

# Topological Spaces

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## 1 Topological Spaces

**Definition.** A **topology** on a set  $X$  is a collection  $\mathcal{T}$  of subsets of  $X$  having the following properties:

(1)  $\emptyset$  and  $X$  are in  $\mathcal{T}$

(2) For any subcollection of  $\mathcal{T}$ , indexed by set  $I$ , we have:  $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$

(3) For any finite subcollection of  $\mathcal{T}$  with  $n$  elements, we have:  $\bigcap_{i=1}^n U_i \in \mathcal{T}$

A set for which a topology  $\mathcal{T}$  is specified is called a **topological space**. And the element of  $\mathcal{T}$  is called **Open Set**

With the element of  $\mathcal{T}$  is defined as open set, we could say a topology is a collection of subsets of  $X$  such that  $\emptyset$  and  $X$  itself are open and satisfies that arbitrary unions and finite intersections of open sets are open. We often write set  $X$  and its topology  $\mathcal{T}$  as the ordered pair:  $(X, \mathcal{T})$ . And when we say "Let  $X$  be open sets", that means we defined a topology on  $X$  and  $\mathcal{T}$  consists the subsets mentioned above.

**EXAMPLE.** If  $X$  is any set, the collection of all subsets of  $X$  is a topology on  $X$ , called **discrete topology**. The collection which has only  $\emptyset$  and  $X$  itself is called **trivial topology**.

**EXAMPLE.** Let  $X$  be a set; let  $\mathcal{T}_f$  be the collection of all subsets  $U$  of  $X$  such that  $X - U$  is either finite or all of  $X$ . Then  $\mathcal{T}_f$  is a topology of  $X$ , called **finite complement topology**. Note that  $\text{varnothing} = U - U$  is finite and  $U = U - \emptyset$ , therefore we have  $\emptyset$  and  $U$  belong to  $\mathcal{T}_f$ . Let  $\{U_\alpha\}$  be a subcollection of  $\mathcal{T}$  indexed by  $I$ . Then we have:

$$X - \bigcup U_\alpha = \bigcap (X - U_\alpha)$$

Since each  $X - U_\alpha$  is finite, we have  $X - \bigcup U_\alpha$  is finite. If  $U_1, \dots, U_n \in \mathcal{T}_f$ . Then:

$$X - \bigcap_{i=1}^n U_i = \bigcup_{i=1}^n (X - U_i)$$

Since each  $X - U_i$  is finite, the finite union of sets with finite cardinal numbers are also finite. Thus  $\bigcap_{i=1}^n U_i \in \mathcal{T}_f$

In conclusion,  $\mathcal{T}_f$  is a topology on set  $X$ .

EXAMPLE. Let  $X$  be a set and  $\mathcal{T}$  a topology on  $X$ . If  $Y$  is a subset of  $U$ . We define the following collection:

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

It is easy to see that  $\mathcal{T}_Y$  is a topology on  $Y$ :

$$\emptyset = \emptyset \cap Y \quad Y = X \cap Y$$

If  $\{V_\alpha\}$  is a subcollection of  $\mathcal{T}_Y$ , then each  $V_\alpha$  could be written as  $U_\alpha \cap Y$ , we have:

$$\bigcup V_\alpha = \bigcup (U_\alpha \cap Y) = (\bigcup U_\alpha) \cap Y$$

Note that  $\bigcup U_\alpha$  is in  $\mathcal{T}$ , hence we have  $\bigcup V_\alpha \in \mathcal{T}_Y$ .

If  $V_i = U_i \cap Y, i = 1, 2, \dots, n$  is a finite collection of  $\mathcal{T}_Y$ . Then:

$$\bigcap_{i=1}^n V_i = \bigcap_{i=1}^n (U_i \cap Y) = (\bigcap_{i=1}^n U_i) \cap Y$$

Note that  $\bigcap_{i=1}^n U_i \in \mathcal{T}$ , thus we have  $\bigcap_{i=1}^n V_i \in \mathcal{T}_Y$ . The above new collection consists of the intersection of  $Y$  and open sets are called **subspace topology**, and therefore,  $Y$  is a topological space.

**REMARK.** It is easy to see that a set could be assigned with different topology. A typical example is the discrete topology and trivial topology of the same set  $X$ . These two topology represents different topological structure. A trivial topology looks like a steel while discrete topology is fine enough to make generate any subset of  $X$ .

