MATH 239 - Introduction to Combinatorics

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Chapter 1

Tutorials

1.1 Tutorial 1

Problem 1. Give a combinatorial proof that the number of subsets of $\{1, \ldots, n\}$ with even cardinality is the same as the number with odd cardinality.

Solution.

Let X denote any subset of $\{1, \ldots, n\}$. The corresponding set Y will be:

$$Y = \begin{cases} X \cup \{1\}, & \text{if } 1 \notin X \\ X \setminus \{1\}, & \text{if } 1 \in X \end{cases}$$

Problem 2. Let n be a positive integer. Give a combinatorial proof of the identity

$$\sum_{i=0}^{n} i \binom{n}{i} = n2^{n-1}.$$

Solution.

Suppose we have a group of n people.

RHS: Choose a committee of size i, then choose one of the i committee members to be a leader. There are $\binom{n}{i}$ ways to pick the members of the committee and i ways to choose the leader, which yields $\sum\limits_{i=0}^{n} i\binom{n}{i}$.

LHS: Pick a leader, then from the remaining $(n-1)$ people we choose them to either be in or out of the committee. There are n ways to pick the leader and 2^{n-1} ways to pick the remaining committee members.

Thus, since we are counting the same object twice in two different ways, we have that

$$\sum_{i=0}^{n} i \binom{n}{i} = n2^{n-1}.$$

Problem 3. For any integers n, k, r where $n \geqslant k \geqslant r \geqslant 0$, give a combinatorial proof of the following identity.

$$\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}.$$

Solution.

Suppose we have a group of n people, with a k-person committee and a r-person subcommittee.

RHS: Choose the committee in $\binom{n}{k}$ ways, then choose the subcommittee from the committee in $\binom{k}{r}$ ways, which yields $\binom{n}{k}\binom{k}{r}$.

LHS: Choose the r subcommittee members in $\binom{n}{r}$ ways, then fill in the remaining $(k-r)$ committee members from the remaining $(n-r)$ people, which yields $\binom{n}{r}\binom{n-r}{k-r}$.

Thus, since we are counting the same object twice in two different ways, we have that

$$\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}.$$

Problem 4. Let $n \geqslant 5$ be an integer. Give a combinatorial proof of the following identity

$$\sum_{k=5}^{n}\binom{k-1}{4} = \sum_{m=3}^{n-2}\binom{m-1}{2}\binom{n-m}{2}.$$

(Hint: Both sides are equal to $\binom{n}{5}$.)

Solution.

1.2 Tutorial 2

Problem 1. Use the negative binomial theorem and substitutions to give a formula for the coefficient of x^n in $(1-3x)^{-1} + 2(1-2x)^{-2}$.

Solution. Recall:

THEOREM 1.2.1 (Negative Binomial Theorem)**.** *Let $m, k \in \mathbb{Z}_{\geqslant 0}$, then*

$$(1-x)^{-k} = \sum_{m\geqslant 0}\binom{m+k-1}{k-1}x^m$$

$$(1-3x)^{-1} = \sum_{m\geqslant 0}\binom{m+1-1}{1-1}(3x)^m = \sum_{m\geqslant 0}3^m x^m$$
$$\implies [x^n](1-3x)^{-1} = 3^n$$

$$2(1-2x)^{-2} = 2\sum_{m\geqslant 0}\binom{m+2-1}{2-1}(2x)^m = 2\sum_{m\geqslant 0}(m+1)2^m x^m$$
$$\implies [x^n]2(1-2x)^{-2} = 2(n+1)2^n = (n+1)2^{n+1}$$

Combining,

$$[x^n]\left[(1-3x)^{-1} + 2(1-2x)^{-2}\right] = 3^n + (n+1)2^{n+1}$$

Problem 2. Let $F(x) = x + x^2 + \cdots$ and let $G(x) = 1 + 3x + 2x^2$. Compute the coefficient of x^n in $G(F(x))$.

