NOTE ON TYPE THEORIES

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Notation 0.1. We employ the following notations.

- (2,1)-categories are denoted by bf symbols: $\mathbf{C}, \mathbf{A}, \mathbf{E}, \dots$ or bb symbols: $\mathbb{I}, \mathbb{D}, \mathbb{A}, \dots$
- **Set** is the category of sets.
- **Cat** is the large (2,1)-category of categories.
- There is a fully faithful (2,1)-functor $\mathtt{disc}\colon \mathbf{Set} \hookrightarrow \mathbf{Cat}$.
- $\mathfrak{C}at$ is the large 2-category of categories.
- Δ^1 is the 1-simplex seen as a category. $\mathbf{C}^{\Delta^1}$ is the arrow category of $\mathbf{C}$.
- $\mathbf{C}_{/A}$ and $\mathbf{C}_{A/}$ are over and under categories respectively.
- $\mathtt{cod}$ and $\mathtt{dom}$ mean codomain and domain respectively. They often have as their type $\mathbf{C}^{\Delta^1} \longrightarrow \mathbf{C}$, $\mathbf{C}_{/A} \longrightarrow \mathbf{C}$, or $\mathbf{C}_{A/} \longrightarrow \mathbf{C}$.
- By a ***replete class of morphisms*** of $\mathbf{C}$, we mean a replete full subcategory of $\mathbf{C}^{\Delta^1}$. ■

1. Dependent type theories in terms of display maps

See also [nLa].

Definition 1.1 (Strict Categories)**.** We write **StrCat** for the (1-)category of categories, and an object in **StrCat** is called a ***strict category***. For each strict category $\mathbf{C}$, we write $\mathsf{Obj}(\mathbf{C})$ for its underlying object. By *object* of $\mathbf{C}$, we mean an element in $\mathsf{Obj}(\mathbf{C})$. ■

Remark 1.2. You can see a strict category as a category equipped with its data of *objects*; we can define a ***strict category*** as a category $\mathbf{C}$ equipped with a set $\mathsf{Obj}(\mathbf{C})$ and a essentially surjective functor towards $\mathbf{C}$. They form a full sub-(2,1)-category of $\mathbf{Cat}^{\Delta^1}$, which is indeed equivalent to **StrCat**. ■

Definition 1.3 (See [KL18, KL21, nLa].)**.** A ***contextual category*** $\mathbf{T}$ consists of the following data.

- A strict category $\mathbf{T}$.
- An object ε of $\mathbf{T}$ which is a terminal object.
- A class[1] $\mathbf{dis_T}$ of ***display maps*** such that for each object $\Gamma \in \mathsf{Obj}(\mathbf{T})$, there exists a unique path of display maps $\Gamma \twoheadrightarrow \cdots \twoheadrightarrow \varepsilon$.
- For each display map $A = (\Gamma.A \twoheadrightarrow \Gamma)$ and a morphism $\vec{t}\colon \Delta \longrightarrow \Gamma$ there is a canonical pullback square

$$\begin{tikzcd} \Delta.A[\vec{t}/\vec{x}] \arrow[r, "q(\vec{t},p)"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d, two heads, "p"] \\ \Delta \arrow[r, "\vec{t}"'] & \Gamma \end{tikzcd}$$

  whose left side is a display map, which is denoted by $A[\vec{t}/\vec{x}]$ and called the ***substitution*** of $\vec{t}$ in A. Moreover, those substitutions are strictly functorial: $A[\vec{t}/\vec{x}][\vec{s}/\vec{y}] = A[\vec{t} \circ \vec{s}/\vec{y}]$ and $q(\vec{t}, A) \circ q(\vec{s}, A[\vec{t}/\vec{x}]) = q(\vec{t} \circ \vec{s}, A)$.

We write **CwC** for the category of contextual categories and functors preserving those structures. ■

Notation 1.4. Let $\mathbf{T}$ be a contextual category.

