Analysis

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preface

The latest version: https://github.com/HoyanMok/NotesOnMathematics/tree/master/Analysis
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Contents

Part I

Mathematical Analysis

Chapter 1

Metric Space and Continuous Map

§1 Metric Space

Definition 1.1 (Metric)**.** A function
$$d\colon X^2 \to \mathbb{R}$$
$\forall x, y, z \in X$ satisfying:

a) $d(x,y) = 0 \leftrightarrow x = y$;
b) $d(x,y) = d(y,x)$ (symmetry);
c) $d(x,z) \le d(x,y) + d(y,z)$ (triangle inequality),

is called a ***metric*** or ***distance*** in X. Such X is said to be equiped with a metric d, (X,d) is called a ***metric space***. If the metric defined over X is definite, we just simply call the X the metric space.

Some examples:

- We can define $\mathbb{R}^n_p := (\mathbb{R}^n, d_p)$, where
$$d_p(x,y) := \left(\sum_{i\in n} \left|x^i - y^i\right|^p\right)^{1/p}, \tag{1-1}$$
while
$$d_\infty(x,y) := \max_{i\in n} \left|x^i - y^i\right|. \tag{1-2}$$

- Similarly we can define metric spaces as $(C[a,b], d_p)$ or simplified $C_p[a,b]$.
$$d_p(f,g) = \left(\int_a^b \left|f-g\right|^p \mathrm{d}x\right)^{1/p}. \tag{1-3}$$
while $C_\infty[a,b]$ is called a ***Chebyshev metric***, where the metric is defined as $d_\infty(f,g) := \max_{x\in[a,b]} |f(x) - g(x)|$.

- On equivalence class $\tilde{\mathfrak{R}}[a,b]$ over $\mathfrak{R}[a,b]$ similar metric can be defined. Functions are considered equicalent if they are equal up to a null set.

Lemma 1 (Quadruple inequality)**.** *Let (X, d) be a metric space.*

$$\forall a, b, u, v \in X, \ \left|d(a,b) - d(u,v)\right| \leq d(a,u) + d(b,v) \tag{1-4}$$

Proof. Without loss of generality, we assume that $d(a,b) > d(u,v)$. According to the triangle inequality (see def. 1.1), $d(a,b) \leq d(a,u) + d(u,v) + d(v,b)$, which is to prove. □

Definition 1.2 (δ-ball)**.** Let (X,d) be a metric space, and $\delta \in \mathbb{R}_+$, $a \in X$. A set

$$B(a;\delta) = \{x \in X \mid d(a,x) < \delta\}$$

is then called a ***ball*** with a centre at $a \in X$ and a radius of δ, or a ***ball*** of point a.

Definition 1.3 (Open set)**.** An ***open set*** $G \in 2^X$ in a metric space (X,d) is a set that satisfies: $\forall x \in G$, $\exists \delta \in \mathbb{R}_+$, s.t. $B(X,\delta) \in 2^G$.

Definition 1.4 (Closed set)**.** A ***closed set*** $F \in 2^X$ in a metric space (X,d) is a set that satisfies: $X - F$ is an open set in (X,d).

A ***closed ball*** $\overline{B}(X,\delta) := \{x \in X \mid d(a,x) \leq r\}$ is an example of closed sets in (X,d).

Proposition 1. a) An infinite union of open sets is an open set.
b) A definite intersection of open sets is an open set.
c) A definite union of closed sets is a closed set.
d) An infinite intersection of closed sets is a closed set.

Proof. Let $\forall \alpha \in A$, G_α be open sets.

a) $\forall x \in \bigcup_{\alpha\in A} G_\alpha$, $\exists \alpha \in A$ s.t. $x \in G_\alpha$. Since G_α is open, $\exists \delta \in \mathbb{R}_+$ s.t. $B(X,\delta) \subset G_\alpha \subset \bigcup_{\alpha\in A} G_\alpha$.

b) Let G_1, G_2 be open sets in (X,d). $\forall a \in G_1 \cap G_2$, $\exists \delta_1, \delta_2 \in \mathbb{R}_+$ s.t. $B(a;\delta_1) \subset G_1$, $B(a;\delta_2) \subset G_2$. Without loss of generality, let $\delta_1 \geq \delta_2$, therefore $a \in B(a;\delta_1) \cap B(a;\delta_2) = B(a;\delta_2) \subset G_1 \cap G_2$.

c) Just consider $\complement_X \left(\bigcap_{\alpha\in A} F_\alpha\right) = \bigcup_{\alpha\in A} \complement_X(F_\alpha)$ and a).

d) Similarly, $\complement_X (F_1 \cup F_2) = \complement_X(F_1) \cap \complement_X(F_2)$.

