NOTE ON TYPE THEORIES

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[Cis19]

Notation 0.1. We employ the following notations.

- (2,1)-categories are denoted by bf symbols: $\mathbf{C}, \mathbf{A}, \mathbf{E}, \ldots$ or bb symbols: $\mathbb{I}, \mathbb{D}, \mathbb{A}, \ldots$
- **Set** is the category of sets.
- **Cat** is the large (2,1)-category of categories.
- There is a fully faithful (2,1)-functor $\mathtt{disc}\colon \mathbf{Set} \hookrightarrow \mathbf{Cat}$.
- $\mathfrak{C}at$ is the large 2-category of categories.
- Δ^1 is the 1-simplex seen as a category. $\mathbf{C}^{\Delta^1}$ is the arrow category of $\mathbf{C}$.
- $\mathbf{C}_{/A}$ and $\mathbf{C}_{A/}$ are over and under categories respectively.
- $\mathtt{cod}$ and $\mathtt{dom}$ mean codomain and domain respectively. They often have as their type $\mathbf{C}^{\Delta^1} \longrightarrow \mathbf{C}$, $\mathbf{C}_{/A} \longrightarrow \mathbf{C}$, or $\mathbf{C}_{A/} \longrightarrow \mathbf{C}$.
- By a ***replete class of morphisms*** of $\mathbf{C}$, we mean a replete full subcategory of $\mathbf{C}^{\Delta^1}$.
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1. Dependent type theories in terms of display maps

Definition 1.1. A ***display map category*** $\mathbf{C} = (\mathbf{C}, \mathbf{dis}_{\mathbf{C}})$ is a pair of a category $\mathbf{C}$ and a replete class $\mathbf{dis}_{\mathbf{C}}$ of morphisms satisfying the following conditions. Arrows in $\mathbf{dis}_{\mathbf{C}}$ are called ***display maps*** of $\mathbf{C}$.

- $\mathbf{C}$ has a terminal object[1].
- Let $h\colon \Delta \longrightarrow \Gamma$ and $f\colon A \twoheadrightarrow \Gamma$ be morphisms in $\mathbf{C}$ such that f is a display map. Then there is a pullback square

(1)
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, two heads, "f"] \\ \Delta \arrow[r, "h"'] & \Gamma \end{tikzcd}$$

  in $\mathbf{C}$ such that the left side is also a display map. ■

Definition 1.2. Suppose we are given a display map category $\mathbf{C}$ and an object $\Gamma \in \mathbf{C}$. We define a diplay map category $\mathbf{C}(\Gamma)$ as follows.

- The underlying category is the full subcategory of the over category $\mathbf{C}_{/\Gamma}$ spanned by diplay maps in $\mathbf{C}$.
- A display map in $\mathbf{C}(\Gamma)$ is a morphism $f \in \mathbf{C}(\Gamma)$ such that $\mathtt{dom}(f)$ is a display map in $\mathbf{C}$. ■

Proposition 1.3. The above definition indeed gives a display map category. ♦

Definition 1.4. Define the 2-category $\mathfrak{C}w\mathfrak{D}$ of display map categories as follows.

- 0-cells are display map categories.
- 1-cells are functors preserving the terminal object, display maps, and pullbacks of the forms (1).
- A 2-cell $\alpha\colon F \Longrightarrow G\colon \mathbf{C} \longrightarrow \mathbf{D}$ is a natural transformation such that naturality squares at display maps are pullback squares; i.e., for each display map $f\colon A \twoheadrightarrow \Gamma$ in $\mathbf{C}$, the naturality square

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\lrcorner", very near start] & GA \arrow[d, "G(f)"] \\ F\Gamma \arrow[r, "\alpha_\Gamma"'] & G\Gamma \end{tikzcd}$$

Date: April 17, 2024.

[1] In some literature, this condition is omitted for the definition of display map category.

is a pullback square in $\mathbf{D}$. ■

Definition 1.5. A display map category is ***democratic*** if for each object $\Gamma \in \mathbf{C}$, there exists a sequence of display maps from Γ to the terminal object. We write $\mathbf{CwD}^{\mathsf{dm}}$ for the full sub 2-category of $\mathfrak{C}w\mathfrak{D}$ spanned by democratic display map categories. ■

Proposition 1.6. $\mathbf{CwD}^{\mathsf{dm}}$ is a (2,1)-category. ♦

Definition 1.7. A ***clan*** $\mathbf{C} = (\mathbf{C}, \mathbf{fib}_{\mathbf{C}})$ is a display map category satisfying the following conditions. Display maps (i.e., arrows in $\mathbf{fib}_{\mathbf{C}}$) are called ***fibrations*** of $\mathbf{C}$.

