

Analysis

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preface

The latest version: <https://github.com/HoyanMok/NotesOnMathematics/tree/master/Analysis>
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Part I

Mathematical Analysis

Chapter 1

Metric Space and Continuous Map

§1 Metric Space

Definition 1.1 (Metric). A function

$$d: X^2 \rightarrow \mathbb{R}$$

$\forall x, y, z \in X$ satisfying:

- a) $d(x, y) = 0 \leftrightarrow x = y$;
- b) $d(x, y) = d(y, x)$ (symmetry);
- c) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality),

is called a **metric** or **distance** in X . Such X is said to be equipped with a metric d , (X, d) is called a **metric space**. If the metric defined over X is definite, we just simply call the X the metric space.

Some examples:

- We can define $(\mathbb{R}^n; d_p)$, where

$$d_p(x, y) := \left(\sum_{i \in n} |x^i - y^i|^p \right)^{1/p}, \quad (1-1)$$

while

$$d_\infty(x, y) := \max_{i \in n} |x^i - y^i|. \quad (1-2)$$

- Similarly we can define metric spaces as $(C[a, b]; d_p)$ or simplified $C_p[a, b]$.

$$d_p(f, g) = \left(\int_a^b |f - g|^p dx \right)^{1/p}. \quad (1-3)$$

while $C_\infty[a, b]$ is called a **Chebyshev metric**, where the metric is defined as $d_\infty(f, g) := \max_{x \in [a, b]} |f(x) - g(x)|$.

- On equivalence class $\tilde{\mathfrak{R}}[a, b]$ over $\mathfrak{R}[a, b]$ similar metric can be defined. Functions are considered equivalent if they are equal up to a null set.

Lemma 1 (Quadruple inequality). *Let (X, d) be a metric space.*

$$\forall a, b, u, v \in X, \quad |d(a, b) - d(u, v)| \leq d(a, u) + d(b, v) \quad (1-4)$$

Proof. Without loss of generality, we assume that $d(a, b) > d(u, v)$. According to the triangle inequality (see def. 1.1), $d(a, b) \leq d(a, u) + d(u, v) + d(v, b)$, which is to prove. \square

Definition 1.2 (δ -ball). Let (X, d) be a metric space, and $\delta \in \mathbb{R}_+$, $a \in X$. A set

$$[Badelta]B(a; \delta) = \{x \in X \mid d(a, x) < \delta\}$$

is then called a **ball** with a centre at $a \in X$ and a radius of δ , or a **ball** of point a .

Definition 1.3 (Open set). An **open set** $G \in 2^X$ in a metric space (X, d) is a set that satisfies: $\forall x \in G, \exists \delta \in \mathbb{R}_+$, s.t. $B(x, \delta) \in 2^G$.

Definition 1.4 (Closed set). A **closed set** $F \in 2^X$ in a metric space (X, d) is a set that satisfies: $X - F$ is an open set in (X, d) .

A **closed ball** $[\overline{B}delta]\overline{B}(X, \delta) := \{x \in X \mid d(a, x) \leq r\}$ is an example of closed sets in (X, d) .

Proposition 1. a) An infinite union of open sets is an open set.

b) A definite intersection of open sets is an open set.

c) A definite union of closed sets is a closed set.

d) An infinite intersection of closed sets is a closed set.

Proof. Let $\forall \alpha \in A$, G_α be open sets.

a) $\forall x \in \bigcup_{\alpha \in A} G_\alpha, \exists \alpha \in A$ s.t. $x \in G_\alpha$. Since G_α is open, $\exists \delta \in \mathbb{R}_+$ s.t. $B(x, \delta) \subset G_\alpha \subset \bigcup_{\alpha \in A} G_\alpha$.

b) Let G_1, G_2 be open sets in (X, d) . $\forall a \in G_1 \cap G_2, \exists \delta_1, \delta_2 \in \mathbb{R}_+$ s.t. $B(a; \delta_1) \subset G_1, B(a; \delta_2) \subset G_2$. Without loss of generality, let $\delta_1 \geq \delta_2$, therefore $a \in B(a; \delta_1) \cap B(a; \delta_2) = B(a; \delta_2) \subset G_1 \cap G_2$.

c) Just consider $\mathbb{C}_X(\bigcap_{\alpha \in A} F_\alpha) = \bigcup_{\alpha \in A} \mathbb{C}_X(F_\alpha)$ and a).

d) Similarly, $\mathbb{C}_X(F_1 \cup F_2) = \mathbb{C}_X(F_1) \cap \mathbb{C}_X(F_2)$. \square

Definition 1.5 (Neighbourhood). If $x \in X$ is an element of an open set, then such open set is called a **neighbourhood** of point x in X , denoted by $[Ux]U(x)$. The collection of all neighbourhoods of x can be denoted by $\mathcal{U}(x)$.

