MATH 138 - Calculus 2

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Contents

Contents 1

1 Integration 2
- 1.2 Riemann Sums and the Definite Integral . . . 2
- 1.3 Properties of the Definite Integral . . . 5
  - 1.3.1 Geometric Interpretation of the Integral . . . 6
- 1.4 Average Value of a Function . . . 7
  - 1.4.1 Geometric Interpretation . . . 7
- 1.5 The Fundamental Theorem of Calculus Part 1 . . . 8
- 1.6 The Fundamental Theorem of Calculus Part 2 . . . 10
- 1.7 Change of Variable . . . 13
  - 1.7.1 Substitution and Definite Integrals . . . 15

2 Techniques of Integration 17
- 2.1 Trigonometric Substitution . . . 17
- 2.2 Integration by Parts . . . 20
- 2.3 Partial Fractions . . . 22
- 2.4 Improper Integrals . . . 25
  - 2.4.1 Type I . . . 26

Chapter 1

Integration

1.2 Riemann Sums and the Definite Integral

To begin with, our goal is to develop methods for determining the area under a curve.

We know we can approximate the area using rectangles (or other geometric shapes), but we want the **exact** area. For this, we will need **Riemann sums**.

DEFINITION 1.2.1: Partition

A **partition**, P, for the interval $[a, b]$ is a finite sequence of increasing numbers of the form

$$a = t_0 < \cdots < t_{n-1} = b$$

This partition subdivides the interval $[a, b]$ into n subintervals:

$$[t_0, t_1], \ldots, [t_{n-1}, t_n]$$

REMARK 1.2.2

These subintervals may **not** all have the same length.

DEFINITION 1.2.3: Length

Denote the **length** of the i^{th} subinterval, $[t_{i-1}, t_i]$, by Δt_i; that is, $\Delta t_i = t_i - t_{i-1}$.

DEFINITION 1.2.4: Norm

The **norm** of a partition is the length of the widest subinterval:

$$\|P\| = \max(\Delta t_1, \ldots, \Delta t_n)$$

DEFINITION 1.2.5: Riemann Sum

Given a bounded function f on $[a, b]$, a partition P of $[a, b]$, and a set $\{c_1, \ldots, c_n\}$ where $c_i \in [t_{i-1}, t_i]$, then a **Riemann Sum** for f with respect to P is

$$S = \sum_{i=1}^{n} f(c_i)\Delta t_i$$

Again, we want the **exact** area, and for that we will need to use infinitely many points!

But we do need to make sure that the norm of our partitions is getting smaller, and that the area we get doesn't depend on the choice of Riemann Sum.

DEFINITION 1.2.6: Integrable

We say that f is **integrable** on $[a, b]$ if there exists a unique number $I \in \mathbb{R}$ such that if whenever $\{P_n\}$ is a sequence of partitions with $\lim\limits_{n\to\infty} \|P_n\| = 0$ and $\{S_n\}$ is any sequence of Riemann Sums associated to the P_n's, we have $\lim\limits_{n\to\infty} S_n = I$.

In this case, we call I the **integral of f over** $[a, b]$ and denote it by

$$\int_a^b f(x)\,dx$$

where a, b are the bounds of integration, $f(x)$ is the integrand, x is the variable of integration. The complete object is called a definite integral.

It represents the exact (signed) area under f.

REMARK 1.2.7

The variable of integration is a **dummy variable** since we can change it into whatever we wand and it won't change the value of the integral; that is,

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \text{etc.}$$

This looks **horrible** to compute in practice (and it is). The good news is if f is continuous, it's not so bad! (still bad though)

THEOREM 1.2.8: Integrability Theorem for Continuous Functions

Let f be continuous on $[a, b]$. Then f is integrable on $[a, b]$.

Proof of: 1.2.8

Beyond the scope of this course.

This is fantastic! This means that we can **choose** any collection of Riemann Sums we want when computing the integral of a continuous function!

Let's examine a "nice" choice: one where the partition is regular and where we just pick the c_i's to be the right-hand endpoints!

DEFINITION 1.2.9: Regular n-partition

For the interval $[a, b]$, the **regular n-partition** where all n subintervals have the same length; that is,

$$\Delta t = \frac{b-a}{n} \quad \text{and} \quad t_i = t_0 + i\Delta t$$

DEFINITION 1.2.10: Regular right-hand Riemann Sum

Using this, we define the **regular right-hand Riemann Sum** by taking $c_i = t_i$ for all i:

$$S_n = \sum_{i=1}^{n} f(t_i)\Delta t = \sum_{i}^{n} f(t_i)\left(\frac{b-a}{n}\right)$$

REMARK 1.2.11

We can also define the regular left-hand Riemann Sum.

Now, we can write a nicer formula for integrating continuous functions!

If f is continuous, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\left(\frac{b-a}{n}\right)$$

EXAMPLE 1.2.12

Evaluate

$$\int_0^4 x + x^3\,dx$$

Since $f(x) = x + x^3$ is continuous, we can use the above formula.
In our case: $\frac{b-a}{n} = \frac{4}{n}$, and $t_i = 0 + \frac{4i}{n} = \frac{4i}{n}$.
So, $f(t_i) = \frac{4i}{n} + \frac{64i^3}{n^3}$. Then, we get:

$$\begin{aligned}
\int_0^4 x + x^3\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4i}{n} + \frac{64i^3}{n^3}\right)\left(\frac{4}{n}\right) && (1.1)\\
&= \lim_{n\to\infty} \frac{16}{n^2}\sum_{i=1}^{n} i + \frac{256}{n^4}\sum_{i=1}^{n} i^3 && (1.2)\\
&= \lim_{n\to\infty} \frac{16}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{256}{n^4}\left(\frac{n^2(n+1)^2}{4}\right) && (1.3)\\
&= \lim_{n\to\infty} \frac{8n+8}{n} + 64\left(\frac{n^2+2n+1}{n^2}\right) && (1.4)\\
&= 8 + 64 && (1.5)\\
&= 72 && (1.6)
\end{aligned}$$

where from 1.2 to 1.3 we used both of the following:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

REMARK 1.2.13

The theorem also holds for functions that are bounded and have finitely many discontinuities.

1.3 Properties of the Definite Integral

Since a definite integral is the limit of a sequence, many limit laws also hold!

