Contents

Part I

Basic Geometry

Chapter 1

Geometry in Regions of a Space

§1 Co-odinate and its transformation

Definition 1.1 (Jacobian)**.** A transformation of co-odinate from $\boldsymbol{x}$ to $\boldsymbol{y}$

$$\boldsymbol{y}(\boldsymbol{x}) = y^i(x^j)\hat{\boldsymbol{e}}_i = y(x^j).$$

Its ***Jacobian***:

$$\boldsymbol{J} = \left(\frac{\partial y^i}{\partial x^j}\right) \tag{1-1}$$

A ***vector*** $\boldsymbol{u}$ at point $\boldsymbol{x}_0$ under such transformation would follow:

$$v^i = \left.\frac{\partial y^i}{\partial x^j}\right|_{\boldsymbol{x}_0} u^i \tag{1-2}$$

$$\text{i.e.} \qquad \boldsymbol{v} = \boldsymbol{J}_0\boldsymbol{u}$$

A linear form (***covector***) $\ell\colon \boldsymbol{x} \mapsto \ell(\boldsymbol{x}) = l_i x^i$ under such transformation would follow:

$$l'_i \,\mathrm{d}y^i = l_j \,\mathrm{d}x^j \qquad \Rightarrow \qquad l'_i = \left.\frac{\partial x^j}{\partial y_i}\right|_{\boldsymbol{x}_0} l_j \tag{1-3}$$

$$\text{i.e.} \qquad \boldsymbol{l}' = \boldsymbol{l}\boldsymbol{J}_0^{-1}$$

A linear transformation $\mathscr{L}\colon \boldsymbol{x} \mapsto \boldsymbol{L}\boldsymbol{x}$ where $\boldsymbol{L} = (L^i{}_j)_{i,j\in n}$ under such transformation would follow:

$$\begin{aligned}\mathrm{d}y\left(\mathscr{L}^i(\boldsymbol{x})\right) &= (L')^i{}_j \,\mathrm{d}y^j \\ &= \left.\frac{\partial y^i}{\partial x^k}\right|_{\boldsymbol{x}_0} \mathrm{d}\mathscr{L}^k(\boldsymbol{x}) = \left.\frac{\partial y^i}{\partial x^k}\right|_{\boldsymbol{x}_0} L^k{}_h \,\mathrm{d}x^h = \left.\frac{\partial y^i}{\partial x^k}\right|_{\boldsymbol{x}_0} L^k{}_h \left.\frac{\partial x^h}{\partial y_j}\right|_{\boldsymbol{x}_0} \mathrm{d}y^j\end{aligned}$$

$$(L')^i{}_j = \left.\frac{\partial y^i}{\partial x^k}\right|_{\boldsymbol{x}_0} L^k{}_h \left.\frac{\partial x^h}{\partial y_j}\right|_{\boldsymbol{x}_0} \qquad \text{or} \qquad \boldsymbol{L}' = \boldsymbol{J}_0\boldsymbol{L}\boldsymbol{J}_0^{-1} \tag{1-4}$$

A bilinear form $\mathscr{B}\colon \boldsymbol{x} \mapsto \boldsymbol{x}^{\mathrm{T}}\boldsymbol{b}\boldsymbol{x} = x^{\,i} b_{ij} x^j$:

$$b'_{ij}\,\mathrm{d}y^{\,i}\,\mathrm{d}y^j = b'_{ij}\left.\frac{\partial y^{\,i}}{\partial x^{\,k}}\right|_{\boldsymbol{x}_0}\left.\frac{\partial y^j}{\partial x^h}\right|_{\boldsymbol{x}_0}\mathrm{d}x^{\,k}\,\mathrm{d}x^h = b_{kh}\,\mathrm{d}x^{\,k}\,\mathrm{d}x^h \quad\Rightarrow\quad b'_{ij} = \left.\frac{\partial x^{\,k}}{\partial y^{\,h}}\right|_{\boldsymbol{x}_0} b_{kh}\left.\frac{\partial x^h}{\partial y^j}\right|_{\boldsymbol{x}_0}$$

$$\text{i.e.} \qquad \boldsymbol{b}' = (\boldsymbol{J}_0^{-1})^{\mathrm{T}}\boldsymbol{b}\boldsymbol{J}_0^{-1} \tag{1-5}$$

§2 Riemannian and Pseudo-Riemannnian Spaces

Definition 2.1 (Riemannian metric)**.** A ***Riemannian metric*** $\boldsymbol{G}$ is a smooth, positive-definite quadrutic form defined on a finite-dimensional vector space over $\mathbb{R}$.

