$.

4.2 Applications of the Mean Value Theorem

We will see that the MVT has LOTS of applications!

4.2.1 Antiderivatives

DEFINITION 4.2.1: Antiderivative (Primitive) of Real Function

Let F be a real function which is continuous on $[a,b]$ and differentiable on (a,b).
Let f be a real function which is continuous on (a,b).

Let

$$\forall x \in (a,b) : F'(x) = f(x),$$

where F' denotes the derivative of F with respect to x.

Then F is a **antiderivative** of f, and is denoted:

$$F = \int f(x)\,\mathrm{d}x.$$

We call $\int f(x)\,\mathrm{d}x$ the **indefinite integral** of f, where $f(x)$ is the **integrand**.

EXAMPLE 4.2.2

$F(x) = \frac{x^2}{2}$ is an antiderivative of $f(x) = x$ since $F'(x) = x$.

- Q: Are antiderivatives unique?
- A: No! $\frac{x^2}{2}$, $\frac{x^2}{2} + 7$, $\frac{x^2}{2} - e$ are all antiderivatives of $f(x) = x$.
- Q: Do different antiderivatives differ by a constant?
- A: Yes! Let's work towards showing this.

THEOREM 4.2.3: Zero Derivative implies Constant Function

Let f be a real function which is continuous on $[a, b]$ and differentiable on (a, b).
Suppose that:

$$\forall x \in (a, b) : f'(x) = 0.$$

Then f is constant on $[a, b]$.

Proof: By MVT,

$$\exists c \in (a, x) : f'(c) = \frac{f(x) - f(a)}{x - a}.$$

But by our supposition:

$$\forall x \in (a, b) : f'(x) = 0$$

which means:

$$\forall x \in (a, b) : f(x) - f(a) = 0$$

and hence:

$$\forall x \in (a, b) : f(x) = f(a).$$

This tells us that the family of antiderivatives for the function $f(x) = 0$ is all the constant functions $f(x) = c \in \mathbb{R}$.

THEOREM 4.2.4: Antiderivative Theorem

If $f'(x) = g'(x)$ for all $x \in I$, then there exists $\alpha \in \mathbb{R}$ such that $f(x) = g(x) + \alpha$ for all $x \in I$.

Proof: Suppose $f'(x) = g'(x)$ for all $x \in I$. Define $h(x) = f(x) - g(x)$. Then $h'(x) = f'(x) - g'(x) = 0$ for all $x \in I$. By Theorem 4.2.3, there exists $\alpha \in \mathbb{R}$ such that $h(x) = \alpha$ for all $x \in I$. Therefore, $f(x) - g(x) = \alpha \implies f(x) = g(x) + \alpha$ for all $x \in I$.

Additional Notes on Antiderivatives

EXAMPLE 4.2.5

$\int x \, \mathrm{d}x = x^2/2 + C$, $\int x^2 \, \mathrm{d}x = x^3/3 + C$.

THEOREM 4.2.6: Integral of Power

$$\forall n \in \mathbb{R}_{\neq -1} : \int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + C.$$

Proof: Easy exercise (just differentiate the RHS).

Also, if F is an antiderivative of f and G is an antiderivative of g, then $\alpha F + \beta G$ is an antiderivative of $\alpha f + \beta g$ since

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(\alpha F(x) + \beta G(x)\big) = \alpha f(x) + \beta g(x).$$

More generally,

$$\int \alpha_1 f_1(x) + \cdots + \alpha_n f_n(x)\,\mathrm{d}x = \alpha_1 \int f_1(x)\,\mathrm{d}x + \cdots + \alpha_n \int f_n(x)\,\mathrm{d}x.$$

Some Basic Indefinite Integrals

- $\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C.$
- $\int e^x\,\mathrm{d}x = e^x + C.$
- $\int \sin x\,\mathrm{d}x = -\cos(x) + C.$
- $\int \cos x\,\mathrm{d}x = \sin(x) + C.$
- $\int \sec^2 x\,\mathrm{d}x = \tan x + C.$
- $\int \frac{1}{1+x^2}\,\mathrm{d}x = \arctan(x) + C.$
- $\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arcsin(x) + C.$
- $\int \frac{-1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arccos(x) + C.$

4.2.2 Increasing Function Theorem

The sign of $f'(x)$ gives us more info about $f(x)$!

THEOREM 4.2.7: Derivative of Monotone Function

Let f be a real function which is continuous on $[a, b]$ and differentiable on (a, b).

$$\begin{aligned}
\forall x \in (a,b) : f'(x) > 0 &\implies f \textit{ is strictly increasing on } [a,b].\\
\forall x \in (a,b) : f'(x) \geq 0 &\implies f \textit{ is non-decreasing on } [a,b].\\
\forall x \in (a,b) : f'(x) < 0 &\implies f \textit{ is strictly decreasing on } [a,b].\\
\forall x \in (a,b) : f'(x) \leq 0 &\implies f \textit{ is non-increasing on } [a,b].
\end{aligned}$$

Proof: We prove the first one, noting that the rest are similar. Let $c, d \in [a, b]$ with $c < d$. Then f satisfies the conditions of the MVT on $[c, d]$. Hence:

$$\exists \xi \in (c,d) : f'(\xi) = \frac{f(d) - f(c)}{d - c}$$

Let f be such that

$$\forall x \in (a,b) : f'(x) > 0.$$

Then:

$$f'(\xi) > 0$$

and hence:

$$f(d) > f(c)$$

Thus f is strictly increasing $[a, b]$.

We will use this theorem when we look at curve sketching.

- Q: Is the converse true?

- A: Not always: $f(x) = x^3$ is increasing everywhere, but $f'(0) = 0$. Also, f' may not exist! So all we can guarantee is that $f'(x) \geq 0$ (when it exists).

4.2.3 Functions with Bounded Derivatives

What can we say about a function f if all we know are the bounds on its derivative?

Say $m \leq f'(x) \leq M$ for $x \in (a,b)$ and say f is continuous on $[a,b]$, so we can apply MVT. Pick $x \in [a,b]$. Then apply MVT to f on $[a,x]$: $\exists c \in (a,x)$ such that

$$f'(c) = \frac{f(x) - f(a)}{x - a},$$

but $f'(c) \in [m, M]$, so

$$\begin{aligned} & m \leq \frac{f(x) - f(a)}{x - a} \leq M \\ \rightsquigarrow\ & m(x-a) \leq f(x) - f(a) \leq M(x-a) \\ \rightsquigarrow\ & f(a) + m(x-a) \leq f(x) \leq f(a) + M(x-a). \end{aligned}$$

So the graph of f lies between the lines $f(a) + m(x-a)$ and $f(a) + M(x-a)$. This leads us to the following theorem.

