# NOTE ON TYPE THEORIES

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[Cis19]

**Notation 0.1.** We employ the following notations.

- (2,1)-categories are denoted by bf symbols: $\mathbf{C}, \mathbf{A}, \mathbf{E}, \dots$ or bb symbols: $\mathbb{I}, \mathbb{D}, \mathbb{A}, \dots$
- $\mathbf{Set}$ is the category of sets.
- $\mathbf{Cat}$ is the large (2,1)-category of categories.
- There is a fully faithful (2,1)-functor $\mathtt{disc}\colon \mathbf{Set} \hookrightarrow \mathbf{Cat}$.
- $\mathfrak{C}at$ is the large 2-category of categories.
- $\Delta^1$ is the 1-simplex seen as a category. $\mathbf{C}^{\Delta^1}$ is the arrow category of $\mathbf{C}$.
- $\mathbf{C}_{/A}$ and $\mathbf{C}_{A/}$ are over and under categories respectively.
- $\mathtt{cod}$ and $\mathtt{dom}$ mean codomain and domain respectively. They often have as their type $\mathbf{C}^{\Delta^1} \longrightarrow \mathbf{C}$, $\mathbf{C}_{/A} \longrightarrow \mathbf{C}$, or $\mathbf{C}_{A/} \longrightarrow \mathbf{C}$.
- By a ***replete class of morphisms*** of $\mathbf{C}$, we mean a replete full subcategory of $\mathbf{C}^{\Delta^1}$.
- $\mathbf{WSub}$ is the replete full sub (2,1)-category of $\mathbf{Cat}^{\Delta^1}$ consisting of wide subcategory inclusions; i.e., faithful functors that induce equivalences on core groupoids.
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## 1. Dependent type theories in terms of display maps

**Definition 1.1.** A ***display map category*** $\mathbf{C} = (\mathbf{C}, \mathbf{dis}_{\mathbf{C}})$ is a pair of a category $\mathbf{C}$ and a replete class $\mathbf{dis}_{\mathbf{C}}$ of morphisms satisfying the following conditions. Arrows in $\mathbf{dis}_{\mathbf{C}}$ are called ***display maps*** of $\mathbf{C}$.

- $\mathbf{C}$ has a terminal object.
- Let $h\colon \Delta \longrightarrow \Gamma$ and $f\colon A \twoheadrightarrow \Gamma$ be morphisms in $\mathbf{C}$ such that $f$ is a display map. Then there is a pullback square
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, two heads, "f"] \\ \Delta \arrow[r, "h"'] & \Gamma \end{tikzcd}$$
in $\mathbf{C}$ such that the left side is also a display map. ■

**Definition 1.2.** A display map category is ***democratic*** if for each object $\Gamma \in \mathbf{C}$, there exists a sequence of display maps from $\Gamma$ to the terminal object. ■

**Definition 1.3.** A ***clan*** $\mathbf{C} = (\mathbf{C}, \mathbf{fib}_{\mathbf{C}})$ is a display map category satisfying the following conditions. Display maps (i.e., arrows in $\mathbf{fib}_{\mathbf{C}}$) are called ***fibrations*** of $\mathbf{C}$.

- For each object $A \in \mathbf{C}$, the unique morphism $A \longrightarrow 1$ towards the terminal object is a fibraion.
- $\mathbf{fib}_{\mathbf{C}}$ is closed under composition. ■

**Definition 1.4.** A ***strict category*** is a category equipped with its wide subcategory that is a set. In other words, the (2,1)-category $\mathbf{StrCat}$ of strict categories is defined by the following pullback square.
$$\begin{tikzcd} \mathbf{StrCat} \arrow[r] \arrow[dd, "\mathsf{Obj}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{WSub} \arrow[d, hook] \\ & \mathbf{Cat}^{\Delta^1} \arrow[d, "\mathtt{dom}"] \\ \mathbf{Set} \arrow[r, "\mathtt{disc}"'] & \mathbf{Cat} \end{tikzcd}$$

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The functor on the left is locally fully faithful and surjective on objects. In particular, **StrCat** is a category. ■

**Definition 1.5.** A ***contextual category*** $\mathbf{C} = (\mathbf{C}, \mathbf{dis}_{\mathbf{C}}, \langle\rangle, \ell)$ consists of the following data.

- A strict category $\mathbf{C}$.
- A structure $\mathbf{dis}_{\mathbf{C}}$ of a display map category for the underlying category of $\mathbf{C}$.
- An object $\langle\rangle$ of $\mathbf{C}$ which is a terminal object in the underlying category of $\mathbf{C}$.
- A function $\ell\colon \mathsf{Obj}(\mathbf{C}) \longrightarrow \mathbb{N}$ satisfying the following conditions.
  - $\ell^{-1}(0) = \{\langle\rangle\}$.
  - The canonical function
  $$(\Gamma, A, f) \mapsto A\colon \coprod_{\Gamma \in \ell^{-1}(n)} \{(A, f) \mid f\colon A \twoheadrightarrow \Gamma\} \longrightarrow \mathsf{Obj}(\mathbf{C})$$
  is a monomorphism and $\ell^{-1}(n+1)$ coincides with its image. ■

## 2. Uemura's logical framework

**Definition 2.1.** A ***representable map category*** $\mathbf{R} = (\mathbf{R}, \mathbf{rep}_{\mathbf{C}})$ is a display map category satisfying the following conditions. Display maps (i.e., arrows in $\mathbf{rep}_{\mathbf{C}}$) are called ***representable maps*** of $\mathbf{R}$.

- $\mathbf{R}$ is finitely complete.
- For each representable map $f\colon X \twoheadrightarrow Y$, the pullback functor $f^*\colon \mathbf{R}_{/Y} \longrightarrow \mathbf{R}_{/X}$ has a right adjoint $f_*$. ■

**Theorem 2.2.** shoumei no aidani claim ireru yatu ♦

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**Theorem 2.3.** Saigo ni claim kuru taipu. ♦

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## References

[Cis19] D.-C. Cisinski. *Higher Categories and Homotopical Algebra*, volume 180 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2019. doi:10.1017/9781108588737.

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