Algebraic Topology

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Chapter 1

Homotopy and Fundamental Group

§1 Homotopy

Definition 1.1 (Homotopy)**.** $f, g \in C(X, Y)$. If $\exists H \in C(X \times [0,1], Y)$ s.t. $H(x, 0) = f(x)$, $H(x, 1) = g(x)$, then we say f and g are ***homotopic***, denoted by $f \simeq g \colon X \to Y$ or just $X \to Y$. H is called a ***homotopy*** between f and g, denoted by $H \colon f \simeq g$ or $f \simeq_H g$.

For $t \in [0,1]$, $h_t \colon X \to Y; x \mapsto H(x,t)$ is called a t***-slice***.

If f is homotopic to a constant mapping, we say that f is ***null-homotopic***.

A ***linear homotopy*** is a homotopy between two functions to $Y \subseteq \mathbb{R}^n$ that change linearly, i.e.

$$H(x,t) = (1-t)f(x) + tg(x).$$

Theorem 1.1 (Maps to convex set are homotopic)**.** $f, g \in C(X, Y)$. *If Y is a convex set in $\mathbb{R}^n$, then $f \simeq g$.*

Proof. Consider linear homotopy. □

Theorem 1.2. *Homotopic relation is an equivalence relation.*

Proof. *reflexity.* $f \simeq f$, just take $H(x,t) = f(x)$ for any t (Such homotopy is called a ***constant homotopy***).

Symmetry. $f \simeq g$ then $g \simeq f$. Just take $\bar{H}(x,t) = H(x, 1-t)$ (Here $\bar{H}$ is called the inverse of H).

Transivity. $f \simeq g \wedge g \simeq h \to f \simeq h$. Let

$$H_1 H_2(x, 2t) = \begin{cases} H_1(x, 2t) & t \in [0, 1/2], \\ H_2(x, 2t-1) & t \in [1/2, 1]. \end{cases}$$

We can see that $H_1 H_2$ is also a homotopy (see Theorem **??** in Point Set Topology) □

Hence, we can define ***homotopy classes*** on $C(X,Y)$, denoted by $[X,Y]$.

As you might expect after reading the proof of Theorem 1.2, the homotopies between mappings within a homotopy class form a group.

Theorem 1.3 (Composition of homotopies)**.** $f_1 \simeq f_2 \colon X \to Y$, $g_1 \simeq g_2 \colon Y \to Z$, *then* $g_1 \circ f_1 \simeq g_2 \circ f_2 \colon X \to Z$.

Proof i. Let $F \colon f_1 \simeq f_2$, $G \colon g_1 \simeq g_2$. Define:

$$\boldsymbol{F} \colon X \times [0,1] \to Y \times [0,1]; (x,t) \mapsto (F(x,t), t).$$

It can be verified tht $G \circ \boldsymbol{F} \colon g_1 \circ f_1 \simeq g_2 \circ g_2 \colon X \to Z$. □

Proof ii. Let $F \colon f_1 \simeq f_2$, $G \colon g_1 \simeq g_2$.

We can verify that $H_1 \colon (x,t) \mapsto g_1 \circ F(x,t)$ is a homotopy between $g_1 \circ f_1$ and $g_1 \circ f_2$; Similarly $H_2 \colon g_1 \circ f_2 \simeq g_2 \circ f_2$ can be defined.

Now consider $H = H_1 H_2$, or in detailed,

$$H(x,t) = \begin{cases} g_1 \circ F(x, 2t) & (x,t) \in X \times [0, 1/2] \\ G(f_2(x), 2t-1). & (x,t) \in X \times [1/2, 1] \end{cases}$$

$\square$

Lemma 1 (Identity map in convex space is null-homotopic)**.** *$X \subset \mathbb{R}^n$ is a convex space. $\forall x_0 \in X$, $\mathrm{id}_X \simeq (x \mapsto x_0)$.*

Proof. The linear homotopy can be constructed as:

$$H_{x_0}(x,t) = tx + (1-t)x_0.$$

$\square$

Theorem 1.4 (Continuous mappings from a convex set are null-homotopic)**.** *$X \subseteq \mathbb{R}^n$ is a convex set. $\forall f \in C(X,Y)$, f is null-homotopic.*

Proof. Let $H_{x_0}(x,t) = tx + (1-t)x_0$. Then, any $f\colon X \to Y$ can be written as $f = f \circ \mathrm{id}_X$, hence $f \simeq f \circ H_{x_0}(x,1) = (x \mapsto f(x_0))$, which means f is null-homotopic. $\square$