Solution.

$$G(F(x)) = 1 + 3(x + x^2 + \cdots) + 2(x + x^2 + \cdots)^2$$

$$\begin{aligned}
3(x+x^2+\cdots) &= 3x(1+x+x^2+\cdots)\\
&= \frac{3x}{1-x}\\
&= 3x\sum_{m\geqslant 0} x^m
\end{aligned}$$

$$\begin{aligned}
2(x+x^2+\cdots)^2 &= 2\left[x(1+x+x^2+\cdots)\right]^2\\
&= 2\left(\frac{x}{1-x}\right)^2\\
&= 2x^2(1-x)^{-2}\\
&= 2x^2\sum_{m\geqslant 0}\binom{m+2-1}{2-1}x^m\\
&= 2x^2\sum_{m\geqslant 0}(m+1)x^m
\end{aligned}$$

Computing coefficients,

$$[x^n]1 = \begin{cases} 0, & \text{if } n \geqslant 1\\ 1, & \text{if } n = 0\end{cases}$$

$$\begin{aligned}
[x^n]3(x+x^2+\cdots) &= [x^n]3x\sum_{m\geqslant 0}x^m\\
&= [x^{n-1}]3\sum_{m\geqslant 0}x^m\\
&= \begin{cases} 3, & \text{if } n\geqslant 1\\ 0, & \text{if } n=0\end{cases}
\end{aligned}$$

$$\begin{aligned}
[x^n]2(x+x^2+\cdots)^2 &= [x^n]2x^2\sum_{m\geqslant 0}(m+1)x^m\\
&= [x^{n-2}]2\sum_{m\geqslant 0}(m+1)x^m\\
&= \begin{cases} 2n-2, & \text{if } n\geqslant 2\\ 0, & \text{if } n\in\{0,1\}\end{cases}
\end{aligned}$$

Thus,

$$[x^n]G(F(x)) = \begin{cases} 2n+1, & \text{if } n\geq 2\\ 3, & \text{if } n=1\\ 1, & \text{if } n=0\end{cases}$$

Problem 3. Show that if $F(x) = a_0 + a_1x + a_2x^2 + \cdots$. Then

$$F(x)(1-x) = a_0 + (a_1-a_0)x + (a_2-a_1)x^2 + \cdots$$

and

$$F(x)(1-x)^{-1} = \sum_{n=0}^{\infty} c_n x^n,$$

where $c_n = a_0 + a_1 + \cdots + a_n$.

Solution.

Part 1.

$$\begin{aligned} F(x)(1-x) &= (a_0 + a_1 x + a_2 x^2 + \cdots)(1-x) \\ &= a_0 + (a_1 - a_0)x + (a_2 - a_1)x^2 - a_2 x^3 - a_3 x^4 + \cdots \end{aligned}$$

as required.

Part 2.

We know that $(1-x)^{-1} = 1 + x + x^2 + \cdots$, thus

$$\begin{aligned} F(x)(1-x)^{-1} &= F(x) + xF(x) + x^2 F(x) + \cdots \\ &= \sum_{i \geqslant 0} x^i F(x) \\ &= \sum_{i \geqslant 0} [x^{n-i}] F(x) \\ &= \sum_{i=0}^{n} a_{n-i} \text{ for } 0 \leq k = n - i \leq n \\ &= \sum_{k=0}^{n} a_k \end{aligned}$$

where

$$[x^{n-i}]F(x) = \begin{cases} a_{n-i}, & \text{if } i \leqslant n \\ 0, & \text{if } i > n \end{cases}$$

Problem 4. Show that for $k \geqslant 1$ and $n \geqslant 1$, we have

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{n-i+k-1}{k-1} = 0,$$

where we interpret $\binom{j}{i} = 0$ when $j < i$. Hint: Look at $1 = (1-x)^k (1-x)^{-k}$) and compute the coefficient of x^n in both sides.