THEOREM 4.2.8: Bounded Derivative Theorem (BDT)

Let f be a real function which is continuous on $[a,b]$ and differentiable on (a,b).
Suppose $\forall x \in (a,b) : m \leq f'(x) \leq M$. Then:

$$\forall x \in [a,b] : f(a) + m(x-a) \leq f(x) \leq f(a) + M(x-a).$$

EXAMPLE 4.2.9

Prove $\sqrt{66} \in (8 + 1/9, 8 + 1/8)$.

Solution. Let $f(x) = \sqrt{x}$ so $f'(x) = \frac{1}{2\sqrt{x}}$. Note that f is continuous on $[64, 66]$ and differentiable on $(64, 66)$. Also, if $x \in [64, 66]$, it is clear that $64 \leq x \leq 81$, so

$$f'(x) = \frac{1}{2\sqrt{x}} \in \left[\frac{1}{18}, \frac{1}{16}\right].$$

By the BDT, we get:

$$\sqrt{64} + \frac{1}{18}(x - 64) \leq \sqrt{x} \leq \sqrt{64} + \frac{1}{16}(x - 64).$$

So, at $x = 66$:

$$\sqrt{64} + \frac{1}{18}(2) \leq \sqrt{66} \leq \sqrt{64} + \frac{1}{16}(2) \rightsquigarrow 8 + \frac{1}{9} \leq \sqrt{66} \leq 8 + \frac{1}{8}.$$

EXAMPLE 4.2.10

If $f(12) = 2$ and $1 \leq f'(x) \leq 3$ for all $x \in \mathbb{R}$, find an interval for $f(20)$.

Solution. BDT says $f(12) + 1(x - 12) \leq f(x) \leq f(12) + 3(x - 12)$. So, at $x = 20$:

$$2 + 8 \leq f(20) \leq 2 + 24 \rightsquigarrow 10 \leq f(20) \leq 26.$$

4.2.4 Comparing Functions Using Their Derivatives

If we know the relative sizes of two functions' derivatives, we can also compare the sizes of the functions!

THEOREM 4.2.11

Assume f, g are continuous at $x = a$ with $f(a) = g(a)$.

(1) If both f, g are differentiable for $x > a$, and if $f'(x) \leq g'(x)$, then

$$\forall x > a : f(x) \leq g(x).$$

(2) If both f, g are differentiable for $x < a$ and if $f'(x) \leq g'(x)$, then

$$\forall x < a : f(x) \geq g(x).$$

Proof of (1): Suppose f, g are continuous at $x = a$ and differentiable for $x > a$, and $f'(x) \leq g'(x)$ for all $x > a$.
Define $h(x) = g(x) - f(x)$, then h is also continuous at $x = a$ and differentiable for $x > a$. Also,

$$\forall x > a : h'(x) = g'(x) - f'(x) \geq 0.$$

So, by MVT, we can find $c \in (a, x)$ such that

$$0 \leq h'(c) = \frac{h(x) - h(a)}{x - a}.$$

But $h(a) = 0$ and $x - a > 0$, so $h(x) \geq 0$ too; that is,

$$h(x) = g(x) - f(x) \geq 0 \rightsquigarrow g(x) \geq f(x)$$

for all $x > a$.

REMARK 4.2.12

Note that if $f'(x) < g'(x)$, then we get $f(x) < g(x)$ for $x > a$.

EXAMPLE 4.2.13

Prove that $\forall x \in \mathbb{R}_{>0} : x - \frac{1}{2}x^2 < \ln(1 + x) < x$.

Proof: Let $f(x) = x - \frac{1}{2}x^2$, $g(x) = \ln(1 + x)$, and $h(x) = x$. Then $f(0) = g(0) = h(0) = 0$ and

$$f'(x) = 1 - x, \quad g'(x) = \frac{1}{1 + x}, \quad h'(x) = 1.$$

If $x > 0$, then $g'(x) = \frac{1}{1+x} < 1 = h'(x)$.
Also, if $x > 0$, then

$$\begin{aligned} & 1 - x^2 < 1 \\ \rightsquigarrow\ & (1 + x)(1 - x) < 1 \\ \rightsquigarrow\ & 1 - x < \frac{1}{1 + x} \\ \rightsquigarrow\ & f'(x) < g'(x). \end{aligned}$$

Therefore, for $x > 0$, $f'(x) < g'(x) < h'(x)$. Apply the theorem twice (with strict inequalities) to get

$$\forall x \in \mathbb{R}_{>0} : x - \frac{1}{2}x^2 < \ln(1 + x) < x.$$

EXERCISE 4.2.14

By comparing derivatives and Squeeze Theorem, prove that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

4.3 L'Hôpital's Rule

First, we worked with limits. Then, we used limits to define derivatives. Now, we come full-circle and show how derivatives can be used to help solve limits!

Recall: the first thing we do when solving limits is to check where each of the component functions go. If we get a number, $\pm\infty$, or DNE, we are done! (May also need the Squeeze Theorem).

But, if we get an indeterminate form:

$$\frac{0}{0}, \pm\frac{\infty}{\infty}, 0\cdot\infty, \infty-\infty, 1^{\infty}, \infty^{0}, 0^{0},$$

we need to do more work. Let's see how L'Hôpital's Rule can help us in each case!

THEOREM 4.3.1: L'Hôpital's Rule (LHR)

Let f and g be real functions which are differentiable on an open interval I, and let $a \in \overline{\mathbb{R}}$.
Let:

$$\forall x \in I : g'(x) \neq 0.$$

Let:

$$\lim_{x\to a}\frac{f(x)}{g(x)} \textit{ be of type } \frac{0}{0} \textit{ or } \pm\frac{\infty}{\infty}.$$

Let:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = L \in \overline{\mathbb{R}}.$$

Then:

$$\lim_{x\to a}\frac{f'(x)}{g'(x)} = L.$$

REMARK 4.3.2

(a) The rule applies to $a \in \overline{\mathbb{R}}$, i.e., $a \in \mathbb{R}$ and $a = \pm\infty$, and one-sided limits.
(b) You can apply the rule multiple times, but make sure after each application that you verify your limit is of type $0/0$ or $\pm\infty/\infty$.
(c) We will use $\overset{\text{L'R}}{=}$ to denote a step which we apply l'Hôpital's Rule.

