Category Theory

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Preface

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Chapter 1

Categories

§1 Categories

Definition 1.1: Category

A ***category*** $\mathcal{C}$ consists of three ingredients:

1. A *class* $\mathrm{obj}(\mathcal{C})$, called the ***objects***;
2. For any $A, B \in \mathrm{obj}(\mathcal{C})$, a set of ***morphisms*** $\mathrm{Hom}(A, B)$;
3. A function $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$, called the ***composition***, for any $A, B, C \in \mathrm{obj}(\mathcal{C})$, denoted as $(f, g) \mapsto gf$,

and they follow the following axioms:

(i) If $(A, B) \neq (A', B')$, then $\mathrm{Hom}(A, B) \cap \mathrm{Hom}(A', B') = \varnothing$;

(ii) *Associativity*: the composition is associative, i.e. $h(gf) = (hg)f$;

(iii) *Identity*: For any $A \in \mathrm{obj}(\mathcal{C})$, there is an identity morphism $\mathrm{id}_A \in \mathrm{Hom}(A, A)$, such that $f\,\mathrm{id}_A = f = \mathrm{id}_A\, f$, for any $B \in \mathrm{obj}(\mathcal{C})$ and $f \in \mathrm{Hom}(A, B)$.

A morphism can be shown by:

$$A \xrightarrow{\quad f \quad} B$$

Examples of categories: Set, Grp, Ab, Top, Ord, Ring, Mod, ...
If $\mathrm{obj}(\mathcal{C})$ is a set, then $\mathcal{C}$ is called a ***small category***.
If $(X, \leq)$ is a preorder set, then $\forall x, y \in X$,

$$\mathrm{Hom}(x, y) = \begin{cases} \varnothing & x > y, \\ \{(x, y)\} & x \leq y, \end{cases} \tag{1-1}$$

and $(y, z)(x, y) = (x, z)$. With this we can say that X is a category. The morphism (x, y) is also denoted by i_y^x.

Definition 1.2: Isomorphism
Let $\mathcal{C}$ be a category and $A, B \in \mathrm{obj}(\mathcal{C})$, $f \in \mathrm{Hom}(A, B)$. If $\exists g \in \mathrm{Hom}(B, A)$ s.t. $gf = \mathrm{id}_B$ and $fg = \mathrm{id}_A$, then f is called an ***isomorphism*** from A to B. g is called the ***inverse*** of f.

§2 Constructions of Categories from Existing Ones

Definition 2.1: Subcategory
We say $\mathcal{S}$ a ***subcategory*** of $\mathcal{C}$, if

(i) $\mathrm{obj}(\mathcal{S}) \subseteq \mathrm{obj}(\mathcal{C})$;

(ii) $\forall A, B \in \mathrm{obj}(\mathcal{S})$, $\mathrm{Hom}_{\mathcal{S}}(A, B) \subseteq \mathrm{Hom}_{\mathcal{C}}(A, B)$;

(iii) $\forall A, B, C \in \mathrm{obj}(\mathcal{S})$,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

then gf is the same in both $\mathrm{Hom}_{\mathcal{S}}(A, C)$ and $\mathrm{Hom}_{\mathcal{C}}(A, C)$,

(iv) $\forall A \in \mathrm{obj}(\mathcal{S})$, $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{S}}(A, A)$ is the same in $\mathrm{Hom}_{\mathcal{C}}(A, A)$.

Definition 2.2: Full subcategory

Let $\mathcal{S}$ be a subcategory of $\mathcal{C}$. If $\forall A, B \in \mathrm{obj}(\mathcal{S})$, $\mathrm{Hom}_{\mathcal{S}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)$, then $\mathcal{S}$ is called a ***full subcategory*** of $\mathcal{C}$.

Definition 2.3: Generated full subcategory

For any subclass $S \subseteq \mathrm{obj}(\mathcal{C})$, one can find a full subcategory $\mathcal{S}$ of $\mathcal{C}$ s.t. $\mathrm{obj}(\mathcal{S}) = S$, which is called the full subcategory generated by S.

