

# Definition of Group

## 1.1

Write a careful proof that every group is the group of isomorphisms of a groupoid. In particular, every group is the group of automorphisms of some object in some category

*Proof.* Let  $G$  be a group, we define a category  $\mathbf{C}$  as follows:

- $\text{Obj}(\mathbf{C}) = \{*\}$
- $\text{Hom}(*, *) = \{g \mid g \in G\}$

We prove the fore-defined structure does form a category:

- **Composition of Morphisms** There is a function as follows:

$$\begin{aligned}\text{Hom}(*, *) \times \text{Hom}(*, *) &\rightarrow \text{Hom}(*, *) \\ (g, h) &\mapsto gh\end{aligned}$$

This composition law explicitly satisfies associativity.

- **Identity**  $1_G \in \text{Hom}(*, *)$  is the identity.

Also, for any  $g \in \text{Hom}(*, *)$ , there exists  $g^{-1} \in \text{Hom}(*, *)$  such that  $gg^{-1} = g^{-1}g = 1_G$ . Thus, every morphism in  $\text{Hom}(*, *)$  is an isomorphism and  $\mathbf{C}$  is a groupoid.  $\square$

## 1.4

Suppose that  $g^2 = e$  for all elements  $g$  of a group  $G$ ; prove that  $G$  is commutative.

*Proof.* For any  $g, h \in G$ , we have:

$$gh = g^{-1}h^{-1} = (hg)^{-1} = hg$$

Which indicates  $G$  is commutative  $\square$

## 1.7

Prove Corollary 1.11:

Let  $g$  be an element of finite order, and let  $N \in \mathbb{Z}$ . Then:

$$g^N = e \Leftrightarrow N \text{ is a multiple of } |g|$$

*Proof.* ( $\Rightarrow$ ) According to Lemma 1.10

( $\Leftarrow$ )

$$g^N = (g^{|g|})^{\frac{N}{|g|}} = (e_G)^{\frac{N}{|g|}} = e_G$$

□

## 1.8

Let  $G$  be a finite **abelian** group, with exactly one element  $f$  of order 2. Prove that  $\prod_{g \in G} g = f$

*Proof.* Since  $G$  is abelian, the product of all elements of  $G$  is well-defined, that is to say, the results is irrelevant to the multiplication order.

Thus, we have:

$$\prod_{g \in G} g = (a_1 a_1^{-1})(a_2 a_2^{-1}) \cdots (a_n a_n^{-1}) f e_G = f$$

□

**Note** The original problem has no abelian condition, which is a false proposition: Consider  $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ , which is a non-commutative group and only  $-1$  has an order of 2. However, the product of all elements in  $Q_8$  may generate different results:

$$1ijk(-1)(-i)(-j)(-k) = 1$$

$$1i(-i)j(-j)k(-k)(-1) = -1$$

## 1.9

Let  $G$  be a finite group, of order  $n$ , and let  $m$  be the number of elements  $g \in G$  of order exactly 2. Prove that  $n - m$  is odd. Deduce that if  $n$  is even then  $G$  necessarily contains elements of order 2.

*Proof.* All elements can be make pair with its inverse, thus:

$$G = \bigcup \{a_i, a_i^{-1}\}$$

For those elements which have order greater than 2,  $a_i$  and  $a_i^{-1}$  are different. Thus we have:  $n = m + 2k + 1$  where  $k$  is the number of pair where element has order greater than 2.

This shows that  $n - m = 2k + 1$  is an odd value. If  $n$  is even, then  $m$  is certainly greater than 0, meaning there are elements has order equals to 2.  $\square$

### 1.11

Prove that for all  $g, h$  in a group  $G$ ,  $|gh| = |hg|$

*Proof.* We prove that for  $n \in \mathbb{N}^+$ ,  $(gh)^n = e \iff (hg)^n = e$

$$\begin{aligned} (gh)^n = e &\iff (gh)(gh) \cdots (gh) = e \\ &\iff g(hg)^{n-1}h = e \\ &\iff (hg)^{n-1}h = g^{-1} \\ &\iff (hg)^n = e \end{aligned}$$

Thus we have:  $|hg| \mid |gh|$  and  $|gh| \mid |hg|$ , indicating  $|gh| = |hg|$   $\square$

### 1.12

In the group of invertible  $2 \times 2$  matrices, consider

$$g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad h = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

Verify that  $|g| = 4$ ,  $|h| = 3$ , and  $|gh| = \infty$

*Proof.* It is easy to show that  $g^2 = -I$ , thus  $|g| = 4$ . For  $h$  we have:

$$h^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad h^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus,  $|h| = 3$ .  $gh = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ , it's not hard to verify that  $(gh)^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$  (By induction), which indicates  $gh$  has no finite order.  $\square$

**Note** If  $g$  and  $h$  are commutative, then  $|gh| \leq lcm(|g|, |h|)$ . However, for a non-commutative group, there is no general result for the order of  $gh$ .

## 1.14

prove that if  $g$  and  $h$  commute, and  $\gcd(|g|, |h|) = 1$ , then  $|gh| = |g||h|$

*Proof.* If  $(gh)^t = e, t \in \mathbb{N}^+$  then:  $g^t = h^{-t}$ . We have:

$$g^{t|h|} = h^{-t|h|} = e \Rightarrow |g| \mid t|h| \Rightarrow |g| \mid t$$

since  $\gcd(|g|, |h|) = 1$ . Also,  $|h| \mid t$  and  $|g||h| \mid t$  because  $\gcd(|g|, |h|) = 1$ . Note that  $(gh)^{|g||h|} = e$  we have:  $|gh| \mid |g||h|$ . By the above fact, we have  $|g||h| \mid |gh|$ . Thus we have:  $|gh| = |g||h|$ .  $\square$

## Examples of groups

### 2.1

One can associate an  $n \times n$  matrix  $M_\sigma$  with a permutation  $\sigma \in S_n$ , by letting the entry at  $(i, \sigma(i))$  be 1, and letting all other entries be 0. For example, the matrix corresponding to the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \in S_3$$

would be

$$M_\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Prove that, with this notation,

$$M_{\sigma\tau} = M_\sigma M_\tau$$

for all  $\sigma, \tau \in S_n$ , where the product on the right is the ordinary product of matrices.

*Proof.*

$$\begin{aligned} M_\sigma M_\tau(i, j) &= \sum_{k=1}^n M_\sigma(i, k) M_\tau(k, j) \\ &= \sum_{\substack{1 \leq k \leq n \\ \sigma(i)=k, \tau(k)=j}} 1 \end{aligned}$$

Only when  $\tau \circ \sigma(i) = j$  would makes this item equals to 1, thus  $M_\sigma M_\tau(i, j) = M_{\sigma\tau}(i, j)$ . It's done.  $\square$

## 2.2

Prove that if  $d \leq n$ , then  $S_n$  contains elements of order  $d$ .

*Proof.* The permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & d-1 & d & d+1 & \cdots & n \\ 2 & 3 & 4 & \cdots & d & 1 & d+1 & \cdots & n \end{pmatrix}$$

is obviously an element has an order of  $d$ .  $\square$

## 2.6

For every positive integer  $n$  construct a group containing two elements  $g, h$  such that  $|g| = 2$ ,  $|h| = 2$ , and  $|gh| = n$ .

*Proof.*  $D_{2n}$  satisfies this condition.  $\square$

## 2.7

Find all elements of  $D_{2n}$  that commute with every other element.

## 2.12

Prove that there are no integers  $a, b, c$  such that  $a^2 + b^2 = 3c^2$ .

*Proof.* Let  $(a, b, c)$  be the smallest tuple that satisfies  $a^2 + b^2 = 3c^2$  then we have:

$$a^2 + b^2 = [0]_3$$

There is only one possible way to achieve this:  $a = [0]_3, b = [0]_3$ . Let  $a = 3a', b = 3b'$  then we have:  $3(a'^2 + b'^2) = c^2$ , indicating  $c = [0]_3$ . Let  $c = 3c'$  would incur  $a'^2 + b'^2 = 3c'^2$  and we have a solution  $(a', b', c')$  which is smaller than  $(a, b, c)$ , a contradiction.  $\square$

## 2.13

Prove that if  $\gcd(m, n) = 1$ , then there exist integers  $a$  and  $b$  such that

$$am + bn = 1$$

Conversely, prove that if  $am + bn = 1$  for some integers  $a$  and  $b$ , then  $\gcd(m, n) = 1$

*Proof.*  $[m]_n$  is an generator of  $\mathbb{Z}/n\mathbb{Z}$ . Thus, there exists some positive integer  $a$  such that:  $a[m]_n = [1]_n$ , i.e  $[am]_n = [1]_n$ . Further, we have:  $am - 1 = b'n$  for some  $b' \in \mathbb{N}$ . which is:  $am - b'n = 1$ , Let  $b = -b'$ , the equation holds.

