

Differential Geometry

Hoyan Mok

December 1, 2022

Contents

Contents	i
I Domestic Differential Geometry	1
1 Manifolds	2
2 Scalar and Vector Fields	3
§1 Scalar Fields	3
§2 Vector Fields	3
§3 Covariant and Contravariant	4
§4 Components of Vector Fields	5
§5 Lie Bracket	6
3 Differential Forms	8
§6 1-forms	8
§7 Components of 1-Forms	9
§8 k -Forms	10
§9 Exterior Derivative	11
4 Metric	12
§10 Pseudo-Riemannian Metric	12
§11 Volume Form	14

§12	Hodge Star Operator	15
§13	Metric and Coordinates	15
5	De Rham Theory	16
§14	Closed and Exact 1-Forms	16
§15	Stokes' Theorem	16
§16	De Rham Cohomology	16
6	Bundles and Connections	17
§17	Fibre Bundles	17
§18	Vector Bundles	18
§19	Constructions of Vector Bundles	19
§20	Connections	19
§21	Parallel Transport	21
7	Curvature	22
8	Pseudo-Riemannian Geometry	23
	bibliography	24
	Symbol List	25
	Index	26

Part I

Domestic Differential Geometry

Chapter 1

Manifolds

Chapter 2

Scalar and Vector Fields

§1 Scalar Fields

Definition 1.1 (Scalar Field). Let M be a smooth manifold, $f \in C^{(\infty)}(M)$ is called a ***scalar field***.

The scalar field over a manifold, form an algebra.

§2 Vector Fields

Definition 2.1 (vector field). A ***vector field*** v over manifold M is a $C^{(\infty)}(M) \rightarrow C^{(\infty)}(M)$ map that satisfies

- (a) $\forall f, g \in C^{(\infty)}(M), \forall \lambda, \mu \in \mathbb{R}, v(\lambda f + \mu g) = \lambda v(f) + \mu v(g)$
(*linearity*).
- (b) $\forall f, g \in C^{(\infty)}(M), v(fg) = v(f)g + fv(g)$

The space of all vector fields on M is denoted by $\text{Vect}(M)$

Definition 2.2 (tangent vector). Let v be a vector field over M , p be a point on M . The tangent vector v_p at p is defined as a $C^{(\infty)}(M) \rightarrow C^{(\infty)}(M)$ map that satisfies

$$v_p(f) = v(f)(p). \quad (2-1)$$

The collection of tangent vectors at p is called the **tangent space** at p , denoted by $T_p M$.

The derivative of a path $\gamma: [0, 1] \rightarrow M$ (or $\mathbb{R} \rightarrow M$) in a smooth manifold is defined as:

$$\begin{aligned} \gamma'(t) &: C^{(\infty)}(M) \rightarrow \mathbb{R}; \\ \gamma'(t)(f) &= \frac{d}{dt} f \circ \gamma(t) \end{aligned} \quad (2-2)$$

We can see that $\gamma'(t) \in T_{\gamma(t)} M$.

Let a path $\gamma: \mathbb{R} \rightarrow M$ follows a vector field (a velocity field), that is

$$\gamma'(t) = v_{\gamma(t)}, \quad (2-3)$$

then we call γ the **integral curve** through $p := \gamma(0)$ of the vector field v .

Definition 2.3. Suppose v is an integrable vector field. Let $\varphi_t(p)$ be the point at time t on the integral curve through p .

$$\varphi_t: M \rightarrow M \quad (2-4)$$

is then called a **flow** generated by v .

$$\frac{d}{dt} \varphi_t(p) = v_{\varphi_t(p)}. \quad (2-5)$$

§3 Covariant and Contravariant

Definition 3.1 (pullback). Let f be a scalar field over N , $\varphi \in C^{(\infty)}(M, N)$. Then the **pullback** of f by φ

$$\varphi^*: C^{(\infty)}(N) \rightarrow C^{(\infty)}(M), \quad (3-1)$$

is defined as

$$\varphi^* f = f \circ \varphi \in C^{(\infty)}(M). \quad (3-2)$$

Fields that are pullbacked are **covariant** fields.

