Projective and Injective Modules

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1.Projective Modules

Definition 1. *A module P over a ring R is said to be* ***projective****, if given any diagram of R-module homomorphisms*

$$\begin{tikzcd} & P \arrow[d, "f"] & \\ A \arrow[r, "g"] & B \arrow[r] & 0 \end{tikzcd}$$

with the bottom row exact, there exists an R-module homomorphism $h : P \to A$ such that the diagram

$$\begin{tikzcd} & P \arrow[dl, "h"'] \arrow[d, "f"] & \\ A \arrow[r, "g"] & B \arrow[r] & 0 \end{tikzcd}$$

is commutative.

note The bottom row exact means homomorphism g is an epimorphism. An R-module P is said to be projective means for any epimorphism $g : A \to B$ and R-module homomorphism $f : P \to B$ there exists $h : P \to A$ such that:$f = gh$

Theorem 1. *Every free module* F *over a ring* R *with identity is projective.*

Proof. Let F be a free module over ring R and F is free on set X. Let $\iota : X \to F$. For any pair of R-module A,B with $g : A \to B$ an epimorphism and $f : F \to B$ a homomorphism

$$\begin{tikzcd} & F \arrow[d, "f"] & \\ A \arrow[r, "g"] & B \arrow[r] & 0 \end{tikzcd}$$

$f(\iota(x)) \in B, x \in X$. Since g is an epimorphism, there exists some $a_x \in A$ such that $g(a_x) = f(\iota(x))$.Let $h' : X \to A, x \mapsto a_x$ then h' induces a homomorphism $h : F \to A$.Hence we have: $h(\iota(x)) = a_x$ and $g(h(\iota(x))) = g(a_x) = f(\iota(x))$.This holds for any $x \in X$. Hence $gh\iota = f\iota$.We have following diagram

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "\phi"] & F \arrow[d, "f"] \\ & A \end{tikzcd}$$

According to the property of free module: For any homomorphism $\phi : X \to A$,there exists unique homomorphism $f : F \to A$ such that $\phi = f\iota$. Here let $\phi = f\iota = gh\iota$. We have: $f = gh$ $\square$

Corollary 1. *Every module A over a ring R is the homomorphism image of a projective R-module*

Proof. Every module A(over R) is homomorphism image of a free module over R. By Theorem1, every free module is a projective module. $\square$