NOTE ON AWFS

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Notation 0.1. We employ the following notations.

- The arrow category of a category $\mathbf{C}$ is denoted by $\mathbf{C}^{\mathbf{2}}$.
- For each class S of morphisms in $\mathbf{C}$, we write $\bar{\mathbf{S}}$ for the corresponding full subcategory of $\mathbf{C}^{\mathbf{2}}$.
- Given two morphisms f, g, we write $f \perp g$ if the lifting problem solves uniquely; i.e., the canonical function $\mathbf{C}(\mathtt{cod}(f), \mathtt{dom}(g)) \longrightarrow \mathbf{C}^{\mathbf{2}}(f, g)$ is a bijection.
- We write $\mathbf{P}\mathrm{mor}(\mathbf{C})$ for the (large) poset of classes of morphisms in $\mathbf{C}$. This is isomorphic to the (large) poset of full subcategories of $\mathbf{C}^{\mathbf{2}}$. ■

1. Orthogonal factorisation systems

Let $\mathbf{C}$ be a category.

1.1. Orthogonality for classes of morphisms.

Definition 1.1. A class S of morphisms is called ***reflective*** / ***coreflective*** / ***replete*** if the full subcategory $\bar{\mathbf{S}}$ of $\mathbf{C}^{\mathbf{2}}$ is reflective, coreflective, and replete respectively. ■

Definition 1.2. Define a monotone function

$$- \perp -\colon \mathbf{P}\mathrm{mor}(\mathbf{C})^{\mathrm{op}} \times \mathbf{P}\mathrm{mor}(\mathbf{C})^{\mathrm{op}} \longrightarrow \mathbf{2}$$

by $\mathrm{S} \perp \mathrm{T} := \forall s \in \mathrm{S},\ \forall t \in \mathrm{T},\ s \perp t$. One can easily check $\mathrm{S} \perp -$ and $- \perp \mathrm{S}$ preserve limits for any $\mathrm{S} \in \mathbf{P}\mathrm{mor}(\mathbf{C})$, and hence they are representable. We write ${}^{\perp}(-)\colon \mathbf{P}\mathrm{mor}(\mathbf{C}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{P}\mathrm{mor}(\mathbf{C})^{\mathrm{op}} : (-)^{\perp}$ for the induced Galois connection. ■

Definition 1.3. An ***orthogonal system*** (E, M) is a pair of classes of morphisms satisfying ${}^{\perp}\mathrm{M} = \mathrm{E}$ and $\mathrm{E}^{\perp} = \mathrm{M}$. ■

Proposition 1.4. ♦

1.2. Wide class of morphisms as subdouble category.

Definition 1.5. A ***(strict) double category*** $\mathbb{S}$ is a category internal to $\mathbf{Cat}$. In particular, $\mathbb{S}$ consists of the following data.

$$\begin{tikzcd} \bar{\mathbf{H}}\,\mathbb{S} \,{}_{\mathsf{bs}}\!\times_{\mathsf{ts}} \bar{\mathbf{H}}\,\mathbb{S} \arrow[r, shift left=2] \arrow[r, "\fatsemi"] \arrow[r, shift right=2] & \bar{\mathbf{H}}\,\mathbb{S} \arrow[r, shift left=2, "\mathsf{bs}"] \arrow[r, shift right=2, "\mathsf{ts}"'] & \mathbf{H}\,\mathbb{S} \arrow[l, "\mathtt{id}"'] \end{tikzcd}$$

A ***(strict) double functor*** $F\colon \mathbb{S} \longrightarrow \mathbb{S}'$ is a pair of functors $\mathbf{H}\,F\colon\ \mathbf{H}\,\mathbb{S} \longrightarrow \mathbf{H}\,\mathbb{S}'$ and $\bar{\mathbf{H}}\,F\colon\ \bar{\mathbf{H}}\,\mathbb{S} \longrightarrow \mathbf{H}\,\mathbb{S}'$ making the obvious diagram commute. We write $\mathbf{SDbl}$ for the category of double categories and double functors.

