Differential Geometry

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Part I

Domestic Differential Geometry

Chapter 1

Manifolds

Chapter 2

Scalar and Vector Fields

§1 Scalar Fields

Definition 1.1 (Scalar Field)**.** Let M be a smooth manifold, $f \in C^{(\infty)}(M)$ is called a **scalar field**.

The scalar field over a manifold, form an algebra.

§2 Vector Fields

Definition 2.1 (vector field)**.** A **vector field** v over manifold M is a $C^{(\infty)}(M) \to C^{(\infty)}(M)$ map that satisfies

(a) $\forall f, g \in C^{(\infty)}(M),\ \forall \lambda, \mu \in \mathbb{R},\ v(\lambda f + \mu g) = \lambda v(f) + \mu v(g)$ (*linearity*).

(b) $\forall f, g \in C^{(\infty)}(M),\ v(fg) = v(f)g + fv(g)$

The space of all vector fields on M is denoted by $\text{Vect}(M)$

Definition 2.2 (tangent vector)**.** Let v be a vector field over M, p be a point on M. The tangent vector v_p at p is defined as a $C^{(\infty)}(M) \to C^{(\infty)}(M)$ map that satisfies

$$v_p(f) = v(f)(p). \tag{2-1}$$

The collection of tangent vectors at p is called the ***tangent space*** at p, denoted by $T_p M$.

The derivative of a path $\gamma \colon [0,1] \to M$ (or $\mathbb{R} \to M$) in a smooth manifold is defined as:

$$\gamma'(t) \colon C^{(\infty)}(M) \to \mathbb{R};$$
$$\gamma'(t)(f) = \frac{\mathrm{d}}{\mathrm{d}t} f \circ \gamma(t) \tag{2-2}$$

We can see that $\gamma'(t) \in T_{\gamma(t)} M$.

Let a path $\gamma \colon \mathbb{R}$ follows a vector field (a velocity field), that is

$$\gamma'(t) = v_{\gamma(t)}, \tag{2-3}$$

then we call γ the ***integral curve*** through $p := \gamma(0)$ of the vector field v.

Definition 2.3. Suppose v is an integrable vector field. Let $\varphi_t(p)$ be the point at time t on the integral curve through p.

$$\varphi_t \colon M \to M \tag{2-4}$$

is then called a ***flow*** generated by v.

$$\frac{\mathrm{d}}{\mathrm{d}t} \varphi_t(p) = v_{\varphi_t(p)}. \tag{2-5}$$

§3   Covariant and Contravariant

Definition 3.1 (pullback)**.** Let f be a scalar field over N, $\varphi \in C^{(\infty)}(M, N)$. Then the ***pullback*** of f by φ

$$\varphi^* \colon C^{(\infty)}(N) \to C^{(\infty)}(M), \tag{3-1}$$

is defined as

$$\varphi^* f = f \circ \varphi \in C^{(\infty)}(M). \tag{3-2}$$

Fields that are pullbacked are ***covariant*** fields.

Definition 3.2 (pushforward)**.** Let v_p be a tangent vector of M at p, $\varphi \in C^{(\infty)}(M, N)$, $q = \varphi(p)$. Then the ***pushforward*** of v_p by φ

$$\varphi_* \colon T_p M \to T_q N, \tag{3-3}$$

is defined as

$$(\varphi_* v)_q(f) = v_p(\varphi^* f). \tag{3-4}$$

Note that the pushforward of a vector field can only be obtained when φ is a diffeomorphism.

Fields that are pushforwarded are ***contravariant*** fields.

Mathematicians and physicists might have disagreement on whether a tangent vector is covariant or contravariant. This is because of that physicists might consider the coordinates (v^μ) of a tangent vector as a vector field, instead of linear combination of bases ∂_μ.

§4   Components of Vector Fields

Let $\varphi \colon U \to \mathbb{R}^n$ be a chart of M ($U \subset M$).

