Differential Geometry

Hoyan Mok

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Domestic Differential Geometry

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Manifolds

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Scalar and Vector Fields

§1 Scalar Fields

Definition 1.1 (Scalar Field)**.** Let M be a smooth manifold, $f \in C^{(\infty)}(M)$ is called a ***scalar field***.

The scalar field over a manifold, form an algebra.

§2 Vector Fields

Definition 2.1 (vector field)**.** A ***vector field*** v over manifold M is a $C^{(\infty)}(M) \to C^{(\infty)}(M)$ map that satisfies

(a) $\forall f, g \in C^{(\infty)}(M)$, $\forall \lambda, \mu \in \mathbb{R}$, $v(\lambda f + \mu g) = \lambda v(f) + \mu v(g)$ (*linearity*).

(b) $\forall f, g \in C^{(\infty)}(M)$, $v(fg) = v(f)g + fv(g)$

The space of all vector fields on M is denoted by $\mathrm{Vect}(M)$

Definition 2.2 (tangent vector)**.** Let v be a vector field over M, p be a point on M. The tangent vector v_p at p is defined as a $C^{(\infty)}(M) \to C^{(\infty)}(M)$ map that satisfies

$$v_p(f) = v(f)(p). \tag{2-1}$$

The collection of tangent vectors at p is called the ***tangent space*** at p, denoted by T_pM.

The derivative of a path $\gamma\colon [0,1] \to M$ (or $\mathbb{R} \to M$) in a smooth manifold is defined as:

$$\begin{aligned} \gamma'(t)\colon C^{(\infty)}(M) \to \mathbb{R}; \\ \gamma'(t)(f) = \frac{\mathrm{d}}{\mathrm{d}t} f \circ \gamma(t) \end{aligned} \tag{2-2}$$

We can see that $\gamma'(t) \in T_{\gamma(t)}M$.

Let a path $\gamma\colon \mathbb{R}$ follows a vector field (a velocity field), that is

$$\gamma'(t) = v_{\gamma(t)}, \tag{2-3}$$

then we call γ the ***integral curve*** through $p := \gamma(0)$ of the vector field v.

Definition 2.3. Suppose v is an integrable vector field. Let $\varphi_t(p)$ be the point at time t on the integral curve through p.

$$\varphi_t\colon M \to M \tag{2-4}$$

is then called a ***flow*** generated by v.

$$\frac{\mathrm{d}}{\mathrm{d}t} \varphi_t(p) = v_{\varphi_t(p)}. \tag{2-5}$$

§3 Covariant and Contravariant

Definition 3.1 (pullback)**.** Let f be a scalar field over N, $\varphi \in C^{(\infty)}(M,N)$. Then the ***pullback*** of f by φ

$$\varphi^*\colon C^{(\infty)}(N) \to C^{(\infty)}(M), \tag{3-1}$$

is defined as

$$\varphi^* f = f \circ \varphi \in C^{(\infty)}(M). \tag{3-2}$$

Fields that are pullbacked are ***covariant*** fields.

Definition 3.2 (pushforward)**.** Let v_p be a tangent vector of M at p, $\varphi \in C^{(\infty)}(M, N)$, $q = \varphi(p)$. Then the ***pushforward*** of v_p by φ

$$\varphi_* \colon T_pM \to T_qN, \tag{3-3}$$

is defined as

$$(\varphi_* v)_q(f) = v_p(\varphi^* f). \tag{3-4}$$

Note that the pushforward of a vector field can only be obtained when φ is a diffeomorphism.

Fields that are pushforwarded are ***contravariant*** fields.

Mathematicians and physicists might have disagreement on whether a tangent vector is covariant or contravariant. This is because of that physicists might consider the coordinates (v^μ) of a tangent vector as a vector field, instead of linear combination of bases ∂_μ.

§4 Components of Vector Fields

Let $\varphi \colon U \to \mathbb{R}^n$ be a chart of M ($U \subset M$).

Let $p \in U$, $\varphi(p) = x = (x^\mu)$ ($\mu = 0, \ldots, n-1$). Locally, a function $f \in C^{(\infty)}(M)$ can be written as

$$(\varphi^{-1})^* f = f \circ \varphi^{-1} \colon \mathbb{R}^n \to \mathbb{R}, \tag{4-1}$$

and a vector field $v \in \mathrm{Vect}(M)$ can be written as

$$(\varphi_* v)_x = \varphi_* v_p \colon C^{(\infty)}(\mathbb{R}^n) \to \mathbb{R}, \tag{4-2}$$

or

$$\varphi_* v \in \mathrm{Vect}(\mathbb{R}^n) \tag{4-3}$$

Since $T_x\mathbb{R}^n \cong \mathbb{R}^n$ is a linear space, one can find a basis for $T_x\mathbb{R}^n$ as

$$\partial_\mu \colon C^{(\infty)}(\mathbb{R}^n) \to C^{(\infty)}(\mathbb{R}^n), \tag{4-4}$$

and $(\varphi_* v)_x = v^\mu(x)\partial_\mu$.

Pushing forward $v^\mu(x)\partial_\mu$ by φ^{-1} one obtains v.

In an abuse of symbols, one may just omit the pullback and pushforward, and refer to the f and v by $(\varphi^{-1})^* f$ and $\varphi_* v$.

Consider another chart $\psi \colon U \to \mathbb{R}^n$ of M, and

$$y = \psi(p), \quad (\psi_* v)_x = u^\mu \partial_\mu, \tag{4-5}$$

where we have chosen the same basis in $T_y\mathbb{R}^n$ as in $T_x\mathbb{R}^n$.

We would like to know how to relate v^μ and u^μ i.e. we want to know how the components of v transforms under a coordinate transformation $\tau = \psi \circ \varphi^{-1}$.

Consider any $f \in C^{(\infty)}(M)$,

$$v(f) = \varphi_* v((\varphi^{-1})_* f) = \psi_* v((\psi^{-1})_* f) \tag{4-6}$$

$\Rightarrow$

$$u^\mu \partial_\mu (f \circ \psi^{-1}) = v^\mu \partial_\mu (f \circ \varphi^{-1}) = v^\mu \partial_\mu (f \circ \psi^{-1} \circ \tau) = v^\mu {\tau'}^\nu_{\ \mu} \partial_\nu (f \circ \psi^{-1}) \tag{4-7}$$

$\Rightarrow$

$$u^\mu = v^\nu {\tau'}^\mu_{\ \nu}, \tag{4-8}$$

where

$${\tau'}^\mu_{\ \nu} = \frac{\partial y^\mu}{\partial x^\nu}. \tag{4-9}$$

§5 Lie Bracket

Definition 5.1 (Lie bracket)**.** Let $v, w \in \mathrm{Vect}(M)$, then the ***Lie bracket*** of v and w is defined as

$$[v, w] \colon C^{(\infty)}(M) \to C^{(\infty)}(M);\ f \mapsto v \circ w(f) - w \circ v(f). \tag{5-1}$$

The Lie bracket is an antisymmetric bilinear map[1], and an important property of the Lie bracket is the Leibniz rule:

$$[v, w](fg) = [v, w](f)g + f[v, w](g). \tag{5-2}$$

Another important property of the Lie bracket is the Jacobi identity:

$$[v, [w, u]] + [w, [u, v]] + [u, [v, w]] = 0. \tag{5-3}$$

[1]Note that it is not $C^{(\infty)}$-linear

Chapter 3

Differential Forms

§6 1-forms

Definition 6.1 (1-form)**.** A ***1-form*** ω on M is a $\mathrm{Vect}(M) \to C^{(\infty)}(M)$ which satisfies that

(a) $\forall v, w \in \mathrm{Vect}(M)$, $\forall f, g \in C^{(\infty)}(M)$,

$$\omega(fv + gw) = f\omega(v) + g\omega(w). \tag{6-1}$$

The space of all 1-forms on M is denoted as $\Omega^1(M)$, which is a module over $C^{(\infty)}(M)$.

The operator d, when given a $C^{(\infty)}(M)$ function (which is called a ***0-form***), would give a 1-form:

$$(\mathrm{d}f)(v) = v(f). \tag{6-2}$$

This is called the ***exterior derivative*** or ***differential*** of f.

The ***cotangent vector*** or ***covector*** is similar as the tangent vector:

$$\omega_p(v_p) = \omega(v)(p). \tag{6-3}$$

The space of cotangent vectors at p on M is denoted by T_p^*M.

1-forms are covariant, that is, if $\varphi\colon M \to N$, then the pushforward of a 1-form ω by φ is

$$(\varphi^*\omega)_p(v_p) = \omega_q(\varphi_* v_p), \tag{6-4}$$

where $\varphi(p) = q$.

Theorem 6.1. *$f \in C^{(\infty)}(N)$, $\varphi\colon M \to N$ is differential, then*

$$\varphi^*(\mathrm{d}f) = \mathrm{d}(\varphi^* f). \tag{6-5}$$

§7 Components of 1-Forms

Let $\varphi\colon U \to \mathbb{R}^n$ be a chart of M ($U \subset M$).