□

Definition 1.5 (Neighbourhood)**.** If $x \in X$ is an element of an open set, then such open set is called a ***neighbourhood*** of point x in X, denoted by *[Ux]*$U(x)$. The collection of all neighbourhoods of x can be denoted by $\mathscr{U}(x)$.

Definition 1.6 (Interior point)**.** Let $x \in X$, $E \subset X$.
a) If $\exists U(x) \subset E$, x is called an ***interior point*** of E.
b) If $\exists U(x) \subset X - E$, x is called an ***exterior point*** of E.
c) If x isn't an interior point nor exterior point of E, it is called a ***boundary point*** of E. The set of boundary points is called ***boundary***, denoted by *[partialE]*∂E.

Definition 1.7 (Limit point)**.** $a \in X$, $E \subset X$. If $\forall U(a)$, card$\left(E \cap U(a)\right) = \infty$, a is called a ***limit point*** of E.

Definition 1.8 (Closure)**.** The intersections of $E \subset X$ and set of all its limit points is called the ***closure*** of E, denoted by $\overline{E}$.

Theorem 1.1. *Let $F \in 2^X$. F is a closed set in X $\leftrightarrow$ $\overline{F} = F$.*

Proof. $\rightarrow$: $\complement_X(F)$ is open, hence its elements are all its interior points. Therefore $\overline{F} - F = \overline{F} \cup \complement_X(F) = \varnothing$, also we know that $F \subset \overline{F}$, hence $F = \overline{F}$.

$\leftarrow$: $F = \overline{F}$ means that $\forall x \in \complement_X(F)$, x is not a boundary of F, which implies that x is an interior point of $X - F$. Therefore $X - F$ is open while F is closed. □

Theorem 1.2. *$\overline{E}$ is always closed.*

Proof. $\forall x \in X - \overline{E}$, since it is not an element of the set E nor its limit points, $\exists U(x)$ s.t. $U(x) \cap \overline{E} = \varnothing$, which implies that x is an extorior point of E, therefore $\overline{E}$ is closed. □

Theorem 1.3. $\overline{E} = \overline{\overline{E}}$.

Proof. Since $\overline{E}$ is closed, its complement is open, which implies that its elements are all exterior points of $\overline{E}$, therefore $\overline{E}$ has contained all of its limit points. □

Definition 1.9. (Metric subspace) We called $(X'; d')$ a ***subspace*** of (X, d) when $X' \subset X$ and $\forall x, y \in X'$, $d'(x, y) = d(x, y)$.

§2 Topological Space

Definition 2.1 (Topology)**.** We say X is epuiped with a ***topology*** if we assigned a $\mathscr{T} \subset 2^X$, with the following propoties:

a) $\varnothing \in \mathscr{T}$; $X \in \mathscr{T}$.

b) $\big(\forall \alpha \in A, G_\alpha \in \mathscr{T}\big) \rightarrow \bigcup_{\alpha \in A} G_\alpha \in \mathscr{T}$.

c) $\forall G_1, G_2 \in \mathscr{T}$, $G_1 \cap G_2 \in \mathscr{T}$.

We call $(X, \mathscr{T})$ a ***topological space***, and sometimes we might simply call X the topological space.

These conditions is the intrinsic propoties of the open sets we have defined in the metric space[1]. The topology consisting of all the open sets defined in the metric space $(\mathbb{R}; d_2)$ is called the ***standard topology*** of the n-dimension Euclidean space.

Definition 2.2 (Open set)**.** Topology $\mathscr{T}$'s elements are called ***open sets***, and their complements are called ***closed sets***.

Definition 2.3 (Base)**.** Let $(X, \mathscr{T})$ be a topological space, and $\mathfrak{B} \subset 2^X$. If $\forall G \in \mathscr{T}$, $\exists \{B_\alpha\}_{\alpha \in A} \in 2^{\mathfrak{B}}$ s.t. $\bigcup_{\alpha \in A} B_\alpha = G$, we called $\mathfrak{B}$ a (topological or open) ***base*** of the topology $\mathscr{T}$.

Definition 2.4 (Weight)**.** The smallest possible cardinity of a base of a topology is called the ***weight*** of the topological space.

Definition 2.5 (Neighbourhood)**.** If $x \in G$ and $G \in \mathscr{T}$, then G is a ***neighbourhood*** of x in topological space $(X, \mathscr{T})$.

