

# Differential Geometry

Hoyan Mok

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## Part I

# Domestic Differential Geometry

# Chapter 1

# Manifolds

# Chapter 2

## Scalar and Vector Fields

### §1 Scalar Fields

**Definition 1.1** (Scalar Field). Let  $M$  be a smooth manifold,  $f \in C^{(\infty)}(M)$  is called a ***scalar field***.

The scalar field over a manifold, form an algebra.

### §2 Vector Fields

**Definition 2.1** (vector field). A ***vector field***  $v$  over manifold  $M$  is a  $C^{(\infty)}(M) \rightarrow C^{(\infty)}(M)$  map that satisfies

- (a)  $\forall f, g \in C^{(\infty)}(M), \forall \lambda, \mu \in \mathbb{R}, v(\lambda f + \mu g) = \lambda v(f) + \mu v(g)$   
(*linearity*).
- (b)  $\forall f, g \in C^{(\infty)}(M), v(fg) = v(f)g + fv(g)$

The space of all vector fields on  $M$  is denoted by  $\text{Vect}(M)$

**Definition 2.2** (tangent vector). Let  $v$  be a vector field over  $M$ ,  $p$  be a point on  $M$ . The tangent vector  $v_p$  at  $p$  is defined as a  $C^{(\infty)}(M) \rightarrow C^{(\infty)}(M)$  map that satisfies

$$v_p(f) = v(f)(p). \quad (2-1)$$

The collection of tangent vectors at  $p$  is called the **tangent space** at  $p$ , denoted by  $T_p M$ .

The derivative of a path  $\gamma: [0, 1] \rightarrow M$  (or  $\mathbb{R} \rightarrow M$ ) in a smooth manifold is defined as:

$$\begin{aligned} \gamma'(t) &: C^{(\infty)}(M) \rightarrow \mathbb{R}; \\ \gamma'(t)(f) &= \frac{d}{dt} f \circ \gamma(t) \end{aligned} \quad (2-2)$$

We can see that  $\gamma'(t) \in T_{\gamma(t)} M$ .

Let a path  $\gamma: \mathbb{R} \rightarrow M$  follows a vector field (a velocity field), that is

$$\gamma'(t) = v_{\gamma(t)}, \quad (2-3)$$

then we call  $\gamma$  the **integral curve** through  $p := \gamma(0)$  of the vector field  $v$ .

**Definition 2.3.** Suppose  $v$  is an integrable vector field. Let  $\varphi_t(p)$  be the point at time  $t$  on the integral curve through  $p$ .

$$\varphi_t: M \rightarrow M \quad (2-4)$$

is then called a **flow** generated by  $v$ .

$$\frac{d}{dt} \varphi_t(p) = v_{\varphi_t(p)}. \quad (2-5)$$

### §3 Covariant and Contravariant

**Definition 3.1** (pullback). Let  $f$  be a scalar field over  $N$ ,  $\varphi \in C^{(\infty)}(M, N)$ . Then the **pullback** of  $f$  by  $\varphi$

$$\varphi^*: C^{(\infty)}(N) \rightarrow C^{(\infty)}(M), \quad (3-1)$$

is defined as

$$\varphi^* f = f \circ \varphi \in C^{(\infty)}(M). \quad (3-2)$$

Fields that are pullbacked are **covariant** fields.

**Definition 3.2** (pushforward). Let  $v_p$  be a tangent vector of  $M$  at  $p$ ,  $\varphi \in C^{(\infty)}(M, N)$ ,  $q = \varphi(p)$ . Then the **pushforward** of  $v_p$  by  $\varphi$

$$\varphi_*: T_p M \rightarrow T_q N, \quad (3-3)$$

is defined as

$$(\varphi_* v)_q(f) = v_p(\varphi^* f). \quad (3-4)$$

Note that the pushforward of a vector field can only be obtained when  $\varphi$  is a diffeomorphism.

Fields that are pushforwarded are **contravariant** fields.

Mathematicians and physicists might have disagreement on whether a tangent vector is covariant or contravariant. This is because of that physicists might consider the coordinates  $(v^\mu)$  of a tangent vector as a vector field, instead of linear combination of bases  $\partial_\mu$ .