Let $p \in U$, $\varphi(p) = x = (x^\mu)$ ($\mu = 0, \ldots, n-1$). Locally a 1-form $\omega \in \Omega^1(M)$ can be written as

$$(\varphi^{-1})^*\omega \in T_x^*\mathbb{R}^n. \tag{7-1}$$

A natural way to impose a basis $\mathrm{d}x^\mu$ in $T_x^*\mathbb{R}^n$ is

$$\mathrm{d}x^\mu(\partial_\nu) = \delta^\mu_\nu, \tag{7-2}$$

and $(\varphi^{-1})^*\omega = \omega_\mu(x)\,\mathrm{d}x^\mu$.

Now by the definition of 1-form:

$$\omega_\mu\,\mathrm{d}x^\mu(v^\nu\partial_\nu) = v^\nu\omega_\mu\delta^\mu_\nu = v^\mu\omega_\mu. \tag{7-3}$$

By the transformation rule of components of a vector, one have

$$\tau'^{\nu}{}_{\mu}\alpha_\nu = \omega_\mu, \tag{7-4}$$

where $\psi\colon U \to \mathbb{R}^n$, $(\psi^{-1})_*\omega = \alpha_\mu\,\mathrm{d}x^\mu$, $\tau = \psi \circ \varphi^{-1}$.

§8 k-Forms

Definition 8.1. If we assign an antisymmetric multilinear k-form $\omega_p \in \bigotimes_{i\in k} T_p^* M$ to each point $p \in M$, we say we have a ***k-form*** on M.

The collection of all k-forms is denoted by $\Omega^k(M)$, and $\Omega(M) := \bigcup_{k\in\mathbb{N}} \Omega^k(M)$.

Theorem 8.1 (Dimension of forms)**.** *If M is an nD manifold, then the dimension of $\Omega^k(M)$ is $\frac{n!}{k!(n-k)!}$ ($k \leq n$), and 0 for $k > n$; The dimension of $\Omega(M)$ is 2^n.*

Definition 8.2 (Wedge product)**.** The ***wedge product*** $\wedge$ is defined as a binary operater that takes a k-form and ℓ-form and gives a $(k+\ell)$-forms, satisfying $\forall\alpha \in \Omega^k(M)$, $\forall\beta \in \Omega^\ell(M)$:

(a) (Associativity) $\forall\gamma \in \Omega^m(M)$,

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma). \tag{8-1}$$

(b) (Supercommutativity)

$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha. \tag{8-2}$$

(c) (Distributiveness) $\forall\gamma \in \Omega^\ell(M)$,

$$\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma. \tag{8-3}$$

(d) (Bilinearity over $C^{(\infty)}(M)$) $\forall f \in C^{(\infty)}(M)$,

$$(f\alpha) \wedge \beta = f(\alpha \wedge \beta). \tag{8-4}$$

(e) (Naturality) If $\varphi\colon M \to N$ is a smooth map, then the pullback of a form by φ can be given by repeatingly applying ($\forall\gamma \in \Omega^\ell(M)$)

$$\begin{aligned} \varphi^*(\beta + \gamma) &= \varphi^*\alpha + \varphi^*\beta \\ \varphi^*(\alpha \wedge \beta) &= \varphi^*\alpha \wedge \varphi^*\beta, \end{aligned} \tag{8-5}$$

while the pullback of a 0-form and a 1-form agree with what we have already defined before.

By convention if $f \in C^{(\infty)}(M)$ then

$$f \wedge \omega =: f\omega. \tag{8-6}$$

It can be shown that any k-form ω can be written as

$$(\varphi^{-1})^*\omega = \frac{\omega_{\mu_1 \cdots \mu_k}}{n!} \bigwedge_{i=1}^{k} \mathrm{d}x^{\mu_i}, \tag{8-7}$$

where $\varphi\colon M \to \mathbb{R}^n$ is a chart.

Definition 8.3 (Interior product)**.** Let $v \in \Gamma(TM)$, we can define the ***interior product*** $i_v\colon \Omega^k(M) \to \Omega^{k-1}(M)$ by: $\forall \omega \in \Omega^k(M)$, $\forall v_i$ $(i \in k-1)$:

$$i_v(\omega)(v_0, \ldots, v_{k-2}) = \omega(v, v_0, \ldots, v_{k-2}). \tag{8-8}$$

Specially, if $k = 0$, then $i_v(\omega) = 0$.

Theorem 8.2. *$\forall v \in \Gamma(TM)$,*

1. *i_v is a $C^{(\infty)}(M)$-linear function;*
2. *$\forall \alpha \in \Omega^k(M)$, $\forall \beta \in \Omega(M)$,*

$$i_v(\alpha \wedge \beta) = i_v(\alpha) \wedge \beta + (-1)^k \alpha \wedge i_v(\beta). \tag{8-9}$$

§9 Exterior Derivative

Definition 9.1 (Exterior derivative)**.** The ***exterior derivative*** d is defined as a linear operator that takes a k-form and gives a $(k+1)$-form, satisfying $\forall \alpha \in \Omega^k(M)$, $\forall \beta \in \Omega^\ell(M)$:

(a) (Linearity) $\forall \lambda, \mu \in \mathbb{R}$, $\forall \gamma \in \Omega^\ell(M)$,

$$\mathrm{d}(\lambda\beta + \mu\gamma) = \lambda\,\mathrm{d}\alpha + \mu\,\mathrm{d}\beta. \tag{9-1}$$

(b) (Leibniz rule)

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta. \tag{9-2}$$

(c)

$$\mathrm{d}^2\omega = 0. \tag{9-3}$$

(d) (Naturality) If $\varphi\colon M \to N$ is a smooth map, then

$$\varphi^* \,\mathrm{d}\omega = \mathrm{d}\varphi^*\omega. \tag{9-4}$$

§10 Derivation ana Antiderivation

Definition 10.1 (Derivation)**.** A map $\theta\colon \Omega(M) \to \Omega(M)$ is called a ***derivation of degree*** k ($k \in \mathbb{Z}$) if $\forall p \in \mathbb{N}$, $\theta[\Omega^p(M)] \subset \Omega^{p+k}(M)$ and, θ is a homomorphism of the $\mathbb{R}$-exterior algebra. Or, explicitly, θ is $\mathbb{R}$-linear, and $\forall \alpha, \beta \in \Omega(M)$,

$$\theta(\alpha \wedge \beta) = \theta(\alpha) \wedge \beta + \alpha \wedge \theta(\beta). \tag{10-1}$$

Definition 10.2 (Antiderivation)**.** A map $\theta\colon \Omega(M) \to \Omega(M)$ is called an ***antiderivation of degree*** k ($k \in \mathbb{Z}$) if i) $\forall p \in \mathbb{N}$, $\theta[\Omega^p(M)] \subset \Omega^{p+k}(M)$, ii) θ is $\mathbb{R}$-linear, iii) $\forall \alpha \in \Omega^p(M)$, $\beta \in \Omega(M)$,

$$\theta(\alpha \wedge \beta) = \theta(\alpha) \wedge \beta + (-1)^p \alpha \wedge \theta(\beta). \tag{10-2}$$

Chapter 4

Metric

§11 Pseudo-Riemannian Metric

Definition 11.1 (Psedudo-Riemannian metric)**.** Let M be a manifold. A ***pseudo-Riemannian metric*** or simply ***metric*** g on a manifold M is a field ($g \in \Gamma(T^*M \otimes T^*M)$) that $\forall p \in M$,

$$g_p \colon T_p^*M \times T_p^*M \to \mathbb{R}, \tag{11-1}$$

is a bilinear form satisfying the following properties:

(a) (Symmetry) $\forall u, v \in T_pM$,

$$g_p(u, v) = g_p(v, u). \tag{11-2}$$

(b) (Non-degenerate)

$$u \mapsto g_p(u, -) \colon T_pM \to T_p^*M \tag{11-3}$$

is an isomorphism.

(c) (Bilinearity) $\forall p \in M$, $\forall u, v \in T_pM$, $\forall \lambda, \mu \in \mathbb{R}$,

$$g_p(\lambda u + \mu v, w) = \lambda g_p(u, w) + \mu g_p(v, w). \tag{11-4}$$

(d) (Smoothness) If $v, u \in \mathrm{Vect}(M)$, then

$$p \mapsto g_p(v_p, u_p) \in C^{(\infty)}(M). \tag{11-5}$$

Given a metric, $\forall p \in M$, we can always find an orthonormal basis $\{e_\mu\}$ of T_pM such that

$$g_p(e_\mu, e_\nu) = \mathrm{sign}(\mu)\delta_{\mu\nu}, \tag{11-6}$$

where $\mathrm{sign}(\mu) = \pm 1$. Conventionally we order the basis such that $\mathrm{sign}(\mu) = 1$ for $\mu \in s$ and $\mathrm{sign}(\mu) = -1$ for $\mu - s \in n - s$, and say that the metric has ***signature*** $(s, n-s)$.