[1] See proposition 1

For example, we define an equivalence relation $\sim$ in $C(\mathbb{R};\mathbb{R})$. If $f, g \in C(\mathbb{R};\mathbb{R})$, at point $a \in \mathbb{R}$:

$$f \sim_a g \ \leftrightarrow\ \exists U(a)\big(\forall x \in U(a),\ f(x) = g(x)\big)\,. \tag{2-1}$$

By collecting all of the continuous functions that are euivalent to f, we call f define a ***germ*** at point a, denoted by f_a. If $f \in C(\mathbb{R};\mathbb{R})$ is defined in $U(a)$, then we can call $\{f_x \mid x \in U(a)\}$ a neighbourhood of germ f_a. Class of neighbourhoods of each f_x constructs a base of topological space $(C(\mathbb{R};\mathbb{R});\mathscr{T})$, where $\mathscr{T}$ is made of the sets of germs of continuous function in $C(\mathbb{R};\mathbb{R})$.

Definition 2.6 (Hausdorff space)**.** We call a topological space $(X, \mathscr{T})$ a ***Hausdorff space***, ***separated space*** or T_2 ***space***, if $\forall x, y \in X,\ x \neq y \to \big(\exists U(x), U(y) \text{ s.t. } U(x) \cap U(y) = \varnothing\big)$[2].

Definition 2.7 (Dense set)**.** $E \subset X$ is a ***dense set*** in the topological space $(X, \mathscr{T})$, if $\forall x \in X$, $\forall U(x)$, $U(x) \cap E \neq \varnothing$.

Definition 2.8 (Separable)**.** If there is a *countable* dense set in topological space $(X, \mathscr{T})$, then $(X, \mathscr{T})$ is ***separable***.

We can also define interior points, exterior points, boundary points, limit points in topological space as in metric space.

Definition 2.9 (Topological subspace)**.** Each subset Y of X equiped with topology $\mathscr{T}$ can be given a ***subspace topology*** $\mathscr{T}_Y$ whose elements G_Y are intersections of the subset with an open set G in $(X, \mathscr{T})$ i.e. $\forall G_Y \in \mathscr{T}_Y$, $\exists G \in \mathscr{T}$ s.t. $G_Y = G \cap Y$. Subsets equiped with such topology construct a ***topological subspace*** $(Y, \mathscr{T}_Y)$.

If two topology $\mathscr{T}_1, \mathscr{T}_2$ are defined on the same X, $\mathscr{T}_1$ is said to be ***stronger*** than $\mathscr{T}_2$ if $\mathscr{T}_1 \subsetneqq \mathscr{T}_2$.

§3 Compact Set

Definition 3.1 (Open cover)**.** Let $(X, \mathscr{T})$ be a topological space, $K \in 2^X$ and $\mathit{\Omega} \in 2^{\mathscr{T}}$. We call $\mathit{\Omega}$ to be an ***open cover*** over K, if $K \subset \cup\mathit{\Omega}$. If there are two open covers $\mathit{\Omega}$, $\mathit{\Omega}'$ over K, and $\mathit{\Omega}' \subset \mathit{\Omega}$, we say that $\mathit{\Omega}'$ is a ***subcover*** of $\mathit{\Omega}$.

Definition 3.2 (Compact set)**.** A set $K \in 2^X$ in topological space $(X, \mathscr{T})$ is called a ***compact set*** if each of its open covers has a *finite* subcover.

Specially, $\varnothing$ is compact.

Theorem 3.1. *A set $K \subset X$ is compact in $(X, \mathscr{T})$* **iff** *K is compact in $(K, \mathscr{T}_K)$ itself.*

This theorem tells a truth that whether K is compact or not doesn't dependent on the topological space it's in. This fact can be easily proved: we just need to notice that every open set G_K in $(K, \mathscr{T}_K)$ is an intersection of an open set G in $(X, \mathscr{T})$ and K.

Theorem 3.2 (Compact $\to$ closed (Hausdorff))**.** *If K is compact in a Hausdorff space $(X, \mathscr{T})$*[3]*, then K is a closed set in $(X, \mathscr{T})$.*

[2] This definition is also called ***Hausdorff axiom*** or ***separation axiom***.

[3] See definition 2.6.

Proof. Let x_0 be a limit point of K, which means $\forall U(x_0)$,

$$\operatorname{card} U(x_0) \cap K \notin \mathbb{N}.$$

Assume that $x_0 \notin K$. In a Hausdorff space, $\forall x \in K - \{x_0\}$, $\exists U(x)$ s.t. $U(x) \cap U(x_0) = \varnothing$. Such $U(x)$ construct an open cover $\mathit{\Omega} = \{U(x) | x \in K\} \subset 2^K$. Since K is compact, $\exists \mathit{\Omega}' \subset \mathit{\Omega}$ s.t. $\operatorname{card} \Omega \in \mathbb{N}$.

$$(\cup \Omega') \cap U(x_0) = \left(\bigcup_{k=1}^{n} U_k\right) \cap U(x_0) = \bigcup_{k=1}^{n} (U_k \cap U(x_0)) = \varnothing\,.$$

Since $K \subset \cup\Omega'$, x_0 is an exterior point of K, which leads to a contradiction.