Theorem 1.5 (Constant mappings to a path-connected space belong to one homotopy class)**.** *If Y is a path-connected space, $y_0 \in Y$, then $[X,Y] = [x \mapsto y_0]$ (i.e. homotopy class of constant mapping to $\{y_0\}$)*

Proof. Let $f_1(x) = y_1$, $f_2(x) = y_2$ be two constant mappings, a is a path from y_1 to y_2. Then the homotopy between f_1 and f_2 can be defined as:

$$H(x,t) = a(t).$$

$\square$

Definition 1.2 (Homotopy relative to a set)**.** Let $A \subseteq X$, $H\colon f \simeq g$. If $\forall a \in A$, $\forall t \in [0,1]$, $f(a) = g(a) = H(a,t)$, we say that f and g are ***homotopic relative to*** A, denoted by $H\colon f \simeq g \operatorname{rel} A$.

We can have parallel results as Theorem 1.2 and Theorem 1.3:

Theorem 1.6. *Given $A \subseteq X$, $\simeq \operatorname{rel} A$ is an equivalence relation in $C(X,Y)$.*

Theorem 1.7 (Composition of relative homotopies)**.** *$f_1 \simeq f_2 \colon X \to Y \operatorname{rel} A$, $g_1 \simeq g_2 \colon Y \to Z \operatorname{rel} B$, and $f_1(A) \subset B$, then $g_1 \circ f_1 \simeq g_2 \circ f_2 \colon X \to Z$.*

Definition 1.3 (Fixed-endpoint Homotopy)**.** Let a, b be two paths in X. If $a \simeq b \operatorname{rel}\{0,1\}$, we say that a and b are ***fixed-endpoint homotopic***. The paths in X modulus fixed-point homotopy is denoted by $[X]$, called the ***path classes***. The path class which a belongs to is denoted by $\langle a \rangle$.

§2 Fundamental Group

Fundamental group of a topological space at a point is the path classes at this point. We need to introduce the multiplicative structure of path classes.

Theorem 2.1. *Let a, b, c, d be four paths in X.*

$$\begin{aligned} a \simeq b \operatorname{rel}\{0,1\} \ &\leftrightarrow \ \bar{a} \simeq \bar{b} \operatorname{rel}\{0,1\}, \\ a \simeq b \operatorname{rel}\{0,1\} \wedge c \simeq d \operatorname{rel}\{0,1\} \wedge a(1) = c(0) \ &\to \ ac \simeq bd \operatorname{rel}\{0,1\}. \end{aligned} \tag{2-1}$$

Definition 2.1 (Inverse and product of path classes)**.** $\alpha, \beta \in [X]$, $a \in \alpha$, $b \in \beta$. $b(0) = a(1)$. We define $\alpha^{-1} := \langle \bar{a} \rangle$ to be the ***inverse*** of the path class α, and $\alpha\beta := \langle ab \rangle$ to be the ***product*** of the two path classes α and β.

While the product of paths does not obey associativity, we have:

Theorem 2.2 (Associativity of product of path classes)**.** *$\alpha, \beta, \gamma \in [X]$. $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ (if they are productible).*

Proof. Consider $\forall a \in \alpha$, $\forall b \in \beta$, $\forall c \in \gamma$.

Let

$$\begin{aligned} \tilde{a}(t) &= t/3, \\ \tilde{b}(t) &= t/3 + 1/3, \\ \tilde{c}(t) &= t/3 + 2/3. \end{aligned} \tag{2-2}$$

$\tilde{a}, \tilde{b}$ and $\tilde{c}$ are three paths in $[0,1]$, and $\tilde{a}(\tilde{b}\tilde{c}) \simeq (\tilde{a}\tilde{b})\tilde{c} \operatorname{rel} \{0,1\}$, since $[0,1]$ is convex, therefore there is a linear homotopy between the two product paths.

Now Let $f\colon [0,1] \to X$ be

$$f(t) = \begin{cases} a(3t), & t \in [0,1/3]; \\ b(3t-1), & t \in [1/3,2/3]; \\ c(3t-2), & t \in [2/3,1]. \end{cases}$$

$a(bc) = f \circ \tilde{a}(\tilde{b}\tilde{c}) \simeq f \circ (\tilde{a}\tilde{b})\tilde{c} = (ab)c\mathrm{rel}\{0,1\}$, by Theorem 1.3. □

Theorem 2.3 (Identity-like properties of point path)**.** $\alpha \in [X]$. *Let the initial and the terminal point of* α *be* x_0 *and* x_1*.* (i) $\alpha^{-1}\alpha = \langle t \mapsto x_1 \rangle$, $\alpha\alpha^{-1} = \langle t \mapsto x_0 \rangle$*;* (ii) $\alpha\langle t \mapsto x_0 \rangle = \alpha = \langle t \mapsto x_1 \rangle \alpha$*.*

Proof. Note that $\mathrm{id}_{[0,1]}$ is a path in the convex set $[0,1]$. □

For now path classes are not closed under production.

