# Differential Geometry

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# Scalar and Vector Fields

## §1 Scalar Fields

**Definition 1.1** (Scalar Field)**.** Let $M$ be a smooth manifold, $f \in C^{(\infty)}(M)$ is called a ***scalar field***.

The scalar field over a manifold, form an algebra.

## §2 Vector Fields

**Definition 2.1** (vector field)**.** A ***vector field*** $v$ over manifold $M$ is a $C^{(\infty)}(M) \to C^{(\infty)}(M)$ map that satisfies

(a) $\forall f, g \in C^{(\infty)}(M)$, $\forall \lambda, \mu \in \mathbb{R}$, $v(\lambda f + \mu g) = \lambda v(f) + \mu v(g)$ (*linearity*).

(b) $\forall f, g \in C^{(\infty)}(M)$, $v(fg) = v(f)g + fv(g)$

The space of all vector fields on $M$ is denoted by $\mathrm{Vect}(M)$

**Definition 2.2** (tangent vector)**.** Let $v$ be a vector field over $M$, $p$ be a point on $M$. The tangent vector $v_p$ at $p$ is defined as a $C^{(\infty)}(M) \to C^{(\infty)}(M)$ map that satisfies

$$v_p(f) = v(f)(p). \tag{2-1}$$

The collection of tangent vectors at $p$ is called the ***tangent space*** at $p$, denoted by $T_pM$.

The derivative of a path $\gamma\colon [0,1] \to M$ (or $\mathbb{R} \to M$) in a smooth manifold is defined as:

$$\begin{aligned} \gamma'(t)\colon C^{(\infty)}(M) &\to \mathbb{R}; \\ \gamma'(t)(f) &= \frac{\mathrm{d}}{\mathrm{d}t} f \circ \gamma(t) \end{aligned} \tag{2-2}$$

We can see that $\gamma'(t) \in T_{\gamma(t)}M$.

Let a path $\gamma\colon \mathbb{R}$ follows a vector field (a velocity field), that is

$$\gamma'(t) = v_{\gamma(t)}, \tag{2-3}$$

then we call $\gamma$ the ***integral curve*** through $p := \gamma(0)$ of the vector field $v$.

**Definition 2.3.** Suppose $v$ is an integrable vector field. Let $\varphi_t(p)$ be the point at time $t$ on the integral curve through $p$.

$$\varphi_t\colon M \to M \tag{2-4}$$

is then called a ***flow*** generated by $v$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(p) = v_{\varphi_t(p)}. \tag{2-5}$$

# §3 Covariant and Contravariant

**Definition 3.1** (pullback)**.** Let $f$ be a scalar field over $N$, $\varphi \in C^{(\infty)}(M,N)$. Then the ***pullback*** of $f$ by $\varphi$

$$\varphi^*\colon C^{(\infty)}(N) \to C^{(\infty)}(M), \tag{3-1}$$

is defined as

$$\varphi^* f = f \circ \varphi \in C^{(\infty)}(M). \tag{3-2}$$

Fields that are pullbacked are ***covariant*** fields.

**Definition 3.2** (pushforward)**.** Let $v_p$ be a tangent vector of $M$ at $p$, $\varphi \in C^{(\infty)}(M, N)$, $q = \varphi(p)$. Then the ***pushforward*** of $v_p$ by $\varphi$

$$\varphi_* \colon T_p M \to T_q N, \tag{3-3}$$

is defined as

$$(\varphi_* v)_q(f) = v_p(\varphi^* f). \tag{3-4}$$

Note that the pushforward of a vector field can only be obtained when $\varphi$ is a diffeomorphism.

Fields that are pushforwarded are ***contravariant*** fields.

Mathematicians and physicists might have disagreement on whether a tangent vector is covariant or contravariant. This is because of that physicists might consider the coordinates $(v^\mu)$ of a tangent vector as a vector field, instead of linear combination of bases $\partial_\mu$.

## §4 Components of Vector Fields

Let $\varphi\colon U \to \mathbb{R}^n$ be a chart of $M$ ($U \subset M$).

Let $p \in U$, $\varphi(p) = x = (x^\mu)$ ($\mu = 0, \ldots, n-1$). Locally, a function $f \in C^{(\infty)}(M)$ can be written as

$$(\varphi^{-1})^* f = f \circ \varphi^{-1} \colon \mathbb{R}^n \to \mathbb{R}, \tag{4-1}$$

and a vector field $v \in \mathrm{Vect}(M)$ can be written as

$$(\varphi_* v)_x = \varphi_* v_p \colon C^{(\infty)}(\mathbb{R}^n) \to \mathbb{R}, \tag{4-2}$$

or

$$\varphi_* v \in \mathrm{Vect}(\mathbb{R}^n) \tag{4-3}$$

Since $T_x\mathbb{R}^n \cong \mathbb{R}^n$ is a linear space, one can find a basis for $T_x\mathbb{R}^n$ as

$$\partial_\mu \colon C^{(\infty)}(\mathbb{R}^n) \to C^{(\infty)}(\mathbb{R}^n), \tag{4-4}$$

and $(\varphi_* v)_x = v^\mu(x)\partial_\mu$.

Pushing forward $v^\mu(x)\partial_\mu$ by $\varphi^{-1}$ one obtains $v$.

In an abuse of symbols, one may just omit the pullback and pushforward, and refer to the $f$ and $v$ by $(\varphi^{-1})^* f$ and $\varphi_* v$.

Consider another chart $\psi \colon U \to \mathbb{R}^n$ of $M$, and

$$y = \psi(p), \quad (\psi_* v)_x = u^\mu \partial_\mu, \tag{4-5}$$

where we have chosen the same basis in $T_y\mathbb{R}^n$ as in $T_x\mathbb{R}^n$.

We would like to know how to relate $v^\mu$ and $u^\mu$ i.e. we want to know how the components of $v$ transforms under a coordinate transformation $\tau = \psi \circ \varphi^{-1}$.

Consider any $f \in C^{(\infty)}(M)$,

$$v(f) = \varphi_* v((\varphi^{-1})_* f) = \psi_* v((\psi^{-1})_* f) \tag{4-6}$$

$\Rightarrow$

$$u^\mu \partial_\mu (f \circ \psi^{-1}) = v^\mu \partial_\mu (f \circ \varphi^{-1}) = v^\mu \partial_\mu (f \circ \psi^{-1} \circ \tau) = v^\mu {\tau'}^\nu_{\ \mu} \partial_\nu (f \circ \psi^{-1}) \tag{4-7}$$

$\Rightarrow$

$$u^\mu = v^\nu {\tau'}^\mu_{\ \nu}, \tag{4-8}$$

where

$${\tau'}^\mu_{\ \nu} = \frac{\partial y^\mu}{\partial x^\nu}. \tag{4-9}$$

# §5 Lie Bracket

**Definition 5.1** (Lie bracket)**.** Let $v, w \in \mathrm{Vect}(M)$, then the ***Lie bracket*** of $v$ and $w$ is defined as

$$[v, w] \colon C^{(\infty)}(M) \to C^{(\infty)}(M);\ f \mapsto v \circ w(f) - w \circ v(f). \tag{5-1}$$

The Lie bracket is an antisymmetric bilinear map[1], and an important property of the Lie bracket is the Leibniz rule:

$$[v,w](fg) = [v,w](f)g + f[v,w](g). \tag{5-2}$$

Another important property of the Lie bracket is the Jacobi identity:

$$[v,[w,u]] + [w,[u,v]] + [u,[v,w]] = 0. \tag{5-3}$$

[1] Note that it is not $C^{(\infty)}$-linear

# Chapter 3

# Differential Forms

## §6 1-forms

**Definition 6.1** (1-form)**.** A ***1-form*** $\omega$ on $M$ is a $\mathrm{Vect}(M) \to C^{(\infty)}(M)$ which satisfies that

(a) $\forall v, w \in \mathrm{Vect}(M)$, $\forall f, g \in C^{(\infty)}(M)$,

$$\omega(fv + gw) = f\omega(v) + g\omega(w). \tag{6-1}$$

The space of all 1-forms on $M$ is denoted as $\Omega^1(M)$, which is a module over $C^{(\infty)}(M)$.