**Definition.** Suppose  $\mathcal{T}$  and  $\mathcal{T}'$  are two topologies on a given set  $X$ . If  $\mathcal{T} \subset \mathcal{T}'$  ( $\mathcal{T} \subsetneq \mathcal{T}'$ ), we say that  $\mathcal{T}'$  is **finer** (**strictly finner**) than  $\mathcal{T}$ , or  $\mathcal{T}$  is **coarser** (**stricly coarser**) than  $\mathcal{T}'$ . We say  $\mathcal{T}$  is **comparable** with  $\mathcal{T}'$  if either  $\mathcal{T} \subset \mathcal{T}'$  or  $\mathcal{T}' \subset \mathcal{T}$ .

Sometimes we also say that  $\mathcal{T}'$  is larger than  $\mathcal{T}$  or  $\mathcal{T}$  is smaller than  $\mathcal{T}'$ , but not as vivid as finer.

## 2 Closed Sets and Limit Point

### 2.1 Closed Set

**Definition.** Let  $(X, \mathcal{T})$  be a topological space. We say a subset  $A$  of  $X$  is **closed** if  $X - A$  is open.

EXAMPLE. Let  $(X, \mathcal{T})$  be a topological space and  $\mathcal{T}$  be the discrete topology, then any subset of  $X$  is a closed set. On the other hand, let  $\mathcal{T}$  be trivial topology, then any subset that is neither  $\emptyset$  nor  $X$  is neither open nor closed.

EXAMPLE. Let  $(\mathbb{R}^2, \mathcal{T})$  be a topological space and  $\mathcal{T}$  generated by all open ball. And consider the set:

$$\{(x, y) \mid x \geq 0, y \geq 0\}$$

The set is closed as its complement is:

$$(-\infty, 0) \times \mathbb{R} \cup \mathbb{R} \times (-\infty, 0)$$

And each of them are open.

**EXAMPLE.** Let  $(\mathbb{R}, \mathcal{T})$  be a topological space with topology  $\mathcal{T}$  consists of all open sets under the metric space  $(\mathbb{R}, d)$ . Consider  $Y = [0, 1] \cup (2, 3)$  and the subspace topology. We claim that  $[0, 1]$  is an open set of  $Y$ , because  $[0, 1] = (-1, \frac{3}{2}) \cap Y$ . Similarly,  $(2, 3)$  is also open in  $Y$ . And the complement of each of them is another interval, therefore  $[0, 1]$  and  $(2, 3)$  are both open and closed.

**REMARK.** By these three examples, we could see that a subset of  $X$  can be open, or closed, or both, or neither. In addition, we see that a subset is open(closed) or not depends on the whole space you consider:  $[0, 1]$  in EXAMPLE3 is not open in  $\mathbb{R}$  but open in  $Y$ .  $(2, 3)$  is not closed in  $\mathbb{R}$  but closed in  $Y$ .

One might think that a single point is a closed set. This wrong intuition comes from the special topological space  $\mathbb{R}$ . But the following example shows exactly that a single point might not be a closed set:

$$X = \{0, 1\}, \mathcal{T} = \{\{0\}, \{0, 1\}, \emptyset\}$$

Then  $\{0\}$  is not a closed set as  $\{1\}$  is not open. However,  $\{0\}$  is dense in  $X$  as its closure equals to  $X$ , which we will talk about later.

**Theorem 1.** Let  $X$  be a topology space. Then the following conditions hold:

- (1)  $\emptyset$  and  $X$  are closed
- (2) For any collection of closed set  $\{V_\alpha \mid \alpha \in I\}$ , we have  $\bigcap_{\alpha \in I} V_\alpha$  is closed
- (3) The intersection of any finite many closed sets are closed.

**Proof.** (1) is trivial with  $\emptyset = X - X$  and  $X = X - \emptyset$ .

As for (2), notice that :

$$\bigcap_{\alpha \in I} V_\alpha = \bigcap_{\alpha \in I} U_\alpha^c = \left( \bigcup_{\alpha \in I} U_\alpha \right)^c$$

where  $U_\alpha$  is an open set. And we denote  $X - U_\alpha$  with  $U_\alpha^c$ . (3) follows the same way with the fact that:

$$\bigcup_{i=1}^n V_\alpha = \bigcup_{i=1}^n U_\alpha^c = \left( \bigcap_{i=1}^n U_\alpha \right)^c$$

**Theorem 2.** Let  $Y$  be a subspace of  $X$ . Then a set  $A$  is closed in  $Y$  if and only if it equals the intersection of a closed set of  $X$  with  $Y$ .

