Completion of a Metric Space

Written by Hosein Rahnama

Definition 1. A mapping $i : M \to N$ is said to be an isometry iff it is surjective and for every $x, y \in M$ we have $d_M(x, y) = d_N(i(x), i(y))$.

Definition 2. A mapping $h : M \to N$ is said to be a homeomorphism iff it is a continuous bijection which also has a continuous inverse. Homeomorphisms are also called topological mappings.

Theorem 1. *If $i : M \to N$ is an isometry then it is injective and uniformly continuous. Furthermore, it has an inverse which is also an isometry. This implies that an isometry is a homeomorphism.*

Definition 3. Two metric spaces (M, d_M) and (N, d_N) are said to be isometric, if there exists an isometry $i : M \to N$ between them.

Theorem 2. *An isometry preserves all topological and metric related properties. More specifically, if $i : M \to N$ is an isometry and $A \subset M$ then openness, closedness, compactness, connetedness or path-connectedness of A implies that its image $i(A)$ has the same topological property. In addition, for metric related properties of A, we have that if A is bounded or complete then its image $i(A)$ is also bounded or complete. Furthermore, if $a : \mathbb{N} \to M$ is a convergent or Cauchy sequence then its image sequence $i \circ a : \mathbb{N} \to N$ is also convergent or Cauchy.*

Definition 4. A completion of a metric space (M, d_M) is a complete metric space (C, d_C) which has a metric subspace (N, d_N) which is dense in C and is isometric with M. It is in this sense that we say C is the smallest complete metric space containing M.

The following two lemmas are frequently used in the proof of the completion theorem.

Lemma 3. *Let $K \in \mathbb{N}$, $a : \mathbb{N} \to \mathbb{R}$ and $b : \mathbb{N} \to \mathbb{R}$ be convergent sequences. If for all $n \geq K$ we have $a_n > b_n$ then $\lim_{n\to\infty} a_n \geq \lim_{n\to\infty} b_n$. That is we can take limit from both sides of an strict inequality and replace it with a non-strict one.*

Lemma 4. *Let $w, x, y, z \in M$. Then we have the inequality $|d(w, x) - d(y, z)| \leq d(w, y) + d(x, z)$.*

Theorem 5. ***Completion of a metric space**. Every metric space has a completion. Furthermore, this completion is unique up to an isometry. This means that any two completions are isometric. This is called the universal property of a completion.*

Proof. The proof is a little bit lengthy so let us first sketch different steps of the proof as follows.

1. Let $\mathscr{C}$ to be the set of all Cauchy sequences in M. Define the relation of being co-Cauchy on $\mathscr{C}$. Show that this relation is an equivalence relation on $\mathscr{C}$. Define C as the set of all resulting equivalence classes. Show that the mapping $d_C : C \times C \to \mathbb{R}$ defined by $d_C([a], [b]) := \lim_{n\to\infty} d_M(a_n, b_n)$ is well-defined and a metric on C.

2. Consider the mapping $i : M \to i(M) \subset C$ which takes every point $x \in M$ to the equivalence class $[a]$ corresponding to the constant sequence $a : \mathbb{N} \to M$ defined as $a_n = x$. This makes sense since every constant sequence is Cauchy. Verify that M and $i(M)$ are isometric and $i(M)$ is dense in C that is $\operatorname{clr} i(M) = C$. Show that C is complete. Verify that $i(M) = C$ if and only if M is complete, meaning that this procedure is trivial when our original space is complete.

3. The last step is to show that every two completions are isometric. Let (C, d_C) and (E, d_E) be any two completions. Then, there are isometries $i : M \to i(M) \subset C$ and $j : M \to j(M) \subset E$ such that $\operatorname{clr} i(M) = C$ and $\operatorname{clr} j(M) = E$. Verify that $i(M)$ and $j(M)$ are isometric by the map $\mathfrak{i} := j \circ i^{-1} : i(M) \to j(M)$. Take any equivalence class $[a] \in C$ and let $\mathcal{A} : \mathbb{N} \to i(M)$ be a sequence of equivalence classes converging to $[a]$. Define $f([a]) := \lim_{n\to\infty}(\mathfrak{i} \circ \mathcal{A})_n$. Show that $f : C \to E$ is well-defined and is an isometry. Verify that $f|_{i(M)} = \mathfrak{i}$. Furthermore, for every other isometry $g : C \to E$ having the property $g|_{i(M)} = \mathfrak{i}$ we have $f = g$, meaning that $f : C \to E$ is unique in this sense.