Definition 3.2 (pushforward). Let v_p be a tangent vector of M at p , $\varphi \in C^{(\infty)}(M, N)$, $q = \varphi(p)$. Then the **pushforward** of v_p by φ

$$\varphi_*: T_p M \rightarrow T_q N, \quad (3-3)$$

is defined as

$$(\varphi_* v)_q(f) = v_p(\varphi^* f). \quad (3-4)$$

Note that the pushforward of a vector field can only be obtained when φ is a diffeomorphism.

Fields that are pushforwarded are **contravariant** fields.

Mathematicians and physicists might have disagreement on whether a tangent vector is covariant or contravariant. This is because of that physicists might consider the coordinates (v^μ) of a tangent vector as a vector field, instead of linear combination of bases ∂_μ .

§4 Components of Vector Fields

Let $\varphi: U \rightarrow \mathbb{R}^n$ be a chart of M ($U \subset M$).

Let $p \in U$, $\varphi(p) = x = (x^\mu)$ ($\mu = 0, \dots, n-1$). Locally, a function $f \in C^{(\infty)}(M)$ can be written as

$$(\varphi^{-1})^* f = f \circ \varphi^{-1}: M \rightarrow \mathbb{R}, \quad (4-1)$$

and a vector field $v \in \text{Vect}(M)$ can be written as

$$(\varphi_* v)_x = \varphi_* v_p: C^{(\infty)}(\mathbb{R}^n) \rightarrow \mathbb{R}, \quad (4-2)$$

or

$$\varphi_* v \in \text{Vect}(\mathbb{R}^n) \quad (4-3)$$

Since $T_x \mathbb{R}^n \cong \mathbb{R}^n$ is a linear space, one can find a basis for $T_x \mathbb{R}^n$ as

$$\partial_\mu : C^{(\infty)}(\mathbb{R}^n) \rightarrow C^{(\infty)}(\mathbb{R}^n), \quad (4-4)$$

and $(\varphi_* v)_x = v^\mu(x) \partial_\mu$.

Pushing forward $v^\mu(x) \partial_\mu$ by φ^{-1} one obtains v .

In an abuse of symbols, one may just omit the pullback and pushforward, and refer to the f and v by $(\varphi^{-1})^* f$ and $\varphi_* v$.

Consider another chart $\psi : U \rightarrow \mathbb{R}^n$ of M , and

$$y = \psi(p), \quad (\psi_* v)_x = u^\mu \partial_\mu, \quad (4-5)$$

where we have chosen the same basis in $T_y \mathbb{R}^n$ as in $T_x \mathbb{R}^n$.

We would like to know how to relate v^μ and u^μ i.e. we want to know how the components of v transforms under a coordinate transformation $\tau = \psi \circ \varphi^{-1}$.

Consider any $f \in C^{(\infty)}(M)$,

$$v(f) = \varphi_* v((\varphi^{-1})^* f) = \psi_* v((\psi^{-1})^* f) \quad (4-6)$$

\Rightarrow

$$u^\mu \partial_\mu (f \circ \psi^{-1}) = v^\mu \partial_\mu (f \circ \varphi^{-1}) = v^\mu \partial_\mu (f \circ \psi^{-1} \circ \tau) = v^\mu \tau'^\nu_\mu \partial_\nu (f \circ \psi^{-1}) \quad (4-7)$$

\Rightarrow

$$u^\mu = v^\nu \tau'^\mu_\nu, \quad (4-8)$$

where

$$\tau'^\mu_\nu = \frac{\partial y^\mu}{\partial x^\nu}. \quad (4-9)$$

§5 Lie Bracket

Definition 5.1 (Lie bracket). Let $v, w \in \text{Vect}(M)$, then the *Lie bracket* of v and w is defined as

$$[v, w] : C^{(\infty)}(M) \rightarrow C^{(\infty)}(M); \quad f \mapsto v \circ w(f) - w \circ v(f). \quad (5-1)$$

The Lie bracket is an antisymmetric bilinear map¹, and an important property of the Lie bracket is the Leibniz rule:

$$[v, w](fg) = [v, w](f)g + f[v, w](g). \quad (5-2)$$

Another important property of the Lie bracket is the Jacobi identity:

$$[v, [w, u]] + [w, [u, v]] + [u, [v, w]] = 0. \quad (5-3)$$

¹Note that it is not $C^{(\infty)}$ -linear

Chapter 3

Differential Forms

§6 1-forms

Definition 6.1 (1-form). A **1-form** ω on M is a $\text{Vect}(M) \rightarrow C^{(\infty)}(M)$ which satisfies that

$$(a) \quad \forall v, w \in \text{Vect}(M), \forall f, g \in C^{(\infty)}(M),$$

$$\omega(fv + gw) = f\omega(v) + g\omega(w). \quad (6-1)$$

The space of all 1-forms on M is denoted as $\Omega^1(M)$, which is a module over $C^{(\infty)}(M)$.