Top_2 is the full subcategory of Top that is generated by the class of all Hausdorff spaces.

Definition 2.4: Opposite category

Let $\mathcal{C}$ be a category. Then $\mathcal{C}^{\mathrm{op}}$ is the category that:

1. $\mathrm{obj}(\mathcal{C}^{\mathrm{op}}) = \mathrm{obj}(\mathcal{C})$;
2. $\forall A, B \in \mathrm{obj}(\mathcal{C})$,
$$\mathrm{Hom}_{\mathcal{C}}(A, B) = \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(B, A). \tag{2-1}$$

Definition 2.5: Product category

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. The ***product category*** $\mathcal{C} \times \mathcal{D}$ is the category that:

1. $$\mathrm{obj}(\mathcal{C} \times \mathcal{D}) = \mathrm{obj}(\mathcal{C}) \times \mathrm{obj}(\mathcal{D}), \tag{2-2}$$
that is, the objects are pairs (C, D) where $C \in \mathrm{obj}(\mathcal{C})$ and

$D \in \text{obj}(\mathcal{D})$[a];

2.
$$\text{Hom}_{\mathcal{C}\times\mathcal{D}}((C,D),(C',D')) = \text{Hom}_{\mathcal{C}}(C,C') \times \text{Hom}_{\mathcal{D}}(D,D'); \tag{2-3}$$

3. The composition is defined as
$$(f,g)(f',g') = (ff',gg'); \tag{2-4}$$

4. The identity morphism is defined as
$$\text{id}_{(C,D)} = (\text{id}_C, \text{id}_D). \tag{2-5}$$

[a]The cartesian product of two classes A, B can be understood as a predicate $\varphi_{A\times B}((a,b)) = \varphi_A(a) \wedge \varphi_B(b)$, where φ_A and φ_B are the predicates that define the classes A and B.

§3 Definitions of Different Categories

Chapter 2

Functors

§4 Functors

Definition 4.1: Functor

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A ***functor*** $F : \mathcal{C} \to \mathcal{D}$ is a function that satisfies the following axioms:

(i) $\forall A \in \mathrm{obj}(\mathcal{C})$, $F(A) \in \mathrm{obj}(\mathcal{D})$;

(ii) $\forall A, B \in \mathrm{obj}(\mathcal{C}), \forall f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$;

(iii) $\forall A, B, C \in \mathrm{obj}(\mathcal{C})$,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

then $F(gf) = F(g)F(f)$.

(iv) $\forall A \in \mathrm{obj}(\mathcal{C})$, $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

We can restate some definition using functors

Theorem 4.1 (Subcategory, in language of functors). *Let $\mathcal{C}$ and $\mathcal{S}$ be two categories, $\mathcal{S} \subseteq \mathcal{C}$. If the inclusion $I: \mathcal{S} \to \mathcal{C}$ is a functor, then $\mathcal{S}$ is a subcategory of $\mathcal{C}$.*

The ***identity functor*** from $\mathcal{C}$ to itself is $1_{\mathcal{C}}: \mathcal{C} \to \mathcal{C}$ s.t. $\forall C, D \in \mathcal{C}$, $\forall f \in \mathrm{Hom}(C, D)$,

$$1_{\mathcal{C}}(C) = C, \quad 1_{\mathcal{C}}(f) = f. \tag{4-1}$$

One can define the ***diagonal functor*** $\Delta: \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ as $\Delta(X) = (X, X)$, $\Delta(f) = (f, f)$.

Theorem 4.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. $F: \mathcal{C} \to \mathcal{D}$ is a functor. $\forall A, B \in \mathrm{obj}(\mathcal{C})$, if $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is an isomorphism, then $F(f)$ is an isomorphism.*

Definition 4.2: Hom

Let $\mathcal{C}$ be a category and $A \in \mathrm{obj}(\mathcal{C})$. The ***Hom functor*** $F_A: \mathcal{C} \to \mathsf{Set}$ is defined as

$$\begin{aligned} &F_A(B) = \mathrm{Hom}(A, B), \\ &F_A(f): \ \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C); \ h \mapsto fh. \end{aligned} \tag{4-2}$$