**Proof.** Consider the subspace topology of  $Y$  and let  $V_Y$  is a closed set under such subspace topology. Then we have  $V_Y = Y - U_Y$  for some open set  $U_Y$  in  $Y$ . With the definition of subspace topology, we have  $U_Y = U \cap Y$  with  $U$  an open set in  $X$ . Then  $V_Y = Y - U_Y = Y - U \cap Y = Y - U = Y \cap (X - U)$  where  $(X - U)$  is closed in  $X$ . Therefore if  $V_Y$  is closed in  $Y$ , then  $V_Y$  is intersection of  $Y$  and a closed set in  $X$ .

On the other hand, if  $V_Y = Y \cap V$  for some closed set  $V$  of  $X$ . We have  $V_Y = Y \cap (X - U) = Y - U = Y - (Y \cap U)$ , which is closed in  $Y$ .

**REMARK.** General speaking, a set that is closed in a subspace may not be closed in the larger topological space. For example, let  $X = \mathbb{R}$  and open set consists of conventional open set in  $\mathbb{R}$ . Consider the subspace  $Y$  generated by the intersection of  $[0, 1)$  and  $X$ . Then

notice that  $[0, \frac{1}{2})$  is open in  $Y$  as  $[0, \frac{1}{2}) = (-1, \frac{1}{2}) \cap [0, 1)$ . Therefore  $Y - [0, \frac{1}{2}) = [\frac{1}{2}, 1)$  is closed in  $Y$ , however, it's not closed in  $\mathbb{R}$ .

But we have the following theorem explained the so called "transitivity" of closed property:

**Theorem 3.** *Let  $Y$  be a subspace of  $X$ . If  $A$  is closed in  $Y$  and  $Y$  is closed in  $X$ , then  $A$  is closed in  $X$ .*

**Proof.** By theorem 2,  $A = Y \cap V$  with some  $V$  closed in  $X$ . Therefore,  $A$  is closed in  $X$  by the fact that the intersection of two closed sets is closed.

## 2.2 Limit Point and Closure

**Definition.** Let  $X$  be a topological space and  $A$  a subset of  $X$ . An element  $x$  of  $X$  is said to be **limitpoint** of  $A$  if: for every open set  $U$  that contains  $x$ ,  $U \cap A \neq \emptyset$  or  $\{x\}$ .

**Definition.** Let  $A$  be a subset of the topological space  $X$ ; let  $A'$  be the set of all limit points of  $A$ , we define the closure of  $A$  as the union of  $A$  and  $A'$ , denoted by  $\bar{A}$ . Which is:

$$\bar{A} = A \cup A'$$

**Theorem 4.** *Let  $X$  be a topological space. Then  $A$  is closed in  $X$  if and only if:  $\bar{A} = A$*

**Proof.**  $\Leftarrow$ : If  $\bar{A} = A$ , we need to show that  $A$  is closed, or to show that  $X - A$  is open. For any element  $x \in X - A$ ,  $x$  is neither an element of  $A$  nor the limit point of  $A$ .  $x \notin A'$  means there is some open set  $U$  that contains  $x$  but  $U \cap A = \emptyset$  or  $\{x\}$ . Now that  $x \notin A$ , we have  $U \cap A = \emptyset$ . For any  $x \in X - A$ , we have such open set  $U_x$ . And thus:

$$X - A = \bigcup_{x \in (X - A)} U_x$$

is union of open set in  $X$ , therefore an open set. Hence we have  $A$  is closed.

$\Rightarrow$ : If  $A$  is closed. To prove  $A = \bar{A}$ , we only need to show that  $A' \subset A$ , which is: any limit point of  $A$  is in  $A$ . Suppose  $x$  is a limit point of  $A$  but  $x \in X - A$ . Then notice that  $X - A$  is an open set that contains  $x$  but  $(X - A) \cap A = \emptyset$ , which contradicts the definition of limit point. Therefore any limit point of  $A$  is in  $A$ , and hence  $A = \bar{A}$ .

**Theorem 5.** *Let  $X$  be a topological space and  $A$  a subset of  $X$ , then  $\bar{A}$  is the smallest closed set that contains  $A$ .*

**Proof.** The proof are divided into two parts:

(i)  $\bar{A}$  is closed.

