

# NOTE ON TYPE THEORIES

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**Notation 0.1.** We employ the following notations.

- (2,1)-categories are denoted by bf symbols:  $\mathbf{C}, \mathbf{A}, \mathbf{E}, \dots$  or bb symbols:  $\mathbb{I}, \mathbb{D}, \mathbb{A}, \dots$
- $\mathbf{Set}$  is the category of sets.
- $\mathbf{Cat}$  is the large (2,1)-category of categories.
- There is a fully faithful (2,1)-functor  $\mathbf{disc}: \mathbf{Set} \hookrightarrow \mathbf{Cat}$ .
- $\mathbf{Cat}$  is the large 2-category of categories.
- $\Delta^1$  is the 1-simplex seen as a category.  $\mathbf{C}^{\Delta^1}$  is the arrow category of  $\mathbf{C}$ .
- $\mathbf{C}_{/A}$  and  $\mathbf{C}_{A/}$  are over and under categories respectively.
- $\mathbf{cod}$  and  $\mathbf{dom}$  mean codomain and domain respectively. They often have as their type  $\mathbf{C}^{\Delta^1} \rightarrow \mathbf{C}, \mathbf{C}_{/A} \rightarrow \mathbf{C}$ , or  $\mathbf{C}_{A/} \rightarrow \mathbf{C}$ .
- By a *replete class of morphisms* of  $\mathbf{C}$ , we mean a replete full subcategory of  $\mathbf{C}^{\Delta^1}$ . ■

## 1. DEPENDENT TYPE THEORIES IN TERMS OF DISPLAY MAPS

See also [nLa].

**Definition 1.1** (Strict Categories). We write  $\mathbf{StrCat}$  for the (1-)category of categories, and an object in  $\mathbf{StrCat}$  is called a *strict category*. For each strict category  $\mathbf{C}$ , we write  $\mathbf{Obj}(\mathbf{C})$  for its underlying object. By *object* of  $\mathbf{C}$ , we mean an element in  $\mathbf{Obj}(\mathbf{C})$ . ■

**Remark 1.2.** One can see a strict category as a category equipped with its data of *objects*; a *strict category* might be defined as a category  $\mathbf{C}$  equipped with a set  $\mathbf{Obj}(\mathbf{C})$  and a essentially surjective functor towards  $\mathbf{C}$ . They form a full sub-(2,1)-category of  $\mathbf{Cat}^{\Delta^1}$ , which is indeed equivalent to  $\mathbf{StrCat}$ . ■

**Definition 1.3** (See [KL18, KL21, nLa]). A *contextual category*  $\mathbf{T}$  consists of the following data.

- A strict category  $\mathbf{T}$ .
- An object  $\varepsilon$  of  $\mathbf{T}$  which is a terminal object.
- A class<sup>1</sup>  $\mathbf{dis}_{\mathbf{T}}$  of *display maps* such that for each object  $\Gamma \in \mathbf{Obj}(\mathbf{T})$ , there exists a unique path of display maps  $\Gamma \twoheadrightarrow \dots \twoheadrightarrow \varepsilon$ .
- For each display map  $A = (\Gamma.A \twoheadrightarrow \Gamma)$  and a morphism  $\vec{t}: \Delta \rightarrow \Gamma$  there is a canonical pullback square

$$\begin{array}{ccc} \Delta.A[\vec{t}/\vec{x}] & \xrightarrow{q(\vec{t}, p)} & \Gamma.A \\ \downarrow & \lrcorner & \downarrow p \\ \Delta & \xrightarrow{\vec{t}} & \Gamma \end{array}$$

whose left side is a display map, which is denoted by  $A[\vec{t}/\vec{x}]$  and called the *substitution* of  $\vec{t}$  in  $A$ . Moreover, those substitutions are strictly functorial:  $A[\vec{t}/\vec{x}][\vec{s}/\vec{y}] = A[\vec{t} \circ \vec{s}/\vec{y}]$  and  $q(\vec{t}, A) \circ q(\vec{s}, A[\vec{t}/\vec{x}]) = q(\vec{t} \circ \vec{s}, A)$ .

We write  $\mathbf{CwC}$  for the category of contextual categories and functors preserving those structures. ■

**Notation 1.4.** Let  $\mathbf{T}$  be a contextual category.

- A *context* in  $\mathbf{T}$  is an object in  $\mathbf{T}$ .

