Topological Spaces

October 26th, 2020

1 Topological Spaces

Definition. A **topology** on a set X is a collection $\mathcal{T}$of subsets of X having the following properties:

(1) $\varnothing$ and X are in $\mathcal{T}$

(2) For any subcollection of $\mathcal{T}$, indexed by set I, we have: $\bigcup\limits_{\alpha \in I} U_\alpha \in \mathcal{T}$

(3) For any finite subcollection of $\mathcal{T}$ with n elements, we have: $\bigcap\limits_{i=1}^{n} U_i \in \mathcal{T}$

A set for which a topology $\mathcal{T}$ is specified is called a **topological space**. And the element of $\mathcal{T}$ is called **Open Set**

With the element of $\mathcal{T}$ is defined as open set, we could say a topology is a collection of subsets of X such that $\varnothing$ and X itself are open and satisfies that arbitrary unions and finite intersections of open sets are open. We often write set X and its topology $\mathcal{T}$ as the ordered pair: $(X, \mathcal{T})$. And when we say :"Let XXX be open sets", that means we defined a topology on X and $\mathcal{T}$ consists the subsets mentioned above.

EXAMPLE. If X is any set, the collection of all subsets of X is a topology on X, called **discrete topology**. The collection which has only $\varnothing$ and X itself is called **trivial topology**.

EXAMPLE. Let X be a set; let $\mathcal{T}_f$ be the collectino of all subsets U of X such that $X-U$ is either finite or all of X. Then $\mathcal{T}_f$ is a topology of X, called ***finite complement topology***. Note that $varnothing = U - U$ is finite and $U = U - \varnothing$, therefore we have $\varnothing$ and U belong to $\mathcal{T}_f$. Let $\{U_\alpha\}$ be a subcollection of $\mathcal{T}$ indexed by I. Then we have:

$$X - \bigcup U_\alpha = \bigcap (X - U_\alpha)$$

Since each $X - U_\alpha$ is finite, we have $X - \bigcup U_\alpha$ is finite. If $U_1, ..., U_n \in \mathcal{T}_f$. Then:

$$X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_i)$$

Since each $X - U_i$ is finite, the finite union of sets with finite cardinal numbers are also finite. Thus $\bigcap\limits_{i=1}^{n} U_i \in \mathcal{T}_f$

In conclusion, $\mathcal{T}_f$ is a topology on set X.

EXAMPLE. Let X be a set and $\mathcal{T}$ a topology on X. If Y is a subset of U. We define the following collection:

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathrm{T}\}$$

It is easy to see that $\mathcal{T}_Y$ is a topology on Y:

$$\varnothing = \varnothing \cap Y \quad Y = X \cap Y$$

If $\{V_\alpha\}$ is a subcollection of $\mathcal{T}_Y$, then each V_α could be written as $U_\alpha \cap Y$, we have:

$$\bigcup V_\alpha = \bigcup (U_\alpha \cap Y) = (\bigcup U_\alpha) \cap Y$$

Note that $\bigcup U_\alpha$ is in $\mathcal{T}$,hence we have $\bigcup V_\alpha \in \mathcal{T}_Y$.
If $V_i = U_i \bigcap Y, i = 1, 2, ..., n$ is a finite collection of $\mathcal{T}_Y$. Then:

$$\bigcap_{i=1}^{n} V_i = \bigcap_{i=1}^{n} (U_i \cap Y) = (\bigcap_{i=1}^{n} U_i) \bigcap Y$$

Note that $\bigcap\limits_{i=1}^{n} U_i \in \mathcal{T}$, thus we have $\bigcap\limits_{i=1}^{n} V_i \in \mathcal{T}_Y$. The above new collection consists of the intersection of Y and open sets are called ***subspace topology***, and therefore, Y is a topological space.

REMARK. It is easy to see that a set could be assigned with different topology. A typical example is the discrete topology and trivial topology of the same set X. These two topology represents different topological structure. A trivial topology looks like a steel while discrete topology is fine enough to make generate any subset of X.