Hence $x_0 \in K$. $\overline{K} = K$. □

Theorem 3.3. *Each decreasing* **nested sequences** *of non-empty compact sets has a non-empty limit, i.e.* $\forall (K_n)_{n\in\mathbb{N}} \in \mathscr{P}(X)^{\mathbb{N}}$ *s.t.* $\forall n \in \mathbb{N}_+$, $K_n \supset K_{n+1} \wedge K_n \neq \varnothing \wedge$ *(K_n is compact):* $K_n \downarrow K \neq \varnothing$.

Proof. Assume that $K = \varnothing$. Compact subsets of K_1 are all colsed, while their complements are all open. An open cover $\mathit{\Omega}$ can be constructed as $\{K_1 - K_n \mid n \in \mathbb{N}_+\}$. Since K_1 is compact, there would be a finite subcover $\mathit{\Omega}' \subset \mathit{\Omega}$, notice that $(X - K_n)_{n\in\mathbb{N}}$ is also a nested sequence, there must be one single $X - K_{n_0} \in \mathit{\Omega}'$ that covers K_1, which means $K_{n_0} = \varnothing$ contradicting that $\forall n \in \mathbb{N}_+$, K_n is non-empty. □

Theorem 3.4. *A Closed subset F of a compact set K is also compact.*

Proof. If $\mathit{\Omega}_F \subset 2^K$ is an open cover of F. Notice that $K - F$ is open, $\mathit{\Omega} = (\cup \mathit{\Omega}_F) \cap \{K - F\}$ constructs an open cover over over K. Since K is compact there must be a finite cover $\mathit{\Omega}' \subset \mathit{\Omega}$ which obviously also covers over F. □

The following propoties of compact sets are about topological spaces induced from metric spaces.

Definition 3.3 (net)**.** (X, d) is a metric space, $E \in 2^X$. E is called an ***ε-net*** if $\forall x \in X, \exists e \in E$, $d(e, x) < \varepsilon$.

Theorem 3.5 (Finite ε-net exists)**.** *If (K, d) is a compact metric space, then $\forall \varepsilon \in \mathbb{R}_+$, $\exists$* finite *ε-net in (K, d).*

Proof. For each point $x \in K$, find it a $B(x, \varepsilon)$, of which an infinite cover Ω over K is made. Since K is compact, there exists a finite subcover $\mathit{\Omega}' = \{B(x_i, \varepsilon)\}_{i\in n}$ $(n \in \mathbb{N}_+)$. Therefore $\{x_i\}_{i\in n}$ is a finite ε-net in K. □

Theorem 3.6 (Sequentially compact)**.** *A metric space (K, d) is compact* **iff** *it is* **sequentially compact**, *that is, $\forall (x_n)_{n\in\mathbb{N}} \in K^{\mathbb{N}}$, it has a convergent subsequence $(x_{k_n})_{n\in\mathbb{N}}$ ($k_n \in \mathbb{N}$; $k_{n+1} > k_n$) whose limit $a \in K$.*

To prove Theorem 3.6, we need to prove two lemmata first.

Lemma 2. *If (K, d) is sequentially compact, then $\forall \varepsilon \in \mathbb{R}_+$, $\exists$ finite ε-net in (K, d).*

Proof. Assume that $\exists \varepsilon_0 \in \mathbb{R}_+$, there were no finite ε_0-net in (K, d). Define such sequence: $(x_n)_{n\in\mathbb{N}}$ s.t. $\forall n \in \mathbb{N}\ \forall k \in n$, $d(x_n, x_k) \geq \varepsilon_0$ (There would always be a next one since there exists no finite ε_0-net or $\{B(x_n; \varepsilon_0)\}_{n\in N}$ gives such). It has no convergent subsequence: if there were a $(x_{k_n})_{n\in\mathbb{N}}$ convergent to $a \in K$, $\exists N, M \in \mathbb{N}_+$, $d(x_N, x_M) \leq d(x_N, a) + d(x_M, a) \leq \varepsilon_0$, which lead to a contradictary. □

Lemma 3. *If (K, d) is sequentially compact then every nested sequence of closed non-empty sets $\{F_n\}_{n\in\mathbb{N}}$ in K have a non-empty intersection.*

Proof. Let $(x_{k_n})_{n\in\mathbb{N}}$ be a convergent subsequence of $(x_n)_{n\in\mathbb{N}}$, where $\forall n \in \mathbb{N}$, $x_n \in F_n$. Let a be the limit of $(x_{k_n})_{n\in\mathbb{N}}$.