THEOREM 1.3.1: Properties of Integrals

If f is integrable on $[a,b]$, then:

(1) For any $c \in \mathbb{R}$,

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

(2)

$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

(3) If $m \leq f(x) \leq M$ for $x \in [a,b]$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

(4) If $0 \leq f(x)$ for $x \in [a,b]$, then

$$0 \leq \int_a^b f(x)\,dx$$

(5) If $f(x) \leq g(x)$ for $x \in [a,b]$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

(6) If $|f|$ is integrable on $[a,b]$, then

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

Proof of: 1.3.1

- (1) and (2) follow from limit laws for sequences.
- (3) implies (4).
- (1), (2), and (4) imply (5).
- (6) follows from the triangle inequality.

We will now prove (3).
Suppose $m \leq f(x) \leq M$ and partition the interval

$$a = t_0 < \cdots < t_n = b$$

Note that

$$\sum_{i=1}^{n} \Delta t = \frac{b-a}{n}(n) = b-a$$

Then, since $m \leq f(x) \leq M$, we get

$$m(b-a) = \sum_{i=1}^{n} m\Delta t \leq \sum_{i=1}^{n} f(t_i)\Delta t \leq \sum_{i=1}^{n} M\Delta t = M(b-a)$$

So, taking limits gives

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

DEFINITION 1.3.2: More Properties

(I) If $f(a)$ is defined, then

$$\int_a^a f(x)\,dx = 0$$

(II) If f is integrable on $[a,b]$, then

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

THEOREM 1.3.3

If f is integrable on an interval I containing a,b,c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Proof of: 1.3.3

Trivial.

REMARK 1.3.4

c does **not** need to be between a and b!

1.3.1 Geometric Interpretation of the Integral

So far, we have only examined positive functions, but we should note that $\int_a^b f(x)\,dx$ returns the **signed** area between f and the x-axis. That is, if $f(x) \leq 0$, then $\int_a^b f(x)\,dx \leq 0$ too.

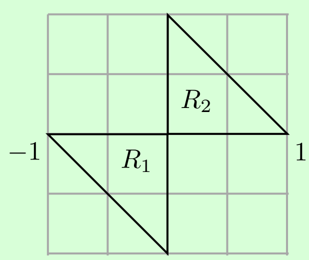
So, in general, $\int_a^b$ is the area under f that lies above the x-axis **minus** the area above the graph of f that lies below the x-axis.

EXAMPLE 1.3.5

$$\int_{-1}^1 x\,dx = R_2 - R_1$$

but $R_2 = R_2$, so

$$\int_{-1}^1 x\,dx = 0$$



REMARK 1.3.6

If we are lucky, we can use geometric formulas to evaluate integrals (see pg 26–28 in the notes). However, we are almost never this lucky…

1.4 Average Value of a Function

DEFINITION 1.4.1: Average Value

If f is continuous on $[a, b]$, the **average value** of f on $[a, b]$ is defined as

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

1.4.1 Geometric Interpretation

Proof of: 1.4.2

If f is continuous on $[a, b]$, EVT says there exists $m, M \in \mathbb{R}$ such that

$$m \leq f(x) \leq M$$

for $x \in [a, b]$ and $f(c_1) = m$, $f(c_2) = M$ for some $c_1, c_2 \in [a, b]$.
Also, we know

$$\begin{aligned} m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a) \implies & m \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq M \\ \iff & f(c_1) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq f(c_2) \end{aligned}$$

IVT says there exists c between c_1 and c_2, so that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

THEOREM 1.4.2: Average Value Theorem (AVT)

Assume f is continuous on $[a, b]$. There exists $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

REMARK 1.4.3

Note that this theorem holds even if $b < a$ since

$$\begin{aligned} f(c) &= \frac{1}{a-b}\int_b^a f(x)dx \\ &= \frac{1}{a-b}\left(-\int_a^b f(x)\,dx\right) \\ &= \frac{1}{b-a}\int_a^b f(x)\,dx \end{aligned}$$

The big problem we face now is that evaluating $\int_a^b f(x)\,dx$ is monstrously difficult for all but the simplest of functions...

IF ONLY THERE WAS A BETTER WAY!

(spoilers: there's a better way! It's the Fundamental Theorem of Calculus!)

1.5 The Fundamental Theorem of Calculus Part 1

The FTC is, essentially, a simple derivative rule. But its consequences are very valuable. The reason is that it provides the link between integral calculus and differential calculus!

We start with integral functions: let f be continuous on $[a, b]$.

Define

$$G(x) = \int_a^x f(t)\,dt$$

for $x \in [a, b]$.

What is $G(x)$? it's the function that returns the signed area under f from a to x.

EXAMPLE 1.5.1

$f(x) = x$ on $[0, 5]$.

$$\begin{aligned} G(x) &= \int_0^x t\,dt \\ &= \frac{1}{2}(\text{base})(\text{height}) \\ &= \frac{1}{2}(x)(x) \\ &= \frac{x^2}{2} \end{aligned}$$

Wait a minute! $G'(x) = x = f(x)$! Is this always true?!

THEOREM 1.5.2: Fundamental Theorem of Calculus I (FTC I)

If f is continuous on an open interval I containing $x = a$, and if

$$G(x) = \int_a^x f(t)\,dt$$

Then G is differentiable for all $x \in I$ and $G'(x) = f(x)$; that is,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Proof of: 1.5.2

Let f be continuous on I, $G(x) = \int_a^x f(t)\,dt$, and fix $x_0 \in I$.
Let $\varepsilon > 0$ be given. Since f is continuous at x_0, there exists a $\delta > 0$ such that if $0 < |c - x_0| < \delta$, then

$$|f(c) - f(x_0)| < \varepsilon$$

Let $0 < |x - x_0| < \delta$. Then,

$$\begin{aligned}\frac{G(x) - G(x_0)}{x - x_0} &= \frac{\int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt}{x - x_0} \\ &= \frac{\int_a^{x_0} f(t)\,dt + \int_{x_0}^x f(t)\,dt - \int_a^{x_0} f(t)\,dt}{x - x_0} \\ &= \frac{1}{x - x_0}\int_{x_0}^x f(t)\,dt\end{aligned}$$

The AVT says there exists c between x and x_0 such that

$$f(c) = \frac{1}{x - x_0}\int_{x_0}^x f(t)\,dt$$

Since $0 < |x - x_0| < \delta$, we get $0 < |c - x_0| < \delta$ too, so

$$\left|\frac{G(x) - G(x_0)}{x - x_0} - f(x_0)\right| = |f(c) - f(x_0)| < \varepsilon$$

This says

$$G'(x_0) = \lim_{x \to x_0}\frac{G(x) - G(x_0)}{x - x_0} = f(x_0)$$

EXAMPLE 1.5.3

$$\frac{d}{dx}\int_5^x \sin(t^2)\,dt = \sin(x^2)$$

since $f(t) = \sin(t^2)$ is continuous by FTC I.