Definition 1.6 (Interior point). Let $x \in X, E \subset X$.

a) If $\exists U(x) \subset E$, x is called an **interior point** of E .

b) If $\exists U(x) \subset X - E$, x is called an **exterior point** of E .

c) If x isn't an interior point nor exterior point of E , it is called a **boundary point** of E . The set of boundary points is called **boundary**, denoted by $[partialE]\partial E$.

Definition 1.7 (Limit point). $a \in X, E \subset X$. If $\forall U(a), \text{card}(E \cap U(a)) = \infty$, a is called a **limit point** of E .

Definition 1.8 (Closure). The intersections of $E \subset X$ and set of all its limit points is called the *closure* of E , denoted by \overline{E} .

Theorem 1.1. Let $F \in 2^X$. F is a closed set in $X \Leftrightarrow \overline{F} = F$.

Proof. \rightarrow : $\mathcal{C}_X(F)$ is open, hence its elements are all its interior points. Therefore $\overline{F} - F = \overline{F} \cup \mathcal{C}_X(F) = \emptyset$, also we know that $F \subset \overline{F}$, hence $F = \overline{F}$.

\leftarrow : $F = \overline{F}$ means that $\forall x \in \mathcal{C}_X(F)$, x is not a boundary of F , which implies that x is an interior point of $X - F$. Therefore $X - F$ is open while F is closed. \square

Theorem 1.2. \overline{E} is always closed.

Proof. $\forall x \in X - \overline{E}$, since it is not an element of the set E nor its limit points, $\exists U(x)$ s.t. $U(x) \cap \overline{E} = \emptyset$, which implies that x is an exterior point of E , therefore \overline{E} is closed. \square

Theorem 1.3. $\overline{E} = \overline{\overline{E}}$.

Proof. Since \overline{E} is closed, its complement is open, which implies that its elements are all exterior points of \overline{E} , therefore \overline{E} has contained all of its limit points. \square

Definition 1.9. (Metric subspace) We called $(X'; d')$ a **subspace** of (X, d) when $X' \subset X$ and $\forall x, y \in X'$, $d'(x, y) = d(x, y)$.

§2 Topological Space

Definition 2.1 (Topology). We say X is equipped with a **topology** if we assigned a $\mathcal{T} \subset 2^X$, with the following properties:

- a) $\emptyset \in \mathcal{T}$; $X \in \mathcal{T}$.
- b) $(\forall \alpha \in A, G_\alpha \in \mathcal{T}) \rightarrow \bigcup_{\alpha \in A} G_\alpha \in \mathcal{T}$.
- c) $\forall G_1, G_2 \in \mathcal{T}, G_1 \cap G_2 \in \mathcal{T}$.

We call (X, \mathcal{T}) a **topological space**, and sometimes we might simply call X the topological space.

These conditions are the intrinsic properties of the open sets we have defined in the metric space¹. The topology consisting of all the open sets defined in the metric space $(\mathbb{R}; d_2)$ is called the **standard topology** of the n -dimension Euclidean space.

Definition 2.2 (Open set). Topology \mathcal{T} 's elements are called **open sets**, and their complements are called **closed sets**.

Definition 2.3 (Base). Let (X, \mathcal{T}) be a topological space, and $\mathfrak{B} \subset 2^X$. If $\forall G \in \mathcal{T}, \exists \{B_\alpha\}_{\alpha \in A} \in 2^{\mathfrak{B}}$ s.t. $\bigcup_{\alpha \in A} B_\alpha = G$, we called \mathfrak{B} a (topological or open) **base** of the topology \mathcal{T} .

Definition 2.4 (Weight). The smallest possible cardinality of a base of a topology is called the **weight** of the topological space.

Definition 2.5 (Neighbourhood). If $x \in G$ and $G \in \mathcal{T}$, then G is a **neighbourhood** of x in topological space (X, \mathcal{T}) .

