NOTE ON TYPE THEORIES

KEISUKE HOSHINO

[Cis19]

Notation 0.1. We employ the following notations.

- (2,1)-categories are denoted by bf symbols: $\mathbb{C}, \mathbb{A}, \mathbb{E}, \ldots$ or bb symbols: $\mathbb{I}, \mathbb{D}, \mathbb{A}, \ldots$
- **Set** is the category of sets.
- Cat is the large (2,1)-category of categories.
- There is a fully faithful (2,1)-functor disc: Set \hookrightarrow Cat.
- *Cat* is the large 2-category of categories.
- Δ^1 is the 1-simplex seen as a category. \mathbf{C}^{Δ^1} is the arrow category of \mathbf{C} .
- $\mathbf{C}_{/A}$ and $\mathbf{C}_{A/}$ are over and under categories respectively.
- cod and dom mean codomain and domain respectively. They often have as their type $\mathbf{C}^{\Delta^1} \longrightarrow \mathbf{C}$, $\mathbf{C}_{/A} \longrightarrow \mathbf{C}$, or $\mathbf{C}_{A/} \longrightarrow \mathbf{C}$.
- By a *replete class of morphisms* of C, we mean a replete full subcategory of C^{Δ^1} .
- **WSub** is the replete full sub (2,1)-category of \mathbf{Cat}^{Δ^1} consisting of wide subcategory inclusions; i.e., faithful functors that induce equivalences on core groupoids.

1. Dependent type theories in terms of display maps

Definition 1.1. A *display map category* $\mathbf{C} = (\mathbf{C}, \mathbf{dis}_{\mathbf{C}})$ is a pair of a category \mathbf{C} and a replete class $\mathbf{dis}_{\mathbf{C}}$ of morphisms satisfying the following conditions. Arrows in $\mathbf{dis}_{\mathbf{C}}$ are called *display maps* of \mathbf{C} .

- C has a terminal object.
- Let $h: \Delta \longrightarrow \Gamma$ and $f: A \longrightarrow \Gamma$ be morphisms in ${\bf C}$ such that f is a display map. Then there is a pullback square

$$\begin{array}{ccc}
\cdot & \longrightarrow & A \\
\downarrow & & \downarrow f \\
\Delta & \longrightarrow & \Gamma
\end{array}$$

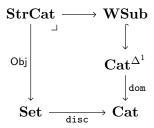
in C such that the left side is also a display map.

Definition 1.2. A display map category is *democratic* if for each object $\Gamma \in \mathbb{C}$, there exists a sequence of display maps from Γ to the terminal object.

Definition 1.3. A $clan\ \mathbf{C} = (\mathbf{C}, \mathbf{fib_C})$ is a display map category satisfying the following conditions. Display maps (i.e., arrows in $\mathbf{fib_C}$) are called fibrations of \mathbf{C} .

- For each object $A \in \mathbb{C}$, the unique morphism $A \to 1$ towards the terminal object is a fibraion.
- $\mathbf{fib}_{\mathbf{C}}$ is closed under composition.

Definition 1.4. A *strict category* is a category equipped with its wide subcategory that is a set. In other words, the (2,1)-category **StrCat** of strict categories is defined by the following pullback square.



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The functor on the left is locally fully faithful and surjective on objects. In particular, StrCat is a category.

Definition 1.5. A *contextual category* $C = (C, dis_C, \langle \rangle, \ell)$ consists of the following data.

- A strict category **C**.
- A structure $\operatorname{dis}_{\mathbf{C}}$ of a display map category for the underlying category of \mathbf{C} .
- An object $\langle \rangle$ of **C** which is a terminal object in the underlying category of **C**.
- A function $\ell \colon \mathsf{Obj}(\mathbf{C}) \longrightarrow \mathbb{N}$ satisfying the following conditions.

 - $-\ell^{-1}(0) = \{\langle\rangle\}.$ The canonical function

$$(\Gamma,A,f)\mapsto A\colon \coprod_{\Gamma\in\ell^{-1}(n)}\{(A,f)\,|\,f\colon A \twoheadrightarrow \Gamma\} \longrightarrow \mathsf{Obj}(\mathbf{C})$$

is a monomorphism and $\ell^{-1}(n+1)$ coincides with its image.

2. Uemura's logical framework

 $\textbf{Definition 2.1.} \ \text{A } \textit{representable map category } \mathbf{R} = (\mathbf{R}, \mathbf{rep_{C}}) \text{ is a display map category satisfying}$ the following conditions. Display maps (i.e., arrows in $\mathbf{rep}_{\mathbf{C}}$) are called *representable maps* of \mathbf{R} .

- **R** is finitely complete.
- For each representable map $f: X \longrightarrow Y$, the pullback functor $f^*: \mathbf{R}_{/Y} \longrightarrow \mathbf{R}_{/X}$ has a right adjoint f_* .

Theorem 2.2. shoumei no aidani claim ireru yatu

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References

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Email address: hoshinok@kurims.kyoto-u.ac.jp

RESEARCH INSTITUTE OF MATHEMATICAL SCIENCE, KYOTO UNIVERSITY

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