The operator d , when given a $C^{(\infty)}(M)$ function (which is called a **0-form**), would give a 1-form:

$$(df)(v) = v(f). \quad (6-2)$$

This is called the **exterior derivative** or **differential** of f .

The **cotangent vector** or **covector** is similar as the tangent vector:

$$\omega_p(v_p) = \omega(v)(p). \quad (6-3)$$

The space of cotangent vectors at p on M is denoted by T_p^*M .

1-forms are covariant, that is, if $\varphi: M \rightarrow N$, then the pushforward of a 1-form ω by φ is

$$(\varphi^*\omega)_p(v_p) = \omega_q(\varphi_*v_p), \quad (6-4)$$

where $\varphi(p) = q$.

Theorem 6.1. $f \in C^{(\infty)}(N)$, $\varphi: M \rightarrow N$ is differential, then

$$\varphi^*(df) = d(\varphi^*f). \quad (6-5)$$

§7 Components of 1-Forms

Let $\varphi: U \rightarrow \mathbb{R}^n$ be a chart of M ($U \subset M$).

Let $p \in U$, $\varphi(p) = x = (x^\mu)$ ($\mu = 0, \dots, n-1$). Locally a 1-form $\omega \in \Omega^1(M)$ can be written as

$$(\varphi^{-1})^*\omega \in T_x^*\mathbb{R}^n. \quad (7-1)$$

A natural way to impose a basis dx^μ in $T_x^*\mathbb{R}^n$ is

$$dx^\mu(\partial_\nu) = \delta_\nu^\mu, \quad (7-2)$$

and $(\varphi^{-1})^*\omega = \omega_\mu(x) dx^\mu$.

Now by the definition of 1-form:

$$\omega_\mu dx^\mu(v^\nu \partial_\nu) = v^\nu \omega_\mu \delta_\nu^\mu = v^\mu \omega_\mu. \quad (7-3)$$

By the transformation rule of components of a vector, one have

$$\tau'^\nu{}_\mu \alpha_\nu = \omega_\mu, \quad (7-4)$$

where $\psi: U \rightarrow \mathbb{R}^n$, $(\psi^{-1})_*\omega = \alpha_\mu dx^\mu$, $\tau = \psi \circ \varphi^{-1}$.

§8 k -Forms

Definition 8.1. If we assign an antisymmetric multilinear k -form $\omega_p \in \bigotimes_{i \in k} T_p^* M$ to each point $p \in M$, we say we have a k -**form** on M .

The collection of all k -forms is denoted by $\Omega^k(M)$, and $\Omega(M) := \bigcup_{k \in \mathbb{N}} \Omega^k(M)$.

Theorem 8.1 (Dimension of forms). *If M is an nD manifold, then the dimension of $\Omega^k(M)$ is $\frac{n!}{k!(n-k)!}$ ($k \leq n$), and 0 for $k > n$; The dimension of $\Omega(M)$ is 2^n .*

Definition 8.2 (Wedge product). The **wedge product** \wedge is defined as a binary operator that takes a k -form and ℓ -form and gives a $(k + \ell)$ -forms, satisfying $\forall \alpha \in \Omega^k(M), \forall \beta \in \Omega^\ell(M)$:

(a) (Associativity) $\forall \gamma \in \Omega^m(M)$,

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma). \quad (8-1)$$

(b) (Supercommutativity)

$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha. \quad (8-2)$$

(c) (Distributiveness) $\forall \gamma \in \Omega^\ell(M)$,

$$\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma. \quad (8-3)$$

(d) (Bilinearity over $C^{(\infty)}(M)$) $\forall f \in C^{(\infty)}(M)$,

$$(f\alpha) \wedge \beta = f(\alpha \wedge \beta). \quad (8-4)$$

(e) (Naturality) If $\varphi: M \rightarrow N$ is a smooth map, then the pullback of a form by φ can be given by repeatedly applying ($\forall \gamma \in \Omega^\ell(M)$)

$$\begin{aligned} \varphi^*(\beta + \gamma) &= \varphi^*\alpha + \varphi^*\beta \\ \varphi^*(\alpha \wedge \beta) &= \varphi^*\alpha \wedge \varphi^*\beta, \end{aligned} \quad (8-5)$$

while the pullback of a 0-form and a 1-form agree with what we have already defined before.