Definition 2.2 (Fundamental group)**.** $x_0 \in X$. The path classes of loops at x_0 (paths that have both endpoints at x_0), equiped with production, is the ***fundamental group*** of X at x_0, denoted by $\pi_1(X, x_0)$.

Definition 2.3 (Homomorphism induced by continuous function)**.** $f \in C(X,Y)$, $x_0 \in X$. We define

$$f_\pi\colon [X] \to [Y], \quad \langle a \rangle \mapsto \langle f \circ a \rangle$$

where a is a path in X.

The limitation of f_π on $\pi_1(X, x_0)$ is said to be a ***homomorphism induced by*** f.

For simplicity, we would write such homomorphism by f_π (without explicitly referring limitation).

Theorem 2.4 (Isomorphism induced by homeomrphism)**.** *Let* f *be a homeomorphism from* X *to* Y*, then* $\forall x_0 \in X$*,* f_π *is an isomorphism from* $\pi_1(X, x_0)$ *to* $\pi_1(Y, f(x_0))$*.*

Proof.

$$
\begin{aligned}
f^{-1} \circ f = \mathrm{id}_X \;&\to\; (f^{-1})_\pi \circ f_\pi = \mathrm{id}_{\pi_1(X,x_0)};\\
f \circ f^{-1} = \mathrm{id}_Y \;&\to\; f_\pi \circ (f^{-1})_\pi = \mathrm{id}_{\pi_1(Y,f(x_0))},
\end{aligned}
\tag{2-3}
$$

therefore $(f^{-1})_\pi$ is the inverse of f_π. An invertible homomorphism is an isomorphism. □

Theorem 2.5 (Fundamental group of product space)**.** $x_0 \in X$, $y_0 \in Y$.

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Proof. Let $j_X \in C(X \times Y, X)$ and $j_Y \in C(X \times Y, Y)$ be projections ($j_X(x, y) = x$, $j_Y(x, y) = y$), and define a homomorphism

$$
\begin{array}{llll}
\varphi\colon & \pi_1(X \times Y, (x_0, y_0)) & \to & \pi_1(X, x_0) \times \pi_1(Y, y_0);\\
& \gamma & \mapsto & ((j_X)_\pi(\gamma), (j_Y)_\pi(\gamma)).
\end{array}
\tag{2-4}
$$

φ is a *monomorphism.* Let $\langle c\rangle \in \ker\varphi$ i.e.

$$\varphi(\langle c\rangle) = (\langle t \mapsto x_0\rangle, \langle t \mapsto y_0\rangle).$$

Let

$$H_X : j_X \circ c \simeq t \mapsto x_0 \text{ rel } \{0\}, \quad H_Y : j_Y \circ c \simeq t \mapsto y_0 \text{ rel } \{0\}.$$

The homotopy between c and $t \mapsto (x_0, y_0)$ is defined as

$$F\colon [0,1]^2 \to X \times Y; (t, s) \mapsto (H_X(t, s), H_Y(t, s)).$$

φ is an *epimorphism.* $\forall\langle a\rangle \in \pi_1(X, x_0)$ and $\forall\langle b\rangle \in \pi_1(Y, y_0)$. $c\colon t \mapsto (a(t), b(t)) \in C([0, 1], X \times Y)$. $\langle c\rangle \in \varphi^{-1}(\{(\langle a\rangle, \langle b\rangle)\})$. □

Theorem 2.6 (Fundamental groups of path connected space at different points are isomorphic)**.** *X is path connected,* $x_1, x_2 \in X$. $\pi_1(X, x_1) \cong \pi_1(X, x_2)$.

Proof. $\langle a\rangle \in \pi_1(X,x_1)$, $\langle b\rangle \in \pi_1(X,x_2)$, $\langle c\rangle$ is a path class with initial point x_1 and terminal point x_2.

