Modules and Homorphism

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Definition 1. *Let* R *be a ring, an (left)* **R − module** *(denoted by* A*) is an abelian group with a function* $R \times A \to A$ *satifies* $\forall r, s \in R, \forall a, b \in A$, *the following conditons holds:*

$$r(a + b) = ra + rb$$
$$(r + s)a = ra + sa$$
$$(rs)a = r(sa)$$

note:

  (i) Let R be a ring with identity, and A satifies:$1_R a = a, \forall a \in A$,then A is called **unitary R-module**

  (ii) If R is a division ring and A is an **unitary R-module**, then A is called **vector space**

Corollary 1. $\forall r \in R, a \in A$,*we have:*

  (i) $r0_A = 0_A, 0_r a = 0_A$

  (ii) $-ra = (-r)a = r(-a)$

  (iii) $n(ra)=(nr)a=r(na)$

Proof. the proof is trivial $\qquad\qquad\square$

Definition 2. *Let* R *be a ring and* A, B *be* R-*module. A* **R − module homorphism** f *is an abelian group homomorphism* $A \to B$ *satifies:*$\forall a, b \in A, r \in R$*:*

$$f(a + b) = f(a) + f(b), f(ra) = rf(a)$$

if f is an abelian group **monomorphism**(*resp.***epimorphism**, **isomorphism**) *then f is called an* R-*module* **monomorphism**(*resp.***epimorphism**, **isomorphism**)*. The kernel of f is the kernel of f as an abelian group homomorphism:* $\ker f = \{a \in A | f(a) = 0_B\}$

note:

  (i) f is monomorphism if and only if ker $f=0_A$

  (ii) f is isomorphism if and only if there is an R-module $g : B \to A$ such that:$fg = 1_B, gf = 1_A$

  (iii) $f(0_A) = 0_B$

Definition 3. *Let* R *be a ring and* A *be an* R-*module. A submodule of* A*, say* B*, is a subset of* A*,satifies:* $\forall a, b \in A, r \in R$*:*
$$a - b \in B, ra \in B$$

In other words, B *is a subgroup of* A *and is closed under the map. It's obviously that* B *is an* R-*module itself. A submodule of a vector space is called a subspace.*

EXAMPLES

(i) Let $f : A \to B$ be an R-module homorphism, then $\ker f$ is a submodule of A and $\text{Im} f$ is a submodule of B

(ii) Let I be a left ideal of R, A an R-module, S a nonempty subset of A. Define IS as follows:

$$IS = \{\sum_{i=1}^{n} r_i s_i \mid r_i \in I, s_i \in S, n \in \mathbb{N}^*\}$$

then IS is a submodule of A

(iii) Let A be an R-module and $A_i, i \in I$ is a family of submodules of A. Then $\cap_{i \in I} A_i$ is a submodule of A

Definition 4. *Let* R *be a ring,* A *a* R*-module.* X *is a nonempty set of* A. **A submodule generatedby X** *is the intersection of all submodules that contains* X. *Let* B *is the submodule generated by X. If X is finite, then B is called* **finitely generated***; If* $X = \{a\}$, *then B is called* **cyclic submodule.** *Let* $B_i, i \in I$ *be a family of submodules of A, the submodule generated by* $\cup_{i \in I} B_i$ *is called the* **sum** *of submodules* $B_i, i \in I$.

REMARK Submodule generated by X is the smallest submodule that contains X. In other words, Let B be the submodule of A generated by X and C is any submodule of A that contains X, we must have: $B \subset C$.
To prove this, we only need to notice that $B = \cap_{X \subset C} C$. For any submodule that contains X, it must on the right side.

Theorem 1. *Let* R *be a ring,* A *an* R*-module,* X *a subset of* A, $\{B_i \mid i \in I\}$ *a family of submodules of* A *and* $a \in$ A. *Let* $\text{Ra} = \{ra \mid r \in R\}$.