By convention if $f \in C^{(\infty)}(M)$ then

$$f \wedge \omega =: f\omega. \quad (8-6)$$

It can be shown that any k -form ω can be written as

$$(\varphi^{-1})^*\omega = \frac{\omega_{\mu_1 \cdots \mu_k}}{k!} \bigwedge_{i=1}^k dx^{\mu_i}, \quad (8-7)$$

where $\varphi: M \rightarrow \mathbb{R}^n$ is a chart.

§9 Exterior Derivative

Definition 9.1 (Exterior derivative). The *exterior derivative* d is defined as a linear operator that takes a k -form and gives a $(k+1)$ -form, satisfying $\forall \alpha \in \Omega^k(M), \forall \beta \in \Omega^\ell(M)$:

(a) (Linearity) $\forall \lambda, \mu \in \mathbb{R}, \forall \gamma \in \Omega^\ell(M),$

$$d(\lambda\beta + \mu\gamma) = \lambda d\beta + \mu d\gamma. \quad (9-1)$$

(b) (Leibniz rule)

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta. \quad (9-2)$$

(c)

$$d^2\omega = 0. \quad (9-3)$$

(d) (Naturality) If $\varphi: M \rightarrow N$ is a smooth map, then

$$\varphi^* d\omega = d\varphi^*\omega. \quad (9-4)$$

Chapter 4

Metric

§10 Pseudo-Riemannian Metric

Definition 10.1 (Pseudo-Riemannian metric). Let M be a manifold. A ***pseudo-Riemannian metric*** or simply ***metric*** g on a manifold M is a field ($g \in \Gamma(T^*M \otimes T^*M)$) that $\forall p \in M$,

$$g_p: T_p^*M \times T_p^*M \rightarrow \mathbb{R}, \quad (10-1)$$

is a bilinear form satisfying the following properties:

(a) (Symmetry) $\forall u, v \in T_p^*M$,

$$g_p(u, v) = g_p(v, u). \quad (10-2)$$

(b) (Non-degenerate)

$$u \mapsto g_p(u, -): T_p^*M \rightarrow T_p^*M \quad (10-3)$$

is an isomorphism.

(c) (Bilinearity) $\forall p \in M, \forall u, v \in T_p^*M, \forall \lambda, \mu \in \mathbb{R}$,

$$g_p(\lambda u + \mu v, w) = \lambda g_p(u, w) + \mu g_p(v, w). \quad (10-4)$$

(d) (Smoothness) If $v, u \in \text{Vect}(M)$, then

$$p \mapsto g_p(v_p, u_p) \in C^{(\infty)}(M). \quad (10-5)$$

Given a metric, $\forall p \in M$, we can always find an orthonormal basis $\{e_\mu\}$ of $T_p M$ such that

$$g_p(e_\mu, e_\nu) = \text{sign}(\mu)\delta_{\mu\nu}, \quad (10-6)$$

where $\text{sign}(\mu) = \pm 1$. Conventionally we order the basis such that $\text{sign}(\mu) = 1$ for $\mu \in s$ and $\text{sign}(\mu) = -1$ for $\mu - s \in n - s$, and say that the metric has **signature** $(s, n - s)$.

If $\gamma: [0, 1] \rightarrow M$ is a smooth path and $\forall t, s \in [0, 1]$,

$$g(\gamma'(t), \gamma'(t))g(\gamma'(s), \gamma'(s)) \geq 0, \quad (10-7)$$

then we can define the arclength of γ as

$$\int_0^1 \sqrt{|g(\gamma'(t), \gamma'(t))|} dt \quad (10-8)$$

if the integral converges.

The metric gives an **inner product** on $\text{Vect}(M)$:

$$\langle u, v \rangle := g(u, v). \quad (10-9)$$

The metric also gives a way to relate a vector field v to a 1-form ω . If v and ω satisfies: $\forall u \in \text{Vect}(M)$,

$$g(v, u) = \omega(u), \quad (10-10)$$

then we say that v is the corresponding vector field of ω , and ω is the corresponding 1-form of v .