The operator d, when given a $C^{(\infty)}(M)$ function (which is called a ***0-form***), would give a 1-form:

$$(\mathrm{d}f)(v) = v(f). \tag{6-2}$$

This is called the ***exterior derivative*** or ***differential*** of $f$.

The ***cotangent vector*** or ***covector*** is similar as the tangent vector:

$$\omega_p(v_p) = \omega(v)(p). \tag{6-3}$$

The space of cotangent vectors at $p$ on $M$ is denoted by $T_p^*M$.

1-forms are covariant, that is, if $\varphi\colon M \to N$, then the pushforward of a 1-form $\omega$ by $\varphi$ is

$$(\varphi^*\omega)_p(v_p) = \omega_q(\varphi_* v_p), \tag{6-4}$$

where $\varphi(p) = q$.

**Theorem 6.1.** *$f \in C^{(\infty)}(N)$, $\varphi\colon M \to N$ is differential, then*

$$\varphi^*(\mathrm{d}f) = \mathrm{d}(\varphi^* f). \tag{6-5}$$

# §7 Components of 1-Forms

Let $\varphi\colon U \to \mathbb{R}^n$ be a chart of $M$ ($U \subset M$).

Let $p \in U$, $\varphi(p) = x = (x^\mu)$ ($\mu = 0, \dots, n-1$). Locally a 1-form $\omega \in \Omega^1(M)$ can be written as

$$(\varphi^{-1})^*\omega \in T_x^*\mathbb{R}^n. \tag{7-1}$$

A natural way to impose a basis $\mathrm{d}x^\mu$ in $T_x^*\mathbb{R}^n$ is

$$\mathrm{d}x^\mu(\partial_\nu) = \delta^\mu_\nu, \tag{7-2}$$

and $(\varphi^{-1})^*\omega = \omega_\mu(x)\,\mathrm{d}x^\mu$.

Now by the definition of 1-form:

$$\omega_\mu\,\mathrm{d}x^\mu(v^\nu\partial_\nu) = v^\nu\omega_\mu\delta^\mu_\nu = v^\mu\omega_\mu. \tag{7-3}$$

By the transformation rule of components of a vector, one have

$${\tau'}^\nu{}_\mu\alpha_\nu = \omega_\mu, \tag{7-4}$$

where $\psi\colon U \to \mathbb{R}^n$, $(\psi^{-1})_*\omega = \alpha_\mu\,\mathrm{d}x^\mu$, $\tau = \psi \circ \varphi^{-1}$.

# §8 $k$-Forms

**Definition 8.1.** If we assign an antisymmetric multilinear $k$-form $\omega_p \in \bigotimes_{i\in k} T_p^* M$ to each point $p \in M$, we say we have a ***k-form*** on $M$.

The collection of all $k$-forms is denoted by $\Omega^k(M)$, and $\Omega(M) := \bigcup_{k\in\mathbb{N}} \Omega^k(M)$.

**Theorem 8.1** (Dimension of forms)**.** *If $M$ is an $nD$ manifold, then the dimension of $\Omega^k(M)$ is $\frac{n!}{k!(n-k)!}$ ($k \le n$), and $0$ for $k > n$; The dimension of $\Omega(M)$ is $2^n$.*

**Definition 8.2** (Wedge product)**.** The ***wedge product*** $\wedge$ is defined as a binary operater that takes a $k$-form and $\ell$-form and gives a $(k+\ell)$-forms, satisfying $\forall \alpha \in \Omega^k(M)$, $\forall \beta \in \Omega^\ell(M)$:

(a) (Associativity) $\forall \gamma \in \Omega^m(M)$,

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma). \tag{8-1}$$

(b) (Supercommutativity)

$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha. \tag{8-2}$$

(c) (Distributiveness) $\forall \gamma \in \Omega^\ell(M)$,

$$\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma. \tag{8-3}$$

(d) (Bilinearity over $C^{(\infty)}(M)$) $\forall f \in C^{(\infty)}(M)$,

$$(f\alpha) \wedge \beta = f(\alpha \wedge \beta). \tag{8-4}$$

(e) (Naturality) If $\varphi\colon M \to N$ is a smooth map, then the pullback of a form by $\varphi$ can be given by repeatingly applying ($\forall \gamma \in \Omega^\ell(M)$)

$$\begin{aligned} \varphi^*(\beta+\gamma) &= \varphi^*\alpha + \varphi^*\beta \\ \varphi^*(\alpha \wedge \beta) &= \varphi^*\alpha \wedge \varphi^*\beta, \end{aligned} \tag{8-5}$$

while the pullback of a 0-form and a 1-form agree with what we have already defined before.

By convention if $f \in C^{(\infty)}(M)$ then

$$f \wedge \omega =: f\omega. \tag{8-6}$$

It can be shown that any $k$-form $\omega$ can be written as

$$(\varphi^{-1})^* \omega = \frac{\omega_{\mu_1 \cdots \mu_k}}{n!} \bigwedge_{i=1}^{k} \mathrm{d}x^{\mu_i}, \tag{8-7}$$

where $\varphi \colon M \to \mathbb{R}^n$ is a chart.

**Definition 8.3** (Interior product)**.** Let $v \in \Gamma(TM)$, we can define the ***interior product*** $i_v \colon \Omega^k(M) \to \Omega^{k-1}(M)$ by: $\forall \omega \in \Omega^k(M)$, $\forall v_i$ $(i \in k-1)$:

$$i_v(\omega)(v_0, \ldots, v_{k-2}) = \omega(v, v_0, \ldots, v_{k-2}). \tag{8-8}$$

Specially, if $k = 0$, then $i_v(\omega) = 0$.

**Theorem 8.2.** *$\forall v \in \Gamma(TM)$,*

1. *$i_v$ is a $C^{(\infty)}(M)$-linear function;*
2. *$\forall \alpha \in \Omega^k(M)$, $\forall \beta \in \Omega(M)$,*

$$i_v(\alpha \wedge \beta) = i_v(\alpha) \wedge \beta + (-1)^k \alpha \wedge i_v(\beta). \tag{8-9}$$

# §9 Exterior Derivative

**Definition 9.1** (Exterior derivative)**.** The ***exterior derivative*** d is defined as a linear operator that takes a $k$-form and gives a $(k+1)$-form, satisfying $\forall \alpha \in \Omega^k(M)$, $\forall \beta \in \Omega^\ell(M)$:

(a) (Linearity) $\forall \lambda, \mu \in \mathbb{R}$, $\forall \gamma \in \Omega^\ell(M)$,

$$\mathrm{d}(\lambda\beta + \mu\gamma) = \lambda\,\mathrm{d}\alpha + \mu\,\mathrm{d}\beta. \tag{9-1}$$

(b) (Leibniz rule)