## §4 Components of Vector Fields

Let  $\varphi: U \rightarrow \mathbb{R}^n$  be a chart of  $M$  ( $U \subset M$ ).

Let  $p \in U$ ,  $\varphi(p) = x = (x^\mu)$  ( $\mu = 0, \dots, n-1$ ). Locally, a function  $f \in C^{(\infty)}(M)$  can be written as

$$(\varphi^{-1})^* f = f \circ \varphi^{-1}: M \rightarrow \mathbb{R}, \quad (4-1)$$

and a vector field  $v \in \text{Vect}(M)$  can be written as

$$(\varphi_* v)_x = \varphi_* v_p: C^{(\infty)}(\mathbb{R}^n) \rightarrow \mathbb{R}, \quad (4-2)$$

or

$$\varphi_* v \in \text{Vect}(\mathbb{R}^n) \quad (4-3)$$



Since  $T_x \mathbb{R}^n \cong \mathbb{R}^n$  is a linear space, one can find a basis for  $T_x \mathbb{R}^n$  as

$$\partial_\mu : C^{(\infty)}(\mathbb{R}^n) \rightarrow C^{(\infty)}(\mathbb{R}^n), \quad (4-4)$$

and  $(\varphi_* v)_x = v^\mu(x) \partial_\mu$ .

Pushing forward  $v^\mu(x) \partial_\mu$  by  $\varphi^{-1}$  one obtains  $v$ .

In an abuse of symbols, one may just omit the pullback and pushforward, and refer to the  $f$  and  $v$  by  $(\varphi^{-1})^* f$  and  $\varphi_* v$ .

Consider another chart  $\psi : U \rightarrow \mathbb{R}^n$  of  $M$ , and

$$y = \psi(p), \quad (\psi_* v)_x = u^\mu \partial_\mu, \quad (4-5)$$

where we have chosen the same basis in  $T_y \mathbb{R}^n$  as in  $T_x \mathbb{R}^n$ .

We would like to know how to relate  $v^\mu$  and  $u^\mu$  i.e. we want to know how the components of  $v$  transforms under a coordinate transformation  $\tau = \psi \circ \varphi^{-1}$ .

Consider any  $f \in C^{(\infty)}(M)$ ,

$$v(f) = \varphi_* v((\varphi^{-1})^* f) = \psi_* v((\psi^{-1})^* f) \quad (4-6)$$

$\Rightarrow$

$$u^\mu \partial_\mu (f \circ \psi^{-1}) = v^\mu \partial_\mu (f \circ \varphi^{-1}) = v^\mu \partial_\mu (f \circ \psi^{-1} \circ \tau) = v^\mu \tau'^\nu_\mu \partial_\nu (f \circ \psi^{-1}) \quad (4-7)$$

$\Rightarrow$

$$u^\mu = v^\nu \tau'^\mu_\nu, \quad (4-8)$$

where

$$\tau'^\mu_\nu = \frac{\partial y^\mu}{\partial x^\nu}. \quad (4-9)$$

## §5 Lie Bracket

**Definition 5.1** (Lie bracket). Let  $v, w \in \text{Vect}(M)$ , then the **Lie bracket** of  $v$  and  $w$  is defined as

$$[v, w] : C^{(\infty)}(M) \rightarrow C^{(\infty)}(M); \quad f \mapsto v \circ w(f) - w \circ v(f). \quad (5-1)$$

The Lie bracket is an antisymmetric bilinear map<sup>1</sup>, and an important property of the Lie bracket is the Leibniz rule:

$$[v, w](fg) = [v, w](f)g + f[v, w](g). \quad (5-2)$$

Another important property of the Lie bracket is the Jacobi identity:

$$[v, [w, u]] + [w, [u, v]] + [u, [v, w]] = 0. \quad (5-3)$$

---

<sup>1</sup>Note that it is not  $C^{(\infty)}$ -linear

# Chapter 3

## Differential Forms

### §6 1-forms

**Definition 6.1** (1-form). A **1-form**  $\omega$  on  $M$  is a  $\text{Vect}(M) \rightarrow C^{(\infty)}(M)$  which satisfies that

$$(a) \quad \forall v, w \in \text{Vect}(M), \forall f, g \in C^{(\infty)}(M),$$

$$\omega(fv + gw) = f\omega(v) + g\omega(w). \quad (6-1)$$

The space of all 1-forms on  $M$  is denoted as  $\Omega^1(M)$ , which is a module over  $C^{(\infty)}(M)$ .