Let us carry out the steps in more detail. We say that two Cauchy sequences $a : \mathbb{N} \to M$ and $b : \mathbb{N} \to M$ are co-Cauchy iff $\lim_{n\to\infty} d_M(a_n, b_n) = 0$ and write $a \sim b$. This is an equivalence relation. It is reflexive since $\lim_{n\to\infty} d_M(a_n, a_n) = \lim_{n\to\infty} 0 = 0$. It is symmetric because symmetry of d_M implies $\lim_{n\to\infty} d_M(b_n, a_n) = \lim_{n\to\infty} d_M(a_n, b_n) = 0$. Furthermore, suppose that $\lim_{n\to\infty} d_M(a_n, b_n) = 0$ and $\lim_{n\to\infty} d_M(b_n, c_n) = 0$. By the positive-definitness and triangle inequality we have that $0 \leq d_M(a_n, c_n) \leq d_M(a_n, b_n) + d_M(b_n, c_n)$ and by Lemma 3 we are allowed to take limits that gives us $\lim_{n\to\infty} d_M(a_n, c_n) = 0$ which means the co-Cauchy relation is transitive. Equivalence relations are just a mathematical way to consider a group of elements of a set as identical. Define $\mathscr{C}$ to be the set of all Cauchy sequences in M, that is

$$\mathscr{C} = \{a : \mathbb{N} \to M \,|\, \forall \epsilon > 0,\ \exists K \in \mathbb{N},\ \forall m, n \geq K \implies d_M(a_n, a_m) < \epsilon\} \tag{1}$$

and let $\sim$ be the equivalence relation of being co-Cauchy on $\mathscr{C}$. An equivalence class is given by $[a] = \{b \in \mathscr{C} \,|\, b \sim a\}$. Let our candidate for a completion to be the set of all these equivalence classes

$$C := \frac{\mathscr{C}}{\sim} = \{[a] \,|\, a \in \mathscr{C}\}. \tag{2}$$

This equivalence relation breaks $\mathscr{C}$ into disjoint equivalence classes, which literally means that

$$\mathscr{C} = \bigsqcup_{[a] \in C} [a]. \tag{3}$$

This whole procedure means that we are inclined to see all co-Cauchy sequences as identical in our candidate space C. The metric d_C we introduced above is natural in the sense that we are just measuring the distance of the tails of Cauchy sequences. First let us verify the well-definedness of d_C. We must show that the limit introduced in its definition exists and it does not depend on the choice of a representative of the equivalence classes. The sequence $d_M \circ (a \times b) : \mathbb{N} \to \mathbb{R}$ is Cauchy since by Lemma 4 we have

$$|d_M(a_m, b_m) - d_M(a_n, b_n)| \leq d_M(a_m, a_n) + d_M(b_m, b_n) \tag{4}$$

and $a : \mathbb{N} \to \mathbb{R}$, $b : \mathbb{N} \to \mathbb{R}$ are Cauchy. Completeness of $\mathbb{R}$ implies that $d_M \circ (a \times b) : \mathbb{N} \to \mathbb{R}$ converges so the limit $\lim_{n\to\infty} d_M(a_n, b_n)$ exists. Let $c : \mathbb{N} \to \mathbb{R}$ and $d : \mathbb{N} \to \mathbb{R}$ be two sequences such that $c \in [a]$ and $d \in [b]$ then $L_1 := \lim_{n\to\infty} d_M(a_n, b_n) = \lim_{n\to\infty} d_M(c_n, d_n) =: L_2$. This means that if you want to measure the distance between the tails of two Cauchy sequences you can replace them with their co-Cauchy counterparts. This is intuitively obvious! To make this precise, note that

$$\begin{aligned} |L_1 - L_2| &\leq |L_1 - d_M(a_n, b_n)| + |d_M(a_n, b_n) - d_M(c_n, d_n)| + |d_M(c_n, d_n) - L_2| \\ &\leq |d_M(a_n, b_n) - L_1| + \big(d_M(a_n, c_n) + d_M(b_n, d_n)\big) + |d_M(c_n, d_n) - L_2|\,, \end{aligned} \tag{5}$$

