

MATH 237 - Calculus 3

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Last updated: October 28, 2019

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1 Q

1.1 Definition (Directional Derivative)

The *directional derivative* of $f(x, y)$ at a point (a, b) in the direction of a unit vector $\mathbf{u} = (u_1, u_2)$ defined by

$$D_{\mathbf{u}}f(a, b) = \left. \frac{d}{ds} f(a + su_1, b + su_2) \right|_{s=0}$$

provided the derivative exists.

1.2 Theorem

If $f(x, y)$ is differentiable at (a, b) and $\mathbf{u} = (u_1, u_2)$ is a unit vector, then

$$D_{\mathbf{u}}f(a, b) = \nabla f(a, b) \cdot \mathbf{u}$$

Remark 1. Be careful to check the condition of Theorem before applying it. If f is not differentiable at (a, b) , then we must apply the definition of the directional derivative.

Remark 2. If we choose $\mathbf{u} = \mathbf{i} = (1, 0)$ or $\mathbf{u} = \mathbf{j} = (0, 1)$, then the directional derivative is equal to the partial derivatives f_x or f_y respectively.

1.3 Theorem

If $f(x, y)$ is differentiable at (a, b) , and $\nabla f(a, b) \neq (0, 0)$, then the largest value of $D_{\mathbf{u}}f(a, b)$ is $\|\nabla f(a, b)\|$, and occurs when \mathbf{u} is in the direction of $\nabla f(a, b)$.

1.4 Theorem

If $f(x, y) \in C^1$ in a neighborhood of (a, b) and $\nabla f(a, b) \neq (0, 0)$, then $\nabla f(a, b)$ is orthogonal to the level curve $f(x, y) = k$ through (a, b) .

1.5 Theorem

If $f(x, y, z) \in C^1$ in a neighborhood of (a, b, c) and $\nabla f(a, b, c) \neq (0, 0, 0)$, then $\nabla f(a, b, c)$ is orthogonal to the level curve $f(x, y, z) = k$ through (a, b, c) .

1.6 Definition (2nd degree Taylor polynomial)

The *second degree Taylor polynomial* $P_{2,(a,b)}$ of $f(x, y)$ at (a, b) is given by

$$P_{2,(a,b)}(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + \frac{1}{2}[f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(a, b)(y - b)^2]$$

1.7 Theorem

If $f''(x)$ exists on $[a, x]$, then there exists a number c between a and x such that

$$f(x) = f(a) + f'(a)(x - a) + R_{1,a}(x)$$

where

$$R_{1,a}(x) = \frac{1}{2}f''(c)(x - a)^2$$

1.8 Theorem (Taylor's Theorem)

If $f(x, y) \in C^2$ in some neighborhood $N(a, b)$ of (a, b) , then for all $(x, y) \in N(a, b)$ there exists a point (c, d) on the line segment joining (a, b) and (x, y) such that

$$f(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + R_{1,(a,b)}(x, y)$$

where

$$R_{1,(a,b)}(x, y) = \frac{1}{2}[f_{xx}(c, d)(x - a)^2 + 2f_{xy}(c, d)(x - a)(y - b) + f_{yy}(c, d)(y - b)^2]$$

Remark 3. Like the one variable case, Taylor's Theorem for $f(x, y)$ is an existence theorem. That is, it only tells us that the point (c, d) exists, but not how to find it.

Remark 4. The most important thing about the error term $R_{1,(a,b)}(x, y)$ is not its explicit form, but rather its dependence on the magnitude of the displacement $\|(x, y) - (a, b)\|$. We state the result as a Corollary.

1.9 Corollary

If $f(x, y) \in C^2$ in some closed neighborhood $N(a, b)$ of (a, b) , then there exists a positive constant M such that

$$R_{1,(a,b)}(x, y) \leq M\|(x, y) - (a, b)\|^2$$

for all $(x, y) \in N(a, b)$.

1.10 Taylor's Theorem of order k

If $f(x, y) \in C^{k+1}$ at each point on the line segment joining (a, b) and (x, y) , then there exists a point (c, d) on the line segment between (a, b) and (x, y) such that

$$f(x, y) = P_{k,(a,b)}(x, y) + R_{k,(a,b)}(x, y)$$

where

$$R_{k,(a,b)}(x, y) = \frac{1}{(k+1)!} |(x-a)D_1 + (y-b)D_2|^{k+1} f(c, d)$$

1.11 Corollary

If $f(x, y) \in C^k$ in some neighborhood of (a, b) then

$$\lim_{(x,y) \rightarrow (a,b)} \frac{|f(x, y) - P_{k,(a,b)}(x, y)|}{\|(x, y) - (a, b)\|^k} = 0$$

1.12 Corollary

If $f(x, y) \in C^{k+1}$ in some closed neighborhood $N(a, b)$ of (a, b) , then there exists a constant $M > 0$ such that

$$|f(x, y) - P_{k,(a,b)}(x, y)| \leq M \|(x, y) - (a, b)\|^{k+1}$$

for all $(x, y) \in N(a, b)$.

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2.1 Definition (Local Maximum and Minimum)

A point (a, b) is a *local maximum point* for $f(x, y)$ if

$$f(x, y) \leq f(a, b)$$

for all (x, y) in some neighborhood of (a, b) .