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta. \tag{9-2}$$

(c)

$$\mathrm{d}^2\omega = 0. \tag{9-3}$$

(d) (Naturality) If $\varphi\colon M \to N$ is a smooth map, then

$$\varphi^* \, \mathrm{d}\omega = \mathrm{d}\varphi^*\omega. \tag{9-4}$$

# §10 Derivation ana Antiderivation

**Definition 10.1** (Derivation)**.** A map $\theta\colon \Omega(M) \to \Omega(M)$ is called a ***derivation of degree*** $k$ ($k \in \mathbb{Z}$) if $\forall p \in \mathbb{N}$, $\theta[\Omega^p(M)] \subset \Omega^{p+k}(M)$ and, $\theta$ is a homomorphism of the $\mathbb{R}$-exterior algebra. Or, explicitly, $\theta$ is $\mathbb{R}$-linear, and $\forall \alpha, \beta \in \Omega(M)$,

$$\theta(\alpha \wedge \beta) = \theta(\alpha) \wedge \beta + \alpha \wedge \theta(\beta). \tag{10-1}$$

**Definition 10.2** (Antiderivation)**.** A map $\theta\colon \Omega(M) \to \Omega(M)$ is called an ***antiderivation of degree*** $k$ ($k \in \mathbb{Z}$) if i) $\forall p \in \mathbb{N}$, $\theta[\Omega^p(M)] \subset \Omega^{p+k}(M)$, ii) $\theta$ is $\mathbb{R}$-linear, iii) $\forall \alpha \in \Omega^p(M)$, $\beta \in \Omega(M)$,

$$\theta(\alpha \wedge \beta) = \theta(\alpha) \wedge \beta + (-1)^p \alpha \wedge \theta(\beta). \tag{10-2}$$

# Chapter 4

# Metric

## §11 Pseudo-Riemannian Metric

**Definition 11.1** (Psedudo-Riemannian metric)**.** Let $M$ be a manifold. A ***pseudo-Riemannian metric*** or simply ***metric*** $g$ on a manifold $M$ is a field ($g \in \Gamma(T^*M \otimes T^*M)$) that $\forall p \in M$,

$$g_p \colon T_p^*M \times T_p^*M \to \mathbb{R}, \tag{11-1}$$

is a bilinear form satisfying the following properties:

(a) (Symmetry) $\forall u, v \in T_pM$,

$$g_p(u, v) = g_p(v, u). \tag{11-2}$$

(b) (Non-degenerate)

$$u \mapsto g_p(u, -) \colon T_pM \to T_p^*M \tag{11-3}$$

is an isomorphism.

(c) (Bilinearity) $\forall p \in M$, $\forall u, v \in T_pM$, $\forall \lambda, \mu \in \mathbb{R}$,

$$g_p(\lambda u + \mu v, w) = \lambda g_p(u, w) + \mu g_p(v, w). \tag{11-4}$$

(d) (Smoothness) If $v, u \in \mathrm{Vect}(M)$, then

$$p \mapsto g_p(v_p, u_p) \in C^{(\infty)}(M). \tag{11-5}$$

Given a metric, $\forall p \in M$, we can always find an orthonormal basis $\{e_\mu\}$ of $T_pM$ such that

$$g_p(e_\mu, e_\nu) = \mathrm{sign}(\mu)\delta_{\mu\nu}, \tag{11-6}$$

where $\mathrm{sign}(\mu) = \pm 1$. Conventionally we order the basis such that $\mathrm{sign}(\mu) = 1$ for $\mu \in s$ and $\mathrm{sign}(\mu) = -1$ for $\mu - s \in n - s$, and say that the metric has ***signature*** $(s, n-s)$.

If $\gamma\colon [0,1] \to M$ is a smooth path and $\forall t, s \in [0,1]$,

$$g(\gamma'(t), \gamma'(t))g(\gamma'(s), \gamma'(s)) \geq 0, \tag{11-7}$$

then we can define the arclength of $\gamma$ as

$$\int_0^1 \sqrt{|g(\gamma'(t), \gamma'(t))|}\,\mathrm{d}t \tag{11-8}$$

if the integral converges.

The metric gives an ***inner product*** on $\mathrm{Vect}(M)$:

$$\langle u, v\rangle := g(u, v). \tag{11-9}$$

The metric also gives a way to relate a vector field $v$ to a 1-form $\omega$. If $v$ and $\omega$ satisfies: $\forall u \in \mathrm{Vect}(M)$,

$$g(v, u) = \omega(u), \tag{11-10}$$

then we say that $v$ is the corresponding vector field of $\omega$, denoted by $v = \omega^\sharp$, and $\omega$ is the corresponding 1-form of $v$, denoted by $\omega = v^\flat$.

We can also define the ***inner product*** on $\Omega^1(M)$ by

$$\langle \alpha, \beta\rangle = \langle a, b\rangle, \tag{11-11}$$

where $a$ and $b$ is the corresponding vector fields of $\alpha$ and $\beta$.

The ***inner product***[1] on $\Omega^k(M)$ is defined by induction with

$$\langle \bigwedge_{i\in k} \alpha_i, \bigwedge_{i\in k} \beta_i \rangle = \det(\langle \alpha_i, \beta_j \rangle)_{i,j\in k}. \tag{11-12}$$

Hence, if $\{e_\mu\}$ is an orthonormal basis (field) of $T_pM$, while the corresponding covectors are $\{f^\mu\}$ ($f^\mu(e_\nu) = \delta^\mu_\nu$) then

$$\langle \bigwedge_{i\in k} f^{\mu_i}, \bigwedge_{i\in k} f^{\mu_i} \rangle = \prod_{i\in k} \operatorname{sign}(\mu_i). \tag{11-13}$$

Specially, when $f, g \in \Omega^0(M) = C^{(\infty)}(M)$,

$$\langle f, g \rangle = fg. \tag{11-14}$$

# §12 Volume Form

Notice that if $M$ is an $n$D manifold, $\dim \Omega^n(M) = 1$, meaning at $p \in M$, $\{\omega_p \mid \omega \in \Omega^n(M)\}$ can be labelled by a parametre $\lambda_p \in \mathbb{R}$. If we have a basis $\{f^\mu\}$ of $T^*_pM$ (or corresponding vectors $\{e_\mu\}$), then

$$\{\omega_p \mid \omega \in \Omega^n(M)\} = \lambda_p \bigwedge_{\mu\in n} f^\mu. \tag{12-1}$$

If there were another basis $\{g^\mu\}$ of $T^*_pM$ (or corresponding vectors $\{h_\mu\}$), and the transformation between the two bases is given by

$$Pe^\mu = f^\mu, \tag{12-2}$$

where $P \in \operatorname{Aut}(T^*_pM)$. When $\det P > 0$, we say that $\{f^\mu\}$ and $\{g^\mu\}$ have the same ***orientation***.

**Definition 12.1** (Volume form)**.** Let $M$ be an orientable manifold. If $\forall p \in M$, we find an oriented orthonormal basis $\{f_\mu\}$ of $T^*_pM$ at point $p$, then the ***volume form*** vol is defined by

$$\bigwedge_{\mu\in n} f_\mu = \operatorname{vol}_p. \tag{12-3}$$

[1] This inner product makes $\Omega_p(M)$ for each $p \in M$, yet not for $\Omega(M)$. The full inner product requires integration over $M$.

# §13 Hodge Star Operator

**Definition 13.1** (Hodge Star Operator)**.** Let $M$ be an orientable manifold. The ***Hodge star operator*** $\star$ is defined by the linear map

$$\star\colon \Omega^k(M) \to \Omega^{n-k}(M), \tag{13-1}$$

$\forall \alpha, \beta \in \Omega^k(M)$,

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \operatorname{vol}. \tag{13-2}$$

We call $\star\omega$ the ***dual*** of $\omega$.

