Modules and Homomorphism

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Definition 1. *Let* R *be a ring, an (left)* $\mathbf{R-module}$ *(denoted by* A*) is an abelian group with a function* $R \times A \to A$ *satifies* $\forall r, s \in \mathrm{R}, \forall a, b \in \mathrm{A}$*, the following conditons holds:*

$$r(a+b) = ra + rb$$
$$(r+s)a = ra + sa$$
$$(rs)a = r(sa)$$

note:

(i) Let R be a ring with identity, and A satifies:$1_R a = a, \forall a \in \mathrm{A}$,then A is called **unitary R-module**

(ii) If R is a division ring and A is an **unitary R-module**, then A is called **vector space**

Corollary 1. $\forall r \in \mathrm{R}, a \in \mathrm{A}$*,we have:*

(i) $r0_A = 0_A, 0_r a = 0_A$

(ii) $-ra = (-r)a = r(-a)$

(iii) n(ra)=(nr)a=r(na)

Proof. the proof is trivial ☐

Definition 2. *Let* R *be a ring and* A, B *be* R*-module. A* $\mathbf{R-module\ homomorphism}$ f *is an abelian group homomorphism* $A \to B$ *satifies:*$\forall a, b \in \mathrm{A}, r \in \mathrm{R}$*:*

$$f(a+b) = f(a) + f(b), f(ra) = rf(a)$$

if f *is an abelian group* **monomorphism**(*resp.***epimorphism**, **isomorphism**) *then* f *is called an* R*-module* **monomorphism**(*resp.***epimorphism**, **isomorphism**)*. The kernel of* f *is the kernel of* f *as an abelian group homomorphism:* $\ker f = \{a \in A | f(a) = 0_B\}$

note:

(i) f is monomorphism if and only if $\ker f$=0_A

(ii) f is isomorphism if and only if there is an R-module $g : B \to A$ such that:$fg = 1_B, gf = 1_A$

(iii) $f(0_A) = 0_B$

Definition 3. *Let* R *be a ring and* A *be an* R*-module. A submodule of* A*, say* B*, is a subset of* A*,satifies:* $\forall a, b \in \mathrm{A}, r \in \mathrm{R}$*:*

$$a - b \in \mathrm{B}, ra \in \mathrm{B}$$

In other words, B *is a subgroup of* A *and is closed under the map. It's obviously that* B *is an* R*-module itself. A submodule of a vector space is called a subspace.*

EXAMPLES

(i) Let $f : A \to B$ be an R-module homomorphism, then $\ker f$ is a submodule of A and $\mathrm{Im} f$ is a submodule of B

(ii) Let I be a left ideal of R, A an R-module, S a nonempty subset of A.Define IS as follows:

$$IS = \{\sum_{i=1}^{n} r_i s_i | r_i \in I, s_i \in S, n \in \mathbb{N}^*\}$$

then IS is a submodule of A

(iii) Let A be an R-module and $A_i, i \in I$ is a family of submodules of A.Then $\cap_{i \in I} A_i$ is a submodule of A

Definition 4. *Let* R *be a ring,* A *a* R*-module.* *X is a nonempty set of* A*.* **A submodule generatedby X** *is the intersection of all submodules that contains* X*.Let B is the submodule generated by X. If X is finite, then B is called* **finitely generated***; If* $X = \{a\}$*,then B is called* **cyclic submodule***.Let* $B_i, i \in I$ *be a family of submodules of A, the submodule generated by* $\cup_{i \in I} B_i$ *is called the* **sum** *of submodules* $B_i, i \in I$*.*

REMARK Submodule generated by X is the smallest submodule that contains X. In other words, Let B be the submodule of A generated by X and C is any submodule of A that contains X, we must have: $B \subset C$.
To prove this, we only need to notice that $B = \cap_{X \subset C} C$. For any submodule that contains X, it must on the right side.

Theorem 1. *Let* R *be a ring,* A *an* R*-module,* X *a subset of* A*,* $\{B_i \mid i \in I\}$ *a family of submodules of* A *and* $a \in$ A*.* *Let* $\mathrm{Ra} = \{ra \mid r \in R\}$*.*

(i) Ra *is a submodule of* A

(ii) The cyclic submodule C generated by $\{a\}$ *is* $\{ra + na \mid r \in R, n \in \mathbb{Z}\}$

(iii) The submodule generated by X *is*

$$\{\sum_{i=1}^{n} r_i a_i + \sum_{j=1}^{m} s_j b_j \mid r_i \in \mathrm{R}, n, m \in \mathbb{N}^*, a_i, b_j \in X, s_j \in \mathbb{Z}\}$$

Proof. (i) $\forall ra, sa \in \mathrm{Ra}, \forall t \in \mathrm{R}$,we have:

$$ra - sa = (r - s)a \in \mathrm{Ra},\ t(sa) = (ts)a \in \mathrm{Ra}$$

According to the definiton of submodule, Ra is a submodule of A.