EXAMPLE 1.5.4

If

$$G(x) = \frac{d}{dx}\int_5^{x^2} \sin(t^2)\,dt$$

then

$$\int_5^{x^2} \sin(t^2)\,dt = G(x^2)$$

so

$$\begin{aligned}\frac{d}{dx}\int_5^{x^2} \sin(t^2)\,dt &= \frac{d}{dx}G(x^2)\\ &= G'(x^2)\cdot 2x\\ &= f(x^2)\cdot 2x\\ &= \sin(x^4)\cdot 2x\end{aligned}$$

We will see a more general formula next week!

1.6 The Fundamental Theorem of Calculus Part 2

It seems like integrating is the opposite operation to differentiation, and it is! We can use antiderivatives to evaluate integrals as we will see. But first, let's quickly recap what we know about antidifferentiation.

DEFINITION 1.6.1: Anti-derivative

Given a function f, an **antiderivative** of f is a function F such that $F'(x) = f(x)$.

REMARK 1.6.2

Antiderivatives are not unique!

EXAMPLE 1.6.3

For $f(x) = 2x$,

- $F_1(x) = x^2$
- $F_2(x) = x^2 + 4$
- $F_3(x) = x^2 - \pi$

are all antiderivatives of $f(x)$.

DEFINITION 1.6.4: Indefinite integral

The collection of all antiderivatives of $f(x)$ is denoted by $\int f(x)\,dx$ and

$$\int f(x)\,dx = F(x) + C$$

where $C \in \mathbb{R}$ and F is any antiderivative. This is called the **indefinite integral**.

REMARK 1.6.5

By the antiderivative theorem, we know any two antiderivatives of f differ by a constant.

Here are a bunch of antiderivatives:

- $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ if $n \neq -1$
- $\int \frac{1}{x}\,dx = \ln|x| + C$

- $\int e^x \, dx = e^x + C$
- $\int \sin(x) \, dx = -\cos(x) + C$
- $\int \cos(x) \, dx = \sin(x) + C$
- $\int \sec^2(x) \, dx = \tan(x) + C$
- $\int \frac{1}{1+x^2} \, dx = \arctan x + C$
- $\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin(x) + C$
- $\int -\frac{1}{\sqrt{1-x^2}} \, dx = \arccos(x) + C$
- $\int \sec(x)\tan(x) \, dx = \sec(x) + C$
- $\int a^x \, dx = \frac{a^x}{\ln(a)} + C$ for $a > 0$

By FTC I, we know every continuous function has an antiderivative, but how can we use them to actually evaluate definite integrals? Well...

THEOREM 1.6.6: Fundamental Theorem of Calculus II (FTC II)

If f is continuous on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(x) \, dx = F(b) - F(a) = \big[F(x)\big]_a^b$$

Proof of: 1.6.6

Let F be any antiderivative of f and define

$$G(x) = \int_a^x f(t) \, dt$$

By FTC I, we know $G'(x) = f(x)$ as well, so by the antiderivative theorem, $G(x) = F(x) + C$ for some $C \in \mathbb{R}$.
But then,

$$G(b) - G(a) = [F(b) + C] - [F(a) + C] = F(b) - F(a)$$

Also,

$$\begin{aligned} \int_a^b f(t) \, dt &= G(b) \\ &= G(b) - G(a) && \text{since } G(a) = 0 \\ &= F(b) - F(a) \end{aligned}$$

Now, we can evaluate definite integrals without Riemann Sums!

EXAMPLE 1.6.7

(i)

$$\begin{aligned}\int_1^3 x^2 + x\,dx &= \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_1^3 \\ &= \left(\frac{3^3}{3} + \frac{3^2}{2}\right) - \left(\frac{1}{3} + \frac{1}{2}\right) \\ &= \frac{27}{2} - \frac{5}{6} \\ &= \frac{38}{3}\end{aligned}$$

(ii)

$$\begin{aligned}\int_0^{2\pi} \sin(x)\,dx &= [-\cos(x)] \\ &= -\cos(2\pi) + \cos(0) \\ &= 1 + 1 \\ &= 0\end{aligned}$$

This makes sense since the signed area is zero.

(iii)

$$\begin{aligned}\int_2^8 \frac{x^2 + 2x + 1}{x}\,dx &= \int_2^8 x + 2 + \frac{1}{x}\,dx \\ &= \left[\frac{x^2}{2} + 2x + \ln|x|\right]_2^8 \\ &= [32 + 16 + \ln(8)] - [2 + 4 + \ln(2)] \\ &= 42 + \ln(8) - \ln(2) \\ &= 42 + \ln(4)\end{aligned}$$

This is fantastic! We are only limited by our ability to find antiderivatives! As we will see, finding antiderivatives is hard in general, but in the next couple of weeks we will learn a few techniques.

But first, let's look at the extended version of FTC I:

COROLLARY 1.6.8: Extended Version of the Fundamental Theorem of Calculus

If f is continuous, and g, h are both differentiable, then

$$\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(t)\,dt\right] = f(h(x))h'(x) - f(g(x))g'(x)$$

(also called the Leibniz Formula).

Proof of: 1.6.8

Let F be an antiderivative of f, then by FTC II:

$$\int_{g(x)}^{h(x)} f(t)\,dt = F(h(x)) - F(g(x))$$

for each x. So,

$$\begin{aligned}\frac{d}{dx}\Big[\int_{g(x)}^{h(x)} f(t)\,dt\Big] &= \frac{d}{dx}\big[F(h(x)) - F(g(x))\big] \\ &= F'(h(x))h'(x) - F'(g(x))g'(x) \\ &= f(h(x))h'(x) - f(g(x))g'(x)\end{aligned}$$

EXAMPLE 1.6.9

$$\frac{d}{dx}\int_{5x}^{\ln(x)} \cos(t^2 - 3t)\,dt = \cos\big[\ln(x)^2 - 3\ln(x)\big]\cdot\frac{1}{x} - \cos(25x^2 - 15x)\cdot 5$$

1.7 Change of Variable

The first integration technique we will examine is the reverse chain rule: Change of Variable, also called Substitution.