Let's examine the various types!

Type $0/0$ or $\pm\infty/\infty$

Apply LHR directly!

EXAMPLE 4.3.3

- $\lim_{x\to 2} \dfrac{2-x}{\sqrt{2}-\sqrt{x}}$ type $\dfrac{0}{0}$

  $\overset{\text{L'R}}{=} \lim_{x\to 2} \dfrac{-1}{-\frac{1}{2\sqrt{x}}}$

  $= \lim_{x\to 2} 2\sqrt{x}$

  $= 2\sqrt{2}.$

- $\lim_{x\to\infty} \dfrac{x^3-2x+7}{3x^3+x^2+x+1}$ type $\dfrac{\infty}{\infty}$

  $\overset{\text{L'R}}{=} \lim_{x\to\infty} \dfrac{3x^2-2}{9x^2+2x+1}$ type $\dfrac{\infty}{\infty}$

  $\overset{\text{L'R}}{=} \lim_{x\to\infty} \dfrac{6x}{18x+2}$ type $\dfrac{\infty}{\infty}$

  $= \lim_{x\to\infty} \dfrac{6}{18}$

  $= \dfrac{1}{3}.$

- $\lim_{x\to 0} \dfrac{\tan x}{x}$ type $\dfrac{0}{0}$

  $\overset{\text{L'R}}{=} \lim_{x\to 0} \dfrac{\sec^2 x}{1}$

  $= 1.$

- $\lim_{x\to 0^+} \dfrac{\ln x}{x} = -\infty$ is not an indeterminate form, so we can't use LHR. Simply,

$$\lim_{x\to 0^+} \frac{\ln x}{x} = \left(\lim_{x\to 0^+} \frac{1}{x}\right)\left(\lim_{x\to 0^+} \ln x\right) = (\infty)(-\infty) = -\infty.$$

Type $0 \cdot \infty$

The trick: divide by the reciprocal of one of them!

$$fg = \frac{f}{1/g}.$$

EXAMPLE 4.3.4

- $\lim_{x\to 0^+} x\ln x$ type $0\cdot -\infty$

  $= \lim_{x\to 0^+} \dfrac{\ln x}{1/x}$ type $-\dfrac{\infty}{\infty}$

  $\overset{\text{L'R}}{=} \lim_{x\to 0^+} \dfrac{1/x}{-1/x^2}$

  $= \lim_{x\to 0^+} (-x)$

  $= 0.$

- $\lim_{x\to 0^+} xe^{1/x}$ type $0\cdot\infty$

  $= \lim_{x\to 0^+} \dfrac{e^{1/x}}{1/x}$ type $\dfrac{\infty}{\infty}$

  $\overset{\text{L'R}}{=} \lim_{x\to 0^+} \dfrac{e^{1/x}(-1/x^2)}{-1/x^2}$

  $= \lim_{x\to 0^+} e^{1/x}$

  $= \infty.$
- $\lim_{x\to\infty} xe^{-x}$ type $\infty\cdot 0$

  $= \lim_{x\to\infty} \dfrac{x}{e^x}$ type $\dfrac{\infty}{\infty}$

  $\overset{\text{L'R}}{=} \lim_{x\to\infty} \dfrac{1}{e^x}$

  $= 0.$

  Alternative argument: x grows asymptotically slower than e^x as $x\to\infty$.

Type $\infty-\infty$

Combine the terms into a single term somehow (rationalize, factor, simplify, etc.).

EXAMPLE 4.3.5

- $\lim_{x\to\pi/2^-} \sec(x)-\tan(x)$ type $\infty-\infty$

  $= \lim_{x\to\pi/2^-} \dfrac{1}{\cos x}-\dfrac{\sin x}{\cos x}$

  $= \lim_{x\to\pi/2^-} \dfrac{1-\sin x}{\cos x}$ type $\dfrac{0}{0}$

  $\overset{\text{L'R}}{=} \lim_{x\to\pi/2^-} \dfrac{-\cos x}{-\sin x}$

  $= \dfrac{0}{1}$

  $= 0.$
- $\lim_{x\to\infty} \ln(x)-\ln(3x+1)$ type $\infty-\infty$

  $= \lim_{x\to\infty} \ln(\dfrac{x}{3x+1})$

  $= \ln\left(\lim_{x\to\infty} \dfrac{x}{3x+1}\right)$ since $\ln x$ is continuous at $x=\dfrac{1}{3}$; type $\dfrac{\infty}{\infty}$

  $\overset{\text{L'R}}{=} \ln\left(\lim_{x\to\infty} \dfrac{1}{3}\right)$

  $= \ln\left(\dfrac{1}{3}\right).$

Type 1^∞, 0^0, ∞^0

In this case, write

$$f(x)^{g(x)} = e^{\ln\left(f(x)^{g(x)}\right)} = e^{g(x)\ln(f(x))} = \exp\left\{g(x)\ln\big(f(x)\big)\right\},$$

then the exponent will be type $0\cdot\infty$. You can pass the limit through $\exp\{\,\cdot\,\}$ since e^x is continuous on $\mathbb{R}$.

EXAMPLE 4.3.6

- $$\begin{aligned}
&\lim_{x\to 0^+} x^x && \text{type } 0^0\\
&= \lim_{x\to 0^+} e^{x\ln x}\\
&= \exp\left\{\lim_{x\to 0^+} x\ln x\right\} && \text{we did this earlier; type } 0\cdot\infty\\
&= e^0\\
&= 1.
\end{aligned}$$

- $$\begin{aligned}
&\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x && \text{type } 1^\infty\\
&= \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)} && \text{type } \infty\cdot 0\\
&= \exp\left\{\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)\right\}\\
&= \exp\left\{\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{1/x}\right\} && \text{type } \frac{0}{0}\\
&\overset{\text{L'R}}{=} \exp\left\{\lim_{x\to\infty}\frac{1+\frac{1}{x}\cdot\frac{-1}{x^2}}{-1/x^2}\right\}\\
&= \exp\left\{\lim_{x\to\infty}\frac{1}{1+1/x}\right\}\\
&= e^1\\
&= e.
\end{aligned}$$

