Topological Spaces

October 26th, 2020

1 Topological Spaces

Definition. A **topology** on a set X is a collection $\mathcal{T}$of subsets of X having the following properties:

(1) $\varnothing$ and X are in $\mathcal{T}$

(2) For any subcollection of $\mathcal{T}$, indexed by set I, we have: $\bigcup\limits_{\alpha \in I} U_\alpha \in \mathcal{T}$

(3) For any finite subcollection of $\mathcal{T}$ with n elements, we have: $\bigcap\limits_{i=1}^{n} U_i \in \mathcal{T}$

A set for which a topology $\mathcal{T}$ is specified is called a **topological space**. And the element of $\mathcal{T}$ is called **Open Set**

With the element of $\mathcal{T}$ is defined as open set, we could say a topology is a collection of subsets of X such that $\varnothing$ and X itself are open and satisfies that arbitrary unions and finite intersections of open sets are open. We often write set X and its topology $\mathcal{T}$ as the ordered pair: $(X, \mathcal{T})$. And when we say :"Let XXX be open sets", that means we defined a topology on X and $\mathcal{T}$ consists the subsets mentioned above.

EXAMPLE. If X is any set, the collection of all subsets of X is a topology on X, called **discrete topology**. The collection which has only $\varnothing$ and X itself is called **trivial topology**.

EXAMPLE. Let X be a set; let $\mathcal{T}_f$ be the collectino of all subsets U of X such that $X-U$ is either finite or all of X. Then $\mathcal{T}_f$ is a topology of X, called ***finite complement topology***. Note that $varnothing = U - U$ is finite and $U = U - \varnothing$, therefore we have $\varnothing$ and U belong to $\mathcal{T}_f$. Let $\{U_\alpha\}$ be a subcollection of $\mathcal{T}$ indexed by I. Then we have:

$$X - \bigcup U_\alpha = \bigcap (X - U_\alpha)$$

Since each $X - U_\alpha$ is finite, we have $X - \bigcup U_\alpha$ is finite. If $U_1, ..., U_n \in \mathcal{T}_f$. Then:

$$X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_i)$$

Since each $X - U_i$ is finite, the finite union of sets with finite cardinal numbers are also finite. Thus $\bigcap\limits_{i=1}^{n} U_i \in \mathcal{T}_f$

In conclusion, $\mathcal{T}_f$ is a topology on set X.

EXAMPLE. Let X be a set and $\mathcal{T}$ a topology on X. If Y is a subset of U. We define the following collection:

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathrm{T}\}$$

It is easy to see that $\mathcal{T}_Y$ is a topology on Y:

$$\varnothing = \varnothing \cap Y \quad Y = X \cap Y$$

If $\{V_\alpha\}$ is a subcollection of $\mathcal{T}_Y$, then each V_α could be written as $U_\alpha \cap Y$, we have:

$$\bigcup V_\alpha = \bigcup (U_\alpha \cap Y) = (\bigcup U_\alpha) \cap Y$$

Note that $\bigcup U_\alpha$ is in $\mathcal{T}$,hence we have $\bigcup V_\alpha \in \mathcal{T}_Y$.
If $V_i = U_i \bigcap Y, i = 1, 2, ..., n$ is a finite collection of $\mathcal{T}_Y$. Then:

$$\bigcap_{i=1}^{n} V_i = \bigcap_{i=1}^{n} (U_i \cap Y) = (\bigcap_{i=1}^{n} U_i) \bigcap Y$$

Note that $\bigcap\limits_{i=1}^{n} U_i \in \mathcal{T}$, thus we have $\bigcap\limits_{i=1}^{n} V_i \in \mathcal{T}_Y$. The above new collection consists of the intersection of Y and open sets are called ***subspace topology***, and therefore, Y is a topological space.

REMARK. It is easy to see that a set could be assigned with different topology. A typical example is the discrete topology and trivial topology of the same set X. These two topology represents different topological structure. A trivial topology looks like a steel while discrete topology is fine enough to make generate any subset of X.

Definition. Suppose $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T} \subset \mathcal{T}' (\mathcal{T} \subsetneqq \mathcal{T}')$,we say that $\mathcal{T}'$ is ***finer***(***strictly finner***) than $\mathcal{T}$, or $\mathcal{T}$ is ***coarser***(***stricly coarser***) than $\mathcal{T}'$. We say $\mathcal{T}$ is ***comparable*** with $\mathcal{T}'$ if either $\mathcal{T} \subset \mathcal{T}'$ or $\mathcal{T}' \subset \mathcal{T}$

Sometimes we also say that $\mathcal{T}'$ is larger than $\mathcal{T}$ or $\mathcal{T}$ is smaller than $\mathcal{T}'$, but not as vivid as finer.

2 Closed Sets and Limit Point

Definition. Let $(X, \mathcal{T})$ be a topological space. We say a subset A of X is ***closed*** if $X - A$ is open.

EXAMPLE. Let $(X, \mathcal{T})$ be a topological space and $\mathcal{T}$ be the discrete topology, then any subset of X is a closed set. On the other hand, let $\mathcal{T}$ be trivial topology, then any subset that is neither $\varnothing$ nor X is neither open nor closed.

EXAMPLE. Let $(\mathbb{R}^2, \mathcal{T})$ be a topological space and $\mathcal{T}$ generated by all open ball. And consider the set:

$$\{(x, y) \mid x \geq 0, y \geq 0\}$$

The set is closed as its complement is:

$$(-\infty, 0) \times \mathbb{R} \cup \mathbb{R} \times (-\infty, 0)$$

And each of them are open.

EXAMPLE. Let $(\mathbb{R}, \mathcal{T})$ be a topological space with topology $\mathcal{T}$ consists of all open sets under the metric space $(\mathbb{R}, \mathrm{d})$. Consider $Y = [0,1] \cup (2,3)$ and the subspace topology.
We claim that $[0,1]$ is an open set of Y, because $[0,1] = (-1, \frac{3}{2}) \cap Y$. Similarly, $(2,3)$ is also open in Y. And the complement of each of them is another interval, therefore $[0,1]$ and $(2,3)$ are both open and closed.

REMARK. By these three examples, we could see that a subset of X can be open, or closed, or both, or neither. In addtition, we see that a subset is open(closed) or not depends on the whole space you consider: $[0,1]$ in EXAMPLE3 is not open in $\mathbb{R}$ but open in Y.$(2,3)$ is not closed in $\mathbb{R}$ but closed in Y.

Theorem 1. *Let X be a topology space. Then the following conditions hold:*
(1) $\varnothing$ and X are closed
(2) For any collection of closed set $\{V_\alpha \mid \alpha \in I\}$, we have $\bigcap_{\alpha \in I} V_\alpha$ is closed
(3) The intersection of any finite many closed sets are closed.

Proof.