

Topological Spaces

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1 Topological Spaces

Definition. A **topology** on a set X is a collection \mathcal{T} of subsets of X having the following properties:

- (1) \emptyset and X are in \mathcal{T}
- (2) For any subcollection of \mathcal{T} , indexed by set I , we have: $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$
- (3) For any finite subcollection of \mathcal{T} with n elements, we have: $\bigcap_{i=1}^n U_i \in \mathcal{T}$

A set for which a topology \mathcal{T} is specified is called a **topological space**. And the element of \mathcal{T} is called **Open Set**

With the element of \mathcal{T} is defined as open set, we could say a topology is a collection of subsets of X such that \emptyset and X itself are open and satisfies that arbitrary unions and finite intersections of open sets are open. We often write set X and its topology \mathcal{T} as the ordered pair: (X, \mathcal{T}) . And when we say "Let X be open sets", that means we defined a topology on X and \mathcal{T} consists the subsets mentioned above.

EXAMPLE. If X is any set, the collection of all subsets of X is a topology on X , called **discrete topology**. The collection which has only \emptyset and X itself is called **trivial topology**.

EXAMPLE. Let X be a set; let \mathcal{T}_f be the collection of all subsets U of X such that $X - U$ is either finite or all of X . Then \mathcal{T}_f is a topology of X , called **finite complement topology**. Note that $\text{varnothing} = U - U$ is finite and $U = U - \emptyset$, therefore we have \emptyset and U belong to \mathcal{T}_f . Let $\{U_\alpha\}$ be a subcollection of \mathcal{T} indexed by I . Then we have:

$$X - \bigcup U_\alpha = \bigcap (X - U_\alpha)$$

Since each $X - U_\alpha$ is finite, we have $X - \bigcup U_\alpha$ is finite. If $U_1, \dots, U_n \in \mathcal{T}_f$. Then:

$$X - \bigcap_{i=1}^n U_i = \bigcup_{i=1}^n (X - U_i)$$

Since each $X - U_i$ is finite, the finite union of sets with finite cardinal numbers are also finite. Thus $\bigcap_{i=1}^n U_i \in \mathcal{T}_f$

In conclusion, \mathcal{T}_f is a topology on set X .

EXAMPLE. Let X be a set and \mathcal{T} a topology on X . If Y is a subset of U . We define the following collection:

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

It is easy to see that \mathcal{T}_Y is a topology on Y :

$$\emptyset = \emptyset \cap Y \quad Y = X \cap Y$$

If $\{V_\alpha\}$ is a subcollection of \mathcal{T}_Y , then each V_α could be written as $U_\alpha \cap Y$, we have:

$$\bigcup V_\alpha = \bigcup (U_\alpha \cap Y) = (\bigcup U_\alpha) \cap Y$$

Note that $\bigcup U_\alpha$ is in \mathcal{T} , hence we have $\bigcup V_\alpha \in \mathcal{T}_Y$.

If $V_i = U_i \cap Y, i = 1, 2, \dots, n$ is a finite collection of \mathcal{T}_Y . Then:

$$\bigcap_{i=1}^n V_i = \bigcap_{i=1}^n (U_i \cap Y) = (\bigcap_{i=1}^n U_i) \cap Y$$

Note that $\bigcap_{i=1}^n U_i \in \mathcal{T}$, thus we have $\bigcap_{i=1}^n V_i \in \mathcal{T}_Y$. The above new collection consists of the intersection of Y and open sets are called **subspace topology**, and therefore, Y is a topological space.

REMARK. It is easy to see that a set could be assigned with different topology. A typical example is the discrete topology and trivial topology of the same set X . These two topology represents different topological structure. A trivial topology looks like a steel while discrete topology is fine enough to make generate any subset of X .

Definition. Suppose \mathcal{T} and \mathcal{T}' are two topologies on a given set X . If $\mathcal{T} \subset \mathcal{T}'$ ($\mathcal{T} \subsetneq \mathcal{T}'$), we say that \mathcal{T}' is **finer** (**strictly finner**) than \mathcal{T} , or \mathcal{T} is **coarser** (**stricly coarser**) than \mathcal{T}' . We say \mathcal{T} is **comparable** with \mathcal{T}' if either $\mathcal{T} \subset \mathcal{T}'$ or $\mathcal{T}' \subset \mathcal{T}$.

Sometimes we also say that \mathcal{T}' is larger than \mathcal{T} or \mathcal{T} is smaller than \mathcal{T}' , but not as vivid as finer.

2 Closed Sets and Limit Point

Definition. Let (X, \mathcal{T}) be a topological space. We say a subset A of X is **closed** if $X - A$ is open.

EXAMPLE. Let (X, \mathcal{T}) be a topological space and \mathcal{T} be the discrete topology, then any subset of X is a closed set. On the other hand, let \mathcal{T} be trivial topology, then any subset that is neither \emptyset nor X is neither open nor closed.

EXAMPLE. Let $(\mathbb{R}^2, \mathcal{T})$ be a topological space and \mathcal{T} generated by all open ball. And consider the set:

$$\{(x, y) \mid x \geq 0, y \geq 0\}$$

The set is closed as its complement is:

$$(-\infty, 0) \times \mathbb{R} \cup \mathbb{R} \times (-\infty, 0)$$

And each of them are open.

