Chapter 5
Consistency, Zero Stability, and the Dahlquist Equivalence Theorem

In Chapter 2 we discussed convergence of numerical methods and gave an experimental method for finding the rate of convergence (aka, global order of accuracy). When we devise a numerical method, however, we prefer to make a claim about the convergence properties *before* implementing it. In this chapter we will describe the conditions under which a numerical scheme can be claimed to be convergent.

18 Self-Assessment

Before reading this chapter, you may wish to review...

- stability [Unified S/S II Lecture 12 Notes]
- stability of difference equations [Wikipedia: Control Theory, Stability]
- basics of difference equations [6.003 Lecture 2 Slides]

After reading this chapter you should be able to...

- evaluate the consistency of a numerical method
- evalute the zero stability of a numerical method
- describe the relationship between consistency, stability, and convergence

19 Zero stability

A numerical method is *zero stable* if the solution remains bounded as $\Delta t \to 0$ for finite final time T. Recall the general form for an s-step multi-step method as given in Definition 1:

$$v^{n+1} + \sum_{i=1}^{s} \alpha_i v^{n+1-i} = \Delta t \sum_{j=0}^{s} \beta_j f^{n+1-j}. \tag{34}$$

In the limit $\Delta t \to 0$ we have

$$v^{n+1} + \sum_{i=1}^{s} \alpha_i v^{n+1-i} = 0. \tag{35}$$

This recurrence relationship determines the characteristic, or unforced, behavior of the multi-step method. The method is zero stable if all solutions to (35) remain bounded.

Definition 1 (Zero stability). A multi-step method is zero stable if all solutions to

$$v^{n+1} + \sum_{i=1}^{s} \alpha_i v^{n+1-i} = 0$$

remain bounded as $n \to \infty$.

To determine if a method is zero stable, we assume that the solution to the recurrence has the following form,

$$v^n = v^0 z^n, \tag{36}$$

where the superscript in the z^n term is in fact a power. Note: z can be a complex number. If the recurrence relationship has solutions with $|z| > 1$, then the multi-step method would be unstable. For our purposes, a multi-step method with a root of $|z| = 1$ is zero-stable provided the root is not repeated. (Note that in general, we need to be careful with the case of $|z| = 1$.)

Example 1. In Example 2, the most accurate two-step, explicit method was found to be,

$$v^{n+1} + 4v^n - 5v^{n-1} = \Delta t \left(4f^n + 2f^{n-1}\right).$$

We will determine if this algorithm is stable. The recurrence relationship is,

$$v^{n+1} + 4v^n - 5v^{n-1} = 0.$$

Then, substitution of $v^n = v^0 z^n$ gives,

$$z^{n+1} + 4z^n - 5z^{n-1} = 0.$$

Factoring this relationship gives,

$$z^{n-1}\left(z^2 + 4z - 5\right) = z^{n-1}(z-1)(z+5) = 0.$$

Thus, the recurrence relationship has roots at $z = 1$, $z = -5$, and $z = 0$ ($n-1$ of these roots). The root at $z = -5$ will grow unbounded as n increases so this method is not stable.

To demonstrate the lack of convergence for this method (due to its lack of stability), we again consider the solution of $u_t = -u^2$ with $u(0) = 1$. These results are shown in Figure 1. These results clearly show the instability. Note that the solution oscillates as is expected since the large parasitic root is negative ($z = -5$). Furthermore, decreasing Δt from 0.1 to 0.05 only causes the instability to manifest itself in shorter time (though the same number of steps). Clearly the method will not converge because of this instability.

20 Consistency

A numerical method is *consistent* if the multi-step discretization satisfies the governing equation in the limit $\Delta t \to 0$.

Definition 2 (Consistency). A numerical method is *consistent* if its local truncation error τ has the property

$$\lim_{\Delta t \to 0} \frac{\tau}{\Delta t} = 0. \tag{37}$$

Let $v^{n+1} = g(v^{n+1}, v^n, \ldots, t^{n+1}, t^n, \ldots, \Delta t)$ represent the numerical method as we have seen before. Then, if we write the ratio $\tau/\Delta t$, we find

$$\frac{\tau}{\Delta t} = \frac{g(u^{n+1}, u^n, \ldots, t^{n+1}, t^n, \ldots, \Delta t) - u^{n+1}}{\Delta t},$$

and substitute the Taylor series of u^{n+1},

$$\frac{\tau}{\Delta t} = \frac{g(u^{n+1}, u^n, \ldots, t^{n+1}, t^n, \ldots, \Delta t) - (u^n + \Delta t u_t^n + \mathcal{O}(\Delta t^2))}{\Delta t}.$$

We can rewrite this in terms of the slope of the numerical method

$$\frac{\tau}{\Delta t} = \frac{g(u^{n+1}, u^n, \ldots, t^{n+1}, t^n, \ldots, \Delta t) - u^n}{\Delta t} - u_t^n + \mathcal{O}(\Delta t)).$$