The special case is when $k = 0$,

$$\star f = f \operatorname{vol}, \tag{13-3}$$

and $k = n$,

$$\star(f \operatorname{vol}) = f \prod_{\mu \in n} \operatorname{sign}(\mu) = (-1)^{n-s} f \tag{13-4}$$

if the signature of the metric is $(s, n-s)$.

The Hodge star operator is also called the ***Hodge duality*** because:

**Theorem 13.1.** $\forall \alpha \in \Omega^p(M)$,

$$\star \star \alpha = (-1)^{p(n-p)} \alpha \operatorname{sign}(g), \tag{13-5}$$

*where* $\operatorname{sign}(g) := \det g / |\det g|$.

**Definition 13.2** (Codifferential)**.** Let $M$ be an orientable manifold. The ***codifferential*** $\delta$ is defined by

$$\delta\colon \Omega^k(M) \to \Omega^{k-1}(M), \tag{13-6}$$

$\forall \alpha \in \Omega^k(M)$,

$$\delta\alpha = \star \mathrm{d} \star \alpha. \tag{13-7}$$

**Definition 13.3** (Laplacian)**.** The ***Laplacian*** $\square$ is defined by

$$\square := \mathrm{d} \circ \delta + \delta \circ \mathrm{d}. \tag{13-8}$$

**Theorem 13.2.**

$$\Box \circ \star = \star \circ \Box, \tag{13-9}$$

$$\Box \circ \delta = \delta \circ \Box, \tag{13-10}$$

$$\Box \circ \mathrm{d} = \mathrm{d} \circ \Box. \tag{13-11}$$

# §14 Metric and Coordinates

# Chapter 5

# DeRham Theory

## §15 Closed and Exact 1-Forms

**Definition 15.1** (Closed and exact forms)**.** Consider $\mathrm{d}\colon \Omega(M) \to \Omega(M)$. The differential forms in $\ker \mathrm{d}$ is said to be ***closed***, and the differential forms in $\mathrm{d}(\Omega(M))$ is said to be ***exact***.

For closed form:

$$\mathrm{d}\omega = 0, \tag{15-1}$$

For exact form:

$$\exists \alpha \in \Omega(M), \quad \omega = \mathrm{d}\alpha, \tag{15-2}$$

where $\alpha$ is often called *potential*.

We want to study, given two points $p$, $q$ that are located in the same arcwise connected component of $M$, and a smooth path $\gamma\colon [0,1] \to M$ s.t. $\gamma(0) = p$, $\gamma(1) = q$, for a closed 1-form $E$,

$$\phi(p,q) := -\int_\gamma E := -\int_0^1 E_{\gamma(t)}(\gamma'(t))\ \mathrm{d}t. \tag{15-3}$$

We want to know that how $\phi$ depends on the choice of $\gamma$.

Assumes that there are two smooth paths $\gamma_1$ and $\gamma_2$ connecting $p$ and $q$, and a fix-ends smooth homotopy $H\colon [0,1]\times[0,1]\to M$ s.t.

$$H(0,t)=\gamma_1(t),\quad H(1,t)=\gamma_2(t),\quad H(s,0)=p,\quad H(s,1)=q. \tag{15-4}$$

By choosing proper charts (if there is no chart that can cover the whole path, we break the path into pieces),

$$\begin{aligned} I_s=\int_{H(s,-)} E &= \int_0^1 E_{H(s,t)}(H'(s,t))\,\mathrm{d}t \\ &= \int_0^1 E_\mu[H(s,t)]\partial_t H^\mu(s,t)\,\mathrm{d}t, \end{aligned} \tag{15-5}$$

where $H'(s,t)$ is the tangent vector of $H(s,-)$ at $t$.

$$\begin{aligned} \frac{\mathrm{d}I_s}{\mathrm{d}s} &= \frac{\mathrm{d}}{\mathrm{d}s}\int_0^1 E_\mu[H(s,t)]\partial_t H^\mu(s,t)\,\mathrm{d}t \\ &= \int_0^1 \left(\partial_s E_\mu[H(s,t)]\partial_t H^\mu + E_\mu[H(s,t)]\partial_s\partial_t H^\mu\right)\mathrm{d}t \\ &= \partial_s\left(E_\mu(H(s,t))H^\mu(s,t)\right)\Big|_{t=0}^{t=1} \\ &\quad + \int_0^1 \left(\partial_s E_\mu[H(s,t)]\partial_t H^\mu - \partial_t E_\mu[H(s,t)]\partial_s H^\mu\right)\mathrm{d}t \\ &= \partial_s(E_\mu(q)q^\mu - E_\mu(p)p^\mu) \\ &\quad + \int_0^1 \partial_\nu E_\mu\left(\partial_s H^\nu\partial_t H^\mu - \partial_t H_\nu\partial_s H^\mu\right)\,\mathrm{d}t \\ &= \int (\mathrm{d}E)_{\mu\nu}\partial_s H^\mu\partial_t H = 0. \end{aligned} \tag{15-6}$$

Now we have proven that if $\gamma_1$ and $\gamma_2$ are homotopic, then the integral for $\phi(p,q)$ is the same.

Then, if $M$ is simply connected, then a closed form $E$ is also exact, and

$$E=-\,\mathrm{d}\phi(p,-). \tag{15-7}$$

# §16 Stokes' Theorem

# §17 DeRham Cohomology

We have shown that, if the manifold is simply connected, then a closed 1-form must also be exact. The study of whether a closed form is exact is called the ***deRham cohomology***.

Since $\mathrm{d} \circ \mathrm{d} = 0$, we know that

$$\mathrm{d}(\Omega(M)) \subset \mathrm{d}(\ker \mathrm{d}). \tag{17-1}$$

The space of exact $p$-forms is denoted by $B^p(M)$ and the space of closed $p$-forms is denoted by $Z^p(M)$.

**Definition 17.1** (DeRham cohomology)**.** The $p$-th ***deRham cohomology*** of $M$ is defined as

$$H^p(M) = Z^p(M)/B^p(M). \tag{17-2}$$

Every element of $H^p(M)$ is a ***cohomologous class***:

$$[\omega] = \{\omega' \in Z^p(M) \mid \omega - \omega' \in B^p(M)\}. \tag{17-3}$$

For $p = 0$, $B^0(M) = \{0\}$ (there is no $(-1)$-form), and $H^0(M) = Z^0(M)$, where $Z^0(M)$ is made of $f$ that is constant in every connected components of $M$. Let $\chi_i$ be the characteristic function of $M$'s $i$th connected components $M_i$ (we assume that $\{M_i\}$ is finite)

$$H^0(M) = Z^0(M) = \left\{f \middle| f = x^i \chi_i\right\} \cong \mathbb{R}^n, \tag{17-4}$$

where $n$ is the number of connected components of $M$.

# Chapter 6

# Bundles and Connections

## §18 Fibre Bundles

**Definition 18.1** (Bundle)**.** A ***bundle*** is a triple $(E, \pi, B)$, where $\pi\colon E \to B$ is a surjective map. $E$ is called the ***total space***, $\pi$ is called the ***projection map***, and $B$ is called the ***base space***.

A bundle $(E, \pi, B)$ can be denoted as $\pi\colon E \to B$ or $E \xrightarrow{\pi} B$ .

**Definition 18.2** (Fibre)**.** For $p \in B$, $\pi^{-1}(\{p\})$ is the ***fibre*** over $b$.

**Definition 18.3** (Subbundle)**.** Let $\pi\colon E \to B$ be a bundle. $F \subset E$, $C \subset B$, $\rho\colon F \to C$. If $\pi|_C = \rho$, then $\rho\colon F \to C$ is called a ***subbundle*** of $\pi\colon E \to B$.

