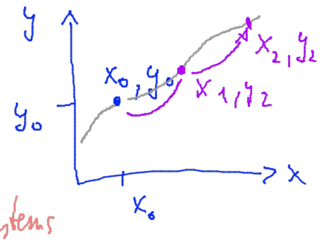
Dynamical systems: odes $\boxed{\frac{dy}{dx} = f(x,y)}$

time-1 mappings

$$y_{n+1} = f(x_n, y_n)$$

$\longrightarrow$ solve analytically or numerically for initial cond x_0



Here: We will solely work with Hamiltonian Systems

$$(*) \quad \frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{pmatrix}$$

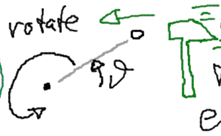
where the Hamiltonian $H(q,p,t) = T(p) + V(q,t)$

Examples: very special "Kicked" systems

Force $= -\frac{\partial}{\partial q} V$ "Conservative"

Kicked rotor

$$V(q,t) = \sum_{n\in\mathbb{Z}} K \cdot \sin\vartheta \cdot \delta(t - n\cdot T)$$



without any potential

then: add a kick: every time $t = 0, T, 2T, \ldots$

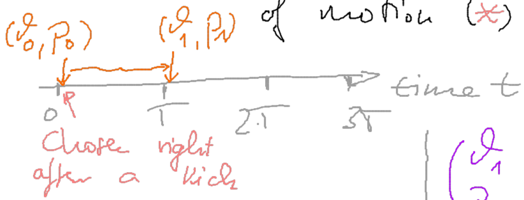
amount of torque applied is $\sim K\cdot\sin\vartheta$ where K - kicking strength

$$H = \frac{p^2}{2} + K\sum_{n\in\mathbb{Z}} \sin\vartheta\, \delta(t-n)$$

when we set $T = 1$

$\Rightarrow$ only one external Parameter: K

turns out: you can integrate the equations of motion $(*)$ for one period of the driving



$$\begin{pmatrix} \vartheta_1 \\ p_1 \end{pmatrix} = \begin{pmatrix} \vartheta_0 + \pi \cdot p_0 \\ p_0 + \frac{K}{\pi}\sin(\vartheta_0 + \pi\cdot p_0) \end{pmatrix}$$

$$M_K : \underbrace{\mathbb{R}\backslash_{[0,2\pi)}}_{\vartheta_n} \times \underbrace{\mathbb{R}}_{p_n} \to \mathbb{R}\backslash_{[0,2\pi)} \times \mathbb{R} \qquad = M_K\left(\begin{pmatrix} \vartheta_0 \\ p_0 \end{pmatrix}\right)$$

$\hat{=}$ $[0,2\pi)$ & periodic or call it "real numbers modulo 2π"

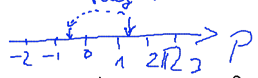
So we have: $\mathcal{M}_K : \mathbb{R}\backslash_{[0,2\pi)} \times \mathbb{R} \circlearrowleft$

where this is modulo 2π

$$\begin{pmatrix}\vartheta_0\\ p_0\end{pmatrix} \longmapsto \mathcal{M}_K\left(\begin{pmatrix}\vartheta_0\\ p_0\end{pmatrix}\right) = \begin{pmatrix}\vartheta_0 + \pi p_0\\ p_0 + \frac{K}{\pi}\sin(\vartheta_0 + \pi p_0)\end{pmatrix}$$

It turns out we can do a similar "modulo" for p:

they are the same

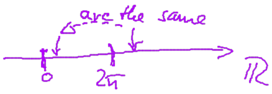


$$\mathcal{M}_K : \mathbb{R}\backslash_{[0,2\pi)} \times \mathbb{R}\backslash_{[-1,1]}$$

$\Rightarrow$ due to only looking just after the kick, we can restrict p like this

$$\mathbb{R}\backslash_{[-1,1]} = \{x \in \mathbb{R} \text{ where } x \sim y \text{ if } \exists n \in \mathbb{Z} \text{ such that } y = 2n + x\}$$

are the same



Why? Example:

$\vartheta_0 = \frac{\pi}{4}$ and $p = \frac{1}{2} \Rightarrow \vartheta_1 = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3}{4}\pi$

but if $p = 2\frac{1}{2} = \frac{5}{2} \Rightarrow \vartheta_1 = \frac{\pi}{4} + \pi \cdot \frac{5}{2} = \frac{11}{4}\pi = 2\frac{3}{4}\pi \mathrel{\hat=} \frac{3}{4}\pi$

NB: turning much faster!