We can also define the **inner product** on $\Omega^1(M)$ by

$$\langle \alpha, \beta \rangle = \langle a, b \rangle, \quad (10-11)$$

where a and b is the corresponding vector fields of α and β .

The **inner product** on $\Omega^k(M)$ is defined by induction with

$$\langle \bigwedge_{i \in k} \alpha_i, \bigwedge_{i \in k} \beta_i \rangle = \det(\langle \alpha_i, \beta_j \rangle)_{i,j \in k}. \quad (10-12)$$

Hence, if $\{e_\mu\}$ is an orthonormal basis (field) of $T_p M$, while the corresponding covectors are $\{f^\mu\}$ ($f^\mu(e_\nu) = \delta^\mu_\nu$) then

$$\langle \bigwedge_{i \in k} f^{\mu_i}, \bigwedge_{i \in k} f^{\mu_i} \rangle = \prod_{i \in k} \text{sign}(\mu_i). \quad (10-13)$$

Specially, when $f, g \in \Omega^0(M) = C^{(\infty)}(M)$,

$$\langle f, g \rangle = fg. \quad (10-14)$$

§11 Volume Form

Notice that if M is an n D manifold, $\dim \Omega^n(M) = 1$, meaning at $p \in M$, $\{\omega_p \mid \omega \in \Omega^n(M)\}$ can be labelled by a parametre $\lambda_p \in \mathbb{R}$. If we have a basis $\{f^\mu\}$ of $T_p^* M$ (or corresponding vectors $\{e_\mu\}$), then

$$\{\omega_p \mid \omega \in \Omega^n(M)\} = \lambda_p \bigwedge_{\mu \in n} f^\mu. \quad (11-1)$$

If there were another basis $\{g^\mu\}$ of $T_p^* M$ (or corresponding vectors $\{h_\mu\}$), and the transformation between the two bases is given by

$$P e^\mu = f^\mu, \quad (11-2)$$

where $P \in \text{Aut}(T_p^* M)$. When $\det P > 0$, we say that $\{f^\mu\}$ and $\{g^\mu\}$ have the same **orientation**.

Definition 11.1 (Volume form). Let M be an orientable manifold. If $\forall p \in M$, we find an oriented orthonormal basis $\{f_\mu\}$ of $T_p^* M$ at point p , then the **volume form** vol is defined by

$$\bigwedge_{\mu \in n} f_\mu = \text{vol}_p. \quad (11-3)$$

§12 Hodge Star Operator

Definition 12.1 (Hodge Star Operator). Let M be an orientable manifold. The **Hodge star operator** \star is defined by the linear map

$$\star: \Omega^k(M) \rightarrow \Omega^{n-k}(M), \quad (12-1)$$

$$\forall \alpha, \beta \in \Omega^k(M), \quad \alpha \wedge \star \beta = \langle \alpha, \beta \rangle \text{vol}. \quad (12-2)$$

We call $\star \omega$ the **dual** of ω .

The special case is when $k = 0$,

$$\star f = f \text{vol}, \quad (12-3)$$

and $k = n$,

$$\star(f \text{vol}) = f \prod_{\mu \in n} \text{sign}(\mu) = (-1)^{n-s} f \quad (12-4)$$

if the signature of the metric is $(s, n - s)$.

§13 Metric and Coordinates

Chapter 5

De Rham Theory

§14 Closed and Exact 1-Forms

§15 Stokes' Theorem

§16 De Rham Cohomology

Chapter 6

Bundles and Connections

§17 Fibre Bundles

Definition 17.1 (Bundle). A **bundle** is a triple (E, π, B) , where $\pi: E \rightarrow B$ is a surjective map. E is called the **total space**, π is called the **projection map**, and B is called the **base space**.

A bundle (E, π, B) can be denoted as $\pi: E \rightarrow B$ or $E \xrightarrow{\pi} B$.

Definition 17.2 (Fibre). For $p \in B$, $\pi^{-1}(\{p\})$ is the **fibre** over b .

Definition 17.3 (Subbundle). Let $\pi: E \rightarrow B$ be a bundle. $F \subset E$, $C \subset B$, $\rho: F \rightarrow C$. If $\pi|_F = \rho$, then $\rho: F \rightarrow C$ is called a **subbundle** of $\pi: E \rightarrow B$.

