

# STAT 443 - Forecasting

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# Chapter 1

## TODO

### 1.1 What is a time series?

In classical statistics, we normally consider  $X_1, \dots, X_n \in \mathbf{R}^p$ , a **simple random sample**.

In particular,

- (1)  $X_1, \dots, X_n$  are i.i.d. (independent and identically distributed)
- (2)  $X_i \sim F_\theta$  which is a common distribution characterized by  $\theta$ .

Examples:

1.  $X_i \stackrel{\text{iid}}{\sim} \mathcal{N}(\mu, \sigma^2)$ , and we wish to estimate and perform inference on  $\mu$  and  $\sigma^2$ .
2.  $X_i = \begin{bmatrix} Y_i \\ Z_i \end{bmatrix}$  where  $Y_i$  is a dependent variable, and  $Z_i$  is an independent variable. Perhaps we happen to observe  $Y_i$  and  $Z_i$  in pairs, and we stop at a model:

$$Y_i = \beta^\top Z_i + \varepsilon_i \quad \text{where } \varepsilon_i \stackrel{\text{iid}}{\sim} \mathcal{N}(0, \sigma_\varepsilon^2)$$

#### REMARK 1.1.1

The relationship between  $Y_i$  and  $Z_i$  doesn't depend on  $i$ , it only depends through the common parameter  $\beta$ , and it assumes that  $\varepsilon$  has fixed variance for each  $i$ .

3. In such settings, one is typically interested in:
  - (a) Prediction: how based on the data, can we predict how these variables would behave in the future?
  - (b) Inference: how do we use the data to try to estimate and understand better the underlying mechanism which generates the data? For example, a linear model or simple Gaussian model.

#### DEFINITION 1.1.2: Time Series

We say  $X_1, \dots, X_T$  is an (observed) **time series** of length  $T$  if  $X_t$  denotes an observation obtained at time  $t$ .

In particular, the observations are ordered in time.

#### DEFINITION 1.1.3: Real-valued

If  $X_t \in \mathbf{R}$ , we say  $X_1, \dots, X_T$  is a **real-valued** or **scalar** time series.

**DEFINITION 1.1.4: Multivariate**

If  $X_t \in \mathbb{R}^p$ , we say  $X_1, \dots, X_T$  is a **multivariate** or **vector-valued** time series.

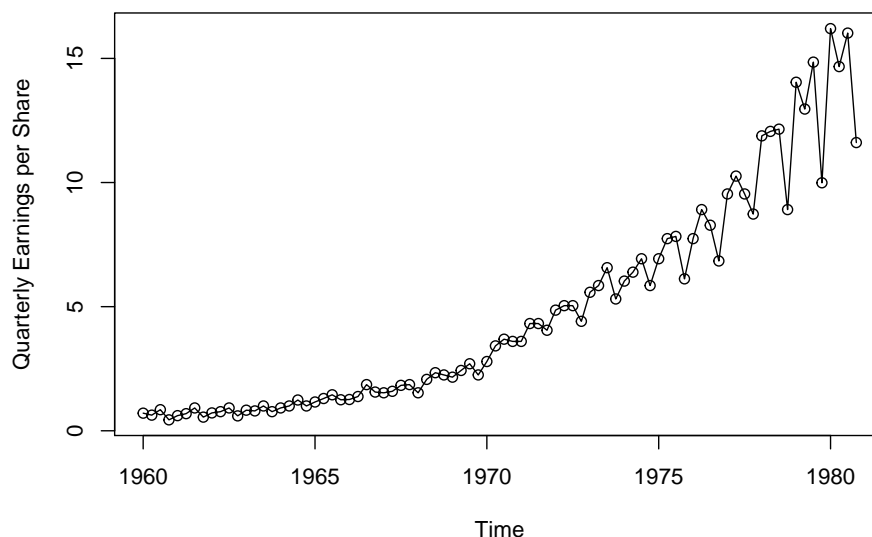


Figure 1.1: Quarterly Johnson and Johnson Earnings

Observe that in Figure 1.1:

- The earnings are steadily increasing over time.
- There is heterogeneity in the variance over time.