(ii) First we need to show that $C = \{ra + na \mid r \in R, n \in \mathbb{Z}\}$ itself is a submoduel of A. The reason is as follows: $\forall r_1, r_2, s \in \mathrm{R}, n_1, n_2 \in \mathbb{Z}$:

$$(r_1 a + n_1 a) - (r_2 a + n_2 a) = (r_1 - r_2)a + (n_1 - n_2)a \in C$$
$$s(r_1 a + n_1 a) = (sr_1)a + s(n_1 a) = (sr_1)a + (n_1 s)a = (sr_1 + n_1 s)a \in C$$

Hence C is a submodule of A that contains $\{a\}$. Besides, for any submodule B_i that contains $\{a\}$, it's obviously $ra \in B_i, r \in R$ and $na \in B_i, n \in \mathbb{Z}$. Hence $C \subset B_i$.
Let B be the submodule generated by X. Then $B = \cap_{i \in I} B_i$, $B \subset C$ because C is a submodule of A contains X, hence one of B_i. $C \subset B$ is trivial since $C \subset B_i$ hence $C \subset \cap_{i \in I} B_i = B$. Therefore, $B = C$.

(iii) The method to proving (iii) is the same as the method used in (ii).

□

REMARK In Theorem 1(ii), if R is a ring with identity and C an unitary module over R. The submodule generated by $\{a\}$ is Ra as $na = (n1_R)a, n1_R \in R$.

Theorem 2. *Let* B *be a submodule of a module* A *over a ring* R*. Then the quotient group* A/B *is an* R*-module with the action of* R *on* A/B *given by:*

$$\mathrm{r}(\mathrm{a}+\mathrm{B})=\mathrm{ra}+\mathrm{B}$$

The map $\pi : \mathrm{A} \to \mathrm{A}/\mathrm{B}$ *given by* $\mathrm{a} \mapsto \mathrm{a}+\mathrm{B}$ *is an* R*-module epimorphism with* $\ker\pi = \mathrm{B}$

Proof. First, we will show the ring acts on A/B is well-defined: Let $a+B = a'+B$, hence $a-a' \in B$.For any $r \in R$, we have $r(a-a') \in B$ as B is a submodule of A.Hence we have $ra+B = ra'+B$,which means $r(a+B) = r(a'+B)$. Therefore the action is well-defined.

Second, we will show A/B is an R-module with action given above. A/B is iteself an abelian group. For any $r, s \in R$, we have:

$$(r+s)(b+B) = (r+s)b+B = (rb+sb)+B = (rb+B)+(sb+B) = r(b+B)+s(b+B)$$

For any $a+B, b+B \in A/B, r \in R$,we have:

$$r((a+B)+(b+B)) = r(a+b+B) = r(a+b)+B = (ra+rb)+B = (ra+B)+(rb+B) = r(a+B)+r(b+B)$$

The associative law is easy to prove.Thus A/B is an R-module with action given above. □

Theorem 3. *(isomorphism theorems)*

(i) Let A, B be R-module and $f : A \to B$ an R-module homomorphism.Then we have:

$$A/\ker f \cong \mathrm{Im} f$$

If f is an epimorphism then $A/\ker \cong B$

(ii) Let B and C be submodules of a module A over a ring R.Then:

$$B/(B\cap C) \cong (B+C)/C$$

(iii) Let B and C be submodules of a module A over a ring R.If $C \subset B$,then B/C is a submodule of A/C,and:

$$(A/C)/(B/C) \cong A/B$$

Proof. Proofs of the theorem is the same as those in the condtions of group and ring. □

Theorem 4. *Let R be a ring and $\{A_i \mid i \in I\}$ a nonempty family of R-modules,$\prod_{i\in I} A_i$ the direct product of the abelian group A_i and $\sum_{i\in I}$ the direct sum of the abelian group A_i.*

(i) $\prod_{i\in I} A_i$ is an R-module with the action of R given by $r\{a_i\} = \{ra_i\}$

(ii) $\sum_{i\in I} A_i$ is a submodule of $\prod_{i\in I} A_i$

(iii) For each $k \in I$, the canonical projection $\pi_k : \prod A_i \to A_k$ is an R-module epimorphism.