If there are  $a, b$  such that  $am + bn = 1$  then  $\gcd(m, n)$  is a divisor of left side, thus a divisor of 1. Then  $\gcd(m, n)$  has to be 1.  $\square$

## 2.15

Let  $n > 0$  be an odd integer.

- Prove that if  $\gcd(m, n) = 1$ , then  $\gcd(2m + n, 2n) = 1$ .
- Prove that if  $\gcd(r, 2n) = 1$ , then  $\gcd(\frac{r+n}{2}, n) = 1$
- Conclude that the function  $[m]_n \rightarrow [2m + n]_{2n}$  is a bijection between  $(\mathbb{Z}/n\mathbb{Z})^*$  and  $(\mathbb{Z}/2n\mathbb{Z})^*$

The number  $\phi(n)$  of elements of  $(\mathbb{Z}/n\mathbb{Z})^*$  is Euler's  $\phi$ -function. The reader has just proved that if  $n$  is odd, then  $\phi(2n) = \phi(n)$ . Much more general formulas will be given later on (cf. Exercise V.6.8)

*Proof.* (1) Let  $d = \gcd(2m + n, 2n)$  then  $d \mid 2(2m + n) - 2n$ , which is  $d \mid 4m$ . Thus:  $d \mid \gcd(4m, 2n)$ . Note that  $\gcd(m, n) = 1$ , then  $\gcd(4m, 2n) = 2\gcd(2m, n) = 2$ . Thus  $d = 1$  or  $d = 2$ . Note that  $2m + n$  is odd, then  $d = 1$ .

(2) Let  $d = \gcd(\frac{r+n}{2}, n)$ , then  $d \mid 2 \times \frac{r+n}{2} - n$ , that is  $d \mid r$ . Then  $d \mid n$  indicates  $d \mid \gcd(r, n)$ . Thus  $d = 1$ .

(3) According to (1),  $\gcd(m, n) = 1$  indicates  $\gcd(2m + n, 2n) = 1$ , thus the element  $[2m + n]_{2n} \in (\mathbb{Z}/2n\mathbb{Z})^*$ . Next we will verify that this function is well-defined.

If  $[m_1]_n = [m_2]_n$  then  $n \mid (m_2 - m_1) \Rightarrow 2n \mid (2m_2 - 2m_1) \Rightarrow 2n \mid ((2m_2 + n) - (2m_1 + n))$ . Thus,  $[2m_2 + n]_{2n} = [2m_1 + n]_{2n}$ . This indicates the function is well-defined.

If  $[2m_1 + n]_{2n} = [2m_2 + n]_{2n}$  then we have  $2n \mid ((2m_2 + n) - (2m_1 + n))$ , which is  $2n \mid 2(m_2 - m_1)$ , and further  $n \mid (m_2 - m_1)$ , indicating  $[m_2]_n = [m_1]_n$ . Thus, this function is injective.

For any  $[2m + n]_{2n} \in (\mathbb{Z}/2n\mathbb{Z})^*$ , we have  $f([m]_n) = [2m + n]_{2n}$ . According to (2),  $\gcd(\frac{2m+n+n}{2}, n) = 1$ , which is  $\gcd(m + n, n) = 1 \Rightarrow \gcd(m, n) = 1$ . Thus,  $[m]_n \in (\mathbb{Z}/n\mathbb{Z})^*$  and  $f$  is surjective.

In conclusion,  $f$  is both injective and surjective, thus bijective.  $\square$

# The Category Grp

## 3.3

Show that if  $G, H$  are abelian groups, then  $G \times H$  satisfies the universal property for coproducts in **Ab**

*Proof.* Let  $\tau_G$  and  $\tau_H$  satisfies  $\tau_G(g) = (g, 0_H)$  and  $\tau_H(h) = (0_G, h)$ . We have to show that the following commutative graph exists:

$$\begin{array}{ccccc}
 & & A & & \\
 & \nearrow f_G & \uparrow \exists! f & \nwarrow f_H & \\
 G & \xrightarrow{\tau_G} & G \times H & \xleftarrow{\tau_H} & H
 \end{array}$$

We define  $f$  as follows:

$$f : G \times H \rightarrow A, \quad (g, h) \mapsto f_G(g) + f_H(h)$$

We show that  $f$  is an homomorphism:

$$\begin{aligned}
 f((g_1, h_1) + (g_2, h_2)) &= f((g_1 + g_2, h_1 + h_2)) = f_G(g_1 + g_2) + f_H(h_1 + h_2) \\
 &= f_G(g_1) + f_G(g_2) + f_H(h_1) + f_H(h_2) \\
 &= (f_G g_1 + f_H(h_1)) + (f_G g_2 + f_H(h_2)) \\
 &= f(g_1, h_1) + f(g_2, h_2)
 \end{aligned}$$

And we show that  $f$  is unique. if  $f'$  satisfies the above commutative diagram, then we have:

$$\begin{aligned}
 f'(g, h) &= f'(g, 0_H) + f'(0_G, h) = f'(\tau_G(g)) + f'(\tau_H(h)) \\
 &= (f' \tau_G)(g) + (f' \tau_H)(h) \\
 &= f_G(g) + f_H(h) = f(g, h)
 \end{aligned}$$

Thus,  $f$  is unique. And by the definition of coproduct,  $G \times H$  is the coproduct of  $G$  and  $H$  in category **Ab**. □

## 3.4

Let  $G, H$  be groups, and assume that  $G \cong H \times G$ . Can you conclude that  $H$  is trivial.

*Solution* No,  $H$  might be non-trivial group. The following example:

$$2\mathbb{Z} \times \mathbb{Z}_2 \cong \mathbb{Z} \cong \mathbb{Z}_2$$

indicates that  $H = \mathbb{Z}_2$  is not a trivial group. We construct homomorphisms as follows:

$$\begin{aligned} f : 2\mathbb{Z} \times \mathbb{Z}_2 &\longrightarrow \mathbb{Z} \\ ([a], 2k) &\mapsto 2k + a, a = 0, 1 \end{aligned}$$

Then it is easy to verify that  $f$  is bijective.  $\forall x = ([a], 2k_1), y = ([b], 2k_2)$ .

$$f(x + y) = f([a + b], 2k_1 + 2k_2) = 2k_1 + 2k_2 + (a + b) = f(x) + f(y)$$

Thus,  $f$  is an homomorphism, therefore,  $2\mathbb{Z} \times \mathbb{Z}_2 \cong \mathbb{Z}$ . The right part,  $2\mathbb{Z} \cong \mathbb{Z}$  is trivial.

### 3.5

Prove that  $\mathbb{Q}$  is not the direct product of two nontrivial groups

*Proof.* Proof by contradiction, say  $\mathbb{Q}$  is the direct product of two groups  $\mathbb{Q} \cong G \times H$ , say that  $G$  is nontrivial. We prove that  $\pi_G$  is injective by proving no other element is mapped to be  $0_G$  except for  $0 \in \mathbb{Q}$

Suppose that  $\pi_G\left(\frac{m}{n}\right) = 0_G$ . We have:  $\pi_G(m) = n\pi_G\left(\frac{m}{n}\right) = nm\pi_G(1) = 0_G$ . Thus  $\pi_G(1) = 0_G$ . Which means  $\pi_G(\mathbb{Z}) = \{0_G\}$ .

Thus, for any  $\frac{a}{b} \in \mathbb{Q}$ , we have:  $0_G = \pi_G(a) = b\pi_G\left(\frac{a}{b}\right) \Rightarrow \pi_G\left(\frac{a}{b}\right) = 0_G$ , which means  $\pi_G(\mathbb{Q}) = \{0_G\}$ . Note that  $\pi_G$  is surjective and  $G$  is nontrivial, we have above assumption failed, that is to say, no element  $\frac{a}{b}$  satisfies  $\pi_G\left(\frac{a}{b}\right) = 0_G$ , which means  $\pi_G$  is injective.