**Definition 18.4** (Section)**.** A ***section*** is a map $s\colon B \to E$ such that

$$p \circ s = \mathrm{id}_B\,. \tag{18-1}$$

All sections of a bundle $\pi\colon E \to B$ is denoted as $\Gamma(E)$.

**Definition 18.5** (Fibre bundle)**.** A ***fibre bundle*** $(E, \pi, B, F)$ is a bundle $\pi\colon E \to B$, where $E$, $B$, $F$ are topology spaces, and $\pi$ is a continuous map, and $\forall p \in B$, $\exists U \in \mathscr{U}(p)$ s.t.

$$\varphi\colon \pi^{-1}(U) \to U \times F, \tag{18-2}$$

is a homeomorphism and $\pi_1 \circ \varphi = \pi$. $\pi_1$ is defined as $\pi_1(p, q) = p$.

A fibre bundle can be denoted as the exact sequence

$$F \longrightarrow E \xrightarrow{\ \pi\ } B \tag{18-3}$$

The last condition is called the ***local triviality condition***. $F$ is called the ***standard fibre***

If $E = B \times F$, then $(E, \pi, B, F)$ is called a ***trivial fibre bundle***.

**Definition 18.6** (Morphism)**.** Let $\pi\colon E \to B$, $\rho\colon F \to C$ be two fibre bundles. A ***morphism*** $(\varphi, \psi)$ is a pair of two continuous maps such that

$$\begin{array}{ccc} E & \xrightarrow{\ \psi\ } & F \\ \Big\downarrow{\scriptstyle\pi} & & \Big\downarrow{\scriptstyle\rho} \\ B & \xrightarrow{\ \varphi\ } & C \end{array} \tag{18-4}$$

commutes.

# §19 Vector Bundles

**Definition 19.1** (Vector bundle)**.** A ***vector bundle*** is a fibre bundle $(E, \pi, B, F)$, where $F$ is a vector space, and the local trivialisation $\varphi\colon \pi^{-1}(U) \to U \times F$ ($U$ is a neibourhood of $p \in B$) satisfies that $\forall x \in U$, $\forall v \in F$,

$$\begin{aligned} F &\to \pi^{-1}(\{x\}) \\ v &\mapsto \varphi^{-1}(x, v) \end{aligned} \tag{19-1}$$

is a linear isomorphism (***fibrewise linear***).

**Definition 19.2** (Morphism (vector bundle))**.** The morphism between two vector bundles $(E, \pi, B, F)$ and $(E', \pi', B', F')$ is a morphism $(\varphi, \psi)$ such that $\forall x \in B$,

$$\psi_* \colon \pi^{-1}(\{x\}) \to (\pi')^{-1}(\{\varphi(x)\}) \tag{19-2}$$

is a linear homomorphism.

**Definition 19.3** (Smooth vector bundle)**.** A ***smooth vector bundle*** is a vector bundle $(E, \pi, B, F)$, where the projection $\pi \colon E \to B$ and the local trivialisation $\varphi \colon \pi^{-1}(U) \to U \times F$ are smooth.

**Definition 19.4** (Tangent bundle)**.** The ***tangent bundle*** $TM$ is the smooth vector bundle over an $n$D smooth manifold $M$ with the standard fibre $T_pM = \mathbb{R}^n$.

A vector field $v \in \mathrm{Vect}(M)$ is the smooth section of the tangent bundle $\Gamma(TM)$.

**Definition 19.5** (Cotangent bundle)**.** The ***cotangent bundle*** of an $n$D manifold $M$, denoted by $T^*M$, is the smooth vector bundle over with the standard fibre $T_p^*M = (\mathbb{R}^n)^*$.

A 1-form $\omega \in \Omega^1(M)$ is the smooth section of the cotangent bundle $\Gamma(T^*M)$.

# §20 Constructions of Vector Bundles

We use local trivialisation to destruct a vector bundle into trivial bundles. We can also construct a vector bundle by "gluing" trivial bundles. We must make sure that in the intersections of bases, we must make sure that they are compactible by introducing *transition functions* to relate points on the fibres. Naturally, transition functions make a group structure.

**Definition 20.1** ($G$-bundle)**.** Consider an open cover $\mathcal{U} = \{U_i \mid i \in I\}$ of the manifold $M$. For each $i \in I$, there is a trivial vector bundle $U_i \times V \xrightarrow{\pi_i} U_i$ with vector fibre $V$. $\rho\colon G \to \mathrm{GL}(V)$ is a representation of $G$ on $V$.

For any $p \in M$, if $p \in \bigcap_{j\in J} U_j$ $(J \subset I)$, then $\pi^{-1}(\{p\})$ is identified by a equivalence class in $\bigsqcup_{j\in J} \pi_j^{-1}(p)$ where two points are equivalent if they are related by the transformation

$$\begin{aligned}\rho_*(g_{jj'}(p))\colon U_j \times V &\to U_{j'} \times V;\\ (p,v) &\mapsto (p, \rho(g_{jj'}(p)v)),\end{aligned} \tag{20-1}$$

where the ***transition functions*** $g_{ij} \in G$ satisfy that:

1. $g_{ii} = 1$;
2. $g_{ij}g_{jk}g_{ki} = 1$.

The bundle $E \xrightarrow{\pi} M$ is then called the $G$***-bundle***, where $G$ is the ***gauge group***.

# §21 Connections

**Definition 21.1** (Connection)**.** A ***connection*** on a smooth vector bundle $(E, \pi, M, F)$ is map

$$D\colon \Gamma(TM) \times \Gamma(E) \to \Gamma(E), \tag{21-1}$$

that satisfies the following conditions: $\forall v, w \in \Gamma(TM)$, $\forall s, t \in \Gamma(E)$, $\forall f \in C^{(\infty)}(M)$,

(a) $D_v(s + t) = D_v s + D_v t$;

(b) $D_v(fs) = v(f)s + fD_v s$;

(c) $D_{v+w}s = D_v s + D_w s$;

(d) $D_{fv}s = fD_v s$.

When a vector field $v \in \Gamma(TM)$ is given to the connection $D$, the map $D_v : \Gamma(E) \to \Gamma(E)$ is called the ***covariant derivative*** with respect to $v$.

**Definition 21.2** (Vector potential)**.** A ***vector potential*** $A$ is an $\mathrm{End}(E)$-valued 1-form, that is

$$A \in \Gamma(\mathrm{End}(E) \otimes T^*M), \tag{21-2}$$

where $\mathrm{End}(E) \cong E \otimes E^*$ can be considered as a vector bundle over $M$ with the standard fibre $\mathrm{End}(E_p) \cong E_p \otimes E_p^*$ $(p \in E)$.

Locally if $s \in \Gamma(E)$ we can have a trivialisation $\varphi\colon E|_U \to U \times F$ $(U \subset M)$. If we assign a basis $\{f_i\}_{i \in m}$ for the $m$D standard fibre $F$, then

$$s = s^i e_i := s^i \varphi^{-1}(f_i), \quad s^i \in C^{(\infty)}(U), \tag{21-3}$$

where we can call $\{s^i\}_{i\in m}$ the ***components of the section*** $s$. With this specific normalisation, one can define that

$$D_v^0 s = v(s^i) e_i \tag{21-4}$$

where $D^0$ is called the ***standard flat connection*** (which depends on trivialisation).

