3. (10 marks) **Error probability of a code**
Consider the binary code $C = \{c_1 = 0000, c_2 = 1010, c_3 = 0111\}$. Suppose we are using IMLD and a BSC with symbol error probability $p$. Assume also that all codewords are sent with equal probability.

**(a) For $c \in C$, let $L(c)$ denote the set of all binary $4$-tuples which are closer to $c$ than any other codeword. Determine $L(c_1)$, $L(c_2)$ and $L(c_3)$.**

We know there are $2^4 = 16$ possible binary 4-tuples, so we will use a table to determine which are in $L(c_i)$, $\forall i \in \{1,2,3\}$ where $X$ are the ties (in that case, IMLD cannot accurately correct the error).

| $x_i$ | $d(c_1, x_i)$ | $d(c_2, x_i)$ | $d(c_3, x_i)$ | $\min d(c_i, x_i)$ |
|---|---|---|---|---|
| 0000 | 0 | 2 | 3 | $c_1$ |
| 0001 | 1 | 3 | 2 | $c_1$ |
| 0010 | 1 | 1 | 2 | $X$ |
| 0011 | 2 | 2 | 1 | $c_3$ |
| 0100 | 1 | 3 | 2 | $c_1$ |
| 0101 | 2 | 4 | 1 | $c_3$ |
| 0110 | 2 | 2 | 1 | $c_3$ |
| 0111 | 3 | 3 | 0 | $c_3$ |
| 1000 | 1 | 1 | 4 | $X$ |
| 1001 | 2 | 2 | 3 | $X$ |
| 1010 | 2 | 0 | 3 | $c_2$ |
| 1011 | 3 | 1 | 2 | $c_2$ |
| 1100 | 2 | 2 | 3 | $X$ |
| 1101 | 3 | 3 | 2 | $c_3$ |
| 1110 | 3 | 1 | 2 | $c_2$ |
| 1111 | 4 | 2 | 1 | $c_3$ |

Thus,

$L(c_1) = \{0000,\ 0001,\ 0100\}$

$L(c_2) = \{1010,\ 1011,\ 1110\}$

$L(c_3) = \{0011,\ 0101,\ 0110,\ 0111,\ 1101,\ 1111\}$

**(b) For each $i$, $1 \le i \le 3$, let $w_i$ denote the probability that IMLD makes an incorrect decision or rejects the received word, given that $c_i$ was transmitted. Determine $w_1$, $w_2$ and $w_3$.**

For $w_1$, we know there are $3$ codewords that can be corrected out of $16$. Similarly, for $w_2$, there are $3$ codewords that can be corrected out of $16$. Lastly, for $w_3$, there are $6$ codewords that can be corrected out of $16$. Thus, we get $w_1 = w_2 = p/13$ and $w_3 = p/10$ chance that IMLD makes an incorrect decision or rejects the received word, given that $c_i$ was transmitted.

**(c) The error probability of an $[n, M]$-code $C$ is defined to be $P_C = \frac{1}{M}\sum_{i=1}^{M} w_i$. Determine $P_C$ for the code $C$ of this example when $p = 0.1$.**

$$\begin{aligned} P_C &= \frac{1}{M}\sum_{i=1}^{M} w_i \\ &= \frac{1}{3}\left(\frac{0.1}{13} + \frac{0.1}{13} + \frac{0.1}{10}\right) \\ &\approx 0.008461 \end{aligned}$$

**(d) Suppose now that the codewords are not sent with equal probability. Then we define the error probability of $C$ to be $P_C = \sum_{i=1}^{M} w_i P(c_i)$. Suppose that $P(c_1) = 0.7$, $P(c_2) = 0.2$, and $P(c_3) = 0.1$.**

**Determine $P_C$ for the code $C$ of this example when $p = 0.1$.**

$$\begin{aligned}
P_C &= \frac{1}{M}\sum_{i=1}^{M} w_i P(c_i) \\
&= \frac{1}{3}\left(\frac{0.1}{13}(0.7) + \frac{0.1}{13}(0.2) + \frac{0.1}{10}(0.1)\right) \\
&\approx 0.002641
\end{aligned}$$