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<sup>1</sup>A *class of morphisms* in a strict category  $\mathbf{T}$  is just a subgraph of the *underlying graph* of  $\mathbf{T}$ , which is defined by pulling back  $\mathbf{T}^{\Delta^1} \rightarrow \mathbf{T} \times \mathbf{T}$  along  $\mathbf{Obj}(\mathbf{T}) \times \mathbf{Obj}(\mathbf{T}) \rightarrow \mathbf{T} \times \mathbf{T}$ .

- A **type**  $A$  over a context  $\Gamma$  in  $\mathbf{C}$  is a display map  $A = (\Gamma.A \rightarrow \Gamma)$ . We write

$$\Gamma \vdash A \text{ type}$$

if  $A$  is a type over  $\Gamma$ .

- A **term**  $t$  of a type  $A$  over a context  $\Gamma$  is a section of the display map  $\Gamma.A \rightarrow \Gamma$ . We write

$$\Gamma \vdash t : A$$

if  $t$  is a term of  $A$ . Moreover, if we have  $\Gamma = \Delta.X$ , then we also write

$$\Delta, x : X \vdash t(x) : A(x)$$

for this judgement.

- We write

$$\Gamma \vdash A \doteq A' \text{ type} \quad \text{and} \quad \Gamma \vdash t \doteq t' : A$$

if  $A$  and  $A'$  are equal as display maps and  $t$  and  $t'$  are equal as sections. We say types (or terms) are **judgementally equal** if such an equality holds.

- (Biased product of contexts.) Let  $\Delta \in \mathbf{C}$  be a context and  $\Gamma = \varepsilon.A_0 \cdots .A_n$  be another context. (Note that such a sequence  $A_0, \dots, A_n$  of types uniquely exists for any context  $\Gamma$ .) We define  $\Delta.\Gamma = \Delta.A_0 \cdots .A_n$  inductively as follows.
  - $\Delta.\varepsilon := \Delta$ . There is a canonical projection  $p_{-1} : \Delta \rightarrow \varepsilon$ .
  - Suppose we have constructed  $\Delta.A_0 \cdots .A_i$  and a projection  $p_i : \Delta.A_0 \cdots .A_i \rightarrow \varepsilon.A_0 \cdots .A_i$ . Then we define  $\Delta.A_0 \cdots .A_i.A_{i+1}$  and  $p_{i+1}$  as the following substitution.

$$\begin{array}{ccc} \Delta.A_0 \cdots .A_i.A_{i+1} & \xrightarrow{p_{i+1}} & \varepsilon.A_0 \cdots .A_i.A_{i+1} \\ \downarrow & \lrcorner & \downarrow \\ \Delta.A_0 \cdots .A_i & \xrightarrow{p_i} & \varepsilon.A_0 \cdots .A_i \end{array}$$

- (Morphisms of contexts.) Let  $\Delta, \Gamma$  be contexts. A morphism  $\vec{t} : \Delta \rightarrow \Gamma$  can be seen as a tuple of terms. For example, when we have  $\Gamma = \varepsilon.A_0.A_1$ , the morphism  $\vec{t}$  is determined by the following terms  $t_0, t_1$ .

$$\begin{array}{ccccccc} \Delta & \xrightarrow{t_1} & \Delta.A_1[t_0/x_0] & \longrightarrow & \Delta.\Gamma & \longrightarrow & \varepsilon.A_0.A_1 = \Gamma \\ & \searrow & \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ & & \Delta & \xrightarrow{t_0} & \Delta.A_0 & \longrightarrow & \varepsilon.A_0 \\ & & & \searrow & \downarrow & \lrcorner & \downarrow \\ & & & & \Delta & \longrightarrow & \varepsilon \end{array}$$