It can be verified that

$$g_c\colon \pi_1(X,x_1) \to \pi_2(X,x_2); \langle a\rangle \mapsto \langle \bar{c}ac\rangle \tag{2-5}$$

is a homomorphism. Same as $g_{\bar{c}}(\langle b\rangle) = cb\bar{c}$.

$$\begin{aligned} g_c \circ g_{\bar{c}}(\langle b\rangle) &= \langle \bar{c}cb\bar{c}c\rangle = \mathrm{id}_{\pi_1(X,x_2)};\\ g_{\bar{c}} \circ g_c(\langle a\rangle) &= \langle c\bar{c}ac\bar{c}\rangle = \mathrm{id}_{\pi_1(X,x_1)}, \end{aligned} \tag{2-6}$$

therefore g_c is an isomorphism. □

With Theorem 2.6, we can write the fundamental group of a path-connected space X by $\pi_1(X)$.

For different path-connected branches, a topological space can have different fundamental groups, while they are isomorphic within one branch.

Theorem 2.7. *Let X and Y be two topological spaces, $f_1 \simeq f_2\colon X \to Y$, $x_0 \in X$, $f_1(x_0) = y_1$, $f_2(x_0) = y_2$; If $\exists c \in C([0,1],Y)$ s.t. $c(0) = y_1$, $c(1) = y_2$, and g_c is defined as in Eq. (2-5), then $g_c \circ f_{1,\pi} = f_{2,\pi}$.*

$$\begin{tikzcd} & \pi_1(Y,y_1) \arrow[dd, "g_c"] \\ \pi_1(X,x_0) \arrow[ur, "f_{1,\pi}"] \arrow[dr, "f_{2,\pi}"'] & \\ & \pi_1(Y,y_2) \end{tikzcd}$$

Proof. □

Definition 2.4 (Simply connected)**.** If the fundamental group of a path connected space X is trivial i.e. $\pi_1(X) \cong \{1\}$, we say that X is ***simply connected***.

Theorem 2.8 (Convex set is simply connected)**.** *If $X \subset \mathbb{R}^n$ is convex, then X is simply connected.*

Proof. $x_0 \in X$, $a \in C([0,1], X)$ s.t. $a(0) = a(1) = x_0$. $H_{a,x_0}(s,t) = (1-t)a(s) + tx_0$. □

§3 Examples of Fundamental Groups

3.1 S^1

Definition 3.1 (Lift)**.** Let X, Y, Z be three topological spaces, and $f \in C(X, Z)$, $p \in C(Y, Z)$. If $\tilde{f} \in C(X, Y)$, s.t. $f = p \circ \tilde{f}$, we say that $\tilde{f}$ is a ***lift*** of f.

$$\begin{tikzcd} & X \arrow[dl, "\tilde{f}"'] \arrow[dr, "f"] & \\ Y \arrow[rr, "p"] & & Z \end{tikzcd}$$

In some case, given f and p, $\tilde{f}$ might do not exist.

Lemma 2 (Lift of path)**.** $a \in C([0,1], S^1)$, $p\colon \mathbb{R} \to S^1; x \mapsto \mathrm{e}^{2\pi x \mathrm{i}}$. *Let $t_0 \in \mathbb{R}$ s.t. $p(x_0) = a(0)$. There exists a unique lift $\tilde{a} \in C([0,1], \mathbb{R})$ of a s.t. $\tilde{a}(0) = x_0$.*

$$\begin{tikzcd} & {[0,1]} \arrow[dl, "\tilde{a}"'] \arrow[dr, "a"] & \\ \mathbb{R} \arrow[rr, "p"] & & S^1 \end{tikzcd}$$

Proof. *Existence.* The collection of open sets that the images under a do not cover S^1, $\{(\alpha_i, \beta_i) \cap [0,1] \mid a_i, b_i \in \mathbb{R}^I \wedge S^1 \subsetneq a((\alpha_i, \beta_i))\}$, is a cover of $[0,1]$ by the definition of continuity. Since $[0,1]$ is compact, there exists a finite subcover $\{(\alpha_i, \beta_i) \cap [0,1] \mid a_i, b_i \in \mathbb{R}^n \wedge S^1 \subsetneq a((\alpha_i, \beta_i))\}$, where $n \in \mathbb{N}$. By dividing these open intevals into closed intevals that has no inner points intersecting, we can get $\Omega = \{I_k := [t_i, t_{i+1}] \mid k \in m\}$ (This can be done by sorting α_i and β_i).

The mapping p is locally homeomorphic i.e. there exists $[x_i, x_i'] \subset \mathbb{R}$ s.t. $p_i := p|_{[x_i, x_i']} \colon [x_i, x_i'] \to a(I_i)$ is a homeomorphism (and $p_i(x_i) = a(t_i)$), therefore $\tilde{a}_i := p_i^{-1} \circ a$ is a lift of $a_i := a|_{I_i}$.