The operator  $d$ , when given a  $C^{(\infty)}(M)$  function (which is called a **0-form**), would give a 1-form:

$$(df)(v) = v(f). \quad (6-2)$$

This is called the **exterior derivative** or **differential** of  $f$ .

The **cotangent vector** or **covector** is similar as the tangent vector:

$$\omega_p(v_p) = \omega(v)(p). \quad (6-3)$$

The space of cotangent vectors at  $p$  on  $M$  is denoted by  $T_p^*M$ .

1-forms are covariant, that is, if  $\varphi: M \rightarrow N$ , then the pushforward of a 1-form  $\omega$  by  $\varphi$  is

$$(\varphi^*\omega)_p(v_p) = \omega_q(\varphi_*v_p), \quad (6-4)$$

where  $\varphi(p) = q$ .

**Theorem 6.1.**  $f \in C^{(\infty)}(N)$ ,  $\varphi: M \rightarrow N$  is differential, then

$$\varphi^*(df) = d(\varphi^*f). \quad (6-5)$$

## §7 Components of 1-Forms

Let  $\varphi: U \rightarrow \mathbb{R}^n$  be a chart of  $M$  ( $U \subset M$ ).

Let  $p \in U$ ,  $\varphi(p) = x = (x^\mu)$  ( $\mu = 0, \dots, n-1$ ). Locally a 1-form  $\omega \in \Omega^1(M)$  can be written as

$$(\varphi^{-1})^*\omega \in T_x^*\mathbb{R}^n. \quad (7-1)$$

A natural way to impose a basis  $dx^\mu$  in  $T_x^*\mathbb{R}^n$  is

$$dx^\mu(\partial_\nu) = \delta_\nu^\mu, \quad (7-2)$$

and  $(\varphi^{-1})^*\omega = \omega_\mu(x) dx^\mu$ .

Now by the definition of 1-form:

$$\omega_\mu dx^\mu(v^\nu \partial_\nu) = v^\nu \omega_\mu \delta_\nu^\mu = v^\mu \omega_\mu. \quad (7-3)$$

By the transformation rule of components of a vector, one have

$$\tau'^\nu{}_\mu \alpha_\nu = \omega_\mu, \quad (7-4)$$

where  $\psi: U \rightarrow \mathbb{R}^n$ ,  $(\psi^{-1})_*\omega = \alpha_\mu dx^\mu$ ,  $\tau = \psi \circ \varphi^{-1}$ .

## §8 $k$ -Forms

**Definition 8.1.** If we assign an antisymmetric multilinear  $k$ -form  $\omega_p \in \bigotimes_{i \in k} T_p^* M$  to each point  $p \in M$ , we say we have a  $k$ -**form** on  $M$ .

The collection of all  $k$ -forms is denoted by  $\Omega^k(M)$ , and  $\Omega(M) := \bigcup_{k \in \mathbb{N}} \Omega^k(M)$ .

**Theorem 8.1** (Dimension of forms). *If  $M$  is an  $nD$  manifold, then the dimension of  $\Omega^k(M)$  is  $\frac{n!}{k!(n-k)!}$  ( $k \leq n$ ), and 0 for  $k > n$ ; The dimension of  $\Omega(M)$  is  $2^n$ .*

**Definition 8.2** (Wedge product). The **wedge product**  $\wedge$  is defined as a binary operator that takes a  $k$ -form and  $\ell$ -form and gives a  $(k + \ell)$ -forms, satisfying  $\forall \alpha \in \Omega^k(M), \forall \beta \in \Omega^\ell(M)$ :

(a) (Associativity)  $\forall \gamma \in \Omega^m(M)$ ,

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma). \quad (8-1)$$

(b) (Supercommutativity)

$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha. \quad (8-2)$$

(c) (Distributiveness)  $\forall \gamma \in \Omega^\ell(M)$ ,

$$\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma. \quad (8-3)$$

(d) (Bilinearity over  $C^{(\infty)}(M)$ )  $\forall f \in C^{(\infty)}(M)$ ,

$$(f\alpha) \wedge \beta = f(\alpha \wedge \beta). \quad (8-4)$$

(e) (Naturality) If  $\varphi: M \rightarrow N$  is a smooth map, then the pullback of a form by  $\varphi$  can be given by repeatedly applying ( $\forall \gamma \in \Omega^\ell(M)$ )

$$\begin{aligned} \varphi^*(\beta + \gamma) &= \varphi^*\alpha + \varphi^*\beta \\ \varphi^*(\alpha \wedge \beta) &= \varphi^*\alpha \wedge \varphi^*\beta, \end{aligned} \quad (8-5)$$

while the pullback of a 0-form and a 1-form agree with what we have already defined before.