Thus  $H$  must be trivial, otherwise,  $\pi_G(g_1, h_1) = g_1 = \pi_G(g_1, h_2)$  indicates that  $\pi_G$  is not injective.  $\square$

### 3.6

Consider the product of the cyclic groups  $C_2, C_3$ :  $C_2 \times C_3$ . By Exercise 3.3, this group is a coproduct of  $C_2$  and  $C_3$  in **Ab**. Show that it is not a coproduct of  $C_2$  and  $C_3$  in **Grp**, as follows:

- find injective homomorphisms  $C_2 \rightarrow S_3, C_3 \rightarrow S_3$ ;



- arguing by contradiction, assume that  $C_2 \times C_3$  is a coproduct of  $C_2, C_3$ , and deduce that there would be a group homomorphism  $C_2 \times C_3 \rightarrow S_3$  with certain properties;
- show that there is no such homomorphism

*Proof.* The injective homomorphism is:

$$f_{C_2} : C_2 \rightarrow S_3$$

$$[0]_2 \mapsto \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, [1]_2 \mapsto \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

and

$$f_{C_3} : C_3 \rightarrow S_3$$

$$[0]_3 \mapsto \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, [1]_3 \mapsto \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, [2]_3 \mapsto \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

According to the definition of coproduct, the following diagram holds

$$\begin{array}{ccccc} & & S_3 & & \\ & \nearrow f_{C_2} & \uparrow \exists! f & \nwarrow f_{C_3} & \\ C_2 & \xrightarrow{\tau_{C_2}} & C_2 \times C_3 & \xleftarrow{\tau_{C_3}} & C_3 \end{array}$$

The homomorphism  $f : C_2 \times C_3 \rightarrow S_3$  satisfies  $f\tau_{C_2} = f_{C_2}$  and  $f\tau_{C_3} = f_{C_3}$ . We prove that such  $f$  does not exist: We write  $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$  as  $a$  and  $b$  for simplicity: thus we must have:

$$f([0]_2, [0]_3) = \mathbf{1}_{S_3}, f([1]_2, [0]_3) = a, f([0]_2, [1]_3) = b, f([0]_2, [1]_3) = b^2$$

And we have:

$$ab = f([1]_2, [0]_3) + f([0]_2, [1]_3) = f([1]_2, [1]_3)$$

and

$$(ab)(ab) = f([1]_2, [1]_3)f([1]_2, [1]_3) = f([0]_2, [2]_3) = b^2$$

This indicates  $abab = b^2 \Rightarrow ba = a^{-1}b = ab$ . However,  $ab = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ ,  $ba = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$  thus  $ab \neq ba$ . Then such  $f$  does not exist. We assert that  $C_2 \times C_3$  is not the coproduct of  $C_2$  and  $C_3$  in category **Grp**.  $\square$

## Group Homomorphisms

### 4.1

Check that the function  $\pi_m^n$  defined in 4.1 is well-defined, and makes the diagram commute. Verify that it is a group homomorphism. Why is the hypothesis  $m \mid n$  necessary?

*Proof.*  $\pi_m^n$  is well-defined: if  $[a_1]_n = [a_2]_n$  then  $n \mid a_1 - a_2$ , thus  $m \mid a_1 - a_2$  as  $m \mid n$ . We have  $[a_1]_m = [a_2]_m$  and  $\pi_m^n([a_1]_n) = \pi_m^n([a_2]_n)$ . The function has nothing to do with the representators. This is a homomorphism because:

$$\pi_m^n([a]_n + [b]_n) = \pi_m^n([a + b]_n) = [a + b]_m = [a]_m + [b]_m = \pi_m^n([a]_n) + \pi_m^n([b]_n)$$

The hypothesis  $m \mid n$  is necessary because if  $m \nmid n$  we may fail to show that  $\pi_m^n$  is well-defined. One example is to use  $m = 4, n = 3$ . Then  $\pi_m^n$  is not well-defined, we have:

$$\pi_3^4([12]_4) = [12]_3 = [0]_3;$$

$$\pi_3^4([8]_4) = [8]_3 = [2]_3 \neq [0]_3$$

□

### 4.2

Show that the homomorphism  $\pi_2^4 \times \pi_2^4 : C_4 \rightarrow C_2 \times C_2$  is not an isomorphism. In fact, is there any nontrivial isomorphism  $C_4 \rightarrow C_2 \times C_2$ ?

*Solution* No, there is no such isomorphism between  $C_4$  and  $C_2 \times C_2$ . The reason is that  $C_4$  has one element of order 4, which is  $[1]_4$ , however, each element of  $C_2 \times C_2$  has order 1 or 2.

### 4.3

Prove that a group of order  $n$  is isomorphic to  $\mathbb{Z}/n\mathbb{Z}$  if and only if it contains an element of order  $n$ .

*Proof.* ( $\Rightarrow$ ) If group  $G$  with order of  $n$  is isomorphic to  $\mathbb{Z}/n\mathbb{Z}$  then  $G$  must have an element of order  $n$ , which is  $f^{-1}([1]_n)$ . Here  $f$  is the isomorphism from  $G$  to  $\mathbb{Z}/n\mathbb{Z}$ .

( $\Leftarrow$ ) If group  $G$  with order  $n$  has an element with order of  $n$ , say  $g$ . Then  $\langle g \rangle = G$ . We define the homomorphism  $f : G \rightarrow \mathbb{Z}/n\mathbb{Z}$  as follows:  $g^k \mapsto [k]_n$ . It is obvious to see that  $f$  is an isomorphism. □

#### 4.4

Prove that no two of the groups  $(\mathbb{Z}, +)$ ,  $(\mathbb{Q}, +)$ ,  $(\mathbb{R}, +)$  are isomorphic to one another. Can you decide whether  $(\mathbb{R}, +)$ ,  $(\mathbb{C}, +)$  are isomorphic to one another.

*Proof.*  $(\mathbb{Z}, +)$  and  $(\mathbb{Q}, +)$  are not isomorphic to  $(\mathbb{R}, +)$  because they even do not have the same cardinality.

$(\mathbb{Z}, +) \not\cong (\mathbb{Q}, +)$ :

Suppose  $f$  is an isomorphism from  $(\mathbb{Z}, +)$  to  $(\mathbb{Q}, +)$ , let  $f(1) = g \in \mathbb{Q}$ . Then we have  $\mathbb{Q}$  is generated by  $g$  as  $\frac{a}{b} = f(n) = nf(1) = ng$  for some  $n$ . Let  $g = \frac{a}{b}$  and  $a, b$  relatively prime, then have:  $\frac{na}{b} = \frac{1}{p}$ . We have:  $pna = b$ . note that  $\gcd(a, b) = 1$ , then we must have  $a = 1$ . And  $np = b$ . We pick  $p$  a prime that is relatively prime to  $b$ . Then  $np = b$  can not be true.  $\square$

#### 4.5

Prove that the groups  $(\mathbb{R} \setminus \{0\}, \times)$  and  $(\mathbb{C} \setminus \{0\}, \times)$  are not isomorphic.

*Proof.* If  $(\mathbb{R} \setminus \{0\}, \times)$  is isomorphic to  $(\mathbb{C} \setminus \{0\}, \times)$  let the isomorphism be  $f$ . and let  $f(1) = 1$  and let  $f(i) = g$  Consider  $f(-1)$ , we have:

$$f(-1)^2 = f((-1)^2) = f(1) = 1$$

Then we have  $f(-1) = 1$  or  $f(-1) = -1$ , note that  $f$  is an isomorphism, we must have  $f(-1) = -1$ . Further we have:  $f(i)^2 = f(i^2) = f(-1) = -1$ . However, no such element in  $\mathbb{R}$  makes this true. Thus, we have show that  $(\mathbb{R} \setminus \{0\}, \times) \not\cong (\mathbb{C} \setminus \{0\}, \times)$ .  $\square$

#### 4.6

We have seen that  $(\mathbb{R}, +)$  and  $(\mathbb{R}^{>0}, \times)$  are isomorphic (Example 4.4). Are the groups  $(\mathbb{Q}, +)$  and  $(\mathbb{Q}^{>0}, \times)$  isomorphic?

*Solution*

#### 4.7

Let  $G$  be a group. Prove that the function  $G \rightarrow G$  defined by  $g \mapsto g^{-1}$  is a homomorphism if and only if  $G$  is abelian. Prove that  $g \mapsto g^2$  is a homomorphism if and only if  $G$  is abelian.

*Proof.*  $g \mapsto g^{-1}$  is an homomorphism iff  $f(ab) = f(a)f(b)$  holds for any  $a, b \in G$ . This is true if and only if  $a^{-1}b^{-1} = b^{-1}a^{-1}$  for any  $a, b \in G$ . And  $a^{-1}b^{-1} = b^{-1}a^{-1} \iff ba = ab$  by taking inverse at both sides. Thus we have  $g \mapsto g^{-1}$  if and only if  $G$  is abelian.

$g \mapsto g^2$  is an homomorphism iff  $f(ab) = f(a)f(b)$  holds for any  $a, b \in G$ . This is true if and only if  $(ab)(ab) = a^2b^2 \iff ab = ba$  for any  $a, b \in G$ .  $\square$

## 4.8

Let  $G$  be a group, and  $g \in G$ . Prove that the function  $\gamma_g : G \rightarrow G$  defined by  $(\forall a \in G) : \gamma_g(a) = gag^{-1}$  is an automorphism of  $G$ . (The automorphisms  $\gamma_g$  are called ‘inner’ automorphisms of  $G$ .) Prove that the function  $G \rightarrow \text{Aut}(G)$  defined by  $g \mapsto \gamma_g$  is a homomorphism. Prove that this homomorphism is trivial if and only if  $G$  is abelian.