Assume that $a \notin \bigcap_{n\in\mathbb{N}} F_n$, in a metric space, $\exists U(a) \in \mathscr{U}(a)$ s.t. $U(a) \cap \big(\bigcap_{n\in\mathbb{N}} F_n\big) = \varnothing$, therefore $U(a) \cap \big(\bigcap_{n\in\mathbb{N}} F_{k_n}\big) = \varnothing$. But this conflict the fact that $\exists N \in \mathbb{N}$, s.t. $n > N \to x_{k_n} \in U(a)$ while $x_{k_n} \in F_{k_n}$. □

Then we get back to the Theorem 3.6.

Proof. $\to$: If $\operatorname{card}\{x_n\}_{n\in\mathbb{N}} \in \mathbb{N}$, it is obvious; Now we let $\operatorname{card}\{x_n\}_{n\in\mathbb{N}} \notin \mathbb{N}$. We can always find finite $1/k$-net $\{B(a_{k,i}, 1/k)\}_{i\in m}$ (Theorem 3.5, $m \in \mathbb{N}$, $a_i \in K$), for all $k \in \mathbb{N}_+$. For each k, there must be at least one $B(a_{k,i_0}; 1/k)$ (for simplication, we denote a_{k,i_0} by a_k) that includes infinite elements in $(x_n)_{n\in\mathbb{N}}$. $\forall n \in \mathbb{N}_+$ (let $k_0 = 0$), select $x_{k_n} \in B(a_{n,0}; 1/n)$, and $\{\overline{B}(x_n; 1/k)\}$ is a nested sequence of a closed non-empty sets in sequentially compact K, (Lemma 3) $\lim_{n\to\infty} x_{k_n} \in K$.

$\leftarrow$: Assume that there were an open cover $\varOmega$ over K having no finite subcover, $\forall n \in \mathbb{N}_+$, $\exists$ finite $1/n$-net (Lemma 3), in which there would be at least one x_n whose $\overline{B}(x_n; \frac{1}{n})$ can't be covered finitely. Then $\overline{B}(x_n; 1/n) \downarrow B = \{a\}$ (Theorem 3.3) can't be finitely covered by any subcover of $\varOmega$, which means $\varOmega$ can't cover the whole K, leading to the contradiction. □

§4 Connected Set

Definition 4.1 (Connected space)**.** Topological space $(X, \mathscr{T})$ is called ***connected*** if there is no ***open-closed set*** (i.e. both open and closed) besides $\varnothing$ and X itself.

Notice that if $A \in 2^X$ is open-closed, its complement $X - A$ is also open-closed, which means a topological space is connected ***iff*** it is not a union of its two open subsets.

Definition 4.2 (Connected set)**.** Let $(X, \mathscr{T})$ be a topological space. Subset C is said to be ***connected*** if subspace $(C, \mathscr{T}_C)$ is connected.

Theorem 4.1. *Let $(X, \mathscr{T})$ be a topological space, and $\{C_\alpha\}_{\alpha\in A}$ be connected subsets of X. If $\bigcap_{\alpha\in A} C_\alpha \neq \varnothing$, then $\bigcup\limits_{\alpha\in A} C_\alpha$ is also connected.*

Proof. Assume that $C = \bigcup_{\alpha\in A} C_\alpha$ were not connected, $\exists E \in 2^C$ s.t. $E \neq \varnothing$, $E \neq C$ and $E, C - E \in \mathscr{T}_C$. For E is not empty there exists a $\beta \in A$ s.t. $E \cap C_\beta \neq \varnothing$.

Now we show that $C_\beta \subset E$. Suppose that $C_\beta \nsubseteq E$, which implies that $(C - E) \cap C_\beta \neq \varnothing$. $E, C - E, C_\beta \in \mathscr{T}_C$, by the definition of the topology, $E \cap C_\beta, (C - E) \cap C_\beta \in \mathscr{T}_C$. This conflicts to the fact that C_β is connected. Therefore $C_\beta \subset E$.

Hence, there exists a $B \subsetneqq A$, $\bigcup_{\beta\in B} C_\beta = A$. Since C_γ, $\gamma \in A - B$ would have a empty intersection with E, which contradicts $\bigcap_{\alpha\in A} C_\alpha \neq \varnothing$. □

Theorem 4.2. *Connected sets have connected closure.*

Proof. □

Theorem 4.3. *$C \subset \mathbb{R}$ is connected* **iff** $\forall x, z \in C \forall y \in \mathbb{R}(x < y < z) \to y \in C$.