- For each object $A \in \mathbf{C}$, the unique morphism $A \longrightarrow 1$ towards the terminal object is a fibraion.
- $\mathbf{fib}_{\mathbf{C}}$ is closed under composition and contains all isomorphisms. ■

We write **Clan** for the full sub 2-category of $\mathfrak{C}w\mathfrak{D}$ spanned by clans. Since clans are always democratic, this is a (2,1)-category.

Definition 1.8 (Strict Categories)**.** We write **StrCat** for the (1-)category of categories, and an object in **StrCat** is called a ***strict category***. For each strict category $\mathbf{C}$, we write $\mathsf{Obj}(\mathbf{C})$ for its underlying object. By *object* of $\mathbf{C}$, we mean an element in $\mathsf{Obj}(\mathbf{C})$. ■

Remark 1.9. You can see a strict category as a category equipped with its data of *objects*; we can define a ***strict category*** as a category $\mathbf{C}$ equipped with a set $\mathsf{Obj}(\mathbf{C})$ and a essentially surjective functor towards $\mathbf{C}$. They form a full sub-(2,1)-category of $\mathbf{Cat}^{\Delta^1}$, which is indeed equivalent to **StrCat**. ■

Definition 1.10. A ***contextual category*** $\mathbf{C} = (\mathbf{C}, \varepsilon, \mathbf{dis}_{\mathbf{C}})$ consists of the following data.

- A strict category $\mathbf{C}$.
- An object ε of $\mathbf{C}$ which is a terminal object.
- A class[2] $\mathbf{dis}_{\mathbf{C}}$ of ***display maps*** such that
  - for each object $\Gamma \in \mathsf{Obj}(\mathbf{C})$, there exists a unique path of display maps $\Gamma \twoheadrightarrow \cdots \twoheadrightarrow \varepsilon$, and
  - for each display map $A = (\Gamma.A \twoheadrightarrow \Gamma)$ and a morphism $\vec{t}\colon \Delta \longrightarrow \Gamma$ there exits a unique pullback square

$$\begin{tikzcd} \Delta.A[\vec{t}/\vec{x}] \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d, two heads, "p"] \\ \Delta \arrow[r, "\vec{t}"'] & \Gamma \end{tikzcd}$$

  whose left side is a display map, which is denoted by $A[\vec{t}/\vec{x}]$ and called the ***substitution*** of $\vec{t}$ in A.

We write **CwC** for the category of contextual categories and functors preserving those structures. ■

Notation 1.11. Let $\mathbf{C}$ be a contextual category.

- A ***context*** in $\mathbf{C}$ is an object in $\mathbf{C}$.
- A ***type*** A over a context Γ in $\mathbf{C}$ is a display map $A = (\Gamma.A \twoheadrightarrow \Gamma)$. We write
$$\Gamma \vdash A\ \mathsf{type}$$
if A is a type over Γ.
- A ***term*** t of a type A over a context Γ is a section of the display map $\Gamma.A \twoheadrightarrow \Gamma$. We write
$$\Gamma \vdash t : A$$
if t is a term of A.
- We write
$$\Gamma \vdash A \doteq A'\ \mathsf{type} \quad \text{and} \quad \Gamma \vdash t \doteq t' : A$$
if A and A' are equal as display maps and t and t' are equal as sections. We say types (or terms) are ***judgementally equal*** if such an equality holds.
- Let $\Delta \in \mathbf{C}$ be a context and $\Gamma = \varepsilon.A_0.\cdots.A_n$ be another context. (Note that such a sequence $A_0, \cdots, A_n$ of types uniquely exists for any context Γ.) We define $\Delta.\Gamma = \Delta.A_0.\cdots.A_n$ inductively as follows.

[2] A *class of morphisms* in a strict category $\mathbf{C}$ is just a subgraph of the *underlying graph* of $\mathbf{C}$, which is defined by pulling back $\mathbf{C}^{\Delta^1} \longrightarrow \mathbf{C} \times \mathbf{C}$ along $\mathsf{Obj}(\mathbf{C}) \times \mathsf{Obj}(\mathbf{C}) \longrightarrow \mathbf{C} \times \mathbf{C}$.