With time series data, we are typically concerned with the same goals as in classical statistics (prediction and inference). However, in contrast, with time series, the data often exhibit:

(1) Heterogeneity

- Time trends  $\rightarrow \mathbb{E}[X_t] \neq \mathbb{E}[X_{t+h}]$
- Heteroskedasticity  $\rightarrow \mathbb{V}(X_t) \neq \mathbb{V}(X_{t+h})$

In classical statistics, it's assumed that all the observations have the same distribution which is clearly not the case in time series.

(2) Serial Dependence (Serial Correlation)

- Observations that are temporally close appear to depend on each other.

In classical statistics, each successive observation is assumed to be independent which is clearly not the case in time series.

Observe that in Figure 1.2:

- The global temperature is steadily increasing over time.
- There is heterogeneity in the mean over time.
- Heterogeneity in the variance over time is not very apparent.

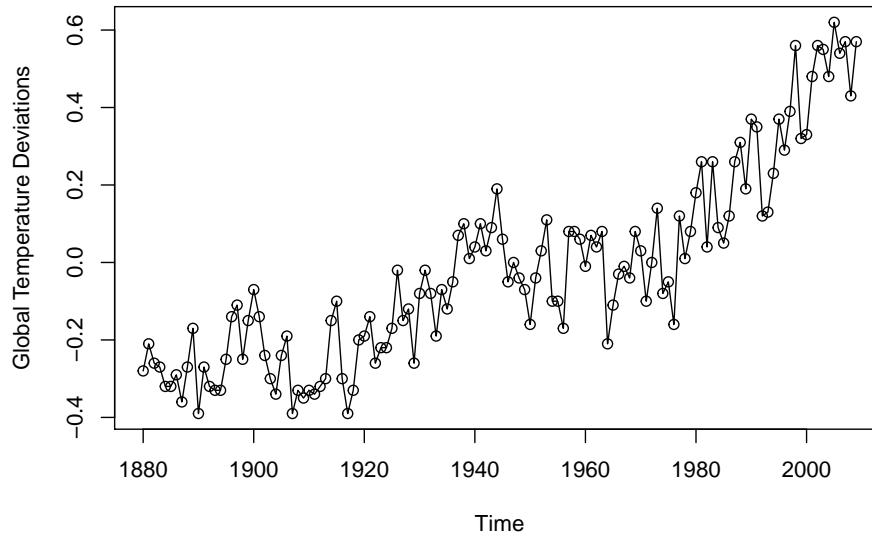


Figure 1.2:  $x_t$  is the deviation of global mean yearly temperature from the mean computed from 1951–1980

- Serial dependence occurs.

#### DEFINITION 1.1.5: Time series (Formal Definition)

We say  $\{X_t\}_{t \in \mathbf{Z}}$  is a **time series** if  $\{X_t : t \in \mathbf{Z}\}$  is a stochastic process indexed by  $\mathbf{Z}$ .

In other words, there is a common probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  so that for all  $X_t : \Omega \rightarrow \mathbf{R}$  is a random variable.

In relation to the original definition, we say  $X_1, \dots, X_T$  is an **observed stretch (realization, simple path)** of length  $T$  from  $\{X_t\}_{t \in \mathbf{Z}}$ .

Formally speaking, we think of a time series as being a little snippet of one long sample path the stochastic process for which would characterize all the serial dependence, time trends, and heteroskedasticity, that exist within a time series.

## 1.2 Basic Principles of Forecasting

Consider a time series of length  $T$ , namely  $X_1, \dots, X_T$ . Based on  $X_1, \dots, X_T$ , we would like to produce a “best guess” for  $X_{T+h}$ :

$$\hat{X}_{T+h} = \hat{X}_{T+h|T} = f_h(X_T, \dots, X_1)$$

#### DEFINITION 1.2.1: Forecast, Horizon

For  $h \geq 1$ , our “best guess”

$$\hat{X}_{T+h} = f_h(X_T, \dots, X_1)$$

is called a **forecast** of  $X_{T+h}$  at **horizon**  $h$ .

There are two primary goals in forecasting:

- Goal 1: Choose  $f_n$  “optimally.” Normally, we or the practitioner have some measure, say  $L(\cdot, \cdot)$ , in mind for determining how “close”  $\hat{X}_{T+h}$  is to the true value,  $X_{T+h}$ . We then wish to choose  $f_h$  so that  $L(X_{T+h}, f_h(X_T, \dots, X_1))$  is minimized.