Solution.

$$[x^n]1 = \begin{cases} 0, & \text{if } n \geqslant 1 \\ 1, & \text{if } n = 0 \end{cases}$$

$$(1-x)^{-k}(1-x)^k = \sum_{j \geqslant 0} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{j+k-1}{k-1} x^{j+i} \qquad \text{Negative Bin. and Bin. Theorem}$$

Let $j + i = n \iff j = n - i$ to compute $[x^n](1-x)^{-k}(1-x)^k$, and we get

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{n-i+k-1}{k-1} = 0$$

for $k \geqslant 1$ and $n \geqslant 1$ as desired.

1.3 Tutorial 3

Problem 1. Consider the set of non-negative integers $\mathbb{N}_0$, but with a non-standard weight function

$$w(a) = \begin{cases} \frac{3}{2}a + 1, \text{ if } a \text{ is even,} \\ 2(a+1), \text{ if } a \text{ is odd.} \end{cases}$$

Find the generating series for $\mathbb{N}_0$ with respect to this weight function and express it as a simplified rational expression.

Solution.

$\mathbb{N}_0 = \mathbb{N}_{\text{even}} \cup \mathbb{N}_{\text{odd}}$

$$\begin{aligned} \Phi_{\mathbb{N}_0}(x) &= \Phi_{\mathbb{N}_{\text{even}}}(x) + \Phi_{\mathbb{N}_{\text{odd}}}(x) \qquad \text{Sum Lemma} \\ &= \sum_{a \text{ even}} x^{3/2a+1} + \sum_{a \text{ odd}} x^{2(a+1)} \\ &= x \sum_{a \text{ even}} x^{3/2a} + x^2 \sum_{a \text{ odd}} x^{2a} \\ &= x \sum_{i \geq 0} (x^{3/2})^{2i} + x^2 \sum_{i \geq 0} (x^2)^{2i+1} \\ &= x \sum_{i \geq 0} (x^3)^i + x^4 \sum_{i \geq 0} (x^4)^i \\ &= \frac{x}{1-x^3} + \frac{x^4}{1-x^4} \end{aligned}$$

Problem 2. Let m, n be positive integers and α, β positive real numbers. Find the generating series for the cartesian product

$$\{1, \ldots, m\} \times \{1, \ldots, n\}$$

with respect to the weight function

$$w(a, b) = \alpha a + \beta b$$

and express it as a simplified rational expression.

Solution.

THEOREM 1.3.1 (Partial Geometric Series).

$$\sum_{i=0}^{k} x^i = \frac{1-x^{k+1}}{1-x}$$

$$\begin{aligned} \Phi_{A \times B}(x) &= \sum_{(a,b) \in A \times B} x^{\alpha a + \beta B} \\ &= \sum_{a \in A} x^{\alpha a} \sum_{b \in B} x^{\beta b} \qquad \text{Product Lemma} \\ &= x^{\alpha + \beta} \sum_{i=1}^{m} (x^a)^i \sum_{j=1}^{n} (x^b)^j \\ &= x^{\alpha+\beta} \left(\frac{x^a(1-x^{am})}{1-x^a} \right) \left(\frac{x^b(1-x^{bm})}{1-x^b} \right) \qquad \text{Partial Geometric Series} \\ &= x^{\alpha+\beta+a+b} \frac{(1-x^{am})(1-x^{bm})}{(1-x^a)(1-x^b)} \end{aligned}$$

Problem 3. Let a, b, n, k be positive integers with $a \leq b$ and $k \leq n$. How many compositions of n with k parts are there in which all parts are elements of $\{a, \ldots, b\}$? Expressing the result as a finite sum $\sum_{i=0}^{k} s_i$ is sufficient.

Rough.

Start with a small example $a = 2,\ b = 4,\ n = 9,\ k = 3$.

Let $C = \{$compositions with 3 parts where each part is 2, 3, or 4$\}$. We want $[x^9]\Phi_C(x)$.

Let $P = \{$parts of value 2, 3, or 4$\}$. $C = P \times P \times P$ (3 parts). Thus, $\Phi_C(x) = (\Phi_P(x))^3$.

$$(\Phi_P(x))^3 = \left(\sum_{i=2}^{4} x^i\right)^3$$

We want $[x^9](x^2 + x^3 + x^4)^3$.