By convention if  $f \in C^{(\infty)}(M)$  then

$$f \wedge \omega =: f\omega. \quad (8-6)$$

It can be shown that any  $k$ -form  $\omega$  can be written as

$$(\varphi^{-1})^*\omega = \frac{\omega_{\mu_1 \cdots \mu_k}}{k!} \bigwedge_{i=1}^k dx^{\mu_i}, \quad (8-7)$$

where  $\varphi: M \rightarrow \mathbb{R}^n$  is a chart.

## §9 Exterior Derivative

**Definition 9.1** (Exterior derivative). The *exterior derivative*  $d$  is defined as a linear operator that takes a  $k$ -form and gives a  $(k+1)$ -form, satisfying  $\forall \alpha \in \Omega^k(M), \forall \beta \in \Omega^\ell(M)$ :

(a) (Linearity)  $\forall \lambda, \mu \in \mathbb{R}, \forall \gamma \in \Omega^\ell(M),$

$$d(\lambda\beta + \mu\gamma) = \lambda d\beta + \mu d\gamma. \quad (9-1)$$

(b) (Leibniz rule)

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta. \quad (9-2)$$

(c)

$$d^2\omega = 0. \quad (9-3)$$

(d) (Naturality) If  $\varphi: M \rightarrow N$  is a smooth map, then

$$\varphi^* d\omega = d\varphi^*\omega. \quad (9-4)$$

# Chapter 4

## Metric

### §10 Pseudo-Riemannian Metric

**Definition 10.1** (Pseudo-Riemannian metric). Let  $M$  be a manifold. A ***pseudo-Riemannian metric*** or simply ***metric***  $g$  on a manifold  $M$  is a field ( $g \in \Gamma(T^*M \otimes T^*M)$ ) that  $\forall p \in M$ ,

$$g_p: T_p^*M \times T_p^*M \rightarrow \mathbb{R}, \quad (10-1)$$

is a bilinear form satisfying the following properties:

(a) (Symmetry)  $\forall u, v \in T_p^*M$ ,

$$g_p(u, v) = g_p(v, u). \quad (10-2)$$

(b) (Non-degenerate)

$$u \mapsto g_p(u, -): T_p^*M \rightarrow T_p^*M \quad (10-3)$$

is an isomorphism.

(c) (Bilinearity)  $\forall p \in M, \forall u, v \in T_p^*M, \forall \lambda, \mu \in \mathbb{R}$ ,

$$g_p(\lambda u + \mu v, w) = \lambda g_p(u, w) + \mu g_p(v, w). \quad (10-4)$$

(d) (Smoothness) If  $v, u \in \text{Vect}(M)$ , then

$$p \mapsto g_p(v_p, u_p) \in C^{(\infty)}(M). \quad (10-5)$$

Given a metric,  $\forall p \in M$ , we can always find an orthonormal basis  $\{e_\mu\}$  of  $T_p M$  such that

$$g_p(e_\mu, e_\nu) = \text{sign}(\mu)\delta_{\mu\nu}, \quad (10-6)$$

where  $\text{sign}(\mu) = \pm 1$ . Conventionally we order the basis such that  $\text{sign}(\mu) = 1$  for  $\mu \in s$  and  $\text{sign}(\mu) = -1$  for  $\mu - s \in n - s$ , and say that the metric has **signature**  $(s, n - s)$ .

If  $\gamma: [0, 1] \rightarrow M$  is a smooth path and  $\forall t, s \in [0, 1]$ ,

$$g(\gamma'(t), \gamma'(t))g(\gamma'(s), \gamma'(s)) \geq 0, \quad (10-7)$$

then we can define the arclength of  $\gamma$  as

$$\int_0^1 \sqrt{|g(\gamma'(t), \gamma'(t))|} dt \quad (10-8)$$

if the integral converges.