Definition. Suppose $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T} \subset \mathcal{T}' (\mathcal{T} \subsetneqq \mathcal{T}')$,we say that $\mathcal{T}'$ is ***finer***(***strictly finner***) than $\mathcal{T}$, or $\mathcal{T}$ is ***coarser***(***stricly coarser***) than $\mathcal{T}'$. We say $\mathcal{T}$ is ***comparable*** with $\mathcal{T}'$ if either $\mathcal{T} \subset \mathcal{T}'$ or $\mathcal{T}' \subset \mathcal{T}$

Sometimes we also say that $\mathcal{T}'$ is larger than $\mathcal{T}$ or $\mathcal{T}$ is smaller than $\mathcal{T}'$, but not as vivid as finer.

2 Closed Sets and Limit Point

2.1 Closed Set

Definition. Let $(X, \mathcal{T})$ be a topological space. We say a subset A of X is ***closed*** if $X - A$ is open.

EXAMPLE. Let $(X, \mathcal{T})$ be a topological space and $\mathcal{T}$ be the discrete topology, then any subset of X is a closed set. On the other hand, let $\mathcal{T}$ be trivial topology, then any subset that is neither $\varnothing$ nor X is neither open nor closed.

EXAMPLE. Let $(\mathbb{R}^2, \mathcal{T})$ be a topological space and $\mathcal{T}$ generated by all open ball. And consider the set:

$$\{(x, y) \mid x \geq 0, y \geq 0\}$$

The set is closed as its complement is:

$$(-\infty, 0) \times \mathbb{R} \cup \mathbb{R} \times (-\infty, 0)$$

And each of them are open.

EXAMPLE. Let $(\mathbb{R}, \mathcal{T})$ be a topological space with topology $\mathcal{T}$ consists of all open sets under the metric space $(\mathbb{R}, \mathrm{d})$. Consider $Y = [0,1] \cup (2,3)$ and the subspace topology.
We claim hat $[0,1]$ is an open set of Y, because $[0,1] = (-1, \frac{3}{2}) \cap Y$. Similarly, $(2,3)$ is also open in Y. And the complement of each of them is another interval, therefore $[0,1]$ and $(2,3)$ are both open and closed.

REMARK. By these three examples, we could see that a subset of X can be open, or closed, or both, or neither. In addtition, we see that a subset is open(closed) or not depends on the whole space you consider: $[0,1]$ in EXAMPLE3 is not open in $\mathbb{R}$ but open in Y.$(2,3)$ is not closed in $\mathbb{R}$ but closed in Y.

Theorem 1. *Let X be a topology space. Then the following conditions hold:*
(1) *$\varnothing$ and X are closed*
(2) *For any collection of closed set $\{V_\alpha \mid \alpha \in I\}$, we have $\bigcap_{\alpha \in I} V_\alpha$ is closed*
(3) *The intersection of any finite many closed sets are closed.*

Proof. (1) is trivial with $\varnothing = X - X$ and $X = X - \varnothing$.
As for (2), notice that :

$$\bigcap_{\alpha \in I} V_\alpha = \bigcap_{\alpha \in I} U_\alpha^c = (\bigcup_{\alpha \in I} U_\alpha)^c$$

where U_α is an open set. And we denote $X - U_\alpha$ with U_α^c. (3) follows the same way with the fact that:

$$\bigcup_{i=1}^{n} V_\alpha = \bigcup_{i=1}^{n} U_\alpha^c = (\bigcap_{i=1}^{n} U_\alpha)^c$$

Theorem 2. *Let Y be a subsapce of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.*

Proof. Consider the subspace topology of Y and let V_Y is a closed set under such subspace topology. Then we have $V_Y = Y - U_Y$ for some open set U_Y in Y. With the definition of subspace topology, we have $U_Y = U \cap Y$ with U an open set in X. Then $V_Y = Y - U_Y = Y - U \cap Y = Y - U = Y \cap (X - U)$ where $(X - U)$ is closed in X. Therefore if V_Y is closed in Y, then V_Y is intersection of Y and a closed set in X.
On the other hand, if $V_Y = Y \cap V$ for some closed set V of X. We have $V_Y = Y \cap (X - U) = Y - U = Y - (Y \cap U)$, which is closed in Y