- A ***context*** in $\mathbf{T}$ is an object in $\mathbf{T}$.
- A ***type*** A over a context Γ in $\mathbf{C}$ is a display map $A = (\Gamma.A \twoheadrightarrow \Gamma)$. We write

$$\Gamma \vdash A \ \mathsf{type}$$

  if A is a type over Γ.

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[1] A *class of morphisms* in a strict category $\mathbf{T}$ is just a subgraph of the *underlying graph* of $\mathbf{T}$, which is defined by pulling back $\mathbf{T}^{\Delta^1} \longrightarrow \mathbf{T} \times \mathbf{T}$ along $\mathsf{Obj}(\mathbf{T}) \times \mathsf{Obj}(\mathbf{T}) \longrightarrow \mathbf{T} \times \mathbf{T}$.

- A ***term*** t of a type A over a context Γ is a section of the display map $\Gamma.A \twoheadrightarrow \Gamma$. We write
$$\Gamma \vdash t : A$$
if t is a term of A. Moreover, if we have $\Gamma = \Delta.X$, then we also write
$$\Delta, x : X \vdash t(x) : A(x)$$
for this judgement.
- We write
$$\Gamma \vdash A \doteq A' \ \mathsf{type} \quad \text{and} \quad \Gamma \vdash t \doteq t' : A$$
if A and A' are equal as display maps and t and t' are equal as sections. We say types (or terms) are ***judgementally equal*** if such an equality holds.
- (Biased product of contexts.) Let $\Delta \in \mathbf{C}$ be a context and $\Gamma = \varepsilon.A_0.\cdots.A_n$ be another context. (Note that such a sequence $A_0, \cdots, A_n$ of types uniquely exists for any context Γ.) We define $\Delta.\Gamma = \Delta.A_0.\cdots.A_n$ inductively as follows.
  - $\Delta.\varepsilon := \Delta$. There is a canonical projection $p_{-1} \colon \Delta \longrightarrow \varepsilon$.
  - Suppose we have constructed $\Delta.A_0.\cdots.A_i$ and a projection $p_i \colon \Delta.A_0.\cdots.A_i \longrightarrow \varepsilon.A_0.\cdots.A_i$. Then we define $\Delta.A_0.\cdots.A_i.A_{i+1}$ and p_{i+1} as the following substitution.
$$\begin{tikzcd}
\Delta.A_0.\cdots.A_i.A_{i+1} \arrow[r, "p_{i+1}"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \varepsilon.A_0.\cdots.A_i.A_{i+1} \arrow[d, two heads] \\
\Delta.A_0.\cdots.A_i \arrow[r, "p_i"] & \varepsilon.A_0.\cdots.A_i
\end{tikzcd}$$
- (Morphisms of contexts.) Let Δ, Γ be contexts. A morphism $\vec{t} \colon \Delta \longrightarrow \Gamma$ can be seen as a tuple of terms. For example, when we have $\Gamma = \varepsilon.A_0.A_1$, the morphism $\vec{t}$ is determined by the following terms t_0, t_1.
$$\begin{tikzcd}
\Delta \arrow[r, "t_1"] \arrow[dr, equal] & \Delta.A_1[t_0/x_0] \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta.\Gamma \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \varepsilon.A_0.A_1 = \Gamma \arrow[d, two heads] \\
& \Delta \arrow[r, "t_0"] \arrow[dr, equal] & \Delta.A_0 \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \varepsilon.A_0 \arrow[d, two heads] \\
& & \Delta \arrow[r] & \varepsilon
\end{tikzcd}$$