- $$\begin{aligned}
&\lim_{x\to\pi/2^-} \sec(x)^{\cos(x)} && \text{type } \infty^0\\
&= \lim_{x\to\pi/2^-} e^{\cos(x)\ln(\sec x)}\\
&= \exp\left\{\lim_{x\to\pi/2^-}\cos(x)\ln(\sec x)\right\} && \text{type } 0\cdot\infty\\
&= \exp\left\{\lim_{x\to\pi/2^-}\frac{\ln(\sec x)}{\sec x}\right\} && \text{type } \frac{\infty}{\infty}\\
&\overset{\text{L'R}}{=} \exp\left\{\lim_{x\to\pi/2^-}\frac{\frac{1}{\sec x}\sec(x)\tan(x)}{\sec(x)\tan(x)}\right\}\\
&= \exp\left\{\lim_{x\to\pi/2^-}\frac{1}{\sec x}\right\}\\
&= e^0\\
&= 1.
\end{aligned}$$

So, in total there are 7 indeterminate forms.

indeterminate form	*method*
$0/0, \infty/\infty$	apply LHR directly
$0\cdot\infty$	$fg = \frac{f}{1/g}$
$\infty-\infty$	combine terms (rationalize, factor, simplify, etc.)
$1^\infty, 0^0, \infty^0$	$fg = \exp\{g\ln(f)\}$.

4.3.1 Interpreting the Second Derivative & Formal Definition of Concavity

While the first derivative told us if our function was increasing or decreasing, the second derivative tells us about concavity!

DEFINITION 4.3.7

The graph of f is **concave upwards** on an interval I if, for every pair of points $a, b \in I$, the secant line joining $(a, f(a))$ to $(b, f(b))$ lies above the graph of $f(x)$.
The graph is **concave downwards** on I if the secant line lies below the graph of f.

We can see if the graph is concave up, then the slope of the tangent line is increasing, that is, f' is increasing. Similarly, if the graph of f is concave down, then f' is decreasing. So we get:

THEOREM 4.3.8

(1) If $f''(x) > 0$ for all $x \in I$, then the graph of f is concave up on I.
(2) If $f''(x) < 0$ for all $x \in I$, then the graph of f is concave down on I.

We also have a name for a point where concavity changes:

DEFINITION 4.3.9

A point c is called an **inflection point** for f if f is continuous at $x = c$ and the concavity of f changes at $x = c$.

If $f''(x)$ is also continuous at $x = c$, then the sign of f'' must change at $x = c$, so by the IVT, we get:

THEOREM 4.3.10

If $f''(x)$ is continuous at $x = c$ and $(c, f(c))$ is an inflection point for f, then $f''(c) = 0$.

REMARK 4.3.11

Note that this is only telling us how to find candidates for inflection points. The converse is false! If $f''(c) = 0$, then that does not mean that $(c, f(c))$ is an inflection point.

EXERCISE 4.3.12

Find a counterexample!

EXAMPLE 4.3.13

Find intervals of concavity and inflection points for $f(x) = x^4 - 6x^2$.

Solution. $f'(x) = 4x^3 - 12x$ and $f''(x) = 12x^2 - 12$. Setting $f''(x) = 0$ yields $x = \pm 1$, let's check:

$$\begin{array}{cccc} & (-\infty, -1] & (-1, 1) & [1, \infty) \\ f'' & + & - & + \\ f & \textit{up} & \textit{down} & \textit{up} \end{array}$$

Therefore, f is concave up on $(-\infty, -1]$ and $[1, \infty)$ and concave down on $[-1, 1]$. Since f'' changes at $x = \pm 1$ and f is continuous at $x = \pm 1$, these are both inflection points.

EXAMPLE 4.3.14

Find intervals of concavity and inflection points for $f(x) = \frac{1}{x}$.

Solution. $f'(x) = -1/x^2$, $f''(x) = 2/x^3$. Note that f'' is undefined at $x = 0$.

$$\begin{array}{ccc} & (-\infty, 0) & (0, \infty) \\ f'' & - & + \\ f & \textit{down} & \textit{up} \end{array}$$

Therefore, f is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$. However, $x = 0$ is <u>not</u> an inflection point since f is continuous at $x = 0$.

4.3.2 Classifying Critical Points: The First and Second Derivative Tests

We know that if c is a local max/min for f, then either $f'(c) = 0$ or $f'(c)$ is undefined, that is, c is a critical point. But not all critical points are local max/mins! So, let's examine two methods for classifying critical points.

Method 1: The First Derivative Test

- Idea: look at the sign of f' on either side of c.
- Say $f'(x) < 0$ for $x \in (a, c)$ and $f'(x) > 0$ for $x \in (c, b)$ ($a < c < b$).
- Then, f is decreasing on (a, c) and increasing on (c, b). This suggests f has a local minimum! Similarly, for local maximums.

Let's collect these results into a theorem!

THEOREM 4.3.15: First Derivative Test

Assume c is a critical point for f and f is continuous at $x = c$. Let $c \in (a, b)$.
(1) $\big(\forall x \in (a, c) : f'(x) > 0\big) \wedge \big(\forall x \in (c, b) : f'(x) < 0\big) \implies$ *f has a local maximum at $x = c$.*
(2) $\big(\forall x \in (a, c) : f'(x) < 0\big) \wedge \big(\forall x \in (c, b) : f'(x) > 0\big) \implies$ *f has a local minimum at $x = c$.*

These are easier to see if you make a table.

$$\begin{array}{ccc} & (a, c) & (c, b) \\ f' & + & - \\ f & \nearrow & \searrow \\ & \multicolumn{2}{c}{\textit{LMAX}} \end{array} \qquad \begin{array}{ccc} & (a, c) & (c, b) \\ f' & - & + \\ f & \searrow & \nearrow \\ & \multicolumn{2}{c}{\textit{LMIN}} \end{array}$$

EXAMPLE 4.3.16

Find the local max/mins of $f(x) = \frac{x^3}{3} - \frac{3x^2}{2} + 2x + 1$.

Solution. $f'(x) = x^2 - 3x + 2 = (x - 2)(x - 1)$. Hence, $f'(x) = 0$ when $x = 1$ and/or $x = 2$; these are both critical points.

$$\begin{array}{cccc} & (-\infty, 1) & (1, 2) & (2, \infty) \\ f' & + & - & + \\ f & \nearrow & \searrow & \nearrow \end{array}$$

So, f has a local max at $x = 1$ and a local min at $x = 2$ by the first derivative test.