① Task: Program $\begin{pmatrix}\vartheta_1\\ p_1\end{pmatrix} = \mathcal{M}_K\left(\begin{pmatrix}\vartheta_0\\ p_0\end{pmatrix}\right) \overset{(*)}{=} \begin{pmatrix}\vartheta_0 + \pi p_0\\ p_0 + \frac{K}{\pi}\sin(\vartheta_0 + \pi p_0)\end{pmatrix}$

where the modulo is applied correctly

that is: use the % operator in MATLAB to achieve $\vartheta_1 \in [0, 2\pi)$ and $p_1 \in [-1, 1]$

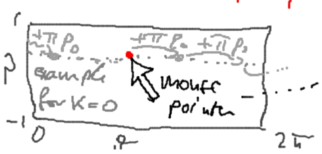
(**) as a check: this modulo must be such that

$$\begin{pmatrix} 0 \to 0 \\ \pi \to \pi \\ 2\pi \to 0 \\ 3\pi \to \pi \\ -\frac{1}{2}\pi \to \frac{3}{2}\pi \end{pmatrix} \qquad \begin{pmatrix} -\frac{1}{2} \to -\frac{1}{2} \\ 0 \to 0 \\ \frac{1}{2} \to \frac{1}{2} \\ \frac{3}{2} \to -\frac{1}{2} \\ -\frac{5}{2} \to -\frac{1}{2} \end{pmatrix}$$

① program (*) including the % from (**)

② Choose $K = 2.4$ and a positive $N = 1000$ and write a small program

window to click in



"click"

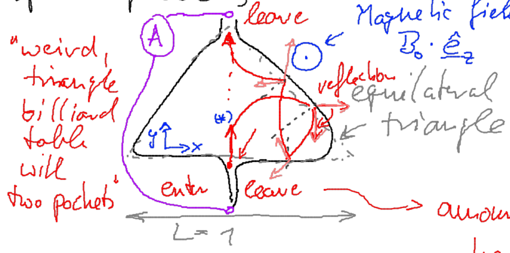
③ Plot (□)

$$\begin{pmatrix}\vartheta_0\\ p_0\end{pmatrix} \to \begin{pmatrix}\vartheta_1\\ p_1\end{pmatrix} \to \begin{pmatrix}\vartheta_2\\ p_2\end{pmatrix} \to \ldots \to \begin{pmatrix}\vartheta_{1000}\\ p_{1000}\end{pmatrix} \ (\square)$$

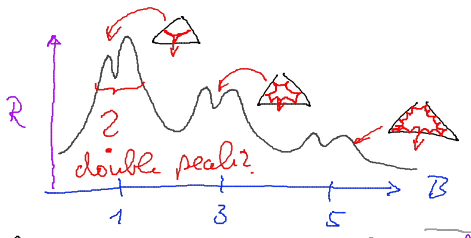
Overall aim of project

transport problems

electrons • e^- in a constant, perpendicular magnetic field confined to 2d and confined to specific region in 2d.



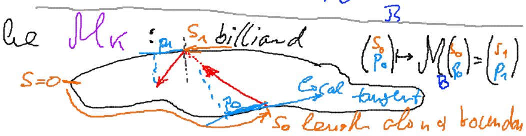
amount of e^- leaving here give rise to electric resistance $R(B)$