EXAMPLE. Let $(\mathbb{R}, \mathcal{T})$ be a topological space with topology \mathcal{T} consists of all open sets under the metric space (\mathbb{R}, d) . Consider $Y = [0, 1] \cup (2, 3)$ and the subspace topology. We claim that $[0, 1]$ is an open set of Y , because $[0, 1] = (-1, \frac{3}{2}) \cap Y$. Similarly, $(2, 3)$ is also open in Y . And the complement of each of them is another interval, therefore $[0, 1]$ and $(2, 3)$ are both open and closed.

REMARK. By these three examples, we could see that a subset of X can be open, or closed, or both, or neither. In addition, we see that a subset is open(closed) or not depends on the whole space you consider: $[0, 1]$ in EXAMPLE3 is not open in \mathbb{R} but open in Y . $(2, 3)$ is not closed in \mathbb{R} but closed in Y .

Theorem 1. Let X be a topology space. Then the following conditions hold:

- (1) \emptyset and X are closed
- (2) For any collection of closed set $\{V_\alpha \mid \alpha \in I\}$, we have $\bigcap_{\alpha \in I} V_\alpha$ is closed
- (3) The intersection of any finite many closed sets are closed.

Proof. (1) is trivial with $\emptyset = X - X$ and $X = X - \emptyset$.

As for (2), notice that :

$$\bigcap_{\alpha \in I} V_\alpha = \bigcap_{\alpha \in I} U_\alpha^c = (\bigcup_{\alpha \in I} U_\alpha)^c$$

where U_α is an open set. And we denote $X - U_\alpha$ with U_α^c . (3) follows the same way with the fact that:

$$\bigcup_{i=1}^n V_\alpha = \bigcup_{i=1}^n U_\alpha^c = (\bigcap_{i=1}^n U_\alpha)^c$$

Theorem 2. Let Y be a subspace of X . Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y .

Proof. Consider the subspace topology of Y and let V_Y is a closed set under such subspace topology. Then we have $V_Y = Y - U_Y$ for some open set U_Y in Y . With the definition of subspace topology, we have $U_Y = U \cap Y$ with U an open set in X . Then $V_Y = Y - U_Y = Y - U \cap Y = Y - U = Y \cap (X - U)$ where $(X - U)$ is closed in X . Therefore if V_Y is closed in Y , then V_Y is intersection of Y and a closed set in X .

On the other hand, if $V_Y = Y \cap V$ for some closed set V of X . We have $V_Y = Y \cap (X - U) = Y - U = Y - (Y \cap U)$, which is closed in Y .

REMARK. General speaking, a set that is closed in a subspace may not be closed in the larger topological space. For example, let $X = \mathbb{R}$ and open set consists of conventional open set in \mathbb{R} . Consider the subspace Y generated by the intersection of $[0, 1)$ and X . Then notice that $[0, \frac{1}{2})$ is open in Y as $[0, \frac{1}{2}) = (-1, \frac{1}{2}) \cap [0, 1)$. Therefore $Y - [0, \frac{1}{2}) = [\frac{1}{2}, 1)$ is closed in Y , however, it's not closed in \mathbb{R} .

But we have the following theorem explained the so called "transitivity" of closed property:

Theorem 3. Let Y be a subspace of X . If A is closed in Y and Y is closed in X , then A is closed in X .

Proof. By theorem 2, $A = Y \cap V$ with V closed in X . Therefore, A is closed in X by the fact that the intersection of two closed sets is closed.

2.1 Limit Point and Closur

Definition. Let X be a topological space and A a subset of X . An element x of X is said to be **limitpoint** of A if: for every open set U that contains x , $U \cap A \neq \emptyset$ or $\{x\}$.

Definition. Let A be a subset of the topological space X ; let A' be the set of all limit points of A , we define the closure of A as the union of A and A' , denoted by \bar{A} . Which is:

$$\bar{A} = A \cup A'$$

Theorem 4. Let X be a topological space. Then A is closed in X if and only if: $\bar{A} = A$

Proof. \Leftarrow : If $\bar{A} = A$, we need to show that A is closed, or to show that $X - A$ is open. For any element $x \in X - A$, x is neither an element of A nor the limit point of A . $x \notin A'$ means there is some open set U that contains x but $U \cap A = \emptyset$ or $\{x\}$. Now that $x \notin A$, we have $U \cap A = \emptyset$. For any $x \in X - A$, we have such open set U_x . And thus:

$$X - A = \bigcup_{x \in (X - A)} U_x$$

is union of open set in X , therefore an open set. Hence we have A is closed.

\Rightarrow : If A is closed. To prove $A = \bar{A}$, we only need to show that $A' \subset A$, which is: any limit point of A is in A . Suppose x is a limit point of A but $x \in X - A$. Then notice that $X - A$ is an open set that contains x but $(X - A) \cap A = \emptyset$, which contradicts the definition of limit point. Therefore any limit point of A is in A , and hence $A = \bar{A}$.