**EXAMPLE 1.2.2**

Most common measure  $L(\cdot, \cdot)$  is mean-squared error (MSE), where

$$L(X, Y) = \mathbb{E}[(X - Y)^2]$$

- Goal 2: Quantify the uncertainty in the forecast. This entails providing some description of how close we expect  $\hat{X}_{T+h}$  to be to  $X_{T+h}$ .

**EXAMPLE 1.2.3: Why is it important to quantify uncertainty?**

Suppose every minute, we flip a coin and denote

- $H \rightarrow 1$
- $T \rightarrow -1$
- $X_t = \text{outcome in minute } t$ , where  $t = 1, \dots, T$ .

This produces a time series of length  $T$ , which is a random sequence of (1)’s and (−1)’s. Note  $\mathbb{E}[X_t] = 0$ . So, if we wish to forecast for  $h \geq 1$ , consider  $\hat{X}_{T+h} = f(X_T, \dots, X_1)$

$$\begin{aligned} L(X_{T+h}, \hat{X}_{T+h}) &= \mathbb{E}[(X_{T+h} - \hat{X}_{T+h})^2] \\ &= \mathbb{E}[X_{T+h}^2] + \mathbb{E}[\hat{X}_{T+h}^2] - 2\mathbb{E}[X_{T+h}\hat{X}_{T+h}] \\ &= \mathbb{E}[X_{T+h}^2] + \mathbb{E}[\hat{X}_{T+h}^2] - 2\mathbb{E}[X_{T+h}]\mathbb{E}[\hat{X}_{T+h}] \\ &= \mathbb{E}[X_{T+h}^2] + \mathbb{E}[\hat{X}_{T+h}^2] \end{aligned}$$

Furthermore, note that  $\mathbb{E}[X_{T+h}^2] = \mathbb{V}(X_t)$  since  $\mathbb{E}[X_{T+h}] = 0$ .

We can write  $\mathbb{E}[X_{T+h}\hat{X}_{T+h}] = \mathbb{E}[X_{T+h}]\mathbb{E}[\hat{X}_{T+h}]$  since  $\hat{X}_{T+h}$  is a function of the data  $X_T, \dots, X_1$ , and hence independent of  $X_{T+h}$ .

We can minimize this by taking  $\hat{X}_{T+h} = 0$ . There’s nothing “wrong” with this forecast. But ideally, we would also be able to say that the sequence appears to be random, and that we don’t expect this forecast to be close to the actual value.

Furthermore, for this basic reason, one can always argue that any forecast that’s not accompanied with some type of quantification of how close we expect the forecast to be, is at very least hard to interpret; at worst, meaningless because it doesn’t describe the accuracy for which we expect the forecast to perform.

How can we quantify the uncertainty in forecasting?

Ideal: The predictive distribution:

$$X_{T+h} \mid X_T, \dots, X_1$$

Excellent: Predictive intervals/sets. For some  $\alpha \in (0, 1)$  find  $I_\alpha$  so that

$$\mathbb{P}(X_{T+h} \in I_\alpha \mid X_T, \dots, X_1) = \alpha$$

( $\alpha = 0.95$ , for example). Often such intervals take the form

$$I_\alpha = (\hat{X}_{T+h} - \hat{\sigma}_h, \hat{X}_{T+h} + \hat{\sigma}_h)$$

Concluding remarks:

1. Estimating predictive distribution leads one towards estimating the joint distribution of

$$X_{T+h}, X_T, \dots, X_1$$

For example, ARMA and ARIMA models.

2. It is important that we acknowledge that some things cannot be predicted!

“It is tough to make predictions, especially about the future.”—Yogi Berra

## 1.3 Definitions of Stationary

Given a time series  $X_1, \dots, X_T$ , we are frequently interested in estimating the joint distribution of

$$X_{T+h}, X_T, \dots, X_1$$

which is useful for forecasting and inference.

