NOTE ON TYPE THEORIES

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Notation 0.1. We employ the following notations.

- (2,1)-categories are denoted by bf symbols: $\mathbf{C}, \mathbf{A}, \mathbf{E}, \dots$ or bb symbols: $\mathbb{I}, \mathbb{D}, \mathbb{A}, \dots$
- $\mathbf{Set}$ is the category of sets.
- $\mathbf{Cat}$ is the large (2,1)-category of categories.
- There is a fully faithful (2,1)-functor $\mathtt{disc}\colon \mathbf{Set} \hookrightarrow \mathbf{Cat}$.
- $\mathfrak{C}at$ is the large 2-category of categories.
- Δ^1 is the 1-simplex seen as a category. $\mathbf{C}^{\Delta^1}$ is the arrow category of $\mathbf{C}$.
- $\mathbf{C}_{/A}$ and $\mathbf{C}_{A/}$ are over and under categories respectively.
- $\mathtt{cod}$ and $\mathtt{dom}$ mean codomain and domain respectively. They often have as their type $\mathbf{C}^{\Delta^1} \longrightarrow \mathbf{C}$, $\mathbf{C}_{/A} \longrightarrow \mathbf{C}$, or $\mathbf{C}_{A/} \longrightarrow \mathbf{C}$.
- By a ***replete class of morphisms*** of $\mathbf{C}$, we mean a replete full subcategory of $\mathbf{C}^{\Delta^1}$.
- $\mathbf{WSub}$ is the replete full sub (2,1)-category of $\mathbf{Cat}^{\Delta^1}$ consisting of wide subcategory inclusions; i.e., faithful functors that induce equivalences on core groupoids.
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1. Dependent type theories in terms of display maps

Definition 1.1. A ***display map category*** $\mathbf{C} = (\mathbf{C}, \mathbf{dis}_{\mathbf{C}})$ is a pair of a category $\mathbf{C}$ and a replete class $\mathbf{dis}_{\mathbf{C}}$ of morphisms satisfying the following conditions. Arrows in $\mathbf{dis}_{\mathbf{C}}$ are called ***display maps*** of $\mathbf{C}$.

- $\mathbf{C}$ has a terminal object.
- Let $h\colon \Delta \longrightarrow \Gamma$ and $f\colon A \twoheadrightarrow \Gamma$ be morphisms in $\mathbf{C}$ such that f is a display map. Then there is a pullback square

(1)
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, two heads, "f"] \\ \Delta \arrow[r, "h"'] & \Gamma \end{tikzcd}$$

in $\mathbf{C}$ such that the left side is also a display map. ■

Definition 1.2. Suppose we are given a display map category $\mathbf{C}$ and an object $\Gamma \in \mathbf{C}$. We define a diplay map category $\mathbf{C}(\Gamma)$ as follows.

- The underlying category is the full subcategory of the over category $\mathbf{C}_{/\Gamma}$ spanned by diplay maps in $\mathbf{C}$.
- A display map in $\mathbf{C}(\Gamma)$ is a morphism $f \in \mathbf{C}(\Gamma)$ such that $\mathtt{dom}(f)$ is a display map in $\mathbf{C}$.

■

Proposition 1.3. The above definition indeed gives a display map category. ♦

Definition 1.4. Define the 2-category $\mathfrak{C}w\mathfrak{D}$ of display map categories as follows.

- 0-cells are display map categories.
- 1-cells are functors preserving the terminal object, display maps, and pullbacks of the forms (1).

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- A 2-cell $\alpha\colon F \Longrightarrow G\colon \mathbf{C} \longrightarrow \mathbf{C}$ is a natural transformation such that naturality squares at display maps are pullback squares; i.e., for each display map $f\colon A \twoheadrightarrow \Gamma$ in $\mathbf{C}$, the naturality square
$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\lrcorner", very near start] & GA \arrow[d, "G(f)"] \\ F\Gamma \arrow[r, "\alpha_\Gamma"'] & G\Gamma \end{tikzcd}$$
is a pullback square in $\mathbf{C}$. ■

Definition 1.5. A display map category is ***democratic*** if for each object $\Gamma \in \mathbf{C}$, there exists a sequence of display maps from Γ to the terminal object. We write $\mathbf{CwD}^{\mathsf{dm}}$ for the full sub 2-category of $\mathfrak{C}w\mathfrak{D}$ spanned by democratic display map categories. ■

Proposition 1.6. $\mathbf{CwD}^{\mathsf{dm}}$ is a (2,1)-category. ♦

Definition 1.7. A ***clan*** $\mathbf{C} = (\mathbf{C}, \mathbf{fib}_{\mathbf{C}})$ is a display map category satisfying the following conditions. Display maps (i.e., arrows in $\mathbf{fib}_{\mathbf{C}}$) are called ***fibrations*** of $\mathbf{C}$.