¹See proposition 1

For example, we define an equivalence relation \sim in $C(\mathbb{R}; \mathbb{R})$. If $f, g \in C(\mathbb{R}; \mathbb{R})$, at point $a \in \mathbb{R}$:

$$f \sim_a g \leftrightarrow \exists U(a) (\forall x \in U(a), f(x) = g(x)). \quad (2-1)$$

By collecting all of the continuous functions that are equivalent to f , we call f define a **germ** at point a , denoted by f_a . If $f \in C(\mathbb{R}; \mathbb{R})$ is defined in $U(a)$, then we can call $\{f_x \mid x \in U(a)\}$ a neighbourhood of germ f_a . Class of neighbourhoods of each f_x constructs a base of topological space $(C(\mathbb{R}; \mathbb{R}); \mathcal{T})$, where \mathcal{T} is made of the sets of germs of continuous function in $C(\mathbb{R}; \mathbb{R})$.

Definition 2.6 (Hausdorff space). We call a topological space (X, \mathcal{T}) a **Hausdorff space**, **separated space** or T_2 **space**, if $\forall x, y \in X, x \neq y \rightarrow (\exists U(x), U(y) \text{ s.t. } U(x) \cap U(y) = \emptyset)$ ².

Definition 2.7 (Dense set). $E \subset X$ is a **dense set** in the topological space (X, \mathcal{T}) , if $\forall x \in X, \forall U(x), U(x) \cap E \neq \emptyset$.

Definition 2.8 (Separable). If there is a *countable* dense set in topological space (X, \mathcal{T}) , then (X, \mathcal{T}) is **separable**.

We can also define interior points, exterior points, boundary points, limit points in topological space as in metric space.

Definition 2.9 (Topological subspace). Each subset Y of X equipped with topology \mathcal{T} can be given a **subspace topology** \mathcal{T}_Y whose elements G_Y are intersections of the subset with an open set G in (X, \mathcal{T}) i.e. $\forall G_Y \in \mathcal{T}_Y, \exists G \in \mathcal{T} \text{ s.t. } G_Y = G \cap Y$. Subsets equipped with such topology construct a **topological subspace** (Y, \mathcal{T}_Y) .

If two topology $\mathcal{T}_1, \mathcal{T}_2$ are defined on the same X , \mathcal{T}_1 is said to be **stronger** than \mathcal{T}_2 if $\mathcal{T}_1 \subsetneq \mathcal{T}_2$.

§3 Compact Set

Definition 3.1 (Open cover). Let (X, \mathcal{T}) be a topological space, $K \in 2^X$ and $\Omega \in 2^{\mathcal{T}}$. We call Ω to be an **open cover** over K , if $K \subset \cup \Omega$. If there are two open covers Ω, Ω' over K , and $\Omega' \subset \Omega$, we say that Ω' is a **subcover** of Ω .

Definition 3.2 (Compact set). A set $K \in 2^X$ in topological space (X, \mathcal{T}) is called a **compact set** if each of its open covers has a *finite* subcover.

Specially, \emptyset is compact.

Theorem 3.1. *Set $K \subset X$ is compact in (X, \mathcal{T}) iff K is compact in (K, \mathcal{T}_K) itself.*

This theorem tells a truth that whether K is compact or not doesn't dependent on the topological space it's in. This fact can be easily proved: we just need to notice that every open set G_K in (K, \mathcal{T}_K) is an intersection of an open set G in (X, \mathcal{T}) and K .

Theorem 3.2 (Compact \rightarrow closed (Hausdorff)). *If K is compact in a Hausdorff space (X, \mathcal{T}) ³, then K is a closed set in (X, \mathcal{T}) .*

²This definition is also called **Hausdorff axiom** or **separation axiom**.

³See definition 2.6.

Proof. Let x_0 be a limit point of K , which means $\forall U(x_0)$,

$$\text{card } U(x_0) \cap K \notin \mathbb{N}.$$

Assume that $x_0 \notin K$. In a Hausdorff space, $\forall x \in K - \{x_0\}$, $\exists U(x)$ s.t. $U(x) \cap U(x_0) = \emptyset$. Such $U(x)$ construct an open cover $\Omega = \{U(x) | x \in K\} \subset 2^K$. Since K is compact, $\exists \Omega' \subset \Omega$ s.t. $\text{card } \Omega' \in \mathbb{N}$.

$$(\cup \Omega') \cap U(x_0) = \left(\bigcup_{k=1}^n U_k \right) \cap U(x_0) = \bigcup_{k=1}^n (U_k \cap U(x_0)) = \emptyset.$$

Since $K \subset \cup \Omega'$, x_0 is an exterior point of K , which leads to a contradiction.

