

NOTE ON TYPE THEORIES

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Notation 0.1. We employ the following notations.

- (2,1)-categories are denoted by bf symbols: $\mathbf{C}, \mathbf{A}, \mathbf{E}, \dots$ or bb symbols: $\mathbb{I}, \mathbb{D}, \mathbb{A}, \dots$
- \mathbf{Set} is the category of sets.
- \mathbf{Cat} is the large (2,1)-category of categories.
- There is a fully faithful (2,1)-functor $\mathbf{disc}: \mathbf{Set} \hookrightarrow \mathbf{Cat}$.
- \mathbf{Cat} is the large 2-category of categories.
- Δ^1 is the 1-simplex seen as a category. \mathbf{C}^{Δ^1} is the arrow category of \mathbf{C} .
- $\mathbf{C}_{/A}$ and $\mathbf{C}_{A/}$ are over and under categories respectively.
- cod and dom mean codomain and domain respectively.
- By a *replete class of morphisms* of \mathbf{C} , we mean a replete full subcategory of \mathbf{C}^{Δ^1} . ■

1. DEPENDENT TYPE THEORIES IN TERMS OF DISPLAY MAPS

See also [nLa].

Definition 1.1 (Strict Categories). We write \mathbf{StrCat} for the (1-)category of categories, and an object in \mathbf{StrCat} is called a *strict category*. For each strict category \mathbf{C} , we write $\text{Obj}(\mathbf{C})$ for its underlying object. By *object* of \mathbf{C} , we mean an element in $\text{Obj}(\mathbf{C})$. ■

Remark 1.2. One can see a strict category as a category equipped with its data of *objects*; a *strict category* might be defined as a category \mathbf{C} equipped with a set $\text{Obj}(\mathbf{C})$ and a essentially surjective functor towards \mathbf{C} . They form a full sub-(2,1)-category of \mathbf{Cat}^{Δ^1} , which is indeed equivalent to \mathbf{StrCat} . ■

Definition 1.3 (See [KL18, KL21, nLa]). A *contextual category* \mathbf{T} consists of the following data.

- A strict category \mathbf{T} .
- An object ε of \mathbf{T} which is a terminal object.
- A class¹ $\mathbf{dis}_{\mathbf{T}}$ of *display maps* such that for each object $\Gamma \in \text{Obj}(\mathbf{T})$, there exists a unique path of display maps $\Gamma \twoheadrightarrow \dots \twoheadrightarrow \varepsilon$.
- For each display map $A = (\Gamma.A \twoheadrightarrow \Gamma)$ and a morphism $\vec{t}: \Delta \rightarrow \Gamma$ there is a canonical pullback square

$$\begin{array}{ccc} \Delta.A[\vec{t}/\vec{x}] & \xrightarrow{q(\vec{t}, p)} & \Gamma.A \\ \downarrow & \lrcorner & \downarrow p \\ \Delta & \xrightarrow{\vec{t}} & \Gamma \end{array}$$

whose left side is a display map, which is denoted by $A[\vec{t}/\vec{x}]$ and called the *substitution* of \vec{t} in A . Moreover, those substitutions are strictly functorial: $A[\vec{t}/\vec{x}][\vec{s}/\vec{y}] = A[\vec{t} \circ \vec{s}/\vec{y}]$ and $q(\vec{t}, A) \circ q(\vec{s}, A[\vec{t}/\vec{x}]) = q(\vec{t} \circ \vec{s}, A)$.

We write \mathbf{CwC} for the category of contextual categories and functors preserving those structures. ■

Notation 1.4. Let \mathbf{T} be a contextual category.

- A *context* in \mathbf{T} is an object in \mathbf{T} .
- A *type* A over a context Γ in \mathbf{C} is a display map $A = (\Gamma.A \twoheadrightarrow \Gamma)$. We write

$$\Gamma \vdash A \text{ type}$$

if A is a type over Γ .

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¹A *class of morphisms* in a strict category \mathbf{T} is just a subgraph of the *underlying graph* of \mathbf{T} , which is defined by pulling back $\mathbf{T}^{\Delta^1} \rightarrow \mathbf{T} \times \mathbf{T}$ along $\text{Obj}(\mathbf{T}) \times \text{Obj}(\mathbf{T}) \rightarrow \mathbf{T} \times \mathbf{T}$.

- A **term** t of a type A over a context Γ is a section of the display map $\Gamma.A \rightarrow \Gamma$. We write

$$\Gamma \vdash t : A$$

if t is a term of A . Moreover, if we have $\Gamma = \Delta.X$, then we also write

$$\Delta, x : X \vdash t(x) : A(x)$$

for this judgement.

- We write

$$\Gamma \vdash A \doteq A' \text{ type} \quad \text{and} \quad \Gamma \vdash t \doteq t' : A$$

if A and A' are equal as display maps and t and t' are equal as sections. We say types (or terms) are **judgementally equal** if such an equality holds.