Since $p_0(t_0) = a(t_0)$, $p_{i+1}(t_i) = p_i(t_i)$, we can define piecewisely the lift of a by $\tilde{a} = \cup\{\tilde{a}_i \mid i \in m\}$.

Uniqueness. Let $\tilde{a}'$ be another lift of a, $p(\tilde{a}'(t) - \tilde{a}(t)) = p \circ \tilde{a}'(t)/p \circ \tilde{a}(t) = a(t)/a(t) = 1$, therefore $\tilde{a}'(t) - \tilde{a}(t) \in \mathbb{Z}$. Since $[0,1]$ is connected, the image of $t \mapsto \tilde{a}'(t) - \tilde{a}(t)$ must be connected, which is possible only if it is constant. $\tilde{a}'(0) = \tilde{a}(0) = x_0$, therefore $\tilde{a} = \tilde{a}'$. □

Notice that we have the freedom to set $\tilde{a}(0) \in \mathbb{Z}$ (the lift is unique after setting that), what really matter is the difference $\tilde{a}(1) - \tilde{a}(0)$. One can proof that $q(a) := \tilde{a}(1) - \tilde{a}(0)$ does not depend on the chose of $\tilde{a}(0) \in \mathbb{Z}$. We call $q(a)$ the ***loop number*** of path a.

Lemma 3 (Two loops that are never antipodal have the same loop number)**.** *Let a, b be two loops at z_0 in S^1. If $\forall t \in [0,1]$, $a(t) \neq -b(t)$, then $q(a) = q(b)$.*

Proof. Choose $\tilde{a}(0) = \tilde{b}(0) = 0$ (if not so, just translate the lift by an integer). In this case, $q(a) = \tilde{a}(1)$, $q(b) = \tilde{b}(1)$.

If $q(a) \neq q(b)$, without loss of generality, $q(a) > q(b)$, then $f := t \mapsto \tilde{a}(t) - \tilde{b}(t)$ is a continuous function from a compact space $[0,1]$ to $\mathbb{R}$, therefore by the connectedness of $[0,1]$, $\exists t_0 \in [0,1]$ s.t. $f(t_0) = 1/2 \in [0, q(a) - q(b)]$, when

$$p \circ \tilde{a}(t_0) + p \circ \tilde{b}(t_0) = \mathrm{e}^{2\pi\mathrm{i}(\tilde{b}(t_0)+1/2)} + \mathrm{e}^{2\pi\mathrm{i}\tilde{b}(t_0)} = 0.$$

□

Lemma 4 (Same loop number iff homotopic ralative to endpoint)**.** *Let a, b be two loops at z_0 in S^1. $a \simeq b \operatorname{rel} \{0\}$ iff $q(a) = q(b)$.*

Proof. $\to$: Let $H\colon a \simeq b \operatorname{rel}\{0\}$, $h_s = t \mapsto H(t,s)$, $f_t = s \mapsto H(t,s)$.

$\forall (t,s) \in [0,1]^2$, $U := \{H(t,s)\mathrm{e}^{\mathrm{i}\theta} \mid \theta \in (-\pi,\pi)\} \in \mathscr{U}_{S^1}(H(t,s))$. Since $f_t \in C([0,1],S^1)$, $\exists V(s) \in \mathscr{U}_{[0,1]}(s)$ s.t. $H(V(s)) \subset U$. Which means, $\forall t \in [0,1]$, $\forall s_1, s2 \in V(s)$, $f_t(s_1) \neq -f_t(s_2)$ or $h_{s_1}(t) \neq -h_{s_2}(t)$. By Lemma 3, $q(h_s) = q(h_{s'})$.

$\Omega = \{V(s) \mid s \in [0,1]\}$ is an open cover of the compact space $[0,1]$, therefore has a finite subcover $\Omega = \{V_i \in \Omega \mid i \in n\}$. In each $V(s_i)$, h_s has the same loop numbers.

We therefore have $q(a) = q(h_0) = q(h_1) = q(b)$.

$\leftarrow$: $H\colon [0,1]^2 \to S^1; (t,s) \mapsto p((1-s)\tilde{a}(t) - s\tilde{b}(t))$. $\square$

Theorem 3.1. $\pi_1(S^1) \cong \mathbb{Z}$.

Proof. $z_0 \in S^1$. Let $Q\colon \pi_1(S^1,z_0) \to \mathbb{Z}; \langle a\rangle \mapsto q(a)$.