Hence $x_0 \in K$. $\overline{K} = K$. □

Theorem 3.3. *Each decreasing nested sequences of non-empty compact sets has a non-empty limit, i.e. $\forall (K_n)_{n \in \mathbb{N}} \in \mathcal{P}(X)^{\mathbb{N}}$ s.t. $\forall n \in \mathbb{N}_+$, $K_n \supset K_{n+1} \wedge K_n \neq \emptyset \wedge (K_n \text{ is compact}): K_n \downarrow K \neq \emptyset$.*

Proof. Assume that $K = \emptyset$. Compact subsets of K_1 are all closed, while their complements are all open. An open cover Ω can be constructed as $\{K_1 - K_n \mid n \in \mathbb{N}_+\}$. Since K_1 is compact, there would be a finite subcover $\Omega' \subset \Omega$, notice that $(X - K_n)_{n \in \mathbb{N}}$ is also a nested sequence, there must be one single $X - K_{n_0} \in \Omega'$ that covers K_1 , which means $K_{n_0} = \emptyset$ contradicting that $\forall n \in \mathbb{N}_+$, K_n is non-empty. □

Theorem 3.4. *Closed subsets F of a compact set K are also compact.*

Proof. If $\Omega_F \subset 2^K$ is an open cover of F . Notice that $K - F$ is open, $\Omega = (\cup \Omega_F) \cap \{K - F\}$ constructs an open cover over K . Since K is compact there must be a finite cover $\Omega' \subset \Omega$ which obviously also covers over F . □

The following properties of compact sets are about topological spaces induced from metric spaces.

Definition 3.3 (net). (X, d) is a metric space, $E \in 2^X$. E is called an ε -**net** if $\forall x \in X, \exists e \in E$, $d(e, x) < \varepsilon$.

Theorem 3.5 (Finite ε -net exists). *If (K, d) is a compact metric space, then $\forall \varepsilon \in \mathbb{R}_+$, \exists finite ε -net in (K, d) .*

Proof. For each point $x \in K$, find it a $B(x, \varepsilon)$, of which an infinite cover Ω over K is made. Since K is compact, there exists a finite subcover $\Omega' = \{B(x_i, \varepsilon)\}_{i \in n}$ ($n \in \mathbb{N}_+$). Therefore $\{x_i\}_{i \in n}$ is a finite ε -net in K . □

Theorem 3.6 (Sequentially compact). *A metric space (K, d) is compact iff it is sequentially compact, that is, $\forall (x_n)_{n \in \mathbb{N}} \in K^{\mathbb{N}}$, it has a convergent subsequence $(x_{k_n})_{n \in \mathbb{N}}$ ($k_n \in \mathbb{N}; k_{n+1} > k_n$) whose limit $a \in K$.*

To prove Theorem 3.6, we need to prove two lemmata first.

Lemma 2. *If (K, d) is sequentially compact, then $\forall \varepsilon \in \mathbb{R}_+$, \exists finite ε -net in (K, d) .*

Proof. Assume that $\exists \varepsilon_0 \in \mathbb{R}_+$, there were no finite ε_0 -net in (K, d) . Define such sequence: $(x_n)_{n \in \mathbb{N}}$ s.t. $\forall n \in \mathbb{N} \forall k \in \mathbb{N}, d(x_n, x_k) \geq \varepsilon_0$ (There would always be a next one since there exists no finite ε_0 -net or $\{B(x_n; \varepsilon_0)\}_{n \in \mathbb{N}}$ gives such). It has no convergent subsequence: if there were a $(x_{k_n})_{n \in \mathbb{N}}$ convergent to $a \in K$, $\exists N, M \in \mathbb{N}_+$, $d(x_N, x_M) \leq d(x_N, a) + d(x_M, a) \leq \varepsilon_0$, which lead to a contradictory. \square

Lemma 3. *If (K, d) is sequentially compact then every nested sequence of closed non-empty sets $\{F_n\}_{n \in \mathbb{N}}$ in K have a non-empty intersection.*

Proof. Let $(x_{k_n})_{n \in \mathbb{N}}$ be a convergent subsequence of $(x_n)_{n \in \mathbb{N}}$, where $\forall n \in \mathbb{N}, x_n \in F_n$. Let a be the limit of $(x_{k_n})_{n \in \mathbb{N}}$.