If $\gamma\colon [0,1] \to M$ is a smooth path and $\forall t, s \in [0,1]$,

$$g(\gamma'(t), \gamma'(t))g(\gamma'(s), \gamma'(s)) \geq 0, \tag{11-7}$$

then we can define the arclength of γ as

$$\int_0^1 \sqrt{|g(\gamma'(t), \gamma'(t))|}\,\mathrm{d}t \tag{11-8}$$

if the integral converges.

The metric gives an ***inner product*** on $\mathrm{Vect}(M)$:

$$\langle u, v\rangle := g(u, v). \tag{11-9}$$

The metric also gives a way to relate a vector field v to a 1-form ω. If v and ω satisfies: $\forall u \in \mathrm{Vect}(M)$,

$$g(v, u) = \omega(u), \tag{11-10}$$

then we say that v is the corresponding vector field of ω, denoted by $v = \omega^\sharp$, and ω is the corresponding 1-form of v, denoted by $\omega = v^\flat$.

We can also define the ***inner product*** on $\Omega^1(M)$ by

$$\langle \alpha, \beta\rangle = \langle a, b\rangle, \tag{11-11}$$

where a and b is the corresponding vector fields of α and β.

The ***inner product***[1] on $\Omega^k(M)$ is defined by induction with

$$\langle \bigwedge_{i\in k} \alpha_i, \bigwedge_{i\in k} \beta_i \rangle = \det(\langle \alpha_i, \beta_j \rangle)_{i,j\in k}. \tag{11-12}$$

Hence, if $\{e_\mu\}$ is an orthonormal basis (field) of T_pM, while the corresponding covectors are $\{f^\mu\}$ ($f^\mu(e_\nu) = \delta^\mu_\nu$) then

$$\langle \bigwedge_{i\in k} f^{\mu_i}, \bigwedge_{i\in k} f^{\mu_i} \rangle = \prod_{i\in k} \operatorname{sign}(\mu_i). \tag{11-13}$$

Specially, when $f, g \in \Omega^0(M) = C^{(\infty)}(M)$,

$$\langle f, g \rangle = fg. \tag{11-14}$$

§12 Volume Form

Notice that if M is an nD manifold, $\dim \Omega^n(M) = 1$, meaning at $p \in M$, $\{\omega_p \mid \omega \in \Omega^n(M)\}$ can be labelled by a parametre $\lambda_p \in \mathbb{R}$. If we have a basis $\{f^\mu\}$ of T^*_pM (or corresponding vectors $\{e_\mu\}$), then

$$\{\omega_p \mid \omega \in \Omega^n(M)\} = \lambda_p \bigwedge_{\mu\in n} f^\mu. \tag{12-1}$$

If there were another basis $\{g^\mu\}$ of T^*_pM (or corresponding vectors $\{h_\mu\}$), and the transformation between the two bases is given by

$$Pe^\mu = f^\mu, \tag{12-2}$$

where $P \in \operatorname{Aut}(T^*_pM)$. When $\det P > 0$, we say that $\{f^\mu\}$ and $\{g^\mu\}$ have the same ***orientation***.

Definition 12.1 (Volume form)**.** Let M be an orientable manifold. If $\forall p \in M$, we find an oriented orthonormal basis $\{f_\mu\}$ of T^*_pM at point p, then the ***volume form*** vol is defined by

$$\bigwedge_{\mu\in n} f_\mu = \operatorname{vol}_p. \tag{12-3}$$

[1]This inner product makes $\Omega_p(M)$ for each $p \in M$, yet not for $\Omega(M)$. The full inner product requires integration over M.

§13 Hodge Star Operator

Definition 13.1 (Hodge Star Operator)**.** Let M be an orientable manifold. The ***Hodge star operator*** $\star$ is defined by the linear map

$$\star\colon \Omega^k(M) \to \Omega^{n-k}(M), \tag{13-1}$$

$\forall \alpha, \beta \in \Omega^k(M)$,

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \operatorname{vol}. \tag{13-2}$$

We call $\star\omega$ the ***dual*** of ω.

The special case is when $k = 0$,

$$\star f = f \operatorname{vol}, \tag{13-3}$$

and $k = n$,

$$\star(f \operatorname{vol}) = f \prod_{\mu \in n} \operatorname{sign}(\mu) = (-1)^{n-s} f \tag{13-4}$$

if the signature of the metric is $(s, n-s)$ (s positives and $n-s$ negatives).

The Hodge star operator is also called the ***Hodge duality*** because:

Theorem 13.1. *$\forall \alpha \in \Omega^p(M)$,*

$$\star\star\alpha = (-1)^{p(n-p)} \alpha \operatorname{sign}(g), \tag{13-5}$$

where $\operatorname{sign}(g) := \det g / |\det g|$.

In local coordinates,

$$\star\alpha = \frac{\varepsilon_{i_0 \cdots i_{n-1}}}{p!} \alpha_{j_0 \cdots j_{p-1}} \prod_{k \in p} g^{i_k j_k} \sqrt{-\det g} \bigwedge_{\ell \in n \setminus p} \mathrm{d}x^{i_\ell}. \tag{13-6}$$

Definition 13.2 (Codifferential)**.** Let M be an orientable manifold. The ***codifferential*** δ is defined by

$$\delta\colon \Omega^k(M) \to \Omega^{k-1}(M), \tag{13-7}$$

$\forall \alpha \in \Omega^k(M)$,

$$\delta\alpha = \star \mathrm{d} \star \alpha. \tag{13-8}$$

Definition 13.3 (Laplacian)**.** The ***Laplacian*** $\Box$ is defined by

$$\Box := \mathrm{d} \circ \delta + \delta \circ \mathrm{d}. \tag{13-9}$$

Theorem 13.2.

$$\Box \circ \star = \star \circ \Box, \tag{13-10}$$

$$\Box \circ \delta = \delta \circ \Box, \tag{13-11}$$

$$\Box \circ \mathrm{d} = \mathrm{d} \circ \Box. \tag{13-12}$$

§14 Metric and Coordinates

Chapter 5

DeRham Theory

§15 Closed and Exact 1-Forms

Definition 15.1 (Closed and exact forms)**.** Consider $\mathrm{d}\colon \Omega(M) \to \Omega(M)$. The differential forms in $\ker \mathrm{d}$ is said to be ***closed***, and the differential forms in $\mathrm{d}(\Omega(M))$ is said to be ***exact***.

For closed form:

$$\mathrm{d}\omega = 0, \tag{15-1}$$

For exact form:

$$\exists \alpha \in \Omega(M), \quad \omega = \mathrm{d}\alpha, \tag{15-2}$$

where α is often called *potential*.

We want to study, given two points p, q that are located in the same arcwise connected component of M, and a smooth path $\gamma\colon [0,1] \to M$ s.t. $\gamma(0) = p$, $\gamma(1) = q$, for a closed 1-form E,

$$\phi(p,q) := -\int_\gamma E := -\int_0^1 E_{\gamma(t)}(\gamma'(t))\,\mathrm{d}t. \tag{15-3}$$

We want to know that how ϕ depends on the choice of γ.

Assumes that there are two smooth paths γ_1 and γ_2 connecting p and q, and a fix-ends smooth homotopy $H\colon [0,1]\times[0,1]\to M$ s.t.

$$H(0,t)=\gamma_1(t),\quad H(1,t)=\gamma_2(t),\quad H(s,0)=p,\quad H(s,1)=q. \tag{15-4}$$

By choosing proper charts (if there is no chart that can cover the whole path, we break the path into pieces),

$$\begin{aligned} I_s = \int_{H(s,-)} E &= \int_0^1 E_{H(s,t)}(H'(s,t))\,\mathrm{d}t \\ &= \int_0^1 E_\mu[H(s,t)]\partial_t H^\mu(s,t)\,\mathrm{d}t, \end{aligned} \tag{15-5}$$

where $H'(s,t)$ is the tangent vector of $H(s,-)$ at t.

$$\begin{aligned} \frac{\mathrm{d}I_s}{\mathrm{d}s} &= \frac{\mathrm{d}}{\mathrm{d}s}\int_0^1 E_\mu[H(s,t)]\partial_t H^\mu(s,t)\,\mathrm{d}t \\ &= \int_0^1 \left(\partial_s E_\mu[H(s,t)]\partial_t H^\mu + E_\mu[H(s,t)]\partial_s\partial_t H^\mu\right)\mathrm{d}t \\ &= \partial_s\,\left(E_\mu(H(s,t))H^\mu(s,t)\right)\Big|_{t=0}^{t=1} \\ &\quad + \int_0^1 \left(\partial_s E_\mu[H(s,t)]\partial_t H^\mu - \partial_t E_\mu[H(s,t)]\partial_s H^\mu\right)\mathrm{d}t \\ &= \partial_s(E_\mu(q)q^\mu - E_\mu(p)p^\mu) \\ &\quad + \int_0^1 \partial_\nu E_\mu\left(\partial_s H^\nu\partial_t H^\mu - \partial_t H_\nu \partial_s H^\mu\right)\,\mathrm{d}t \\ &= \int (\mathrm{d}E)_{\mu\nu}\partial_s H^\mu \partial_t H = 0. \end{aligned} \tag{15-6}$$

Now we have proven that if γ_1 and γ_2 are homotopic, then the integral for $\phi(p,q)$ is the same.

