Compactness of Topological Space

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1 Compact Spaces

Compact Spaces is a kind of special topological spaces. In such a topological space, a local property may be true in the whole space. In mathmatical analysis, we have already seen some compact spaces, for example, the closed interval. A basis but important theorem in analysis says that a continuous function must be bounded in a closed interval. The key point of the proof to this theorem is the concept of *compactness*.

Definition. Let X be topological space. A collection $\mathcal{A}$ of subsets of X is said to be a ***covering*** of X, if their union equals to X.If elements of this collection are all open sets, then $\mathcal{A}$ is said to be an ***open covering*** of X.

Definition. A topological space X is said to be ***compact***, if every open covering of X has finite subcollection that covers X.

Finite subcollection means we can pick up finite many open set to form a new collection. Here are some examples about compact topological spaces.

EXAMPLE. The following subspace of $\mathbb{R}$ is compact:

$$X = \{0\} \cup \{1/n \mid n \in \mathbb{N}^+\}$$

Let $\mathcal{A}$ be an open covering of X, we will pick up finite of them to cover X. Since $0 \in X = \bigcup_{U \in \mathcal{A}} U$, there must be some open set that contains 0, we pick up this open set. Notice that 0 is a limit point of $X \setminus \{0\}$, there are only finite many points not included in this open set. So we can pick finite many open sets to cover them.

EXAMPLE. Consider $\mathbb{R}$ and general topology on $\mathbb{R}$, then $(0, 1]$ is not compact. The reason is that we have an open cover:

$$\mathcal{A} = \{(1/n, 1] \mid n \in \mathbb{N}^+\}$$

but $\mathcal{A}$ has no finite sub-cover.

Definition. Let X be a topological space and Y a subset of X. Y is said to be a ***compact set*** (of X), if any open covering of Y has finite subcover.

REMARK. We say a collection $\mathcal{A}$ of X is a cover of Y, if:

$$Y \subset \bigcup_{U \in \mathcal{A}} U$$

and $\mathcal{A}$ is said to be an open cover iff every elements of $\mathcal{A}$ is an open set.

Different from the definition of *compact space*, a *compact set* specifies the compactness of a subset,but there are no substantial difference between this two definitions. We will demonstrate you a theorem proof(very easy, just follow the definition):

Theorem 1. *Let X be a topological space. Y is a compact set if and only if Y is compact space under subspace topology.*

Now we may not distinguish *compact set* and *compact space* deliberately.

Theorem 2. *Every closed set of a compact space is a compact set, thus a compact space.*

Proof. Let X be a compact space, and Y a closed set of X. We shall see that every open cover of Y has a finite sub-cover.

Let $\mathcal{A} = \{U_\alpha \mid \alpha \in I\}$ is an open cover of Y, st. $Y \subset \bigcup_{\alpha \in I} U_\alpha$. Then $\bigcup_{\alpha \in I} U_\alpha \cup Y^c$ is an open covering of X as Y is closed in $X(Y^c = X \setminus Y)$. Thus there are finite sub-cover of X for X is compact. Let: $\bigcup_{i=1}^{n} U_i = X$, which is also a finite sub-cover of Y. If Y^c is one of these open sets, kick it out, and we get a finite sub-cover from $\mathcal{A}$ for Y, which concludes that Y is compact set in X.

In mathmatical analysis, we have proved so-called "finite-covering theorem" for closed interval, therefore every closed interval of $\mathbb{R}$ is compact. One may naively think that compact set must be closed set. This is not true: Consider $X = \{0, 1, 2\}, \mathcal{T} = \{\varnothing, X, 1, 2\}$. Then $\{1\}$ is compact as there are totally finite open set, however, $\{1\}$ is not closed. But we will see this assertion is true in some more particular space.

Theorem 3. *Every compact set of a Hausdorff space is closed.*

Proof. Let X be a Hausdorff space and Y a compact set of X. We will show that Y is closed. We will prove that every elements not in Y is also not a limit point of Y.