■

Definition 1.5. Let $\mathbf{C}$ be a contextual category. We say $\mathbf{C}$ has ***extensional Σ-types*** if for each context Γ and types $\Gamma \vdash A$ type and $\Gamma.A \vdash B$ type, it satisfies the following condition. There exists a type
$$\Gamma.A.B \vdash \Sigma(A, x.B) \ \mathsf{type}$$
and terms
$$\Gamma.A.B \vdash \mathtt{pair} : \Sigma(A, x.B)$$
$$\Gamma.\Sigma(A, x.B) \vdash \mathtt{prl} : A \quad \text{and} \quad \Gamma.\Sigma(A, x.B) \vdash \mathtt{prr} : B[\mathtt{prl}/x]$$
satisfying the following judgemental equalities;
$$\Gamma.\Sigma(A, x.B) \vdash \mathtt{pair} \circ \langle \mathtt{prl}, \mathtt{prr} \rangle \doteq (p \mapsto p)$$
$$\Gamma.A.B \vdash \mathtt{prl} \circ \mathtt{pair} \doteq (x : A, y : B \mapsto x), \quad \Gamma.A.B \vdash \mathtt{prr} \circ \mathtt{pair} \doteq (x : A, y : B \mapsto y)$$
We write $\mathbf{CwC}_\Sigma$ for the full subcategory of $\mathbf{CwC}$ consisting of those with extensional Σ-types. ■

Definition 1.6. A ***display map category*** $\mathbf{C} = (\mathbf{C}, \mathbf{dis}_\mathbf{C})$ is a pair of a category $\mathbf{C}$ and a replete class $\mathbf{dis}_\mathbf{C}$ of morphisms satisfying the following conditions. Arrows in $\mathbf{dis}_\mathbf{C}$ are called ***display maps*** of $\mathbf{C}$.

- $\mathbf{C}$ has a terminal object[2].

[2]In some literature, this condition is omitted for the definition of display map category.

- Let $h\colon \Delta \longrightarrow \Gamma$ and $f\colon A \twoheadrightarrow \Gamma$ be morphisms in $\mathbf{C}$ such that f is a display map. Then there is a pullback square

(1)
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, two heads, "f"] \\ \Delta \arrow[r, "h"'] & \Gamma \end{tikzcd}$$

in $\mathbf{C}$ such that the left side is also a display map. ■

Definition 1.7. Suppose we are given a display map category $\mathbf{C}$ and an object $\Gamma \in \mathbf{C}$. We define a diplay map category $\mathbf{C}(\Gamma)$ as follows.

- The underlying category is the full subcategory of the over category $\mathbf{C}_{/\Gamma}$ spanned by diplay maps in $\mathbf{C}$.
- A display map in $\mathbf{C}(\Gamma)$ is a morphism $f \in \mathbf{C}(\Gamma)$ such that $\mathtt{dom}(f)$ is a display map in $\mathbf{C}$. ■

Proposition 1.8. The above definition indeed gives a display map category. ♦

Definition 1.9. Define the 2-category $\mathfrak{C}w\mathfrak{D}$ of display map categories as follows.

- 0-cells are display map categories.
- 1-cells are functors preserving the terminal object, display maps, and pullbacks of the forms (1).
- A 2-cell $\alpha\colon F \Longrightarrow G\colon \mathbf{C} \longrightarrow \mathbf{D}$ is a natural transformation such that naturality squares at display maps are pullback squares; i.e., for each display map $f\colon A \twoheadrightarrow \Gamma$ in $\mathbf{C}$, the naturality square

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] \arrow[dr, phantom, very near start, "\lrcorner"] & GA \arrow[d, "G(f)"] \\ F\Gamma \arrow[r, "\alpha_\Gamma"'] & G\Gamma \end{tikzcd}$$

is a pullback square in $\mathbf{D}$. ■

Definition 1.10. A display map category is ***democratic*** if for each object $\Gamma \in \mathbf{C}$, there exists a sequence of display maps from Γ to the terminal object. We write $\mathbf{CwD}^{\mathsf{dm}}$ for the full sub 2-category of $\mathfrak{C}w\mathfrak{D}$ spanned by democratic display map categories. ■

Proposition 1.11. $\mathbf{CwD}^{\mathsf{dm}}$ is a (2,1)-category. ♦

Definition 1.12 ([Joy17])**.** A ***clan*** $\mathbf{C} = (\mathbf{C}, \mathbf{fib}_{\mathbf{C}})$ is a display map category satisfying the following conditions. Display maps (i.e., arrows in $\mathbf{fib}_{\mathbf{C}}$) are called ***fibrations*** of $\mathbf{C}$.