$\forall \langle a\rangle, \langle b\rangle \in \pi_1(S^1,z_0)$, choose $\tilde{a}(1) = \tilde{b}(0)$,

$$\begin{aligned} Q(\langle a\rangle\langle b\rangle) = Q(\langle ab\rangle) &= q(ab) \\ &= \tilde{b}(1) - \tilde{a}(0) = \tilde{b}(1) - \tilde{b}(0) + \tilde{a}(1) - \tilde{a}(0) = q(a) + q(b) \\ &= Q(\langle a\rangle) + Q(\langle b\rangle), \end{aligned} \tag{3-1}$$

which means Q is a homomorphism.

By Lemma 4, Q is a monomorphism. $\forall n \in \mathbb{Z}$, $Q(\langle t \mapsto \mathrm{e}^{2\pi n t \mathrm{i}}\rangle) = n$, therefore Q is also an epimorphism. $\square$

3.2 S^n, $n > 2$

The situation for S^n is much simpler:

Theorem 3.2. *$\forall n \in \mathbb{N}$, if $n \geq 2$, then S^n is simply connected.*

Proof. Let $x_0 \in S^n$, and a be a loop at x_0 in S^n. $x \in S^n$ and $x \neq x_0$. Embed S^n into $\mathbb{R}^{n+1}$ and let $B(x;\delta)$ be a $(n+1)$-D ball with radius δ around x that $x_0 \notin B(x;\delta)$.

$a^{-1}(B(x;\delta) \cap S^n)$ is a collection of open, disjoint intervals in $[0,1]$, which can be considered as an open cover of $a^{-1}(\{x\})$, which is compact. Let the finite subcover of $a^{-1}(\{x\})$ be $\{(\alpha_i,\beta_i)\cap[0,1] \mid \alpha_i,\beta_i\in\mathbb{R}, i\in m\}$, where $m\in\mathbb{N}$.

Let $P\colon \mathbb{R}^{n+1}\times\mathbb{R}^{n+1}\times\mathbb{R}\to\mathbb{R}^{n+1}$ be:

$$P(y,y_0,r)=\frac{y-y_0}{\|y-y_0\|}r+y_0,$$

which means project y to the sphere with radius r around y_0.

Now we define the loop b that go as:

$$b(t)=\begin{cases} a(t), & t\notin(\alpha_i,\beta_i), \forall i\in m;\\ P[P(a(t),x,\delta),0,1], & t\in(\alpha_i,\beta_i)-a^{-1}(\{x\}), \exists i\in m;\\ \lim\limits_{\substack{t'\to t\\ t'\in(\alpha_i,\beta_i)-a^{-1}(\{x\})}} b(t'), & t\in a^{-1}(\{x\})\cap(\alpha_i,\beta_i), \exists i\in m,\end{cases}$$

and the homotopy between a and b can be written as

$$H\colon [0,1]^2\to S^n; (t,s)\mapsto P[(1-s)a(t)+sb(t),0,1].$$

Since b is a loop in $S^n-\{x\}$, while $S^n-\{x\}\cong\mathbb{R}^n$ (by stereographic projection), which is simply connected, we know that b is homotopic to $t\mapsto x_0$ i.e. null-homotopic. $\square$

By Theorem 2.5, the fundamenal group of $T^2:=S^1\times S^1$ is $\pi_1(T^2)\cong\mathbb{Z}\times\mathbb{Z}$, which is not isomorphic to S^2, therefore $T^2\ncong S^2$.

§4 Homotopy Types

Definition 4.1 (Homotopy type)**.** If $\exists f\in C(X,Y)$, $\exists g\in C(Y,X)$ s.t.

$$g\circ f\simeq \mathrm{id}_X,\quad f\circ g\simeq\mathrm{id}_Y, \tag{4-1}$$

then we say X and Y are ***homotopy equivalent***, or they are of the same ***homotopy type***, denoted by $X\simeq Y$. f is called a ***homotopy map*** or a ***homotopy equivalence*** from X to Y, and g is called a ***homotopy inverse*** of f.

An inverse of a homotopy map is not unique.
Some examples of spaces having same homotopy types:

- $\mathbb{R} \simeq \mathbb{R}^n$ $(n \in \mathbb{N}_+)$.
- $X \times [0,1] \simeq X$.

Theorem 4.1. *If $X \simeq Y$ and f is a homotopy map from X to Y, $f(x_0) = y_0$, then f_π is an isomorphism from $\pi_1(X, x_0)$ to $\pi_1(Y, y_0)$.*

Proof. □

Spaces with the simplest homotopy type are contractable spaces.

Definition 4.2 (Contractable space)**.** If $X \simeq \{x\}$, we call X a ***contractable space***.

