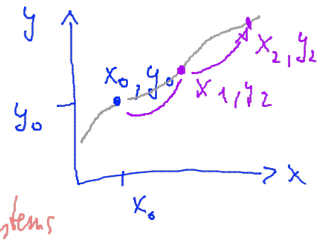
Dynamical systems: odes $\frac{dy}{dx} = f(x,y)$

→ solve analytically or numerically for initial cond x_0

time-1 mappings

$$y_{n+1} = f(x_n, y_n)$$



Here: We will solely work with Hamiltonian Systems

$$(*) \quad \frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{pmatrix}$$

where the Hamiltonian $H(q,p,t) = T(p) + V(q,t)$

Examples: very special "Kicked" systems

Force $= -\frac{\partial}{\partial q} V$ "conservative"

Kicked rotor

$$V(q,t) = \sum_{n\in\mathbb{Z}} K\cdot\sin\vartheta\cdot\delta(t-n\cdot T)$$

without any potential

then: add a kick: every time $t = 0, T, 2T, \dots$

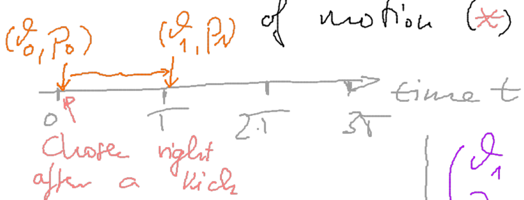
amount of torque applied is $\sim K\cdot\sin\vartheta$ where K - kicking strength

$$H = \frac{p^2}{2} + K\sum_{n\in\mathbb{Z}} \sin\vartheta\,\delta(t-n)$$

when we set $T = 1$

⇒ only one external Parameter: K

turns out: you can integrate the equations of motion (*) for one period of the driving



$$\begin{pmatrix} \vartheta_1 \\ p_1 \end{pmatrix} = \begin{pmatrix} \vartheta_0 + T\cdot p_0 \\ p_0 + \frac{K}{T}\sin(\vartheta_0 + T\cdot p_0) \end{pmatrix} = M_K\left(\begin{pmatrix} \vartheta_0 \\ p_0 \end{pmatrix}\right)$$

$$M_K : \underbrace{\mathbb{R}\backslash_{[0,2\pi)}}_{\vartheta_n} \times \underbrace{\mathbb{R}}_{p_n} \to \mathbb{R}\backslash_{[0,2\pi)} \times \mathbb{R}$$

$\hat{=}$ $[0, 2\pi)$ & periodic or call it "real numbers modulo 2π"

So we have: $\mathcal{M}_K : \mathbb{R}\backslash_{[0,2\pi)} \times \mathbb{R} \circlearrowleft$

where this is modulo 2π

$$\begin{pmatrix}\vartheta_0\\ p_0\end{pmatrix} \longmapsto \mathcal{M}_K\left(\begin{pmatrix}\vartheta_0\\ p_0\end{pmatrix}\right) = \begin{pmatrix}\vartheta_0 + \pi p_0\\ p_0 + \frac{K}{\pi}\sin(\vartheta_0 + \pi p_0)\end{pmatrix}$$

It turns out we can do a similar "modulo" for p:

$$\mathcal{M}_K : \mathbb{R}\backslash_{[0,2\pi)} \times \mathbb{R}\backslash_{[-1,1]}$$

$\mathbb{R}\backslash_{[0,2\pi)}$: 0 and 2π are the same on $\mathbb{R}$

$\mathbb{R}\backslash_{[-1,1]} = \{x \in \mathbb{R}$ where $x \sim y$ if $\exists n \in \mathbb{Z}$ such that $y = 2n + x\}$

they are the same: $-2, -1, 0, 1, 2, 3$ on the p axis

$\Rightarrow$ due to only looking just after the kick, we can restrict ? Like this ?? (illegible)

Why? Example: $\vartheta_0 = \frac{\pi}{4}$ and $p = \frac{1}{2} \Rightarrow \vartheta_1 = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3}{4}\pi$

but if $p = 2\frac{1}{2} = \frac{5}{2} \Rightarrow \vartheta_1 = \frac{\pi}{4} + \pi \cdot \frac{5}{2} = \frac{11}{4}\pi = 2\frac{3}{4}\pi \mathrel{\hat{=}} \frac{3}{4}\pi$

NB: turning much faster!