(iv) For each $k \in I$, the canonical injection $\iota_k : A_k \to \sum A_i$ is an R-module monomorphism.

Proof. (i) $\prod_{i\in I} A_i$ is itself an abelian group.For any $r, s \in R$, $\{a_i\}, \{b_i\} \in \prod_{i\in I} A_i$ we have:

$$\begin{aligned} r(\{a_i\}+\{b_i\}) &= r(\{a_i+b_i\}) = \{r(a_i+b_i)\} \\ &= \{ra_i+rb_i\} = \{ra_i\}+\{rb_i\} \text{ (by definition of plus in direct product)} \\ &= r\{a_i\}+r\{b_i\} \\ (r+s)\{a_i\} &= \{(r+s)a_i\} = \{ra_i+sa_i\} \\ &= \{ra_i\}+\{sa_i\} \\ &= r\{a_i\}+s\{a_i\} \\ (rs)\{a_i\} &= \{(rs)a_i\} = \{r(sa_i)\} = r\{sa_i\} = r(s\{a_i\}) \end{aligned}$$

Thus $\prod_{i\in I} A_i$ is an R-module.

(ii) $\sum_{i\in I} A_i$ consists of those elements $\{a_i\}$ with only finite number of a_k are not 0_{A_k}. Thus $\sum_{i\in I} A_i$ is obviously a subset of $\prod_{i\in I} A_i$. For any $\{a_i\},\{b_i\} \in \sum_{i\in I} A_i$:

$$\{a_i\} - \{b_i\} = \{a_i - b_i\}$$

It's trivial that $\{a_i - b_i\}$ has at most $n_1 + n_2$ elements are not 0, where n_1 is the number of elements in $\{a_i\}$ that are not 0 and similar for n_2.Hence $\{a_i - b_i\} \in \sum_{i\in I} A_i$
For any $r \in R$, we have:

$$r\{a_i\} = \{ra_i\}$$

$\{ra_i\}$ has the same number of non-zero elements as $\{a_i\}$ does.Hence $\{ra_i\} \in \sum_{i\in I} A_i$. Therefore, $\sum_{i\in I} A_i$ is a submodule of $\prod_{i\in I} A_i$.

(iii) Canonical projection $\pi_k : \prod_{i\in I} A_i \to A_k, \{a_i\} \mapsto a_k$ satisfies:

$$\pi_k(\{a_i\} + \{b_i\}) = \pi_k(\{a_i + b_i\}) = a_k + b_k = \pi_k(\{a_i\}) + \pi_k(\{b_i\})$$
$$\pi_k(r\{a_i\}) = \pi_k(\{ra_i\}) = (ra_i)_k = ra_k = r\pi_k(\{a_i\})$$

Thus π_k is an R-module homomorphism. It's obviously that π_k is epimorphism since for each $a_k \in A_k$, we have:$\pi_k(\mathbf{a'_k}) = a_k$ where $\mathbf{a'_k}$ is the element with only the k^{th} element is a_k and others are 0.

(iv) Canonical injection $\iota_k : A_k \to \sum_{i\in I} A_i, a_k \mapsto \mathbf{a_k}$ where $\mathbf{a_k}$ is the element with k^{th} element is a_k and others are 0. ι_k is easily to be proved as an R-module homomorphism. And it's trivial that $\ker \iota_k = 0_{A_k}$. Therefore ι_k is monomorphism. □

Theorem 5. *If* R *is a ring,* $\{A_i \mid i \in I\}$ *a family of* R*-modules,*C *an* R*-module,and* $\{\phi_i : \mathrm{C} \to \mathrm{A}_i \mid i \in I\}$ *a family of* R*-module homomorphisms, then there is a unique* R*-module homomorphism* $\phi : \mathrm{C} \to \prod_{i\in I} A_i$ *such that* $\pi_i\phi = \phi_i, \forall i \in I$*.Hence* $\prod_{i\in I} A_i$ *is the product of* $\{A_i \mid i \in I\}$ *in the category of* R*-modules.*