Then, if M is simply connected, then a closed form E is also exact, and

$$E = -\,\mathrm{d}\phi(p,-). \tag{15-7}$$

§16 Stokes' Theorem

§17 DeRham Cohomology

We have shown that, if the manifold is simply connected, then a closed 1-form must also be exact. The study of whether a closed form is exact is called the ***deRham cohomology***.

Since $\mathrm{d} \circ \mathrm{d} = 0$, we know that

$$\mathrm{d}(\Omega(M)) \subset \mathrm{d}(\ker \mathrm{d}). \tag{17-1}$$

The space of exact p-forms is denoted by $B^p(M)$ and the space of closed p-forms is denoted by $Z^p(M)$.

Definition 17.1 (DeRham cohomology)**.** The p-th ***deRham cohomology*** of M is defined as

$$H^p(M) = Z^p(M)/B^p(M). \tag{17-2}$$

Every element of $H^p(M)$ is a ***cohomologous class***:

$$[\omega] = \{\omega' \in Z^p(M) \mid \omega - \omega' \in B^p(M)\}. \tag{17-3}$$

For $p = 0$, $B^0(M) = \{0\}$ (there is no (-1)-form), and $H^0(M) = Z^0(M)$, where $Z^0(M)$ is made of f that is constant in every connected components of M. Let χ_i be the characteristic function of M's ith connected components M_i (we assume that $\{M_i\}$ is finite)

$$H^0(M) = Z^0(M) = \left\{f \middle| f = x^i \chi_i\right\} \cong \mathbb{R}^n, \tag{17-4}$$

where n is the number of connected components of M.

Chapter 6

Bundles and Connections

§18 Fibre Bundles

Definition 18.1 (Bundle)**.** A ***bundle*** is a triple (E, π, B), where $\pi\colon E \to B$ is a surjective map. E is called the ***total space***, π is called the ***projection map***, and B is called the ***base space***.

A bundle (E, π, B) can be denoted as $\pi\colon E \to B$ or $E \xrightarrow{\pi} B$.

Definition 18.2 (Fibre)**.** For $p \in B$, $\pi^{-1}(\{p\})$ is the ***fibre*** over b.

Definition 18.3 (Subbundle)**.** Let $\pi\colon E \to B$ be a bundle. $F \subset E$, $C \subset B$, $\rho\colon F \to C$. If $\pi|_C = \rho$, then $\rho\colon F \to C$ is called a ***subbundle*** of $\pi\colon E \to B$.

Definition 18.4 (Section)**.** A ***section*** is a map $s\colon B \to E$ such that

$$p \circ s = \mathrm{id}_B\,. \tag{18-1}$$

All sections of a bundle $\pi\colon E \to B$ is denoted as $\Gamma(E)$.

Definition 18.5 (Fibre bundle)**.** A ***fibre bundle*** (E, π, B, F) is a bundle $\pi\colon E \to B$, where E, B, F are topology spaces, and π is a continuous map, and $\forall p \in B$, $\exists U \in \mathscr{U}(p)$ s.t.

$$\varphi\colon \pi^{-1}(U) \to U \times F, \tag{18-2}$$

is a homeomorphism and $\pi_1 \circ \varphi = \pi$. π_1 is defined as $\pi_1(p, q) = p$.

A fibre bundle can be denoted as the exact sequence

$$F \longrightarrow E \xrightarrow{\pi} B \tag{18-3}$$

The last condition is called the ***local triviality condition***. F is called the ***standard fibre***

If $E = B \times F$, then (E, π, B, F) is called a ***trivial fibre bundle***.

Definition 18.6 (Morphism)**.** Let $\pi\colon E \to B$, $\rho\colon F \to C$ be two fibre bundles. A ***morphism*** (φ, ψ) is a pair of two continuous maps such that

$$\begin{tikzcd} E \arrow[r, "\psi"] \arrow[d, "\pi"] & F \arrow[d, "\rho"] \\ B \arrow[r, "\varphi"] & C \end{tikzcd} \tag{18-4}$$

commutes.

§19 Vector Bundles

Definition 19.1 (Vector bundle)**.** A ***vector bundle*** is a fibre bundle (E, π, B, F), where F is a vector space, and the local trivialisation $\varphi\colon \pi^{-1}(U) \to U \times F$ (U is a neibourhood of $p \in B$) satisfies that $\forall x \in U$, $\forall v \in F$,

$$\begin{aligned} F &\to \pi^{-1}(\{x\}) \\ v &\mapsto \varphi^{-1}(x, v) \end{aligned} \tag{19-1}$$

is a linear isomorphism (***fibrewise linear***).

Definition 19.2 (Morphism (vector bundle))**.** The morphism between two vector bundles (E, π, B, F) and (E', π', B', F') is a morphism (φ, ψ) such that $\forall x \in B$,

$$\psi_* \colon \pi^{-1}(\{x\}) \to (\pi')^{-1}(\{\varphi(x)\}) \tag{19-2}$$

is a linear homomorphism.

Definition 19.3 (Smooth vector bundle)**.** A ***smooth vector bundle*** is a vector bundle (E, π, B, F), where the projection $\pi \colon E \to B$ and the local trivialisation $\varphi \colon \pi^{-1}(U) \to U \times F$ are smooth.

Definition 19.4 (Tangent bundle)**.** The ***tangent bundle*** TM is the smooth vector bundle over an nD smooth manifold M with the standard fibre $T_pM = \mathbb{R}^n$.

A vector field $v \in \mathrm{Vect}(M)$ is the smooth section of the tangent bundle $\Gamma(TM)$.

Definition 19.5 (Cotangent bundle)**.** The ***cotangent bundle*** of an nD manifold M, denoted by T^*M, is the smooth vector bundle over with the standard fibre $T_p^*M = (\mathbb{R}^n)^*$.

A 1-form $\omega \in \Omega^1(M)$ is the smooth section of the cotangent bundle $\Gamma(T^*M)$.

§20 Constructions of Vector Bundles

We use local trivialisation to destruct a vector bundle into trivial bundles. We can also construct a vector bundle by "gluing" trivial bundles. We must make sure that in the intersections of bases, we must make sure that they are compactible by introducing *transition functions* to relate points on the fibres. Naturally, transition functions make a group structure.

Definition 20.1 (G-bundle)**.** Consider an open cover $\mathcal{U} = \{U_i \mid i \in I\}$ of the manifold M. For each $i \in I$, there is a trivial vector bundle $U_i \times V \xrightarrow{\pi_i} U_i$ with vector fibre V. $\rho\colon G \to \mathrm{GL}(V)$ is a representation of G on V.

For any $p \in M$, if $p \in \bigcap_{j\in J} U_j$ $(J \subset I)$, then $\pi^{-1}(\{p\})$ is identified by a equivalence class in $\bigsqcup_{j\in J} \pi_j^{-1}(p)$ where two points are equivalent if they are related by the transformation

$$\begin{aligned} \rho_*(g_{jj'}(p))\colon U_j \times V &\to U_{j'} \times V; \\ (p, v) &\mapsto (p, \rho(g_{jj'}(p)v)), \end{aligned} \tag{20-1}$$

where the ***transition functions*** $g_{ij} \in G$ satisfy that:

1. $g_{ii} = 1$;
2. $g_{ij}g_{jk}g_{ki} = 1$.

The bundle $E \xrightarrow{\pi} M$ is then called the ***G-bundle***, the element of which is denoted as $[p, v_p]$ for some $v_p \in U_i$, where G is the ***gauge group***.

One can show that the G-bundles are also vector bundles.

Consider transformations of the sections of the G-bundle. If $T\colon E_p \to E_p$ can be expressed by *some* $g \in G$ s.t.

$$T([p, v_p]) = [p, \rho(g)v_p], \tag{20-2}$$

then we say T ***lives in*** G. Similarly we can define when T lives in $\mathfrak{g}$.

Notice that we do not specify which $g \in G$ corresponds to T, because we have the freedom to choose the v_p as the representative of the equivalence class, and for different v_p, we have different $g \in G$.

If, $\forall p \in M$, we have $T_p\colon E_p \to E_p$ that T_p lives in G, we call T a ***gauge transformation***. The set of all gauge transformations is denoted as $\mathcal{G}$[1].

[1] physicists call it the ***gauge group***, as opposite to G.

§21 Connections

Definition 21.1 (Connection)**.** A ***connection*** on a smooth vector bundle (E, π, M, F) is map

$$D\colon \Gamma(TM) \times \Gamma(E) \to \Gamma(E), \tag{21-1}$$

that satisfies the following conditions: $\forall v, w \in \Gamma(TM)$, $\forall s, t \in \Gamma(E)$, $\forall f \in C^{(\infty)}(M)$,

(a) $D_v(s + t) = D_v s + D_v t$;

(b) $D_v(fs) = v(f)s + fD_v s$;

(c) $D_{v+w}s = D_v s + D_w s$;

(d) $D_{fv}s = fD_v s$.