§5 Retractability

Definition 5.1 (Retractability)**.** $A \subset X$, $i\colon A \to X$ is an ***inclusion*** from A to X, meaning $\forall a \in A$, $i(a) = a$. If $\exists r \in C(X, A)$ s.t. $r \circ i = \mathrm{id}_A$, then A is called a ***retract*** of X, r is a ***retraction***, and X is said to be ***retractable***.

Definition 5.2 (Deformation retractability)**.** $A \subset X$, $i\colon A \to X$ is an inclusion. If $\exists r \in C(X, A)$ s.t. $r \circ i = \mathrm{id}_A \wedge H\colon i \circ r \simeq \mathrm{id}_X$, then A is called a ***deformation retract*** of X, H is a ***deformation retraction*** of X, and X is said to be ***deformation retractable***.

Theorem 5.1 (Spaces are homotopically equivalent to their deformation retracts)**.** *If $X \simeq Y$ and X is a deformation retract of Y, then $X \simeq Y$. And, the retraction $r\colon Y \to X$ and the inclusion $i\colon X \to Y$ are homotopy inverse to each other.*

Theorem 5.2 (Contractable space can be deformatively retracted to all its points)**.** *If X is a contractable space, $\forall x \in X$, $\{x\}$ is a deformation retract of X.*

Definition 5.3 (Strong deformation retractability)**.** $A \subset X$, $i\colon A \to X$ is an inclusion. If $\exists r \in C(X, A)$ s.t. $r \circ i = \mathrm{id}_A \wedge H\colon i \circ r \simeq \mathrm{id}_X \,\mathrm{rel} A$, then A is called a ***strong deformation retract*** of X, H is a ***strong deformation retraction*** of X, and X is said to be ***strongly deformation retractable***.

Some examples:

- $X \times [0, 1]$ has strong deformation retracts $X \times \{t\}$ for each $t \in [0, 1]$.
- S^{n-1} is a strong deformation retract of $\mathbb{R}^n \backslash \{0\}$.
- Topological cone $CX = X \times [0, 1]/X \times \{1\}$ has a strong deformation retract at the tip of the cone i.e. $X \times \{1\}$.
- Möbius belt can be strong-deformatively retracted to the circle which is the centre line of the belt.

Chapter 2

Van-Kampen Theorem

§6 Free Abelian Group and Finitely Generated Group

In this section, we only talk about Abelian groups, and their multiplications are called "addition", i.e. $(G, +)$.

Definition 6.1 (Free Abelian group)**.** Let $(F, +)$ be an Abelian group. If $\exists A \subset F$ s.t. $\forall f \in F$, $\exists! n_f \colon A \to \mathbb{Z}$ s.t.

$$f = \sum_{a \in A} n_f(a)a, \quad \operatorname{card}\{a \in A \mid n_f(a) \neq 0\} \in \mathbb{N},$$

then we call F a ***free Abelian group***, A is a ***basis*** of F.

In plain words, all elements in F can be uniquely decided by finite integer-linear combinations of the elements in A. Notice that A can be infinite.

Typical free Abelian groups are integer vectors groups $\mathbb{Z}^n$ $(n \in \mathbb{N}_+)$, while Z

Theorem 6.1 (Homomorphism induced by any function of basis to a group)**.** *Let F be a free Abelian group, A be a basis of F, G is another Abelian group. $\forall f\colon A \to G$, $\exists! \varphi \in \mathrm{Hom}(F, G)$ s.t. $\forall a \in A$, $\varphi(a) = f(a)$.*

Proof. If $x = \sum_{i \in m} n_i a_i \in F$ ($n_i \in \mathbb{Z}$, $m \in \mathbb{N}$, $a_i \in A$), then

$$\varphi(x) = \sum_{i \in m} n_i f(a_i).$$

□

Definition 6.2 (Finitely generated Abelian group)**.** $(F, +)$ is an Abelian group. If $\exists A \subset F$ s.t. $\operatorname{card} A \in \mathbb{Z}$ and $\forall f \in F$, $\exists n_f\colon A \to \mathbb{Z}$ s.t.

$$f = \sum_{a \in A} n_f(a) a,$$

then F is called a ***finitely generated Abelian group***, A ***generates*** F. A is a ***generating set*** of F.