- (Biased product of contexts.) Let $\Delta \in \mathbf{C}$ be a context and $\Gamma = \varepsilon.A_0 \cdots A_n$ be another context. (Note that such a sequence A_0, \dots, A_n of types uniquely exists for any context Γ .) We define $\Delta.\Gamma = \Delta.A_0 \cdots A_n$ inductively as follows.
 - $\Delta.\varepsilon := \Delta$. There is a canonical projection $p_{-1} : \Delta \rightarrow \varepsilon$.
 - Suppose we have constructed $\Delta.A_0 \cdots A_i$ and a projection $p_i : \Delta.A_0 \cdots A_i \rightarrow \varepsilon.A_0 \cdots A_i$. Then we define $\Delta.A_0 \cdots A_i.A_{i+1}$ and p_{i+1} as the following substitution.

$$\begin{array}{ccc} \Delta.A_0 \cdots A_i.A_{i+1} & \xrightarrow{p_{i+1}} & \varepsilon.A_0 \cdots A_i.A_{i+1} \\ \downarrow & \lrcorner & \downarrow \\ \Delta.A_0 \cdots A_i & \xrightarrow{p_i} & \varepsilon.A_0 \cdots A_i \end{array}$$

- (Morphisms of contexts.) Let Δ, Γ be contexts. A morphism $\vec{t} : \Delta \rightarrow \Gamma$ can be seen as a tuple of terms. For example, when we have $\Gamma = \varepsilon.A_0.A_1$, the morphism \vec{t} is determined by the following terms t_0, t_1 .

$$\begin{array}{ccccccc} \Delta & \xrightarrow{t_1} & \Delta.A_1[t_0/x_0] & \longrightarrow & \Delta.\Gamma & \longrightarrow & \varepsilon.A_0.A_1 = \Gamma \\ & \searrow & \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ & & \Delta & \xrightarrow{t_0} & \Delta.A_0 & \longrightarrow & \varepsilon.A_0 \\ & & & \searrow & \downarrow & \lrcorner & \downarrow \\ & & & & \Delta & \longrightarrow & \varepsilon \end{array}$$

■

Definition 1.5. Let \mathbf{C} be a contextual category. We say \mathbf{C} has **extensional Σ -types** if for each context Γ and types $\Gamma \vdash A \text{ type}$ and $\Gamma.A \vdash B \text{ type}$, it satisfies the following condition. There exists a type

$$\Gamma \vdash \Sigma(A, x.B) \text{ type}$$

and terms

$$\Gamma.A.B \vdash \text{pair} : \Sigma(A, x.B)$$

$$\Gamma.\Sigma(A, x.B) \vdash \text{prl} : A \quad \text{and} \quad \Gamma.\Sigma(A, x.B) \vdash \text{prr} : B[\text{prl}/x]$$

satisfying the following judgemental equalities;

$$\Gamma.\Sigma(A, x.B) \vdash \text{pair} \circ \langle \text{prl}, \text{prr} \rangle \doteq (p \mapsto p)$$

$$\Gamma.A.B \vdash \text{prl} \circ \text{pair} \doteq (x : A, y : B \mapsto x), \quad \Gamma.A.B \vdash \text{prr} \circ \text{pair} \doteq (x : A, y : B \mapsto y)$$

We write \mathbf{CwC}_Σ for the full subcategory of \mathbf{CwC} consisting of those with extensional Σ -types. ■

Definition 1.6. A **display map category** $\mathbf{C} = (\mathbf{C}, \mathbf{dis}_\mathbf{C})$ is a pair of a category \mathbf{C} and a replete class $\mathbf{dis}_\mathbf{C}$ of morphisms satisfying the following conditions. Arrows in $\mathbf{dis}_\mathbf{C}$ are called **display maps** of \mathbf{C} .

- \mathbf{C} has a terminal object².

²In some literature, this condition is omitted for the definition of display map category.

- Let $h: \Delta \rightarrow \Gamma$ and $f: A \rightarrow \Gamma$ be morphisms in \mathbf{C} such that f is a display map. Then there is a pullback square

$$(1) \quad \begin{array}{ccc} \cdot & \xrightarrow{\quad} & A \\ \downarrow & \lrcorner & \downarrow f \\ \Delta & \xrightarrow{h} & \Gamma \end{array}$$

in \mathbf{C} such that the left side is also a display map. \blacksquare

Definition 1.7. Suppose we are given a display map category \mathbf{C} and an object $\Gamma \in \mathbf{C}$. We define a display map category $\mathbf{C}(\Gamma)$ as follows.

- The underlying category is the full subcategory of the over category \mathbf{C}/Γ spanned by display maps in \mathbf{C} .
- A display map is a morphism f in $\mathbf{C}(\Gamma)$ that is a display map in \mathbf{C} . \blacksquare

Proposition 1.8. The above definition indeed gives a display map category. \blacklozenge

Definition 1.9. Define the 2-category \mathfrak{CwD} of display map categories as follows.