Let $p \in U$, $\varphi(p) = x = (x^\mu)$ ($\mu = 0, \dots, n-1$). Locally, a function $f \in C^{(\infty)}(M)$ can be written as

$$(\varphi^{-1})^* f = f \circ \varphi^{-1} \colon M \to \mathbb{R}, \tag{4-1}$$

and a vector field $v \in \mathrm{Vect}(M)$ can be written as

$$(\varphi_* v)_x = \varphi_* v_p \colon C^{(\infty)}(\mathbb{R}^n) \to \mathbb{R}, \tag{4-2}$$

or

$$\varphi_* v \in \mathrm{Vect}(\mathbb{R}^n) \tag{4-3}$$

Since $T_x\mathbb{R}^n \cong \mathbb{R}^n$ is a linear space, one can find a basis for $T_x\mathbb{R}^n$ as

$$\partial_\mu \colon C^{(\infty)}(\mathbb{R}^n) \to C^{(\infty)}(\mathbb{R}^n), \tag{4-4}$$

and $(\varphi_* v)_x = v^\mu(x)\partial_\mu$.

Pushing forward $v^\mu(x)\partial_\mu$ by φ^{-1} one obtains v.

In an abuse of symbols, one may just omit the pullback and pushforward, and refer to the f and v by $(\varphi^{-1})^* f$ and $\varphi_* v$.

Consider another chart $\psi \colon U \to \mathbb{R}^n$ of M, and

$$y = \psi(p), \quad (\psi_* v)_x = u^\mu \partial_\mu, \tag{4-5}$$

where we have chosen the same basis in $T_y\mathbb{R}^n$ as in $T_x\mathbb{R}^n$.

We would like to know how to relate v^μ and u^μ i.e. we want to know how the components of v transforms under a coordinate transformation $\tau = \psi \circ \varphi^{-1}$.

Consider any $f \in C^{(\infty)}(M)$,

$$v(f) = \varphi_* v((\varphi^{-1})_* f) = \psi_* v((\psi^{-1})_* f) \tag{4-6}$$

$\Rightarrow$

$$u^\mu \partial_\mu(f \circ \psi^{-1}) = v^\mu \partial_\mu(f \circ \varphi^{-1}) = v^\mu \partial_\mu(f \circ \psi^{-1} \circ \tau) = v^\mu \tau'^\nu_{\ \mu} \partial_\nu(f \circ \psi^{-1}) \tag{4-7}$$

$\Rightarrow$

$$u^\mu = v^\nu \tau'^\mu_{\ \nu}, \tag{4-8}$$

where

$$\tau'^\mu_{\ \nu} = \frac{\partial y^\mu}{\partial x^\nu}. \tag{4-9}$$

§5 Lie Bracket

Definition 5.1 (Lie bracket)**.** Let $v, w \in \text{Vect}(M)$, then the ***Lie bracket*** of v and w is defined as

$$[v, w] \colon C^{(\infty)}(M) \to C^{(\infty)}(M); \ f \mapsto v \circ w(f) - w \circ v(f). \tag{5-1}$$

The Lie bracket is an antisymmetric bilinear map[1], and an important property of the Lie bracket is the Leibniz rule:

$$[v,w](fg) = [v,w](f)g + f[v,w](g). \qquad (5\text{-}2)$$

Another important property of the Lie bracket is the Jacobi identity:

$$[v,[w,u]] + [w,[u,v]] + [u,[v,w]] = 0. \qquad (5\text{-}3)$$

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[1]Note that it is not $C^{(\infty)}$-linear

Chapter 3

Differential Forms

§6   1-forms

Definition 6.1 (1-form)**.** A ***1-form*** ω on M is a $\mathrm{Vect}(M) \to C^{(\infty)}(M)$ which satisfies that

(a) $\forall v, w \in \mathrm{Vect}(M)$, $\forall f, g \in C^{(\infty)}(M)$,

$$\omega(fv + gw) = f\omega(v) + g\omega(w). \tag{6-1}$$

The space of all 1-forms on M is denoted as $\Omega^1(M)$, which is a module over $C^{(\infty)}(M)$.

The operator d, when given a $C^{(\infty)}(M)$ function (which is called a ***0-form***), would give a 1-form:

$$(\mathrm{d}f)(v) = v(f). \tag{6-2}$$

This is called the ***exterior derivative*** or ***differential*** of f.