Proof. The R-module homomorphism is easy to see:

$$\phi : \mathrm{C} \to \prod_{i\in I} A_i, c \mapsto \{\phi_i(c)\}_{i\in I}$$

ϕ is easy to be proved as an R-module homomorphism. Hence we have:$\pi_k\phi(c) = \pi_k(\{\phi_i(c)\}_{i\in I}) = phi_k(c), c \in \mathrm{C}, k \in I$. Thus we have $\pi_i\phi = \phi_i, i \in I$.
To prove the uniqueness of ϕ, let f be another R-module homomorphism $f : C \to \prod_{i\in I} A_i$ with $\pi_i f = \phi_i, i \in I$.We need to prove that $\phi = f$. If there is some $c \in \mathrm{C}$ such that $f(c) \neq \phi(c)$,then $f(c)$ and $\phi(c)$ have at lease one position with different elements,let's say the k^{th} element. Then we have:$\pi_k(\phi(c)) \neq \pi_k(f(c))$, which means $\phi_k(c) \neq \phi_k(c)$. This is obviously not gonna happen. Therefore we must have $\phi = f$. □

Theorem 6. *If* R *is a ring,* $\{A_i \mid i \in I\}$ *a family of* R*-modules,*D *an* R*-module,and* $\{\psi_i : A_i \to \mathrm{D} \mid i \in I\}$ *a family of* R*-module homomorphisms, then there is a unique* R*-module homomorphism* $\psi : \sum_{i\in I} A_i \to \mathrm{D}$ *such that* $\psi\iota_i = \psi_i, \forall i \in I$*.Hence* $\prod_{i\in I} A_i$ *is the coproduct of* $\{A_i \mid i \in I\}$ *in the category of* R*-modules.*

Proof. The R-module homomorphism ψ is easy to see:

$$\psi : \sum_{i\in I} A_i \to D, \{a_i\}_{i\in I} \mapsto \sum_{i\in I} \psi_i(a_i)$$

Here $\sum_{i\in I}\psi(a_i)$ means we add finite many nonzero elements together. ψ is easy to be seen as an R-module homomorphism. And it's easy to prove that $\psi\iota_i = \psi_i$.
To prove the uniqueness of ψ, let f be another R-module homomorphism with $f\iota_i = \psi_i$. Then for any $\{a_i\} \in \sum_{i\in I} A_i$, we have:

$$f(\{a_i\}) = f(\sum_{i\in I}\mathbf{a_i}) = f(\sum_{i\in I}\iota_i(a_i)) = \sum_{i\in I}(f\iota_i)(a_i) = \sum_{i\in I}\psi_i(a_i)$$

Thus $f = \psi$.We have proved the uniqueness of ψ $\square$

Theorem 7. *Let* R *be a ring and* $\mathrm{A_1, A_2, ..., A_n}$ R*-modules. Then* $\mathrm{A} \cong \mathrm{A_1} \bigoplus \mathrm{A_2} \bigoplus ... \bigoplus \mathrm{A_n}$ *if and only if for each* $i = 1, 2, ..., n$ *there are* R*-module homomorphism* $\pi_i : \mathrm{A} \to \mathrm{A_i}$ *and* $\iota_i : \mathrm{A}_i \to \mathrm{A}$ *such that:*

(i) $\pi_i\iota_i = 1_{A_i}$ *for* $i = 1, 2, ..., n$
(ii) $\pi_j\iota_i = 0$ *for* $j \neq i$
(iii) $\iota_1\pi_1 + \iota_2\pi_2 + ... + \iota_n\pi_n = 1_A$

Proof. ($\Rightarrow$) If $\mathrm{A} \cong \mathrm{A_1} \bigoplus \mathrm{A_2} \bigoplus ... \bigoplus \mathrm{A_n}$, let π_i, ι_i be the canonical projection and injection. It's easy to prove that π_i, ι_i satify condtions(i)-(iii)

($\Leftarrow$) If π_i, ι_i satisfy (i)-(iii). Let π_i', ι_i' be the canonical projection and injection between $\mathrm{A_1} \bigoplus \mathrm{A_2} \bigoplus ... \bigoplus \mathrm{A_n}$ and $\mathrm{A_i}$.Let $\phi : A_1 \bigoplus A_2 \bigoplus ... \bigoplus A_n \to A$ be given by $\phi = \iota_1\pi_1' + \iota_2\pi_2' + ... + \iota_n\pi_n'$ and $psi : A \to A_1 \bigoplus A_2 \bigoplus ... \bigoplus A_n$ by $\psi = \iota_1'\pi_1 + \iota_2'\pi_2 + ... + \iota_n'\pi_n$. Then it's easy to verify that $\phi\psi = 1_A$ and $\psi\phi = 1_{A_1 \bigoplus A_2 \bigoplus ... \bigoplus A_n}$. Therefore $A \cong A_1 \bigoplus A_2 \bigoplus ... \bigoplus A_n$. $\square$