1 Math 239 Tutorial # 2 (Shayla)

Problem 1. Use the negative binomial theorem and substitutions to give a formula for the coefficient of x^n in $(1-3x)^{-1} + 2(1-2x)^{-2}$.

$$(1-x)^{-k} = \sum_{n \geq 0} \binom{n+k-1}{k-1} x^n$$

Solution.

Hint: Start with $[x^n](1-3x)^{-1}$

$x \to 3x$, $k \to 1$

$$(1-3x)^{-1} = \sum_{n \geq 0} \binom{n+1-1}{1-1} (3x)^n = \sum_{n \geq 0} \binom{n}{0} 3^n x^n = \sum_{n \geq 0} 3^n x^n$$

$$(1-2x)^{-2} = \sum_{n \geq 0} \binom{n+2-1}{2-1} (2x)^n = \sum_{n \geq 0} \binom{n+1}{1} 2^n x^n = \sum_{n \geq 0} (n+1) 2^{n+1}$$

$$[x^n](\text{expr.}) = 3^n + (n+1)2^{n+1}$$

Problem 2. Let $F(x) = x + x^2 + \cdots$ and let $G(x) = 1 + 3x + 2x^2$. Compute the coefficient of x^n in $G(F(x))$.

Solution.

$$G(F(x)) = 1 + 3(x + x^2 + \cdots) + 2(x + x^2 + \cdots)^2$$

$$\begin{aligned}
(x + x^2 + \cdots)^2 &= \left(\frac{1}{1-x} - 1\right)^2 \\
&= \left(\frac{x}{1-x}\right)^2 \\
&= x^2(1-x)^{-2} \\
&= x^2 \sum_{n \geq 0} \binom{n+2-1}{2-1} x^n \\
&= x^2 \sum_{n \geq 0} (n+1) x^n
\end{aligned}$$

$$[x^{n-2}] \sum_{n \geq 0} \binom{n+1}{1} x^n \qquad [x^n] \sum_{n \geq 0} (n+1) x^{n+2}$$

$$[x^{n-2}] \sum_{n \geq 0} \binom{n+1}{1} x^n = n - 1 \text{ where } m = n - 2 \geq 0$$

$$[x^n] 2(x + x^2 + \cdots)^2 = \begin{cases} 2(n-1), \; n \geq 2 \\ 0, \; n = 0, 1 \end{cases}$$

$$[x^n] 1 = \begin{cases} 0, \; n > 0 \\ 1, \; n = 0 \end{cases}$$

$$[x^n] 3(x + x^2 + \cdots) = \begin{cases} 3, \; n \geq 1 \\ 0, \; n = 0 \end{cases}$$

Thus,

$$= \begin{cases} 2n+1,\ n \geq 2 \\ 3,\ n = 1 \\ 1,\ n = 0 \end{cases}$$

Problem 3. Show that if $F(x) = a_0 + a_1x + a_2x^2 + \cdots$. Then

$$F(x)(1-x) = a_0 + (a_1 - a_0)x + (a_2 - a_1)x^2 + \cdots$$

and

$$F(x)(1-x)^{-1} = \sum_{n=0}^{\infty} c_n x^n,$$

where $c_n = a_0 + a_1 + \cdots + a_n$.

Solution.

Note that $F(x)(1-x) = F(x) - xF(x)$

We know that $(1-x)^{-1} = 1 + x + x^2 + \cdots$, thus

$$\begin{aligned} F(x)(1-x)^{-1} &= F(x) + xF(x) + x^2F(x) + \cdots \\ &= \sum_{i \geq 0} x^i F(x) \\ &= \sum_{i \geq 0} [x^{n-i}]F(x) \\ &= \sum_{i \geq 0}^{n} a_{n-i} \text{ for } 0 \leq k = n-i \leq n \\ &= \sum_{k=0}^{n} a_k \end{aligned}$$

where $[x^{n-i}]F(x) = a_{n-i} = \begin{cases} a_{n-i},\ i \leq n \\ 0,\ i > n \end{cases}$

Problem 4. Show that for $k \geq 1$ and $n \geq 1$, we have

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{n-i+k-1}{k-1} = 0,$$

where we interpret $\binom{j}{i} = 0$ when $j < i$. Hint: Look at $1 = (1-x)^k(1-x)^{-k})$ and compute the coefficient of x^n in both sides.

Solution.

$[x^n]1 = 0$

$[x^n](1-x)^k(1-x)^{-k}$; coefficient of x^i in $(1-x)^k$ and x^{n-i} for $(1-x)^{-k}$; add them up.

$$\begin{aligned} &= \sum_{i=0}^{n} [x^i](1-x)^{-k}[x^{n-i}](1-x)^{-k} \\ &= \sum_{i=0}^{n} (-1)^i \binom{k}{i} \binom{n-i+k-1}{k-1} \text{ by Bin. \& NB .thm} \\ &= \sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{n-i+k-1}{k-1} \end{aligned}$$