(ii) Every closed set that contains  $A$  must contain  $\bar{A}$ .

For (ii), we only need to show that every closed set that contains  $A$  must contain the limit point of  $A$ . This is easy to show: Let  $B$  a closed set that contains  $A$  and  $x$  a limit point of  $A$ , then  $x$  must be a limit point of  $B$  as  $A \subset B$ . By theorem 4 and the fact that  $B$  is closed, we have:  $x \in \bar{B} = B$ . Therefore,  $\bar{A} \subset B$

For (i), we only need to show that  $\bar{\bar{A}} = \bar{A}$  by theorem 4. which is concluded as the following lemma.

**Lemma 6.** Let  $X$  be a topological space and  $A$  a subset of  $X$ , then  $\bar{\bar{A}} = \bar{A}$ .

**Proof.**  $\bar{A} \subset \bar{\bar{A}}$  according to the definition of closure. As for the other side, we need to show that the limit point of  $\bar{A}$  is in  $\bar{A}$ .

If  $x$  is a limit point of  $\bar{A}$ . If  $x \in A$ , we're done. Otherwise let  $U$  be any open set that contains  $x$ , we have:

$$U \cap \bar{A} \neq \emptyset, \{x\}$$

We claim that  $x$  is a limit point of  $A$ , by claiming that  $U \cap A \neq \emptyset$  (of course it can't be  $\{x\}$  as  $x \notin A$ ).

- (i) If  $U \cap A \neq \emptyset$ , we're done.
- (ii) Otherwise  $U \cap A = \emptyset$  but  $U \cap A' \neq \emptyset$ .  $U \cap A' \neq \emptyset$  shows that there is some point, say  $y$ , is a limit point of  $A$ , and  $y \notin A$ . Therefore  $U \cap A \neq \emptyset$  as  $U$  is an open set containing  $y$ , this contradicts that assumption that  $U \cap A = \emptyset$

In conclusion,  $U \cap A \neq \emptyset$  and thus  $x$  is a limit point of  $A$  by definition.

Both sides contains the other side, therefore we have:  $\bar{\bar{A}} = \bar{A}$ .

By using the result of lemma 6, we may draw the conclusion of theorem 5 as explained in the proof.

**REMARK.** The name "closure" means that  $\bar{A}$  remains constant under the map by mapping a set of topological space into the union of  $A$  and  $A'$ . Or, as explained in theorem 5, closure is the smallest closed set that contains  $A$ . Further more, we can easily prove the closure of  $A$  has an equivalent definition:

$$\bar{A} = \bigcap_{A \subset V, V \text{ closed}} V$$

So far, we have actually given two ways of explaining what is a closed set is. One by clarifying the relationship between open set and closed set; and the other by using the definition of limit point. Theorem 4, 5 and lemma 6 has showed the equivalence of these two expression, and we conclude it as:

**Theorem 7.** Let  $X$  be a topological space, a subset  $A$  of  $X$  is closed iff every limit point of  $A$  is in  $A$ .

**Proof.** Omitted, see theorem 4.

The following theorem describes the closure of a subset in subspace.

**Theorem 8.** Let  $X$  be a topological space and  $Y$  a subspace of  $X$ ; let  $A$  be a subset of  $Y$ ; let  $\bar{A}$  denote the closure of  $A$  in  $X$ , Then the closure of  $A$  in  $Y$  equals  $\bar{A} \cap Y$ .

**Proof.** By theorem 5 and its remark, we know that the closure of  $A$  in  $Y$ , denoted by  $\bar{A}_Y$ , equals to the intersection of all closed set in  $Y$  that contains  $A$ . Note that any closed set in  $Y$  equals to the intersection of  $Y$  and a closed set in  $X$ . Therefore we have:

$$\begin{aligned} \bar{A}_Y &= \bigcap_{A \subset V_Y, V_Y \text{ closed in } Y} V_Y \\ &= \bigcap_{A \subset V \cap Y, V \text{ closed in } X} (V \cap Y) \\ &= \left( \bigcap_{A \subset V, V \text{ closed in } X} V \right) \cap Y \\ &= \bar{A}_X \cap Y \end{aligned}$$

**REMARK.** The equivalence between the second line and the third line is easy to prove with the following claim:

$$A \subset V, V \text{ closed in } X \Leftrightarrow A \subset V \cap Y, V \cap Y \text{ closed in } Y$$

A question is that whether the following proposition is true:

**Proposition.** Let  $X$  be a topological space and  $Y$  is a subspace of  $X$ ;  $A$  is a subset of  $X$  (not  $Y$ ), then:

$$\overline{A \cap Y} = \bar{A}_X \cap Y$$

Unfortunately, this proposition is false. But we have the left side subsets the right side.

