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Differential Geometry

Hoyan Mok

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Manifolds

Chapter 2

Scalar and Vector Fields

2.1 Scalar Fields

[Scalar Field] Let M be a smooth manifold, $f \in C^{(\infty)}(M)$ is called a scalar field.

The scalar field over a manifold, form an algebra.

2.2 Vector Fields

[vector field] A vector field v over manifold M is a $C^{(\infty)}(M) \to C^{(\infty)}(M)$ map that satisfies [label=()]$\forall f, g \in C^{(\infty)}(M)$, $\forall \lambda, \mu \in R$, $v(\lambda f + \mu g) = \lambda v(f) + \mu v(g)$ (*linearity*). $\forall f, g \in C^{(\infty)}(M)$, $v(fg) = v(f)g + fv(g)$

The space of all vector fields on M is denoted by $[Vect(M)](M)$

[tangent vector] Let v be a vector field over M, p be a point on M. The tangent vector v_p at p is defined as a $C^{(\infty)}(M) \to C^{(\infty)}(M)$ map that satisfies

$$v_p(f) = v(f)(p). \tag{2.1}$$

The collection of tangent vectors at p is called the tangent space at p, denoted by $[TpM]T_pM$.

The derivative of a path $\gamma\colon [0,1] \to M$ (or $R \to M$) in a smooth manifold is defined as:

$$\gamma'(t)\colon C^{(\infty)}(M) \to R; \gamma'(t)(f) = tf \circ \gamma(t) \tag{2.2}$$

We can see that $\gamma'(t) \in T_{\gamma(t)}M$.

2.3 Covariant and Contravariant

[pullback] Let f be a scalar field over M, $\varphi \in C^{(\infty)}(M, N)$. Then the pullback of f by φ is defined as

$$\varphi^* f = f \circ \varphi \in C^{(\infty)}(N). \tag{2.3}$$

Fields that are pullbacked are covariant fields.

[pushforward] Let v_p be a tangent vector of M at p, $\varphi \in C^{(\infty)}(M, N)$, $q = \varphi(p)$. Then the pushforward of v_p by φ is defined as

$$(\varphi_* v)_q(f) = v_p(\varphi^* f). \tag{2.4}$$

Note that the pushforward of a vector field can only be obtained when φ is a diffeomorphism.

Fields that are pushforwarded are contravariant fields.

Mathematicians and physicists might have disagreement on whether a tangent vector is covariant or contravariant. This is because of that physicists might consider the coordinates (v^μ) of a tangent vector as a vector field, instead of linear combination of bases ∂_μ.

2.4 Flows

Let a path $\gamma\colon R$ follows a vector field (a velocity field), that is

$$\gamma'(t) = v_{\gamma(t)}, \tag{2.5}$$

then we call γ the integral curve through $p := gamma(0)$ of the vector field v.

Suppose v is an integrable vector field. Let $\varphi_t(p)$ be the point at time t on the integral curve through p.

$$\varphi_t\colon M \to M \tag{2.6}$$

is then called a flow generated by v.

$$t\varphi_t(p) = v_{\varphi_t(p)}. \tag{2.7}$$

Chapter 3

Differential Forms

3.1 1-forms

[1-form] A 1-form ω on M is a $(M) \to C^{(\infty)}(M)$ which satisfies that

[label=()]$\forall v, w \in (M)$, $\forall f, g \in C^{(\infty)}(M)$,

$$\omega(fv + gw) = f\omega(v) + g\omega(w). \tag{3.1}$$

The space of all 1-forms on M is denoted as $[Omega1(M)]^1(M)$.

The operator , when given a $C^{(\infty)}(M)$ function (which is called a 0-form), would give a 1-form:

$$(f)(v) = v(f). \tag{3.2}$$

This is called the exterior derivative or differential of f.

The cotangent vector or covector is similar as the tangent vector:

$$\omega_p(v_p) = \omega(v)(p). \tag{3.3}$$

The space of cotangent vectors at p on M is denoted by $[TpastM]T_p^*M$.

1-forms are contravariant, that is, if $\varphi\colon M \to N$, then

$$(\varphi^*\omega_q)(v_p) = \omega_q(\varphi_* v_p), \tag{3.4}$$

where $\varphi(p) = q$.

[heading=bibliography, title=bibliography]

Here listed the important symbols used in these notes [symbol]