where we used the triangle inequality and Lemma 4. Using Lemma 3, taking limits from both sides and using the continuity of absolute value $|\cdot|$ leads us to $L_1 = L_2$, as desired. The next step is to justify that d_C is a metric. Let $a : \mathbb{N} \to M$, $b : \mathbb{N} \to M$, $c : \mathbb{N} \to M$ be Cauchy sequences. It is clear that $d_M(a_n, b_n) \geq 0$ and using Lemma 3 gives us $d_C([a], [b]) \geq 0$ so d_C is positive. It is also definite since we have

$$[a] = [b] \iff a \sim b \iff \lim_{n\to\infty} d_M(a_n, b_n) = 0 \iff d_C([a], [b]) = 0. \tag{6}$$

It is symmetric since d_M is symmetric

$$d_C([a], [b]) = \lim_{n\to\infty} d_M(a_n, b_n) = \lim_{n\to\infty} d_M(b_n, a_n) = d_C([b], [a]). \tag{7}$$

It also obeys the triangle inequality since d_M obeys this inequality and by Lemma 3 we can take limits from both side of an inequality to obtain

$$d_M(a_n, c_n) \leq d_M(a_n, b_n) + d_M(b_n, c_n) \implies d_C([a], [c]) \leq d_C([a], [b]) + d_C([b], [c]). \tag{8}$$

Now, consider the mapping introduced in step 2. Choose any two elements of M, say x and y. Let $a : \mathbb{N} \to M$ and $b : \mathbb{N} \to M$ be two constant sequences such that $a_n = x$ and $b_n = y$. By construction of i, we have $i(x) = [a]$ and $i(y) = [b]$. Then, it is easily verified that i is an isometry

$$d_C(i(x), i(y)) = d_C([a], [b]) = \lim_{n\to\infty} d_M(a_n, b_n) = \lim_{n\to\infty} d_M(x, y) = d_M(x, y). \tag{9}$$

Furthermore, it is trivial that $i : M \to i(M)$ is surjective. Thus M and $i(M)$ are isometric. By Theorem 2, this means that these two metric space are indistinguishable and $i(M)$ is a copy of M, having all original information about M, living inside C. Next, we show that $i(M)$ is dense in C. Let us see what does it really mean. We should have the set equality $\operatorname{clr} i(M) = C$. This is equivalent to $\operatorname{clr} i(M) \subset C$ and $C \subset \operatorname{clr} i(M)$. The former follows immediately from $i(M) \subset C$ and the latter is equivalent to that every equivalence class $[a] \in C$ is a limit point of $i(M)$. This means that every open ball $B_C([a], r)$ contains an element of $i(M)$. More precisely, we want to prove that

$$\forall [a] \in C,\ \forall r > 0,\ \exists [b] \in i(M),\ d_C([b], [a]) < r. \tag{10}$$

This is equivalent to

$$\forall [a] \in C,\ \forall r > 0,\ \exists y \in M,\ \lim_{n\to\infty} d_M(y, a_n) < r. \tag{11}$$

Since $a : \mathbb{N} \to M$ is a Cauchy sequence then there is a K such that for all $n, m \geq K$ we have $d_M(a_m, a_n) < \frac{r}{2}$. Take $y = a_K$ so that for $n \geq K$ we have $d_M(y, a_n) < \frac{r}{2}$. Using Lemma 3, we can pass to limits $\lim_{n\to\infty} d_M(y, a_n) \leq \frac{r}{2} < r$.

We are ready to show that C is complete. For this purpose, we should consider an arbitrary Cauchy sequence of equivalence classes. To adopt a notation consistent with our previous development, let $\mathsf{a} : \mathbb{N} \times \mathbb{N} \to M$ be a sequence of sequences or a double sequence. Clearly, $\mathsf{a}_{m,n}$ is in M and by $\mathsf{a}_{m,\cdot}$ we mean the sequence obtained by puting m in the first argument of a. Keeping this in mind, $\mathcal{A} : \mathbb{N} \to C$, defined by $\mathcal{A}_n := [\mathsf{a}_{n,\cdot}]$, is a sequence of equivalence classes and $\mathcal{A}_n$ is an equivalence class itself. Since $i(M)$ is dense in C, for each $\mathcal{A}_n \in C$ there is a $\mathcal{B}_n \in i(M)$ such that $d_C(\mathcal{A}_n, \mathcal{B}_n) < \frac{1}{n}$ for all $n \in \mathbb{N}$. We can show that $\mathcal{B} : \mathbb{N} \to i(M)$ is Cauchy. Indeed, we have