Definition 17.4 (Section). A **section** is a map $s: B \rightarrow E$ such that

$$p \circ s = \text{id}_B. \quad (17-1)$$

All sections of a bundle $\pi: E \rightarrow B$ is denoted as $\Gamma(E)$.

Definition 17.5 (Fibre bundle). A **fibre bundle** (E, π, B, F) is a bundle $\pi: E \rightarrow B$, where E, B, F are topology spaces, and π is a continuous map, and $\forall p \in B, \exists U \in \mathcal{U}(p)$ s.t.

$$\varphi: \pi^{-1}(U) \rightarrow U \times F, \quad (17-2)$$

is a homeomorphism and $\pi_1 \circ \varphi = \pi$. π_1 is defined as $\pi_1(p, q) = p$.

A fibre bundle can be denoted as the exact sequence

$$F \longrightarrow E \xrightarrow{\pi} B \quad (17-3)$$

The last condition is called the **local triviality condition**. F is called the **standard fibre**

If $E = B \times F$, then (E, π, B, F) is called a **trivial fibre bundle**.

Definition 17.6 (Morphism). Let $\pi: E \rightarrow B, \rho: F \rightarrow C$ be two fibre bundles. A **morphism** (φ, ψ) is a pair of two continuous maps such that

$$\begin{array}{ccc} E & \xrightarrow{\psi} & F \\ \downarrow \pi & & \downarrow \rho \\ B & \xrightarrow{\varphi} & C \end{array} \quad (17-4)$$

commutes.

§18 Vector Bundles

Definition 18.1 (Vector bundle). A **vector bundle** is a fibre bundle (E, π, B, F) , where F is a vector space, and the local trivialisation $\varphi: \pi^{-1}(U) \rightarrow U \times F$ (U is a neighbourhood of $p \in B$) satisfies that $\forall x \in U, \forall v \in F$,

$$\begin{aligned} F &\rightarrow \pi^{-1}(\{x\}) \\ v &\mapsto \varphi^{-1}(x, v) \end{aligned} \quad (18-1)$$

is a linear isomorphism (**fibrewise linear**).

Definition 18.2 (Morphism (vector bundle)). The morphism between two vector bundles (E, π, B, F) and (E', π', B', F') is a morphism (φ, ψ) such that $\forall x \in B$,

$$\psi_*: \pi^{-1}(\{x\}) \rightarrow (\pi')^{-1}(\{\varphi(x)\}) \quad (18-2)$$

is a linear homomorphism.

Definition 18.3 (Smooth vector bundle). A **smooth vector bundle** is a vector bundle (E, π, B, F) , where the projection $\pi: E \rightarrow B$ and the local trivialisation $\varphi: \pi^{-1}(U) \rightarrow U \times F$ are smooth.

Definition 18.4 (Tangent bundle). The **tangent bundle** TM is the smooth vector bundle over an n D smooth manifold M with the standard fibre $T_p M = \mathbb{R}^n$.

A vector field $v \in \text{Vect}(M)$ is the smooth section of the tangent bundle $\Gamma(TM)$.

Definition 18.5 (Cotangent bundle). The **cotangent bundle** of an n D manifold M , denoted by T^*M , is the smooth vector bundle over with the standard fibre $T_p^*M = (\mathbb{R}^n)^*$.

A 1-form $\omega \in \Omega^1(M)$ is the smooth section of the cotangent bundle $\Gamma(T^*M)$.

§19 Constructions of Vector Bundles

Definition 19.1 (Duality).

§20 Connections

Definition 20.1 (Connection). A **connection** on a smooth vector bundle (E, π, M, F) is map

$$D: \Gamma(TM) \times \Gamma(E) \rightarrow \Gamma(E), \quad (20-1)$$

that satisfies the following conditions: $\forall v, w \in \Gamma(TM), \forall s, t \in \Gamma(E), \forall f \in C^{(\infty)}(M)$,

- (a) $D_v(s + t) = D_v s + D_v t$;
- (b) $D_v(fs) = v(f)s + fD_v s$;
- (c) $D_{v+w}s = D_v s + D_w s$;
- (d) $D_{fv}s = fD_v s$.

When a vector field $v \in \Gamma(TM)$ is given to the connection D , the map $D_v: \Gamma(E) \rightarrow \Gamma(E)$ is called the **covariant derivative** with respect to v .