NB: radius of circles here(*) $\sim \frac{1}{B}$

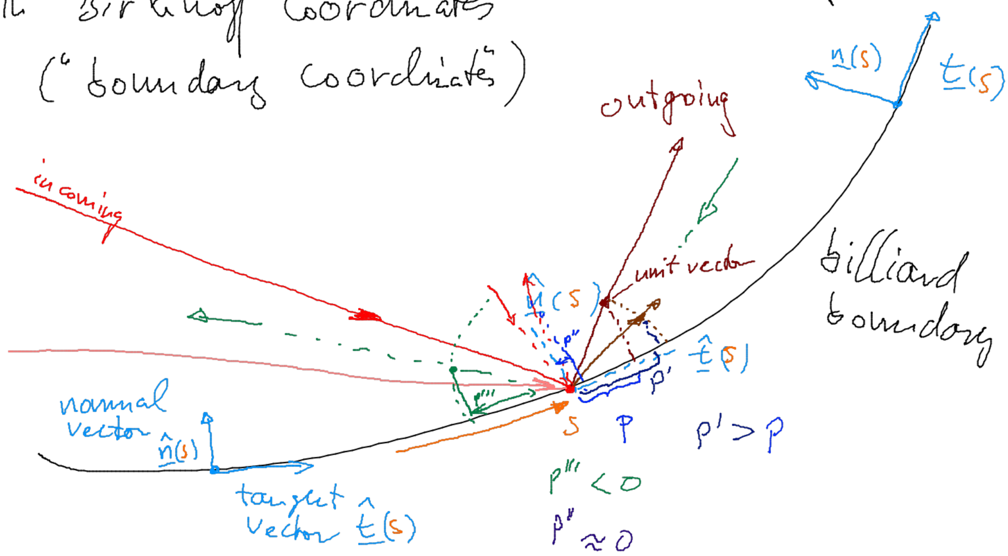
Link to time-1 maps like M_K:

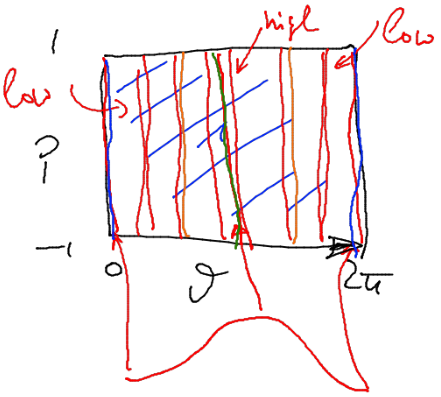
Birkhoff-coordinates
(special type of Poincaré sections)

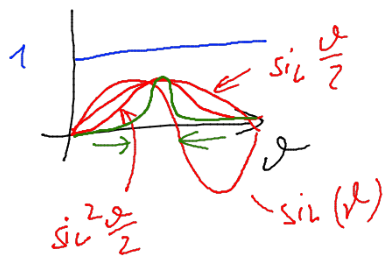


$\binom{s_0}{p_0} \mapsto M\binom{s_0}{p_0} = \binom{s_1}{p_1}$

Meaning of s & p for M-boundary map in Birkhoff coordinates ("boundary coordinates")

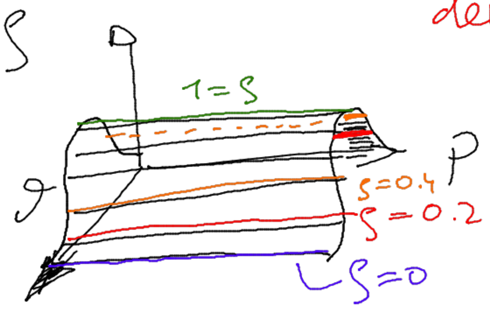






$\sin^2 \vartheta/2 = \rho(\vartheta, p)$

$\sin^{10}\frac{\vartheta}{2}$



density $\rho : [0, 2\pi) \times (-1, 1) \longrightarrow [0, \infty)$

for example

$(\vartheta, p) \longmapsto \rho(\vartheta, p) = 1$

or another example

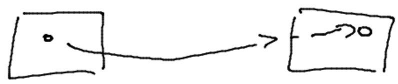
$\rho(\vartheta, p) = \sin^2 \frac{\vartheta}{2}$

As a first step in understanding how densities are mapped, we have to create some examples!