(i) Ra *is a submodule of* A

(ii) The cyclic submodule C generated by $\{a\}$ *is* $\{ra + na \mid r \in R, n \in \mathbb{Z}\}$

(iii) The submodule generated by X *is*

$$\{\sum_{i=1}^{n} r_i a_i + \sum_{j=1}^{m} s_j b_j \mid r_i \in \text{R}, n, m \in \mathbb{N}^*, a_i, b_j \in X, s_j \in \mathbb{Z}\}$$

Proof.   (i) $\forall ra, sa \in \text{Ra}, \forall t \in \text{R}$, we have:

$$ra - sa = (r - s)a \in \text{Ra}, \ t(sa) = (ts)a \in \text{Ra}$$

According to the definiton of submodule, Ra is a submodule of A.

(ii) First we need to show that $C = \{ra + na \mid r \in R, n \in \mathbb{Z}\}$ itself is a submoduel of A. The reason is as follows: $\forall r_1, r_2, s \in \text{R}, n_1, n_2 \in \mathbb{Z}$:

$$(r_1 a + n_1 a) - (r_2 a + n_2 a) = (r_1 - r_2)a + (n_1 - n_2)a \in C$$
$$s(r_1 a + n_1 a) = (sr_1)a + s(n_1 a) = (sr_1)a + (n_1 s)a = (sr_1 + n_1 s)a \in C$$

Hence C is a submodule of A that contains $\{a\}$. Besides, for any submodule B_i that contains $\{a\}$, it's obviously $ra \in B_i, r \in R$ and $na \in B_i, n \in \mathbb{Z}$. Hence $C \subset B_i$.
Let B be the submodule generated by X. Then $B = \cap_{i \in I} B_i$, $B \subset C$ because C is a submodule of A contains X, hence one of B_i. $C \subset B$ is trivial since $C \subset B_i$ hence $C \subset \cap_{i \in I} B_i = B$. Therefore, $B = C$.

(iii) The method to proving (iii) is the same as the method used in (ii).

$\square$

REMARK In Theorem 1(ii), if R is a ring with identity and C an unitary module over R. The submodule generated by $\{a\}$ is Ra as $na = (n1_R)a, n1_R \in R$.

Theorem 2. *Let* B *be a submodule of a module* A *over a ring* R. *Then the quotient group* A/B *is an* R-*module with the action of* R *on* A/B *given by:*

$$r(a + B) = ra + B$$

The map $\pi : A \to A/B$ *given by* $a \mapsto a + B$ *is an* R-*module epimorphism with* $\ker \pi = B$

Proof. First, we will show the ring acts on A/B is well-defined: Let $a + B = a' + B$, hence $a - a' \in B$. For any $r \in R$, we have $r(a - a') \in B$ as B is a submodule of A. Hence we have $ra + B = ra' + B$, which means $r(a + B) = r(a' + B)$. Therefore the action is well-defined.

Second, we will show A/B is an R-module with action given above. A/B is iteself an abelian group. For any $r, s \in R$, we have:

$$(r + s)(b + B) = (r + s)b + B = (rb + sb) + B = (rb + B) + (sb + B) = r(b + B) + s(b + B)$$

For any $a + B, b + B \in A/B, r \in R$, we have:

$$r((a+B)+(b+B)) = r(a+b+B) = r(a+b)+B = (ra+rb)+B = (ra+B)+(rb+B) = r(a+B)+r(b+B)$$

The associative law is easy to prove. Thus A/B is an R-module with action given above. $\square$

Theorem 3. *(isomorphism theorems)*

(i) *Let* A, B *be* R-*module and* $f : A \to B$ *an* R-*module homorphism. Then we have:*

$$A/\ker f \cong \operatorname{Im} f$$

   If f *is an epimorphism then* $A/\ker \cong B$

(ii) *Let* B *and* C *be submodules of a module* A *over a ring* R. *Then:*

$$B/(B \cap C) \cong (B + C)/C$$

(iii) *Let* B *and* C *be submodules of a module* A *over a ring* R. *If* $C \subset B$, *then* B/C *is a submodule of* A/C, *and:*
$$(A/C)/(B/C) \cong A/B$$

Proof. Proofs of the theorem is the same as those in the condtions of group and ring. $\square$

Theorem 4. *Let* R *be a ring and* $\{A_i \mid i \in I\}$ *a nonempty family of* R-*modules,* $\prod_{i \in I} A_i$ *the direct product of the abelian group* A_i *and* $\sum_{i \in I}$ *the direct sum of the abelian group* A_i.