The rule is:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

that is, we "substitute" $u = g(x)$.

Proof of: Change of Variable (Sketch)

et f and g be functions and let h be an antiderivative of f, so $h'(x) = f(x)$.
Let $H(x) = h(g(x))$, so

$$H'(x) = h'(g(x))g'(x) = f(g(x))g'(x)$$

so $h(g(x))$ is an antiderivative of $f(g(x))g'(x)$.
Therefore,

$$\begin{aligned}\int f(g(x))g'(x)dx &= h(g(x)) + C && \text{for some } c\in\mathbb{R} \\ &= h(u) + C && \text{if } u = g(x) \\ &= \int f(u)\,du && \text{if } u = g(x)\end{aligned}$$

So, if $u = g(x)$, then $du = g'(x)\,dx$.

General strategy: let $u = \ldots$, $du = \ldots\, d(x)$, then solve for dx, substitute in u and dx, try to transform the integral into one in terms of only u.

Good choices for u:

- $u =$ a function whose derivative is present.
- $u =$ base of an ugly power
- $u =$ function inside another function (that is, inside $\sin$ / $\cos$ / $\ln$, or in the exponent of e).

EXAMPLE 1.7.1

(i)

$$\begin{aligned}\int \frac{\ln(x)}{x}\,dx &= \int \frac{u}{x} x\,du \qquad\qquad u = \ln(x) \iff du = \frac{1}{x}\,dx\\ &= \int u\,du\\ &= \frac{u^2}{2} + C\\ &= \frac{[\ln(x)]^2}{2} + C\end{aligned}$$

(ii)

$$\begin{aligned}\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx &= \int \frac{\cos(u)}{u} 2u\,du \qquad\qquad u = \sqrt{x} \iff du = \frac{1}{2\sqrt{x}}\,dx\\ &= 2\int \cos(u)\,du\\ &= 2\sin(u) + C\\ &= 2\sin(\sqrt{x}) + C\end{aligned}$$

(iii) Don't forget to eliminate all of the x's!

$$\begin{aligned}\int \frac{x^2}{\sqrt{x+1}}\,dx &= \int \frac{(u-1)^2}{\sqrt{u}}\,du \qquad\qquad u = x+1 \iff du = dx\\ &= \int \frac{u^2-2u+1}{\sqrt{u}}\,du\\ &= \int u^{3/2} - 2u^{1/2} + u^{-1/2}\,du\\ &= \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + 2u^{1/2} + C\\ &= \frac{2}{5}(x+1)^{5/2} - \frac{4}{3}(x+1)^{3/2} + 2(x+1)^{1/2} + C\end{aligned}$$

(iv)

$$\begin{aligned}\int \sin^6(x)\cos(x)\,dx &= \int u^6\,du \qquad\qquad u = \sin(x) \iff du = \cos(x)\,dx\\ &= \frac{u^7}{7} + C\\ &= \frac{\sin^7(x)}{7} + C\end{aligned}$$

(v)

$$\begin{aligned}\int xe^{5x^2}\,dx &= \int \frac{xe^u}{10x}\,du \qquad\qquad u = 5x^2 \iff du = 10x\,dx\\ &= \frac{1}{10}\int e^u\,du\\ &= \frac{e^u}{10} + C\\ &= \frac{e^{5x^2}}{10} + C\end{aligned}$$

1.7.1 Substitution and Definite Integrals

Q: What should we do with the limits of integration when making a substitution?

A: We should change them as well!

THEOREM 1.7.2: Change of Variable

If $g'(x)$ is continuous on $[a,b]$ and $f(x)$ is continuous between $g(a)$ and $g(b)$, then

$$\int_{x=a}^{x=b} f(g(x))g'(x)\,dx = \int_{u=g(a)}^{u=g(b)} f(u)\,du$$

Proof of: 1.7.2

Let $h(u)$ be an antiderivative of $f(u)$. Then $h(g(x))$ is an antiderivative of $f(g(x))g'(x)$.
By FTC II,

$$\int_a^b f(g(x))g'(x)\,dx = h(g(b)) - h(g(a))$$

But also,

$$\int_{g(a)}^{g(b)} f(u)\,du = h(g(b)) - h(g(a))$$

so we get

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

EXAMPLE 1.7.3

(i)

$$\begin{aligned}\int_0^1 e^x\cos(e^x)\,dx &= \int_1^e \frac{u\cos(u)}{u}\,du & u = e^x \iff du = e^x\,dx\\ &= \int_1^e \cos(u)\,du\\ &= \big[\sin(u)\big]_1^e\\ &= \sin(e) - \sin(1)\end{aligned}$$

(ii)

$$\begin{aligned}\int_0^1 \frac{x^3}{1+x^4}\,dx &= \int_1^2 \frac{x^3}{u\cdot 4x^3}\,du & u = 1+x^4 \iff du = 4x^3\,dx\\ &= \frac{1}{4}\int_1^2 \frac{1}{u}\,du\\ &= \left[\frac{\ln|u|}{4}\right]_1^2\\ &= \frac{\ln(2)}{4} - \frac{\ln(1)}{4}\\ &= \frac{\ln(2)}{4}\end{aligned}$$

REMARK 1.7.4

You can also leave the limits of integration in terms of x as long as you make it clear and don't forget to switch back to x at the end before plugging numbers in!

EXAMPLE 1.7.5: Tricky Change of Variable

$$\begin{aligned}\int \sec(x)\,dx &= \int \sec(x)\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx\\ &= \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx\\ &= \int \frac{1}{u}\,du\\ &= \ln|u|+C\\ &= \ln|\sec(x)+\tan(x)|+C\end{aligned}$$

We made the substitution $u = \sec(x)+\tan(x) \iff du = \sec(x)\tan(x)+\sec^2(x)\,dx$.

REMARK 1.7.6

The trick used in Example 1.7.5 only works for $\sec(x)$ and $\csc(x)$, so it's not useful to memorize.

Chapter 2

Techniques of Integration

2.1 Trigonometric Substitution

Sometimes, changing x into a trigonometric function can simplify an integral!

There are three situations where this is useful: say $\alpha \in \mathbb{R}$.