for example

$\rho(\vartheta, p) = \sin^{10}(\vartheta/2) \cdot \cos^{10}(\pi \cdot p/2)$

Standard Map $\qquad \mathcal{M}_K : [0,2\pi) \times (-1,1) \longrightarrow [0,2\pi) \times (-1,1)$



$\downarrow$

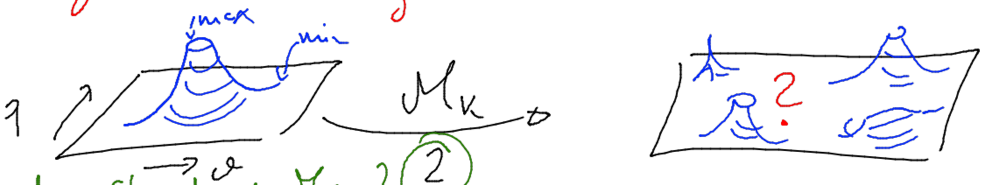
apply this to where the user clicked $\longrightarrow$ $\boxed{\text{one}}$ initial condition $\longrightarrow$ iterate it using $\mathcal{M}_K$

Now: change perspective: Do not think of single trajectories (ϑ_0, p_0) but of ensemble (i.e. many of them)

will all end up somewhere

$N=1$

More abstractly: densities of initial conditions



Aim: understand what happens to densities under Standard Map? (2)

Next steps:

(1) define a density function

$$\varrho : [0, 2\pi) \times (-1, 1) \longrightarrow [0, \infty)$$

$$(\vartheta, p) \longmapsto \varrho(\vartheta, p) = \underline{\sin^{10}\left(\frac{\vartheta}{2}\right) \cdot \cos^{10}\left(\pi \frac{p}{2}\right)}$$

(2) plot it in the same coordinate system: a contour plot of (1)

(3) Find out: what is the inverse of the Standard Map

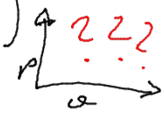
$$\mathcal{M}_K \begin{pmatrix} \vartheta \\ p \end{pmatrix} = \begin{pmatrix} \vartheta + \pi p \\ p + \frac{K}{\pi} \sin(\vartheta + \pi p) \end{pmatrix}$$

$$\mathcal{M}_K^{-1} \begin{pmatrix} \vartheta \\ p \end{pmatrix} = \begin{pmatrix} ? \end{pmatrix}$$ such that for arbitrary K $$\mathcal{M}_K\left(\mathcal{M}_K^{-1}\begin{pmatrix} \vartheta \\ p \end{pmatrix}\right) = \begin{pmatrix} \vartheta \\ p \end{pmatrix}$$

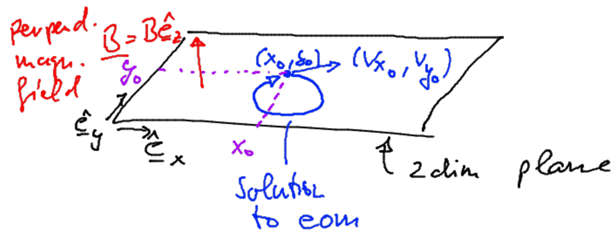
(4) Define a new function $\varrho_1 = \varrho \circ \mathcal{M}_K^{-1}$

that means $\varrho_1(\vartheta, p) = \varrho\left(\mathcal{M}^{-1}_{K_\vartheta}(\vartheta, p), \mathcal{M}^{-1}_{K_p}(\vartheta, p)\right)$

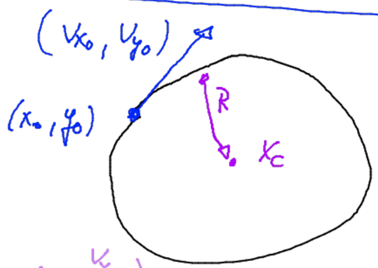
(5) Do a <u>contour plot</u> like (2) but for (4)



Equations of motion



$$\boxed{\begin{aligned} \dot{x} &= v_x & \dot{v}_x &= 2\sqrt{2}B\, v_y \\ \dot{y} &= v_y & \dot{v}_y &= -2\sqrt{2}B\, v_x \end{aligned}}\ \text{eom}$$