■

**Definition 1.5.** Let  $\mathbf{C}$  be a contextual category. We say  $\mathbf{C}$  has **extensional  $\Sigma$ -types** if for each context  $\Gamma$  and types  $\Gamma \vdash A \text{ type}$  and  $\Gamma.A \vdash B \text{ type}$ , it satisfies the following condition. There exists a type

$$\Gamma.A.B \vdash \Sigma(A, x.B) \text{ type}$$

and terms

$$\Gamma.A.B \vdash \text{pair} : \Sigma(A, x.B)$$

$$\Gamma.\Sigma(A, x.B) \vdash \text{prl} : A \quad \text{and} \quad \Gamma.\Sigma(A, x.B) \vdash \text{prr} : B[\text{prl}/x]$$

satisfying the following judgemental equalities;

$$\Gamma.\Sigma(A, x.B) \vdash \text{pair} \circ \langle \text{prl}, \text{prr} \rangle \doteq (p \mapsto p)$$

$$\Gamma.A.B \vdash \text{prl} \circ \text{pair} \doteq (x : A, y : B \mapsto x), \quad \Gamma.A.B \vdash \text{prr} \circ \text{pair} \doteq (x : A, y : B \mapsto y)$$

We write  $\mathbf{CwC}_\Sigma$  for the full subcategory of  $\mathbf{CwC}$  consisting of those with extensional  $\Sigma$ -types. ■

**Definition 1.6.** A **display map category**  $\mathbf{C} = (\mathbf{C}, \mathbf{dis}_\mathbf{C})$  is a pair of a category  $\mathbf{C}$  and a replete class  $\mathbf{dis}_\mathbf{C}$  of morphisms satisfying the following conditions. Arrows in  $\mathbf{dis}_\mathbf{C}$  are called **display maps** of  $\mathbf{C}$ .

- $\mathbf{C}$  has a terminal object<sup>2</sup>.

<sup>2</sup>In some literature, this condition is omitted for the definition of display map category.

- Let  $h: \Delta \rightarrow \Gamma$  and  $f: A \rightarrow \Gamma$  be morphisms in  $\mathbf{C}$  such that  $f$  is a display map. Then there is a pullback square

$$(1) \quad \begin{array}{ccc} \cdot & \xrightarrow{\quad} & A \\ \downarrow & \lrcorner & \downarrow f \\ \Delta & \xrightarrow{h} & \Gamma \end{array}$$

in  $\mathbf{C}$  such that the left side is also a display map. ■

**Definition 1.7.** Suppose we are given a display map category  $\mathbf{C}$  and an object  $\Gamma \in \mathbf{C}$ . We define a display map category  $\mathbf{C}(\Gamma)$  as follows.

- The underlying category is the full subcategory of the over category  $\mathbf{C}/\Gamma$  spanned by display maps in  $\mathbf{C}$ .
- A display map is a morphism  $f$  in  $\mathbf{C}(\Gamma)$  that is a display map in  $\mathbf{C}$ . ■

**Proposition 1.8.** The above definition indeed gives a display map category. ◆

**Definition 1.9.** Define the 2-category  $\mathfrak{CwD}$  of display map categories as follows.

- 0-cells are display map categories.
- 1-cells are functors preserving the terminal object, display maps, and pullbacks of the forms (1).
- A 2-cell  $\alpha: F \Rightarrow G: \mathbf{C} \rightarrow \mathbf{D}$  is a natural transformation such that naturality squares at display maps are pullback squares; i.e., for each display map  $f: A \rightarrow \Gamma$  in  $\mathbf{C}$ , the naturality square

$$\begin{array}{ccc} FA & \xrightarrow{\alpha_A} & GA \\ F(f) \downarrow & \lrcorner & \downarrow G(f) \\ F\Gamma & \xrightarrow{\alpha_\Gamma} & G\Gamma \end{array}$$

is a pullback square in  $\mathbf{D}$ . ■

**Definition 1.10.** A display map category is *democratic* if for each object  $\Gamma \in \mathbf{C}$ , there exists a sequence of display maps from  $\Gamma$  to the terminal object. We write  $\mathbf{CwD}^{\text{dm}}$  for the full sub 2-category of  $\mathfrak{CwD}$  spanned by democratic display map categories. ■

**Proposition 1.11.**  $\mathbf{CwD}^{\text{dm}}$  is a (2,1)-category. ◆

**Definition 1.12** ([Joy17]). A *clan*  $\mathbf{C} = (\mathbf{C}, \mathbf{fib}_{\mathbf{C}})$  is a display map category satisfying the following conditions. Display maps (i.e., arrows in  $\mathbf{fib}_{\mathbf{C}}$ ) are called *fibrations* of  $\mathbf{C}$ .