The joint distribution is a feature of the process  $\{X_t\}_{t \in \mathbf{Z}}$

$$X_1, \dots, X_T \xrightarrow{\text{infer}} \{X_t\}_{t \in \mathbf{Z}}$$

- $X_1, \dots, X_T$ : Observed data.
- $\{X_t\}_{t \in \mathbf{Z}}$ : Stochastic process.

Worst case:  $X_t \sim F_t$ , where  $F_t$  is a *changing* function of  $t$ . If so, it is hard to pool the data  $X_1, \dots, X_T$ , to estimate  $F_t$ . If **serial dependence** occurs; that is, if the distribution of  $(X_t, X_{t+h})$  depends strongly on  $t$ , then we have a similar problem in estimating e.g.  $\text{Cov}(X_t, X_{t+h})$ .

### DEFINITION 1.3.1: Strictly stationary

We say that a time series  $\{X_t\}_{t \in \mathbf{Z}}$  is **strictly stationary (strongly stationary)** if for each  $k \geq 1$ ,  $i_1, \dots, i_k, h \in \mathbf{Z}$ ,

$$(X_{i_1}, \dots, X_{i_k}) \stackrel{d}{=} (X_{i_1+h}, \dots, X_{i_k+h})$$

If we look at the  $k$ -dimensional joint distribution  $(X_{i_1}, \dots, X_{i_k})$  of the series at points  $i_1, \dots, i_k$ , then **strict stationary means this is shift-invariant**. That is, shifting the window on which you view the data, does not change its distribution. This implies that if  $F_t = \text{CDF of } X_t$ , then  $F_t = F_{t+h} = F$  that is, all variables have a common distribution function.

### DEFINITION 1.3.2: Mean function

For a time series  $\{X_t\}_{t \in \mathbf{Z}}$ , with  $\mathbb{E}[X_t^2] < \infty$  for all  $t \in \mathbf{Z}$ , we denote the **mean function** of the time series as

$$\mu_t = \mathbb{E}[X_t]$$

### DEFINITION 1.3.3: Autocovariance function, Lag

The **autocovariance** function of the time series  $\{X_t\}_{t \in \mathbf{Z}}$  is defined as

$$\gamma(t, s) = \mathbb{E}[(X_t - \mu_t)(X_s - \mu_s)] = \text{Cov}(X_t, X_s)$$

### DEFINITION 1.3.4: Weakly stationary, Lag

We say that a time series  $\{X_t\}_{t \in \mathbf{Z}}$  is **weakly stationary** if  $\mathbb{E}[X_t] = \mu$  (does not depend on  $t$ ), and if

$$\gamma(t, s) = f(|t - s|)$$

that is,  $\gamma(t, s)$  is a function of  $|t - s|$ . In this case, we usually write

$$\gamma(h) = \text{Cov}(X_t, X_{t+h})$$

and we call the input  $h$  the **lag** parameter.

Additional terminology:

- The property when  $\mathbb{E}[X_t] = \mu$  does not depend on  $t$  is often called the **first order stationary**.
- The property when  $\gamma(t, s) = \gamma(|t - s|)$  only depends on the lag  $|t - s|$  is called the **second order stationary**.
- For a second order stationary process,

$$\begin{aligned}\gamma(h) &= \text{Cov}(X_t, X_{t+h}) \\ &= \text{Cov}(X_{t-h}, X_{t-h+h}) & t \rightarrow (t-h) \\ &= \text{Cov}(X_t, X_{t-h}) \\ &= \gamma(-h)\end{aligned}$$

Since  $\gamma(h) = \gamma(-h)$ , we normally, we only record  $\gamma(h)$  for  $h \geq 0$ .

## 1.4 White Noise and Stationary Examples

### DEFINITION 1.4.1: Strong white noise

We say  $\{X_t\}_{t \in \mathbb{Z}}$  is a **strong white noise** if  $\mathbb{E}[X_t] = 0$  and the  $\{X_t\}_{t \in \mathbb{Z}}$  are i.i.d.

### DEFINITION 1.4.2: Weak white noise

We say  $\{X_t\}_{t \in \mathbb{Z}}$  is a **weak white noise** if  $\mathbb{E}[X_t] = 0$  and

$$\gamma(t, s) = \text{Cov}(X_t, X_s) = \begin{cases} \sigma^2 & |t - s| = 0 \\ 0 & |t - s| > 0 \end{cases}$$

### DEFINITION 1.4.3: Gaussian white noise

We say  $\{X_t\}_{t \in \mathbb{Z}}$  is a **Gaussian white noise** if  $X_t \stackrel{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2)$ .