A ***horizontally full subdouble category*** of $\mathbb{S}$ is a double category equipped with a double functor F towards $\mathbb{S}$ such that both $\mathbf{H}\,F$ and $\bar{\mathbf{H}}\,F$ are full subcategory-inclusions. A horizontally full subdoble category is called ***wide*** if $\mathbf{H}\,F$ is an identitity and $\bar{\mathbf{H}}\,F$ defines a replete subcategory. We write $\mathbf{hfSub}(\mathbb{S})$ for the poset of horizontally full subdouble categories, and $\mathbf{whfSub}(\mathbb{S})$ for its subposet consisting of wide ones. ■

Example 1.6. Finite ordinals $([2], [1], [0])$ form a category object in $\mathbf{Cat}^{\mathrm{op}}$, and hence the internal hom functor $(-_2)^{(-_1)}\colon \mathbf{Cat}^{\mathrm{op}} \times \mathbf{Cat} \longrightarrow \mathbf{Cat}$ induces a functor $\mathbb{S}\mathrm{q}\colon \mathbf{Cat} \longrightarrow \mathbf{SDbl}$ since $\mathbf{C}^{(-)}\colon \mathbf{Cat}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ preserves limits for each $\mathbf{C} \in \mathbf{Cat}$.

For each $\mathbf{C} \in \mathbf{Cat}$, vertical arrows and horizontal arrows in $\mathbb{S}\mathrm{q}(\mathbf{C})$ are morphisms in $\mathbf{C}$, while a (unique) cell exists for a square in $\mathbb{S}\mathrm{q}(\mathbf{C})$ if and only if it is a commutative square in $\mathbf{C}$. ■

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Definition 1.7. A ***wide subcategory*** $\mathbf{W}$ of $\mathbf{C}$ is a replete subcategory of $\mathbf{C}$ whose inclusion $\mathbf{W} \hookrightarrow \mathbf{C}$ is (essentially) surjective. ■

Proposition 1.8 (Subcategory axiom)**.** There is an isomorphism of posets

$$\mathbf{Sub}(\mathbf{C}) \cong \mathbf{hfSub}(\mathbb{S}\mathrm{q}(\mathbf{C})),$$

where $\mathbf{Sub}(\mathbf{C})$ is the poset of subcategories of $\mathbf{C}$. Moreover, it restricts to another isomorphism

$$\mathbf{wSub}(\mathbf{C}) \cong \mathbf{whfSub}(\mathbb{S}\mathrm{q}(\mathbf{C})),$$

where $\mathbf{wSub}(\mathbf{C})$ is the poset of wide subcategories. ◆

Proof. A subcategory of $\mathbf{C}$ and a horizontally full subdouble category of $\mathbb{S}\mathrm{q}(\mathbf{C})$ specify the same data: a class of arrows S such that S is closed under composition, and for each $f \in \mathrm{S}$, the identities on $\mathtt{dom}(f)$ and $\mathtt{cod}(f)$ are in S. The subcategory and the subdouble category are wide simultaneously if and only if S contains all isomorphisms in $\mathbf{C}$. □

Notation 1.9. We say a class S of morphisms is ***wide*** if it is closed under composition and contains isomorphisms. We write $\mathbf{S}$ and $\mathbb{S}$ for the corresponding wide subcategory and wide horizontally full subdouble category. Note that $\bar{\mathbf{S}}$ coincide with $\bar{\mathbf{H}}\,\mathbb{S}$. ■

1.3. Functorial factorisation.

Definition 1.10. A ***functorial factorisation*** on $\mathbf{C}$ is a factorisation of the canonical natural transformation $\mathtt{dom} \Longrightarrow \mathtt{cod} \colon \mathbf{C^2} \longrightarrow \mathbf{C}$. In other words, a functorial factorisation $F = (\mathsf{E}, \bar{\mathsf{L}}, \bar{\mathsf{R}})$ consists of the following data.

- A functor $\mathsf{E} \colon \mathbf{C^2} \longrightarrow \mathbf{C}$.
- Two natural transformations $\mathtt{dom} \overset{\bar{\mathsf{L}}}{\Longrightarrow} \mathsf{E} \overset{\bar{\mathsf{R}}}{\Longrightarrow} \mathtt{cod}$ whose composite is the canonical natural transformation $\mathtt{dom} \Longrightarrow \mathtt{cod}$.

We write $\mathsf{L} \colon \mathbf{C^2} \longrightarrow \mathbf{C^2}$ and $\mathsf{R} \colon \mathbf{C^2} \longrightarrow \mathbf{C^2}$ for the functors obtained from $\bar{\mathsf{L}}$ and $\bar{\mathsf{R}}$ respectively by the natural isomorphism $\big[\mathbf{2}, [\mathbf{C^2}, \mathbf{C}]\big] \cong \big[\mathbf{C^2}, \mathbf{C^2}\big]$. ■

References

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