The Hom functor is also denoted by $\mathrm{Hom}(A, -)$. We call the $F_A(f) =: \mathrm{Hom}(A, f)$ the ***induced map***, and denote it by f_*

$$f_* h = fh. \tag{4-3}$$

Definition 4.3: Faithful functor

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A ***faithful functor*** $F: \mathcal{C} \to \mathcal{D}$ is a functor that satisfies $\forall A, B \in \mathrm{obj}(\mathcal{C})$,

$$i: \ \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B)); \ f \mapsto F(f) \tag{4-4}$$

is injective.

Definition 4.4: Concrete category
Let $\mathcal{C}$ be a category. $\mathcal{C}$ is called a ***concrete category*** if there exists a faithful functor $F\colon \mathcal{C} \to \mathsf{Set}$.

§5 Contravariant Functors

Definition 5.1: Contravariant functor
Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A ***contravariant functor*** $F : \mathcal{C} \to \mathcal{D}$ is a function that satisfies the following axioms:

(i) $\forall A \in \mathrm{obj}(\mathcal{C}),\ F(A) \in \mathrm{obj}(\mathcal{D})$;

(ii) $\forall A, B \in \mathrm{obj}(\mathcal{C}), \forall f \in \mathrm{Hom}_{\mathcal{C}}(A, B),\ F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(B), F(A))$;

(iii) $\forall A, B, C \in \mathrm{obj}(\mathcal{C})$,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

then $F(gf) = F(f)F(g)$.

(iv) $\forall A \in \mathrm{obj}(\mathcal{C}),\ F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

To distinguish functors from contravariant functors, we sometimes call the functors ***covariant functors***.

$-^{\mathrm{op}}\colon \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ is a contravariant functor.

Definition 5.2: Contravariant Hom
Let $\mathcal{C}$ be a category and $A \in \mathrm{obj}(\mathcal{C})$. The ***contravariant Hom functor*** $F_A\colon \mathcal{C} \to \mathsf{Set}$ is defined as

$$\begin{aligned} &F_A(B) = \mathrm{Hom}(B, A), \\ &F_A(f)\colon\ \mathrm{Hom}(B, A) \to \mathrm{Hom}(C, A);\ h \mapsto hf. \end{aligned} \tag{5-1}$$

The contravariant Hom functor is also denoted by $\mathrm{Hom}(-, A)$. We call the $F_A(f) =: \mathrm{Hom}(f, A)$ the ***induced map***, and denote it by f^*

$$f^* h = hf. \tag{5-2}$$

§6 Diagrams

Definition 6.1: Diagram

A ***diagram*** in a category $\mathcal{C}$ is a functor $D\colon \mathcal{D} \to \mathcal{C}$ where $\mathcal{D}$ is a small category.

We have already seemed drawn diagrams like

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"] \arrow[dr, bend right=60, "h'"'] & B \arrow[d, "g"] \\ & C \end{tikzcd} \tag{6-1}$$

where $A, B, C \in D(\mathrm{obj}(\mathcal{D}))$, and each arrow from one to another is a morphism in the image of morphism in $\mathcal{D}$ under D e.g. $\exists D_A, D_B \in \mathrm{obj}(\mathcal{D})$ s.t.

$$f \in D(\mathrm{Hom}_{\mathcal{D}}(D_A, D_B)) \subseteq \mathrm{Hom}_{\mathcal{C}}(A, B). \tag{6-2}$$

Definition 6.2: Path

A ***path*** in a category $\mathcal{C}$ is a functor $P\colon n+1 \to \mathcal{C}$ where $n+1$ is considered as a preorder with morphism defined in Eq. (1-1).

Conventionally we denote a path as:

$$P_0 \xrightarrow{f_0} P_1 \xrightarrow{f_1} P_2 \longrightarrow \cdots \longrightarrow P_{n-1} \xrightarrow{f_{n-1}} P_n \tag{6-3}$$

A ***simple path*** is a path such that $\forall i, j \in n+1$, $P_i = P_j \to i = j$.