Figure 1.3 is a Gaussian *white* noise series. **White** comes from spectral analysis, in which a white noise series shares the same spectral properties as white light: all periodicities occur with equal strength.

### EXAMPLE 1.4.4

Suppose  $\{W_t\}_{t \in \mathbb{Z}}$  is a strong white noise, then  $\mathbb{E}[W_t] = 0$ ; that is, it doesn't depend on  $t$ .

$$\gamma(t, s) = \text{Cov}(W_t, W_s) = \mathbb{E}[W_t W_s] = \begin{cases} \sigma_W^2 & |t - s| = 0 \\ 0 & |t - s| > 0 \end{cases}$$

only depends on  $|t - s|$ .

$\{W_t\}_{t \in \mathbb{Z}}$  is **weakly stationary**. Furthermore,  $\{W_t\}_{t \in \mathbb{Z}}$  is **strictly stationary**. Let  $k \geq 1$  with  $i_1 < \dots < i_k$  and  $h \in \mathbb{Z}$ , then

$$\begin{aligned}\mathbb{P}(W_{i_1} \leq t_1, \dots, W_{i_k} \leq t_k) &= \prod_{j=1}^k \mathbb{P}(W_{i_j} \leq t_j) && \text{independence} \\ &= \prod_{j=1}^k \mathbb{P}(W_{i_j+h} \leq t_j) \\ &= \mathbb{P}(W_{i_1+h} \leq t_1, \dots, W_{i_k+h} \leq t_k)\end{aligned}$$



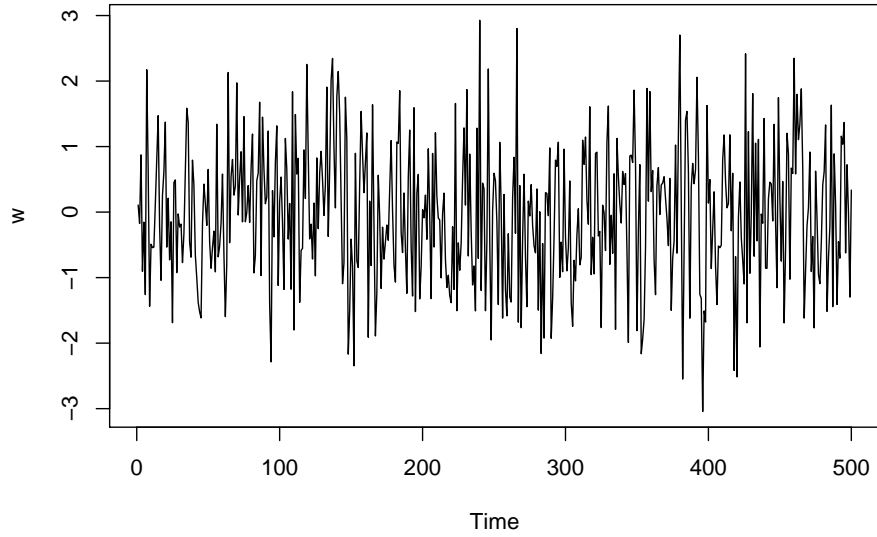


Figure 1.3: Gaussian White Noise of Length 500

**EXAMPLE 1.4.5**

Suppose  $\{W_t\}_{t \in \mathbb{Z}}$  is a strong white noise. Define  $X_t = W_t + \theta W_{t-1}$  for  $\theta \in \mathbb{R}$ . Since  $\{W_t\}_{t \in \mathbb{Z}}$  is a strong white noise, we have  $\mathbb{E}[W_t] = 0$  for all  $t$ , and so we have  $\mathbb{E}[X_t] = \mathbb{E}[W_t + \theta W_{t-1}] = \mathbb{E}[W_t] + \theta \mathbb{E}[W_{t-1}] = 0$  which is first order stationary.

$$\gamma(t, s) = \text{Cov}(X_t, X_s) = \begin{cases} (1 + \theta^2)\sigma_W^2 & |t - s| = 0 \\ \theta\sigma_W^2 & |t - s| = 1 \\ 0 & |t - s| > 0 \end{cases}$$