Method 2: The Second Derivative Test

Suppose $f'(c) = 0$, so c is a critical point of f. Then, the tangent line to the graph of f at $x = c$ is horizontal. Suppose also that $f''(c) < 0$. Then, the tangent line sits above the graph since the graph is concave down! So, $x = c$ is a local max! This is the second derivative test! We get a similar result for $f''(c) > 0$.

THEOREM 4.3.17: Second Derivative Test

Let $f'(c) = 0$, and let f'' be continuous at $x = c$.
(1) $f''(c) < 0 \implies f$ has a local maximum at $x = c$.
(2) $f''(c) > 0 \implies f$ has a local minimum at $x = c$.
(3) $f''(c) = 0 \implies$ no information.

EXAMPLE 4.3.18

$f(x) = \frac{x^3}{3} + 3x^2 - 7x + 3.$

Solution. $f'(x) = x^2 + 6x - 7 = (x - 1)(x + 7)$. Hence, $f'(x) = 0$ when $x = -7$ and/or $x = 1$. Note that $f''(x) = 2x + 6$ is continuous on $\mathbb{R}$.

- $f''(-7) = -8 < 0 \implies x = -7$ is a local maximum.
- $f''(1) = 8 > 0 \implies x = 1$ is a local minimum.

REMARK 4.3.19

You can use either the first or second derivative test to classify critical points, whichever you prefer!

4.4 Curve Sketching

To sketch $f(x)$:

(1) Find the domain of f.

(2) Find all the intercepts (x-int ($y = 0$) and y-int ($x = 0$)).

(3) (a) Find all vertical asymptotes ($\div 0$, $\ln$)

 (b) Find all horizontal asymptotes ($\lim_{x\to\pm\infty} f(x)$).

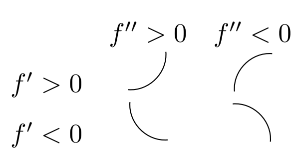
(4) Find $f'(x)$ and any critical points ((x, y) coordinates).

(5) Find $f''(x)$ and solve $f''(x) = 0$, find any points where $f''(x)$ does not exist ((x, y) coordinates).

(6) Test all intervals for increasing/decreasing, concavity, inflection points, local extrema.

(7) Plot the interest points and asymptotes on a graph.

(8) Connect the dots using the following:



EXAMPLE 4.4.1

Sketch $f(x) = x^4 - 16x^2$ with calculus.

Solution.

(1) Domain: $\mathbb{R}$.

(2)
- x-int ($y=0$): $0 = x^4 - 16x^2 = x^2(x-4)(x+4) \implies x = 0, \pm 4$. Points: $(0,0)$, $(4,0)$, $(-4,0)$.
- y-int ($x=0$): $(0,0)$.

(3) No VA's, no HA's.

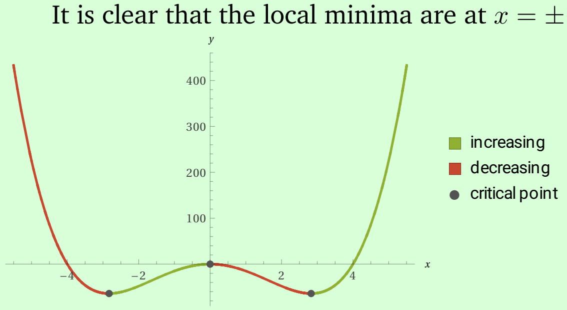
(4) $f'(x) = 4x^3 - 32x = 4x(x-\sqrt{8})(x+\sqrt{8}) = 0$ if $x = 0, \pm\sqrt{8}$ (all critical points).

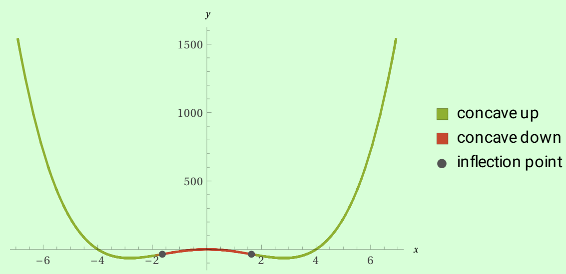
(5) $f''(x) = 12x^2 - 32 = 4(3x^2 - 8) = 0$ if $x = \pm\sqrt{\frac{8}{3}}$. Points: $(\pm\sqrt{\frac{8}{3}}, -\frac{320}{9})$.

(6) The test:

<table>
<tr><td></td><td>$(-\infty, -\sqrt{8})$</td><td>$(-\sqrt{8}, -\sqrt{\frac{8}{3}})$</td><td>$(-\sqrt{\frac{8}{3}}, 0)$</td><td>$(0, \sqrt{\frac{8}{3}})$</td><td>$(\sqrt{\frac{8}{3}}, \sqrt{8})$</td><td>$(\sqrt{8}, \infty)$</td></tr>
<tr><td>f''</td><td colspan="2">$+$</td><td colspan="2">$-$</td><td colspan="2">$+$</td></tr>
<tr><td>f'</td><td>$-$</td><td colspan="2">$+$</td><td colspan="2">$-$</td><td>$+$</td></tr>
<tr><td>f</td><td>$\searrow$</td><td>$\nearrow$</td><td>$\nearrow$</td><td>$\searrow$</td><td>$\searrow$</td><td>$\nearrow$</td></tr>
<tr><td>Shape</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

It is clear that the local minima are at $x = \pm\sqrt{8}$ and the local maximum is at $x = 0$.





Hence, f is:
- Increasing on $(-\sqrt{8}, 0)$ and $(\sqrt{8}, \infty)$;
- Decreasing on $(-\infty, -\sqrt{8})$ and $(0, \sqrt{8})$;
- Concave up on $(-\infty, -\sqrt{\frac{8}{3}})$ and $(\sqrt{\frac{8}{3}}, \infty)$;
- Concave down on $(-\sqrt{\frac{8}{3}}, \sqrt{\frac{8}{3}})$.

EXAMPLE 4.4.2

Sketch $f(x) = e^{-x^2}$ with calculus.

Solution.

(1) Domain: $\mathbb{R}$.

(2) y-int ($x=0$): $y = 1$; no x-int.