*Proof.* First we show that  $\gamma_g$  is an homomorphism: for any  $a, b \in G$  we have:

$$\gamma_g(ab) = g(ab)g^{-1} = (gag^{-1})(gbg^{-1}) = \gamma_g(a)\gamma_g(b)$$

Thus  $\gamma_g$  is an homomorphism.  $\gamma_g$  has an inverse:  $\gamma_{g^{-1}}$ . We have:  $\gamma_g\gamma_{g^{-1}}(a) = \gamma_g(g^{-1}ag) = g(g^{-1}ag)g^{-1} = a$  for any  $a \in G$ . Thus,  $\gamma_g\gamma_{g^{-1}} = I_G$ . Similarly,  $\gamma_{g^{-1}}\gamma_g = I_G$ . Thus  $\gamma_g$  has inverse and therefore a bijection, this indicates  $\gamma_g$  is an isomorphism.

Let  $f : G \rightarrow \text{Aut}(G), g \mapsto \gamma_g$  be the function mentioned above. We shall prove that this function is actually an homomorphism:  $f(ab) = \gamma_{ab}$  and we have:  $\gamma_{ab}(g) = (ab)^{-1}gab = b^{-1}(a^{-1}ga)b = \gamma_a \circ \gamma_b(g)$  for all  $g \in G$ . Thus we have  $f(ab) = \gamma_{ab} = \gamma_a \circ \gamma_b = f(a)f(b)$ . Therefore  $f$  is an homomorphism. If  $G$  is abelian then all  $f(g) = \gamma_g = I_G$ , thus is trivial.  $\square$

## 4.9

Prove that if  $m, n$  are positive integers such that  $\gcd(m, n) = 1$ , then  $C_{mn} \cong C_m \times C_n$ .

*Proof.* The homomorphism  $\pi_m^{mn} \times \pi_n^{mn} : C_{mn} \rightarrow C_m \times C_n$  is defined as follows:

$$[a]_{mn} \mapsto ([a]_m, [a]_n)$$

and is an homomorphism as  $\pi_m^{mn}$  and  $\pi_n^{mn}$  are homomorphisms. We shall show that this function is bijection. First it is injective: if  $f([a]_{mn}) = f([b]_{mn})$  then  $([a]_m, [a]_n) = ([b]_m, [b]_n)$  which means:  $m \mid a - b$  and  $n \mid a - b$ . Further we

have  $mn \mid a - b$  because  $\gcd(m, n) = 1$ . Thus  $[a]_{mn} = [b]_{mn}$  and this indicates  $f$  is injective.

For surjective property, note that  $\gcd(m, n) = 1$  indicates there exist some  $x, y$  such that  $xm - ny = 1$ . Then we have  $x$  satisfies  $xm = ny + 1$ , we call  $\mathbf{x} = [xm]_{mn}$ , we have  $f(\mathbf{x}) = ([0]_m, [1]_n)$ . Similarly, we will have such  $\mathbf{y}$  satisfying  $f(\mathbf{y}) = ([1]_m, [0]_n)$ . For any  $([a]_m, [b]_n) \in C_m \times C_n$  we have:  $([a]_m, [b]_n) = ([a]_m, [0]_n) + ([0]_m, [b]_n) = af(\mathbf{x}) + bf(\mathbf{y}) = f(a\mathbf{x} + b\mathbf{y})$ . Thus  $f$  is surjective and  $f$  is bijective.

In conclusion, we have  $f$  to be group homomorphism and bijection. Thus  $f$  is a group isomorphism.  $\square$

## 4.10

Let  $p \neq q$  be odd prime integers; show that  $(\mathbb{Z}/pq\mathbb{Z})^*$  is not cyclic.

*Proof.* Suppose that  $(\mathbb{Z}/pq\mathbb{Z})^*$  is cyclic. Then we have the order of  $\square$

## 4.11

In due time we will prove the easy fact that if  $p$  is a prime integer then the equation  $x^d = 1$  can have at most  $d$  solutions in  $\mathbb{Z}/p\mathbb{Z}$ . Assume this fact, and prove that the multiplicative group  $G = (\mathbb{Z}/p\mathbb{Z})^*$  is cyclic

*Proof.* Let the maximum order of elements in  $(\mathbb{Z}/p\mathbb{Z})^*$  be  $d$ , we show that  $d$  must be  $p$ .

If  $d \leq p - 2$ , say  $g$  has order  $d$ , then for every element  $h \in (\mathbb{Z}/p\mathbb{Z})^*$  we have  $|h| \mid d$ . Otherwise, the element  $gh$  will have order of  $\text{lcm}(|h|, d) > d$ , contradicts the assumption that  $d$  is the maximum order.

Thus we have  $g^d = 1$  for every element in  $\mathbb{Z}/p\mathbb{Z}$ , which means  $x^d = 1$  has  $p - 1$  solutions, contradicts the assumption. Thus, we have  $d = p - 1$  and  $\mathbb{Z}/p\mathbb{Z}$  is cyclic.  $\square$

**NOTE** This proof can not be used to proof a general  $(\mathbb{Z}/n\mathbb{Z})^*, n \in \mathbb{N}^+$  is cyclic(though this proposition is false). The assumption  $x^d = 1$  has at most  $d$  solutions is constrained within  $\mathbb{Z}/p\mathbb{Z}$ , not generalized group.

## 4.14

Prove that the order of the group of automorphisms of a cyclic group  $C_n$  is the number of positive integers  $r < n$  that are *relatively prime to*  $n$ .

*Proof.*  $C_n$  is generated by  $[1]_n$ , so any automorphism from  $C_n$  to  $C_n$  is determined by the image of  $[1]_n$ . To make this homomorphism  $f$  bijective, we must make  $f([1]_n)$  also be a generator. Thus the number of elements in  $\text{Aut}_{\mathbf{Grp}}(C_n)$  is determined by the number of generators in  $C_n$ , which is the number of positive number that is relatively prime to  $n$ . We formally prove this as followed:

Let  $f \in \text{Aut}_{\mathbf{Grp}}(C_n)$ , consider  $f([1]_n)$ . Notice that  $f$  is isomorphism, thus we have  $|f([1]_n)|$  has order  $n$  (proposition 4.8), thus  $|f([1]_n)|$  is relatively prime to  $n$  (The representator of  $f([1]_n)$ ).

On the contrary, if  $[m]_n, \gcd(m, n) = 1$ , we define  $f([1]_n) = [m]_n$ , it derives an isomorphism from  $C_n$  to  $C_n$ . Thus, we have established a map from  $\text{Aut}_{\mathbf{Grp}}(C_n)$  to the set of numbers that are relatively prime to  $n$ , denoted as  $S$ . This map is injective as each  $f$  maps  $[1]_n$  to different elements in  $S$ , and is surjective by the construction described above. Thus, it is bijection and they have the same cardinality.  $\square$

## 4.15

Compute the group of automorphisms of  $(\mathbb{Z}, +)$ . Prove that if  $p$  is prime, then  $\text{Aut}_{\mathbf{Grp}}(C_p) \cong C_{p-1}$ . (Use Exercise 4.11.)

*Proof.* There are only two elements in  $\text{Aut}_{\mathbf{Grp}}(\mathbb{Z}, +)$ : The identity and the isomorphism that maps 1 to  $-1$ .

To prove  $\text{Aut}_{\mathbf{Grp}}(C_p) \cong C_{p-1}$ , we show that  $\text{Aut}_{\mathbf{Grp}}(C_p) \cong (\mathbb{Z}/p\mathbb{Z})^*$  and leverage the result of exercise 4.11.