We obtain these calculations as follows:

- $|t - s| = 0$ .

$$\mathbb{E}[(W_t + \theta W_{t-1})^2] = \mathbb{E}[W_t^2] + \theta^2 \mathbb{E}[W_{t-1}^2] + 2\mathbb{E}[\theta W_t W_{t-1}] = (1 + \theta^2)\sigma_W^2$$

- $t = s + 1$  (or  $s = t + 1$ ).

$$\mathbb{E}[(W_{s+1} + \theta W_s)(W_s + \theta W_{s-1})] = \theta \mathbb{E}[W_s^2] = \theta \sigma_W^2$$

since  $W_{s+1}$  is independent of  $W_s$  and  $W_{s-1}$ . The calculation is easy to verify.

- $|t - s| > 1$ .  $W_t + \theta W_{t-1}$  is independent of  $W_s + \theta W_{s-1}$ .

$\{X_t\}_{t \in \mathbf{Z}}$  is also strictly stationary. Suppose  $k \geq 1$ ,  $i_1, \dots, i_k, h \in \mathbf{Z}$  with  $i_1 < \dots < i_k$ , then

$$\begin{aligned} \mathbb{P}(X_{i_1} \leq t_1, \dots, X_{i_k} \leq t_k) &= \mathbb{P}(W_{i_1} + \theta W_{i_1-1} \leq t_1, \dots, W_{i_k} + \theta W_{i_k-1} \leq t_k) \\ &= \mathbb{P}\left(\begin{bmatrix} W_{i_1-1} \\ W_{i_1} \\ \vdots \\ W_{i_k} \end{bmatrix} \in B\right) \\ &= \mathbb{P}\left(\begin{bmatrix} W_{i_1-1+h} \\ \vdots \\ W_{i_k+h} \end{bmatrix} \in B\right) \\ &= \mathbb{P}(X_{i_1+h} \leq t_1, \dots, X_{i_k+h} \leq t_k) \end{aligned}$$

where  $B$  is some subset of  $\mathbf{R}^{i_k - i_1 + 1}$ , and hence is shift-invariant.

#### DEFINITION 1.4.6: Bernoulli shift

Suppose  $\{\varepsilon_t\}_{t \in \mathbf{Z}}$  is a strong white noise. If  $X_t = g(\varepsilon_t, \varepsilon_{t-1}, \dots)$  for some function  $g : \mathbf{R}^\infty \rightarrow \mathbf{R}$ , we say that  $\{X_t\}_{t \in \mathbf{Z}}$  is a **Bernoulli shift**.

#### THEOREM 1.4.7

If  $\{X_t\}_{t \in \mathbf{Z}}$  is a Bernoulli shift, then  $\{X_t\}_{t \in \mathbf{Z}}$  is strictly stationary.

#### REMARK 1.4.8

Norbert Wiener conjectured that **every** stationary sequence is a Bernoulli shift, which is not true. The truth is, almost every one is.

#### EXERCISE 1.4.9

Suppose  $\{W_t\}_{t \in \mathbf{Z}}$  is a strong white noise. The **two-sided random walk** is defined as

$$X_t = \sum_{i=0}^t W_i + \sum_{i=t}^{-1} W_i$$

Show that  $\{X_t\}_{t \in \mathbf{Z}}$  is first order stationary, but  $\{X_t\}_{t \in \mathbf{Z}}$  is not second order stationary.

## 1.5 Weak versus Strong Stationary

Sadly,  $\{X_t\}_{t \in \mathbf{Z}}$  is strictly stationary does not imply  $\{X_t\}_{t \in \mathbf{Z}}$  is weakly stationary.

#### EXAMPLE 1.5.1

Suppose  $X_t \stackrel{\text{iid}}{\sim}$  Cauchy Random Variables; that is,

$$\mathbb{P}(X_t \leq s) = \int_{-\infty}^s \frac{1}{\pi(1+x^2)} dx$$

Then,  $\mathbb{E}[X_t]$  does not exist, and hence  $\{X_t\}_{t \in \mathbf{Z}}$  cannot be weakly stationary. However,  $\{X_t\}_{t \in \mathbf{Z}}$  is strictly stationary in this case since  $\{X_t\}_{t \in \mathbf{Z}}$  is a strong white noise.

