NOTE ON AWFS

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Notation 0.1. We employ the following notations.

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1. Orthogonal factorisation systems

Let $\mathbf{C}$ be a category.

Definition 1.1. A ***functorial factorisation*** on $\mathbf{C}$ is a factorisation of the canonical natural transformation $\mathtt{dom} \Longrightarrow \mathtt{cod} \colon \mathbf{C}^{\mathbf{2}} \longrightarrow \mathbf{C}$. In other words, a functorial factorisation $F = (E, \eta, \varepsilon)$ consists of the following data.

- A functor $E \colon \mathbf{C}^{\mathbf{2}} \longrightarrow \mathbf{C}$.
- Two natural transformations $\mathtt{dom} \overset{\eta}{\Longrightarrow} E \overset{\varepsilon}{\Longrightarrow} \mathtt{cod}$ whose composite is the canonical natural transformation $\mathtt{dom} \Longrightarrow \mathtt{cod}$. ■

Definition 1.2. A ***(strict) double category*** $\mathbb{S}$ is a category internal to $\mathbf{Cat}$. In particular, $\mathbb{S}$ consists of the following data.

$$\mathbf{H}_1 \mathbb{S} \,{}_{\mathtt{bs}}\!\times_{\mathtt{ts}} \mathbf{H}_1 \mathbb{S} \;\begin{smallmatrix} \longrightarrow \\ -\,\text{\textfrak{;}}\!\rightarrow \\ \longrightarrow \end{smallmatrix}\; \mathbf{H}_1 \mathbb{S} \;\begin{smallmatrix} \xrightarrow{\mathtt{bs}} \\ \leftarrow \mathtt{id} - \\ \xrightarrow[\mathtt{ts}]{} \end{smallmatrix}\; \mathbf{H}\,\mathbb{S}$$

A ***(strict) double functor*** $F \colon \mathbb{S} \longrightarrow \mathbb{S}'$ is a pair of functors $\mathbf{H}\,F \colon \mathbf{H}\,\mathbb{S} \longrightarrow \mathbf{H}\,\mathbb{S}'$ and $\mathbf{H}_1 F \colon \mathbf{H}_1 \mathbb{S} \longrightarrow \mathbf{H}\,\mathbb{S}'$ making the obvious diagram commute. We write $\mathbf{SDbl}$ for the category of double categories and double functors.

A ***horizontally full subdouble category*** of $\mathbb{S}$ is a double category equipped with a double functor F towards $\mathbb{S}$ such that both $\mathbf{H}\,F$ and $\mathbf{H}_1 F$ are full subcategory inclusions. A horizontally full subdoble category is called ***wide*** if $\mathbf{H}\,F$ is an identitity and $\mathbf{H}_1 F$ is a replete subcategory inclusion. We write $\mathbf{HFSub}(\mathbb{S})$ for the poset of horizontally full subdouble categories, and $\mathbf{WHFSub}(\mathbb{S})$ for its subposet consisting of wide ones. ■

Example 1.3. (Nerves of) simplices $([2],[1],[0])$ form a category object in $\mathbf{Cat}^{\mathsf{op}}$, and hence the internal hom functor $(-_2)^{(-_1)} \colon \mathbf{Cat}^{\mathsf{op}} \times \mathbf{Cat} \longrightarrow \mathbf{Cat}$ induces a functor $\mathbb{S}\mathrm{q} \colon \mathbf{Cat} \longrightarrow \mathbf{SDbl}$ since $\mathbf{C}^{(-)} \colon \mathbf{Cat}^{\mathsf{op}} \longrightarrow \mathbf{Cat}$ preserves limits for each $\mathbf{C} \in \mathbf{Cat}$.

For each $\mathbf{C} \in \mathbf{Cat}$, vertical arrows and horizontal arrows in $\mathbb{S}\mathrm{q}(\mathbf{C})$ are morphisms in $\mathbf{C}$, while a (unique) cell exists for a square in $\mathbb{S}\mathrm{q}(\mathbf{C})$ if and only if it is a commutative square in $\mathbf{C}$. ■

Proposition 1.4 (Subcategory axiom)**.** There is an isomorphism of posets

$$\mathbf{Sub}(\mathbf{C}) \cong \mathbf{HFSub}(\mathbb{S}\mathrm{q}(\mathbf{C})),$$

where $\mathbf{Sub}(\mathbf{C})$ is the poset of subcategories of $\mathbf{C}$. Moreover, it restricts to another isomorphism

$$\mathbf{WSub}(\mathbf{C}) \cong \mathbf{WHFSub}(\mathbb{S}\mathrm{q}(\mathbf{C})),$$

where $\mathbf{WSub}(\mathbf{C})$ is the poset of wide subcategories. ♦

Definition 1.5. A ***wide subcategory*** $\mathbf{W}$ of $\mathbf{C}$ is a replete subcategory of $\mathbf{C}$ whose inclusion $\mathbf{W} \hookrightarrow \mathbf{C}$ is (essentially) surjective. ■

References

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