(i) $\prod_{i \in I} A_i$ *is an* R-*module with the action of* R *given by* $r\{a_i\} = \{ra_i\}$

(ii) $\sum_{i \in I} A_i$ *is a submodule of* $\prod_{i \in I} A_i$

(iii) *For each* $k \in I$, *the canonical projection* $\pi_k : \prod A_i \to A_k$ *is an* R-*module epimorphism.*

(iv) *For each* $k \in I$, *the canonical injection* $\iota_k : A_k \to \sum A_i$ *is an* R-*module monomorphism.*

Proof. (i) $\prod_{i \in I} A_i$ is itself an abelian group. For any $r, s \in R, \{a_i\}, \{b_i\} \in \prod_{i \in I} A_i$ we have:

$$
\begin{aligned}
r(\{a_i\} + \{b_i\}) &= r(\{a_i + b_i\}) = \{r(a_i + b_i)\} \\
&= \{ra_i + rb_i\} = \{ra_i\} + \{rb_i\} \text{ (by definition of plus in direct product)} \\
&= r\{a_i\} + r\{b_i\} \\
(r + s)\{a_i\} &= \{(r + s)a_i\} = \{ra_i + sa_i\} \\
&= \{ra_i\} + \{sa_i\} \\
&= r\{a_i\} + s\{a_i\} \\
(rs)\{a_i\} &= \{(rs)a_i\} = \{r(sa_i)\} = r\{sa_i\} = r(s\{a_i\})
\end{aligned}
$$

Thus $\prod_{i \in I} A_i$ is an R-module.

(ii) $\sum_{i \in I} A_i$ consists of those elements $\{a_i\}$ with only finite number of a_k are not 0_{A_k}. Thus $\sum_{i \in I} A_i$ is obviously a subset of $\prod_{i \in I} A_i$. For any $\{a_i\}, \{b_i\} \in \sum_{i \in I} A_i$:

$$\{a_i\} - \{b_i\} = \{a_i - b_i\}$$

It's trivial that $\{a_i - b_i\}$ has at most $n_1 + n_2$ elements are not 0, where n_1 is the number of elements in $\{a_i\}$ that are not 0 and similar for n_2. Hence $\{a_i - b_i\} \in \sum_{i \in I} A_i$
For any $r \in R$, we have:

$$r\{a_i\} = \{ra_i\}$$

$\{ra_i\}$ has the same number of non-zero elements as $\{a_i\}$ does. Hence $\{ra_i\} \in \sum_{i \in I} A_i$. Therefore, $\sum_{i \in I} A_i$ is a submodule of $\prod_{i \in I} A_i$.

(iii) Canonical projection $\pi_k : \prod_{i \in I} A_i \to A_k, \{a_i\} \mapsto a_k$ satifies:

$$\pi_k(\{a_i\} + \{b_i\}) = \pi_k(\{a_i + b_i\}) = a_k + b_k = \pi_k(\{a_i\}) + \pi_k(\{b_i\})$$
$$\pi_k(r\{a_i\}) = \pi_k(\{ra_i\}) = (ra_i)_k = ra_k = r\pi_k(\{a_i\})$$

Thus π_k is an R-module homorphism. It's obviously that π_k is epimorphism since for each $a_k \in A_k$, we have:$\pi_k(\mathbf{a_k'}) = a_k$ where $\mathbf{a_k'}$ is the element with only the kth element is a_k and others are 0.

(iv) Canonical injection $\iota_k : A_k \to \sum_{i \in I} A_i, a_k \mapsto \mathbf{a_k}$ where $\mathbf{a_k}$ is the element with k^{th} element is a_k and others are 0. ι_k is easily to be proved as an R-module homorphism. And it's trivial that $\ker \iota_k = 0_{A_k}$. Therefore ι_k is monomorphism. $\qquad \square$