A diagram D ***commutes*** iff $A, B \in D(\mathrm{obj}\,\mathcal{D})$, the compositions of morphisms in any two simple paths from A to B are the same.

§7 Natural transformations

Definition 7.1: Natural transformation

Let $\mathcal{C}, \mathcal{D}$ be two categories and $F, G\colon \mathcal{D} \to \mathcal{C}$ be functors. A ***natural transformation*** $\alpha\colon F \to G$ is *one-parametre family of morphisms* in $\mathcal{D}$:

$$\alpha\colon \operatorname{obj}(\mathcal{C}) \to \{\operatorname{Hom}(F(A), G(A)) \mid A \in \operatorname{obj}(\mathcal{C})\};\ A \mapsto \alpha_A, \quad (7\text{-}1)$$

such that $\forall A, B \in \operatorname{obj}(\mathcal{C})$, $\forall f \in \operatorname{Hom}(A, B)$, the following diagram commutes:

$$\begin{tikzcd} F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\alpha_B"] & G(B) \end{tikzcd} \quad (7\text{-}2)$$

or,

$$\alpha_B F(f) = G(f)\alpha_A. \quad (7\text{-}3)$$

All natural transformations between two functors $F, G\colon \mathcal{C} \to \mathcal{D}$ is denoted as $\operatorname{Nat}(F, G)$. However, $\operatorname{Nat}(F, G)$ can only be considered as an object in our metalanguage, since it does not even have to be a class.

A ***natural isomorphism*** is a natural transformation $\alpha\colon F \to G$ such that $\forall A \in \operatorname{obj}(\mathcal{C})$, α_A is an isomorphism.

Natural transformations can compose, and for any functor, there exists an identity natural isomorphism.

You can define the contravariant version of natrual transformation too.

Theorem 7.1 (Yoneda lemma)**.** *Let $\mathcal{C}$ be a category and $F\colon \mathcal{C} \to \mathsf{Set}$ be a functor. Then, $\forall A \in \operatorname{obj}(\mathcal{C})$, $\operatorname{Nat}(\operatorname{Hom}_{\mathcal{C}}(A, -), G)$ is a set*[1]*, and there exists a bijection*

$$y\colon \operatorname{Nat}(\operatorname{Hom}_{\mathcal{C}}(A, -), G) \to F(A) \quad (7\text{-}4)$$

[1] Why?

s.t.

$$y(\tau) = \tau_A(1_A). \tag{7-5}$$

Proof. $\tau \in \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(A,-),G)$ means $\forall A, B \in \mathrm{obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\tau_B"] \arrow[d, "\varphi_*"] & F(B) \arrow[d, "F(\varphi)"] \\ \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[r, "\tau_C"] & F(C) \end{tikzcd} \tag{7-6}$$

commutes. Setting $B = A$ we have

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(A,A) \arrow[r, "\tau_A"] \arrow[d, "\varphi_*"] & F(A) \arrow[d, "F(\varphi)"] \\ \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[r, "\tau_C"] & F(C) \end{tikzcd} \tag{7-7}$$

which gives

$$F(\varphi)\tau_A(1_A) = \tau_C\varphi_*(1_A) = \tau_C(\varphi). \tag{7-8}$$

Injectivity Now assuming there exists another natural transformation $\sigma\colon \mathrm{Hom}_{\mathcal{C}}(A,-) \to G$ such that $\sigma_A(1_A) = \tau_A(1_A)$, we have $\forall C \in \mathrm{obj}(\mathcal{C})$,

$$\sigma_C(\varphi) = F(\varphi)\sigma_A(1_A) = F(\varphi)\tau_A(1_A) = \tau_C(\varphi), \tag{7-9}$$

i.e. $y(\tau) = y(\sigma) \to \tau = \sigma$, or in plain words, y is an injection.