The proof of exercise 4.14 shows that there is a bijection from  $\text{Aut}_{\mathbf{Grp}}(C_p)$  to  $(\mathbb{Z}/p\mathbb{Z})^*$  by  $[m]_n \mapsto f_{[m]_n}, \gcd(m, n) = 1$ , where  $f_{[m]_n}$  is the automorphism derived by  $f_{[m]_n}([1]_n) = [m]_n$ . We show that this map, namely  $\phi$  is an homomorphism:

$$\phi([m_1]_n \times [m_2]_n) = \phi([m_1 m_2]_n) = f_{[m_1 m_2]_n} = f_{[m_1]_n} \circ f_{[m_2]_n}$$

The last  $=$  is true by checking the image of  $[1]_n$  under  $f_{[m_1 m_2]_n}$  and  $f_{[m_1]_n} \circ f_{[m_2]_n}$ . In conclusion, we have the map  $\phi$  is both a homomorphism and bijection. Thus,  $\text{Aut}_{\mathbf{Grp}}(C_p) \cong (\mathbb{Z}/p\mathbb{Z})^* \cong C_{p-1}$ .  $\square$

## 4.16

Prove *Wilson's theorem*: a positive integer  $p$  is prime if and only if

$$(p-1)! \equiv -1 \pmod{p}$$

*Proof.* ( $\Rightarrow$ ) If  $p$  is a prime, then  $(\mathbb{Z}/p\mathbb{Z})^*$  is cyclic, let  $g \in (\mathbb{Z}/p\mathbb{Z})^*$  be the elements with order  $p - 1$ , then we have:

$$(p - 1)! \equiv gg^2 \dots g^{p-1} \equiv g^{\frac{p(p-1)}{2}} \pmod{p}$$

Note that we have  $g^{p-1} \equiv 1 \pmod{p}$  and  $g^{\frac{p-1}{2}} \equiv -1 \pmod{p}$  because the order of  $g$  is exactly  $p - 1$ . We have:

$$g^{\frac{p(p-1)}{2}} = g^{\frac{(p-1)^2}{2}} g^{\frac{p-1}{2}} \equiv g^{\frac{p-1}{2}} \equiv -1 \pmod{p}$$

The proof is done.

( $\Leftarrow$ ) Suppose  $p$  is not a prime and  $d$  is a divisor of  $p$ . Then we have:  $(p - 1)! \equiv -1 \pmod{d}$ . However,  $d < p$  indicates  $d \mid d!$  and  $d! \mid (p - 1)!$ , thus we have:  $(p - 1)! \equiv 0 \pmod{d}$ , a contradiction.  $\square$

## 5. Free Group

### 5.1

Does the category  $\mathcal{F}^A$  defined in 5.2 have final objects? If so, what are they.

*Solution* It has, the object  $(G, j)$  where  $G$  is trivial group and  $j$  is a set-function satisfies:  $a \mapsto 1_G, \forall a \in A$  is a final object in  $\mathcal{F}^A$ . It's obvious that any other object in this category has a morphism to this object, namely the trivial homomorphism. Note that final object in a category is the same up to isomorphism, thus, these are all possible final objects.

### 5.2

### 5.3

Use the universal property of free groups to prove that the map  $j : A \rightarrow F(A)$  is injective, for all sets  $A$ .

*Proof.* The universal property indicates that the following commutative diagram holds for any objects  $(G, j_2)$ :

$$\begin{array}{ccc} F(A) & \xrightarrow{\varphi} & G \\ \uparrow j & \nearrow j_1 & \\ A & & \end{array}$$

Specifically, let  $j_1$  be injective set-function, we must have  $j_1 = \varphi \circ j$ , the fact that  $j_1$  is injective indicates  $j$  is injective. The difficulty is to show that such  $j_1$  and  $G$  exists. □

## 5.5

Verify explicitly that  $H^{\oplus A}$  is a group.

*Proof.*  $H^{\oplus A}$  is a subset of  $H^A$  that consists of set-functions only has finitely many “non-zero” images. For  $\alpha_1, \alpha_2 \in H^{\oplus A}$ , we have  $\alpha_1 + \alpha_2 \in H^{\oplus A}$  by defining:

$$(\alpha_1 + \alpha_2)(a) = \alpha_1(a) + \alpha_2(a)$$

Note that  $\alpha_1$  and  $\alpha_2$  has at most finitely many non-zero images, thus  $\alpha_1 + \alpha_2$  has only finitely many non-zero images. Further, we have the zero element:  $\mathbf{0} : a \mapsto 0_H$  and addition inverse:  $-\alpha : a \mapsto -\alpha(a)$ . Thus  $H^{\oplus A}$  is a group. The commutativity of  $H$  also indicates that  $H^{\oplus A}$  is an abelian group. □

## 5.6

Prove that the group  $F(\{x, y\})$  (visualized in Example 5.3) is a coproduct  $\mathbb{Z} * \mathbb{Z}$  of  $\mathbb{Z}$  by itself in the category **Grp**.

*Proof.* There is a explicit proof to show that  $F(\{x, y\})$  is the coproduct of  $\mathbb{Z}$  and  $\mathbb{Z}$ : We have the following diagram:

$$\begin{array}{ccc} \mathbb{Z} & & \\ & \searrow \iota_1 & \\ & & F(\{x, y\}) \\ & \nearrow \iota_2 & \\ \mathbb{Z} & & \end{array}$$

$\iota_1$  and  $\iota_2$  are homomorphisms derived by defining  $\iota_1(1) = x$  and  $\iota_2(1) = y$ . Then for any other group  $G$  and  $f_1, f_2$  we have to prove the next diagram holds:



$$\begin{array}{ccccc}
& & G & & \\
& \nearrow f_1 & \uparrow \exists! \varphi & \nwarrow f_2 & \\
\mathbb{Z} & \xrightarrow{\iota_1} & F(\{x, y\}) & \xleftarrow{\iota_2} & \mathbb{Z}
\end{array}$$

Define  $\varphi$  such that  $\varphi(x) = f_1(1)$  and  $\varphi(y) = f_2(1)$ . Then we have such  $\varphi$  is a homomorphism and is unique. Thus, the free group on  $\{x, y\}$  is a coproduct of  $\mathbb{Z}$  and  $\mathbb{Z}$ .  $\square$

## 5.7

Extend the result of Exercise 5.6 to free groups  $F(\{x_1, \dots, x_n\})$  and to free abelian groups  $F^{ab}(\{x_1, \dots, x_n\})$

*Solution* The Extended result is that:  $F(\{x_1, \dots, x_n\})$  is the coproduct of  $n$   $\mathbb{Z}$  in category **Grp** and is a coproduct of  $n$   $\mathbb{Z}$  in category **Ab**.

## 5.8

Still more generally, prove that  $F(A \sqcup B) = F(A) * F(B)$  and that  $F^{ab}(A \sqcup B) = F^{ab}(A) \oplus F^{ab}(B)$  for all sets  $A, B$ .

*Proof.* We will only prove the fact that  $F(A \sqcup B) = F(A) * F(B)$ . In this question, we can only use the universal property. To prove that  $F(A \sqcup B)$  is the coproduct of  $F(A)$  and  $F(B)$ , we first construct the “injection” homomorphism: Here is the diagram:

$$\begin{array}{ccc}
A & \xrightarrow{i_A} & F(A) \\
\downarrow \iota_A & & \downarrow I_{F(A)} \\
A \sqcup B & \xrightarrow{i_{A \sqcup B}} & F(A \sqcup B) \\
\uparrow \iota_B & & \uparrow I_{F(B)} \\
B & \xrightarrow{i_B} & F(B)
\end{array}$$

Note that the set-function  $i_{A \sqcup B} \circ \iota_A$  (or  $i_{A \sqcup B} \circ \iota_B$ ) is a function from  $A$  (or  $B$ ) to  $F(A \sqcup B)$ , according to the universal property of  $F(A)$ , there exists a

unique homomorphism  $I_{F(A)}$  (or  $I_{F(B)}$ ) such that  $I_{F(A)} \circ i_A = i_{A \sqcup B} \circ \iota_A$  and  $I_{F(B)} \circ i_B = i_{A \sqcup B} \circ \iota_B$ . We prove that  $(F(A \sqcup B), I_{F(A)}, I_{F(B)})$  is a coproduct of  $F(A)$  and  $F(B)$ .

Say  $G$  is another group with homomorphism  $f_{F(A)} : F(A) \rightarrow G$  and  $f_{F(B)} : F(B) \rightarrow G$ . Then we have:

$$\begin{array}{ccc}
 A & \xrightarrow{i_A} & F(A) \\
 \downarrow \iota_A & & \downarrow f_{F(A)} \\
 A \sqcup B & \xrightarrow{f} & G \\
 \uparrow \iota_B & & \uparrow f_{F(B)} \\
 B & \xrightarrow{i_B} & F(B)
 \end{array}$$

Note that  $A \sqcup B$  is a coproduct of  $A$  and  $B$ , then there is a set function  $f$  such that  $f \circ \iota_A = f_{F(A)} \circ i_A$  and  $f \circ \iota_B = f_{F(B)} \circ i_B$ .

According to the universal property of  $F(A \sqcup B)$ , there exists some  $\varphi$  such that the following diagram commutes:

$$\begin{array}{ccc}
 A \sqcup B & \xrightarrow{i_{A \sqcup B}} & F(A \sqcup B) \\
 & \searrow f & \downarrow \varphi \\
 & & G
 \end{array}$$

We have to prove that  $f_{F(A)} = \varphi \circ I_{F(A)}$  and  $f_{F(B)} = \varphi \circ I_{F(B)}$  and such  $\varphi$  is unique. We only prove that  $f_{F(A)} = \varphi \circ I_{F(A)}$  due to similarity.