Proof. $\to$: Assume that there were such $y \in \mathbb{R}$ that $\exists x, z \in C$, $x < y < z$ but $y \notin C$. $\{x \in C \mid x < y\}$ and $\{x \in C \mid x > y\}$ are open in C for they are intersection of open sets in $\mathbb{R}$ and C. Since they're each other's complement, they are both open-closed, which conflicts to the definition of a connected set.

$\leftarrow$: It can be proved that $(\inf C, \sup C) \subset C$. Assume that there were an open-closed proper subset $E \neq \varnothing$ contained in C. Find two points $x \in E$, $z \in C - E$. Without loss of generality, let $x < z$. Since E and $C - E$ are closed, $c_1 = \inf\big(E \cap [a, b]\big) \in E$ while $c_2 = \inf\big((C - E) \cap [a, b]\big) \in C - E$. However $E \cap (C - E) = \varnothing$, hence $c_1 < c_2$, which means $(c_1, c_2) \cap E = \varnothing$. Here's the contradiction. □

Definition 4.3 (Locally connected)**.** A topological space $(X, \mathscr{T})$ is said to be ***locally connected*** if $\forall x \in X$, $\exists U(x)$ s.t. $U(x)$ is connected.

§5 Complete Metric Spaces

We now take a closer look at one of the most important examples of metric spaces: complete spaces.

Definition 5.1 (Cauchy sequence)**.** A sequence $(x_n)_{n \in \mathbb{N}}$ of points in a metric space (X, d) is called a ***fundamental sequence*** or ***Cauchy sequence*** if $\forall \varepsilon \in \mathbb{R}_+$, $\exists N \in \mathbb{N}$ s.t. as long as $m, n > N$, $d(x_n, x_m) < \varepsilon$.

Definition 5.2 (complete space)**.** A metric space (X, d) is ***complete*** if any Cauchy sequence of its points is convergent.

For example, a metric space $C_\infty[a, b]$ is complete while $C_1[a, b]$ isn't. The proof see [1, p. 22].

Let us consider an incomplete space $\mathbb{Q}_1$, which is a subspace of the complete space $\mathbb{R}_1$. If $\mathbb{R}_1$ is the smallest complete space containing $\mathbb{Q}_1$, we can say that we have achieved a ***completion*** of $\mathbb{Q}_1$. However, the term "smallest" hasn't been properly defined yet.

Definition 5.3 (completion)**.** If a metric space (X, d) is a subspace of a complete metric space (Y, d) and everywhere dense in it, we call the latter one the ***completion*** of (X, d).

We need to confirm that such completion is the smallest and unique. So we introduce:

Definition 5.4 (isometry)**.** If there exists a ***isometry*** $f\colon X_1 \to X_2$ when (X_1, d_1) and (X_2, d_2) are both metric space, i.e. f is a bijective and $\forall a, b \in X_1$, $d_2\big(f(a), f(b)\big) = d_1(a, b)$, then these two metric spaces are ***isometric***.

This relation is reflexive (id_X), symmetric (f^{-1}), and transitive ($f \circ g$), so it is a equivalence relation, denoted by $\sim$. We shall consider isometric spaces as identical, when only discussing within metric topological topics.

Theorem 5.1. *If metirc spaces (Y_1, d_1) and (Y_2, d_2) are both completions of (X, d), then they are isometric.*

Proof. Between two completions such isometry $f\colon Y_1 \to Y_2$ can be defined: if $x_1, x_2 \in X$,

$$d_2\big(f(x_1), f(x_2)\big) = d\big(f(x_1), f(x_2)\big) = d(x_1, x_2) = d_1(x_1, x_2).$$

For each $y_1 \in Y_1 - X_1$, a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ can be found in the nested sequence of balls centered in y_1. It is obvious that $(x_n)_{n\in\mathbb{N}}$ is also fundamental in Y_2, limitting to $y_2 \in Y_2$.

Differently selected sequences of points $(x'_n)_{n\in\mathbb{N}}$ won't limit to a different y'_2, namely $d(x_n, x'_n)$ shall converge to 0, or the fact that the radii of balls converge to 0 would be violated.

Let $f(y_1) = y_2$.

a) For each $y_2 \in Y_2 - X$, there always exists a Cauchy sequence converging to it, which implies that f is a surjection.
b) On the other hand, we shall notice that $\forall y'_1, y''_1 \in Y_1 - X$,

$$d_1(y'_1, y''_1) = \lim_{n\to\infty} d(x'_n, x''_n) = d_2(y'_2, y''_2)$$

while $(x'_n)_{n\in\mathbb{N}}$ and $(x''_n)_{n\in\mathbb{N}}$ are both Cauchy sequence. This equality proved that f is a injection.