- For each object $A \in \mathbf{C}$, the unique morphism $A \longrightarrow 1$ towards the terminal object is a fibraion.
- $\mathbf{fib}_{\mathbf{C}}$ is closed under composition. ■

We write **Clan** for the full sub 2-category of $\mathfrak{C}w\mathfrak{D}$ spanned by clans. Since clans are always democratic, this is a (2,1)-category.

Definition 1.8. A ***strict category*** is a category equipped with its wide subcategory that is a set. In other words, the (2,1)-category **StrCat** of strict categories is defined by the following pullback square.
$$\begin{tikzcd} \mathbf{StrCat} \arrow[r] \arrow[dd, "\mathsf{Obj}"'] \arrow[ddr, phantom, "\lrcorner", very near start] & \mathbf{WSub} \arrow[d, hook] \\ & \mathbf{Cat}^{\Delta^1} \arrow[d, "\mathtt{dom}"] \\ \mathbf{Set} \arrow[r, "\mathtt{disc}"'] & \mathbf{Cat} \end{tikzcd}$$
The functor on the left is locally fully faithful and surjective on objects. In particular, **StrCat** is a category. ■

Definition 1.9. A ***contextual category*** $\mathbf{C} = (\mathbf{C}, \mathbf{dis}_{\mathbf{C}}, \langle\rangle, \ell)$ consists of the following data.

- A strict category $\mathbf{C}$.
- A structure $\mathbf{dis}_{\mathbf{C}}$ of a display map category for the underlying category of $\mathbf{C}$.
- An object $\langle\rangle$ of $\mathbf{C}$ which is a terminal object in the underlying category of $\mathbf{C}$.
- A function $\ell\colon \mathsf{Obj}(\mathbf{C}) \longrightarrow \mathbb{N}$ satisfying the following conditions.
  - $\ell^{-1}(0) = \{\langle\rangle\}$.
  - The canonical function
$$(\Gamma, A, f) \mapsto A\colon \coprod_{\Gamma \in \ell^{-1}(n)} \{(A, f) \mid f\colon A \twoheadrightarrow \Gamma\} \longrightarrow \mathsf{Obj}(\mathbf{C})$$
    is a monomorphism and $\ell^{-1}(n+1)$ coincides with its image.

We write **CwC** for the category of contextual categories and functors preserving those structures. ■

Notation 1.10. Let $\mathbf{C}$ be a contextual category.

- A ***context*** in $\mathbf{C}$ is an object in $\mathbf{C}$.
- A ***type*** A over a context Γ in $\mathbf{C}$ is a display map $A = (\Gamma.A \twoheadrightarrow \Gamma)$.
-

■

Theorem 1.11. There exists an 2-adjunction
$$\begin{tikzcd} \mathbf{CwC} \arrow[r, bend left, "|-|"] \arrow[r, phantom, "\perp"] & \mathfrak{C}w\mathfrak{D} \arrow[l, bend left, "\mathsf{ctx}"] \end{tikzcd}$$

that restricts to a biequivalence

$$\begin{tikzcd} \mathbf{CwC} \arrow[r, bend left, "|-|"] \arrow[r, phantom, "\simeq"] & \mathbf{CwD}^{\mathsf{dm}} \arrow[l, bend left, "\mathsf{ctx}"] \end{tikzcd}$$

♦

2. Uemura's logical framework

Definition 2.1. A ***representable map category*** $\mathbf{R} = (\mathbf{R}, \mathbf{rep}_{\mathbf{R}})$ is a display map category satisfying the following conditions. Display maps (i.e., arrows in $\mathbf{rep}_{\mathbf{R}}$) are called ***representable maps*** of $\mathbf{R}$.

- $\mathbf{R}$ is finitely complete.
- For each representable map $f\colon X \twoheadrightarrow Y$, the pullback functor $f^*\colon \mathbf{R}_{/Y} \longrightarrow \mathbf{R}_{/X}$ has a right adjoint f_*. ■

Theorem 2.2. shoumei no aidani claim ireru yatu ♦

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Theorem 2.3. Saigo ni claim kuru taipu. ♦

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References

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