Theorem 6.2 (Finitely generated iff quotient of a free Abelian group)**.** *F is an Abelian group. F is finitely generated $\leftrightarrow$ there exists a free Abelian group H, whose basis is finite, $\exists j\colon H \to F$ s.t. j is an epimorphism.*

Definition 6.3 (Direct sum of Abelian group)**.** Let H_i ($i \in n$) be Abelian subgroups of H. If $\forall h \in H$, $\exists! h_i$ ($i \in n$) s.t.

$$h = \sum_{i \in n} h_i,$$

then we say that H is a ***direct sum*** of H_i ($i \in n$), denoted as:

$$H = \bigoplus_{i \in n} H_i.$$

If H_i are not subgroup of H and $H \cong \bigoplus_{i\in n} H_i$, H is also called a direct sum of H_i $(i \in n)$. In order to avoid confusion, this is called an outer direct sum.

The following theorem is very useful to construct a direct sum:

Theorem 6.3. *H_1, H_2, H are Abelian groups. If $H = H_1 + H_2$ (this is the Abelian group version of $H = H_1H_2$) and $H_1 \cap H_2 = \{0\}$, then $H = H_1 \oplus H_2$.*

Theorem 6.4. *Let $j\colon H \to F$ be an epimorphism, F is a free Abelian group.*

$$H \cong \ker j \oplus F.$$

We define some concepts that are very familar in vector spaces:

Definition 6.4 (Independence and basis)**.** Let H be an Abelian group, A is a subset of H. If $\forall n\colon A \to \mathbb{Z}$,

$$\sum_{a\in A} n(a)a = 0 \;\to\; \forall a \in A,\; n(a) = 0,$$

then A is an ***independent*** set. And if A generates H, we call it a ***basis*** of H.

Theorem 6.5. *Let H be an Abelian group, and there exists a basis of H. All bases of H have same cardinality.*

§7 Free Product of Groups

Definition 7.1. Let G and H be groups. The ***free product*** $G * H$ is defined as a string of alternative gs and hs from $G\backslash\{1_G\}$ and $H\backslash\{1_H\}$, that is

$$\begin{aligned} g_0h_0\cdots g_nh_n, \quad \text{or,} \quad g_0h_0\cdots g_nh_ng_{n+1}, \\ \text{or,} \quad h_0g_1h_1\cdots g_nh_n, \quad \text{or,} \quad h_0g_1h_1\cdots g_nh_ng_{n+1}, \end{aligned} \tag{7-1}$$

and the string that has zero length, denoted by $1 \in G * H$.

The product of two strings in $G*H$ is either concatenation (when ends are from different groups) or multiplication (when ends are from the same group).

§8 Van-Kampen Theorem

Chapter 3

Covering Space

§9 Covering space

Definition 9.1 (Even cover)**.** Let $p\colon E \to B$ be a continuous surjective, $\mathscr{T}_E$ is the topology of E, U be an open subset of B, I be an index set. If $\exists \langle V_i \rangle_{i \in I} \in \mathscr{T}_E^I$ s.t.

$$p^{-1}(U) = \coprod_{i \in I} V_i,$$

and $p|_{U_i} : U_i \to U$ is a homeomorphism from U_i to U, then U is said to be evenly covered by p. Each U_i is called a ***sheet*** or a ***slice***.

Definition 9.2 (Covering space)**.** Let $p\colon E \to B$ be a continuous surjective. If $\forall b \in B$, $\exists U \in \mathscr{U}(b)$ (***evenly covered neighbourhood***) s.t. U is evenly covered by p, then we call (E, p) a ***covering space***, p is a ***covering map***, B is the ***base space***.

Many authors ([4]) impose path connectivity and local path connectivity onto E and B.

Theorem 9.1. *Let (E, p) be a covering space. p is an open mapping.*

Proof. Let G be an open set in E. $\forall b \in p(G)$, $\exists U \in \mathscr{U}(b)$ s.t. U is evenly covered by p.

Choose $e \in p^{-1}(b)$, which should be contained in a sheet $V \subset p^{-1}(U)$. $V \cap G$ is open, and since $p|_V$ is a homoemorphism, $p(V \cap G) \subset U \cap p(G) \subset p(G)$ is an open set contained in G which b belongs to.

Therefore, $p(G)$ is open. □

Theorem 9.2 (Restriction of a covering map)**.** *Let (E, p) be a covering space onto B, $B_0 \subset B$, $E_0 = p^{-1}(B_0)$. $(E_0, p|_{E_0})$ is a covering space onto B_0.*

Theorem 9.3 (Product of covering maps)**.** *Let (E, p) and (F, q) be covering spaces onto B and C. The* **product** *of (E, p) and (F, q) (E and F, or p and q)*

$$\begin{aligned} p \times q \colon E \times F &\to B \times C; \\ (e, f) &\mapsto (p(e), q(f)), \end{aligned} \tag{9-1}$$

is also a covering map (onto $B \times C$).

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Symbol List

Here listed the important symbols used in these notes

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