Note that  $I_{F(A)} \circ i_A = i_{A \sqcup B} \circ \iota_A$ , we have:  $\varphi \circ I_{F(A)} \circ i_A = \varphi \circ i_{A \sqcup B} \circ \iota_A = (\varphi \circ i_{A \sqcup B}) \circ \iota_A = f \circ \iota_A = f_{F(A)} \circ i_A$  that is  $(\varphi \circ I_{F(A)}) \circ i_A = f_{F(A)} \circ i_A$ .

In the following diagram:

$$\begin{array}{ccccc}
 & & F(A) & & \\
 & \swarrow \varphi \circ I_{F(A)} & \uparrow i_A & \searrow f_{F(A)} & \\
 G & \xleftarrow{(\varphi \circ I_{F(A)}) \circ i_A} & A & \xrightarrow{f_{F(A)} \circ i_A} & G
 \end{array}$$

According to the universal property of  $F(A)$ , we must have:  $\varphi \circ I_{F(A)} = f_{F(A)}$  due to the uniqueness. To prove the uniqueness of  $\varphi$ , we assume that  $\varphi'$  satisfies  $\varphi' \circ I_{F(A)} = f_{F(A)}$  (same for  $B$ ), we have  $\varphi' \circ I_{F(A)} \circ i_A = f_{F(A)} \circ i_A$ . The left side equals to  $\varphi' \circ (i_{A \sqcup B} \circ \iota_A)$ , thus we have:  $(\varphi' \circ i_{A \sqcup B}) \circ \iota_A = f_{F(A)} \circ i_A$ . According to the universal property of  $A \sqcup B$ , we have  $f = \varphi' \circ i_{A \sqcup B} \Rightarrow \varphi \circ i_{A \sqcup B} = \varphi' \circ i_{A \sqcup B}$ . And we are done.  $\square$

## 6. Subgroups

### 6.2

Prove that the set of  $2 \times 2$  matrices

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

with  $a, b, d$  in  $\mathbb{C}$  is a subgroup of  $\text{GL}_2(\mathbb{C})$ . More generally, prove that the set of  $n \times n$  complex matrices  $(a_{ij})_{1 \leq i, j \leq n}$  with  $a_{ij} = 0$  for  $i > j$ , and  $a_{11} \cdots a_{nn} \neq 0$ , is a subgroup of  $\text{GL}_n(\mathbb{C})$ . (These matrices are called ‘upper triangular’, for evident reasons.)

*Proof.* Let  $A$  denote the set comprises matrix described in this question, then for any  $a, b \in A$ , we have:

$$ab^{-1} = \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} \times \frac{1}{ad} \begin{pmatrix} d_2 & -b_2 \\ 0 & a_2 \end{pmatrix} = \frac{1}{ad} \begin{pmatrix} a_1d_2 & b_1a_2 - a_1b_2 \\ 0 & d_1a_2 \end{pmatrix}$$

And  $(a_1d_2)(d_1a_2) = (a_1d_1)(a_2d_2) \neq 0$ . Thus we have  $ab^{-1} \in A$  and  $A$  is a subgroup of  $\text{GL}_2(\mathbb{C})$ .

For a more general case, we show that the multiplication of two ‘upper triangular’ matrix is still ‘upper triangular’ and the inverse of an ‘upper trivial’ matrix is still upper trivial.

If  $A$  and  $B$  are ‘upper triangular’ matrixes, then for  $AB$  we have:

$$(AB)_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$$

For  $i > j$ , note that:

$$a_{ik}b_{kj} = \begin{cases} 0, a_{ik} = 0, i > k \\ 0, b_{kj} = 0, k \geq i > j \end{cases}$$

Thus, we have  $(AB)_{ij} = 0$  for  $i > j$ . This indicates that  $AB$  is still ‘upper triangular’.

For the second proposition, we induct on  $n$ : for  $n = 2$ , the case is proved above; Let’s assume this proposition is held for  $n = k$ , and for  $n = k + 1$ , for any ‘upper triangular’ matrix, it could be written as:

$$B = \begin{pmatrix} a_{11} & B_{1 \times k} \\ \mathbf{0}_{k \times 1} & T_{k \times k} \end{pmatrix}$$

where  $a_{11} \neq 0$  and  $T_{k \times k}$  is an ‘upper triangular’ matrix of order  $n$ . We have its inverse as:

$$B^{-1} = \begin{pmatrix} a_{11}^{-1} & -a_{11}^{-1}B_{1 \times k}T_{k \times k}^{-1} \\ \mathbf{0}_{k \times 1} & T_{k \times k}^{-1} \end{pmatrix}$$

According to the assumption that  $T_{k \times k}^{-1}$  is an ‘upper triangular’, we have  $B^{-1}$  is also ‘upper triangular’.

With above two propositions, for any  $a, b \in A_n$ , we have  $ab^{-1}$  is still an ‘upper triangular’ matrix, thus  $ab^{-1} \in A_n$  and the proof is done.  $\square$

### 6.3

Prove that every matrix in  $\text{SU}_2(\mathbb{C})$  may be written in the form

$$\begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$$

where  $a, b, c, d \in \mathbb{R}$  and  $a^2 + b^2 + c^2 + d^2 = 1$ . (Thus,  $\text{SU}_2(\mathbb{C})$  may be realized as a three-dimensional sphere embedded in  $\mathbb{R}^4$ ; in particular, it is simply connected.)

*Proof.* Let  $M \in \text{SU}_2(\mathbb{C})$  and

$$M = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

. We have

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} \bar{x} & \bar{z} \\ \bar{y} & \bar{w} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \bar{x} & \bar{z} \\ \bar{y} & \bar{w} \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

That means:

$$\begin{cases} x\bar{z} + y\bar{w} = 0 \\ z\bar{x} + w\bar{y} = 0 \\ \bar{x}y + \bar{z}w = 0 \\ \bar{y}x + \bar{w}z = 0 \end{cases}$$

$\square$

### 6.5

Let  $G$  be a commutative group, and let  $n > 0$  be an integer. Prove that  $\{g^n \mid g \in G\}$  is a subgroup of  $G$ . Prove that this is not necessarily the case if  $G$  is not commutative

*Proof.* For any  $a, b \in G$ , we have  $a = g^n, b = h^n$  for some  $g, h \in G$ , and  $b^{-1} = (h^{-1})^n$ . Thus:

$$ab^{-1} = g^n(h^{-1})^n = (gh^{-1})^n$$

Note that  $gh^{-1} \in G$ , thus  $ab^{-1} \in \{g^n \mid g \in G\}$ , which means this group is a subgroup of  $G$ . An counter example of the latter assertion would be the permutation group  $S_4$  and let  $n = 2$ .  $\square$

## 6.7

Show that inner automorphisms (cf. Exercise 4.8) form a subgroup of  $\text{Aut}(G)$ ; this subgroup is denoted  $\text{Inn}(G)$ . Prove that  $\text{Inn}(G)$  is cyclic if and only if  $\text{Inn}(G)$  is trivial if and only if  $G$  is abelian

*Proof.* For  $\gamma_a, \gamma_b \in \text{Inn}(G)$ , we have  $\gamma_a \gamma_b^{-1} = \gamma_{ab^{-1}} \in \text{Inn}(G)$ . Thus it is a subgroup of  $\text{Aut}(G)$ .

$\text{Inn}(G)$  is trivial is obviously equivalent to the fact that  $G$  is abelian. If  $\text{Inn}(G)$  is cyclic, then there exists some  $a \in G$  such that for any  $g \in G$ , there exists some  $n \in \mathbb{N}^+$  such that  $\gamma_{a^n} = \gamma_g$ , this indicates  $gag^{-1} = a^n aa^{-n} = a$  and thus  $ga = ag, \forall g \in G$ . Thus we have  $\forall \gamma_g \in \text{Inn}(G), \gamma_g = \gamma_{a^m}$  and  $\forall x \in G, \gamma_{a^m}(x) = x$ , thus  $\gamma_g = \text{Id}_G$ . The proof is done.  $\square$

## 6.9

Prove that every finitely generated subgroup of  $\mathbb{Q}$  is cyclic. Prove that  $\mathbb{Q}$  is not finitely generated

*Proof.* Let  $H < G$  be a finitely generated subgroup and  $H = \langle a_1, a_2, \dots, a_n \rangle$ . We induct on  $n$  to prove that  $H$  is cyclic:

(1) If  $n = 1$  then we have  $F(\{a_1\})$  to be cyclic, thus  $H = \varphi(F(\{a_1\}))$  is also cyclic

(2) Assume for  $n$  this holds, consider  $n + 1$ . Since  $H' = \langle a_1, a_2, \dots, a_n \rangle$  is cyclic, there exists some  $q \in \mathbb{Q}$  such that  $H' = \langle q \rangle$ . Consider  $a_{n+1}$  and  $q$ , let's

say  $a_{n+1}$  and  $q$  both has the form:  $q = \frac{s}{t}, a_{n+1} = \frac{s'}{t}$ . Consider  $q' = \gcd(s, s')$

and we will have both  $q$  and  $a_{n+1}$  be multiple  $\frac{q'}{t}$ . Note that  $\gcd(\frac{s}{q'}, \frac{s'}{q'}) = 1$ .