The ***cotangent vector*** or ***covector*** is similar as the tangent vector:

$$\omega_p(v_p) = \omega(v)(p). \tag{6-3}$$

The space of cotangent vectors at p on M is denoted by T_p^*M.

1-forms are covariant, that is, if $\varphi \colon M \to N$, then the pushforward of a 1-form ω by φ is

$$(\varphi^*\omega)_p(v_p) = \omega_q(\varphi_* v_p), \tag{6-4}$$

where $\varphi(p) = q$.

Theorem 6.1. *$f \in C^{(\infty)}(N)$, $\varphi \colon M \to N$ is differential, then*

$$\varphi^*(\mathrm{d}f) = \mathrm{d}(\varphi^*f). \tag{6-5}$$

§7 Components of 1-Forms

Let $\varphi \colon U \to \mathbb{R}^n$ be a chart of M ($U \subset M$).

Let $p \in U$, $\varphi(p) = x = (x^\mu)$ ($\mu = 0, \ldots, n-1$). Locally a 1-form $\omega \in \Omega^1(M)$ can be written as

$$(\varphi^{-1})^*\omega \in T_x^*\mathbb{R}^n. \tag{7-1}$$

A natural way to impose a basis $\mathrm{d}x^\mu$ in $T_x^*\mathbb{R}^n$ is

$$\mathrm{d}x^\mu(\partial_\nu) = \delta_\nu^\mu, \tag{7-2}$$

and $(\varphi^{-1})^*\omega = \omega_\mu(x)\,\mathrm{d}x^\mu$.

Now by the definition of 1-form:

$$\omega_\mu\,\mathrm{d}x^\mu(v^\nu \partial_\nu) = v^\nu \omega_\mu \delta_\nu^\mu = v^\mu \omega_\mu. \tag{7-3}$$

By the transformation rule of components of a vector, one have

$$\tau'^\nu_\mu \alpha_\nu = \omega_\mu, \tag{7-4}$$

where $\psi \colon U \to \mathbb{R}^n$, $(\psi^{-1})_*\omega = \alpha_\mu\,\mathrm{d}x^\mu$, $\tau = \psi \circ \varphi^{-1}$.

§8 k-**Forms**

Definition 8.1. If we assign an antisymmetric multilinear k-form $\omega_p \in \bigotimes_{i \in k} T_p^* M$ to each point $p \in M$, we say we have a k-***form*** on M.

The collection of all k-forms is denoted by $\Omega^k(M)$, and $\Omega(M) := \bigcup_{k \in \mathbb{N}} \Omega^k(M)$.

Theorem 8.1 (Dimension of forms). *If M is an nD manifold, then the dimension of $\Omega^k(M)$ is $\frac{n!}{k!(n-k)!}$ ($k \leq n$), and 0 for $k > n$; The dimension of $\Omega(M)$ is 2^n.*

Definition 8.2 (Wedge product). The ***wedge product*** $\wedge$ is defined as a binary operater that takes a k-form and ℓ-form and gives a $(k + \ell)$-forms, satisfying $\forall \alpha \in \Omega^k(M)$, $\forall \beta \in \Omega^\ell(M)$:

(a) (Associativity) $\forall \gamma \in \Omega^m(M)$,
$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma). \tag{8-1}$$

(b) (Supercommutativity)
$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha. \tag{8-2}$$

(c) (Distributiveness) $\forall \gamma \in \Omega^\ell(M)$,
$$\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma. \tag{8-3}$$

(d) (Bilinearity over $C^{(\infty)}(M)$) $\forall f \in C^{(\infty)}(M)$,
$$(f\alpha) \wedge \beta = f(\alpha \wedge \beta). \tag{8-4}$$

(e) (Naturality) If $\varphi \colon M \to N$ is a smooth map, then the pullback of a form by φ can be given by repeatingly applying ($\forall \gamma \in \Omega^\ell(M)$)
$$\begin{aligned} \varphi^*(\beta + \gamma) &= \varphi^*\alpha + \varphi^*\beta \\ \varphi^*(\alpha \wedge \beta) &= \varphi^*\alpha \wedge \varphi^*\beta, \end{aligned} \tag{8-5}$$

while the pullback of a 0-form and a 1-form agree with what we have already defined before.