When a vector field $v \in \Gamma(TM)$ is given to the connection D, the map $D_v : \Gamma(E) \to \Gamma(E)$ is called the ***covariant derivative*** with respect to v.

Definition 21.2 (Vector potential)**.** A ***vector potential*** A is an $\mathrm{End}(E)$-valued 1-form, that is

$$A \in \Gamma(\mathrm{End}(E) \otimes T^*M), \tag{21-2}$$

where $\mathrm{End}(E) \cong E \otimes E^*$ can be considered as a vector bundle over M with the standard fibre $\mathrm{End}(E_p) \cong E_p \otimes E_p^*$ $(p \in E)$.

Locally if $s \in \Gamma(E)$ we can have a trivialisation $\varphi\colon E|_U \to U \times F$ $(U \subset M)$. If we assign a basis $\{f_i\}_{i \in m}$ for the mD standard fibre F, then

$$s = s^i e_i := s^i \varphi^{-1}(f_i), \quad s^i \in C^{(\infty)}(U), \tag{21-3}$$

where we can call $\{s^i\}_{i \in m}$ the ***components of the section*** s. With this specific normalisation, one can define that

$$D_v^0 s = v(s^i)e_i \tag{21-4}$$

where D^0 is called the ***standard flat connection*** (which depends on trivialisation).

Theorem 21.1. *Let (E, π, M, F) be a smooth vector bundle. If D is a connection on E, $A \in \Gamma(\mathrm{End}(E)) \otimes T^*M$, then the $D+A$, which defined as*

$$D + A\colon (v, s) \mapsto D_v s + A(v)s, \tag{21-5}$$

is also a connection.

Theorem 21.2. *Let (E, π, M, F) be a smooth vector bundle, and D^0 is the standard flat connection on $U \subset E$ with the trivialisation $\varphi\colon E|_U \to U \times F$. If D is a connection on a (E, π, M, F), then $\exists A \in \Gamma(\mathrm{End}(E|_U)) \otimes T^*U$ s.t.*

$$D|_U = D^0 + A. \tag{21-6}$$

Definition 21.3 (G-connection)**.** Let E be a G-bundle, we define a ***G-connection*** as a connection D on E that

$$D = D^0 + A, \tag{21-7}$$

where in any local coordinates $A = A_\mu \, \mathrm{d}x^\mu$, $A_\mu \in \mathrm{End}(E)$ lives in $\mathfrak{g}$.

Definition 21.4 (Gauge transformation of G-connection)**.** Let E be a G-bundle with G-connection D. If $g \in \mathcal{G}$ is a gauge transformation, then

$$D'_v(s) = gD_v(g^{-1}s) \tag{21-8}$$

is also a G-connection, and we say that D and D' are ***gauge equivalent***.

In local coordinates,

$$A'_\mu = gA_\mu g^{-1} + g\partial_\mu g^{-1}. \tag{21-9}$$

Let $\mathcal{A}$ be the space of all G-connections on E, then we say $\mathcal{A}/\mathcal{G}$ is the space of connections modulo gauge transformation.

Given a connection D on E, we can construct conncections for different structures built upon E.

The ***dual connection*** D^* on E^* is defined as

$$(D^*_v\sigma)(s) = v[\sigma(s)] - \sigma(D_v s), \tag{21-10}$$

where $v \in \Gamma(TM)$, $s \in \Gamma(E)$, $\sigma \in \Gamma(E^*)$.

The ***direct sum of connections*** $D \oplus D'$ on $E \oplus E'$ is defined as

$$(D \oplus D')_v(s \oplus s') = D_v s \oplus D'_v s', \tag{21-11}$$

where $v \in \Gamma(TM)$, $s \in \Gamma(E)$, $s' \in \Gamma(E')$.

The ***tensor product of connections*** $D \otimes D'$ on $E \otimes E'$ is defined as

$$(D \otimes D')_v(s \otimes s') = D_v s \otimes s' + s \otimes D'_v s', \tag{21-12}$$

where $v \in \Gamma(TM)$, $s \in \Gamma(E)$, $s' \in \Gamma(E')$.

Since $\mathrm{End}(E) \cong E \otimes E^*$, the connection D (we use the same symbol for D on E) on $\mathrm{End}(E)$ can be shown to

$$D_v T(s) = D_v(Ts) - T(D_v s), \tag{21-13}$$

where $v \in \Gamma(TM)$, $s \in \Gamma(E)$, $T \in \Gamma(\mathrm{End}(E))$.

§22 Parallel Transport

Definition 22.1 (Parallel transport)**.** Let (E, π, M, F) be a smooth vector bundle, and D is a connection on E. A ***paralell transport*** of $s_0 \in \pi^{-1}(\{p\})$ $(p \in M)$ along a curve $\gamma\colon [0,1] \to M$ is a section $s \in \Gamma(E|_{\gamma([0,1])})$ such that

$$\forall t \in [0,1], \quad D_{\gamma'(t)} s(t) = 0, \quad s(0) = s_0, \tag{22-1}$$

where $s(t) := s_{\gamma(t)}$.

If $s =: v$ is a vector field, the Eq. (22-1) can be rewritten as

$$\frac{\mathrm{d}u \circ \gamma}{\mathrm{d}t}(t) + A[\gamma'(t)]u \circ \gamma(t) = 0, \tag{22-2}$$

which is a 1st order ODE. Given $\gamma_x(0) = x \in M$, there is a unique curve γ_x associated to the vector field u.

We can extend the domain of γ_x to $\mathbb{R}$ (note that $\mathbb{R}$ is diffeomorphic to $(0, 1)$), and define:

$$\phi \colon \mathbb{R} \times M \to M; (t, x) \mapsto \gamma_x(t), \tag{22-3}$$

which is called the ***flow*** of u.

Definition 22.2 (Holonomy)**.** Let D be a connection on a smooth vector bundle (E, π, M, F), γ is a (piecewise) smooth curve in M, with ends $\gamma(0) = p$ and $\gamma(1) = q$. $u_0 \in E_p$. The ***holonomy*** of u along γ is the map

$$H(\gamma, D) \colon E_p \to E_q, \tag{22-4}$$

such that $H(\gamma, D)u_0$ is the end of the parallel transport of u_0 along γ.

It can be shown that $H(\gamma, D)$ is a linear transformation, and it transforms as

$$H(\gamma, D') = g(q)H(\gamma, D)g(p)^{-1} \tag{22-5}$$

under gauge transformation $g \in \mathcal{G}$.

Specially, if γ is a loop ($p = q$), then $H(\gamma, D) \in \mathrm{End}(E)_p$ (and it can be proved to be living in G), and $H(\gamma, D') = g(p)H(\gamma, D)g(p)^{-1}$. Therefore, $\operatorname{tr} H(\gamma, D)$ is a ***gauge invariant***.

Definition 22.3 (Wilson loop)**.** Let D be a connection on a smooth vector bundle (E, π, M, F), γ is a (piecewise) smooth loop in M. The ***Wilson loop*** is defined as

$$W(\gamma, D) := \operatorname{tr} H(\gamma, D). \tag{22-6}$$

Chapter 7

Curvature

Definition 22.4 (Curvature). A ***curvature*** of a connection D on a smooth vector bundle (E, π, M, F) is a section $F \in \Gamma(\mathrm{End}(E) \otimes \Omega^2(M))$ (a $\mathrm{End}(E)$-*valued* 2-*form*) defined as

$$F(v, w)s = D_v D_w s - D_w D_v s - D_{[v,w]} s, \quad v, w \in \Gamma(TM), \quad s \in \Gamma(E). \tag{22-1}$$

If $\forall v, w \in \Gamma(TM)$, $\forall s \in \Gamma(E)$, $F(v, w)s = 0$, then D is called a ***flat connection***.

Consider a local trivialisation $\varphi \colon E|_U \to U \times F$ ($U \subset M$) s.t.

$$s = s^i e_i := s^i \varphi^{-1}(f_i), \tag{22-2}$$

where $s \in \Gamma(E|_U)$, $s^i \in C^{(\infty)}(U)$ and $\{f_i\}_{i \in m}$ is a set of bases of F, and $\sigma \colon U \to \mathbb{R}^n$ is a chart of M, $\sigma_* d_\mu := \partial_\mu$. Notice that

$[\partial_\mu, \partial_\nu] = 0$,

$$\begin{aligned}
&F(v,u)(s^i e_i) = v^\mu u^\nu F(d_\mu, d_\nu)(s^i e_i) \\
&= v^\mu u^\nu [D_\mu(d_\nu(s^i)e_i + s^i A^j_{\nu i} e_j) - D_\nu(d_\mu(s^i)e_i + s^i A^j_{\mu i} e_j)] \\
&= v^\mu u^\nu \big[d_\nu d_\mu(s^i)e_i + d_\nu(s^i)A^j_{\mu i}e_j + d_\mu(s^i A^j_{\nu i})e_j + s^i A^j_{\nu i} A^k_{\mu j} e_k \\
&\quad - d_\mu d_\nu(s^i)e_i - d_\mu(s^i)A^j_{\nu i}e_j - d_\nu(s^i A^j_{\mu i})e_j - s^i A^j_{\mu i} A^k_{\nu j} e_k\big] \\
&= v^\mu u^\nu s^i \big[d_\mu(A^k_{\nu i}) + A^j_{\nu i}A^k_{\mu j} - d_\nu(A^k_{\mu i}) - A^j_{\mu i}A^k_{\nu j}\big] e_k
\end{aligned} \tag{22-3}$$

If we write $F(d_\mu, d_\nu) = F^i{}_{j\mu\nu} e_i \otimes e^j$, then

$$F^i{}_{j\mu\nu} = d_\mu(A^i_{\nu j}) - d_\nu(A^i_{\mu j}) + A^i_{\mu k}A^k_{\nu j} - A^i_{\nu k}A^k_{\mu j}. \tag{22-4}$$

By definition, we have

$$F(u,v) = -F(v,u). \tag{22-5}$$

Theorem 22.1. *Let D be a G-connection on a G-bundle E, F is the curvature of D. If $g \in \mathcal{G}$, F' is the corresponding curvature of $D' = gDg^{-1}$, then $\forall u, v \in \Gamma(TM)$*

$$F'(u,v) = gF(u,v)g^{-1}. \tag{22-6}$$

§23 E-Valued p-Form

Definition 23.1. We define the ***E-valued p-form*** as a section of $E \otimes \bigwedge^p T^*M$, denoted as $\Omega^p_E(M)$.