A point (a, b) is a *local minimum point* for $f(x, y)$ if

$$f(x, y) \geq f(a, b)$$

for all (x, y) in some neighborhood of (a, b) .

2.2 Theorem

Let $f(x, y)$ have continuous partials. If (a, b) is a local maximum or minimum point of f , then

$$\nabla f(a, b) = 0$$

or at least one of f_x, f_y does not exist at (a, b) .

Proof. Let (a, b) be a local maximum or minimum point of f . Fix $x = a$, consider $f(a, y) = z$ (cross section), it has a local maximum/minimum point at $y = b \implies \frac{\partial f}{\partial y}(a, b) = 0$ (or DNE) when $y = b$.

Similarly, $\frac{\partial f}{\partial x}(a, b) = 0$ (or DNE). □

2.3 Definition (Critical Point)

A point (a, b) in the domain of $f(x, y)$ is called a *critical point* of f if $\frac{\partial f}{\partial x}(a, b) = 0$ or $\frac{\partial f}{\partial x}(a, b)$ does not exist, and $\frac{\partial f}{\partial y}(a, b) = 0$ or $\frac{\partial f}{\partial y}(a, b)$ does not exist.

2.4 Examples

Consider $f(x, y) = \sqrt{x^2 + y^2}$ which is a cone (upper half). $(0, 0)$ is a local minimum point.

$$f(x, y) = \sqrt{x^2 + y^2} > 0 = f(0, 0)$$

However, $f_x(0, 0)$ and $f_y(0, 0)$ does not exist.

Consider $g(x, y) = x^2 - y^2$ which is a hyperbolic paraboloid (saddle surface)

$$g_x = 2x$$

$$g_y = 2y$$

So, $(0, 0)$ is the only critical point of g , but

$$g(x, 0) > g(0, 0)$$

$$h(0, y) < h(0, 0)$$

for all $x, y \in \mathbb{R}$, so $(0, 0)$ is neither a local maximum or minimum point. We classify it as a *saddle point*.

To summarize, all critical points are either local maxima, minima or saddle points.

2.5 Example (Finding Critical Points)

Find all critical points of $f(x, y) = xy(1 - x^2 - y^2)$.

$$f_x = y(1 - x^2 - y^2) + xy(-2x) \quad (1)$$

$$= y[(1 - x^2 - y^2) + x(-2x)] \quad (2)$$

$$= y(1 - x^2 - y^2 - 2x^2) \quad (3)$$

$$= y(1 - y^2 - 3x^2) \quad (4)$$

$$(5)$$

Similarly, we get

$$f_y = x(1 - 3y^2 - x^2) = 0 \quad (6)$$

Note that (1) and (6) are both non-linear systems. (6) yields roots $y = 0$ and $y = \pm\sqrt{1 - 3x^2}$ for roots, split into two cases.

Case 1 $y = 0$

Substituting into (6) we get $x(1 - x^2) = 0$, giving $x = -1, 0, 1$. Thus, the corresponding critical points are: $(-1, 0), (0, 0), (1, 0)$.

Case 2 $y = \pm\sqrt{1 - 3x^2}$

Substituting into (6) we get $x(8x^2 - 2) = 0$, giving $x = 0, \frac{1}{2}, -\frac{1}{2}$. To find the corresponding y values, plug the x values into $y = \pm\sqrt{1 - 3x^2}$. Thus, the corresponding critical points are: $(0, 0), \underbrace{(\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, \frac{1}{2})}_{+ \text{ sqrt}}, \underbrace{(\frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2})}_{- \text{ sqrt}},$

We need an analogy to the 2nd derivative test for $y = f(x)$.

$f'' > 0 \rightarrow$ local minimum, $f'' < 0 \rightarrow$ local maximum

Consider the Taylor Series for $f(x, y)$ about (a, b) such that $\nabla f(a, b) = (0, 0)$

$$f(x, y) - f(a, b) \approx \frac{1}{2} [f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(y - b) + f_{yy}(a, b)(y - b)^2] + \underbrace{\ddots}_{\text{H.O.T.}} \quad (7)$$

If x is close to a and y is close to b , then the higher order terms can be neglected. So,

$$f(x, y) - f(a, b) \approx \frac{1}{2} [f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(y - b) + f_{yy}(a, b)(y - b)^2] \quad (8)$$

2.6 Theorem (Second Partial Derivatives Test)

Suppose $f(x, y) \in C^2$ in some neighborhood of (a, b) and that

$$\nabla f(a, b) = 0$$

- (1) If $f(x, y) - f(a, b) > 0$ (positive definite) for all (x, y) near (a, b) , $(x, y) \neq (0, 0) \neq (a, b)$ then (a, b) is a local minimum point of f .
- (2) If $f(x, y) - f(a, b) < 0$ (negative definite) for all (x, y) near (a, b) , $(x, y) \neq (0, 0) \neq (a, b)$ then (a, b) is a local maximum point of f .
- (3) If $f(x, y) - f(a, b) < 0$ for some (x, y) near (a, b) and $f(x, y) - f(a, b) > 0$ for some other (x, y) near (a, b) , then (a, b) is a saddle point (indefinite) of f .