By convention if $f \in C^{(\infty)}(M)$ then

$$f \wedge \omega =: f\omega. \tag{8-6}$$

It can be shown that any k-form ω can be written as

$$(\varphi^{-1})^* \omega = \frac{\omega_{\mu_1 \cdots \mu_k}}{n!} \bigwedge_{i=1}^{k} dx^{\mu_i}, \tag{8-7}$$

where $\varphi \colon M \to \mathbb{R}^n$ is a chart.

§9 Exterior Derivative

Definition 9.1 (Exterior derivative). The ***exterior derivative*** d is defined as a linear operator that takes a k-form and gives a $(k+1)$-form, satisfying $\forall \alpha \in \Omega^k(M)$, $\forall \beta \in \Omega^\ell(M)$:

(a) (Linearity) $\forall \lambda, \mu \in \mathbb{R}$, $\forall \gamma \in \Omega^\ell(M)$,

$$d(\lambda\beta + \mu\gamma) = \lambda\, d\alpha + \mu\, d\beta. \tag{9-1}$$

(b) (Leibniz rule)

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta. \tag{9-2}$$

(c)
$$d^2 \omega = 0. \tag{9-3}$$

(d) (Naturality) If $\varphi \colon M \to N$ is a smooth map, then

$$\varphi^* \, d\omega = d\varphi^* \omega. \tag{9-4}$$

Chapter 4

Metric

§10   Pseudo-Riemannian Metric

Definition 10.1 (Psedudo-Riemannian metric)**.** Let M be a manifold. A **_pseudo-Riemannian metric_** or simply **_metric_** g on a manifold M is a field $(g \in \Gamma(T^*M \otimes T^*M))$ that $\forall p \in M$,

$$g_p \colon T_p^*M \times T_p^*M \to \mathbb{R}, \tag{10-1}$$

is a bilinear form satisfying the following properties:

(a) (Symmetry) $\forall u, v \in T_pM$,

$$g_p(u, v) = g_p(v, u). \tag{10-2}$$

(b) (Non-degenerate)

$$u \mapsto g_p(u, -) \colon T_pM \to T_p^*M \tag{10-3}$$

is an isomorphism.

(c) (Bilinearity) $\forall p \in M$, $\forall u, v \in T_pM$, $\forall \lambda, \mu \in \mathbb{R}$,

$$g_p(\lambda u + \mu v, w) = \lambda g_p(u, w) + \mu g_p(v, w). \tag{10-4}$$

(d) (Smoothness) If $v, u \in \text{Vect}(M)$, then

$$p \mapsto g_p(v_p, u_p) \in C^{(\infty)}(M). \tag{10-5}$$

Given a metric, $\forall p \in M$, we can always find an orthonormal basis $\{e_\mu\}$ of $T_p M$ such that

$$g_p(e_\mu, e_\nu) = \text{sign}(\mu)\delta_{\mu\nu}, \tag{10-6}$$

where $\text{sign}(\mu) = \pm 1$. Conventionally we order the basis such that $\text{sign}(\mu) = 1$ for $\mu \in s$ and $\text{sign}(\mu) = -1$ for $\mu - s \in n - s$, and say that the metric has **signature** $(s, n - s)$.

If $\gamma \colon [0, 1] \to M$ is a smooth path and $\forall t, s \in [0, 1]$,

$$g(\gamma'(t), \gamma'(t))g(\gamma'(s), \gamma'(s)) \geq 0, \tag{10-7}$$

then we can define the arclength of γ as

$$\int_0^1 \sqrt{|g(\gamma'(t), \gamma'(t))|}\, \mathrm{d}t \tag{10-8}$$

if the integral converges.