① Task: Program $\begin{pmatrix}\vartheta_1\\ p_1\end{pmatrix} = \mathcal{M}_K\left(\begin{pmatrix}\vartheta_0\\ p_0\end{pmatrix}\right) \overset{(*)}{=} \begin{pmatrix}\vartheta_0 + \pi p_0\\ p_0 + \frac{K}{\pi}\sin(\vartheta_0 + \pi p_0)\end{pmatrix}$

where the modulo is applied correctly

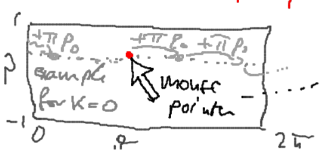
that is: use the % operator in MATLAB to achieve $\vartheta_1 \in [0, 2\pi)$ and $p_1 \in [-1, 1]$

(**) as a check: this modulo must be such that

ϑ		p	
$0 \to 0$		$-\frac{1}{2} \to -\frac{1}{2}$	
$\pi \to \pi$		$0 \to 0$	
$2\pi \to 0$		$\frac{1}{2} \to \frac{1}{2}$	
$3\pi \to \pi$		$\frac{3}{2} \to -\frac{1}{2}$	
$-\frac{1}{2}\pi \to \frac{3}{2}\pi$		$-\frac{5}{2} \to \frac{1}{2}$	

1. program (*) including the % from (**)
2. Choose $K = 2.4$ and a positive $N = 1000$ and write a small program
3. Plot (□)

window to click in

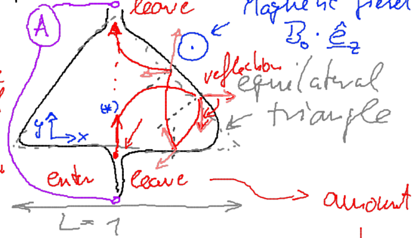


"click"

$$\begin{pmatrix}\vartheta_0\\ p_0\end{pmatrix} \to \begin{pmatrix}\vartheta_1\\ p_1\end{pmatrix} \to \begin{pmatrix}\vartheta_2\\ p_2\end{pmatrix} \to \dots \to \begin{pmatrix}\vartheta_{1000}\\ p_{1000}\end{pmatrix} \ (\square)$$

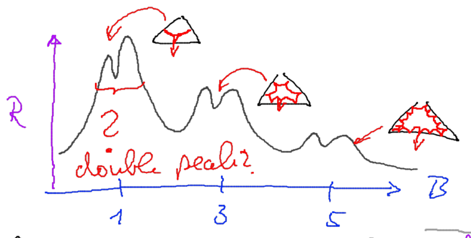
Overall aim of project

transport problems



electrons • e^- in a constant, perpendicular magnetic field confined to 2d and confined to specific region in 2d.

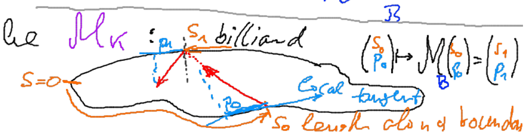
amount of e^- leaving here give rise to electric resistance $R(B)$