(3) No VA's.

$$\lim_{x\to\pm\infty} e^{-x^2} = 0,$$

so HA at $y = 0$ for $x \to \pm\infty$.

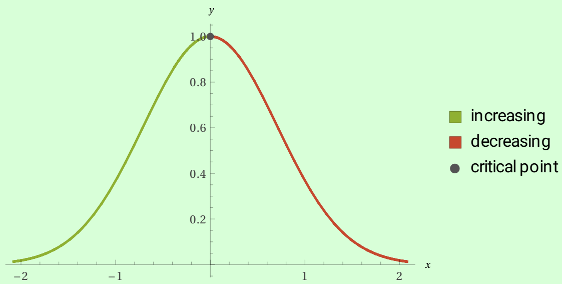
(4) $f'(x) = e^{-x^2}(-2x) = 0$ if $x = 0$. Point is $(0,1)$.

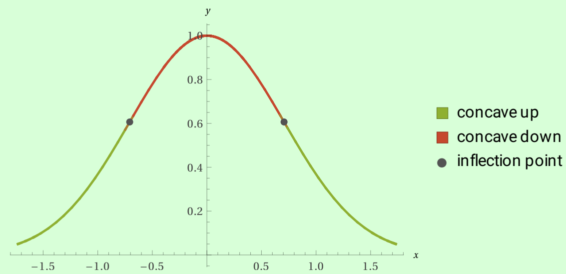
(5) $f''(x) = -2e^{-x^2} + e^{-x^2}(4x^2) = e^{-x^2}(4x^2 - 2) = 0$ if $x = \pm\frac{1}{\sqrt{2}}$. Points: $(\pm\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{e}})$.

(6) The test:

	$(-\infty, -\frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}}, 0)$	$(0, \frac{1}{\sqrt{2}})$	$(\frac{1}{\sqrt{2}}, \infty)$
f''	$+$	$-$		$+$
f'	$+$		$-$	
f	$\nearrow$	$\nearrow$	$\searrow$	$\nearrow$
Shape				

It is clear that the local maximum is at $x = 0$.





Hence, f is:
- Increasing on $(-\infty, 0)$;
- Decreasing on $(0, \infty)$;
- Concave up on $(-\infty, -\frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}}, \infty)$;
- Concave down on $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

EXAMPLE 4.4.3

Sketch $f(x) = \dfrac{x^2}{x^2 - 4}$ with calculus. Note that

$$f'(x) = -\frac{8x}{(x^2-4)^2}, \quad f''(x) = \frac{8(3x^2+4)}{(x^2-4)^3}.$$

Solution.

(1) Domain: $x \neq \pm 2$.

(2) x and y-int ($x = 0$, $y = 0$): $(0, 0)$

(3) VA at $x = \pm 2$.

$$\lim_{x \to \pm\infty} f(x) = 1,$$

so HA at $y = 1$ for $x \to \pm\infty$.

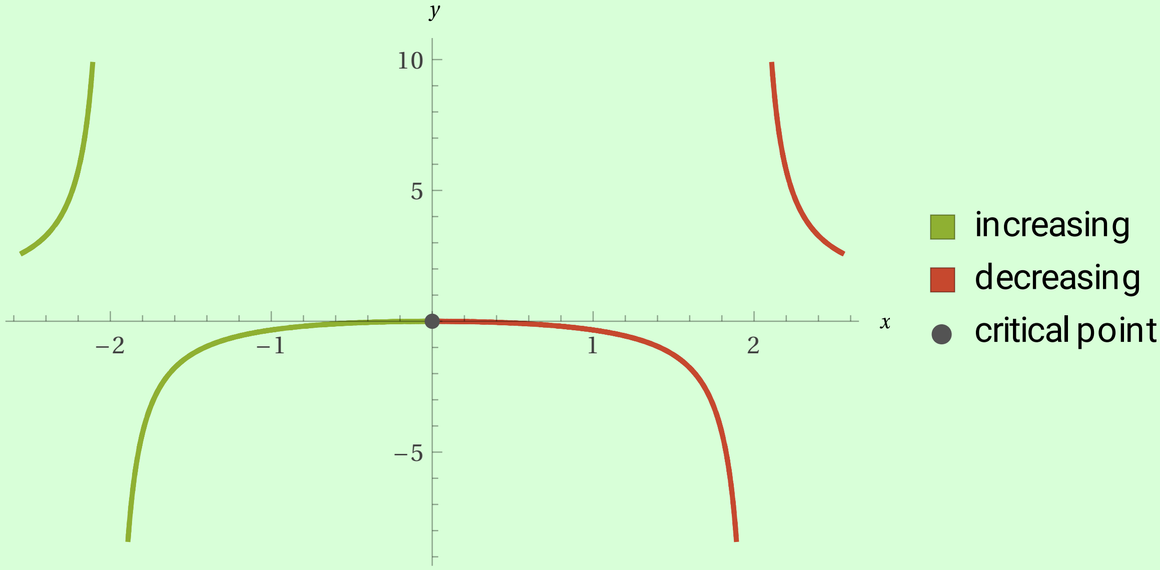
(4) $f'(x) = 0$ when $x = 0$. $f'(x)$ DNE if $x = \pm 2$, but $x = \pm 2$ is not in the domain of the function. Therefore, the only critical point is $x = 0$.

(5) $f''(x) = 0$ never. $f''(x)$ DNE when $x = \pm 2$.

(6) The test:

	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
f''	$+$	$-$		$+$
f'	$+$	$+$	$-$	$-$
f	$\nearrow$	$\nearrow$	$\searrow$	$\searrow$
Shape				

It is clear that the local maximum is at $x = 0$.



Hence, f is:

- Increasing on $(-\infty, -2)$ and $(-2, 0)$;
- Decreasing on $(0, 2)$ and $(2, \infty)$;
- Concave up on $(-\infty, -2)$ and $(2, \infty)$;
- Concave down on $(-2, 2)$.