**Theorem 21.1.** *Let $(E, \pi, M, F)$ be a smooth vector bundle. If $D$ is a connection on $E$, $A \in \Gamma(\mathrm{End}(E)) \otimes T^*M$, then the $D + A$, which defined as*

$$D + A\colon (v, s) \mapsto D_v s + A(v)s, \tag{21-5}$$

*is also a connection.*

**Theorem 21.2.** *Let $(E, \pi, M, F)$ be a smooth vector bundle, and $D^0$ is the standard flat connection on $U \subset E$ with the trivialisation $\varphi\colon E|_U \to U \times F$. If $D$ is a connection on a $(E, \pi, M, F)$, then $\exists A \in \Gamma(\mathrm{End}(E)) \otimes T^*M$ s.t.*

$$D = D^0 + A. \tag{21-6}$$

# §22 Parallel Transport

**Definition 22.1** (Parallel transport)**.** Let $(E, \pi, M, F)$ be a smooth vector bundle, and $D$ is a connection on $E$. A ***paralell transport*** of $s_0 \in \pi^{-1}(\{p\})$ $(p \in M)$ along a curve $\gamma\colon [0,1] \to M$ is a section $s \in \Gamma(E|_{\gamma([0,1])})$ such that

$$\forall t \in [0,1], \quad D_{\gamma'(t)} s(t) = 0, \quad s(0) = s_0, \tag{22-1}$$

where $s(t) := s_{\gamma(t)}$.

If $s =: v$ is a vector field, the Eq. (22-1) can be rewritten as

$$\frac{\mathrm{d}u \circ \gamma}{\mathrm{d}t}(t) + A[\gamma'(t)]u \circ \gamma(t) = 0, \tag{22-2}$$

which is a 1st order ODE. Given $\gamma_x(0) = x \in M$, there is a unique curve $\gamma_x$ associated to the vector field $u$.

We can extend the domain of $\gamma_x$ to $\mathbb{R}$ (note that $\mathbb{R}$ is diffeomorphic to $(0,1)$), and define:

$$\phi\colon \mathbb{R} \times M \to M; (t, x) \mapsto \gamma_x(t), \tag{22-3}$$

which is called the ***flow*** of $u$.

# Chapter 7

# Curvature

**Definition 22.2** (Curvature)**.** A ***curvature*** of a connection $D$ on a smooth vector bundle $(E, \pi, M, F)$ is a section $F \in \Gamma(\mathrm{End}(E) \otimes \Omega^2(M))$ (a End$(E)$-*valued* 2-*form*) defined as

$$F(v,w)s = D_v D_w s - D_w D_v s - D_{[v,w]} s, \quad v, w \in \Gamma(TM), \quad s \in \Gamma(E). \tag{22-1}$$

If $\forall v, w \in \Gamma(TM)$, $\forall s \in \Gamma(E)$, $F(v,w)s = 0$, then $D$ is called a ***flat connection***.

Consider a local trivialisation $\varphi\colon E|_U \to U \times F$ $(U \subset M)$ s.t.

$$s = s^i e_i := s^i \varphi^{-1}(f_i), \tag{22-2}$$

where $s \in \Gamma(E|_U)$, $s^i \in C^{(\infty)}(U)$ and $\{f_i\}_{i \in m}$ is a set of bases of $F$, and $\sigma\colon U \to \mathbb{R}^n$ is a chart of $M$, $\sigma_* d_\mu := \partial_\mu$. Notice that

$[\partial_\mu, \partial_\nu] = 0,$

$$
\begin{aligned}
&F(v,u)(s^i e_i) = v^\mu u^\nu F(d_\mu, d_\nu)(s^i e_i) \\
&= v^\mu u^\nu [D_\mu(d_\nu(s^i)e_i + s^i A^j_{\nu i} e_j) - D_\nu(d_\mu(s^i)e_i + s^i A^j_{\mu i} e_j)] \\
&= v^\mu u^\nu \big[d_\nu d_\mu(s^i)e_i + d_\nu(s^i)A^j_{\mu i} e_j + d_\mu(s^i A^j_{\nu i})e_j + s^i A^j_{\nu i} A^k_{\mu j} e_k \\
&\quad - d_\mu d_\nu(s^i)e_i - d_\mu(s^i)A^j_{\nu i} e_j - d_\nu(s^i A^j_{\mu i})e_j - s^i A^j_{\mu i} A^k_{\nu j} e_k\big] \\
&= v^\mu u^\nu s^i \big[d_\mu(A^k_{\nu i}) + A^j_{\nu i} A^k_{\mu j} - d_\nu(A^k_{\mu i}) - A^j_{\mu i} A^k_{\nu j}\big] e_k
\end{aligned}
\tag{22-3}
$$

# §23 Bianchi Identity

$$[D_u, [D_v, D_w]] + [D_v, [D_w, D_u]] + [D_w, [D_u, D_v]] = 0 \tag{23-1}$$

$$[D_\mu, F_{\nu\lambda}] + [D_\nu, F_{\lambda\mu}] + [D_\lambda, F_{\mu\nu}] = 0 \tag{23-2}$$

# Chapter 8

# Pseudo-Riemannian Geometry

## §24 Tensors

**Definition 24.1** (Tensor)**.** Let $M$ be a smooth manifold. A $(r, s)$-***tensor*** is a smooth section of the tensor product of $r$th tensor power of $TM$ and $s$th tensor power of $T^*M$:

$$t \in \Gamma(TM^{\otimes r} \otimes T^*M^{\otimes s}) =: TM^r_s. \tag{24-1}$$

In local coordinates:

$$t^{\mu_1 \cdots \mu_r}_{\nu_1 \cdots \nu_s} \bigotimes_{k=1}^{r} \partial_{\mu_k} \otimes \bigotimes_{k=1}^{s} \mathrm{d}x^{\nu_k}. \tag{24-2}$$

It is conventional to use the local coordinates form in pseudo-Riemannian geometry, and do not distinguish between a tensor and its components, written in forms of ***abstract indices***, where indices are written just to indicates types and operations on tensors.

And since we can raise and lower indices of a tensor, it is sometimes important to distinguish the orders between covariant and contravariant indices. e.g. $T^{\mu}{}_{\nu} \neq T^{\nu}{}_{\mu}$.

**Raising and Lowering of Indices** We have defined $\omega^\sharp$ for a 1-form and $v^\flat$ for a vector field. Now we can generalise the definition for any $(p, q)$ tensor $T$ that:

$$\begin{aligned}
&T^\sharp \in \Gamma(TM^{\otimes(p+q)}),\\
&T^\sharp(\omega_0, \dots, \omega_{p+q-1}) = T(\omega_0, \dots, \omega_{p-1}, \omega_p^\sharp, \dots, \omega_{p+q-1}^\sharp);\\
&T^\flat \in \Gamma(T^*M^{\otimes(p+q)}),\\
&T^\flat(v_0, \dots, v_{p+q-1}) = T(v_0^\flat, \dots, v_{p-1}^\flat, v_p, \dots, v_{p+q-1}).
\end{aligned} \tag{24-3}$$

We can even raise or lower some instead of all indices in $T$, by writing $T^{\sharp i \flat j}$ or $T^{\sharp\{i_0,\dots\}\flat\{j_0,\dots\}}$.