Also, we define the wedge product of a E-valued p-form and a q-form as

$$\left(\sum_i s_i \otimes \omega_i\right) \wedge \mu := \sum_i s_i \otimes (\omega_i \wedge \mu). \tag{23-1}$$

Definition 23.2 (Exterior covariant derivative)**.** We define the ***exterior covariant derivative*** as a map $\mathrm{d}_D \colon \Omega^p_E(M) \to \Omega^{p+1}_E(M)$ s.t.

$$i_v(\mathrm{d}_D s) = D_v s, \tag{23-2}$$

where i_v is the interior product, D is a connection on E and $s \in \Omega^0_E(M) = \Gamma(E)$, and

$$\mathrm{d}_D \sum_i s_i \otimes \omega_i := \sum_i (\mathrm{d}_D s_i \wedge \omega_i + s \otimes \mathrm{d}\omega_i), \tag{23-3}$$

for $\omega_i \in \Omega^p(M)$.

In local coordinates ($\varphi\colon M \to \mathbb{R}^n$, $\varphi^* \,\mathrm{d}x^\mu = e^\mu$)

$$\frac{1}{p!}\,\mathrm{d}_D(s_{i_1\cdots i_p} \otimes e^{i_1} \wedge \cdots \wedge e^{i_p}) = D_\mu s_{i_1\cdots i_p} \otimes e^\mu \wedge e^{i_1} \wedge \cdots \wedge e^{i_p}. \tag{23-4}$$

Theorem 23.1. *Let $\eta \in \Omega^p_E(M)$ be a E-valued p-form, and $F \in \Omega^2_{\mathrm{End}(E)}(M)$ is the curvature form of connection D on E,*

$$\mathrm{d}_D^2 \eta(u, v, w_0, \ldots, w_{p-1}) = F(u, v)\eta(w_0, \ldots, w_{p-1}). \tag{23-5}$$

We can denote $(u, v, w_0, \ldots, w_{p-1}) \mapsto F(u, v)\eta(w_0, \ldots, w_{p-1})$ as $F \wedge \eta$, therefore

$$\mathrm{d}_D^2 \eta = F \wedge \eta. \tag{23-6}$$

Theorem 23.2 (Gauge transformation of exterior covariant derivative)**.** *Let E be a G-bundle and D is the G-connection on E. $g \in \mathcal{G}$ is a gauge transformation of E, then*

$$\mathrm{d}_{gDg^{-1}} = g\,\mathrm{d}_D g^{-1} \tag{23-7}$$

Proof.

$$\begin{aligned} \mathrm{d}_{gDg^{-1}}(gs)(v, u_0, \ldots, u_{p-1}) &= gD_v g^{-1} gs(u_0, \ldots, u_{p-1}) \\ &= g\,\mathrm{d}_D s(v, u_0, \ldots, u_{p-1}) \end{aligned} \tag{23-8}$$

□

§24 Bianchi Identity

Theorem 24.1 (Bianchi identity)**.** *Given any connection D on E, for the curvature F we have*

$$\mathrm{d}_D F = 0, \tag{24-1}$$

where d_D should be understood as the exterior covariant derivative of D on $\mathrm{End}(E)$*.*

Proof. It can be proved (by calculating in local coordinates) that $\forall \omega \in \Omega^p_{\mathrm{End}(E)}(M)$, $\forall \eta \in \Omega_E(M)$,

$$\mathrm{d}_D(\omega \wedge \eta) = \mathrm{d}_D\omega \wedge \eta + (-1)^p \omega \wedge \mathrm{d}_D\eta. \tag{24-2}$$

Hence,

$$\begin{aligned} \mathrm{d}_D^3\eta = \mathrm{d}_D(F \wedge \eta) &= \mathrm{d}_D F \wedge \eta + (-1)^2 F \wedge \mathrm{d}_D\eta \\ &= \mathrm{d}_D F \wedge \eta + F \wedge \mathrm{d}_D\eta. \end{aligned} \tag{24-3}$$

On the other hand,

$$\mathrm{d}_D^3\eta = F \wedge \mathrm{d}_D\eta. \tag{24-4}$$

$\Rightarrow$

$$\mathrm{d}_D F \wedge \eta = 0, \tag{24-5}$$

for any $\eta \in \Omega_E(M)$. □

In local coordinates, we have

$$D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} + D_\lambda F_{\mu\nu} = 0. \tag{24-6}$$

Or, using the definition of D on $\mathrm{End}(E)$, if written in D on E:

$$[D_\mu, F_{\nu\lambda}] + [D_\nu, F_{\lambda\mu}] + [D_\lambda, F_{\mu\nu}] = 0, \tag{24-7}$$

or

$$[D_u, [D_v, D_w]] + [D_v, [D_w, D_u]] + [D_w, [D_u, D_v]] = 0. \tag{24-8}$$

We can have a different approach. We need several algebraic constructions first.

We define the wedge product of two $\mathrm{End}(E)$-valued forms as

$$\sum_i (S_i \otimes \omega_i) \wedge \sum_j (T_j \otimes \mu_j) := \sum_{i,j} (S_i T_j) \otimes (\omega_i \wedge \mu_j). \tag{24-9}$$

It can be proved that

$$\mathrm{d}_D(\omega \wedge \mu) = \mathrm{d}_D\omega \wedge \mu + (-1)^p \omega \wedge \mathrm{d}_D\mu, \tag{24-10}$$

if ω is a $\mathrm{End}(E)$-valued p-form.

Definition 24.1 (Graded commutator)**.** For $\mathrm{End}(E)$-valued forms ω and μ, we define the ***graded commutator*** as

$$[\omega, \mu] := \omega \wedge \mu - (-1)^{pq} \mu \wedge \omega. \tag{24-11}$$

The graded commutator gives a graded Lie algebra structure on $\Omega^*_{\mathrm{End}(E)}(M)$, with graded antisymmetric:

$$[\omega, \mu] = -(-1)^{pq} [\mu, \omega], \tag{24-12}$$

and ***graded Jacobi identity***:

$$(-1)^{pr} [\omega, [\mu, \nu]] + (-1)^{pq} [\mu, [\nu, \omega]] + (-1)^{qr} [\nu, [\omega, \mu]] = 0. \tag{24-13}$$

Alternative proof for Bianchi identity. Let $\mathrm{d} := \mathrm{d}_{D^0}$ in some local trivialisation of E ($E|_U \cong U \times V$). Since D^0 is flat, we have $\mathrm{d}^2 = 0$.

If $D = D^0 + A$, then for $\omega \in \Omega_E(U)$,

$$\mathrm{d}_D\omega = \mathrm{d}\omega + A \wedge \omega; \tag{24-14}$$

while $T \in \Omega_{\mathrm{End}(E)}(U)$,

$$\mathrm{d}_D T = \mathrm{d}T + [A, T]. \tag{24-15}$$

$\Rightarrow$

$$\mathrm{d}_D^2\omega = \mathrm{d}_D(24\text{-}14) = (\mathrm{d}A + A \wedge A) \wedge \omega. \tag{24-16}$$

$\Rightarrow$

$$F = \mathrm{d}A + A \wedge A. \tag{24-17}$$

$$\mathrm{d}_D F = \mathrm{d}F + [A, F] = \mathrm{d}(A \wedge A) + [A, \mathrm{d}A + A \wedge A] = 0. \tag{24-18}$$

$\square$

Chapter 8

Pseudo-Riemannian Geometry

§25 Tensors

Definition 25.1 (Tensor)**.** Let M be a smooth manifold. A (r, s)-***tensor*** is a smooth section of the tensor product of rth tensor power of TM and sth tensor power of T^*M:

$$t \in \Gamma(TM^{\otimes r} \otimes T^*M^{\otimes s}) =: TM^r_s. \tag{25-1}$$

In local coordinates:

$$t^{\mu_1 \cdots \mu_r}_{\nu_1 \cdots \nu_s} \bigotimes_{k=1}^{r} \partial_{\mu_k} \otimes \bigotimes_{k=1}^{s} \mathrm{d}x^{\nu_k}. \tag{25-2}$$

It is conventional to use the local coordinates form in pseudo-Riemannian geometry, and do not distinguish between a tensor and its components, written in forms of ***abstract indices***, where indices are written just to indicates types and operations on tensors.