$$d_C(\mathcal{B}_m, \mathcal{B}_n) \leq d_C(\mathcal{B}_m, \mathcal{A}_m) + d_C(\mathcal{A}_m, \mathcal{A}_n) + d_C(\mathcal{A}_n, \mathcal{B}_n) < \frac{1}{m} + d_C(\mathcal{A}_m, \mathcal{A}_n) + \frac{1}{n}, \tag{12}$$

where we made use of the triangle inequality and the point mentioned above. Now, let ϵ be given. For the sequence $n \mapsto \frac{1}{n}$ converges to 0, there is a $K_1 \in \mathbb{N}$ such that for all $n \geq K_1$ we have $\frac{1}{n} < \frac{\epsilon}{3}$. Indeed, the existence of K_1 is guaranteed by the Archimedean property of real numbers, which states that for every real number there is an integer, which is greater than it. Consequently, for the real number $\frac{3}{\epsilon}$ there is a K_1 such that $K_1 > \frac{3}{\epsilon}$, which implies that $\frac{1}{n} < \frac{1}{K_1} < \frac{\epsilon}{3}$. In addition, $\mathcal{A} : \mathbb{N} \to C$ is Cauchy so there is a K_2 such that for all $n, m \geq K_2$ we have $d_C(\mathcal{A}_m, \mathcal{A}_n) < \frac{\epsilon}{3}$. Consequently, if we choose $K = \max\{K_1, K_2\}$ then for all $n, m \geq K$ we have $d_C(\mathcal{B}_m, \mathcal{B}_n) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$ which implies that $\mathcal{B} : \mathbb{N} \to i(M)$ is Cauchy. Now, look at the sequence $b : \mathbb{N} \to M$ obtained by $b_n := i^{-1}(\mathcal{B}_n)$ which makes sense since $\mathcal{B}_n \in i(M)$. Since $i^{-1} : i(M) \to M$ is an isometry and $\mathcal{B} : \mathbb{N} \to i(M)$ is Cauchy then $b : \mathbb{N} \to M$ is also Cauchy. Then, there is an equivalence class $[b] \in C$. We claim that $\mathcal{B} : \mathbb{N} \to i(M)$ converges to $[b]$. By construction, $i(b_m) = \mathcal{B}_m$, so the equivalence class $\mathcal{B}_m$ should contain a representative constant sequence whose constant term is b_m. Let us represent $\mathcal{B}_m$ with this sequence so $\mathcal{B}_m = [\mathsf{b}_{m,\cdot}]$ where $\mathsf{b}_{m,n} = b_m$ for all $n \in \mathbb{N}$. Keeping this in mind, we have

$$d_C(\mathcal{B}_m, [b]) = d_C([\mathsf{b}_{m,\cdot}], [b]) = \lim_{n\to\infty} d_M(\mathsf{b}_{m,n}, b_n) = \lim_{n\to\infty} d_M(b_m, b_n), \tag{13}$$

but this can be made sufficiently small since $b : \mathbb{N} \to M$ is Cauchy. More specifically, given ϵ, there is a $K \in \mathbb{N}$ such that for all $n, m \geq K$ we have have $d_M(b_m, b_n) < \frac{\epsilon}{2}$. Again, by using Lemma 3, take limits we respect to n to obtain $\lim_{n\to\infty} d_M(b_m, b_n) \leq \frac{\epsilon}{2} < \epsilon$. Since $\mathcal{B} : \mathbb{N} \to i(M)$ is co-Cauchy with $\mathcal{A} : \mathbb{N} \to C$ and $\mathcal{B} : \mathbb{N} \to i(M)$ converges, $\mathcal{A} : \mathbb{N} \to C$ also converges which implies that every Cauchy sequence in C converges and C is complete.

