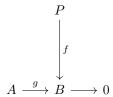
Projective and Injective Modules

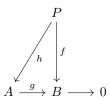
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1. Projective Modules

Definition 1. A module P over a ring R is said to be **projective**, if given any diagram of R-module homomorphisms



with the bottom row exact, there exists an R-module homomorphism $h: P \to A$ such that the diagram

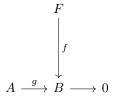


is commutative.

note The bottom row exact means homomorphism g is an epimorphism. An R-module P is said to be projective means for any epimorphism $g:A\to B$ and R-module homomorphism $f:P\to B$ there exists $h:P\to A$ such that: f=gh

Theorem 1. Every free module F over a ring R with identity is projective.

Proof. Let F be a free module over ring R and F is free on set X. Let $\iota: X \to F$. For any pair of R-module A,B with $g: A \to B$ an epimorphism and $f: F \to B$ a homomorphism



 $f(\iota(x)) \in B, x \in X$. Since g is an epimorphism, there exists some $a_x \in A$ such that $g(a_x) = f(\iota(x))$. Let $h': X \to A, x \mapsto a_x$ then h' induces a homomorphism $h: F \to A$. Hence we have: $h(\iota(x)) = a_x$ and $g(h(\iota(x))) = g(a_x) = f(\iota(x))$. This holds for any $x \in X$. Hence $gh\iota = f\iota$. We have following diagram



According to the property of free module: For any homomorphism $\phi: X \to A$, there exists unic homomorphism $f: F \to A$ such that $\phi = f\iota$. Here let $\phi = f\iota = gh\iota$. We have: $f = gh$	Įue □
Corollary 1. Every module A over a ring R is the homomorphism image of a projective R-mod	ule
<i>Proof.</i> Every module A(over R) is homomorphism image of a free module over R. By Theorem every free module is a projective module.	n1,