Using solution of $\boxed{\text{eom}}$:

$$R = R(B) = \underline{\underline{\frac{|V_0|}{2\sqrt{2}|B|}}} \sim \frac{1}{|B|}$$

$$x_c = x_c(x_0, y_0, v_{x_0}, v_{y_0}, B) = \underline{\underline{\left(x_0 + \frac{v_{y_0}}{2\sqrt{2}B},\ y_0 - \frac{v_{x_0}}{2\sqrt{2}B}\right)}}$$

(you solve it for t)

alternatively:

$$\left\{\begin{aligned} x(t) &= A_1 + B_1 \sin(2\sqrt{2}B\,t) + C_1 \cos(2\sqrt{2}B\,t) \\ y(t) &= A_2 + B_2 \sin(2\sqrt{2}B\,t) + C_2 \cos(2\sqrt{2}B\,t) \\ v_x(t) &= 2\sqrt{2}B \cdot B_1 \cos(2\sqrt{2}B\,t) + \dots C_1 \dots \\ v_y(t) &= 2\sqrt{2}B \cdot B_2 \cos(2\sqrt{2}B\,t) + \dots C_2 \dots \end{aligned}\right\}$$

at $t = 0$

$A_1 + C_1 \stackrel{!}{=} x_0$

$A_2 + C_2 \stackrel{!}{=} y_0$

$2\sqrt{2}B \cdot B_1 \stackrel{!}{=} v_{x_0}$

$2\sqrt{2}B \cdot B_2 \stackrel{!}{=} v_{y_0}$

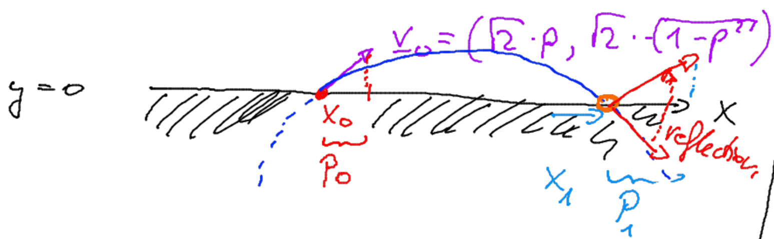
$x(t) = x_c + R\cos(\omega t + \varphi)$

$y(t) = y_c + R\sin(\omega t + \varphi)$

$v_x = \dots,\ v_y = \dots$

with: $\omega := -2\sqrt{2} \cdot B$, $R := \frac{|V_0|}{|\omega|}$, $\varphi = \arctan\left(\frac{y_0 - y_c}{x_0 - x_c}\right) = \arctan(-v_{x_0}/\omega \,/\, v_{y_0}/\omega)$ ⇒ MATLAB "atan2", $(x_c, y_c) = \left(x_0 - \frac{v_{y_0}}{\omega},\ y_0 + \frac{v_{x_0}}{\omega}\right)$

what we need is a boundary map:



input: $x_0,\ p_0 \in (-1,1)$

yield initial cond

$(x_0, 0, \sqrt{2}\cdot p, \sqrt{2}\cdot\sqrt{1-p^2})$

(see first page)

yields circular orbit with radius R and centre x_c

yields another intersection with line $y=0$

do a reflection $\to (x_1, p_1)$

gives us a mapping

$$\tilde{M}_B : \mathbb{R}\times(-1,1) \longrightarrow \mathbb{R}\times(-1,1)$$

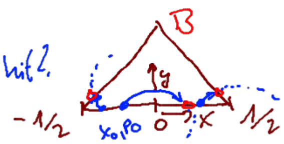
$$(x_0, p_0) \longmapsto \tilde{M}_B(x_0,p_0) = (x_1,p_1)$$

Task: determine $\tilde{M}_B$ $\rightsquigarrow$ Next step is then: do the same but for three walls:

$x_0, p_0 \Rightarrow R, x_c$

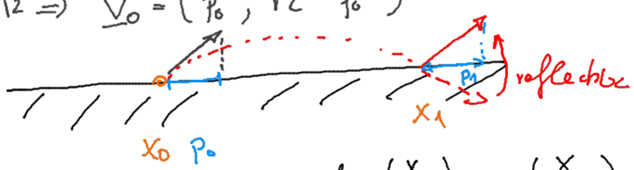
$\Rightarrow$ which wall hit?