In abstract indices, it is conventional to keep the order of the indices including the raised and lowered ones, and abuse the original symbol of the tensor, for example if $T$ is a $(3, 4)$ tensor:

$$(T^{\sharp 4})^{\alpha_0\alpha_1\alpha_2\mu}{}_{\beta_0\beta_2\beta_3} =: T^{\alpha_0\alpha_1\alpha_2}{}_{\beta_0}{}^{\mu}{}_{\beta_2\beta_3} \tag{24-4}$$

Strictly speaking, the tensors after raising or lowering indices might not be the tensors as we defined in Def. 24.1, since it might belong to e.g.

$$TM^{\otimes r} \otimes T^*M \otimes TM^{\otimes s} \otimes T^*M^{\otimes t} \tag{24-5}$$

where the order of the tensor product is not canonical. One way to avoid this is to reorder the indices, but this approach is not conventional to those who use abstract indices. Since we can still consider the "tensors" as multilinear maps, we can include these non-canonical tensors, while, in order to avoid confusion in the order of indices, we will prefer to use the abstract indices form if there is any ambiguity.

**Tensor Product** Let $T_1$, $T_2$ be $(p_1, q_1)$ and $(p_2, q_2)$ tensors, we can have their tensor product:

$$T_1 \otimes T_2 \in TM^{p_1+p_2}_{q_1+q_2}, \tag{24-6}$$

where at each point $p \in M$, the tensor product is but the tensor product of the corresponding multilinear functions.

In abstract indices, we have:

$$\begin{aligned}&(T_1 \otimes T_2)^{\mu_1\cdots\mu_{p_1+p_2}}{}_{\nu_1\cdots\nu_{q_1+q_2}}\\ &= T_1^{\mu_0\cdots\mu_{p_1-1}}{}_{\nu_0\cdots\nu_{q_0-1}} T_2^{\mu_{p_1}\cdots\mu_{p_1+p_2-1}}{}_{\nu_{q_1}\cdots\nu_{q_1+q_2-1}}.\end{aligned} \tag{24-7}$$

**Contractions** The contraction is a generalisation of the inner product of vectors. Let $T$ is a $(p+1, q+1)$ tensor, we can define the $(i, p+j)$ contraction of $T$ as

$$\begin{aligned}&\operatorname{tr}_{(i,p+j)} T\colon T^*M^p \times TM^q \to \mathbb{R}\\ &(\omega_0, \dots, \omega_{i-1}, \omega_{i+1}, \dots, \omega_p, v_0, \dots, v_{j-1}, v_{j+1}, \dots, v_q) \mapsto\\ &\sum_{\mu\in N} T(\omega_0, \dots, \omega_{i-1}, \mathrm{d}x^\mu, \omega_{i+1}, \dots, \omega_p, v_0, \dots, v_{j-1}, \partial_\mu, v_{j+1}, \dots, v_q).\end{aligned} \tag{24-8}$$

The index-free notation can be found at [1].

# §25 Diffeomorphism and Invariance

Let $\phi\colon M \to N$ be a diffeomorphism, we have already known that we have pushforward $\phi_*$ for $(p, 0)$-tensors, and pullback $\phi^*$ for $(0, q)$-tensors. Since $\phi$ is a diffeomorphism, both $\phi^*$ and $\phi_*$ are isomorphisms, and we can generalise the definitions to obtain a pair of isomorphisms:

$$\phi_*\colon TM^p_q \to TN^p_q, \quad \phi^*\colon T^*N^p_q \to T^*M^p_q, \tag{25-1}$$

such that $\phi_* \circ \phi^* = \phi_* \circ \phi^*$, and

$$\begin{aligned}&\phi_* T(\omega_0, \dots, \omega_p, v_0, \dots, v_q)\\ &\quad = T(\phi^*\omega_0, \dots, \phi^*\omega_p, \phi_*^{-1} v_0, \dots, \phi_*^{-1} v_q),\end{aligned} \tag{25-2}$$

$$\begin{aligned}\phi^*T(\omega_0, \dots, \omega_p, v_0, \dots, v_q) \\ = T((\phi^{-1})^*\omega_0, \dots, (\phi^{-1})^*\omega_p, \phi_* v_0, \dots, \phi_* v_q).\end{aligned} \tag{25-3}$$

The special case when $M = N$ ($\phi$ is an endomorphism), if $\phi_* T = T$, then we say that $T$ is ***invariant*** under $\phi$.

**Theorem 25.1.** *Let $\phi\colon M \to N$ be a diffeomorphism, $T \in TM^p_q$, and $S \in TN^r_s$.*

1. *$\phi^*$ and $\phi_*$ are isomorphisms of $\mathbb{R}$-algebras.*
2. *$\phi_*(T \otimes S) = \phi_* T \otimes \phi_* S$, and $\phi^*(T \otimes S) = \phi^* T \otimes \phi^* S$.*

**Theorem 25.2.** *Let $\phi\colon M \to N$ be a* homeomorphism *(differentiable map), then $\forall \alpha, \beta \in \Omega(M)$, we have*

$$\phi^*(\alpha \wedge \beta) = \phi^*\alpha \wedge \phi^*\beta, \tag{25-4}$$

*that is, $\phi^*$ is the induced homeomorphism of the exterior algebra $\Omega(M)$.*

## §26 Lie Derivative

Let $u$ be a vector field on $M$, and $\phi$ be the corresponding flow.

**Definition 26.1** (Lie derivative)**.** Let $T$ be a $(p, q)$ tensor, then the ***Lie derivative*** of $T$ along $u$ is defined as

$$\mathcal{L}_u T = \lim_{t \to 0} \frac{\phi_t^* T - T}{t}. \tag{26-1}$$

**Theorem 26.1.** *$u, v \in \mathrm{Vect}(M)$.*

1. *$\mathcal{L}_{u+v} = \mathcal{L}_u + \mathcal{L}_v$.*
2. *$\mathcal{L}_{[u,v]} = \mathcal{L}_u \mathcal{L}_v - \mathcal{L}_v \mathcal{L}_u =: [\mathcal{L}_u, \mathcal{L}_v]$.*

**Theorem 26.2.** *$u \in \mathrm{Vect}(M)$, $T \in TM^p_q$, $S \in TM^r_s$.*

1. *$\mathcal{L}_u$ is $\mathbb{R}$-linear.*
2. *$\mathcal{L}_u(T \otimes S) = \mathcal{L}_u T \otimes S + T \otimes \mathcal{L}_u S$ (**Leibniz law**).*
3. *$\operatorname{tr}_{(i,j)} \mathcal{L}_u T = \mathcal{L}_u \operatorname{tr}_{(i,j)} T$.*
4. *$\forall f \in C^{(\infty)}(M)$, $\mathcal{L}_u f = u(f)$.*
5. *$\forall v \in \operatorname{Vect}(M)$, $\mathcal{L}_u v = [u, v]$.*

Applying the laws, we can calculate

$$\mathcal{L}_u \omega(v) = u[\omega(v)] - \omega([u, v]), \quad \omega \in \Omega^1(M) \tag{26-2}$$

by $u[\omega(v)] = T_u[\omega(v)] = T_u \operatorname{tr}_{(0,0)}(\omega \otimes v) = T_u\omega(v) + \omega([u, v])$. Similarly:

$$\begin{aligned} \mathcal{L}_u \omega(v_0, \ldots, v_{q-1}) &= u[\omega(v_0, \ldots, v_{q-1})] \\ &\quad - \omega([u, v_0], \ldots, v_{q-1}) - \cdots - \omega(v_0, \ldots, [u, v_{q-1}]). \end{aligned} \tag{26-3}$$