REMARK. General speaking, a set that is closed in a subspace may not be closed in the larger topological space. For example, let $X = \mathbb{R}$ and open set consists of conventional open set in $\mathbb{R}$. Consider the subspace Y generated by the intersection of $[0,1)$ and X. Then notice that $[0, \frac{1}{2})$ is open in Y as $[0, \frac{1}{2}) = (-1, \frac{1}{2}) \cap [0,1)$. Therefore $Y - [0, \frac{1}{2}) = [\frac{1}{2}, 1)$ is closed in Y, however, it's not closed in $\mathbb{R}$.
But we have the following theorem explained the so called "transitivity" of closed property:

Theorem 3. *Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.*

Proof. By theorem 2, $A = Y \cap V$ with V closed in X. Therefore, A is closed in X by the fact that the intersection of two closed sets is closed.

2.2 Limit Point and Closure

Definition. Let X be a topological space and A a subset of X. An element x of X is said to be ***limitpoint*** of A if: for every open set U that contains x, $U \cap A \neq \varnothing$ or $\{x\}$.

Definition. Let A be a subset of the topologival space X; let $A^{'}$ be the set of all limit points of A, we define the closure of A as the union of A and $A^{'}$, denoted by $\bar{A}$. Which is:

$$\bar{A} = A \cup A^{'}$$

Theorem 4. *Let X be a topological space. Then A is closed in X if and only if:$\bar{A} = A$*

Proof. $\Leftarrow$: If $\bar{A} = A$, we need to show that A is closed, or to show that $X - A$ is open. For any element $x \in X - A$, x is neither an element of A nor the limit point of A. $x \notin A^{'}$ means there is some open set U that contains x but $U \cap A = \varnothing$ or $\{x\}$. Now that $x \notin A$, we have $U \cap A = \varnothing$. For any $x \in X - A$, we have such open set U_x. And thus:

$$X - A = \bigcup_{x \in (X-A)} U_x$$

is union of open set in X, therefore an open set. Hence we have A is closed.

$\Rightarrow$: If A is closed. To prove $A = \bar{A}$, we only need to show that $A^{'} \subset A$, which is: any limit point of A is in A. Suppose x is a limit point of A but $x \in X - A$. Then notice that $X - A$ is an open set that contains x but $(X - A) \cap A = \varnothing$, which contradicts the definition of limit point. Therefore any limit point of A is in A, and hence $A = \bar{A}$.

Theorem 5. *Let X be a topological space and A a subset of X, then $\bar{A}$ is the smallest closed set that contains A.*

Proof. The proof are divided into two parts:

(i) $\bar{A}$ is closed.

(ii) Every closed set that contains A must contain $\bar{A}$.

For (ii), we only need to show that every closed set that contains A must contain the limit point of A. This is easy to show: Let B a closed set that contains A and x a limit point of A, then x must be a limit point of B as $A \subset B$. By theorem 4 and the fact that B is closed, we have: $x \in \bar{B} = B$. Therefore, $\bar{A} \subset B$
For (i), we only need to show that $\bar{A} = \bar{\bar{A}}$ by theorem 4. which is concluded as the following lemma.

Lemma 6. Let X be a topological space and A a subset of X, then $\bar{\bar{A}} = \bar{A}$.

Proof. $\bar{A} \subset \bar{\bar{A}}$ according to the definition of closure. As for the other side ,we need to show that the limit point of $\bar{A}$ is in $\bar{A}$.
If x is a limit point of $\bar{A}$. If $x \in A$, we're done. Otherwise let U be any open set that contains x, we have:

$$U \cap \bar{A} \neq \varnothing, \{x\}$$

We claim that x is a limit point of A, by claiming that $U \cap A \neq \varnothing$(of course it can't be $\{x\}$ as $x \notin A$).

(i) If $U \cap A \neq \varnothing$, we're done.