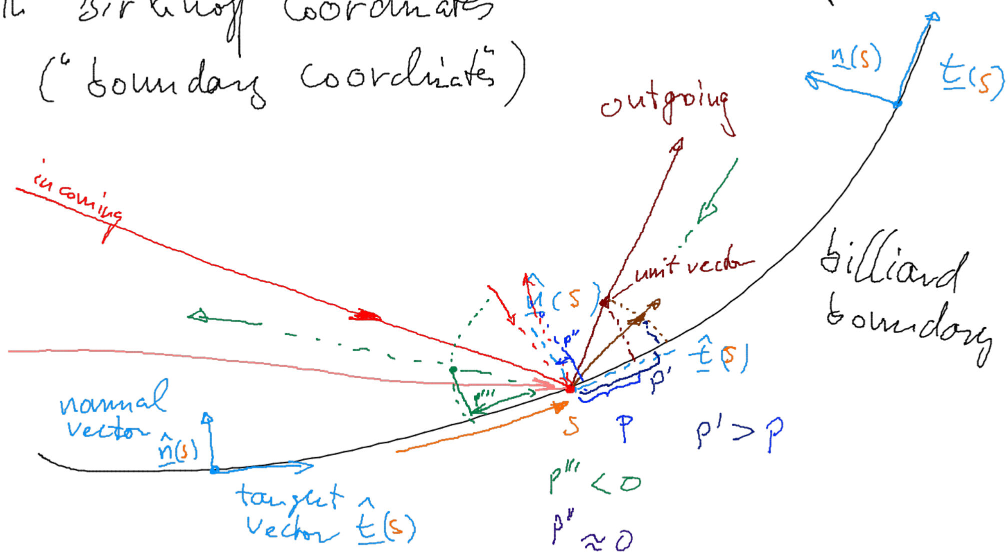
NB: radius of circles here(*) $\sim \frac{1}{B}$

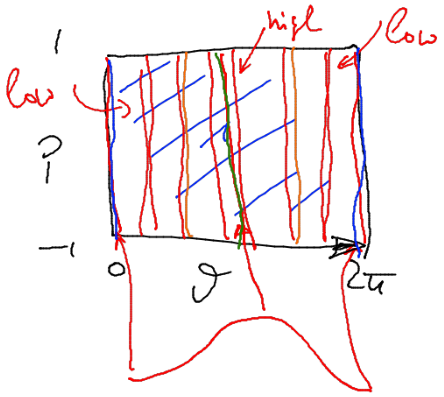
Link to time-1 maps like $\mathcal{M}_K$:

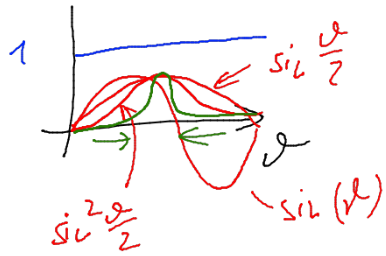
Birkhoff-coordinates
(special type of Poincaré sections)



Meaning of s & p for $\mathcal{M}$-boundary map in Birkhoff coordinates ("boundary coordinates")





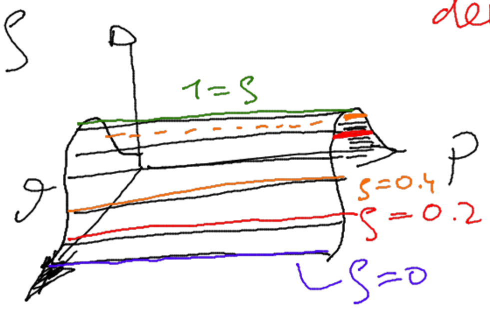


$\sin^2 \vartheta/2 = \rho(\vartheta, p)$

$\sin^{10} \frac{\vartheta}{2}$

density $\rho : [0, 2\pi) \times (-1, 1) \longrightarrow [0, \infty)$

$(\vartheta, p) \longmapsto \rho(\vartheta, p) \overset{\text{for example}}{=} 1$



or another example

$\rho(\vartheta, p) = \sin^2 \frac{\vartheta}{2}$

As a first step in understanding how densities are mapped, we have to create some examples!

for example

$\rho(\vartheta, p) = \sin^{10}(\vartheta/2) \cdot \cos^{10}(\pi \cdot p/2)$

Standard Map $\quad \mathcal{M}_K : [0,2\pi) \times (-1,1) \longrightarrow [0,2\pi) \times (-1,1)$

$\downarrow$

apply this to where the user clicked $\longrightarrow$ $\boxed{\text{one}}$ initial condition $\longrightarrow$ iterate it using $\mathcal{M}_K$