The metric gives an **inner product** on  $\text{Vect}(M)$ :

$$\langle u, v \rangle := g(u, v). \quad (10-9)$$

The metric also gives a way to relate a vector field  $v$  to a 1-form  $\omega$ . If  $v$  and  $\omega$  satisfies:  $\forall u \in \text{Vect}(M)$ ,

$$g(v, u) = \omega(u), \quad (10-10)$$

then we say that  $v$  is the corresponding vector field of  $\omega$ , and  $\omega$  is the corresponding 1-form of  $v$ .

We can also define the **inner product** on  $\Omega^1(M)$  by

$$\langle \alpha, \beta \rangle = \langle a, b \rangle, \quad (10-11)$$

where  $a$  and  $b$  is the corresponding vector fields of  $\alpha$  and  $\beta$ .



The **inner product** on  $\Omega^k(M)$  is defined by induction with

$$\langle \bigwedge_{i \in k} \alpha_i, \bigwedge_{i \in k} \beta_i \rangle = \det(\langle \alpha_i, \beta_j \rangle)_{i,j \in k}. \quad (10-12)$$

Hence, if  $\{e_\mu\}$  is an orthonormal basis (field) of  $T_p M$ , while the corresponding covectors are  $\{f^\mu\}$  ( $f^\mu(e_\nu) = \delta^\mu_\nu$ ) then

$$\langle \bigwedge_{i \in k} f^{\mu_i}, \bigwedge_{i \in k} f^{\mu_i} \rangle = \prod_{i \in k} \text{sign}(\mu_i). \quad (10-13)$$

Specially, when  $f, g \in \Omega^0(M) = C^{(\infty)}(M)$ ,

$$\langle f, g \rangle = fg. \quad (10-14)$$

## §11 Volume Form

Notice that if  $M$  is an  $n$ D manifold,  $\dim \Omega^n(M) = 1$ , meaning at  $p \in M$ ,  $\{\omega_p \mid \omega \in \Omega^n(M)\}$  can be labelled by a parametre  $\lambda_p \in \mathbb{R}$ . If we have a basis  $\{f^\mu\}$  of  $T_p^* M$  (or corresponding vectors  $\{e_\mu\}$ ), then

$$\{\omega_p \mid \omega \in \Omega^n(M)\} = \lambda_p \bigwedge_{\mu \in n} f^\mu. \quad (11-1)$$

If there were another basis  $\{g^\mu\}$  of  $T_p^* M$  (or corresponding vectors  $\{h_\mu\}$ ), and the transformation between the two bases is given by

$$P e^\mu = f^\mu, \quad (11-2)$$

where  $P \in \text{Aut}(T_p^* M)$ . When  $\det P > 0$ , we say that  $\{f^\mu\}$  and  $\{g^\mu\}$  have the same **orientation**.

**Definition 11.1** (Volume form). Let  $M$  be an orientable manifold. If  $\forall p \in M$ , we find an oriented orthonormal basis  $\{f_\mu\}$  of  $T_p^* M$  at point  $p$ , then the **volume form**  $\text{vol}$  is defined by

$$\bigwedge_{\mu \in n} f_\mu = \text{vol}_p. \quad (11-3)$$

## §12 Hodge Star Operator

**Definition 12.1** (Hodge Star Operator). Let  $M$  be an orientable manifold. The **Hodge star operator**  $\star$  is defined by the linear map

$$\star: \Omega^k(M) \rightarrow \Omega^{n-k}(M), \quad (12-1)$$

$$\forall \alpha, \beta \in \Omega^k(M), \quad \alpha \wedge \star \beta = \langle \alpha, \beta \rangle \text{vol}. \quad (12-2)$$

We call  $\star \omega$  the **dual** of  $\omega$ .

The special case is when  $k = 0$ ,

$$\star f = f \text{vol}, \quad (12-3)$$

and  $k = n$ ,

$$\star(f \text{vol}) = f \prod_{\mu \in n} \text{sign}(\mu) = (-1)^{n-s} f \quad (12-4)$$

if the signature of the metric is  $(s, n - s)$ .