We will have  $x, y \in \mathbb{N}$  such that  $\frac{xs}{q'} + \frac{ys'}{q'} = 1$ , by multiplying  $\frac{q'}{t}$  at both sides:

$$\frac{q'}{t} = \frac{xs}{t} + \frac{ys'}{t}$$

This means:  $\frac{q'}{t} \in \langle a_1, a_2, \dots, a_{n+1} \rangle$  and it's obviously that each element can be expressed as multiple of  $\frac{q'}{t}$ . Thus the proposition is true for the case of  $n + 1$ .

In conclusion, we have proved that any finitely generated subgroup of  $\mathbb{Q}$  is cyclic.

$\mathbb{Q}$  is not finitely generated as  $\mathbb{Q}$  is not cyclic.  $\square$

## 6.10

The set of  $2 \times 2$  matrices with integer entries and determinant 1 is denoted  $\text{SL}_2(\mathbb{Z})$ :

$$\text{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \text{such that } a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

Prove that  $\text{SL}_2(\mathbb{Z})$  is generated by the matrices:

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

*Proof.* Using induction, we have  $t^a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ ,  $a \in \mathbb{N}$  and  $s^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ ,  $s^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$   $\square$

## 6.12

Let  $m, n$  be positive integers, and consider the subgroup  $\langle m, n \rangle$  of  $\mathbb{Z}$  they generate. By Proposition 6.9,  $\langle m, n \rangle = d\mathbb{Z}$  for some positive integer  $d$ . What is  $d$ , in relation to  $m, n$ ?

*Proof.* Since  $\langle m, n \rangle = d\mathbb{Z}$ , there exists some  $x, y \in \mathbb{N}$  such that  $xm + yn = d$ . Thus we have  $\gcd(m, n) \mid d$ . On the contrary, note that  $m \in d\mathbb{Z}$  and  $n \in d\mathbb{Z}$ , thus we have  $d \mid m$  and  $d \mid n$ , which indicates  $d \mid \gcd(m, n)$ . Thus we have  $\gcd(m, n) = d\mathbb{Z}$ .  $\square$

## 6.16

Counterpoint to Exercise 6.15: the homomorphism  $\varphi : \mathbb{Z}/3\mathbb{Z} \rightarrow S_3$  given by

$$\varphi([0]) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \varphi([1]) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad \varphi([2]) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

is a monomorphism; show that it has no left-inverse in **Grp**

*Proof.* If there the left-inverse of  $\varphi$  exists, denoted as  $\varphi^{-1}$ , then we must have:  
 $\varphi^{-1}\left(\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\right) = [1]$ ,  $\varphi^{-1}\left(\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\right) = [2]$ , Consider  $\varphi^{-1}((1\ 2))$ ,  $\varphi^{-1}((1\ 3))$ ,  $\varphi^{-1}((2\ 3))$ . We have  $\varphi^{-1}(\text{id}) = \varphi^{-1}((1\ 2)^2) = \varphi^{-1}((1\ 2))^2$  That is:  $\varphi^{-1}((1\ 2))^2 = [0]$ . Thus we must have  $\varphi^{-1}((1\ 2)) = [0]$ . Similarly we have  $\varphi^{-1}((2\ 3)) = [0]$ ,  $\varphi^{-1}((1\ 3)) = [0]$ . However, note that  $[2] = \varphi^{-1}\left(\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\right) = \varphi^{-1}((1\ 2)(1\ 3)) = \varphi^{-1}((1\ 2)) + \varphi^{-1}((1\ 3)) = [0]$ . a contradiction. This means  $\varphi^{-1}$  does not exists in **Grp**.  $\square$

## 1 Quotient groups

### 7.1

List all subgroups of  $S_3$  and determine which subgroups are normal and which are not normal.

*Solution* All subgroups of  $S_3$  are as follows:

- Order of 1:  $\{\text{id}\}$
- Order of 2:  $\{\text{id}, (1\ 2)\}$ ,  $\{\text{id}, (1\ 3)\}$ ,  $\{\text{id}, (2\ 3)\}$
- Order of 3:  $\left\{\text{id}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\right\}$

The normal subgroups are  $\{\text{id}\}$  and  $\left\{\text{id}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\right\}$

### 7.2

Is the *image* of a group homomorphism necessarily a normal subgroup of the target?

*Solution* For abelian group, the proposition is true as any subgroup of an abelian group is normal. Generally speaking, the proposition is not true. The counterpoint would be

$$\begin{aligned} \varphi : \mathbb{Z}/2\mathbb{Z} &\rightarrow S_3 \\ \varphi([0]) &= \text{id}, \quad \varphi([1]) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \end{aligned}$$

And obviously  $\text{Im}(\varphi)$  is not a normal subgroup as Exercise 7.1 claims the only normal subgroup is the subgroup of order 3.

## 7.6

Let  $G$  be a group, and let  $n$  be a positive integer. Consider the relation

$$a \sim b \Leftrightarrow (\exists g \in G) ab^{-1} = g^n$$

- Show that in general  $\sim$  is not an equivalence relation
- Prove that  $\sim$  is an equivalence relation if  $G$  is commutative, and determine the corresponding subgroup of  $G$ .

*Proof.* Note this relation naturally satisfies *symmetricity* and *reflexivity*:  $\forall g \in G, gg^{-1} = \text{id} = \text{id}^n$  and if  $a \sim b$  then  $ab^{-1} = g^n$  for some  $g$ , indicates  $ba^{-1} = (ab^{-1})^{-1} = (g^n)^{-1} = (g^{-1})^n \Rightarrow b \sim a$

To show that  $\sim$  is not an equivalent relation in general, we need to consider a counterpoint of *transitivity*.

If  $G$  is commutative, then *transitivity* is true as: If  $a \sim b, b \sim c$  then  $ab^{-1} = g^n$  and  $bc^{-1} = h^n$ , as a result  $ab^{-1}bc^{-1} = ac^{-1} = g^n h^n = (gh)^n$ , indicating  $a \sim c$ .  $\square$

## 7.7

Let  $G$  be a group,  $n$  a positive integer, and let  $H \subseteq G$  be the subgroup generated by all elements of order  $n$  in  $G$ . Prove that  $H$  is normal.

*Proof.* If  $h$  is an element of order  $n$ , then for any  $g \in G$ , we have  $ghg^{-1} = \gamma_g(h)$  also has an order of  $n$  as  $\gamma_g$  is an isomorphism.

Thus, for any elements of  $H$ , it can be denoted as  $g_1 g_2 \dots g_n$  where  $g_i, i = 1, 2, \dots, n$  is an element of order  $n$ . Then  $\forall g \in G$  we have:  $g(g_1 g_2 \dots g_n)g^{-1} = (gg_1 g^{-1})(gg_2 g^{-1}) \dots (gg_n g^{-1}) \in H$  as  $gg_i g^{-1}$  is an element of order  $n$ . Thus we have proved that  $H$  is normal.  $\square$

## 7.8

Prove that If  $H$  is any subgroup of a group  $G$ , the relation  $\sim_L$  defined by

$$(\forall a, b \in G) : a \sim_L b \iff a^{-1}b \in H$$

is an equivalence relation satisfying  $(\dagger)$

*Proof.* We only need to prove  $\sim_L$  is an equivalent relation and the remaining part is done by the author:

- *reflexivity*:  $\forall g \in G$  we have  $g^{-1}g = e \in H \Rightarrow g \sim_L g$



- *symmetricity*: If  $a \sim_L b$  then  $a^{-1}b \in H \Rightarrow b^{-1}a = (a^{-1}b)^{-1} \in H \Rightarrow b \sim_L a$
- *transitivity*: If  $a \sim_L b$  and  $b \sim_L c$  then  $a^{-1}b, b^{-1}c \in H \Rightarrow a^{-1}c = (a^{-1}b)(b^{-1}c) \in H \Rightarrow a \sim_L c$