**THEOREM 1.5.2**

If  $\{X_t\}_{t \in \mathbb{Z}}$  is strongly stationary and  $\mathbb{E}[X_0^2] < \infty$ , then  $\{X_t\}_{t \in \mathbb{Z}}$  is weakly stationary.

**Proof of: Theorem 1.5.2**

Note that if  $\{X_t\}_{t \in \mathbb{Z}}$  is strictly stationary,

$$(X_t) \stackrel{d}{=} (X_0)$$

so that  $\mathbb{E}[X_t] = \mathbb{E}[X_0]$  which does not depend on  $t$ , and also

$$\mathbb{V}(X_t) = \mathbb{V}(X_0)$$

By the Cauchy-Schwarz inequality,

$$\gamma(t, s) = \text{Cov}(X_t, X_s) \leq \mathbb{V}(X_t^2) < \infty$$

and suppose  $t < s$ ,

$$\text{Cov}(X_t, X_s) = \text{Cov}(X_0, X_{s-t}) = f(|s-t|)$$

since it is shift-invariant, and hence if we shift everything over by  $t$ ,

$$(X_t, X_s) \stackrel{d}{=} (X_{t-t}, X_{s-t}) \stackrel{d}{=} (X_0, X_{s-t})$$

**DEFINITION 1.5.3: Gaussian process**

$\{X_t\}_{t \in \mathbb{Z}}$  is said to be a **Gaussian process (Gaussian time series)** if for each  $k \geq 1$ ,  $i_1 < i_2 < \dots < i_k$  we have

$$(X_{i_1}, \dots, X_{i_k}) \sim \text{MVN}(\boldsymbol{\mu}_k(i_1, \dots, i_k), \boldsymbol{\Sigma}_{k \times k}(i_1, \dots, i_k))$$

$$\boldsymbol{\mu}_k = \begin{bmatrix} \mathbb{E}[X_{i_1}] \\ \vdots \\ \mathbb{E}[X_{i_k}] \end{bmatrix} \quad \boldsymbol{\Sigma}_{k \times k} = \text{Cov}(X_{i_j}, X_{i_r})_{1 \leq j, r \leq k}$$

**THEOREM 1.5.4**

If  $\{X_t\}_{t \in \mathbb{Z}}$  is weakly stationary and a Gaussian process, then  $\{X_t\}_{t \in \mathbb{Z}}$  is strictly stationary.

**Proof of: Theorem 1.5.4**

If  $\{X_t\}_{t \in \mathbb{Z}}$  is weakly stationary, then  $\mathbb{E}[X_t] = \mu$  for all  $t$ .

$$(X_{i_1}, \dots, X_{i_k}) \rightarrow \begin{bmatrix} \mathbb{E}[X_{i_1}] \\ \vdots \\ \mathbb{E}[X_{i_k}] \end{bmatrix} = \begin{bmatrix} \mu \\ \vdots \\ \mu \end{bmatrix} = \boldsymbol{\mu} = \begin{bmatrix} \mathbb{E}[X_{i_1+h}] \\ \vdots \\ \mathbb{E}[X_{i_k+h}] \end{bmatrix}$$

Also,

$$\begin{aligned} \mathbb{V}(X_{i_1}, \dots, X_{i_k}) &= \text{Cov}(X_{i_j}, X_{i_r})_{1 \leq j, r \leq k} \\ &= \text{Cov}(X_0, X_{i_r-i_j})_{1 \leq j, r \leq k} \\ &= \text{Cov}(X_0, X_{i_r+h}, X_{i_r+h-(i_j+h)}) \\ &= \text{Cov}(X_{i_j+h}, X_{i_r+h}) \\ &= \mathbb{V}(X_{i_1+h}, \dots, X_{i_k+h}) \end{aligned}$$

**EXAMPLE 1.5.5**

Using the Gaussian assumption

$$(X_{i_1}, \dots, X_{i_k}) \stackrel{d}{=} \text{MVN}(\boldsymbol{\mu}, \boldsymbol{\Sigma}_{k \times k}) \stackrel{d}{=} (X_{i_1+h}, \dots, X_{i_k+h})$$

Hence  $\{X_t\}_{t \in \mathbb{Z}}$  is strictly stationary in this case.