- For each object $A \in \mathbf{C}$, the unique morphism $A \longrightarrow 1$ towards the terminal object is a fibraion.
- $\mathbf{fib}_{\mathbf{C}}$ is closed under composition and contains all isomorphisms. ■

We write **Clan** for the full sub 2-category of $\mathfrak{C}w\mathfrak{D}$ spanned by clans. Since clans are always democratic, this is a (2,1)-category.

Theorem 1.13. There exists a biadjunction

$$\begin{tikzcd} \mathbf{CwC} \arrow[r, bend left, "|-|"] \arrow[r, phantom, "\perp"] & \mathfrak{C}w\mathfrak{D} \arrow[l, bend left, "\mathsf{cxt}"] \end{tikzcd}$$

that restricts to the following biequivalences;

$$\begin{tikzcd} \mathbf{CwC} \arrow[r, bend left, "|-|"] \arrow[r, phantom, "\simeq"] & \mathbf{CwD}^{\mathsf{dm}} \arrow[l, bend left, "\mathsf{cxt}"] \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{CwC}_\Sigma \arrow[r, bend left, "|-|"] \arrow[r, phantom, "\simeq"] & \mathbf{Clan} \arrow[l, bend left, "\mathsf{cxt}"] \end{tikzcd}$$

In particular, $\mathbf{CwD}^{\mathsf{dm}}$ and **Clan** are essentially 1-categories; i.e., setoid enriched categories. ♦

Sketch of proof. There is a canonical way to see a contextual catgory as a display map category. This indeed induces a 2-functor $|-|\colon \mathbf{CwC} \longrightarrow \mathfrak{C}w\mathfrak{D}$ which is an embedding.

For each display map category $\mathbf{D}$, we construct a contextual category $\mathsf{cxt}(\mathbf{D})$ as follows.

- Firstly, choose a terminal object 1. Then choose a section $\Gamma.-$ of the quotient functor $\mathbf{D}(\Gamma)^{\cong} \longrightarrow \pi_0(\mathbf{D}(\Gamma)^{\cong})$ for each $\Gamma \in \mathbf{D}$.
- An object in $\mathsf{cxt}(\mathbf{D})$ is a tuple $(A_0, A_1, \cdots, A_n)$ $(n \geq 0)$ such that
  - $A_0 \in \pi_0(\mathbf{D}(1))$.
  - $A_{i+1} \in \pi_0(\mathbf{D}(1.A_0.\cdots.A_i))$ for each $0 \leq i < n$.
- A morphism $(A_0, \cdots, A_n) \longrightarrow (B_0, \cdots, B_m)$ is a morphism $1.A_0.\cdots.A_n \longrightarrow 1.B_0.\cdots.B_m$.

$\mathsf{cxt}(\mathbf{D})$ is independent of the choice of the terminal object and the sections of the quotient functors, up to isomorphism of categories. Moreover, one can check the natural equivalence

$$\mathbf{CwC}(\mathbf{T}, \mathsf{cxt}(\mathbf{D})) \simeq \mathfrak{C}w\mathfrak{D}(|\mathbf{T}|, \mathbf{D})$$

of categories, which shows the biadjunction. By definition of $\mathsf{cxt}(\mathbf{D})$, the counit $|\mathsf{cxt}(\mathbf{D})| \longrightarrow \mathbf{D}$ is an embedding, and it is essentially surjective if and only if $\mathbf{D}$ is democratic. This shows the first biequivalence. The second one is checked by confirming the image of clans under cxt has extensional Σ-types. $\square$

2. Logical framework à la Uemura

[Uem23]

Definition 2.1. A ***representable map category*** $\mathbf{R} = (\mathbf{R}, \mathbf{rep}_{\mathbf{R}})$ is a clan satisfying the following conditions. Arrows in $\mathbf{rep}_{\mathbf{R}}$ are called ***representable maps*** of $\mathbf{R}$.

- $\mathbf{R}$ is finitely complete.
- For each representable map $f \colon X \twoheadrightarrow Y$, the pullback functor $f^* \colon \mathbf{R}_{/Y} \longrightarrow \mathbf{R}_{/X}$ has a right adjoint f_*. ∎

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