Solution.

So, in general we have

$$\begin{aligned}\Phi_C(x) &= (\Phi_P(x))^k \\ &= \left(\sum_{i=a}^{b} x^i\right)^k\end{aligned}$$

We want $[x^n]\left(\sum_{i=a}^{b} x^i\right)^k$.

$$\begin{aligned}[x^n]\left(\sum_{i=a}^{b} x^i\right)^k &= [x^n]\,(x^a + \cdots + x^b)^k \\ &= [x^n]\,x^{ak}(1 + \cdots + x^{b-a})^k \\ &= \left[x^{n-ak}\right]\left(\frac{1 - x^{b-a+1}}{1 - x}\right)^k \\ &= \left[x^{n-ak}\right]\left(1 + (-x^{b-a+1})\right)^k (1 - x)^{-k} \\ &= \left[x^{n-ak}\right]\sum_{i \geq 0}\binom{k}{i}(-1)^i x^{i(b-a+1)} \sum_{j \geq 0}\binom{j+k-1}{k-1}x^j \\ &= \left[x^{n-ak}\right]\sum_{i \geq 0}\sum_{j \geq 0}\binom{k}{i}\binom{j+k-1}{k-1}(-1)^i x^{i(b-a+1)+j}\end{aligned}$$

$i(b - a + 1) + j = n - ak$

$$\sum_{i=0}^{\lfloor \frac{n-ak}{b-a+1} \rfloor}\binom{k}{i}\binom{n - ak - i(b-a+1) + k - 1}{k-1}(-1)^i$$

1.4 Tutorial 4

Problem 1. Let S denote the set of strings of the form $\{1\}^*\{0\}^*\{1\}^*\{0\}^*$. Find the generating function for $\Phi_S(x)$, where the weight of a string is given by its length.

Solution.

$\{1\}^* \left(\{0\}\{0\}^*\{1\}\{1\}^*\right) \{0\}^* \cup \{1\}^*\{0\}^*$

Problem 2. Let $S = \{00, 111\}^*$. Find a formula for $\Phi_S(x)$.

Solution.

$$\begin{aligned}\Phi_S(x) &= \Phi_{\{00,111\}^*} \\ &= \frac{1}{1 - \Phi_{\{00,111\}}} \\ &= \frac{1}{1 - (x^2 + x^3)}\end{aligned}$$

Problem 3. Let S be $\{00, 111\}^*$ and let S_n denote the set of strings of length n in S. Give a combinatorial proof that $|S_n| = |S_{n-2}| + |S_{n-3}|$ for $n \geqslant 3$.

Look at $n = 0, ..., 6$.

$n = 2 : |S_2| = 1 \to 00$

$n = 3 : |S_3| = 1 \to 111$

$n = 4 : |S_4| = 1 \to 0000$

$n = 5 : |S_5| = 2 \to 00111, 11100$

$n = 6 : |S_6| = 2 \to 111111, 000000$

Solution.

Let $s \in S_n$, $n \geqslant 3$. What could s start with?

Case 1: $00t$, $t \in S_{n-2}$

Case 2: $111t$, $t \in S_{n-3}$

$f : S_n \to S_{n-2} \cup S_{n-3}$, $\forall s \in S_n$, $f(s) = t$ where

$$s = \begin{cases} 00t, \text{ if } s \text{ starts with } 00 \\ 111t, \text{ if } s \text{ starts with } 111 \end{cases}$$

$g : S_{n-2} \cup S_{n-3} \to S_n$

$$g(t) = \begin{cases} 00t, \ t \in S_{n-2} \\ 111t, \ t \in S_{n-3} \end{cases}$$

Explain how $f(g(t)) = t$ and $g(f(s)) = s$.

Problem 4. Explain why $(\{1\}^*\{0\}^*)$ is ambiguous.

Solution.