The metric gives an **inner product** on $\text{Vect}(M)$:

$$\langle u, v \rangle := g(u, v). \tag{10-9}$$

The metric also gives a way to relate a vector field v to a 1-form ω. If v and ω satisfies: $\forall u \in \text{Vect}(M)$,

$$g(v, u) = \omega(u), \tag{10-10}$$

then we say that v is the corresponding vector field of ω, and ω is the corresponding 1-form of v.

We can also define the **inner product** on $\Omega^1(M)$ by

$$\langle \alpha, \beta \rangle = \langle a, b \rangle, \tag{10-11}$$

where a and b is the corresponding vector fields of α and β.

The ***inner product*** on $\Omega^k(M)$ is defined by induction with

$$\langle \bigwedge_{i\in k} \alpha_i, \bigwedge_{i\in k} \beta_i \rangle = \det((\langle \alpha_i, \beta_j \rangle)_{i,j\in k}. \tag{10-12}$$

Hence, if $\{e_\mu\}$ is an orthonormal basis (field) of T_pM, while the corresponding covectors are $\{f^\mu\}$ ($f^\mu(e_\nu) = \delta^\mu_\nu$) then

$$\langle \bigwedge_{i\in k} f^{\mu_i}, \bigwedge_{i\in k} f^{\mu_i} \rangle = \prod_{i\in k} \operatorname{sign}(\mu_i). \tag{10-13}$$

Specially, when $f, g \in \Omega^0(M) = C^{(\infty)}(M)$,

$$\langle f, g \rangle = fg. \tag{10-14}$$

§11 Volume Form

Notice that if M is an nD manifold, $\dim \Omega^n(M) = 1$, meaning at $p \in M$, $\{\omega_p \mid \omega \in \Omega^n(M)\}$ can be labelled by a parametre $\lambda_p \in \mathbb{R}$. If we have a basis $\{f^\mu\}$ of T_p^*M (or corresponding vectors $\{e_\mu\}$), then

$$\{\omega_p \mid \omega \in \Omega^n(M)\} = \lambda_p \bigwedge_{\mu\in n} f^\mu. \tag{11-1}$$

If there were another basis $\{g^\mu\}$ of T_p^*M (or corresponding vectors $\{h_\mu\}$), and the transformation between the two bases is given by

$$Pe^\mu = f^\mu, \tag{11-2}$$

where $P \in \operatorname{Aut}(T_p^*M)$. When $\det P > 0$, we say that $\{f^\mu\}$ and $\{g^\mu\}$ have the same ***orientation***.

Definition 11.1 (Volume form)**.** Let M be an orientable manifold. If $\forall p \in M$, we find an oriented orthonormal basis $\{f_\mu\}$ of T_p^*M at point p, then the ***volume form*** vol is defined by

$$\bigwedge_{\mu\in n} f_\mu = \operatorname{vol}_p. \tag{11-3}$$

§12 Hodge Star Operator

Definition 12.1 (Hodge Star Operator)**.** Let M be an orientable manifold. The ***Hodge star operator*** $\star$ is defined by the linear map

$$\star \colon \Omega^k(M) \to \Omega^{n-k}(M), \tag{12-1}$$

$\forall \alpha, \beta \in \Omega^k(M)$,

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \operatorname{vol}. \tag{12-2}$$

We call $\star\omega$ the ***dual*** of ω.

The special case is when $k = 0$,

$$\star f = f \operatorname{vol}, \tag{12-3}$$

and $k = n$,

$$\star(f \operatorname{vol}) = f \prod_{\mu \in n} \operatorname{sign}(\mu) = (-1)^{n-s} f \tag{12-4}$$

if the signature of the metric is $(s, n - s)$.

§13 Metric and Coordinates

Chapter 5

De Rham Theory

Chapter 6

Bundles and Connections

Chapter 7

Curvature

Chapter 8

Pseudo-Riemannian Geometry

bibliography

[1]   Javier P. Muniain John C. Baez. *Gauge Fields, Knots and Gravity (Series on Knots and Everything)*. Series on Knots and Everything. World Scientific Publishing Company, 1994. ISBN: 9789810217297,9810217293,9810220340.

Symbol List

Here listed the important symbols used in these notes

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