Assume that $a \notin \bigcap_{n \in \mathbb{N}} F_n$, in a metric space, $\exists U(a) \in \mathcal{U}(a)$ s.t. $U(a) \cap (\bigcap_{n \in \mathbb{N}} F_n) = \emptyset$, therefore $U(a) \cap (\bigcap_{n \in \mathbb{N}} F_{k_n}) = \emptyset$. But this conflict the fact that $\exists N \in \mathbb{N}$, s.t. $n > N \rightarrow x_{k_n} \in U(a)$ while $x_{k_n} \in F_{k_n}$. \square

Then we get back to the Theorem 3.6.

Proof. \rightarrow : If $\text{card}\{x_n\}_{n \in \mathbb{N}} \in \mathbb{N}$, it is obvious; Now we let $\text{card}\{x_n\}_{n \in \mathbb{N}} \notin \mathbb{N}$. We can always find finite $1/k$ -net $\{B(a_{k,i}, 1/k)\}_{i \in m}$ (Theorem 3.5, $m \in \mathbb{N}$, $a_i \in K$), for all $k \in \mathbb{N}_+$. For each k , there must be at least one $B(a_{k,i_0}, 1/k)$ (for simplification, we denote a_{k,i_0} by a_k) that includes infinite elements in $(x_n)_{n \in \mathbb{N}}$. $\forall n \in \mathbb{N}_+$ (let $k_0 = 0$), select $x_{k_n} \in B(a_{n,0}, 1/n)$, and $\{\overline{B}(x_n; 1/k)\}$ is a nested sequence of a closed non-empty sets in sequentially compact K , (Lemma 3) $\lim_{n \rightarrow \infty} x_{k_n} \in K$.

\leftarrow : Assume that there were an open cover Ω over K having no finite subcover, $\forall n \in \mathbb{N}_+$, \exists finite $1/n$ -net (Lemma 3), in which there would be at least one x_n whose $\overline{B}(x_n; \frac{1}{n})$ can't be covered finitely. Then $\overline{B}(x_n; 1/n) \downarrow B = \{a\}$ (Theorem 3.3) can't be finitely covered by any subcover of Ω , which means Ω can't cover the whole K , leading to the contradiction. \square

§4 Connected Set

Definition 4.1. Topological space (X, \mathcal{T}) is called **connected** if there is no **open-closed set** (i.e. both open and closed) besides \emptyset and X itself.

Notice that if $A \subset X$ is open-closed, its complement $X - A$ is also open-closed, which means a topological space is connected **iff** it is not a union of its two open subsets.

Definition 4.2. (X, \mathcal{T}) is a topological space. Subset C is said to be **connected** if subspace $(C; \mathcal{T}_C)$ is connected.

Theorem 4.1. (X, \mathcal{T}) is a topological space. $\forall \alpha \in A$, C_α are connected subsets of X . If $\bigcap_{\alpha \in A} C_\alpha \neq \emptyset$, then $\bigcup_{\alpha \in A} C_\alpha$ is also connected.

Proof. If $C = \bigcup_{\alpha \in A} C_\alpha$ were not connected, $\exists E \subset C$ s.t. $E \neq \emptyset \wedge E \neq C \wedge E, C - E \in \mathcal{T}_C$. For E is not empty there exists a $\beta \in A$ s.t. $E \cap C_\beta \neq \emptyset$. It can be proved that $C_\beta \subset E$.

Suppose that $C_\beta \not\subset E$, which implies that $(C - E) \cap C_\beta \neq \emptyset$. $E, C - E, C_\beta \in \mathcal{T}_C \Rightarrow E \cap C_\beta, (C - E) \cap C_\beta \in \mathcal{T}_C$. This conflicts to the fact that C_β is connected. Therefore $C_\beta \subset E$.

