Compactness of Topological Space

December 1st, 2020

1 Compact Spaces

Compact Spaces is a kind of special topological spaces. In such a topological space, a local property may be true in the whole space. In mathmatical analysis, we have already seen some compact spaces, for example, the closed interval. A basis but important theorem in analysis says that a continuous function must be bounded in a closed interval. The key point of the proof to this theorem is the concept of *compactness*.

Definition. Let X be topological space. A collection $\mathcal{A}$ of subsets of X is said to be a ***covering*** of X, if their union equals to X.If elements of this collection are all open sets, then $\mathcal{A}$ is said to be an ***open covering*** of X.

Definition. A topological space X is said to be ***compact***, if every open covering of X has finite subcollection that covers X.

Finite subcollection means we can pick up finite many open set to form a new collection. Here are some examples about compact topological spaces.

EXAMPLE. The following subspace of $\mathbb{R}$ is compact:

$$X = \{0\} \cup \{1/n \mid n \in \mathbb{N}^+\}$$

Let $\mathcal{A}$ be an open covering of X, we will pick up finite of them to cover X. Since $0 \in X = \bigcup_{U \in \mathcal{A}} U$, there must be some open set that contains 0, we pick up this open set. Notice that 0 is a limit point of $X \setminus \{0\}$, there are only finite many points not included in this open set. So we can pick finite many open sets to cover them.

Definition. Let X be a topological space and Y a subset of X. Y is said to be a ***compact set*** (of X), if any open covering of Y has finite subcover.

REMARK. We say a collection $\mathcal{A}$ of X is a cover of Y, if:

$$Y \subset \bigcup_{U \in \mathcal{A}} U$$

and $\mathcal{A}$ is said to be an open cover iff every elements of $\mathcal{A}$ is an open set.

Different from the definition of *compact space*, a *compact set* specifies the compactness of a subset,but there are no substantial difference between this two definitions. We will demonstrate you a theorem proof(very easy, just follow the definition):

Theorem 1. *Let X be a topological space. Y is a compact set if and only if Y is compact space under subspace topology.*

Now we may not distinguish *compact set* and *compact space* deliberately.