- 0-cells are display map categories.
- 1-cells are functors preserving the terminal object, display maps, and pullbacks of the forms (1).
- A 2-cell $\alpha: F \Rightarrow G: \mathbf{C} \rightarrow \mathbf{D}$ is a natural transformation such that naturality squares at display maps are pullback squares; i.e., for each display map $f: A \rightarrow \Gamma$ in \mathbf{C} , the naturality square

$$\begin{array}{ccc} FA & \xrightarrow{\alpha_A} & GA \\ F(f) \downarrow & \lrcorner & \downarrow G(f) \\ F\Gamma & \xrightarrow{\alpha_\Gamma} & G\Gamma \end{array}$$

is a pullback square in \mathbf{D} . \blacksquare

Definition 1.10. A display map category is *democratic* if for each object $\Gamma \in \mathbf{C}$, there exists a sequence of display maps from Γ to the terminal object. We write \mathbf{CwD}^{dm} for the full sub 2-category of \mathfrak{CwD} spanned by democratic display map categories. \blacksquare

Proposition 1.11. \mathbf{CwD}^{dm} is a (2,1)-category. \blacklozenge

Definition 1.12 ([Joy17]). A *clan* $\mathbf{C} = (\mathbf{C}, \mathbf{fib}_{\mathbf{C}})$ is a display map category satisfying the following conditions. Display maps (i.e., arrows in $\mathbf{fib}_{\mathbf{C}}$) are called *fibrations* of \mathbf{C} .

- For each object $A \in \mathbf{C}$, the unique morphism $A \rightarrow 1$ towards the terminal object is a fibration.
- $\mathbf{fib}_{\mathbf{C}}$ is closed under composition and contains all isomorphisms. \blacksquare

We write \mathbf{Clan} for the full sub 2-category of \mathfrak{CwD} spanned by clans. Since clans are always democratic, this is a (2,1)-category.

Theorem 1.13. There exists a biadjunction

$$\mathbf{CwC} \begin{array}{c} \xrightarrow{|-|} \\ \perp \\ \xleftarrow{\text{cxt}} \end{array} \mathfrak{CwD}$$

that restricts to the following biequivalences;

$$\begin{array}{ccc} \mathbf{CwC} & \begin{array}{c} \xrightarrow{|-|} \\ \simeq \\ \xleftarrow{\text{cxt}} \end{array} & \mathbf{CwD}^{\text{dm}} \\ & & \\ \mathbf{CwC}_{\Sigma} & \begin{array}{c} \xrightarrow{|-|} \\ \simeq \\ \xleftarrow{\text{cxt}} \end{array} & \mathbf{Clan} \end{array}$$

In particular, \mathbf{CwD}^{dm} and \mathbf{Clan} are essentially 1-categories; i.e., setoid enriched categories. \blacklozenge

Sketch of proof. There is a canonical way to see a contextual category as a display map category. This indeed induces a 2-functor $|-|: \mathbf{CwC} \rightarrow \mathfrak{CwD}$ which is an embedding.

For each display map category \mathbf{D} , we construct a contextual category $\mathbf{cxt}(\mathbf{D})$ as follows.

- Firstly, choose a terminal object 1 . Then choose a section $\Gamma.-$ of the quotient functor $\mathbf{D}(\Gamma)^\cong \rightarrow \pi_0(\mathbf{D}(\Gamma)^\cong)$ for each $\Gamma \in \mathbf{D}$.
- An object in $\mathbf{cxt}(\mathbf{D})$ is a tuple (A_0, A_1, \dots, A_n) ($n \geq 0$) such that
 - $A_0 \in \pi_0(\mathbf{D}(1))$.
 - $A_{i+1} \in \pi_0(\mathbf{D}(1.A_0 \cdots A_i))$ for each $0 \leq i < n$.
- A morphism $(A_0, \dots, A_n) \rightarrow (B_0, \dots, B_m)$ is a morphism $1.A_0 \cdots A_n \rightarrow 1.B_0 \cdots B_m$.

$\mathbf{cxt}(\mathbf{D})$ is independent of the choice of the terminal object and the sections of the quotient functors, up to isomorphism of categories. Moreover, one can check the natural equivalence

$$\mathbf{CwC}(\mathbf{T}, \mathbf{cxt}(\mathbf{D})) \simeq \mathfrak{CwD}(|\mathbf{T}|, \mathbf{D})$$

of categories, which shows the biadjunction. By definition of $\mathbf{cxt}(\mathbf{D})$, the counit $|\mathbf{cxt}(\mathbf{D})| \rightarrow \mathbf{D}$ is an embedding, and it is essentially surjective if and only if \mathbf{D} is democratic. This shows the first biequivalence. The second one is checked by confirming the image of clans under \mathbf{cxt} has extensional Σ -types. \square

2. LOGICAL FRAMEWORK À LA UEMURA

[Uem23]

Definition 2.1. A *representable map category* $\mathbf{R} = (\mathbf{R}, \mathbf{rep}_{\mathbf{R}})$ is a clan satisfying the following conditions. Arrows in $\mathbf{rep}_{\mathbf{R}}$ are called *representable maps* of \mathbf{R} .

- \mathbf{R} is finitely complete.
- For each representable map $f: X \twoheadrightarrow Y$, the pullback functor $f^*: \mathbf{R}_Y \rightarrow \mathbf{R}_X$ has a right adjoint f_* . ■

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