Chapter 5

Taylor Polynomials and Taylor's Theorem

5.1 Introduction to Taylor Polynomials and Approximation

Recall the linear approximation of $f(x)$ at $x = a$:

$$L_a^f(x) = f(a) + f'(a)(x - a).$$

Idea: use higher-order derivatives to get a better approximation! Let's find a polynomial $T_{n,a}(x)$ that agrees with $f(x), f'(x), f''(x), \ldots, f^{(n)}(x)$ at $x = a$, say

$$T_{n,a}(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots + c_n(x - a)^n.$$

First, $T_{n,a}(a) = c_0$, and we want $T_{n,a}(a) = f(a)$, so

$$c_0 = f(a).$$

Next, $T'_{n,a}(x) = c_1 + 2c_2(x - a) + \cdots + nc_n(x - a)^{n-1}$, and $T'_{n,a}(a) = c_1$. But, we want $T'_{n,a}(a) = f'(a)$, so

$$c_1 = f'(a).$$

$$T''_{n,a}(x) = 2c_2 + 6c_3(x - a) + \cdots + n(n - 1)c_n(x - a)^{n-2},$$

so $T''_{n,a}(a) = 2c_2$. But we want $T''_{n,a}(a) = f''(a)$, so

$$2c_2 = f''(a) \implies c_2 = \frac{f''(a)}{2}.$$

Keep going!

$$c_3 = \frac{f^{(3)}(a)}{6} = \frac{f^{(3)}(a)}{3!}.$$

In general,

$$c_k = \frac{f^{(k)}(a)}{k!},\ 0 < k \in \mathbb{Z}.$$

DEFINITION 5.1.1

Assume that f is n-times differentiable at $x = a$. The n^{th} **degree Taylor polynomial** for f centred at $x = a$ is:

$$T_{n,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

EXAMPLE 5.1.2

Find $T_{4,0}(x)$ for $f(x) = e^x$.

Solution:

$$\begin{aligned} f(x) = e^x &\implies f(0) = 1, \\ f'(x) = e^x &\implies f'(0) = 1, \\ f''(x) = e^x &\implies f''(0) = 1, \\ f^{(3)}(x) = e^x &\implies f^{(3)}(0) = 1, \\ f^{(4)}(x) = e^x &\implies f^{(4)}(0) = 1. \end{aligned}$$

So,

$$T_{4,0}(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}.$$

In general, the Taylor series expansion at $x = 0$ is:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

It is clear that the larger n is, the better $T_{n,a}(x)$ approximates $f(x)$.

EXAMPLE 5.1.3

Consider $f(x) = \cos x$ (so $f(0) = 1$), we get

$$\begin{aligned} f'(x) = -\sin x &\implies f'(0) = 0, \\ f''(x) = -\cos x &\implies f''(0) = -1, \\ f^{(3)}(x) = \sin x &\implies f^{(3)}(0) = 0, \\ f^{(4)}(x) = \cos x &\implies f^{(4)}(0) = 1. \end{aligned}$$

So, we get $T_{0,0}(x) = T_{1,0}(x) = 1$ and

$$T_{3,0}(x) = 1 - \frac{x^2}{2!}, \quad T_{4,0}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}.$$

REMARK 5.1.4

Since odd derivatives at $x = 0$, only the next even Taylor polynomial changes. This also has to do with the fact that $\cos x$ is an even function.

In general, the Taylor series expansion at $x = 0$ is:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

EXAMPLE 5.1.5

For $f(x) = \ln x$, find $T_{3,1}(x)$.

Solution.

$$\begin{aligned} f(x) = \ln x &\implies f(1) = 0, \\ f'(x) = \frac{1}{x} &\implies f'(1) = 1, \\ f''(x) = -\frac{1}{x^2} &\implies f''(1) = -1, \\ f^{(3)}(x) = \frac{2}{x^3} &\implies f^{(3)}(1) = 2. \end{aligned}$$

So,

$$T_{3,1}(x) = 0 + 1(x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3.$$

5.2 Taylor's Theorem and Errors in Approximations

As for linear approximations, we need a formula that allows us to estimate the size of the error in using the Taylor polynomial to approximate a function.

DEFINITION 5.2.1

Assume that f is n-times differentiable at $x = a$. Let

$$R_{n,a}(x) = f(x) - T_{n,a}(x).$$

$R_{n,a}(x)$ is called the n^{th} **degree Taylor remainder function** for $f(x)$ centred at $x = a$.
Then, we define the **error** in using the Taylor polynomial to approximate f as

$$\epsilon(x) = |R_{n,a}(x)|.$$

Now, we can write a formula for the **remainder**.

THEOREM 5.2.2: Taylor's Theorem

Assume f is $(n+1)$-times differentiable on an interval I containing $x = a$. Let $x \in I$. Then, there exists a point c between a and x such that

$$f(x) - T_{n,a}(x) = R_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

REMARK 5.2.3: Observations of Taylor's Functions

(1) $T_{1,a}(x) = L_a^f(x)$ and

$$|R_{n,a}(x)| = \left|\frac{f''(c)}{2!}\right|(x-a)^2 \le \frac{M}{2}(x-a)^2,$$

which is the linear approximation error!

(2) If $n = 0$, f is differentiable on I, and for $x \in I$, there exists a point c between a and x such that

$$f(x) - T_{0,a}(x) = f(a),$$

so it says

$$f(x) - f(a) = f'(c)(x-a) \implies \frac{f(x)-f(a)}{x-a} = f'(c),$$

which is the MVT! So, Taylor's Theorem is a higher-order version of the MVT.

(3) Again, the theorem doesn't tell us how to find c, but we can find an upper bound on the error, like we did for linear approximations.

THEOREM 5.2.4: Taylor's Inequality

$$|R_{n,a}(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!},$$

where $|f^{(n+1)}(c)| \leq M$ *for all* c *between* a *and* x.

EXAMPLE 5.2.5

Let $f(x) = \sqrt{1+x}$.

(1) Show that $T_{2,0}(x) = 1 + \frac{x}{2} - \frac{x^2}{8}$.

(2) Approximate $\sqrt{1.1}$ using $T_{2,0}(x)$.

(3) Find an upper bound on the error.

Solution.

(1) Exercise.

(2) $\sqrt{1.1} = f(0,1) \approx T_{2,0}(0,1) = 1 + \frac{0.1}{2} - \frac{0.01}{8} = 1 + \frac{1}{20} - \frac{1}{800} = \frac{839}{800}$.