Ambiguous means there are multiple ways to create a string. So, taking ε works since,

$$\begin{aligned}\left(\{1\}^0\{0\}^0\right)^x &= (\varepsilon)^x \\ &= \varepsilon\end{aligned}$$

1.5 Tutorial 5

Problem 1. Let S denote the set of binary strings not containing the string 101 as a substring. Find an unambiguous expression for S, and use it to give a rational expression for $\Phi_S(x)$, weighted by length.

Solution.

$\{0\}^*(\{1\}\{1\}^*\{0\}\{0\}^*)^*\{1\}^*$

$S = \{0\}^*(\{1\}\{1\}^*\{00\}\{0\}^*)^*\{1\}^*\{\varepsilon, 10\}$

$T = \{$binary strings containing exactly one copy of 101 as a suffix$\}$

$$\begin{aligned}
\{\varepsilon\} \cup S\{0,1\} &= S \cup T \\
S\{101\} &= T \cup T\{01\} \\
1 + \Phi_S(x)2x &= \Phi_S(x) + \Phi_T(x) \\
\Phi_S(x)x^3 &= \Phi_T(x) + \Phi_T(x)x^2 \\
\implies 1 + \Phi_S(x)2x - \Phi_S(x) &= \Phi_T(x)
\end{aligned}$$

substituting,

$$\begin{aligned}
\Phi_S(x)x^3 &= 1 + \Phi_S(x)2x - \Phi_S(x) + x^2 + \Phi_S(x)2x^3 - \Phi_S(x)x^2 \\
\implies \Phi_S(x)x^3 &= 1 + \Phi_S(x)2x - \Phi_S(x) + \Phi_S(x)2x^3 - \Phi_S(x)x^2 + x^2 \\
\implies \Phi_S(x)x^3 - \Phi_S(x)2x + \Phi_S(x) - \Phi_S(x)2x^3 + \Phi_S(x)x^2 &= 1 + x^2 \\
\implies -\Phi_S(x)x^3 - \Phi_S(x)2x + \Phi_S(x) + \Phi_S(x)x^2 &= 1 + x^2 \\
\implies \Phi_S(x)(-x^3 - 2x + 1 + x^2) &= 1 + x^2 \\
\implies \Phi_S(x) &= \frac{1 + x^2}{-x^3 + x^2 - 2x + 1}
\end{aligned}$$

where the algebra has been verified with WolframAlpha.

Problem 2. Let S be the set of binary strings with an odd number of blocks. Find an unambiguous recursive decomposition for S, and use it to find a rational expression for $\Phi_S(x)$, weighted by length.

$X = \{$binary strings with odd number of blocks, beginning with $1\}$

$Y = \{$binary strings with odd number of blocks, beginning with $0\}$

$S \cup T = T\{0,1\} \cup S\{0,1\} \cup \{\varepsilon\}$

$S = X \cup Y$

$X = \{1\}\{1\}^*(\{0\}\{0\}^* X \cup \{\varepsilon\}) \to \Phi_X(x) = \frac{x}{1-x}\left(\frac{x}{1-x}\Phi_X(x) + 1\right)$

$Y = \{0\}\{0\}^*(\{1\}\{1\}^* Y \cup \{\varepsilon\})$

$$\Phi_X(x) = \frac{x^2}{(1-x)^2}\Phi_X(x) + \frac{x}{1-x}$$

$$\begin{aligned}
\implies \Phi_X(x) &= \frac{\frac{x}{1-x}}{1 - \frac{x^2}{(1-x)^2}} \\
&= \frac{x(1-x)}{(1-x)^2 - x^2} \\
&= \frac{x - x^2}{1 - 2x}
\end{aligned}$$

$$\Phi_S(x) = \frac{2x - 2x^2}{1 - 2x}$$

Problem 3. Let k and ℓ be non negative, and S be the set of binary strings in which no block of zeros has length greater than k and no blocks of ones has length greater than ℓ. Find an unambiguous recursive decomposition for S, and use it to find a rational expression for $\Phi_S(x)$, weighted by length.