Fix a point of Y^c, say x. For any $y \in Y$, there exists two open set U_y, V_y, such that:$x \in U_y, y \in V_y$ but $U_y \cap V_y = \varnothing$ for X is Hausdorff. $\{V_y \mid y \in Y\}$ is obviously an open covering of Y. Then there are finite sub-cover,say $\{V_{y_1}, V_{y_2}, ..., V_{y_n}\}$. Consider $U = \bigcup_{i=1} U_{y_i}$. Then U is an open set that contains x, and it's easy to prove that $U \cap V_{y_i} = \varnothing, i = 1, 2, ..., n$, thus $U \cap Y = \varnothing$. This indicates that x is not a limit point of Y. Hence, $\bar{Y} = Y$ and Y is closed. The proof is done.

The following theorem is a direct corollary of ***Theorem 2***(not ***Theorem 3***). It discusses what a compact set is in a more special space.

Theorem 4. ***(Heine–Borel theorem)*** *A subset of $\mathbb{R}^n$ is compact if and only if it's closed and bounded*

Proof. We will just give a sketch of proof for this theorem.
($\Rightarrow$): If Y is compact, it must be closed as $\mathbb{R}^n$ is Hausdorff. To see Y is bounded, we consider the distance between any point of Y and the original point $\mathbf{0}$. For any $y \in Y$, there is an open ball $B(y, r_y)$ that contains y but doesn't contain $\mathbf{0}$. Then $Y \subset \bigcup_{y \in Y} B_y$. We can pick finite of these open ball to cover Y, say $B(y_1, r_{y_1}), ..., B(y_n, r_{y_n})$. Then every point of Y must be one of these open balls. Assume $y \in B(y_1, r_{y_1})$ then $d(y, \mathbf{0}) \leq d(y, y_1) + d(y_1, \mathbf{0})$. Note that there are only finite many open ball, then we can let $M = \max r_{y_i}, N = \max d(y_i, \mathbf{0})$. Thus $d(y, \mathbf{0}) \leq M + N$ and the proof is done.
($\Leftarrow$) If Y is bounded, then $Y \subset [a_1, b_1] \times \cdots [a_n, b_n]$ for some a_i, b_i. We denote this cubic with U. Note that Y is also closed in U because $U \cap Y$ is closed in U. We assert that U is compact, and by *theorem 3*, Y is compact.
(There are many methods to proving U is compact. A direct way is use the same technique of proving compactness of closed interval.)

Theorem 5. *The image of a compact space under a continuous function is still compact.*

Proof. To simplify this question, we will prove that a continous function maps a compact set to a compact set.

Let X, Y be topological space and $f : X \to Y$ a continuous function. Let $A \subset X$ a compact set, we will prove that $f(A)$ is a compact set of Y. Consider an open covering of $f(A)$, say $\{U_\alpha \mid \alpha \in I\}$. And consider $\{f^{-1}(U_\alpha) \mid \alpha \in I\}$ in X. Since f is continuous, each $f^{-1}(U_\alpha)$ is open in X. And obviously $A \subset \bigcup\limits_{\alpha \in I} f^{-1}(U_\alpha)$. A has a finite sub-cover, say: $A \subset \bigcup\limits_{i=1}^{n} f^{-1}(U_i)$. Then $f(A) \subset \bigcup\limits_{i=1}^{n} U_i$ and the proof is done.

Theorem 6. *Let $f : X \to Y$ be a bijection continous function. If X is compact and Y is Hausdorff, then f is homeomorphism.*

Proof. For any open set $U \subset X$, we have U^c is closed in X, therefore, compact in X(*theorem 2*). Thus, $f(U^c)$ is compact in Y(*theorem 5*), and therefore closed (*theorem 3*) Note that $f(U^c) = f(U)^c$ as f is bijective, we have $f(U)$ is open. Therefore, $f(U)$ is open if and only if U is open, the proof is done.