Hence there exists a $B \subsetneq A$, $\bigcup_{\beta \in B} C_\beta = A$. Since $C_\gamma, \gamma \in A - B$ would have a empty intersection with E , which contradicts $\bigcap_{\alpha \in A} C_\alpha \neq \emptyset$. \square

Theorem 4.2. *Connected sets have connected closure.*

Proof. □

Theorem 4.3. *$E \subset \mathbb{R}$ is connected iff that if $\forall x, z \in E, y \in \mathbb{R}$ s.t. $x < y < z$, then $y \in E$.*

Proof. \Rightarrow : Assume that there were such $y \in \mathbb{R}$ that $\exists x, z \in E, x < y < z$ but $y \notin E$. $\{x \in E | x < y\}$ and $\{x \in E | x > y\}$ are open in E for they are intersection of open sets in \mathbb{R} and E . Since they're each other's complement, they are both open-closed, which conflict to the definition of connected set.

\Leftarrow : It can be proofed that $(\inf E, \sup E) \subset E$. Assume that there were an open-closed proper subset $E \neq \emptyset$ contained in E . Find two points $x \in E, z \in E - E$. Without loss of generality, let $x < z$. Since E and $E - E$ are closed, $c_1 = \inf\{E \cap [a, b]\} \in E$ while $c_2 = \inf\{(E - E) \cap [a, b]\} \in E - E$. However $E \cap (E - E) = \emptyset \Rightarrow c_1 < c_2$, which means $(c_1, c_2) \cap E = \emptyset$. Here's the contradiction. □

Definition 4.3. A topological space (X, \mathcal{T}) is said to be **locally connected** if $\forall x \in X, \exists U(x)$ s.t. $U(x)$ is connected.

§5 Complete Metric Spaces

We now take a closer look at one of the most important sorts of metric spaces: complete spaces.

Definition 5.1. A sequence $\{x_n | n \in \mathbb{N}\}$ of points of a metric space (X, d) is called a **fundamental** or **Cauchy sequence** if $\forall \varepsilon \in \mathbb{R}_+, \exists N \in \mathbb{N}$ s.t. as long as $m, n > N, d(x_n, x_m) < \varepsilon$.

Definition 5.2. A metric space (X, d) is **complete** if every Cauchy sequence of its points is convergent.

For example, metric space $C_\infty[a, b]$ is complete while $C_1[a, b]$ isn't. Proof see p22, Zorich. Consider incomplete space \mathbb{Q}_1 , which is a subspace of the complete space \mathbb{R}_1 . If \mathbb{R}_1 is the smallest complete space containing \mathbb{Q}_1 , we can say that we have achieved a **completion** of \mathbb{Q}_1 . However, the definition of "completion" hasn't been defined yet.

Definition 5.3. If a metric space (X, d) is a subspace of a complete metric space (Y, d) and everywhere dense in it, we call the latter one the **completion** of (X, d) .

We need to confirm that such completion is the smallest and unique. So we introduce:

Definition 5.4. If there exists a **isometry** $f : X_1 \rightarrow X_2$ when $(X_1; d_1)$ and $(X_2; d_2)$ are both metric space, i.e. f is a bijective and for each $a, b \in X_1, d_2(f(a), f(b)) = d_1(a, b)$, then these two metric space is **isometric**.

This relation is reflexive (e), symmetric (f^{-1}), and transitive ($f \circ g$), so it is a equivalence relation, noted by \sim . We shall consider isometric spaces are identical.

Theorem 5.1. *If metric spaces $(Y_1; d_1)$ and $(Y_2; d_2)$ are both completions of (X, d) , then they are isometric.*

Proof. Such isometry $f : Y_1 \rightarrow Y_2$ can be defined: if $x_1, x_2 \in X$,

$$d_2(f(x_1), f(x_2)) = d(f(x_1), f(x_2)) = d(x_1, x_2) = d_1(x_1, x_2).$$

For each $y_1 \in Y_1 - X_1$, a Cauchy sequence $\{x_n\}$ can be found in the nested sequence of balls centered in y_1 . It is obvious that $\{x_n\}$ is also fundamental in Y_2 , limitting to $y_2 \in Y_2$. Different sequences of points $\{x'_n\}$ selected won't result in a different y'_2 , or $d(x_n, x'_n)$ wouldn't converge to 0, which violate the fact that the radii of balls converge to 0. Let $f(y_1) = y_2$.

a) For each $y_2 \in Y_2 - X$, there always exists a Cauchy sequence converging to it, which implies that f is a surjection.

b) Also notice that $\forall y'_1, y''_1 \in Y_1 - X$,

$$d_1(y'_1, y''_1) = \lim_{n \rightarrow \infty} d(x'_n, x''_n) = d_2(y'_2, y''_2)$$

while $\{x'_n\}$ and $\{x''_n\}$ are both Cauchy sequence. This equality also proofed that f is a injection. \square

Theorem 5.2. *There always exists a completion for every metric space.*

Proof. A isometric space (S_X, d) to the metric space (X, d_X) can be constructed, which consists of constant sequence of points in X . Its completion (S, d) can be defined as Cauchy sequences whose mutual distances' limits are not 0. \square

§6 Continuous Mapping

Let's recall the definition of the limitation.