$T = \{0, 00, \ldots, 0^k\}$, $U = \{1, 11, \ldots, 1^\ell\}$

$(T \cup \{\varepsilon\})(UT)^*(U \cup \{\varepsilon\})$

$$\Phi_T(x) = \frac{x(1 - x^k)}{1 - x}$$

$$\Phi_U(x) = \frac{x(1 - x^\ell)}{1 - x}$$

1.6 Tutorial 6

Problem 1. Let $n \geqslant 0$. Use partial fractions to compute the coefficient of x^n in

$$\frac{x(x-1)}{x^3 + 6x^2 + 11x + 6}$$

Solution.

$$\begin{aligned}
\frac{x(x-1)}{x^3 + 6x^2 + 11x + 6} &= \frac{x^2 - x}{(x+1)(x+2)(x+3)} \\
&= \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3} \\
&= A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2) \\
&= A(x^2 + 5x + 6) + B(x^2 + 4x + 3) + C(x^2 + 3x + 2) \\
&= (6A + 3B + 2C) + (5A + 4B + 3C)x + (A + B + C)x^2
\end{aligned}$$

Equating coefficients, gives three equations and three unknowns:

$$6A + 3B + 2C = 0$$

$$5A + 4B + 3C = -1$$

$$A + B + C = 1$$

$$\left[\begin{array}{ccc|c} 6 & 3 & 2 & 0 \\ 5 & 4 & 3 & -1 \\ 1 & 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

Thus, we have $A = 1$, $B = -6$, $C = 6$.

$$\begin{aligned}
\frac{1}{x+1} + \frac{-6}{x+2} + \frac{6}{x+3} &= (-1)^n - 6[x^n]\frac{1}{x+2} + 6[x^n]\frac{1}{x+3} \\
&= (-1)^n - 6(-2)^n + 6(-3)^n
\end{aligned}$$

Problem 2. Solve the following recurrences:

(a) $a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3} = 0$ for $n \geqslant 3$ with initial conditions $a_0 = 0$, $a_1 = 1$ and $a_2 = 2$.

(b) $a_n - za_{n-2} = 0$ for $n \geqslant 2$, $z \in \mathbb{R}$, with initial conditions $a_0 = 1$ and $a_1 = 2z$.

Solution.

(a) The polynomial is $x^3 - 3x^2 + 3x - 1 = 0 \implies (x-1)^3 = 0$. The multiplicity is 3 so the degree is at most 2. $a_n = 1^n(An^2 + Bn + C)$

$$a_0 = C = 0$$

$$a_1 = A + B + C = 1$$

$$a_2 = 4A + 2B + C = 2$$

$A = 0,\ B = 1,\ C = 0$. Substituting gives,

$$a_n = n$$

(b) The polynomial is $x^2 - z = 0 \implies (x + \sqrt{z})(x - \sqrt{z}) = 0$. The roots are $x = \sqrt{z}$ and $x = -\sqrt{z}$ both with multiplicity of 1, so the degree is at most 0. $a_n = (\sqrt{z})^n A + (-\sqrt{z})^n B$

$$a_0 = A + B = 1$$

$$a_1 = \sqrt{z}A - \sqrt{z}B = 2z$$

Solving, gives $A = 1/2 + \sqrt{z}$ and $B = 1/2 - \sqrt{z}$. Substituting gives,

$$a_n = \sqrt{z}\left(\frac{1}{2} + \sqrt{z}\right) + (-\sqrt{z})\left(\frac{1}{2} - \sqrt{z}\right)$$

Problem 3. Find (linear, homogenous) recurrence equations and initial conditions for

(a) $a_n = (n+1)2^n + (n-1)(-2)^n$

(b) $a_n = 1 + z^n$ where $z \in \mathbb{R} \setminus \{0\}$

Solution.