$\Rightarrow$ intersection?



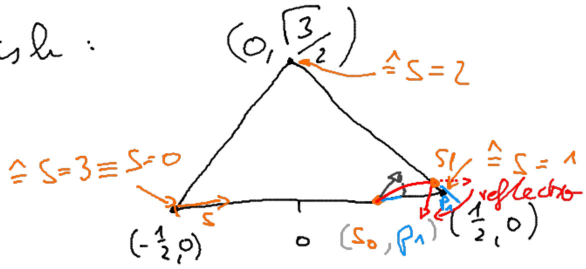
$\Rightarrow$ gives the M_B we are interested in!

Old task was: $|v| = \sqrt{2} \Rightarrow \underline{v_0} = (p_0, \sqrt{2-p_0^2})$



$$M_B \binom{x_0}{p_0} = \binom{x_1}{p_1}$$

New task:

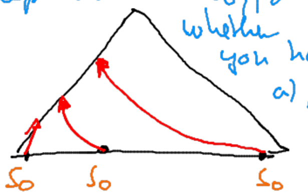


$$\boxed{M_B \binom{s_0}{p_0} = \binom{s_1}{p_1}}$$

3 cases: Assume $\underline{B > 0}$, i.e. all arcs are this way around

assume we start at lower wall

a) depends on s_0, p_0 whether you have a), b), or c)

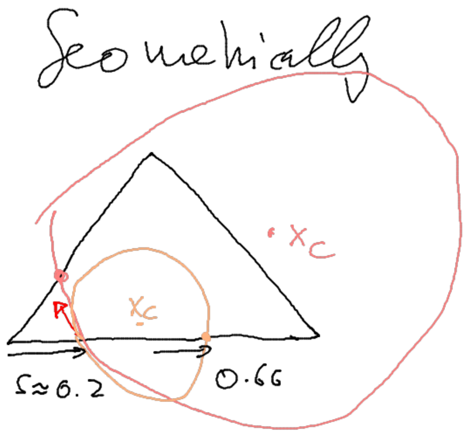


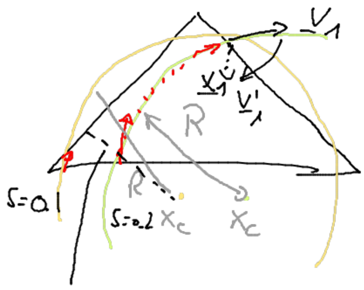
b)

c)

cases a), b), c):

Geometrically





distance wall $- x_c$

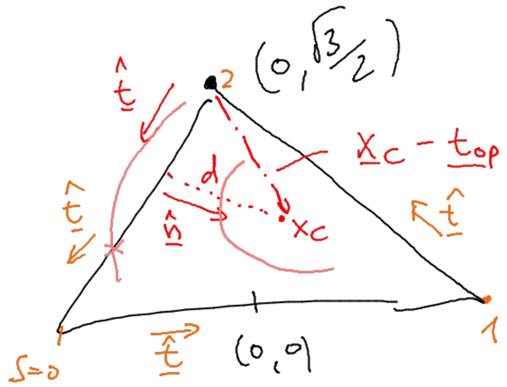
Let $s \in (0,1) \cup (1,2) \cup (2,3)$

Fix $B > 0$: $\left.\begin{array}{l} s \in (0,3) \setminus \{1,2\} \\ p \in (-\sqrt{2}, \sqrt{2}) \end{array}\right\}$ $\boxed{\mathcal{M}_B \binom{s}{p} = ?}$