We have constructed a complete metric space C from our original metric space M. Evidently, if M was already complete then we would expect that this process adds nothing to M. Let us verify this. More precisely, we claim that $i(M) = C$ if and only if M is complete. Suppose that $i(M) = C$, hence $i(M)$ is complete. But $i : M \to i(M)$ is an isometry so M is complete. Conversely, let M be complete. First, we show that $i(M)$ is closed, which is equivalent to saying that it contains all of its limit points. Let $[a] \in C$ be a limit point of $i(M)$. Consequently, there is a sequence $\mathcal{A} : \mathbb{N} \to i(M)$ that converges to $[a]$. As a convergent sequence, $\mathcal{A} : \mathbb{N} \to i(M)$ is Cauchy and its inverse image $i^{-1} \circ \mathcal{A} : \mathbb{N} \to M$ is also Cauchy as $i^{-1} : i(M) \to M$ is an isometry. Since M is complete, $i^{-1} \circ \mathcal{A} : \mathbb{N} \to M$ converges to some point $x \in M$. As every isometry sends convergent sequences to convergent ones, the sequence $i \circ i^{-1} \circ \mathcal{A} = \mathbb{1} \circ \mathcal{A} = \mathcal{A} : \mathbb{N} \to i(M)$ should converge to some $[b] \in i(M)$. But, the limit of a sequence is unique so $[a] = [b]$ and $[a] \in i(M)$, telling us that $i(M)$ is closed. This immediately gives us $i(M) = C$ as we already know that $i(M) = \operatorname{clr} i(M)$ and $\operatorname{clr} i(M) = C$.

Now, we want to show that any other completion is isometric to the one we constructed. This clearly implies that any two other completions are isometric. First note that $\mathfrak{i} := j \circ i^{-1} : i(M) \to j(M)$ is an isometry since composition of isometries is an isometry. More precisely, since $i^{-1} : i(M) \to M$ and $j : M \to j(M)$ are isometries, for every $[a], [b] \in i(M)$ we have that

$$\begin{aligned} d_C([a], [b]) &= d_M\big(i^{-1}([a]), i^{-1}([b])\big) \\ &= d_E\big(j(i^{-1}([a])), j(i^{-1}([b]))\big) \\ &= d_E\big((j \circ i^{-1})([a]), (j \circ i^{-1})([b])\big) \\ &= d_E\big(\mathfrak{i}([a]), \mathfrak{i}([b])\big). \end{aligned} \tag{14}$$

Moreover, $\mathfrak{i} : i(M) \to j(M)$ is a surjection as the composition of surjections is a surjection. These two results imply that $\mathfrak{i} : i(M) \to j(M)$ is an isometry. Take an equivalence class $[a] \in C$. Since $i(M)$ is dense in C, then

$[a]$ is a limit point of $i(M)$ and there is a sequence of equivalence classes $\mathcal{A} : \mathbb{N} \to i(M)$ that converges to $[a]$. As a convergent sequence, $\mathcal{A} : \mathbb{N} \to i(M)$ is Cauchy so its isometric image $\mathfrak{i} \circ \mathcal{A} : \mathbb{N} \to j(M)$ is also Cauchy. Noting that $j(M) \subset E$ and E is complete, $\lim_{n\to\infty}(\mathfrak{i} \circ \mathcal{A})_n \in E$ exists. Define the mapping $f : C \to E$ by this procedure such that $f([a]) = \lim_{n\to\infty}(\mathfrak{i} \circ \mathcal{A})_n$. To show that this mapping is well defined we must show that the limit does not depend on the choice of sequences of equivalence classes converging to $[a]$. For this purpose, suppose $\lim_{n\to\infty} \mathcal{A}_n = [a]$ and $\lim_{n\to\infty} \mathcal{B}_n = [a]$. This implies that $\lim_{n\to\infty} \mathcal{A}_n = \lim_{n\to\infty} \mathcal{B}_n$ which in turn gives $\mathfrak{i}(\lim_{n\to\infty} \mathcal{A}_n) = \mathfrak{i}(\lim_{n\to\infty} \mathcal{B}_n)$. Noting that every isometry is continuous, we can simply take the limits out to obtain $\lim_{n\to\infty} \mathfrak{i}(\mathcal{A}_n) = \lim_{n\to\infty} \mathfrak{i}(\mathcal{B}_n)$, concluding that f is well-defined. Next, we show that $f : C \to E$ is surjective. Now, take any element $u \in E$. As $j(M)$ is dense in E, there is a sequence $\mathcal{U} : \mathbb{N} \to j(M)$ converging to u. The sequence $\mathcal{U} : \mathbb{N} \to j(M)$ is Cauchy and its isometric image $\mathcal{A} := \mathfrak{i}^{-1} \circ \mathcal{U} : \mathbb{N} \to i(M)$ is also Cauchy. Since C is complete $\mathcal{A} : \mathbb{N} \to i(M)$ converges to some $[a] \in C$. Now, we clearly see that

