

Topological Spaces

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1 Topological Spaces

Definition. A **topology** on a set X is a collection \mathcal{T} of subsets of X having the following properties:

- (1) \emptyset and X are in \mathcal{T}
- (2) For any subcollection of \mathcal{T} , indexed by set I , we have: $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$
- (3) For any finite subcollection of \mathcal{T} with n elements, we have: $\bigcap_{i=1}^n U_i \in \mathcal{T}$

A set for which a topology \mathcal{T} is specified is called a **topological space**. And the element of \mathcal{T} is called **Open Set**

With the element of \mathcal{T} is defined as open set, we could say a topology is a collection of subsets of X such that \emptyset and X itself are open and satisfies that arbitrary unions and finite intersections of open sets are open. We often write set X and its topology \mathcal{T} as the ordered pair: (X, \mathcal{T}) . And when we say "Let X be open sets", that means we defined a topology on X and \mathcal{T} consists the subsets mentioned above.

EXAMPLE. If X is any set, the collection of all subsets of X is a topology on X , called **discrete topology**. The collection which has only \emptyset and X itself is called **trivial topology**.

EXAMPLE. Let X be a set; let \mathcal{T}_f be the collection of all subsets U of X such that $X - U$ is either finite or all of X . Then \mathcal{T}_f is a topology of X , called **finite complement topology**. Note that $\text{varnothing} = U - U$ is finite and $U = U - \emptyset$, therefore we have \emptyset and U belong to \mathcal{T}_f . Let $\{U_\alpha\}$ be a subcollection of \mathcal{T} indexed by I . Then we have:

$$X - \bigcup U_\alpha = \bigcap (X - U_\alpha)$$

Since each $X - U_\alpha$ is finite, we have $X - \bigcup U_\alpha$ is finite. If $U_1, \dots, U_n \in \mathcal{T}_f$. Then:

$$X - \bigcap_{i=1}^n U_i = \bigcup_{i=1}^n (X - U_i)$$

Since each $X - U_i$ is finite, the finite union of sets with finite cardinal numbers are also finite. Thus $\bigcap_{i=1}^n U_i \in \mathcal{T}_f$

In conclusion, \mathcal{T}_f is a topology on set X .

EXAMPLE. Let X be a set and \mathcal{T} a topology on X . If Y is a subset of U . We define the following collection:

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

It is easy to see that \mathcal{T}_Y is a topology on Y :

$$\emptyset = \emptyset \cap Y \quad Y = X \cap Y$$

If $\{V_\alpha\}$ is a subcollection of \mathcal{T}_Y , then each V_α could be written as $U_\alpha \cap Y$, we have:

$$\bigcup V_\alpha = \bigcup (U_\alpha \cap Y) = (\bigcup U_\alpha) \cap Y$$

Note that $\bigcup U_\alpha$ is in \mathcal{T} , hence we have $\bigcup V_\alpha \in \mathcal{T}_Y$.

If $V_i = U_i \cap Y, i = 1, 2, \dots, n$ is a finite collection of \mathcal{T}_Y . Then:

$$\bigcap_{i=1}^n V_i = \bigcap_{i=1}^n (U_i \cap Y) = (\bigcap_{i=1}^n U_i) \cap Y$$

Note that $\bigcap_{i=1}^n U_i \in \mathcal{T}$, thus we have $\bigcap_{i=1}^n V_i \in \mathcal{T}_Y$. The above new collection consists of the intersection of Y and open sets are called **subspace topology**, and therefore, Y is a topological space.

REMARK. It is easy to see that a set could be assigned with different topology. A typical example is the discrete topology and trivial topology of the same set X . These two topology represents different topological structure. A trivial topology looks like a steel while discrete topology is fine enough to make generate any subset of X .

Definition. Suppose \mathcal{T} and \mathcal{T}' are two topologies on a given set X . If $\mathcal{T} \subset \mathcal{T}'$ ($\mathcal{T} \subsetneq \mathcal{T}'$), we say that \mathcal{T}' is **finer** (**strictly finner**) than \mathcal{T} , or \mathcal{T} is **coarser** (**stricly coarser**) than \mathcal{T}' . We say \mathcal{T} is **comparable** with \mathcal{T}' if either $\mathcal{T} \subset \mathcal{T}'$ or $\mathcal{T}' \subset \mathcal{T}$.

Sometimes we also say that \mathcal{T}' is larger than \mathcal{T} or \mathcal{T} is smaller than \mathcal{T}' , but not as vivid as finer.

2 Closed Sets and Limit Point

Definition. Let (X, \mathcal{T}) be a topological space. We say a subset A of X is **closed** if $X - A$ is open.

EXAMPLE. Let (X, \mathcal{T}) be a topological space and \mathcal{T} be the discrete topology, then any subset of X is a closed set. On the other hand, let \mathcal{T} be trivial topology, then any subset that is neither \emptyset nor X is neither open nor closed.

EXAMPLE. Let $(\mathbb{R}^2, \mathcal{T})$ be a topological space and \mathcal{T} generated by all open ball. And consider the set:

$$\{(x, y) \mid x \geq 0, y \geq 0\}$$

The set is closed as its complement is:

$$(-\infty, 0) \times \mathbb{R} \cup \mathbb{R} \times (-\infty, 0)$$

And each of them are open.

EXAMPLE. Let $(\mathbb{R}, \mathcal{T})$ be a topological space with topology \mathcal{T} consists of all open sets under the metric space (\mathbb{R}, d) . Consider $Y = [0, 1] \cup (2, 3)$ and the subspace topology. We claim that $[0, 1]$ is an open set of Y , because $[0, 1] = (-1, \frac{3}{2}) \cap Y$. Similarly, $(2, 3)$ is also open in Y . And the complement of each of them is another interval, therefore $[0, 1]$ and $(2, 3)$ are both open and closed.

REMARK. By these three examples, we could see that a subset of X can be open, or closed, or both, or neither. In addition, we see that a subset is open(closed) or not depends on the whole space you consider: $[0, 1]$ in EXAMPLE3 is not open in \mathbb{R} but open in Y . $(2, 3)$ is not closed in \mathbb{R} but closed in Y .