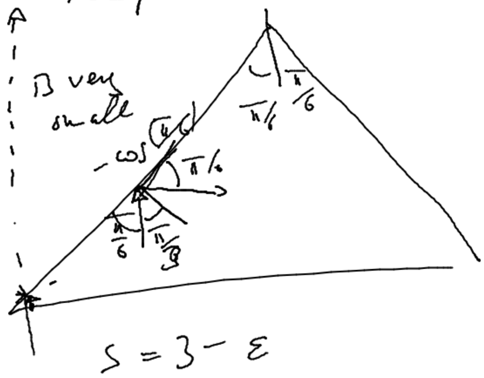
as a first step: only consider $s \in (0,1)$ (others follow from symmetry)

this is what you have to solve geometrically

$$(s,p) \xrightarrow{\text{give you}} \underline{x}_0, \underline{v}_0 \xrightarrow{\text{give you}} \underline{x}_c, R \xrightarrow{\text{gives you new intersection}} \underline{x}_1, \underline{v}_1 \xrightarrow{\text{gives you reflection}} \underline{x}_1, \underline{v}_1' \longrightarrow (s_1, p_1)$$



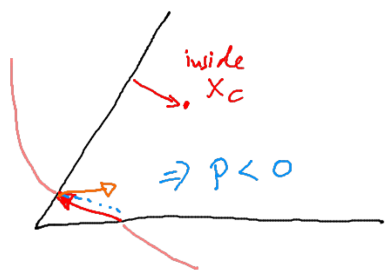
Example

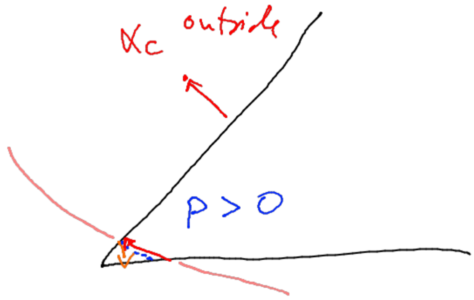


$s = 3 - \varepsilon$

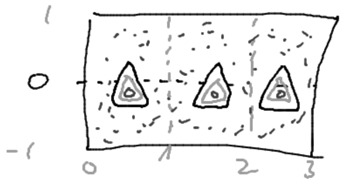
$p = -\cos(\pi/6)$

For the reflection, for example in case (1) check:

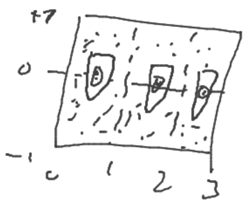




So for $\underline{B \approx 1.1}$ the boundary map has to look like

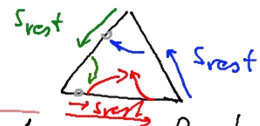


and for $B \approx 0.90$



$$\hat{\underline{t}}\left(\underline{v} - 2\hat{\underline{n}}(\hat{\underline{n}}\underline{v})\right) = \hat{\underline{t}}\underline{v}$$

To allow for $s \in (0,1) \cup (1,2) \cup (2,3)$ we can do the following:



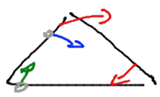
① split s into $s_{integer} + \boxed{s_{rest}}$ via $\boxed{\text{divmod}}$ function

② Do all calculations with $\boxed{s_{rest}} \in (0,1)$

③ at the very end, calculate

$$\underbrace{(s_{new} \boxed{+ s_{integer}}) \boxed{\% \ 3}}_{\text{yields:}}$$

yields:



Regarding Algorithm: Calculate the following

(1) R & x_c - centre of circle

(2) calculate distance $x_c \leftrightarrow$ left wall

↳ if $\boxed{d < R}$ go to (3) ↳ if $\boxed{d \geq R}$ → go to (5)

(3) Calculate s for left wall (↗ formula)

There are two!

make sure to take correct solution ϑ & s

↳ if $s \in (2, 3)$ go to (4) else: go to (5)

(4) Calculate p from ϑ for wall I → return s & p

(5) calculate distance $x_c \leftrightarrow$ right wall

$\boxed{d < R}$ go to (6) else go to (8)

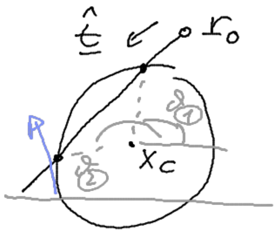
(6) Calculate s for right wall (↗ formula)

take correct solution

↳ if $s \in (1, 2)$ ↳ go to (7) else: go to (8)