Surjectivity $\forall a \in F(A)$, find a morphism $\tau_A\colon \mathrm{Hom}_{\mathcal{C}}(A,A) \to F(A)$ s.t. $\tau_A(1_A) = a$ (this is always possible e.g. we can set τ_A to be the constant function $\mathrm{Hom}_{\mathcal{C}}(A,A) \ni f \mapsto a$). Then, $\forall C \in \mathrm{obj}(\mathcal{C})$, $\forall \varphi \in \mathrm{Hom}_{\mathcal{C}}(A,C)$, define morphism as $\tau_C(\varphi) = F(\varphi)(a)$.

We have not yet proved that τ is natural, so we check if $\psi \in \mathrm{Hom}_{\mathcal{C}}(B,C)$, $\forall \vartheta \in \mathrm{Hom}_{\mathcal{C}}(A,B)$:

$$\tau_C\psi_*(\vartheta) = F(\psi_*\vartheta)(a) = F(\psi\vartheta)(a) = F(\psi)F(\vartheta)(a) = F(\psi)\tau_B(\vartheta). \tag{7-10}$$

□

Chapter 3

Universal Constructions

§8 Universal Properties

Definition 8.1: universal morphism

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor, $D \in \operatorname{obj}(\mathcal{D})$. A ***universal morphism*** from D to F is a unique pair (C, u) where $C \in \operatorname{obj}(\mathcal{C})$, $u \in \operatorname{Hom}_{\mathcal{D}}(X, F(C))$, with the following property (a.k.a. the ***universal property***):

$$\forall C' \in \operatorname{obj}(\mathcal{C})\, \forall f \in \operatorname{Hom}_{\mathcal{D}}(D, F(C'))\ \exists! g \in \operatorname{Hom}_{\mathcal{C}}(C, C')$$

s.t. the following diagram commutes:

$$\begin{tikzcd} D \arrow[r, "u"] \arrow[dr, "f"'] & F(C) \arrow[d, dashed, "F(g)"] & C \arrow[d, dashed, "g"] \\ & F(C') & C' \end{tikzcd} \tag{8-1}$$

The dashed arrow in a diagram indicates that the morphism is unique.

Definition 8.2: universal morphism (dualised)

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor, $D \in \mathrm{obj}(\mathcal{D})$. A ***universal morphism*** from F to D is a unique pair (C, u) where $C \in \mathrm{obj}(\mathcal{C})$, $u \in \mathrm{Hom}_{\mathcal{D}}(F(C), D)$, with the universal property:

$$\forall C' \in \mathrm{obj}(\mathcal{C})\, \forall f \in \mathrm{Hom}_{\mathcal{D}}(F(C'), D)\ \exists! g \in \mathrm{Hom}_{\mathcal{C}}(C', C)$$

s.t. the following diagram commutes:

$$\begin{tikzcd}
D & F(C) \arrow[l, "u"'] & C \\
& F(C') \arrow[ul, "f"] \arrow[u, dashed, "F(g)"] & C' \arrow[u, dashed, "g"]
\end{tikzcd} \tag{8-2}$$

As an example, one tries to define $C_1 \times C_2$ for a category $\mathcal{C}$, which should be a new object. One already has the product category $\mathcal{C} \times \mathcal{C}$ and one can find (C_1, C_2) in $\mathrm{obj}(\mathcal{C} \times \mathcal{C})$. Then, the universal morphism from Δ to $(C_1, C_2) \in \mathrm{obj}(\mathcal{C} \times \mathcal{C})$ is $(C_1 \times C_2, (\pi_1, \pi_2))$, where π_1, π_2 are the projections from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$:

$$\pi_1\colon C_1 \times C_2 \to C_1, \quad \pi_2\colon C_1 \times C_2 \to C_2, \tag{8-3}$$

if the category $\mathcal{C}$ has finite products (i.e. the product $C_1 \times C_2 \in \mathrm{obj}(\mathcal{C})$).

Appendix A

Appendix

Bibliography

[1] Joseph J. Rotman. *An Introduction to Homological Algebra*. Ed. by Joseph J. Rotman. Universitext. New York, NY: Springer, 2009, pp. 1–36. ISBN: 978-0-387-68324-9. DOI: 10.1007/978-0-387-68324-9_1. URL: https://doi.org/10.1007/978-0-387-68324-9_1.

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