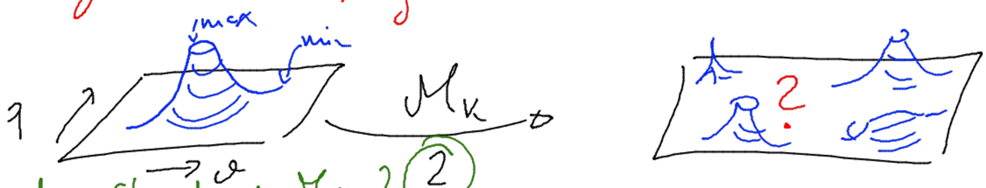
Now: change perspective: Do not think of single trajectories (ϑ_0, p_0) but of ensemble (i.e. many of them)

will all end up somewhere

$N=1$

More abstractly: densities of initial conditions

Aim: understand what happens to densities under Standard Map?



Next steps:

(1) define a density function

$$\varrho : [0,2\pi) \times (-1,1) \longrightarrow [0,\infty)$$

$$(\vartheta,p) \longmapsto \varrho(\vartheta,p) = \underline{\sin^{10}\left(\frac{\vartheta}{2}\right)\cdot\cos^{10}\left(\pi\frac{p}{2}\right)}$$

(2) plot it in the same coordinate system: a contour plot of (1)

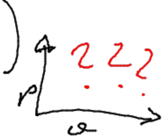
(3) Find out: what is the inverse of the Standard Map

$$M_K\begin{pmatrix}\vartheta\\ p\end{pmatrix} = \begin{pmatrix}\vartheta+\pi p\\ p+\frac{K}{\pi}\sin(\vartheta+\pi p)\end{pmatrix}$$

$$M_K^{-1}\begin{pmatrix}\vartheta\\ p\end{pmatrix} = \begin{pmatrix}?\end{pmatrix}$$ such that $M_K\left(M_K^{-1}\begin{pmatrix}\vartheta\\ p\end{pmatrix}\right) = \begin{pmatrix}\vartheta\\ p\end{pmatrix}$ for arbitrary K

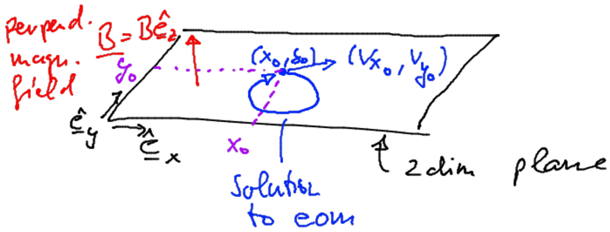
(4) Define a new function $\varrho_1 = \varrho \circ M_K^{-1}$

that means $\varrho_1(\vartheta,p) = \varrho\left(M^{-1}_{K_\vartheta}(\vartheta,p),\, M^{-1}_{K_p}(\vartheta,p)\right)$



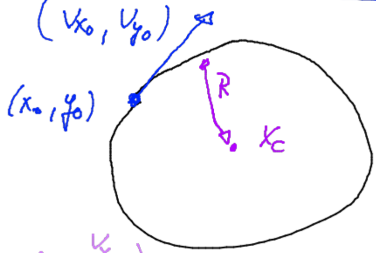
(5) Do a contour plot like (2) but for (4)

Equations of motion



eom

$$\boxed{\begin{aligned} \dot{x} &= v_x & \dot{v}_x &= 2\sqrt{2}B\, v_y \\ \dot{y} &= v_y & \dot{v}_y &= -2\sqrt{2}B\, v_x \end{aligned}}$$