(a) Roots: 2 and -2, $(n+1)$ and $(n-1)$ are both linear $\rightarrow$ multiplicity of 2 for both roots. Thus, $h(x) = (x-2)^2(x+2)^2 = x^4 - 8x^2 + 16$. Therefore, the recurrence relation is:

$$a_n - 0a_{n-1} - 8a_{n-2} + 0a_{n-3} + 16a_{n-4} = 0$$

(b) Roots: 1 and z, both have multiplicity of 1. Thus, $h(x) = (x-1)(x-z) = x^2 - (1+z)x + z$. Therefore, the recurrence relation is:

$$a_n - (1+z)a_{n-1} + za_{n-2} = 0$$

1.7 Tutorial 7

Problem 1. Determine the following coefficient. $[x^{2n}](x/(1-x))^n$.

Solution.

Problem 2. Let S be the class of binary strings in which every block has length of at most 6, every block of zeros has even length, and every block of ones has odd length. Find an unambiguous expression for S, and use it to compute $\Phi_S(x)$.

Solution.

Decomposition strategy (from course notes):

$$\{0\}^*(\{1\}\{1\}^*\{0\}\{0^*\})^*\{1\}^*$$

$B_0 = \{\varepsilon, 00, 0000, 000000\}$

$B_1 = \{\varepsilon, 1, 111, 11111\}$

$$S = B_0(B_1 \setminus \varepsilon\ B_0 \setminus \varepsilon)^* B_1$$
$$\Phi_{B_0}(x) = 1 + x^2 + x^4 + x^6$$
$$\Phi_{B_1}(x) = 1 + x + x^3 + x^5$$
$$\Phi_S(x) = \frac{\Phi_{B_0}(x)\Phi_{B_1}(x)}{1 - (x^2 + x^4 + x^6)(x + x^3 + x^5)}$$

Problem 3. For each n, let a_n denote the number of compositions of n where every part is even, and the number of parts is a multiple of 3. Find an explicit formula for a_n.

Solution.

Let $A = \{(\alpha_1, \alpha_2, \ldots, \alpha_{3m}) : \alpha_i \in \mathbb{N}_{\text{even}}, m \in \mathbb{N}\}$.

$$A = \bigcup_{m \geqslant 0} (\mathbb{N}_{\text{even}})^{3m}(x)$$

$$\Phi_{\mathbb{N}_{\text{even}}} = x^2 + x^4 + \cdots = \frac{x^2}{1 - x^2}$$

$$\begin{aligned}
\Phi_A(x) &= \sum_{m \geqslant 0} \Phi_{\mathbb{N}_{\text{even}}{}^{3m}}(x) && \text{by Sum Lemma} \\
&= \sum_{m \geqslant 0} \Phi_{\mathbb{N}_{\text{even}}}(x)^{3m} && \text{by Product Lemma} \\
&= \sum_{m \geqslant 0} \left(\frac{x^2}{1 - x^2}\right)^{3m} \\
&= \sum_{m \geqslant 0} \left(\frac{x^6}{(1 - x^2)^3}\right)^{m} \\
&= \frac{1}{1 - \frac{x^6}{(1-x^2)^3}} \\
&= \frac{(1 - x^2)^3}{(1 - x^2)^3 - x^6}
\end{aligned}$$

Thus,

$$a_n = [x^n]\frac{(1 - x^2)^3}{(1 - x^2)^3 - x^6}$$

Problem 4. Solve the linear recurrence relation defined by $a_n = 5a_{n-1} - 6a_{n-2}$, with initial conditions $a_0 = 2$ and $a_1 = 5$.

Solution.

Textbook Problem.

$$S_k = \{(A, B) : A, B \subseteq \{1, \ldots, n\}, |A| = |B| = m, |A \cap B| = k\}$$
$$T_k = \{(X, Y, Z) : X, Y, Z \subseteq \{1, \ldots, n\}, |X| = k, |Y| = |Z| = m - k, X \cap Y = X \cap Z = Y \cap Z = \varnothing\}$$

Define a bijection $f : S_k \to T_k$ and it's inverse.

$$f(A, B) = (A \cap B, A \setminus B, B \setminus A)$$

1.8 Tutorial 7

Problem 1. Determine the number of vertices and edges of each of the following graphs.