**EXERCISE 1.5.6**

Prove that if  $\{X_t\}_{t \in \mathbb{Z}}$  is not weakly stationary; that is, either  $\mathbb{E}[X_t]$  depends on  $t$  or  $\gamma(t, s)$  does not depend on the lag, then  $\{X_t\}_{t \in \mathbb{Z}}$  is not strictly stationary.

## 1.6 † Theoretical L2 Framework for Time Series

- $X_t = \lim_{h \rightarrow \infty} X_{h,t}$ . In what sense does this limit exist?
- How “close” are two random variables  $X$  and  $Y$ ?
- Is there a random variable that achieves

$$\inf_{y \in S} d(Y, S)$$

**DEFINITION 1.6.1:  $L^2$  space**

Consider a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . The space  $L^2$  is the set of random variables  $X : \Omega \rightarrow \mathbf{R}$  measurable so that  $\mathbb{E}[X^2] < \infty$ .

**DEFINITION 1.6.2:  $L^2$ -time series**

We say that  $\{X_t\}_{t \in \mathbb{Z}}$  is an  $L^2$ -time series if  $X_t \in L^2$  for all  $t \in \mathbb{Z}$ .

$L^2$  is a Hilbert space when equipped with inner product,  $X, Y \in L^2$ .

$$\langle X, Y \rangle = \mathbb{E}[XY]$$

$\langle \cdot, \cdot \rangle$  is an inner product since it is

- (1) Linear:  $\langle aX + bY, Z \rangle = a\langle X, Z \rangle + b\langle Y, Z \rangle$ .
- (2) “Almost” Positive Definite:  $\langle X, X \rangle = \mathbb{E}[X^2] = 0 \iff X = 0$  almost surely. Which implies  $\mathbb{P}(X = 0) = 1$ .
- (3) Symmetric:  $\langle X, Y \rangle = \langle Y, X \rangle$ .

$L^2$  is complete with this inner product; that is, whenever  $X_n \in L^2$  so that  $\mathbb{E}[(X_n - X_m)^2] \rightarrow 0$  as  $n, m \rightarrow \infty$ , then there exists  $X \in L^2$  so that  $X_n \rightarrow X$ ; that is,  $\mathbb{E}[(X_n - X)^2] \rightarrow 0$ . This follows from the “famous” Riesz-Fischer Theorem.

### 1.6.1 Useful Tools for Time Series

#### (1) Existence of Limits

$$X_{t,n} = \sum_{j=0}^n \Psi_j \varepsilon_{t-j}$$

$\{\varepsilon_t\}_{t \in \mathbb{Z}}$  is a strong white noise. Since for  $n > m$ ,

$$\mathbb{E}[(X_{t,n} - X_{t,m})^2] = \mathbb{E}\left[\left(\sum_{j=m+1}^n \Psi_j \varepsilon_{t-j}\right)^2\right] = \sum_{j=m+1}^n \Psi_j^2 \sigma_\varepsilon^2 \rightarrow 0 \text{ as } n, m \rightarrow \infty$$

only if  $\sum_{j=0}^{\infty} \psi_j^2 < \infty$ , then there **must** exist a random variable  $X_t$  (by the completeness of  $L^2$ ), so that

$$X_t = \lim_{n \rightarrow \infty} X_{t,n} = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$$

(2) **Projection Theorem and Forecasting.** Forecasting can be often cast as finding a random variable  $Y$  among a collection of possible forecasts  $\mathcal{M}$  (e.g.  $\mathcal{M} = \text{Span}(X_T, \dots, X_1)$ ) so that

$$Y = \arg \inf_{Z \in \mathcal{M}} \mathbb{E}[(X_{T+h} - Z)^2]$$

When  $\mathcal{M}$  is a closed linear subspace of  $L^2$ , the Projection Theorem guarantees that such a  $Y$  exists, and it must satisfy

$$\langle X_{T+h} - Y, Z \rangle = 0 \quad \forall Z \in \mathcal{M}$$

must be in the orthogonal complement.