Using solution of $\boxed{\text{eom}}$ (you solve it for t):

$$R = R(B) = \underline{\underline{\frac{|V_0|}{2\sqrt{2}|B|}}} \sim \frac{1}{|B|}$$

$$X_c = X_c(x_0, y_0, V_{x_0}, V_{y_0}, B) = \underline{\underline{\left(x_0 + \frac{V_{y_0}}{2\sqrt{2}B},\ y_0 - \frac{V_{x_0}}{2\sqrt{2}B}\right)}}$$

alternatively:

$$\left\{\begin{aligned} x(t) &= A_1 + B_1 \sin(2\sqrt{2}B\,t) + C_1 \cos(2\sqrt{2}B\,t) \\ y(t) &= A_2 + B_2 \sin(2\sqrt{2}B\,t) + C_2 \cos(2\sqrt{2}B\,t) \\ V_x(t) &= 2\sqrt{2}B \cdot B_1 \cos(2\sqrt{2}B\,t) + \ldots\ C_1 \ldots \\ V_y(t) &= 2\sqrt{2}B \cdot B_2 \cos(2\sqrt{2}B\,t) + \ldots\ C_2 \ldots \end{aligned}\right\}$$

at $t = 0$

$$A_1 + C_1 \overset{!}{=} x_0$$
$$A_2 + C_2 \overset{!}{=} y_0$$
$$2\sqrt{2}B \cdot B_1 \overset{!}{=} V_{x_0}$$
$$2\sqrt{2}B \cdot B_2 \overset{!}{=} V_{y_0}$$

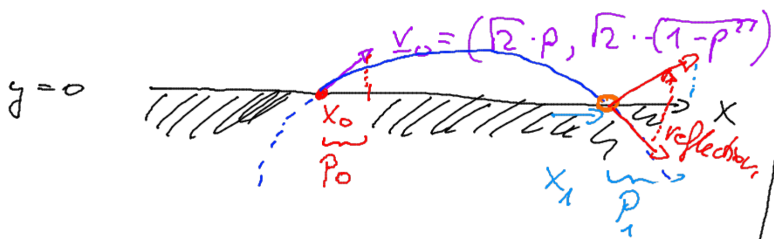
$x(t) = x_c + R\cos(\omega t + \varphi)$

$y(t) = y_c + R\sin(\omega t + \varphi)$

$V_x = \ldots,\ V_y = \ldots$

with: $\omega := -2\sqrt{2}\cdot B$, $R := \frac{|V_0|}{|\omega|}$, $\varphi = \arctan\left(\frac{y_0 - y_c}{x_0 - x_c}\right) \hat{=} \arctan\left(-V_{x_0}/\omega \,/\, V_{y_0}/\omega\right) \Rightarrow$ MATLAB "atan2", $(x_c, y_c) = \left(x_0 - \frac{V_{y_0}}{\omega},\ y_0 + \frac{V_{x_0}}{\omega}\right)$

what we need is a boundary map:



Input: $x_0,\ p_0 \in (-1,1)$

yield initial cons

$(x_0, 0, \sqrt{2}\cdot p, \sqrt{2}\cdot\sqrt{1-p^2})$

(see first page)

yields circular orbit with radius R and centre x_c

yields another intersection with line $y=0$

do a reflection $\rightarrow (x_1, p_1)$

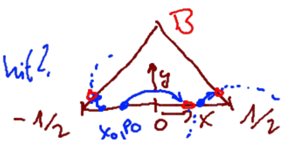
gives us a mapping

$$\tilde{M}_B : \mathbb{R}\times(-1,1) \longrightarrow \mathbb{R}\times(-1,1)$$

$$(x_0,p_0) \longmapsto \tilde{M}_B(x_0,p_0) = (x_1,p_1)$$

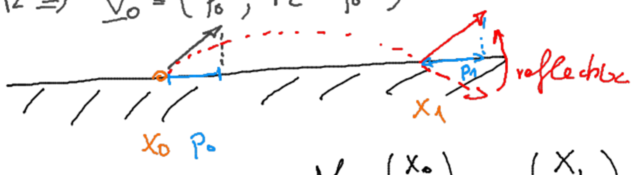
Task: determine $\tilde{M}_B$ $\rightsquigarrow$ Next step is then: do the same but for three walls:

$x_0, p_0 \Rightarrow R, x_c$
$\Rightarrow$ which wall hit?
$\Rightarrow$ intersection?



