Quantum Groups

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Chapter 1

Poisson-Lie Groups and Lie Bialgebras

§1 Poisson Manifolds

Definition 1.1 (Poisson bracket)**.** Let M be a smooth manifold of finite dimension m, $C^{(\infty)}(M)$ be the algebra of smooth real-valued functions on M.

A ***Poisson bracket*** on M is an $\mathbb{R}$-bilinear map

$$\{,\}\colon C^{(\infty)}(M) \times C^{(\infty)}(M) \to C^{(\infty)}(M), \tag{1-1}$$

which satisfies the following conditions:

1. Anti-symmetric:
$$\{f,g\} = -\{g,f\}, \tag{1-2}$$

2. Jacobi identity:

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0, \tag{1-3}$$

3. Leibniz identity:

$$\{f,\{gh\}\} = \{f,g\}h + \{f,h\}g, \tag{1-4}$$

for any $f,g,h \in C^{(\infty)}(M)$.

$X_f = \{f,\}\colon g \mapsto \{f,g\}$ defines a vector field on M, called the ***Hamiltonian vector field***.

$$B_M\colon T^*M \to TM; \mathrm{d}f \mapsto X_f. \tag{1-5}$$

$\exists w_M \in TM^{\otimes 2}$ (***Poisson bivector***), s.t. $\{f,g\} = (\mathrm{d}f \otimes \mathrm{d}g)(w_M)$. In coordinates:

$$\{f,g\} = w^{ij}(x)\partial_i f(x)\partial_j g(x). \tag{1-6}$$

Definition 1.2 (Poisson map)**.** Let N, M be two Poisson manifolds. If $F\colon N \to M$ is smooth and $\forall f,g \in C^{(\infty)}(M)$,

$$\{f,g\}_M \circ F = \{f\circ F, g\circ F\}_N, \tag{1-7}$$

then F is called a ***Poisson map***.

If $y = F(x)$, $x \in N$, in coordinates:

$$\begin{aligned} w_M^{ij}(y)\frac{\partial f}{\partial y^i}(y)\frac{\partial g}{\partial y^i}(y) &= w_N^{k\ell}(x)\frac{\partial f\circ F}{\partial x^k}(x)\frac{\partial g\circ F}{\partial x^\ell}(x) \\ &= w_N^{k\ell}(x)\frac{\partial f}{\partial y^i}(y)\frac{\partial g}{\partial y^j}(y)\frac{\partial F^i}{\partial x^k}\frac{\partial F^j}{\partial x^\ell} \end{aligned}$$

$\Rightarrow$

$$(F'(x)\otimes F'(x))(w_N(x)) = w_M(F(x)). \tag{1-8}$$

Definition 1.3 (Poisson submanifold)**.** Let S be a submanifold of M. The inclusion map $S \hookrightarrow M$ is a Poisson map *iff* $\forall x \in S$, $w_M(x) \in (T_xS)^{\otimes 2}$. In this case, S is called a ***Poisson submanifold*** of M.

Definition 1.4 (Product of Poisson manifolds)**.** Let M and N be Poisson manifolds, their product $M \times N$, with $\{,\}_{M\times N}$ defined as

$$\begin{aligned}\{f,g\}_{M\times N}(x,y) =& \{x \mapsto f(x,y), x \mapsto g(x,y)\}_M(x) \\ &+ \{y \mapsto f(x,y), y \mapsto g(x,y)\}_N(y),\end{aligned} \tag{1-9}$$

is also a Poisson manifold.

$X_f = \{f,\}\colon g \mapsto \{f,g\}$ defines a vector field on M, called the ***Hamiltonian vector field***.

$$B_M\colon T^*M \to TM;\ \mathrm{d}f \mapsto X_f. \tag{1-10}$$

If B is an isomorphism, M is said to be ***symplectic***. Equivalently this means that $\forall x \in M$, $w(x)$ is a non-degenerate bilinear form on T^*_xM. In this case, $(B \otimes B)^{-1}(w) =: \omega \in T^*M \otimes T^*M$ is a non-degenerate 2-form on M.

Definition 1.5 (Symplectic leaves)**.** Let M be a Poisson manifold. Define an equivalence relation $\sim$ as $x \sim y$ *iff* x and y can be joined by a piecewise smooth curve on M, each smooth segment of which is part of an integral curve of a Hamiltonian vector field on M. Then each class in $M/\sim$ is called a ***symplectic leaf***.

Theorem 1.1 (Symplectic leaves are Poisson submanifolds)**.** $\forall L \in M/\sim$ *(defined in Def. 1.5), L is a Poisson submanifold of M.*

Example: Lie-Poisson structure

Definition 1.6 (Lie-Poisson structure)**.** Let $\mathfrak{g}$ be a mD Lie algebra, whose bases are x_i $(i \in m)$, and Lie bracket is defined as

$$[x_i, x_j] = c^k_{ij} x_k. \tag{1-11}$$

We can define the Poisson bracket on $\mathfrak{g}^*$ as

$$\{f_1, f_2\}(\xi) = \langle[(\mathrm{d}f_1)_\xi, (\mathrm{d}f_2)_\xi], \xi\rangle, \tag{1-12}$$

or in coordinates form:

$$\{f_1, f_2\}(\xi) = c^k_{ij} \frac{\partial f_1}{\partial x^i} \frac{\partial f_2}{\partial x^j} \langle \xi, x_k\rangle. \tag{1-13}$$

Note that the bases of $T^*\mathfrak{g}^* \cong (\mathfrak{g}^*)^* \cong \mathfrak{g}$ can be considered as x_i i.e. bases of $\mathfrak{g}$,

$$\mathrm{d}f = \frac{\partial f}{\partial x_i} x_i, \tag{1-14}$$

and where the x_i which partial derivative with respect to are coordinates on $\mathfrak{g}^*$.

Appendix A

Appendix

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Here listed the important symbols used in these notes.

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