Chapter5 Fields and Galois theory Solutions

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Field Extensions

1.

(a) $[F : K] = 1$ if and only if $F = K$.
(b) If $[F : K]$ is a prime, then there are no intermediate fields between F and K
(c) If $u \in F$ has degree n over K, then n divides $[F : K]$

Proof. (a) $\Rightarrow$: If $[F : K] = 1$ then let $\{u\}, u \in F$ be the basis of F. If $u = 0$ then $F = 0$ as every elements in F has the form ku for some $k \in K$.Let f be a map, which is $f : K \to F, k \mapsto ku$. Then it's easy to see that f is injective. By the fact that every element in F has form ku for some $k \in K$, we have f is surjective, hence f is bijective. Therefore $F = K$.

$\Leftarrow$ If $F = K$ then any nonzero element could be the basis of F over K
(b)If there is some intermediate field E between F and K then we have

$$[F : K] = [F : E][E : K]$$

which means $[F : E] = 1$ or $[E : K]$=1 as $[F : K]$ is prime. Therefore we have $F = E$ or $E = K$ by (a).
(c) By the condition, let f be the minimal polynomial of u over K, we have that $1, u, u^2, ..., u^{n-1}$ is a basis of $K(u)$(**Theorem1.6**) Notice that $K(u)$ is an intermediate field between K and F, we have n divides $[F : K]$ by **Theorem1.2**

2.

Give an example of a finitely generation field extension, which is not finite dimensional.

Solution. Consider $\mathbb{Q}(e)$, it's obvious that $\mathbb{Q}(e)$ is a finitly generated extension but $\mathbb{Q}(e)$ is not finite dimensional over $\mathbb{Q}$, otherwise e is algebraic over $\mathbb{Q}$, which is false.

3.

If $u_1, u_2, ..., u_n \in F$ then the field $F(u_1, ..., u_n)$ is isomorphic to the quotient field of the ring $K[u_1, ..., u_n]$.

Proof. Define map between $F(u_1, ..., u_2)$ and the quotient field of $F[u_1, ..., u_n]$ as follows:

$$f : h(u_1, ..., u_n)/k(u_1, ..., u_n) \mapsto (h(u_1, ..., u_n), k(u_1, ..., u_n))$$

It's easy to see that f is an isomorphism.

4.

(a) For any $u_1, ..., u_n \in F$ and any permutation $\sigma \in S_n, K(u_1, ..., u_n) = K(u_{\sigma(1)}, ..., u_{\sigma(n)})$
(b) $K(u_1, ..., u_{n-1})(u_n) = K(u_1, ..., u_{n-1}, u_n)$
(c) State and prove the analogues of (a) and (b) for $K[u_1, ..., u_n]$.
(d) If each u_i is algebraic over K, then $K(u_1, ..., u_n) = K[u_1, ..., u_n]$

Proof. (a) According to the definition and remark after **Theorem1.2**, $K(u_1, ..., u_n)$ is the subfield generated by $F \cup \{u_1, ..., u_n\}$ and $K(u_{\sigma(1)}, ..., u_{\sigma(n)})$ is the subfield generated by $F \cup \{u_{\sigma(1)}, ..., u_{\sigma(n)}\}$. These two sets are equal as σ is bijective.

(b)$K(u_1, ..., u_{n-1})(u_n)$ is a subfield (of F) that contains $u_1, .., u_{n-1}, u_n$, therefore according to the difinition of $K(u_1, ..., u_n)$, we have:

$$K(u_1, ..., u_n) \subset K(u_1, ..., u_{n-1})(u_n)$$

On the other hand, $K(u_1, ..., u_{n-1})(u_n)$ is the subfield generated by $K(u_1, ..., u_{n-1}) \cup \{u_n\}$ Notice that $K(u_1, ..., u_n)$ contains $K(u_1, ..., u_{n-1})$ and u_n, we have:

$$K(u_1, ..., u_{n-1})(u_n) \subset K(u_1, ..., u_n)$$

therefore these two subfield are equal.

(c) The analogues of $K[u_1, ..., u_n]$ are easy to write and prove as long as we replace "subfield" with "subring".

(d) We prove by induction: when $n = 1$ this holds as $K(u) = K[u]$, which is showed in **Theorem1.6**. Let's assume $K(u_1, ..., u_{n-1}) = K[u_1, ..., u_{n-1}]$, then u_n is algebraic over K implies u_n is also algebraic over $K(u_1, ..., u_{n-1})$. We have:

$$K(u_1, ..., u_n) = K(u_1, ..., u_{n-1})(u_n) = K[u_1, ..., u_{n-1}](u_n) = K[u_1, ..., u_{n-1}][u_n] = K[u_1, ..., u_n]$$

The count-down-2 equation follows from the conclusion of adding one algebraic element.

5.

Let L and M be subfields of F and LM their composite.
(a) If $K \subset L \cap M$ and $M = K(S)$ for some $S \subset M$, then $LM = L(S)$.
(b) When is it true that LM is the set theoretic union $L \cup M$
(c) If $E_1, ..., E_n$ are subfields of F, show that

$$E_1 E_2 ... E_n = E_1(E_2(...(E_{n-1}(E_n))...)).$$

Proof. PASS

6.

Every element of $K(x_1, ..., x_n)$ which is not in K is transcendental over K.

Proof. PASS: I feel this question is incorrect

7.

If v is algebraic over $K(u)$ for some $u \in F$ and v is transcendental over K, then u is algebraic over $K(v)$.

Proof. v is algebraic over $K(u)$ means there is some polynomial $f \in K(u)[x]$ such that $f(u) = 0$. We can write this in the following form:

$$\sum_{i=0}^{n} \frac{h_i(u)}{k_i(u)} v^i = 0, h_i(x), k_i(x) \in K[x]$$

. By multiplying $\prod_{i=0}^{n} h_i(u)$ we have:

$$\sum_{i=0}^{n} F_i(u) v^i = 0, F_i(u) = \prod_{j \neq i} k_j(u) h_i(u)$$

If we combine all coefficiences of each u^i together, we will have:

$$\sum_{i=0}^{m} G_i(v) u^i = 0, G_i(x) \in K[x]$$

Notice that $G_i(v) \neq 0, \forall i = 0, ..., m$ as v is transcendental over K. We have u is algebraic over $K(v)$.

8.

If $u \in F$ is algebraic of odd degree over K, then so is u^2 and $K(u) = K(u^2)$

Proof. If u is algebraic over K then $[F(u) : F]$ is finite and equals to the degree of the minimal polynomial of u. It's easy to see that $K(u^2)$ is an intermediate between K and $K(u)$, according to **Theorem1.2** we have $[K(u^2) : K] \mid [K(u) : K]$. Now that $[K(u) : K]$ is odd, so is $[K(u^2) : K]$ and u^2 has odd degree, which shows u^2 is also algebraic over K