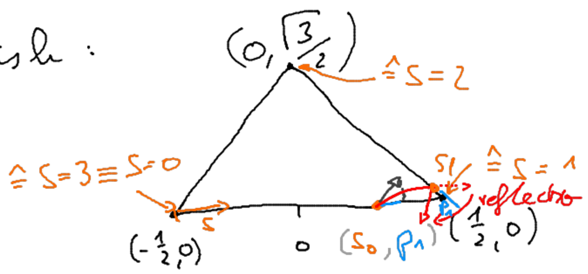
$\Rightarrow$ gives the M_B we are interested in!

Old task was : $|v| = \sqrt{2} \Rightarrow \underline{v_0} = \left(p_0, \sqrt{2 - p_0^2}\right)$



$$M_B \begin{pmatrix} x_0 \\ p_0 \end{pmatrix} = \begin{pmatrix} x_1 \\ p_1 \end{pmatrix}$$

New task :

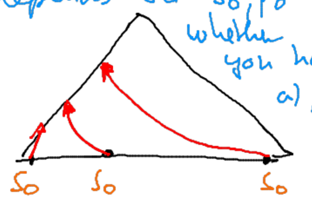


$$\boxed{M_B \begin{pmatrix} s_0 \\ p_0 \end{pmatrix} = \begin{pmatrix} s_1 \\ p_1 \end{pmatrix}}$$

3 cases : Assume $\underline{B > 0}$, i.e. all arcs are this way around

assume we start at lower wall

a) depends on s_0, p_0 whether you have a), b), or c)

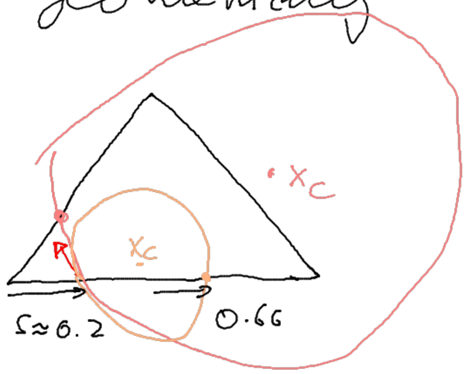


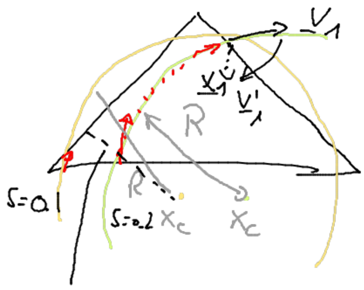
b)

c)

cases a), b), c):

Geometrically





distance wall – x_c

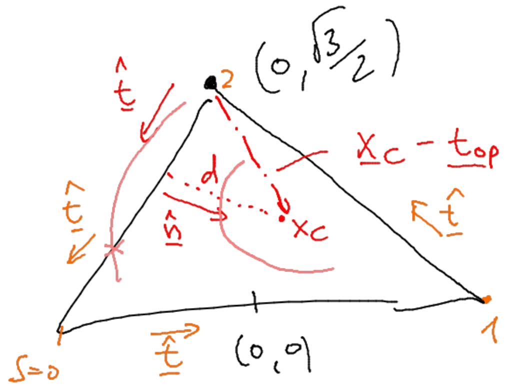
Let $s \in (0,1) \cup (1,2) \cup (2,3)$

Fix $B > 0$ $\left.\begin{array}{l} s \in (0,3) \setminus \{1,2\} \\ p \in (-\sqrt{2}, \sqrt{2}) \end{array}\right\}$ $\boxed{\mathcal{M}_B \binom{s}{p} = ?}$