If you see:	Try substituting:	Range for θ
$\sqrt{a^2 - x^2}$	$x = a\sin(\theta)$	$\theta \in (-\pi/2, \pi/2)$
$\sqrt{a^2 + x^2}$	$x = a\tan(\theta)$	$\theta \in (-\pi/2, \pi/2)$
$\sqrt{x^2 - a^2}$	$x = a\sec(\theta)$	$\theta \in [0, \pi/2] \cup [\pi, 3\pi/2]$

REMARK 2.1.1

- The range for θ is important to ensure that $\sin(\theta)/\tan(\theta)/\sec(\theta)$ are invertible (so we can solve for θ in terms of x, if need be).
- No, you don't need to state the range for θ each time.
- The integrand may need to be simplified before a trigonometric substitution can be made.
- Don't forget to change back to x in an indefinite integral.

EXAMPLE 2.1.2

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2+4}}\,dx &= \int \frac{1}{\sqrt{4\tan^2(\theta)+4}} \cdot 2\sec^2(\theta)\,d\theta && x = 2\tan(\theta) \iff dx = 2\sec^2(\theta)\,d\theta \\
&= \int \frac{\sec^2(\theta)}{\sqrt{\tan^2(\theta+1)}}\,d\theta \\
&= \int \frac{\sec^2(\theta)}{\sqrt{\sec^2(\theta)}}\,d\theta \\
&= \int \frac{\sec^2(\theta)}{|\sec(\theta)|}\,d\theta \\
&= \int \sec(\theta)\,d\theta && \text{since } \sec(\theta) > 0 \\
&= \ln|\sec\theta + \tan(\theta)| + C \\
&= \ln\left|\frac{\sqrt{x^2+y}}{2} + \frac{x}{2}\right| + C
\end{aligned}$$

Where we substituted $x = 2\tan(\theta) \iff \tan(\theta) = \frac{x}{2} \implies \sec(\theta) = \frac{\sqrt{x^2+4}}{2}$ in the last step.

REMARK 2.1.3

When using a trigonometric substitution, the absolute values will **always** go away due to the choice of θ's!

EXAMPLE 2.1.4

(i)

$$\begin{aligned}
\int \frac{\sqrt{9-4x^2}}{x^2}\,dx &= 2\int \frac{\sqrt{9/4 - x^2}}{x^2}\,dx \\
&= 2\int \frac{\sqrt{9/4 - 9/4\sin^2(\theta)}\cdot 3/2\cos(\theta)}{9/4\sin^2(\theta)}\,d\theta \qquad x = 3/2\sin(\theta) \iff dx = 3/2\cos(\theta)\,d\theta \\
&= 2\cdot\frac{3}{2}\cdot\frac{3}{2}\cdot\frac{4}{9}\int \frac{\sqrt{1-\sin^2(\theta)}\cos(\theta)}{\sin^2(\theta)}\,d\theta \\
&= 2\int \frac{|\cos(\theta)|\cos(\theta)}{\sin^2(\theta)}\,d\theta \\
&= 2\int \frac{\cos^2(\theta)}{\sin^2(\theta)}\,d\theta \\
&= 2\int \cot^2(\theta)\,d\theta \\
&= 2[-\cot(\theta) - \theta] + C \\
&= 2\left[-\frac{\sqrt{9-4x^2}}{2x} - \arcsin\left(\frac{2x}{3}\right)\right] + C
\end{aligned}$$

Where we substituted $x = 3/2 \iff 2x/3 = \sin(\theta) \implies \theta = \arcsin(2x/3)$ and $\cot(\theta) = \sqrt{9-4x^2}/2x$ in the last step.

(ii)

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2-4}}\,dx &= \int \frac{2\sec(\theta)\tan(\theta)}{4\sec^2(\theta)\sqrt{4\sec^2(\theta)-4}}\,d\theta \quad x = 2\sec(\theta) \iff dx = 2\sec(\theta)\tan(\theta)\,d\theta \\
&= \frac{1}{4}\int \frac{\tan(\theta)}{\sec(\theta)\sqrt{\sec^2(\theta)-1}}\,d\theta \\
&= \frac{1}{4}\int \frac{\tan(\theta)}{\sec(\theta)\tan(\theta)}\,d\theta \\
&= \frac{1}{4}\int \frac{\tan(\theta)}{\sec(\theta)+\tan(\theta)}\,d\theta \\
&= \frac{1}{4}\int \frac{1}{\sec(\theta)}\,d\theta \\
&= \frac{1}{4}\int \cos(\theta)\,d\theta \\
&= \frac{\sin(\theta)}{4} + C \\
&= \frac{\sqrt{x^2-4}}{4x} + C
\end{aligned}$$

Where we substituted $x = 2\sec(\theta) \implies \sin(\theta) = \frac{x^2-4}{x}$ in the last step.

(iii)

$$\begin{aligned}
\int x\sqrt{x^2-9}\,dx &= \int \frac{x\sqrt{u}}{2x}\,du && u = x^2-9 \iff du = 2x\,dx \\
&= \frac{1}{2}\int \sqrt{u}\,du \\
&= \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C \\
&= \frac{1}{3}\left(x^2-9\right)^{3/2} + C
\end{aligned}$$

(iv)

$$\begin{aligned}
\int_0^3 \frac{x}{(1+x^2)^2}\,dx &= \int_0^{\pi/3} \frac{\tan(\theta)\sec^2(\theta)}{\left[1+\tan^2(\theta)\right]^2}\,d\theta && x = \tan(\theta) \iff dx = \sec^2(\theta)\,d\theta \\
&= \int_0^{\pi/3} \frac{\tan(\theta)\sec^2(\theta)}{\sec^4(\theta)}\,d\theta \\
&= \int_0^{\pi/3} \frac{\tan(\theta)}{\sec^2(\theta)}\,d\theta \\
&= \int_0^{\pi/3} \frac{\sin(\theta)}{\cos(\theta)}\cdot\cos^2(\theta)\,d\theta \\
&= \int_0^{\pi/3} \sin(\theta)\cos(\theta)\,d\theta && u = \sin(\theta) \iff du = \cos(\theta)\,d\theta \\
&= \int_0^{\sqrt{3/2}} u\,du \\
&= \left[\frac{u^2}{2}\right]_0^{3/2} \\
&= \frac{3}{4}
\end{aligned}$$

Exercise: $3-2x-x^2 = 4-(x+1)^2$.