And since we can raise and lower indices of a tensor, it is sometimes important to distinguish the orders between covariant and contravariant indices. e.g. ${T^\mu}_\nu \neq {T^\nu}_\mu$.

Raising and Lowering of Indices We have defined $\omega^\sharp$ for a 1-form and $v^\flat$ for a vector field. Now we can generalise the definition for any (p, q) tensor T that:

$$\begin{aligned}
&T^\sharp \in \Gamma(TM^{\otimes(p+q)}),\\
&T^\sharp(\omega_0, \dots, \omega_{p+q-1}) = T(\omega_0, \dots, \omega_{p-1}, \omega_p^\sharp, \dots, \omega_{p+q-1}^\sharp);\\
&T^\flat \in \Gamma(T^*M^{\otimes(p+q)}),\\
&T^\flat(v_0, \dots, v_{p+q-1}) = T(v_0^\flat, \dots, v_{p-1}^\flat, v_p, \dots, v_{p+q-1}).
\end{aligned} \tag{25-3}$$

We can even raise or lower some instead of all indices in T, by writing $T^{\sharp i \flat j}$ or $T^{\sharp\{i_0,\dots\}\flat\{j_0,\dots\}}$.

In abstract indices, it is conventional to keep the order of the indices including the raised and lowered ones, and abuse the original symbol of the tensor, for example if T is a $(3, 4)$ tensor:

$$(T^{\sharp 4})^{\alpha_0\alpha_1\alpha_2\mu}{}_{\beta_0\beta_2\beta_3} =: T^{\alpha_0\alpha_1\alpha_2}{}_{\beta_0}{}^{\mu}{}_{\beta_2\beta_3} \tag{25-4}$$

Strictly speaking, the tensors after raising or lowering indices might not be the tensors as we defined in Def. 25.1, since it might belong to e.g.

$$TM^{\otimes r} \otimes T^*M \otimes TM^{\otimes s} \otimes T^*M^{\otimes t} \tag{25-5}$$

where the order of the tensor product is not canonical. One way to avoid this is to reorder the indices, but this approach is not conventional to those who use abstract indices. Since we can still consider the "tensors" as multilinear maps, we can include these non-canonical tensors, while, in order to avoid confusion in the order of indices, we will prefer to use the abstract indices form if there is any ambiguity.

Tensor Product Let T_1, T_2 be (p_1, q_1) and (p_2, q_2) tensors, we can have their tensor product:

$$T_1 \otimes T_2 \in TM^{p_1+p_2}_{q_1+q_2}, \tag{25-6}$$

where at each point $p \in M$, the tensor product is but the tensor product of the corresponding multilinear functions.

In abstract indices, we have:

$$\begin{aligned}&(T_1 \otimes T_2)^{\mu_1 \cdots \mu_{p_1+p_2}}{}_{\nu_1 \cdots \nu_{q_1+q_2}}\\&= T_1^{\mu_0 \cdots \mu_{p_1-1}}{}_{\nu_0 \cdots \nu_{q_0-1}} T_2^{\mu_{p_1} \cdots \mu_{p_1+p_2-1}}{}_{\nu_{q_1} \cdots \nu_{q_1+q_2-1}}.\end{aligned} \tag{25-7}$$

Contractions The contraction is a generalisation of the inner product of vectors. Let T is a $(p+1, q+1)$ tensor, we can define the $(i, p+j)$ contraction of T as

$$\begin{aligned}&\operatorname{tr}_{(i,p+j)} T \colon T^*M^p \times TM^q \to \mathbb{R}\\&(\omega_0, \ldots, \omega_{i-1}, \omega_{i+1}, \ldots, \omega_p, v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_q) \mapsto\\&\sum_{\mu \in N} T(\omega_0, \ldots, \omega_{i-1}, \mathrm{d}x^\mu, \omega_{i+1}, \ldots, \omega_p, v_0, \ldots, v_{j-1}, \partial_\mu, v_{j+1}, \ldots, v_q).\end{aligned} \tag{25-8}$$

The index-free notation can be found at [1].

§26 Diffeomorphism and Invariance

Let $\phi \colon M \to N$ be a diffeomorphism, we have already known that we have pushforward ϕ_* for $(p, 0)$-tensors, and pullback ϕ^* for $(0, q)$-tensors. Since ϕ is a diffeomorphism, both ϕ^* and ϕ_* are isomorphisms, and we can generalise the definitions to obtain a pair of isomorphisms:

$$\phi_* \colon TM^p_q \to TN^p_q, \quad \phi^* \colon T^*N^p_q \to T^*M^p_q, \tag{26-1}$$

such that $\phi_* \circ \phi^* = \phi_* \circ \phi^*$, and

$$\begin{aligned}&\phi_* T(\omega_0, \ldots, \omega_p, v_0, \ldots, v_q)\\&\quad = T(\phi^*\omega_0, \ldots, \phi^*\omega_p, \phi_*^{-1} v_0, \ldots, \phi_*^{-1} v_q),\end{aligned} \tag{26-2}$$

$$\begin{aligned}
&\phi^*T(\omega_0, \ldots, \omega_p, v_0, \ldots, v_q) \\
&\quad = T((\phi^{-1})^*\omega_0, \ldots, (\phi^{-1})^*\omega_p, \phi_* v_0, \ldots, \phi_* v_q).
\end{aligned} \tag{26-3}$$

The special case when $M = N$ (ϕ is an endomorphism), if $\phi_* T = T$, then we say that T is ***invariant*** under ϕ.

Theorem 26.1. *Let $\phi\colon M \to N$ be a diffeomorphism, $T \in TM^p_q$, and $S \in TN^r_s$.*

1. *ϕ^* and ϕ_* are isomorphisms of $\mathbb{R}$-algebras.*
2. *$\phi_*(T \otimes S) = \phi_* T \otimes \phi_* S$, and $\phi^*(T \otimes S) = \phi^* T \otimes \phi^* S$.*

Theorem 26.2. *Let $\phi\colon M \to N$ be a* homeomorphism *(differentiable map), then $\forall \alpha, \beta \in \Omega(M)$, we have*

$$\phi^*(\alpha \wedge \beta) = \phi^*\alpha \wedge \phi^*\beta, \tag{26-4}$$

that is, ϕ^* is the induced homeomorphism of the exterior algebra $\Omega(M)$.

§27 Lie Derivative

Let u be a vector field on M, and ϕ be the corresponding flow.

Definition 27.1 (Lie derivative)**.** Let T be a (p, q) tensor, then the ***Lie derivative*** of T along u is defined as

$$\mathcal{L}_u T = \lim_{t \to 0} \frac{\phi_t^* T - T}{t}. \tag{27-1}$$

Theorem 27.1. *$u, v \in \mathrm{Vect}(M)$.*

1. *$\mathcal{L}_{u+v} = \mathcal{L}_u + \mathcal{L}_v$.*
2. *$\mathcal{L}_{[u,v]} = \mathcal{L}_u \mathcal{L}_v - \mathcal{L}_v \mathcal{L}_u =: [\mathcal{L}_u, \mathcal{L}_v]$.*

Theorem 27.2. *$u \in \mathrm{Vect}(M)$, $T \in TM^p_q$, $S \in TM^r_s$.*

1. *$\mathcal{L}_u$ is $\mathbb{R}$-linear.*

2. *$\mathcal{L}_u(T \otimes S) = \mathcal{L}_u T \otimes S + T \otimes \mathcal{L}_u S$ (**Leibniz law**).*

3. *$\operatorname{tr}_{(i,j)} \mathcal{L}_u T = \mathcal{L}_u \operatorname{tr}_{(i,j)} T$.*

4. *$\forall f \in C^{(\infty)}(M)$, $\mathcal{L}_u f = u(f)$.*

5. *$\forall v \in \operatorname{Vect}(M)$, $\mathcal{L}_u v = [u, v]$.*

Applying the laws, we can calculate

$$\mathcal{L}_u \omega(v) = u[\omega(v)] - \omega([u, v]), \quad \omega \in \Omega^1(M) \tag{27-2}$$

by $u[\omega(v)] = T_u[\omega(v)] = T_u \operatorname{tr}_{(0,0)}(\omega \otimes v) = T_u \omega(v) + \omega([u, v])$. Similarly:

$$\begin{aligned}\mathcal{L}_u \omega(v_0, \ldots, v_{q-1}) &= u[\omega(v_0, \ldots, v_{q-1})] \\ &- \omega([u, v_0], \ldots, v_{q-1}) - \cdots - \omega(v_0, \ldots, [u, v_{q-1}]).\end{aligned} \tag{27-3}$$