In conclusion,  $\sim_L$  is an equavalent relation. And it satisfies the following property:

$$a \sim_L b \Rightarrow (\forall g \in G)ga \sim_L gb$$

as:

$$a \sim_L b \Rightarrow a^{-1}b \in H \Rightarrow a^{-1}g^{-1}gb \in H \Rightarrow (ga)^{-1}(gb) \in H \Rightarrow ga \sim_L gb$$

□

## 7.9

State and prove the ‘mirror’ statements of Proposition 7.4 and 7.6, leading to the description of relations satisfying  $(\dagger\dagger)$

The mirror statement of Proposition 7.4 is:

**Mirror of Proposition 7.4** *Let  $\sim$  be an equivalence relation on a group  $G$ , satisfying  $(\dagger\dagger)$ . Then:*

- *the equivalence class of  $e_G$  is a subgroup  $H$  of  $G$ ; and*
- *$a \sim b \iff ab^{-1} \in H \iff Ha = Hb$*

*Proof.* Let  $H$  be the equivalence class of  $e_G$ . If  $a \in H, b \in H, b \in H$ . Then  $a \sim b$ . By  $(\dagger\dagger)$  we have  $ab^{-1} \sim bb^{-1}$ , which is  $ab^{-1} \sim e_G$ . Thus  $ab^{-1} \in H$ . And  $H$  is a subgroup of  $G$ .

$a \sim b \iff ab^{-1} \sim e_G \iff ab^{-1} \in H$ . Then we prove that the equivalence class of  $a$  is  $Ha$ . Let the equivalence class of  $a$  be  $[a]$ , then  $(\forall b \in [a]) : a \sim b \Rightarrow ab^{-1} \sim e_G \Rightarrow ab^{-1} \in H \Rightarrow ab^{-1} = h, h \in H$ . Thus  $b = h^{-1}a \in Ha$ . Thus  $[a] \subseteq Ha$ . Conversely, for any  $ha \in Ha, h \in H$ , we have  $h \sim e_G$  and  $ha \sim e_G a \Rightarrow ha \sim a$ . Thus  $ha \in [a]$ . This indicates  $Ha \subseteq [a]$ . So we have  $[a] = Ha$ .

Thus we have  $a \sim b \iff [a] = [b] \iff Ha = Hb$ . □

**Mirror of Proposition 7.4** *If  $H$  is any subgroup of a group  $G$ , the relation  $\sim_R$  defined by*

$$(\forall a, b \in G) : a \sim_R b \iff ab^{-1} \in H$$

*is an equivalence relation satisfying  $(\dagger\dagger)$ .*

*Proof.* We do not prove that this relation is an equivalence relation but only to prove it satisfies  $(\dagger\dagger)$ .

If  $a \sim b$  then for any  $g \in G$ , we have  $a \sim b \Rightarrow ab^{-1} \in H \Rightarrow a(gg^{-1})b^{-1} \in H \Rightarrow (ag)(g^{-1}b^{-1}) \in H \Rightarrow (ag)(bg)^{-1} \in H \Rightarrow ag \sim bg$ . Thus this equivalence relation satisfies  $(\dagger\dagger)$ .  $\square$

## 7.10

Let  $G$  be a group, and  $H \subseteq G$  a subgroup. With notation as in Exercise 6.7, show that  $H$  is normal in  $G$  if and only if  $\forall \gamma \in \text{Inn}(G), \gamma(H) \subseteq H$ . Conclude that if  $H$  is normal in  $G$  then there is an interesting homomorphism  $\text{Inn}(G) \longrightarrow \text{Aut}(H)$

*Proof.*  $H$  is normal in  $G$  if and only if  $(\forall g \in G) : gH \subseteq Hg \iff (\forall g \in G) : gHg^{-1} \subseteq H \iff (\forall g \in G) : \gamma_g(H) \subseteq H$ . The proof is done.

Note that  $\gamma_g$  constrained on  $H$  is actually an automorphism of  $H$  as  $\gamma_g(H) \subseteq H$ . Define function  $\varphi : \text{Inn}(G) \longrightarrow \text{Aut}(H), \gamma_g \mapsto \gamma_g|_H$ . Is actually an homomorphism. But why is it interesting?  $\square$

## 7.11

Let  $G$  be a group, and let  $[G, G]$  be the subgroup of  $G$  generated by all elements of the form  $aba^{-1}b^{-1}$ . (This is the commutator subgroup of  $G$ ; we will return to it in §IV.3.3.) Prove that  $[G, G]$  is normal in  $G$ .

*Proof.* Note for any  $aba^{-1}b^{-1} \in [G, G]$  and for any  $g \in G$ , we have:  $gaba^{-1}b^{-1}g^{-1} = (gag^{-1})(gbg^{-1})(ga^{-1}g^{-1})(gb^{-1}g^{-1}) = (gag^{-1})(gbg^{-1})(gag^{-1})^{-1}(gbg^{-1})^{-1}$ . Note that  $gag^{-1}, gbg^{-1} \in G$ . Thus  $gaba^{-1}b^{-1}g^{-1} \in [G, G]$ . Note that  $\forall t \in [G, G]$ , it can be written as  $t = g_1g_2 \cdots g_n$  where each  $g_i$  has form  $aba^{-1}b^{-1}$ . Thus for any  $g \in G$ ,  $gtg^{-1} = (gg_1g^{-1})(gg_2g^{-1}) \cdots (gg_ng^{-1})$ . Note that each  $gg_i g^{-1}$  still has form  $aba^{-1}b^{-1}$ . Thus  $gtg^{-1} \in [G, G]$ . Indicates  $[G, G]$  is normal.  $\square$

## 7.12

Let  $F = F(A)$  be a free group, and let  $f : A \longrightarrow G$  be a set-function from the set  $A$  to a commutative group  $G$ . Prove that  $f$  induces a unique homomorphism  $F/[F, F] \rightarrow G$ , where  $[F, F]$  is the commutator subgroup of  $F$  defined in Exercise 7.11. (Use Theorem 7.12.) Conclude that  $F/[F, F] \cong F^{ab}(A)$ .

*Proof.* We first need to prove that  $F/[F, F]$  is abelian. By exercise 7.11, we have  $[F, F]$  is a normal subgroup of  $F$  and  $F/[F, F]$  the quotient group. We

need to prove that for any  $a, b \in F$ , we have  $ab[F, F] = ba[F, F]$  to indicate commutativity.

For any  $t \in [F, F]$ , let  $t = g_1 g_2 \cdots g_n$  where each  $g_i$  has form  $aba^{-1}b^{-1}$ . Then we have  $abt = baa^{-1}b^{-1}abt = (ba)(a^{-1}b^{-1}abt)$ , note that  $a^{-1}b^{-1}ab \in [F, F]$ . Thus  $abt \in ba[F, F]$  and  $ab[F, F] \subseteq ba[F, F]$ . Similarly  $ba[F, F] \subseteq ab[F, F]$ . Thus we have

$$(\forall a, b \in G) : (a[F, F])(b[F, F]) = ab[F, F] = ba[F, F] = (b[F, F])(a[F, F])$$

This indicates  $F/[F, F]$  is commutative.

According to the universal property of  $F(A)$ , the following diagram holds:

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & F \\ & \searrow f & \downarrow \exists! \varphi \\ & & G \end{array}$$

Note that  $\varphi$  we have  $\forall t \in [F, F]$ , we have  $\varphi(t) = \varphi(g_1 g_2 \cdots g_n) = e_G$ , thus  $[F, F] \subseteq \ker \varphi$ . Thus according to theorem 7.12, we have:

$$\begin{array}{ccc} F & \xrightarrow{\varphi} & G \\ & \searrow \pi & \uparrow \exists! \tilde{\varphi} \\ & & F/[F, F] \end{array}$$

Put these two diagram together:

$$\begin{array}{ccccc} A & \xrightarrow{\iota_A} & F & \xrightarrow{\pi} & F/[F, F] \\ & \searrow f & & \nearrow \exists! \tilde{\varphi} & \\ & & G & & \end{array}$$

This indicates that  $F/[F, F] \cong F^{ab}(A)$ . □

## 7.13

Let  $A, B$  be sets, and  $F(A), F(B)$  the corresponding free groups. Assume  $F(A) \cong F(B)$ . If  $A$  is finite, prove that so is  $B$ , and  $A \cong B$ .

*Proof.*  $F(A) \cong F(B) \Rightarrow F(A)/[F(A), F(A)] \cong F(B)/[F(B), F(B)]$  and according to exercise 7.12, we have:

$$F^{ab}(A) \cong F(A)/[F(A), F(A)] \cong F(B)/[F(B), F(B)] \cong F^{ab}(B)$$

The result of exercise 5.10 indicates that  $A \cong B$ .

□