□

Theorem 5.2. *There always exists a completion for every metric space.*

Proof. Let $C_X := \{(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}} \mid \forall\varepsilon \in \mathbb{R}_+, \exists N \in \mathbb{N} \text{ s.t. } \forall n, m \in \mathbb{N}(n > N \wedge m > N \to d_X(x_n, x_m) < \varepsilon)\}$, namely the collections of Cauchy sequences in X.

We say two Cauchy sequences $(x_n)_{n\in\mathbb{N}}$, $(x'_n)_{n\in\mathbb{N}}$ are equivalent (or, we shall say in a complete space, that they have a same limit) if $\lim_{n\to\infty} d(x_n, x'_n) = 0$.

It can be easily proved that such relation is a equivalence relation, and it divides C_X into equivalence classes S.

$\forall (x_n)_{n\in\mathbb{N}}, (x'_n)_{n\in\mathbb{N}} \in C_X$, $\forall\varepsilon \in \mathbb{R}_+$, $\exists N \in \mathbb{N}$ s.t. $\forall n, m \in \mathbb{N}$, as long as $n > N$ and $m > N$ (by Lemma 1):

$$|d_X(x_n, x'_n) - d_X(x_m, x'_m)| \le d_X(x_n, x_m) + d_X(x'_n, x'_m) < 2\varepsilon\,.$$

Hence, $(d(x_n, x'_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}_1$. Since $\mathbb{R}_1$ is a complete space, $\lim_{n\to\infty} d(x_n, x'_n)$ always exists. This fact allows us to introduce[4]:

$$d\colon S^2 \to \mathbb{R};\ \big([(x_n)_{n\in\mathbb{N}}], [(x'_n)_{n\in\mathbb{N}}]\big) \mapsto \lim_{n\to\infty} d(x_n, x'_n)$$

A metric space (S_X, d) isometric to any given metric space (X, d_X) can be constructed, where $S_X := \{[(x)_{n\in\mathbb{N}}] \mid x \in X\}$.

Then we shall show that S is the completion of S_X.

Let $\big([(x^i_n)_{n\in\mathbb{N}}]\big)_{i\in\mathbb{N}}$ be a Cauchy sequence in S. By definition, for any $i \in \mathbb{N}_+$, there exists a N that is large enough such that as long as $j > N$, $k > N$, $d_X(x^i_j, x^i_k) < 1/i$. Choose $a^i := x^i_k$ for such $k > N$, so that $d\big([(a^i)_{n\in\mathbb{N}}], [(x^i_n)_{n\in\mathbb{N}}]\big) < 1/i$.

$\forall\varepsilon \in \mathbb{R}_+$, $\exists N \in \mathbb{N}$ (e.g. we can choose $N = \lfloor 4/\varepsilon \rfloor$) s.t. $\forall n, m \in \mathbb{N}$, $p > N \wedge q > N \ \to$

$$d\big([(x^p_n)_{n\in\mathbb{N}}], [(x^q_n)_{n\in\mathbb{N}}]\big) < \frac{\varepsilon}{2} \wedge d\big([(x^p_n)_{n\in\mathbb{N}}], [(a^p)_{n\in\mathbb{N}}]\big) < \frac{1}{p} \wedge d\big([(x^q_n)_{n\in\mathbb{N}}], [(a^q)_{n\in\mathbb{N}}]\big) < \frac{1}{q}\,,$$

[4]We implicitly use the (countable) axiom of choice: we must find a Cauchy sequence for each equivalence class.

therefore when p, q are great enough, (by the triangle inequality)

$$d\big([(a^p)_{n\in\mathbb{N}}],[(a^q)_{n\in\mathbb{N}}]\big) \le \frac{\varepsilon}{2}+\frac{1}{p}+\frac{1}{q} < \varepsilon\,.$$

So, $[(a^n)_{n\in\mathbb{N}}]$ is a Cauchy sequence, therefore it is an element of S.

By $\lim_{i\to\infty} d\big([(x_n^i)_{n\in\mathbb{N}}],[(a^n)_{n\in\mathbb{N}}]\big) = 0$, we found a limit for the arbitary Cauchy sequence $\big([(x_n^i)_{n\in\mathbb{N}}]\big)_{i\in\mathbb{N}}$ in S.