Given a basis, we usually denote the Riemannian metric by $g_{ij}(\boldsymbol{x})$.

We can define ***arc length*** ℓ and ***inner product*** $\langle,\rangle$ in a ***Riemannian space*** (i.e. a vector space equiped with a Riemannian metric):

$$\ell := \int_{t_1}^{t_2}\sqrt{g_{ij}[\boldsymbol{x}(t)]\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}}\,\mathrm{d}t, \qquad \langle\boldsymbol{u},\boldsymbol{v}\rangle := g_{ij}u^i v^j.$$

We can also introduce the following notation: $u_i := g_{ij}u^j$, which means the linear form $\boldsymbol{v} \mapsto u_i v^i$; and $\mathrm{d}\ell^2 = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$.

Definition 2.2 (Euclidean metric)**.** If a metric $\boldsymbol{G}(\boldsymbol{x})$ is said to be ***Euclidean*** if there exists a coodinates $\boldsymbol{y}(\boldsymbol{x})$ s.t.

$$g_{ij} = \delta_{k\ell}\frac{\partial y^k}{\partial x^i}\frac{\partial y^\ell}{\partial x^j}.$$

Such coodinates $\boldsymbol{y}(\boldsymbol{x})$ are said to be a ***Euclidean coodinates***.

Definition 2.3 (Pseudo-Riemannian metric)**.** A ***pseudo-Riemannian metric*** $\boldsymbol{G}$ is a smooth, indefinite quadrutic form defined on a finite-dimensional vector space over $\mathbb{R}$.

A pseudo-Riemannian metric shall have the following cannonical form at some coodinates:

$$\boldsymbol{G} = \mathrm{diag}(\eta_1^2, \ldots, \eta_p^2, -\eta_{p+1}^2, \ldots, -\eta_n^2)$$

where η

By Sylvester's law of inertia, the index of inertia i.e. the number of positive terms on the caninical form, shall conserve under any coodinate change.

Definition 2.4 (Pseudo-Euclidean metric)**.** If a metric $\boldsymbol{G}(\boldsymbol{x})$ is said to be ***pseudo-Euclidean*** if there exists a coodinates $\boldsymbol{y}(\boldsymbol{x})$ s.t.

$$g_{ij} = \sum_{k=1}^{p}\frac{\partial y^k}{\partial x^i}\frac{\partial y^k}{\partial x^j} - \sum_{\ell=p+1}^{n}\frac{\partial y^\ell}{\partial x^i}\frac{\partial y^\ell}{\partial x^j}.$$

Such coodinates $\boldsymbol{y}(\boldsymbol{x})$ are said to be a ***pseudo-Euclidean coodinates***.

We denote a pseudo-Euclidean space by $\mathbb{R}^n_{p,n-p}$, where p is the index of innertia. Especially, we call $\mathbb{R}^4_{1,3}$ the ***Minkowski space***, since its significance in relativistic mechanics.

§3 The Simplest Groups of Transformations of a Euclidean Space

When we say a ***tansformation*** from Ω to Ω', we refer to a bijective φ s.t. both φ and φ^{-1} are smooth, i.e. a ***diffeomorphism***. If $\Omega = \Omega'$, we would call it the transformation of Ω.

The transformations form a group, we might call it the ***transformation group***.

Definition 3.1 (Isometry)**.** If a transformation φ of Ω, with a metric $\boldsymbol{G}$, satisfies that $\forall \boldsymbol{x} \in \Omega$,

$$g_{ij} = g_{k\ell} \frac{\partial \varphi^k}{\partial x^i} \frac{\partial \varphi^\ell}{\partial x^j},$$

we shall call it a ***isometry***, or a ***motion*** of the given metric.

Theorem 3.1 (Isometry group)**.** *The isometries of a region form a group.*

We might call this group the ***isometries*** or the ***group of motions***. If an isometry preserve the orientation of $\mathbb{R}^n$, we might call it a ***proper isometry***. The proper isometries is a subgroup of the isometries of $\mathbb{R}^n$.

Lemma 1 (Proper isometries of Euclidean space)**.** *Every proper isometry of* $\mathbb{R}^n$ *is either a translation along a vector, or a rotation about a point.*

The proper isometries can be parameteriesed by a special othogonal matrix $\boldsymbol{A} \in \mathrm{SO}(n)$ (2D special othogonal group), and a vector $\boldsymbol{v}$:

$$\boldsymbol{x} \mapsto \boldsymbol{A}\boldsymbol{x} + \boldsymbol{v}.$$

We can find the group a representation by a matrix:

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{v} \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{pmatrix} \tag{3-1}$$

If $\det \boldsymbol{A} = \pm 1$, then we have the parametrisation of the isometry of the Euclidean space $\mathbb{R}^n$.