And, in local coordinates, we have

$$\begin{aligned}\mathcal{L}_u T^{\alpha_0 \cdots \alpha_{p-1}}{}_{\beta_0 \cdots \beta_{q-1}} &= u^\mu T^{\alpha_0 \cdots \alpha_{p-1}}{}_{\beta_0 \cdots \beta_{q-1}, \mu} \\ &- T^{\mu \alpha_1 \cdots \alpha_{p-1}}{}_{\beta_0 \cdots \beta_{q-1}} u^{\alpha_0}{}_{,\mu} \cdots - T^{\alpha_0 \cdots \alpha_{p-2} \mu}{}_{\beta_0 \cdots \beta_{q-1}} u^{\alpha_{p-1}}{}_{,\mu} \\ &+ T^{\alpha_0 \cdots \alpha_{p-1}}{}_{\mu \beta_1 \cdots \beta_{q-1}} u^\mu{}_{,\beta_0} \cdots + T^{\alpha_0 \cdots \alpha_{p-1}}{}_{\beta_0 \cdots \beta_{q-2} \mu} u^\mu{}_{,\beta_{q-1}}.\end{aligned} \tag{27-4}$$

Definition 27.2 (Divergence)**.** Let $u \in \operatorname{Vect}(M)$, then the ***divergence*** of u is defined as

$$\operatorname{div} u = (-1)^{\operatorname{sign}(g)} \star (\mathcal{L}_u \operatorname{vol}) \tag{27-5}$$

Definition 27.3 (Killing field)**.** If $u \in \operatorname{Vect}(M)$ is such that $\mathcal{L}_u g = 0$, then u is called a ***Killing field***. The equation

$$\mathcal{L}_u g = 0, \quad \text{or} \quad u_{(\alpha;\beta)} = 0 \tag{27-6}$$

is called the ***Killing equation***.

Theorem 27.3 (***Cartan's formula***)**.**

$$\mathcal{L}_u|_{\Omega(M)} = \mathrm{d} \circ i_u + i_u \circ \mathrm{d}. \tag{27-7}$$

Corollary 1.

$$\mathcal{L}_u|_{\Omega(M)} \circ \mathrm{d} = \mathrm{d} \circ \mathcal{L}_u|_{\Omega(M)}. \tag{27-8}$$

Corollary 2.

$$i_{[u,v]} = [\mathcal{L}_u|_{\Omega(M)}, i_v]. \tag{27-9}$$

As an application of Cartan's formula, we can prove the following theorem by induction.

Theorem 27.4.

$$\begin{aligned} \mathrm{d}\omega(u_0, \ldots, u_p) &= \sum_{i \in p+1} (-1)^i u_i \omega(u_0, \ldots, u_{i-1}, u_{i+1}, \ldots, u_p) \\ &+ \sum_{\substack{(i,j) \in (p+1)^2 \\ i<j}} (-1)^{i+j} \omega([u_i, u_j], u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{j-1}, u_{j+1}, \ldots, u_p). \end{aligned} \tag{27-10}$$

§28 Levi-Civita Connection

Definition 28.1 (Levi-Civita connection)**.** Let $E \to M$ be a smooth vector bundle, where M is a Riemannian manifold with metric $g \in T^*M \otimes T^*M$. Let $\nabla \in \Gamma(\mathrm{End}(E) \otimes T^*M)$ be a connection on E. Then ∇ is called a ***Levi-Civita connection*** if

$$ug(v, w) = g(\nabla_u v, w) + g(v, \nabla_u w), \tag{28-1}$$

and

$$[v, w] = \nabla_v w - \nabla_w v, \tag{28-2}$$

where $u, v, w \in \Gamma(TM)$.

Since $T(u,v) = \nabla_u v - \nabla_v u - [v,u]$ is called the ***torsion*** of u and v, Eq. (28-2) is called the ***torsion free*** condition. In the torsion free condition is missing, then the connection ∇ is called an ***affine connection***.

In local coordinates:

$$\nabla_\alpha \partial_\beta = \Gamma^\gamma_{\alpha\beta}\partial_\gamma, \tag{28-3}$$

where $\Gamma^\gamma_{\alpha\beta}$ is the ***Christoffel symbol***.

The torsion free condition is equivalent to

$$\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha}. \tag{28-4}$$

For any $T \in \Gamma(TM^{\otimes p} \otimes T^*M^{\otimes q})$, we have

$$\nabla T = T^{\alpha_0\cdots\alpha_{p-1}}{}_{\beta_0\cdots\beta_{q-1};\mu} \bigotimes_{k\in p} \partial_{\alpha_k} \otimes \bigotimes_{\ell\in q} \mathrm{d}x^{\beta_\ell} \otimes \mathrm{d}x^\mu \tag{28-5}$$

$$\begin{aligned} T^{\alpha_0\cdots\alpha_{p-1}}{}_{\beta_0\cdots\beta_{q-1};\mu} = {} & T^{\alpha_0\cdots\alpha_{p-1}}{}_{\beta_0\cdots\beta_{q-1},\mu} \\ & + \sum_{i\in p} \Gamma^{\alpha_i}_{\lambda\mu} T^{\alpha_0\cdots\alpha_{i-1}\lambda\alpha_{i+1}\cdots\alpha_{p-1}}{}_{\beta_0\cdots\beta_{q-1}} \\ & - \sum_{i\in q} \Gamma^\lambda_{\beta_i\mu} T^{\alpha_0\cdots\alpha_{p-1}}{}_{\beta_0\cdots\beta_{i-1}\lambda\beta_{i+1}\cdots\beta_{q-1}}. \end{aligned} \tag{28-6}$$

It is useful to define the generalisation of divergence:

$$\nabla \cdot T = \mathrm{tr}_{(0,q)}(\nabla T) \tag{28-7}$$

if T is a (p,q) tensor.

It can be shown that

$$\nabla \cdot u = \operatorname{div} u = \delta u^\flat, \quad u \in \mathrm{Vect}(M). \tag{28-8}$$

Theorem 28.1. $\forall u \in \Gamma(TM)$,

$$\nabla_u \mathrm{tr}_{(i,j)} = \mathrm{tr}_{(i,j)} \nabla_u. \tag{28-9}$$

Theorem 28.2. $\forall \omega \in \Gamma(T^*M)$,

$$-\mathrm{d}\omega(u,v) = \nabla\omega(u,v) - \nabla\omega(v,u). \tag{28-10}$$

(Notice that $\nabla\omega(u,v) = (\nabla_v\omega)(u)$)

Proof.

$$u[\omega(v)] = ug(\omega^\sharp, v) = \nabla\omega(v,u) + \omega(\nabla_u v) \tag{28-11}$$

$\Rightarrow$ (Theorem 27.4)

$$\begin{aligned} \nabla\omega(v,u) - \nabla\omega(u,v) &= u[\omega(v)] - v[\omega(u)] - \omega([u,v]) \\ &= \mathrm{d}\omega(u,v) \end{aligned} \tag{28-12}$$

□

In fact, the Theorem 28.2 is but a special case of:

Theorem 28.3. $\forall \omega \in \Omega^p(M)$,

$$(-1)^p \,\mathrm{d}\omega(u_0,\dots,u_p) = (p+1) \sum_{\pi \in S_{p+1}} \nabla\omega(u_{\pi(0)},\dots,u_{\pi(p)}). \tag{28-13}$$

§29 Curvatues

The ***Rieman tensor*** Riem is defined as the curvature of the affine connection ∇:

$$\mathrm{Riem}(u,v)w = \nabla_u\nabla_v w - \nabla_v\nabla_u w - \nabla_{[u,v]}w. \tag{29-1}$$

It is conventional to write the components of Riem as

$$R^\mu{}_{\nu\alpha\beta} = \mathrm{d}x^\mu(\mathrm{Riem}(\partial_\alpha,\partial_\beta)\partial_\nu), \tag{29-2}$$

and consider it as a $(1,3)$-tensor.

The trace of the linear map $u \mapsto \mathrm{Riem}(u,v)w$ is defined as $\mathrm{Ric}(v,w)$ where Ric is the ***Ricci tensor***, with components

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}. \tag{29-3}$$

Finnally, the trace of $u \mapsto \mathrm{Ric}(u,-)^{\sharp}$ is called the ***scalar curvature*** or ***Ricci scalar***:

$$R = g^{\mu\nu} R_{\mu\nu} =: \mathrm{tr}_g \,\mathrm{Ric}\,. \tag{29-4}$$

bibliography

[1] Yuri Vyatkin (https://math.stackexchange.com/users/2002/yuri-vyatkin). *Coordinate-free notation for tensor contraction?* Mathematics Stack Exchange. eprint: https://math.stackexchange.com/q/1804213. URL: https://math.stackexchange.com/q/1804213.

[2] Javier P. Muniain John C. Baez. *Gauge Fields, Knots and Gravity (Series on Knots and Everything)*. Series on Knots and Everything. World Scientific Publishing Company, 1994. ISBN: 9789810217297,9810217293,9810220340.

Symbol List

Here listed the important symbols used in these notes

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