We give a theorem about the closure of the union of two subsets and the closure of its intersection:

**Theorem 9.** Let  $X$  be a topological space and  $A, B$  subsets of  $X$ . Then we have:

$$\overline{A \cup B} = \bar{A} \cup \bar{B}, \overline{A \cap B} \subset \bar{A} \cap \bar{B}$$

**Proof.** We first prove that  $\overline{A \cup B} = \bar{A} \cup \bar{B}$ . For any  $x \in \overline{A \cup B}$ , if  $x \in A \cup B$  then  $x$  belongs to the right side. Otherwise if  $x$  is the limit point of  $A \cup B$  and  $x \notin A \cup B$ . We will show that  $x \in \bar{A}$  or  $x \in \bar{B}$ . If  $x \notin \bar{A}$  and  $x \notin \bar{B}$ , then there are some open sets  $W, V$  such that  $x \in W, x \in V$  but  $W \cap A = \emptyset, V \cap B = \emptyset$ . Therefore:  $(W \cap V) \cap (A \cup B) = \emptyset$ , with  $W \cap V$  an open set contains  $x$ , which contradicts the assumption that  $x$  is a limit point of  $A \cup B$ . Therefore,  $x \in \bar{A}$  or  $x \in \bar{B}$  and  $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$ .

For the other inclusion:  $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$ . For  $x \in \bar{A} \cup \bar{B}$ , assume  $x \in \bar{A}$ , then  $x \in A$  or  $x \in \bar{A}$ . If  $x \in A$  then  $x \in A \cup B \subset \overline{A \cup B}$ ; If  $x \in \bar{A}$  then  $x$  is obviously a limit point of  $A \cup B$  as  $A \subset A \cup B$ , which means  $x \in (A \cup B)' \subset \overline{A \cup B}$  and thus  $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$ . Therefore we have the first part of this theorem proved. The second part follows the same way.

**REMARK.**  $\overline{A \cap B} \subsetneq \bar{A} \cap \bar{B}$  might be true. Consider  $X = \mathbb{R}$  and its conventional topology. Let  $A = [0, 1), B = (1, 2]$ , then we have:

$$\overline{A \cap B} = \emptyset \subsetneq \bar{A} \cap \bar{B} = \{1\}$$

## 2.3 Dense Set

**Definition.** Let  $X$  be a topological space and  $A$  a subset of  $X$ .  $A$  is said to be a **dense** set of  $X$  if  $\bar{A} = X$ .

In other words, a set  $A$  is called dense, if any point  $x$  in  $X$  belongs to  $A$  or  $x$  is a limit point of  $A$ . There are many examples of dense set, the most common one, which is frequently mentioned in Mathematical Analysis, is that  $\mathbb{Q}$  is dense in  $\mathbb{R}$ .

**Definition.** Let  $X$  be a topological space, the **interior** of  $A$  is defined as the union of all open sets that is contained in  $A$ , denoted as **Int** $A$ . The element of  $\text{Int}A$  is called **interior point**.

It's easy to see the following relation between  $\text{Int}A$ ,  $\bar{A}$  and  $A$ :

$$\text{Int}A \subset A \subset \bar{A}$$

And further more,  $A$  is open iff  $\text{Int}A = A$ ,  $A$  is closed iff  $\bar{A} = A$ .

**Theorem 10.**  *$X$  is a topological space and  $A$  a subset of  $X$ . The following conditions are equivalent:*

(i)  *$x$  is an interior point of  $A$ .*

(ii) *There is an open set  $U$ , such that:  $x \in U \subset A$*

**Proof.** (i)  $\Rightarrow$  (ii):

$$x \in \text{Int}A \Rightarrow x \in \bigcup_{U \subset A, U \text{ open}} U$$

Therefore, there is an open set in the right side that contains  $x$  and it's done.

(ii)  $\Rightarrow$  (i):

$$x \in U \subset \bigcup_{U \subset A, U \text{ open}} U = \text{Int}A$$