EXAMPLE 2.1.5

Substitution: $x+1 = 2\sin(\theta) \iff dx = 2\cos(\theta)\,d\theta$.

$$\begin{aligned}
\int \frac{x}{(3-2x-x^2)^{3/2}}\,dx &= \int \frac{x}{[4-(x+1)^2]^{3/2}}\,dx \\
&= \int \frac{[2\sin(\theta)-1]\cdot 2\cos(\theta)}{\left[4-4\sin^2(\theta)\right]^{3/2}}\,d\theta \\
&= \frac{1}{4}\int \frac{[2\sin(\theta)-1]\cdot\cos(\theta)}{\cos^2(\theta)^{3/2}}\,d\theta \\
&= \frac{1}{4}\int \frac{2\sin(\theta)}{\cos^2(\theta)} - \frac{1}{\cos^2(\theta)}\,d\theta \\
&= \frac{1}{4}\int 2\tan(\theta)\sec(\theta) - \sec^2(\theta)\,d\theta \\
&= \frac{1}{4}\left[2\sec(\theta)-\tan(\theta)\right] + C \\
&= \frac{1}{4}\left[\frac{4}{\sqrt{4-(x+1)^2}} - \frac{(x+1)}{\sqrt{4-(x+1)^2}}\right] + C
\end{aligned}$$

Where we substituted $x+1 = 2\sin(\theta) \iff \sin(\theta) = {}^{(x+1)}/_{2} \implies \sec(\theta) = {}^{2}/\sqrt{4-(x+1)^2}$ and $\tan(\theta) = {}^{(x+1)}/\sqrt{4-(x+1)^2}$ in the last step.

2.2 Integration by Parts

Let u and v be functions of x. From the product rule, we know

$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating both sides gives:

$$\int \frac{d}{dx}[uv]\,dx = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

Omit dx's to make

$$uv = \int u\,dv - \int v\,du$$

So, we get

$$\int u\,dv = uv - \int v\,du$$

Strategy: When integrating the product of two functions, pick one to integrate (call it dv), and one to differentiate (call it u).

- Pick dv to be the most difficult function you know how to integrate.
- Pick u so that it gets simpler when differentiated.

Or, use ILATE: Pick $u =$ the first function in the list:

- I: Inverse trigonometric functions
- L: Logarithmic functions
- A: Algebraic functions (powers of x)
- T: Trigonometric functions

- E: Exponential functions

EXAMPLE 2.2.1

(i) Let $u = \ln(x)$ and $dv = x^2\,dx$, so we have $du = {}^1/_x\,dx$ and $v = {}^{x^3}/_3$.

$$\begin{aligned}\int x^2 \ln(x)\,dx &= \frac{x^3}{3}\ln(x) - \int \frac{x^3}{3}\cdot\frac{1}{x}\,dx \\ &= \frac{x^3}{3}\ln(x) - \int \frac{x^2}{3}\,dx \\ &= \frac{x^3}{3}\ln(x) - \frac{x^3}{9} + C\end{aligned}$$

(ii) Let $u = x$ and $dv = e^x\,dx$, so we have $du = dx$ and $v = e^x$.

$$\begin{aligned}\int xe^x\,dx &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + C\end{aligned}$$

(iii) Let $u = x$ and $du = dx$, so we have $du = \cos(x)\,dx$ and $v = \sin(x)$.

$$\begin{aligned}\int_0^\pi x\cos(x)\,dx &= \big[x\sin(x)\big]_0^\pi - \int_0^\pi \sin(x)\,dx \\ &= \big[\cos(x)\big]_0^\pi \\ &= \cos(\pi) - \cos(0) \\ &= -1 - 1 \\ &= -2\end{aligned}$$

(iv) Sometimes, we don't want to integrate any part! Let $u = \ln(x)$ and $dv = dx$, so we have $du = {}^1/_x\,dx$ and $v = x$.

$$\begin{aligned}\int \ln(x)\,dx &= x\ln(x) - \int \frac{x}{x}\,dx \\ &= x\ln(x) - x + C\end{aligned}$$

(v) We may need to apply it more than once! Let $u = x^2$ and $dv = \cos(x)\,dx$, so we have $du = 2x\,dx$ and $v = \sin(x)$.

$$\int x^2\cos(x)\,dx = x^2\sin(x) - \int 2x\sin(x)\,dx$$

Let $u = 2x$ and $dv = \sin(x)\,dx$, so we have $du = 2\,dx$ and $v = -\cos(x)$.

$$\begin{aligned}&= x^2\sin(x) - \left[-2x\cos(x) - \int -2\cos(x)\,dx\right] \\ &= x^2\sin(x) + 2x\cos(x) - \int 2\cos(x)\,dx \\ &= x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C\end{aligned}$$

(vi) And sometimes, we don't integrate at all! Let $u = \cos(x)$ and $dv = e^x\,dx$, so we have $du = -\sin(x)\,dx$ and $v = e^x$.

$$\begin{aligned}I &= \int e^x\cos(x)\,dx \\ &= e^x\cos(x) + \int e^x\sin(x)\,dx\end{aligned}$$

Let $u = \sin(x)$ and $dv = e^x\,dx$, so we have $du = \cos(x)\,dx$ and $v = e^x$.

$$= e^x \cos(x) + e^x \sin(x) - \int e^x \cos(x)\,dx$$
$$= e^x \cos(x) + e^x \sin(x) - I$$

So, $2I = e^x \cos(x) + e^x \sin(X)$, therefore

$$I = \frac{e^x \cos(x) + e^x \sin(x)}{2} + C$$

Neat!

(vii) Sometimes, a combination of methods is needed.

$$\begin{aligned}\int x^3 \cos(x^2)\,dx &= \int x^2 \cos(u) \frac{1}{2x}\,du \qquad\qquad u = x^2 \iff du = 2x\,dx \\ &= \frac{1}{2} \int x^2 \cos(u)\,du \\ &= \frac{1}{2} \int u \cos(u)\,du\end{aligned}$$

Now, do integration by parts with some unfortunate (but fine) letter choices! Let $u = u$ and $dv = \cos(u)\,du$, so we have $du = du$ and $v = \sin(u)$.

$$\begin{aligned}&= \frac{1}{2} u \sin(u) - \frac{1}{2} \int \sin(u)\,du \\ &= \frac{1}{2} u \sin(u) + \frac{1}{2} \cos(u) + C \\ &= \frac{1}{2} x^2 \sin(x^2) + \frac{1}{2} \cos(x^2) + C\end{aligned}$$

2.3 Partial Fractions

Partial fractions are useful for evaluating $\int \frac{p(x)}{q(x)}\,dx$ where p and q are polynomials.