(7) calculate p from ϑ for wall II → return s & p

(8) consider Wall 3

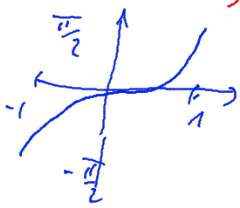


$$\frac{\pi}{2}+\left(\frac{\pi}{2}-\vartheta_{MATLAB}\right) \overset{②}{=} \pi - \vartheta_{MATLAB} \qquad (*)$$

① ϑ_{MATLAB}

Some value $= \sin\vartheta$

$\vartheta = ?$



MATLAB

$\vartheta_{MATLAB} = \mathrm{asin}(\text{some value})$

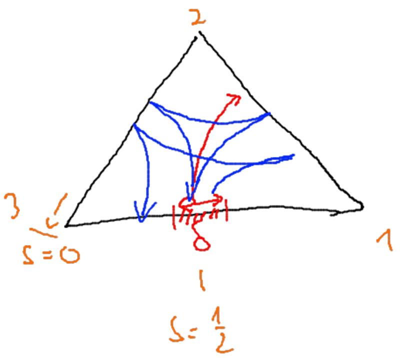
$-\frac{\pi}{2} \geq \vartheta_{MATLAB} > 0$

$$\left.\begin{array}{ll} ① & 2\pi + \vartheta_{MATLAB} \\ ② & \pi - \vartheta_{MATLAB} \end{array}\right\}$$

① $\vartheta_{MATLAB} < 0$

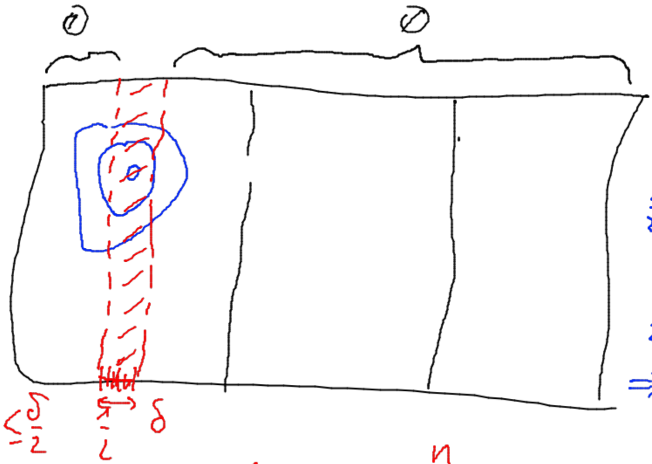
② $\pi - \vartheta_{MATLAB} > 0$ (**)

Task: replace solve with your solution and employ (*) & (**)



density such that

$\neq 0$ if $|s - \frac{1}{2}| \leq \frac{\delta}{2}$



very last thing:

$$\int_{\frac{1}{2}-\frac{\delta}{2}}^{\frac{1}{2}+\frac{\delta}{2}} \int_{-1}^{1} T_\rho^3 \, dp \, ds$$

$\Rightarrow R(B)$

$$\rho(s,p) = \begin{cases} 0 & |s-\frac{1}{2}| > \frac{\delta}{2} \\ f_n\left(\frac{s-(\frac{1}{2}-\frac{\delta}{2})}{\delta}\right) & |s-\frac{1}{2}| \leq \frac{\delta}{2} \end{cases}$$

$\left(\sqrt{x(1-x)}\right)^n = f_n(x)$ $\quad n = 1, 3, 5, 7, \ldots$

$\frac{1}{2}-\frac{\delta}{2}$ $\quad \frac{1}{2}+\frac{\delta}{2}$

i. plot contour of $\rho(s,p)$ (hint: maybe meaningful only on $s \in (\frac{1}{2}-\frac{\delta}{2}, \frac{1}{2}+\frac{\delta}{2})$)

ii. $T\rho = \rho \circ \mathcal{M}^{-1}$ as well as $\boxed{T^3\rho} = \rho \circ \mathcal{M}^{-1} \circ \mathcal{M}^{-1} \circ \mathcal{M}^{-1} \rightarrow$ contour plot