Definition 6.1. A set $\mathcal{B} \subset 2^X$ is called a (**filter**) **base** in X if the following conditions hold:

- a) $\emptyset \notin \mathcal{B}$.
- b) $\forall B_1, B_2 \in \mathcal{B}, \exists B \in \mathcal{B} \text{ s.t. } B \subset B_1 \cap B_2 \subset B_2$.

Introduction of the limits in a topological space is as follows.

Definition 6.2. Let $a \in Y$ be the **limit** over the base $\mathcal{B} \subset 2^{\mathcal{D}(f)}$ of a mapping $f : \mathcal{D}(f) \rightarrow Y$, in which Y is equipped with a topology \mathcal{T} .

$$\lim_{\mathcal{B}} f = a \quad := \quad \forall U(a) \subset Y \exists B \in \mathcal{B} (f(B) \subset U(a)).$$

Such definition is parallel to the definition we have introduced on the limits of real number, hence it basically holds the same propoties.

Definition 6.3. A mapping $f : X \rightarrow Y$, where X, Y is respectively equipped with topology $\mathcal{T}_X, \mathcal{T}_Y$, is said to be **continuous** at $x_0 \in X$ (let $y_0 = f(x_0) \in Y$), if $\forall U(y_0), \exists U(x_0) \text{ s.t. } f(U(x_0)) \subset U(y_0)$. It is **continuous** in X if it is continuous at each point $x \in X$.

The set of continuous mappings from X into Y can be denoted by $C(X, Y)$ or $C(X)$ when Y is clear.

Theorem 6.1 (Criterion for continuity). *(X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) are both topological spaces. A mapping $f : X \rightarrow Y$ is continuous iff $\forall G_Y \in \mathcal{T}_Y, f^{-1}(G_Y) \in \mathcal{T}_X$.*

Proof. \Rightarrow : It is obvious if $f^{-1}(G_Y) = \emptyset$. If $f^{-1}(G_Y) \neq \emptyset$ and $x_0 \in X$, since $f \in C(X, Y)$, for G_Y , $\exists U(x_0)$ s.t. $f(U(x_0)) \subset G_Y$. Also notice that $f(U(x_0)) \subset G_Y \Rightarrow U(x_0) \subset f^{-1}(G_Y)$, therefore $f^{-1}(G_Y)$ is open.

\Leftarrow : $\forall x_0 \in X$, let $y_0 = f(x_0)$, $f^{-1}(U(y_0)) \in \mathcal{T}_X$. Notice that $x_0 \in f^{-1}(U(y_0))$, therefore $f \in C(X, Y)$. \square

Definition 6.4. (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) are both topological spaces. A bijective mapping $f : X \rightarrow Y$ is a **homeomorphism** if $f \in C(X, Y) \wedge f^{-1} \in C(Y, X)$.

Definition 6.5. Two topological spaces (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) are said to be **homeomorphic** if there exists a homeomorphism $f : X \rightarrow Y$.

Homeomorphic topological spaces are identical with respect to their topological properties since the theorem 6.1 has shown that their open sets correspond to each other.

Theorem 6.2. (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) are both topological spaces. $K \subset X$ is a compact set. If $f : X \rightarrow Y \in C(X, Y)$, then $f(K)$ is compact.

Proof. For each open cover $\Omega_Y = \{G_Y \in \mathcal{T}_Y\} \subset \mathcal{T}_Y$ over $f(K)$, $f^{-1}(G_Y) \in \mathcal{T}_X$ (Theorem 6.1). $f(K) \subset \cup \Omega_Y \Rightarrow K \subset f^{-1}(\cup \Omega_Y) = \cup \Omega_X$, where $\Omega_X = \{f^{-1}(G_Y) \mid G_Y \in \Omega_Y\}$ is an open cover over K . Since K is compact, $\exists \Omega'_X \subset \Omega_X$ ($|\Omega'_X| \in \mathbb{N}_+ \wedge K \subset \cup \Omega'_X$), $f(K) \subset f(\cup \Omega'_X)$. $f(G'_X) \in \Omega_Y$, hence $\Omega'_Y = \{f(G'_X) \mid G'_X \in \Omega'_X\}$ is a finite subcover over $f(K)$. \square

Theorem 6.3. $(K; \mathcal{T}_K)$ is a compact space and (Y, \mathcal{T}_Y) is a Hausdorff space. If a bijective $f : K \rightarrow Y \in C(K, Y)$, then it is a homeomorphism.