– $\Delta.\varepsilon := \Delta$. There is a canonical projection $p_{-1} \colon \Delta \longrightarrow \varepsilon$.
– Suppose we have constructed $\Delta.A_0.\cdots.A_i$ and a projection $p_i \colon \Delta.A_0.\cdots.A_i \longrightarrow \varepsilon.A_0.\cdots.A_i$. Then we define $\Delta.A_0.\cdots.A_i.A_{i+1}$ and p_{i+1} as the following substitution.

$$\begin{tikzcd} \Delta.A_0.\cdots.A_i.A_{i+1} \arrow[r, "p_{i+1}"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \varepsilon.A_0.\cdots.A_i.A_{i+1} \arrow[d, two heads] \\ \Delta.A_0.\cdots.A_i \arrow[r, "p_i"] & \varepsilon.A_0.\cdots.A_i \end{tikzcd}$$

■

Theorem 1.12. There exists a biadjunction

$$\begin{tikzcd} \mathbf{CwC} \arrow[r, bend left, "|-|"] \arrow[r, phantom, "\perp"] & \mathfrak{C}w\mathfrak{D} \arrow[l, bend left, "\mathsf{cxt}"] \end{tikzcd}$$

that restricts to a biequivalence

$$\begin{tikzcd} \mathbf{CwC} \arrow[r, bend left, "|-|"] \arrow[r, phantom, "\simeq"] & \mathbf{CwD}^{\mathsf{dm}} \arrow[l, bend left, "\mathsf{cxt}"] \end{tikzcd}$$

In particular, $\mathbf{CwD}^{\mathsf{dm}}$ is essentially a 1-category; i.e., a setoid enriched category. ♦

Sketch of proof. There is a canonical way to see a contextual catgory as a display map category. This indeed induces a 2-functor $|-| \colon \mathbf{CwC} \longrightarrow \mathfrak{C}w\mathfrak{D}$.

For each display map category $\mathbf{D}$, we construct a contextual category $\mathsf{cxt}(\mathbf{D})$ as follows.

- Firstly, choose a terminal object 1. Then choose a section $\Gamma.-$ of the quotient functor $\mathbf{D}(\Gamma)^{\cong} \longrightarrow \pi_0(\mathbf{D}(\Gamma)^{\cong})$ for each $\Gamma \in \mathbf{D}$.
- An object in $\mathsf{cxt}(\mathbf{D})$ is a tuple $(A_0, A_1, \cdots, A_n)$ $(n \geq 0)$ such that
  – $A_0 \in \pi_0(\mathbf{D}(1))$.
  – $A_{i+1} \in \pi_0(\mathbf{D}(1.A_0.\cdots.A_i))$ for each $0 \leq i < n$.
- A morphism $(A_0, \cdots, A_n) \longrightarrow (B_0, \cdots, B_m)$ is a morphism $1.A_0.\cdots.A_n \longrightarrow 1.B_0.\cdots.B_m$.

$\mathsf{cxt}(\mathbf{D})$ is independent of the choice of the terminal object and the sections of the quotient functors, up to isomorphism of categories. Moreover, one can check the natural equivalence

$$\mathbf{CwC}(\mathbf{C}, \mathsf{cxt}(\mathbf{D})) \simeq \mathfrak{C}w\mathfrak{D}(|\mathbf{C}|, \mathbf{D})$$

of categories, which shows the biadjunction. □

2. Logical framework à la Uemura

Definition 2.1. A ***representable map category*** $\mathbf{R} = (\mathbf{R}, \mathbf{rep}_{\mathbf{R}})$ is a clan satisfying the following conditions. Arrows in $\mathbf{rep}_{\mathbf{R}}$ are called ***representable maps*** of $\mathbf{R}$.

- $\mathbf{R}$ is finitely complete.
- For each representable map $f\colon X \twoheadrightarrow Y$, the pullback functor $f^*\colon \mathbf{R}_{/Y} \longrightarrow \mathbf{R}_{/X}$ has a right adjoint f_*. ■

References

[Cis19] D.-C. Cisinski. *Higher Categories and Homotopical Algebra*, volume 180 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2019. doi:10.1017/9781108588737.

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