Modules and Homorphism

July 30, 2020

Definition 1. *Let* R *be a ring, an (left)* $\mathbf{R-module}$ *(denoted by* A*) is an abelian group with a function* $R \times A \to A$ *satifies* $\forall r, s \in \mathrm{R}, \forall a, b \in \mathrm{A}$*, the following conditons holds:*

$$r(a+b) = ra + rb$$
$$(r+s)a = ra + sa$$
$$(rs)a = r(sa)$$

note:

(i) Let R be a ring with identity, and A satifies:$1_R a = a, \forall a \in \mathrm{A}$,then A is called **unitary R-module**

(ii) If R is a division ring and A is an **unitary R-module**, then A is called **vector space**

Corollary 1. $\forall r \in \mathrm{R}, a \in \mathrm{A}$*,we have:*

(i) $r0_A = 0_A, 0_r a = 0_A$

(ii) $-ra = (-r)a = r(-a)$

(iii) *n(ra)=(nr)a=r(na)*

Proof. the proof is trivial □

Definition 2. *Let* R *be a ring and* A, B *be* R*-module. A* $\mathbf{R-module\ homorphism}$ f *is an abelian group homorphism* $A \to B$ *satifies:*$\forall a, b \in \mathrm{A}, r \in \mathrm{R}$*:*

$$f(a+b) = f(a) + f(b), f(ra) = rf(a)$$

if f *is an abelian group* **monomorphism**(*resp.***epimorphism**, **isomorphism**) *then* f *is called an* R*-module* **monomorphism**(*resp.***epimorphism**, **isomorphism**)*. The kernel of* f *is the kernel of* f *as an abelian group homorphism:* $\ker f = \{a \in A | f(a) = 0_B\}$

note:

(i) f is monomorphism if and only if $\ker f$=0_A

(ii) f is isomorphism if and only if there is an R-module $g : B \to A$ such that:$fg = 1_B, gf = 1_A$

(iii) $f(0_A) = 0_B$

Definition 3. *Let* R *be a ring and* A *be an* R*-module. A submodule of* A*, say* B*, is a subset of* A*,satifies:* $\forall a, b \in \mathrm{A}, r \in \mathrm{R}$*:*

$$a - b \in \mathrm{B}, ra \in \mathrm{B}$$

In other words, B *is a subgroup of* A *and is closed under the map. It's obviously that* B *is an* R*-module itself. A submodule of a vector space is called a subspace.*

EXAMPLES

(i) Let $f : A \to B$ be an R-module homorphism, then $\ker f$ is a submodule of A and Imf is a submodule of B

(ii) Let I be a left ideal of R, A an R-module, S a nonempty subset of A.Define IS as follows:

$$IS = \{\sum_{i=1}^{n} r_i s_i | r_i \in I, s_i \in S, n \in \mathbb{N}^*\}$$

then IS is a submodule of A

(iii) Let A be an R-module and $A_i, i \in I$ is a family of submodules of A.Then $\cap_{i\in I} A_i$ is a submodule of A

Definition 4. *Let* R *be a ring,* A *a* R*-module.* X *is a nonempty set of* A. **A submodule generatedby X** *is the intersection of all submodules that contains* X.*Let B is the submodule generated by X. If X is finite, then B is called* **finitely generated***; If* $X = \{a\}$*,then B is called* **cyclic submodule**.*Let* $B_i, i \in I$ *be a family of submodules of A, the submodule generated by* $\cup_{i\in I} B_i$ *is called the* **sum** *of submodules* $B_i, i \in I$.

REMARK Submodule generated by X is the smallest submodule that contains X. In other words, Let B be the submodule of A generated by X and C is any submodule of A that contains X, we must have: $B \subset C$.
To prove this, we only need to notice that $B = \cap_{X\subset C} C$. For any submodule that contains X, it must on the right side.

Theorem 1. *Let* R *be a ring,* A *an* R*-module,* X *a subset of* A, $\{B_i \mid i \in I\}$ *a family of submodules of* A *and* $a \in$ A. *Let* Ra $= \{ra \mid r \in R\}$.

(i) Ra *is a submodule of* A

(ii) The cyclic submodule C generated by $\{a\}$ *is* $\{ra + na \mid r \in R, n \in \mathbb{Z}\}$

(iii) The submodule generated by X *is*

$$\{\sum_{i=1}^{n} r_i a_i + \sum_{j=1}^{m} s_j b_j \mid r_i \in \mathrm{R}, n, m \in \mathbb{N}^*, a_i, b_j \in X, s_j \in \mathbb{Z}\}$$

Proof. (i) $\forall ra, sa \in$ Ra, $\forall t \in$ R,we have:

$$ra - sa = (r - s)a \in \mathrm{Ra},\ t(sa) = (ts)a \in \mathrm{Ra}$$

According to the definiton of submodule, Ra is a submodule of A.

(ii) First we need to show that $C = \{ra + na \mid r \in R, n \in \mathbb{Z}\}$ itself is a submoduel of A. The reason is as follows: $\forall r_1, r_2, s \in$ R, $n_1, n_2 \in \mathbb{Z}$:

$$(r_1 a + n_1 a) - (r_2 a + n_2 a) = (r_1 - r_2)a + (n_1 - n_2)a \in C$$
$$s(r_1 a + n_1 a) = (sr_1)a + s(n_1 a) = (sr_1)a + (n_1 s)a = (sr_1 + n_1 s)a \in C$$

Hence C is a submodule of A that contains $\{a\}$. Besides, for any submodule B_i that contains $\{a\}$, it's obviously $ra \in B_i, r \in R$ and $na \in B_i, n \in \mathbb{Z}$. Hence $C \subset B_i$.
Let B be the submodule generated by X. Then $B = \cap_{i\in I} B_i$, $B \subset C$ because C is a submodule of A contains X, hence one of B_i. $C \subset B$ is trivial since $C \subset B_i$ hence $C \subset \cap_{i\in I} B_i = B$. Therefore, $B = C$.

(iii) The method to proving (iii) is the same as the method used in (ii).

□

REMARK In Theorem 1(ii), if R is a ring with identity and C an unitary module over R. The submodule generated by $\{a\}$ is Ra as $na = (n1_R)a, n1_R \in R$.