Definition 20.2 (Vector potential). A **vector potential** A is an $\text{End}(E)$ -valued 1-form, that is

$$A \in \Gamma(\text{End}(E) \otimes T^*M), \quad (20-2)$$

where $\text{End}(E) \cong E \otimes E^*$ can be considered as a vector bundle over M with the standard fibre $\text{End}(E_p) \cong E_p \otimes E_p^*$ ($p \in E$).

Locally if $s \in \Gamma(E)$ we can have a trivialisation $\varphi: E|_U \rightarrow U \times F$ ($U \subset M$). If we assign a basis $\{f_i\}_{i \in m}$ for the m D standard fibre F , then

$$s = s^i e_i := s^i \varphi^{-1}(f_i), \quad s^i \in C^{(\infty)}(U), \quad (20-3)$$

where we can call $\{s^i\}_{i \in m}$ the **components of the section** s . With this specific normalisation, one can define that

$$D_v^0 s = v(s^i) e_i \quad (20-4)$$

where D^0 is called the **standard flat connection** (which depends on trivialisation).

Theorem 20.1. Let (E, π, M, F) be a smooth vector bundle. If D is a connection on E , $A \in \Gamma(\text{End}(E) \otimes T^*M)$, then the $D + A$, which defined as

$$D + A: (v, s) \mapsto D_v s + A(v)s, \quad (20-5)$$

is also a connection.

Theorem 20.2. *Let (E, π, M, F) be a smooth vector bundle, and D^0 is the standard flat connection on $U \subset E$ with the trivialisation $\varphi: E|_U \rightarrow U \times F$. If D is a connection on a (E, π, M, F) , then $\exists A \in \Gamma(\text{End}(E)) \otimes T^*M$ s.t.*

$$D = D^0 + A. \quad (20-6)$$

§21 Parallel Transport

Definition 21.1 (Parallel transport). Let (E, π, M, F) be a smooth vector bundle, and D is a connection on E . A ***parallel transport*** of $s_0 \in \pi^{-1}(\{p\})$ ($p \in M$) along a curve $\gamma: [0, 1] \rightarrow M$ is a section $s \in \Gamma(E|_{\gamma([0,1])})$ such that

$$\forall t \in [0, 1], \quad D_{\gamma'(t)} s(t) = 0, \quad s(0) = s_0, \quad (21-1)$$

where $s(t) := s_{\gamma(t)}$.

Chapter 7

Curvature

Chapter 8

Pseudo-Riemannian Geometry

bibliography

- [1] Javier P. Muniain John C. Baez. *Gauge Fields, Knots and Gravity (Series on Knots and Everything)*. Series on Knots and Everything. World Scientific Publishing Company, 1994. ISBN: 9789810217297,9810217293,9810220340.

Symbol List

Here listed the important symbols used in these notes

D^0 , 20

d , 11

$\Gamma(E)$, 17

$\Omega^1(M)$, 8

$\Omega^k(M)$, 10

$\Omega(M)$, 10

\star , 15

T^*M , 19

TM , 19

T_p^*M , 9

T_pM , 4

$\text{Vect}(M)$, 3

vol , 14

\wedge , 10

Index

- 0-form, 8
- 1-form, 8
- base space, 17
- bundle, 17
- components of the section, 20
- connection, 19
- contravariant, 5
- cotangent bundle, 19
- cotangent vector, 8
- covariant, 5
- covariant derivative, 20
- covector, 8
- differential, 8
- dual, 15
- exterior derivative, 8, 11
- fibre, 17
- fibre bundle, 18
- fibrewise linear, 18
- flow, 4
- Hodge star operator, 15
- inner product, 13, 14
- integral curve, 4
- k -form, 10
- Lie bracket, 6
- local triviality condition, 18
- metric, 12
- morphism, 18
- orientation, 14
- parallel transport, 21
- projection map, 17
- pseudo-Riemannian metric, 12
- pullback, 4
- pushforward, 5
- scalar field, 3
- section, 17
- signature, 13

- smooth vector bundle, 19
- standard fibre, 18
- standard flat connection, 20
- subbundle, 17
- tangent bundle, 19
- tangent space, 4
- total space, 17
- trivial fibre bundle, 18
- vector bundle, 18
- vector field, 3
- vector potential, 20
- volume form, 14
- wedge product, 10