$G(n,k)$ for each n and k. For integers n and k, let $G(n,k)$ be the graph whose vertices are the k-element subsets of $\{1,\dots,n\}$, where two vertices A and B are adjacent if $|A\cap B|\leqslant 2$.

(a) The graph G_1 whose vertices are the 4-element subsets of $\{1,\dots,8\}$, where two vertices A and B are adjacent if and only if $|A\cap B|\leqslant 2$.

(b) The graph G_2 whose vertices are the binary strings of length n, where two vertices are adjacent if and only if they differ in exactly two positions.

Solution. The number of vertices are 2^n. Let $s\in V(G_2)$. How many other elements is s adjacent to?

$$\deg(s)=\binom{n}{2}$$

Handshake Lemma:

$$2|E|=\sum_{s\in V(G)}\deg(s)=2^n\binom{n}{2}$$

Thus,

$$|E|=2^{n-1}\binom{n}{2}$$

(c) The graph G_3 with vertex set $\{1,2,3,4,5\}\times\{1,2,3,4,5\}$, where two vertices (x,y) and (x',y') are adjacent if and only if $(x-x')^2+(y-y')\leqslant 2$.

Solution. The number of vertices are $|V(G_3)|=5\times 5=25$. Let $(x,y)\in V(G_3)$. What vertices are adjacent to (x,y)?

$$(x+1,y),(x-1,y),(x+1,y-1),(x-1,y-1),(x,y-1),(x+1,y+1),(x-1,y+1),(x,y+1)$$

If $2\leqslant x,y\leqslant 4$,

$$\deg((x,y))=8$$

If $x\in\{1,5\}$, $2\leqslant y\leqslant 5$,

$$\deg((x,y))=5$$

If $x,y\in\{1,5\}$,

$$\deg((x,y))=3$$

Using the Handshake Lemma,

$$\begin{aligned}2|E|&=\sum_{v\in V(G_3)}\deg(v)\\&=8(3\times 3)+5(2\times 3\times 2)+3(2\times 2)\\&=72+60+12\\&=144\end{aligned}$$

Thus,

$$|E|=72$$

Problem 2. Which of the graphs in the previous question are connected? Give a proof either way.

(b) **Solution.** $n=4$, $\binom{n}{2}=\binom{4}{2}=6$. We know 1010 is adjacent to $1001,0011,0000,1100,0110,1111$. Note that the parity is the same in all.

Connected: $\forall x,y\in V(G)$, there exists a path between x and y in G.

Consider 0001 and 1101.

Changing two bits will always leave the parity the same (since we either add 2 to the sum, subtract 2, or add 1 and subtract 1). Therefore, there is no path between a vertex of even parity and a vertex with odd parity. Thus, G_2 is not connected.

(c) **Solution.** Claim: G_3 is connected. Suppose for a contradiction that G_3 is not connected. Then, there exists a $(x, y), (x', y') \in V(G_3)$ such that there is not path between them. WLOG, $x \leqslant x'$ and $y \leqslant y'$ (invert one of the following sequences in each case we have $>$)

$$x, x+1, \ldots, x' \qquad ; \qquad y, y+1, \ldots, y'$$

$$(x, y)\frown(x+1, y)\frown\cdots\frown(x', y)\frown(x', y+1)\frown(x', y')$$

Contradiction.

Problem 3. Prove that every graph on at least two vertices has two vertices of the same degree.

Solution. Suppose for a contradiction that $V = \{v_1, \ldots, v_k\}$ have different degrees. Say d_i is the degree of v_i for each $i \in [1, k]$. WLOG,

$$d_1 < d_2 < \cdots < d_k$$

You can assume that $d_1 \geqslant 1$.

$$d_1 \geqslant 1 \implies d_i \geqslant i \qquad \forall i$$

$$d_k \geqslant k \implies v_k \text{ is adjacent to } k \text{ vertices, but there are only } k-1 \text{ available}$$

contradiction. If $d_1 = 0$, then it doesn't affect the degrees of other vertices so we can remove it from G, and just look at

$$v_2, \ldots, v_k$$