Theorem 3.2 (Mazur-Ulam Theorem)**.** *The isometry group of* $\mathbb{R}^n$ *is the semiproduct of translations* ($\simeq \mathbb{R}^n$) *and the othogonal group* $\mathrm{O}(n)$ *(rotations and rotatory reflections).*

The isometry group can be represented by Eq. 3-1.

Theorem 3.3. $\mathrm{SO}(n)$*, as a Lie group, is continuous.*

Proof. find the rotation a basis so that it take canonical form

$$\begin{pmatrix} \pm 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & \pm 1 & & & & & & \\ & & & \cos\theta_1 & -\sin\theta_1 & & & & \\ & & & \sin\theta_1 & \cos\theta_1 & & & & \\ & & & & & & \ddots & & \\ & & & & & & & \cos\theta_m & -\sin\theta_m \\ & & & & & & & \sin\theta_m & \cos\theta_m \end{pmatrix}$$

Hence by replacing θ_i by $t\theta_i$, we can get the rotation from identity continuously by changing t from 0 to 1. □

The ***Galilean group*** is the group of ***Galilean transformation***, characterising by a rotatory reflection $\boldsymbol{A} \in \mathrm{O}(3)$, a translation $\boldsymbol{x}_0 \in \mathbb{R}^3$ a velocity $\boldsymbol{v} \in \mathbb{R}^3$:

$$x := (ct, \boldsymbol{x}) \mapsto (ct, \boldsymbol{A}\boldsymbol{x} + \boldsymbol{x}_0 - \boldsymbol{v}t).$$

The Galilean group can be represented as:

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{x}_0 & \boldsymbol{v} \\ \boldsymbol{0}^{\mathrm{T}} & 1 & 0 \\ \boldsymbol{0}^{\mathrm{T}} & 0 & 1 \end{pmatrix} \tag{3-2}$$

§4 Curvature and Torsion

Let $\gamma\colon [t_1, t_2] \to \mathbb{R}^3; t \mapsto \boldsymbol{x}(t)$ be a diffeomorphism to a smooth curve in $\mathbb{R}^3$, the velocity and the acceleration are defined as:

$$\boldsymbol{v} := \dot{\boldsymbol{x}} := \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \gamma'(t), \qquad \boldsymbol{a} := \ddot{\boldsymbol{x}} := \frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2} = \gamma''(t).$$

Definition 4.1 (Natural parametre)**.** $s(t)$ is the ***natural parametre*** of the curve defined as:

$$s(t) = \int_0^t |\boldsymbol{v}| \,\mathrm{d}t.$$

We shall denote the velocity and the accelaration by $\boldsymbol{v}_s$ and $\boldsymbol{a}_s$, with natural parametre. To change the parametre, we only need:

$$\mathrm{d}s = |\boldsymbol{v}| \,\mathrm{d}t, \qquad \frac{\mathrm{d}}{\mathrm{d}s} = \frac{1}{|\boldsymbol{v}|}\frac{\mathrm{d}}{\mathrm{d}t}.$$

By differentiating $\boldsymbol{v}_s \cdot \boldsymbol{v}_s = 1$ we know the orthogonality of $\boldsymbol{a}_s$ and $\boldsymbol{v}_s$.

Define three unit vectors for a parameteriesed curve $t \mapsto \boldsymbol{x}(t)$: the tangent vector $\hat{\boldsymbol{t}} := \boldsymbol{v}/|\boldsymbol{v}|$, the (primary) normal $\hat{\boldsymbol{n}} := \frac{\mathrm{d}\hat{\boldsymbol{t}}}{\mathrm{d}s}/|\frac{\mathrm{d}\hat{\boldsymbol{t}}}{\mathrm{d}s}|$ and $\hat{\boldsymbol{b}} = \hat{\boldsymbol{t}} \times \hat{\boldsymbol{n}}$.

$\boldsymbol{v}_s = \hat{\boldsymbol{t}}$, $\hat{\boldsymbol{n}} = \hat{\boldsymbol{a}}_s/|\hat{\boldsymbol{a}}_s|$.

Definition 4.2 (Curvature and torsion)**.** The ***curvature*** κ is the norm of $\boldsymbol{a}_s$, and the torsion is defined as $\tau = \frac{\mathrm{d}\hat{\boldsymbol{n}}}{\mathrm{d}s}$.

