Well-Ordering and Induction

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Axiom 1. Every nonempty subset S of the natural numbers $\mathbb{N}$ has a minimum. This is called the well-ordering principle.

Axiom 2. Let S be a subset of $\mathbb{N}$ such that $1 \in S$ and $n \in S$ implies that $n+1 \in S$. Then S is equal to $\mathbb{N}$. This is called the induction principle.

Theorem 1. *The well-ordering principle is equivalent to the induction principle.*

Proof. Both proofs are by contradiction. First, let us show that the well-ordering principle implies the induction principle. Let S be a subset of $\mathbb{N}$ such that $1 \in S$ and $n \in S$ implies that $n+1 \in S$. We want to prove that $S = \mathbb{N}$. Suppose that it is not true then the complement of S defined as $T = \mathbb{N} - S$ is nonempty, and by the well-ordering principle has a minimum $t_{\min} \in T$. Since $1 \notin T$ then $1 < t_{\min}$ and $t_{\min} - 1 \in \mathbb{N}$. Furthermore, $t_{\min} - 1 < t_{\min}$ which implies that $t_{\min} - 1 \notin T$ since for every $t \in T$ we have $t_{\min} \leq t$. Consequenlty, $t_{\min} - 1$ should belong to the complement of T which is $\mathbb{N} - T = \mathbb{N} - (\mathbb{N} - S) = S$ so $t_{\min} - 1 \in S$. The inductive property of S tell us that $(t_{\min} - 1) + 1 = t_{\min} \in S$, which is in contradiction with $t_{\min} \in T$ and we should have $S = \mathbb{N}$.

Now, let us prove that the induction principle implies the well-ordering principle. Let S be a nonempty subset of $\mathbb{N}$. We want to show that it has a minimum. Suppose that this is not true then $1 \notin S$ otherwise S would have a minimum since for every $s \in S$ we have $1 \leq s$. This means that $S \subset \mathbb{N} - \{1\}$. Now, $2 \notin S$ otherwise 2 would be a minimum for S since for every $s \in S$, we know that $s \in \mathbb{N} - \{1\}$ which implies $2 \leq s$. This leads to $S \subset \mathbb{N} - \{1, 2\}$. Let us define $T = \{t \mid S \subset \mathbb{N} - \{1, 2, \ldots, t\}\}$. We observe that $1 \in T$. Furthermore, if $t \in T$ then $t+1 \in T$. Indeed, $t \in T$ tells us that $S \subset \mathbb{N} - \{1, 2, \ldots, t\}$. We claim that $t+1 \notin S$ otherwise for every $s \in S$ we know that $s \in \mathbb{N} - \{1, 2, \ldots, t\}$, implying that $t+1 \leq s$, which is in contraction with our assumption that S does not have any minimum. By the induction principle we conclude that $T = \mathbb{N}$. Since S is nonempty, there is an $s \in S$. Moreover, $T = \mathbb{N}$ and we have $s \in T$, which leads us to $S \subset \mathbb{N} - \{1, 2, \ldots, s\}$ which implies that $s \notin S$. This is a contradiction so S must have a minimum. ■