(ii) Otherwise $U \cap A = \varnothing$ but $U \cap A^{'} \neq \varnothing$. $U \cap A^{'} \neq \varnothing$ shows that there is some point, say y, is a limit point of A, and $y \notin A$. Therefore $U \cap A \neq \varnothing$ as U is an open set containing y, this contradicts that assumption that $U \cap A = \varnothing$

In conclusion, $U \cap A \neq \varnothing$ and thus x is a limit point of A by definition.

Both sides contains the other side, therefore we have:$\bar{\bar{A}} = \bar{A}$.

By using the result of lemma 6, we may draw the conclusion of theorem 5 as explained in the proof.

REMARK. The name "closure" means that $\bar{A}$ remains constant under the map by mapping a set of topological space into the union of A and $A^{'}$. Or, as explained int theorem 5, closure is the smallest closed set that contains A. Further more, we can easily prove the closure of A has an equvalent definition:

$$\bar{A} = \bigcap_{A \subset V, V \; closed} V$$

So far,we have actually given two ways of explaining what is a closed set is. One by clarifying the relationship between open set and closed set; and the other by using the definition of limit point. Theorem 4, 5 and lemma 6 has showed the equivalence of these two expression, and we conclude it as:

Theorem 7. *Let X be a topological space, a subset A of X is closed iff every limit point of A is in A.*

Proof. Omitted, see theorem 4.

The following theorem describes the closure of a subset in subspace.

Theorem 8. *Let X be a topological space and Y a subspace of X; let A be a subset of Y; let $\bar{A}$ denote the closure of A in X, Then the closure of A in Y equals $\bar{A} \cap Y$.*

Proof. By theorem 5 and its remark, we know that the closure of A in Y, denoted by $\bar{A_Y}$, equals to the insersection of all closed set in Y that contains A. Note that any closed set in Y equals to the intersection of Y and a closed set in X. Therefore we have:

$$\begin{aligned}
\bar{A_Y} &= \bigcap_{A \subset V_Y, V_Y \; closed \; in \; Y} V_Y \\
&= \bigcap_{A \subset V \cap Y, V \; closed \; in \; X} (V \cap Y) \\
&= (\bigcap_{A \subset V, V \; closed \; in \; X} V) \bigcap Y \\
&= \bar{A_X} \bigcap Y
\end{aligned}$$

REMARK. The equivalence between the second line and the third line is easy to prove with the following claim:

$$A \subset V, V \; closed \; in \; X \Leftrightarrow A \subset V \cap Y, V \cap Y \; closed \; in \; Y$$

A question is that whether the following proposition is true:

Proposition. Let X be a topological space and Y is a subspace of X; A is a subset of X(not Y), then:

$$\overline{A \cap Y} = \bar{A}_X \cap Y$$

Unfortunately, this proposition is false. But we have the left side subsets the right side.

2.3 Dense Set

Definition. Let X be a topological space and A a subset of X. A is said to be a ***dense*** set of X if $\bar{A} = X$.

In other words, a set A is called dense, if any point x in X belongs to A or x is a limit point of A. There are many examples of dense set, the most common one, which is frequently mentioned in Mathematical Analysis, is that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Definition. Let X be a topological space, the ***interior*** of A is defined as the union of all open sets that is contained in A, denoted as **IntA**. The element of Int$A$ is called ***interior point***

It's easy to see the following relation between Int$A$, $\bar{A}$ and A:

$$\mathrm{Int}A \subset A \subset \bar{A}$$

And further more, A is open iff Int$A = A$, A is closed iff $\bar{A} = A$.

Theorem 9. *X is a topological space and A a subset of X. The following conditions are equalvalent:*

(i) x is an interior point of A.

(ii) There is an open set U, such that: $x \in U \subset A$

Proof. (i)$\Rightarrow$ (ii):

$$x \in \mathrm{Int}A \Rightarrow x \in \bigcup_{U \subset A, U\ open} U$$

Therefore, there is an open set in the right side that contains x and it's done.
(ii)$\Rightarrow$ (i):

$$x \in U \subset \bigcup_{U \subset A, U\ open} U = \mathrm{Int}A$$