## §13 Metric and Coordinates

## Chapter 5

# De Rham Theory

§14 Closed and Exact 1-Forms

§15 Stokes' Theorem

§16 De Rham Cohomology

# Chapter 6

## Bundles and Connections

### §17 Fibre Bundles

**Definition 17.1** (Bundle). A *bundle* is a triple  $(E, \pi, B)$ , where  $\pi: E \rightarrow B$  is a surjective map.  $E$  is called the *total space*,  $\pi$  is called the *projection map*, and  $B$  is called the *base space*.

A bundle  $(E, \pi, B)$  can be denoted as  $\pi: E \rightarrow B$  or  $E \xrightarrow{\pi} B$ .

**Definition 17.2** (Fibre). For  $p \in B$ ,  $\pi^{-1}(\{p\})$  is the *fibre* over  $b$ .

**Definition 17.3** (Subbundle). Let  $\pi: E \rightarrow B$  be a bundle.  $F \subset E$ ,  $C \subset B$ ,  $\rho: F \rightarrow C$ . If  $\pi|_F = \rho$ , then  $\rho: F \rightarrow C$  is called a *subbundle* of  $\pi: E \rightarrow B$ .

**Definition 17.4** (Section). A *section* is a map  $s: B \rightarrow E$  such that

$$p \circ s = \text{id}_B. \quad (17-1)$$

All sections of a bundle  $\pi: E \rightarrow B$  is denoted as  $\Gamma(E)$ .

**Definition 17.5** (Fibre bundle). A **fibre bundle**  $(E, \pi, B, F)$  is a bundle  $\pi: E \rightarrow B$ , where  $E, B, F$  are topology spaces, and  $\pi$  is a continuous map, and  $\forall p \in B, \exists U \in \mathcal{U}(p)$  s.t.

$$\varphi: \pi^{-1}(U) \rightarrow U \times F, \quad (17-2)$$

is a homeomorphism and  $\pi_1 \circ \varphi = \pi$ .  $\pi_1$  is defined as  $\pi_1(p, q) = p$ .

A fibre bundle can be denoted as the exact sequence

$$F \longrightarrow E \xrightarrow{\pi} B \quad (17-3)$$

The last condition is called the **local triviality condition**.  $F$  is called the **standard fibre**

If  $E = B \times F$ , then  $(E, \pi, B, F)$  is called a **trivial fibre bundle**.

**Definition 17.6** (Morphism). Let  $\pi: E \rightarrow B, \rho: F \rightarrow C$  be two fibre bundles. A **morphism**  $(\varphi, \psi)$  is a pair of two continuous maps such that

$$\begin{array}{ccc} E & \xrightarrow{\psi} & F \\ \downarrow \pi & & \downarrow \rho \\ B & \xrightarrow{\varphi} & C \end{array} \quad (17-4)$$

commutes.

## §18 Vector Bundles

**Definition 18.1** (Vector bundle). A **vector bundle** is a fibre bundle  $(E, \pi, B, F)$ , where  $F$  is a vector space, and the local trivialisation  $\varphi: \pi^{-1}(U) \rightarrow U \times F$  ( $U$  is a neighbourhood of  $p \in B$ ) satisfies that  $\forall x \in U, \forall v \in F$ ,

$$\begin{aligned} F &\rightarrow \pi^{-1}(\{x\}) \\ v &\mapsto \varphi^{-1}(x, v) \end{aligned} \quad (18-1)$$

is a linear isomorphism (**fibrewise linear**).

**Definition 18.2** (Morphism (vector bundle)). The morphism between two vector bundles  $(E, \pi, B, F)$  and  $(E', \pi', B', F')$  is a morphism  $(\varphi, \psi)$  such that  $\forall x \in B$ ,

$$\psi_* : \pi^{-1}(\{x\}) \rightarrow (\pi')^{-1}(\{\varphi(x)\}) \quad (18-2)$$

is a linear homomorphism.

**Definition 18.3** (Tangent bundle). The *tangent bundle*  $TM$  is the vector bundle over a  $n$ D manifold  $M$  with the standard fibre  $T_p M = \mathbb{R}^n$ .

# Chapter 7

## Curvature

## Chapter 8

# Pseudo-Riemannian Geometry



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# Symbol List

Here listed the important symbols used in these notes

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