- For each object  $A \in \mathbf{C}$ , the unique morphism  $A \rightarrow 1$  towards the terminal object is a fibration.
- $\mathbf{fib}_{\mathbf{C}}$  is closed under composition and contains all isomorphisms. ■

We write  $\mathbf{Clan}$  for the full sub 2-category of  $\mathfrak{CwD}$  spanned by clans. Since clans are always democratic, this is a (2,1)-category.

**Theorem 1.13.** There exists a biadjunction

$$\mathbf{CwC} \begin{array}{c} \xrightarrow{|-|} \\ \perp \\ \xleftarrow{\text{cxt}} \end{array} \mathfrak{CwD}$$

that restricts to the following biequivalences;

$$\begin{array}{ccc} \mathbf{CwC} & \begin{array}{c} \xrightarrow{|-|} \\ \simeq \\ \xleftarrow{\text{cxt}} \end{array} & \mathbf{CwD}^{\text{dm}} \\ \\ \mathbf{CwC}_\Sigma & \begin{array}{c} \xrightarrow{|-|} \\ \simeq \\ \xleftarrow{\text{cxt}} \end{array} & \mathbf{Clan} \end{array}$$

In particular,  $\mathbf{CwD}^{\text{dm}}$  and  $\mathbf{Clan}$  are essentially 1-categories; i.e., setoid enriched categories. ◆

*Sketch of proof.* There is a canonical way to see a contextual category as a display map category. This indeed induces a 2-functor  $|-|: \mathbf{CwC} \rightarrow \mathfrak{CwD}$  which is an embedding.

For each display map category  $\mathbf{D}$ , we construct a contextual category  $\mathbf{cxt}(\mathbf{D})$  as follows.

- Firstly, choose a terminal object  $1$ . Then choose a section  $\Gamma.-$  of the quotient functor  $\mathbf{D}(\Gamma)^\cong \rightarrow \pi_0(\mathbf{D}(\Gamma)^\cong)$  for each  $\Gamma \in \mathbf{D}$ .
- An object in  $\mathbf{cxt}(\mathbf{D})$  is a tuple  $(A_0, A_1, \dots, A_n)$  ( $n \geq 0$ ) such that
  - $A_0 \in \pi_0(\mathbf{D}(1))$ .
  - $A_{i+1} \in \pi_0(\mathbf{D}(1.A_0 \cdots A_i))$  for each  $0 \leq i < n$ .
- A morphism  $(A_0, \dots, A_n) \rightarrow (B_0, \dots, B_m)$  is a morphism  $1.A_0 \cdots A_n \rightarrow 1.B_0 \cdots B_m$ .

$\mathbf{cxt}(\mathbf{D})$  is independent of the choice of the terminal object and the sections of the quotient functors, up to isomorphism of categories. Moreover, one can check the natural equivalence

$$\mathbf{CwC}(\mathbf{T}, \mathbf{cxt}(\mathbf{D})) \simeq \mathfrak{CwD}(|\mathbf{T}|, \mathbf{D})$$

of categories, which shows the biadjunction. By definition of  $\mathbf{cxt}(\mathbf{D})$ , the counit  $|\mathbf{cxt}(\mathbf{D})| \rightarrow \mathbf{D}$  is an embedding, and it is essentially surjective if and only if  $\mathbf{D}$  is democratic. This shows the first biequivalence. The second one is checked by confirming the image of clans under  $\mathbf{cxt}$  has extensional  $\Sigma$ -types.  $\square$

## 2. LOGICAL FRAMEWORK À LA UEMURA

[Uem23]

**Definition 2.1.** A *representable map category*  $\mathbf{R} = (\mathbf{R}, \mathbf{rep}_{\mathbf{R}})$  is a clan satisfying the following conditions. Arrows in  $\mathbf{rep}_{\mathbf{R}}$  are called *representable maps* of  $\mathbf{R}$ .

- $\mathbf{R}$  is finitely complete.
- For each representable map  $f: X \twoheadrightarrow Y$ , the pullback functor  $f^*: \mathbf{R}_Y \rightarrow \mathbf{R}_X$  has a right adjoint  $f_*$ . ■

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