Overall idea: break a difficult integrand into many easy ones!

REMARK 2.3.1

We will assume the degree of the denominator is **larger** than the degree of the numerator. If not, use long division first!

Table 2.1: How to Break up Fractions: The Rules

If the denominator has:	Then we write:
(I) Distinct linear factors	One constant per factor
(II) A repeated linear factor	One constant per power
(III) Distinct irreducible quadratic factors	One **linear term** per factor
(IV) Repeated irreducible quadratic factors	One linear term per power

EXAMPLE 2.3.2: Decomposition Practice

(i)

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

(ii)

$$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

(iii)

$$\frac{x^3+x+7}{x^2(x+1)^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{Ex+F}{x^2+1}$$

(iv)

$$\frac{x^2+7}{(x-1)^3(x^2+3)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{Dx+E}{x^2+3} + \frac{Fx+G}{(x^2+3)^3}$$

(v)

$$\frac{x^{10}+5}{(x+1)^3(x^2+1)} = \cdots \text{use long division first, not partial fractions}$$

REMARK 2.3.3: What integrals could we be left with after partial fractions?

C1

$$\int \frac{A}{ax+b}\,dx = \frac{A}{a}\ln|ax+b| + C$$

C2

$$\int \frac{A}{(ax+b)^n}\,dx = \frac{A}{a}\cdot\frac{(ax+b)^{-n+1}}{-n+1}$$

where $n \neq 0, 1$.

C3

$$\frac{Ax+B}{ax^2+bx+c} = \int \frac{Ax}{ax^2+bx+c} + \frac{B}{ax^2+bx+c}\,dx$$

C4

$$\frac{Ax+B}{(ax^2+bx+c)^n}$$

Note for C3 and C4, you may want to complete the square and use a trigonometric substitution. A regular substitution may also work.

EXAMPLE 2.3.4: Partial Fractions (Easy)

Using partial fractions, compute

$$\int \frac{x}{x^2-4x-5}\,dx$$

First, we break it up with partial fractions.

$$\frac{x}{x^2-4x-5} = \frac{x}{(x+1)(x-5)} = \frac{A}{x+1} + \frac{B}{x-5}$$

Multiply both sides by the LHS denominator to get the following.

$$\begin{aligned} x &= (x+1)(x-5)\left[\frac{A}{x+1} + \frac{B}{x-5}\right] \\ x &= A(x-5) + B(x+1) \end{aligned}$$

There are two ways we can solve for A and B.

(i) Linear Algebra!

$$x = Ax - 5A + Bx + B = (A+B)x + (-5A + B)$$

Therefore, $A + B = 1$ and $B - 5A = 0$. Thus, $A = 1/6$ and $B = 5/6$.

(ii) Substitute in "nice" values for x.

$$\begin{aligned} x = 5: &\quad 5 = A(0) + B(6) \\ x = 1: &\quad -1 = A(-6) + B(0) \end{aligned}$$

Thus, $A = 1/6$ and $B = 5/6$.

Either way, we get

$$\frac{x}{(x+1)(x-5)} = \frac{1/6}{x+1} + \frac{5/6}{x-5}$$

So,

$$\begin{aligned} \int \frac{x}{x^2 - 4x - 5}\,dx &= \frac{1}{6}\int \frac{1}{x+1}\,dx + \frac{5}{6}\int \frac{1}{x-5}\,dx \\ &= \frac{1}{6}\ln|x+1| + \frac{5}{6}\ln|x-5| + C \end{aligned}$$

EXAMPLE 2.3.5: Partial Fractions (Slightly Difficult)

$$\int \frac{x+3}{x^4 + 9x^2}\,dx$$

First, we break up with partial fractions.

$$\frac{x+3}{x^4+9x^2} = \frac{x+3}{x^2(x^2+9)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+9}$$

Multiply both sides by $x^2(x^2+9)$ to get the following.

$$\begin{aligned} x + 3 &= x(x^2+9)A + (x^2+9)B + x^2(Cx+D) \\ x + 3 &= Ax^3 + 9Ax + Bx^2 + 9B + Cx^3 + Dx^2 \\ x + 3 &= (A+C)x^3 + (B+D)x^2 + 9Ax + 9B \end{aligned}$$

Therefore, $A + C = 0$, $B + D = 0$, $9A = 1$, and $9B = 3$. Thus, $A = 1/9$, $B = 1/3$, $C = -1/9$, and $D = -1/3$. So,

$$\begin{aligned} \int \frac{x+3}{x^4+9x^2}\,dx &= \frac{1}{9}\int \frac{1}{x}\,dx + \frac{1}{3}\int \frac{1}{x^2}\,dx - \frac{1}{9}\int \frac{x}{x^2+9}\,dx - \frac{1}{3}\int \frac{1}{x^2+9}\,dx \\ &= \frac{1}{9}\ln|x| - \frac{1}{3x} - \frac{1}{9}\int \frac{x}{x^2+9}\,dx - \frac{1}{3}\left[\frac{1}{3}\arctan\left(\frac{x}{3}\right)\right] \end{aligned}$$

where we computed $\int \frac{1}{x^3+3}\,dx$ with remark 2.3.6. For $\int \frac{x}{x^2+9}\,dx$, use a substitution: $u = x^2 + 9 \iff du = 2x\,dx$.

$$\begin{aligned} \int \frac{x}{x^2+9}\,dx &= \int \frac{x}{u}\frac{1}{2x}\,du \\ &= \frac{1}{2}\int \frac{1}{u}\,du \\ &= \frac{\ln|u|}{2} + C \\ &= \frac{\ln|x^2+9|}{2} + C \end{aligned}$$

So, the final answer is:

$$\frac{1}{9}\ln|x| - \frac{1}{3x} - \frac{1}{18}\ln|x^2+9| - \frac{1}{9}\arctan\left(\frac{x}{3}\right) + C$$

REMARK 2.3.6: Useful Identity

$$\int \frac{1}{x^2+k^2}\,dx = \frac{1}{k}\arctan\left(\frac{x}{k}\right) + C$$

EXAMPLE 2.3.7: Partial Fractions (Long Division)

$$\int \frac{x^3-2x}{x^2+3x+2}\,dx$$

Using long division, we get

$$\int x - 3 + \frac{5x+6}{x^2+3x+2}\,dx$$

Now,

$$\frac{5x+6}{x^2+3x+2} = \frac{5x+6}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

Therefore, $5x+6 = A(x+2)+B(x+1)$. Substituting $x=-2$, we get $B=-4$. Substituting $x=-1$, we get $A=1$. Thus, the integral is:

$$\int x - 3 + \frac{1}{x+1} + \frac{4}{x+2}\,dx = \frac{x^2}{2} - 3x + \ln|x+1| + 4\ln|x+2| + C$$

2.4 Improper Integrals

So far, we have only examined integrals of continuous, or at least bounded functions. Let's see how to deal with a more general collection of functions!