$$f([a]) = \lim_{n\to\infty}(\mathfrak{i} \circ \mathcal{A})_n = \lim_{n\to\infty}(\mathfrak{i} \circ \mathfrak{i}^{-1} \circ \mathcal{U})_n = \lim_{n\to\infty}(\mathbb{1} \circ \mathcal{U})_n = \lim_{n\to\infty} \mathcal{U}_n = u. \tag{15}$$

The only thing left to show is that $f : C \to E$ preserves metrics. That's easy! Take a careful look at each step of the following

$$\begin{aligned}
d_E\big(f([a]), f([b])\big) &= d_E\big((\lim_{n\to\infty}(\mathfrak{i} \circ \mathcal{A})_n, \lim_{n\to\infty}(\mathfrak{i} \circ \mathcal{B})_n)\big) \\
&= d_E\big((\lim_{n\to\infty} \mathfrak{i}(\mathcal{A}_n), \lim_{n\to\infty} \mathfrak{i}(\mathcal{B}_n))\big) \\
&= d_E\big(\lim_{n\to\infty}(\mathfrak{i}(\mathcal{A}_n), \mathfrak{i}(\mathcal{B}_n))\big) \\
&= \lim_{n\to\infty} d_E\big(\mathfrak{i}(\mathcal{A}_n), \mathfrak{i}(\mathcal{B}_n)\big) \\
&= \lim_{n\to\infty} d_C(\mathcal{A}_n, \mathcal{B}_n) \\
&= d_C\big(\lim_{n\to\infty}(\mathcal{A}_n, \mathcal{B}_n)\big) \\
&= d_C\big(\lim_{n\to\infty} \mathcal{A}_n, \lim_{n\to\infty} \mathcal{B}_n\big). \\
&= d_C([a], [b]).
\end{aligned} \tag{16}$$

We noted the fact every sequence in a Cartesian product metric space converges if and only if all of its components converge. We also used that d_E and d_C are continuous and that $\mathfrak{i} : i(M) \to j(M)$ is an isometry.

It remains to show that the isometry $f : C \to E$ is unique in the sense described in step 3. Let $[a] \in i(M)$, so we can take a constant sequence $\mathcal{A} : \mathbb{N} \to i(M)$ such that $\mathcal{A}_n = [a]$. According to the definition of f we have

$$f([a]) := \lim_{n\to\infty}(\mathfrak{i} \circ \mathcal{A})_n = \lim_{n\to\infty} \mathfrak{i}(\mathcal{A}_n) = \lim_{n\to\infty} \mathfrak{i}([a]) = \mathfrak{i}([a]), \tag{17}$$

which implies that $f|_{i(M)} = \mathfrak{i}$. Let $g : C \to E$ be any other isometry with this propery, so $g|_{i(M)} = \mathfrak{i}$. It is easily observed that $f = g$. Choose any $[a] \in C \backslash i(M)$ and let $\mathcal{A} : \mathbb{N} \to i(M)$ be a sequence converging to $[a]$. This is possible since $i(M)$ is dense in C, implying that $[a]$ is a limit point of $i(M)$. Then we have

$$f([a]) = f(\lim_{n\to\infty} \mathcal{A}_n) = \lim_{n\to\infty} f(\mathcal{A}_n) = \lim_{n\to\infty} g(\mathcal{A}_n) = g(\lim_{n\to\infty} \mathcal{A}_n) = g([a]), \tag{18}$$

where we made use of $f|_{i(M)} = g|_{i(M)}$ and the continuity of $f : C \to E$ and $g : C \to E$. ■