(3) Note that $f''(x) = \frac{3}{8(1+x)^{5/2}}$ is decreasing on $[0, 0.1]$, so

$$|f''(x)| \leq \frac{3}{8} \text{ for } x \in [0, 0.1] \text{ using } c = 0,$$

so $M = 3/8$ works. Therefore,

$$\epsilon(x) \leq \frac{(3/8)|x|^3}{3!}$$

or

$$\epsilon(0.1) \leq \frac{3}{8}\frac{0.1^3}{3!} = \frac{1}{16}\frac{1}{1000} = \frac{1}{16000}.$$

Additional questions:

- Is $T_{2,0}(x)$ an over or underestimate for $f(x)$ if $x \geq 0$?
- We know

$$f(x) - T_{2,0}(x) = \frac{f^{(3)}(c)}{3!}x^3 = \frac{3}{8(1+c)^{5/3}}\frac{x^3}{3!} \geq 0$$

for $x \geq 0 \implies c \geq 0$. So, $f(x) \geq T_{2,0}(x)$, which means $T_{2,0}(x)$ underestimates $f(x)$ for $x \geq 0$. So, the estimate is a lower bound on the actual value! Therefore,

$$\sqrt{1.1} \in \left[\frac{839}{800}, \frac{839}{800} + \frac{1}{16000}\right].$$

EXAMPLE 5.2.6

Let $f(x) = x^{2/3}$. Find the second-order Taylor polynomial centred at $x = 8$ and find an upper bound on the error if $x \in [5, 11]$.

Solution. Second-order means two derivatives plus one for the error.

$$\begin{aligned} f(x) &= x^{2/3} \implies f(8) = 4, \\ f'(x) &= \frac{2}{3}x^{-1/3} \implies f'(8) = \frac{1}{3}, \\ f''(x) &= -\frac{2}{9}x^{-4/3} \implies f''(8) = -\frac{1}{72}, \\ f^{(3)}(x) &= \frac{8}{27}x^{-7/3}. \end{aligned}$$

So,

$$T_{2,8}(x) = 4 + \frac{1}{3}(x-8) - \frac{1}{144}(x-8)^2.$$

For the error,

$$\epsilon(x) \leq \frac{M|x-8|^3}{3!},$$

where $|f^{(3)}(c)| \leq M$ for $c \in [5, 11]$ (same range as x). Note that

$$\left|f^{(3)}(c)\right| = \left|\frac{8}{27}c^{-7/3}\right|$$

is clearly decreasing, so use $c = 5$ to get

$$M = \frac{8}{27}(5)^{-7/3}.$$

Also, if $x \in [5, 11]$, $|x-8|^3 \leq 3^3 = 27$, so

$$|R_{2,8}(x)| \leq \frac{8}{27}(5)^{-7/3}\frac{27}{3!} = \frac{4}{3}(5)^{-7/3}.$$

We can make the error bound even more general.

THEOREM 5.2.7: Taylor's Approximation Theorem I (TAT I)

If $f^{(k+1)}$ is continuous on an interval I containing $x = a$, then there exists a constant $N > 0$ such that

$$|f(x) - T_{k,a}(x)| \leq N|x-a|^{k+1}$$

or

$$-N|x-a|^{k+1} \leq f(x) - T_{k,a}(x) \leq N|x-a|^{k+1}.$$

Actually, $N = \frac{M}{(k+1)!}$ from the inequality.

Let's see how to use this to solve limits!

EXAMPLE 5.2.8

Evaluate $\lim\limits_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$ using TAT I.

Solution. First, for $f(x) = e^x$,

$$T_{3,0}(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!},$$

and we use TAT I to get

$$-Nx^4 \le e^x - 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!} \le Nx^4$$

where $0 < N \in \mathbb{R}$ for x near 0. Then,

$$-Nx^2 \le \frac{e^x - 1 - x}{x^2} - \frac{1}{2} - \frac{x}{6} \le Nx^2.$$

By the Squeeze Theorem, since $\pm Nx^2 \to 0$ as $x \to 0$,

$$\lim_{x\to 0}\left[\frac{e^x - 1 - x}{x^2} - \frac{1}{2} - \frac{x}{6}\right] = 0.$$

So,

$$\lim_{x\to 0}\frac{e^x - 1 - x}{x^2} = \lim_{x\to 0}\left[\frac{1}{2} + \frac{x}{6}\right] = \frac{1}{2}.$$

Hold on tight for the next example.

EXAMPLE 5.2.9

Evaluate $\lim\limits_{x\to 0}\dfrac{e^{x^4} + \cos(x^2) - 2}{x^4}$ using TAT I (twice).

Solution. First, for $f(u) = e^u$, we know $T_{1,0}(u) = 1 + u$. Also, for $g(u) = \cos(u)$, $T_{3,0}(x) = 1 - \frac{u^2}{2!}$. So, there exists $0 < N_1, N_2 \in \mathbb{R}$ such that

$$-N_1u^2 \le e^u - 1 - u \le N_1u^2,\ u \in (-1,1)$$

$$-N_2u^4 \le \cos(u) - 1 + \frac{u^2}{2!} \le N_2u^4,\ u \in (-1,1).$$

In the first equation, sub $u = x^4$ for $x \in (-1,1)$ (so $u \in (-1,1)$ too):

$$-N_1x^8 \le e^{x^4} - 1 - x^4 \le N_1x^8. \quad (\star)$$

In the second equation, sub $u = x^2$ for $x \in (-1,1)$ (so $u \in (-1,1)$ too):

$$-N_2x^8 \le \cos(x^2) - 1 + \frac{x^4}{2!} \le N_2x^8. \quad (\star\star)$$

Add $(\star)$ and $(\star\star)$:

$$\begin{aligned}
&-(N_1+N_2)x^8 \le e^{x^4} - 1 - x^4 + \cos(x^2) - 1 + \frac{x^4}{2} \le (N_1+N_2)x^8.\\
\implies\ &-(N_1+N_2)x^8 \le e^{x^4} + \cos(x^2) - 2 - \frac{x^4}{2} \le (N_1+N_2)x^8.\\
\implies\ &-(N_1+N_2)x^4 \le \frac{e^{x^4} + \cos(x^2) - 2}{x^4} - \frac{1}{2} \le (N_1+N_2)x^4.
\end{aligned}$$

Using the Squeeze Theorem, we get

$$\lim_{x\to 0}\left[\frac{e^{x^4} + \cos(x^2) - 2}{x^4} - \frac{1}{2}\right] = 0 \implies \lim_{x\to 0}\left[\frac{e^{x^4} + \cos(x^2) - 2}{x^4}\right] = \frac{1}{2}.$$

Remark: You could have also used LHR twice to get the answer.

5.3 Big-O