In particular, we will examine two types:

1) Continuous functions over infinite intervals
2) Functions with infinite discontinuities

In particular:

- Type I: Infinite Intervals. Integrals of the form

$$\int_{-\infty}^{a} f(x)\,dx,\ \int_{a}^{\infty} f(x)\,dx,\ \int_{-\infty}^{\infty} f(x)\,dx$$

- Type II: Infinite Discontinuity. For example,

$$\int_{-1}^{1} \frac{1}{x}\,dx$$

  as there is an issue at $x=0$.

In all cases, the idea is to replace the problematic point with a letter and take a limit.

Let's see them in more detail now!

2.4.1 Type I

We replace the infinite endpoint with a letter and take a limit

- $$\int_{-\infty}^{a} f(x)\,dx = \lim_{b\to-\infty}\int_{b}^{a} f(x)\,dx$$

- $$\int_{\infty}^{a} f(x)\,dx = \lim_{b\to\infty}\int_{a}^{b} f(x)\,dx$$

- $$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{b_1\to-\infty}\int_{b_1}^{0} f(x)\,dx + \lim_{b_2\to\infty}\int_{0}^{b_2} f(x)\,dx$$

Don't use
$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_{-b}^{b} f(x)\,dx$$
This is called the "Cauchy Principal Value" and it is something else!

We say that the integral **converges** if all the limits exist (and are finite). The integral **diverges** if even one limit does not exist (or is $\pm\infty$).

EXAMPLE 2.4.1: Type I Integrals

Evaluate the following or show they diverge.

(i)

$$\begin{aligned}\int_{2}^{\infty}\frac{1}{x^2}\,dx &= \lim_{b\to\infty}\int_{2}^{b}\frac{1}{x^2}\,dx\\ &= \lim_{b\to\infty}\left[-\frac{1}{x}\right]_2^b\\ &= \lim_{b\to\infty}\left(-\frac{1}{b}+\frac{1}{2}\right)\\ &= \frac{1}{2}\end{aligned}$$

Thus, the integral converges.

(ii)

$$\int_{-\infty}^{\infty}\sin(x)\,dx = \lim_{b_1\to-\infty}\int_{b_1}^{0}\sin(x)\,dx + \lim_{b_2\to\infty}\int_{0}^{b_2}\sin(x)\,dx$$

Let's evaluate the first one:

$$\lim_{b_1\to-\infty}\int_{b_1}^{0}\sin(x)\,dx = \lim_{b_1\to-\infty}\left[-\cos(x)\right]_{b_1}^{0} = \lim_{b_1\to-\infty}\left[-\cos(0)+\cos(b_1)\right]$$

which does not exist. Therefore, this integral diverges, there is no need to check the second limit!

(iii)

$$\begin{aligned}
\int_0^\infty \frac{1}{1+x^2}\,dx &= \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx \\
&= \lim_{b\to\infty}\left[\arctan(x)\right]_0^b \\
&= \lim_{b\to\infty}\left[\arctan(b)-\arctan(0)\right] \\
&= \frac{\pi}{2}-0 \\
&= \frac{\pi}{2}
\end{aligned}$$

Thus, the integral converges.

Question: For which $p \in \mathbb{R}$ does $\int_1^\infty \frac{1}{x^p}\,dx$ converge?

Let's find out!

C1 $p > 1$.

$$\begin{aligned}
\lim_{b\to\infty}\int_1^b x^{-p}\,dx &= \lim_{b\to\infty}\left[\frac{x^{-p+1}}{p+1}\right]_1^b \\
&= \lim_{b\to\infty}\left(\frac{b^{-p+1}}{-p+1}-\frac{1^{-p+1}}{-p+1}\right) \\
&= \frac{1}{p-1}
\end{aligned}$$

since $-p+1<0$, so $b^{-p+1}\to 0$. So, the integral converges if $p>1$.

C2 $p < 1$. The calculation is the same as C1, until:

$$\lim_{b\to\infty}\left(\frac{b^{-p+1}}{-p+1}-\frac{1}{-p+1}=\infty\right)$$

since $-p+1>0$, so $b^{-p+1}\to\infty$. So, the integral diverges if $p<1$.

C3 $p = 1$.

$$\begin{aligned}
\lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx &= \lim_{b\to\infty}\left[\ln|x|\right]_1^b \\
&= \lim_{b\to\infty}\left(\ln|b|-\ln|1|\right) \\
&= \infty
\end{aligned}$$

So, the integral diverges if $p=1$.

Therefore, we have proven:

THEOREM 2.4.2: p-Integrals

The improper integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converges if and only if $p>1$.
If $p>1$,

$$\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1}$$

Next, let's examine some properties of Type I improper integrals.

THEOREM 2.4.3: Properties of Type I Improper Integrals

Suppose $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ both converge.

(1) $\int_a^\infty cf(x)\,dx$ converges for any $c \in \mathbb{R}$, and

$$\int_a^\infty cf(x)\,dx = c\int_a^\infty f(x)\,dx$$

(2) $\int_a^\infty f(x) + g(x)\,dx$ converges, and

$$\int_a^\infty f(x) + g(x)\,dx = \int_a^\infty f(x)\,dx + \int_a^\infty g(x)\,dx$$

(3) If $f(x) \leqslant g(x)$ for all $x \geqslant a$, then

$$\int_a^\infty f(x)\,dx \leqslant \int_a^\infty g(x)\,dx$$

(4) If $a < c < \infty$, then $\int_c^\infty f(x)\,dx$ converges, and

$$\int_a^\infty f(x)\,dx = \int_a^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

Evaluating integrals in general is hard, and determining if an improper integral converges may be even harder! However, we do have a way of comparing a difficult integral to a simpler one (for example, a p-Integral!)