Theorem 4.1 (Serret-Frenet formulae for space curve)**.**

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix} \hat{\boldsymbol{t}} \\ \hat{\boldsymbol{n}} \\ \hat{\boldsymbol{b}} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & \\ 0 & & \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{t}} \\ \hat{\boldsymbol{n}} \\ \hat{\boldsymbol{b}} \end{pmatrix}. \tag{4-1}$$

§5 Pseudo-Euclidean Spaces

$\mathbb{R}^4_{1,2}$ (or, more general case, $\mathbb{R}^n_{1,n-1}$) is called the Minkowski space. The metric is always denoted by $\eta_{\mu\nu}$, $\eta_{00} = 1$, $\eta_{ij} = -\delta_{ij}$, $i, j \neq 0$. (By convention, greek index take its value from 0 to $n-1$)

Considering the physics reality, we usually restrict our study to the region where $x_\mu x^\mu = \eta_{\mu\nu} x^\mu x^\nu \geq 0$, that is, the ***time-like*** region bounded by the ***light cone*** (or ***isotropic cone***).

A ***world-line*** is the curve $x(t) := (ct, \boldsymbol{x}(t))$, where $t := x^0/c$. The natural parametre of a world-line is called the ***proper time*** τ (not to be confused with the torsion).

The isomeries of $\mathbb{R}^4_{1,3}$ is called the ***Poincarér group***.

Theorem 5.1. *Any isometry of* $\mathbb{R}^4_{1,3}$ *is an affine transformation.*

Proof. We need to prove that if there points are collinear, the image of them under the isometry is also collinear. After a translation, which is affine, we can set one of the point to be the origin, and the remianing two points are in the same branch (both $t > 0$ or both $t < 0$).

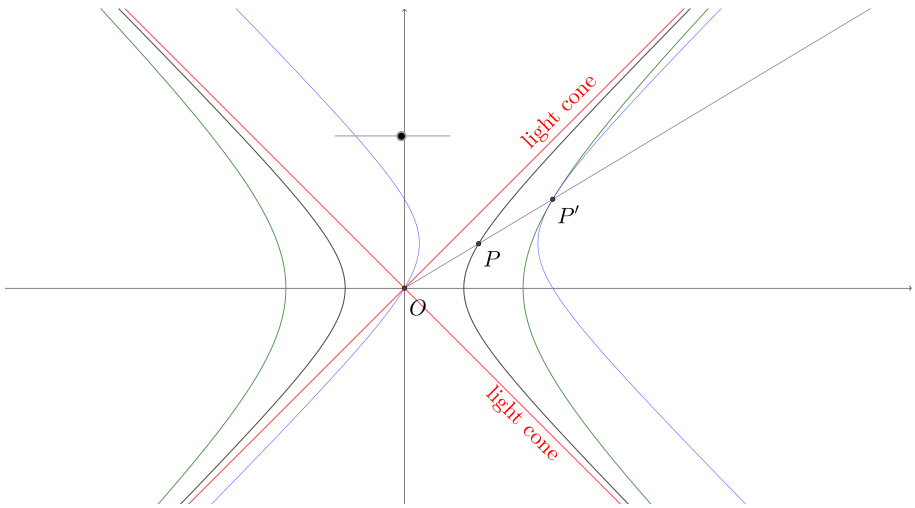

Figure 1.1: Pseudo-spheres in $\mathbb{R}^2_{1,1}$.

A pseudo-sphere centred at a point is a hyperboloid (of two sheets) centred at it and the principal axis is along the x^0-axis. Since the isometry preserve the distance, under which the points must move along the pseudo-sphere centred at origin. As illustrated in Fig. 1.1. If $x_\mu x^\mu = \ell$, and $y^\mu = \lambda x^\mu$, under an isometry, $x \mapsto x'$. The pseudo-sphere centred at x' of radius $(\lambda - 1)\ell$ and thepseudo-sphere centred at origin of radius $\lambda\ell$ are tangent at y', which along with x' and the origin are collinear. $\square$

Bibliography

[1] R.G. Burns et al. *Modern Geometry — Methods and Applications: Part I: The Geometry of Surfaces, Transformation Groups, and Fields*. Graduate Texts in Mathematics. Springer New York, 1991. ISBN: 9780387976631. URL: https://books.google.co.jp/books?id=FC0QFlx12pwC.

Symbol List

Here listed the important symbols used in this notes.

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