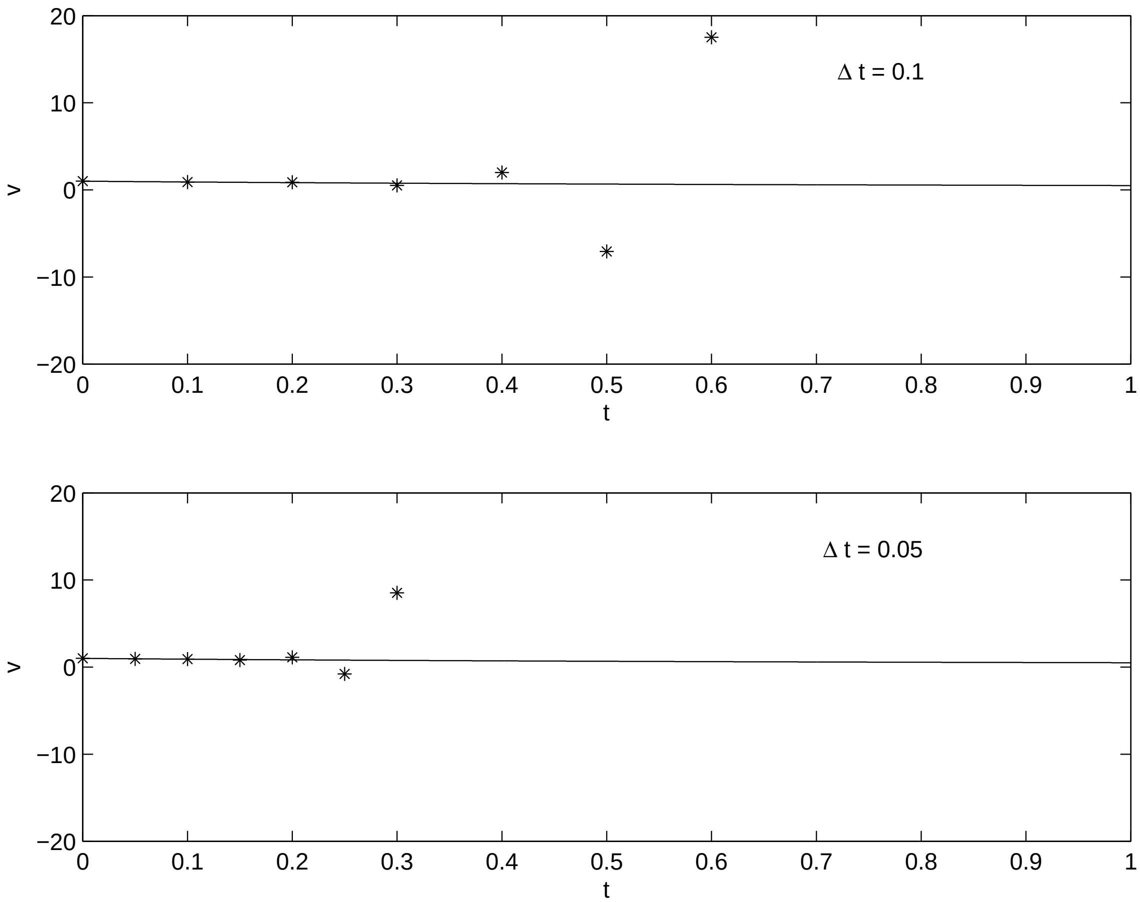

Fig. 3 Most-accurate explicit, two-step multi-step method applied to $\dot{u} = -u^2$ with $u(0) = 1$ with $\Delta t = 0.1$ (upper plot) and 0.05 (lower plot).

If we now take the limit $\Delta t \to 0$, the last term vanishes and we find

$$\lim_{\Delta t \to 0} \frac{\tau}{\Delta t} = \frac{g(u^{n+1}, u^n, \ldots, t^{n+1}, t^n, \ldots, \Delta t) - u^n}{\Delta t} - u_t^n.$$

It follows then that if the numerical method is consistent, we have $\lim_{\Delta t \to 0} \frac{\tau}{\Delta t} = 0$ and therefore

$$\frac{g(u^{n+1}, u^n, \ldots, t^{n+1}, t^n, \ldots, \Delta t) - u^n}{\Delta t} = u_t^n.$$

In other words, the multi-step discretization satisfies the governing equation in the limit as $\Delta t \to 0$.

Thought Experiment For consistency we require $\lim_{\Delta t \to 0} \tau/\Delta t = 0$. Why is it insufficient to require the local truncation error τ to vanish in the limit, $\lim_{\Delta t \to 0} \tau = 0$?

We can extend the definition of consistency directly to a criterion on the order of accuracy of the numerical method. Requiring $\lim_{\Delta t \to 0} \tau/\Delta t = 0$ implies that $\tau = \mathscr{O}(\Delta t^{p+1})$ where $p \geq 1$ since then $\tau/\Delta t = \mathscr{O}(\Delta t^p)$, which only vanishes in the limit if $p \geq 1$. This means that a consistent method must be at least first-order accurate.

Exercise 1. Which of the following numerical methods is consistent?

(a) $v^{n+1} = \frac{1}{2}v^n + \frac{1}{2}\Delta t(f^n + f^{n-1})$
(b) $v^{n+1} = v^n + \Delta t f^{n+1}$
(c) $v^{n+1} = v^n + \frac{1}{2}\Delta t f^n$
(d) $v^{n+1} = v^{n-1}$

21 Dahlquist Equivalence Theorem

The Dahlquist Equivalence Theorem is the primary tool for assessing whether or not a numerical scheme is convergent. Using the concepts of consistency and zero stability alone, we can draw a conclusion about the convergence.

To summarize from above, we have the following concepts:

Consistency: In the limit $\Delta t \to 0$, the method gives a consistent discretization of the ordinary differential equation.
Zero stability: In the limit $\Delta t \to 0$, the method has no solutions that grow unbounded as $N = T/\Delta t \to \infty$.

The Dahlquist Equivalence Theorem guarantees that a method that is consistent and stable is convergent, and also that a convergent method is consistent and stable:

Theorem 1 (Dahlquist Equivalence Theorem). *A multi-step method is convergent if and only if it is consistent and stable.*

Exercise 2. The numerical scheme defined by the equation $v^{n+1} + 4v^n - 5v^{n-1} = 4\Delta t f^n + 2\Delta t f^{n-1}$ is

(a) zero stable, not consistent, and convergent
(b) not zero stable, consistent, and not convergent
(c) zero stable, consistent, and convergent
(d) none of the above