Proof. $\forall F = K - G$ s.t. $G \in \mathcal{T}_K$ is compact (Theorem ??). Hence $f(F)$ is compact (Theorem 6.2), then it is also closed (Theorem ??). This fact shows that f^{-1} is continuous (Theorem 6.1). \square

Theorem 6.4. (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) are both topological spaces. $E \subset X$ is a connected set. If $f : X \rightarrow Y \in C(X, Y)$, then $f(E)$ is also connected.

Proof. Only to notice that the open-closed sets in $(f(E); \mathcal{T}_{f(E)})$ have concurrently open-closed pre-images in $(E; \mathcal{T}_E)$. \square

§7 Contraction

Definition 7.1. A point $a \in X$ is a **fixed point** of a mapping $f : X \rightarrow X$ if $f(a) = a$.

Definition 7.2. Let $(X; d)$ be a metric space. A mapping $f : X \rightarrow X$ is called a **contraction** if $\exists q \in (0, 1) \subset \mathbb{R}$ s.t. $\forall x_1, x_2 \in X$,

$$d(f(x_1), f(x_2)) \leq qd(x_1, x_2). \quad (7-1)$$

Lemma 4. A contraction $f : X \rightarrow X$ is always continuous.

Proof. $\forall x \in X, \forall \varepsilon \in \mathbb{R}_+, \exists \delta < \varepsilon/q$, according to inequality 7-1:

$$f(B(x; \delta)) \subset B(f(x); \varepsilon).$$

\square

Theorem 7.1 (Picard-Banach fixed-point principle or contraction mapping principle).
Let $(X; d)$ be a complete metric space. Each contraction $f : X \rightarrow X$ has a unique fixed point a . Also, $\forall \{x_n\} \subset X$ s.t. $\forall n \in \mathbb{N} (f(x_n) = x_{n+1})$ then $\lim_{n \rightarrow \infty} x_n = a$, and

$$d(x_n, a) \leq \frac{q^n}{1 - q} d(x_1, x_0). \quad (7-2)$$

Proof. By the inequality 7-1:

$$d(x_{n+1}, x_n) \leq q d(x_n, x_{n-1}) \leq \cdots \leq q^n d(x_1, x_0)$$

Therefore, $\forall n, k \in \mathbb{N}$,

$$d(x_{n+k}, x_n) \leq \sum_{i=0}^{k-1} d(x_{n+i+1}, x_{n+i}) \leq \sum_{i=0}^{k-1} q^{n+i} d(x_1, x_0) \leq \frac{q^n}{1 - q} d(x_1, x_0), \quad (7-3)$$

which implies that x_n is a Cauchy sequence in a complete space $(X; d)$, hence it converges to a point $a \in X$.

To prove that a is a fixed point of f , since f is continuous (Lemma 4), just notice that

$$a = \lim_{n \rightarrow \infty} f(x_n) = f(\lim_{n \rightarrow \infty} x_n) = f(a).$$

If there were a second fixed point $a' \in X$ of f , then:

$$0 \leq d(a, a') = d(f(a), f(a')) \leq q d(a, a')$$

which can't be true unless $a = a'$.

By passing to the limit as $k \rightarrow \infty$ in the inequality 7-3, we have the inequality 7-2. \square

Chapter 2

Normed Linear Space and Differential Calculus

§8 Normed Linear Space

Definition 8.1. Let V be a linear space over \mathbb{R} or \mathbb{C} . A function $\| \cdot \| : X \rightarrow \mathbb{R}$ assigning to each vector $\mathbf{x} \in X$ a real number $\|\mathbf{x}\|$ is called a **norm** in the linear space X if:

- a) $\|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$ (nondegeneracy);
- b) $\|\lambda\mathbf{x}\| = |\lambda|\|\mathbf{x}\|$ (homogeneity);
- c) $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ (the triangle inequality).

A linear space with a norm defined on it is called **normed**.

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Symbol List

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