And, in local coordinates, we have

$$\begin{aligned} \mathcal{L}_u T^{\alpha_0 \cdots \alpha_{p-1}}{}_{\beta_0 \cdots \beta_{q-1}} &= u^\mu T^{\alpha_0 \cdots \alpha_{p-1}}{}_{\beta_0 \cdots \beta_{q-1}, \mu} \\ &\quad - T^{\mu \alpha_1 \cdots \alpha_{p-1}}{}_{\beta_0 \cdots \beta_{q-1}} u^{\alpha_0}{}_{,\mu} \cdots - T^{\alpha_0 \cdots \alpha_{p-2} \mu}{}_{\beta_0 \cdots \beta_{q-1}} u^{\alpha_{p-1}}{}_{,\mu} \\ &\quad + T^{\alpha_0 \cdots \alpha_{p-1}}{}_{\mu \beta_1 \cdots \beta_{q-1}} u^\mu{}_{,\beta_0} \cdots + T^{\alpha_0 \cdots \alpha_{p-1}}{}_{\beta_0 \cdots \beta_{q-2} \mu} u^\mu{}_{,\beta_{q-1}}. \end{aligned} \tag{26-4}$$

**Definition 26.2** (Divergence)**.** Let $u \in \operatorname{Vect}(M)$, then the ***divergence*** of $u$ is defined as

$$\operatorname{div} u = (-1)^{\operatorname{sign}(g)} \star (\mathcal{L}_u \operatorname{vol}) \tag{26-5}$$

**Definition 26.3** (Killing field)**.** If $u \in \operatorname{Vect}(M)$ is such that $\mathcal{L}_u g = 0$, then $u$ is called a ***Killing field***. The equation

$$\mathcal{L}_u g = 0, \quad \text{or} \quad u_{(\alpha;\beta)} = 0 \tag{26-6}$$

is called the ***Killing equation***.

**Theorem 26.3** (***Cartan's formula***)**.**

$$\mathcal{L}_u|_{\Omega(M)} = \mathrm{d} \circ i_u + i_u \circ \mathrm{d}. \tag{26-7}$$

**Corollary 1.**

$$\mathcal{L}_u|_{\Omega(M)} \circ \mathrm{d} = \mathrm{d} \circ \mathcal{L}_u|_{\Omega(M)}. \tag{26-8}$$

**Corollary 2.**

$$i_{[u,v]} = [\mathcal{L}_u|_{\Omega(M)}, i_v]. \tag{26-9}$$

As an application of Cartan's formula, we can prove the following theorem by induction.

**Theorem 26.4.**

$$\begin{aligned} \mathrm{d}\omega(u_0, \ldots, u_p) = &\sum_{i \in p+1} (-1)^i u_i \omega(u_0, \ldots, u_{i-1}, u_{i+1}, \ldots, u_p) \\ + &\sum_{\substack{(i,j) \in (p+1)^2 \\ i<j}} (-1)^{i+j} \omega([u_i, u_j], u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{j-1}, u_{j+1}, \ldots, u_p). \end{aligned} \tag{26-10}$$

# §27 Levi-Civita Connection

**Definition 27.1** (Levi-Civita connection)**.** Let $E \to M$ be a smooth vector bundle, where $M$ is a Riemanian manifold with metric $g \in T^*M \otimes T^*M$. Let $\nabla \in \Gamma(\mathrm{End}(E) \otimes T^*M)$ be a connection on $E$. Then $\nabla$ is called a ***Levi-Civita connection*** if

$$ug(v, w) = g(\nabla_u v, w) + g(v, \nabla_u w), \tag{27-1}$$

and

$$[v, w] = \nabla_v w - \nabla_w v, \tag{27-2}$$

where $u, v, w \in \Gamma(TM)$.

Since $T(u,v) = \nabla_u v - \nabla_v u - [v,u]$ is called the ***torsion*** of $u$ and $v$, Eq. (27-2) is called the ***torsion free*** condition.

In local coordinates:

$$\nabla_\alpha \partial_\beta = \Gamma^\gamma_{\alpha\beta} \partial_\gamma, \tag{27-3}$$

where $\Gamma^\gamma_{\alpha\beta}$ is the ***Christoffel symbol***.

The torsion free condition is equivalent to

$$\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha}. \tag{27-4}$$

For any $T \in \Gamma(TM^{\otimes p} \otimes T^*M^{\otimes q})$, we have

$$\nabla T = T^{\alpha_0 \cdots \alpha_{p-1}}{}_{\beta_0 \cdots \beta_{q-1};\mu} \bigotimes_{k \in p} \partial_{\alpha_k} \otimes \bigotimes_{\ell \in q} \mathrm{d}x^{\beta_\ell} \otimes \mathrm{d}x^\mu \tag{27-5}$$

$$\begin{aligned} T^{\alpha_0 \cdots \alpha_{p-1}}{}_{\beta_0 \cdots \beta_{q-1};\mu} = {} & T^{\alpha_0 \cdots \alpha_{p-1}}{}_{\beta_0 \cdots \beta_{q-1},\mu} \\ & + \sum_{i \in p} \Gamma^{\alpha_i}_{\lambda\mu} T^{\alpha_0 \cdots \alpha_{i-1} \lambda \alpha_{i+1} \cdots \alpha_{p-1}}{}_{\beta_0 \cdots \beta_{q-1}} \\ & - \sum_{i \in q} \Gamma^{\lambda}_{\beta_i\mu} T^{\alpha_0 \cdots \alpha_{p-1}}{}_{\beta_0 \cdots \beta_{i-1} \lambda \beta_{i+1} \cdots \beta_{q-1}}. \end{aligned} \tag{27-6}$$

It is useful to define the generalisation of divergence:

$$\nabla \cdot T = \operatorname{tr}_{(0,q)}(\nabla T) \tag{27-7}$$

if $T$ is a $(p,q)$ tensor.

It can be shown that

$$\nabla \cdot u = \operatorname{div} u = \delta u^\flat, \quad u \in \operatorname{Vect}(M). \tag{27-8}$$

**Theorem 27.1.** $\forall u \in \Gamma(TM)$,

$$\nabla_u \operatorname{tr}_{(i,j)} = \operatorname{tr}_{(i,j)} \nabla_u. \tag{27-9}$$

**Theorem 27.2.** $\forall \omega \in \Gamma(T^*M)$,

$$-\mathrm{d}\omega(u,v) = \nabla\omega(u,v) - \nabla\omega(v,u). \tag{27-10}$$

*(Notice that $\nabla\omega(u,v) = (\nabla_v \omega)(u)$)*

***Proof.***

$$u[\omega(v)] = ug(\omega^\sharp, v) = \nabla\omega(v,u) + \omega(\nabla_u v) \tag{27-11}$$

$\Rightarrow$ (Theorem 26.4)

$$\begin{aligned} \nabla\omega(v,u) - \nabla\omega(u,v) &= u[\omega(v)] - v[\omega(u)] - \omega([u,v]) \\ &= \mathrm{d}\omega(u,v) \end{aligned} \tag{27-12}$$

□

In fact, the Theorem 27.2 is but a special case of:

**Theorem 27.3.** $\forall\omega \in \Omega^p(M)$,

$$(-1)^p\,\mathrm{d}\omega(u_0,\ldots,u_p) = (p+1)\sum_{\pi\in S_{p+1}} \nabla\omega(u_{\pi(0)},\ldots,u_{\pi(p)}). \tag{27-13}$$

# bibliography

[1] Yuri Vyatkin (https://math.stackexchange.com/users/2002/yuri-vyatkin). *Coordinate-free notation for tensor contraction?* Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/1804213 (version: 2016-05-29). eprint: https://math.stackexchange.com/q/1804213. URL: https://math.stackexchange.com/q/1804213.

[2] Javier P. Muniain John C. Baez. *Gauge Fields, Knots and Gravity (Series on Knots and Everything)*. Series on Knots and Everything. World Scientific Publishing Company, 1994. ISBN: 9789810217297,9810217293,9810220340.

# Symbol List

Here listed the important symbols used in these notes

# Index