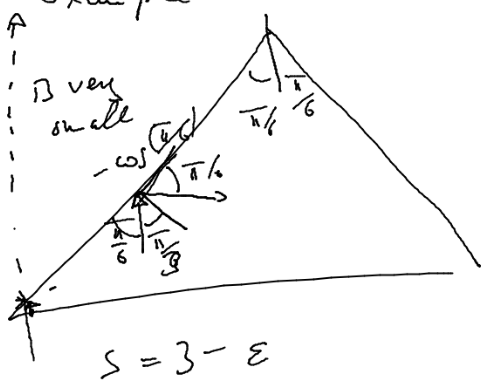
as a first step: only consider $s \in (0,1)$ (others follow from symmetry)

this is what you have to solve geometrically

$$(s, p) \xrightarrow{\text{give you}} \underline{x}_0, \underline{v}_0 \xrightarrow{\text{give you}} \underline{x}_c, R \xrightarrow{\text{gives you new intersection}} \underline{x}_1, \underline{v}_1 \xrightarrow{\text{gives you reflection}} \underline{x}_1, \underline{v}_1' \longrightarrow (s_1, p_1)$$



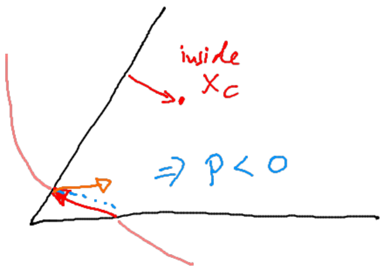
Example

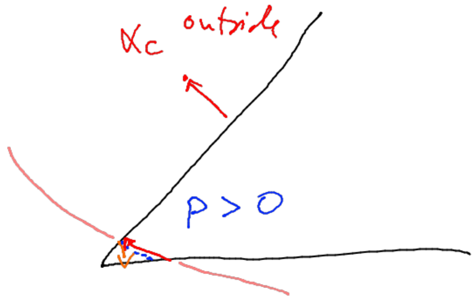


$s = 3 - \varepsilon$

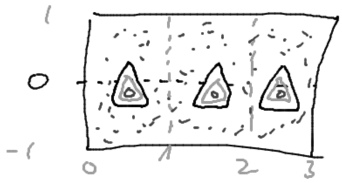
$p = -\cos(\pi/6)$

For the reflection, for example in case 1 check:

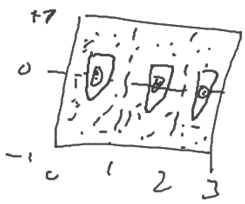




So for $B \approx 1.1$ the boundary map has to look like

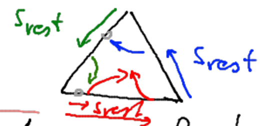


and for $B \approx 0.90$



$$\hat{\underline{t}}\left(\underline{v} - 2\hat{\underline{n}}(\hat{\underline{n}}\underline{v})\right) = \hat{\underline{t}}\underline{v}$$

To allow for $s \in (0,1) \cup (1,2) \cup (2,3)$ we can do the following:



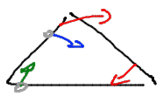
① split s into $s_{integer} + \boxed{s_{rest}}$ via $\boxed{\text{divmod}}$ function

② Do all calculations with $\boxed{s_{rest}} \in (0,1)$

③ at the very end, calculate

$$\underbrace{(s_{new} \boxed{+ s_{integer}})\ \boxed{\%\ 3}}_{\text{yields:}}$$

yields :



Regarding Algorithm: Calculate the following

1. R & x_c - centre of circle 
2. calculate distance $x_c \Leftrightarrow$ left wall 

   ↳ if $\boxed{d < R}$ go to (3) ↳ if $\boxed{d \geq R}$ → go to (5)
3. Calculate S for left wall (↗ formula) 

   There are two!

   ↳ if $s \in (2, 3)$ go to (4) else: go to (5)

   make sure to take correct solution ϑ & s
4. Calculate p from ϑ for wall I → return s & p
5. calculate distance $x_c \Leftrightarrow$ right wall 

   $\boxed{d < R}$ go to (6) else go to (8)
6. calculate s for right wall (↗ formula) 

   take correct solution

   ↳ if $s \in (1, 2)$ ↳ go to (7) else: go to (8)
7. calculate p from ϑ for wall II → return s & p
8. consider Wall 3