Finally, we have to check that S_X is everywhere dense in S. For any aribitary $[(x_n)_{n\in\mathbb{N}}] \in S$, $\forall\varepsilon$, we can always choose a $N \in \mathbb{N}$ great enough so that $[(x_N)_{n\in\mathbb{N}}] \in S_X \cap B\big([(x_n)_{n\in\mathbb{N}}],\varepsilon\big)$. Since every neighbourhood of $[(x_n)_{n\in\mathbb{N}}]$ contains a ball centred at it, we have proved that $\forall U \in \mathscr{U}([(x_n)_{n\in\mathbb{N}}])\big(U\cap S_X \ne \varnothing\big)$. □

Note: We have already seem such technique when we construct the real numbers from the sequences of rational numbers.

§6 Continuous Mapping

Let's recall the definition of the limitation.

Definition 6.1. A set $\mathscr{B} \subset 2^X$ is called a **(filter) base** in X if the following conditions hold:

a) $\varnothing \notin \mathscr{B}$.

b) $\forall B_1, B_2 \in \mathscr{B}$, $\exists B \in \mathscr{B}$ s.t. $B \subset B_1 \cap B_2 \subset B_2$.

Introduction of the limits in a topological space is as follows.

Definition 6.2. Let $a \in Y$ be the ***limit*** over the base $\mathscr{B} \subset 2^{\mathscr{D}(f)}$ of a mapping $f : \mathscr{D}(f) \to Y$, in which Y is epuiped with a topology $\mathscr{T}$.

$$\lim_{\mathscr{B}} f = a \quad := \quad \forall U(a) \subset Y\ \exists B \in \mathscr{B}(f(B) \subset U(a)).$$

Such definition is parallel to the definition we have introduced on the limits of real number, hence it basically holds the same propoties.

Definition 6.3. A mapping $f : X \to Y$, where X,Y is respectively equiped with topology $\mathscr{T}_X$,$\mathscr{T}_Y$, is said to be ***continuous*** at $x_0 \in X$ (let $y_0 = f(x_0) \in Y$), if $\forall U(y_0)$, $\exists U(x_0)$ s.t. $f(U(x_0)) \subset U(y_0)$. It is ***continuous*** in X if it is continuous at each point $x \in X$.

The set of continuous mappings from X into Y can be denoted by $C(X,Y)$ or $C(X)$ when Y is clear.

Theorem 6.1 (**Criterion for continuity**). *$(X,\mathscr{T}_X)$ and $(Y,\mathscr{T}_Y)$ are both topological spaces. A mapping $f : X \to Y$ is continuous* **iff** *$\forall G_Y \in \mathscr{T}_Y$, $f^{-1}(G_Y) \in \mathscr{T}_X$.*

Proof. $\Rightarrow$: It is obvious if $f^{-1}(G_Y) = \varnothing$. If $f^{-1}(G_Y) \ne \varnothing$ and $x_0 \in X$, since $f \in C(X,Y)$, for G_Y, $\exists U(x_0)$ s.t $f(U(x_0)) \subset G_Y$. Also notice that $f(U(x_0)) \subset G_Y \Rightarrow U(x_0) \subset f^{-1}(G_Y)$, therefore $f^{-1}(G_Y)$ is open.

$\Leftarrow$: $\forall x_0 \in X$, let $y_0 = f(x_0)$, $f^{-1}(U(y_0)) \in \mathscr{T}_X$. Notice that $x_0 \in f^{-1}(U(y_0))$, therefore $f \in C(X,Y)$. □

Definition 6.4. $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ are both topological spaces. A bijective mapping $f : X \to Y$ is a ***homeomorphism*** if $f \in C(X, Y) \wedge f^{-1} \in C(Y, X)$.

Definition 6.5. Two topological spaces $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ are said to be ***homeomorphic*** if there exists a homeomorphism $f : X \to Y$.

Homeomorphic topological spaces are identical with respect to their topological propoties since the theorem 6.1 has shown that their open sets correspond to each other.

Theorem 6.2. *$(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ are both topological spaces. $K \subset X$ is a compact set. If $f : X \to Y \in C(X, Y)$, then $f(K)$ is compact.*

Proof. For each open cover $\Omega_Y = \{G_Y \in \mathscr{T}_Y\} \subset \mathscr{T}_Y$ over $f(K)$, $f^{-1}(G_Y) \in \mathscr{T}_X$ (Therem 6.1). $f(K) \subset \cup\Omega_Y \Rightarrow K \subset f^{-1}(\cup\Omega_Y) = \cup\Omega_X$, where $\Omega_X = \{f^{-1}(G_Y) \mid G_Y \in \Omega_Y\}$ is an open cover over K. Since K is compact, $\exists\Omega'_X \subset \Omega_X\, (|\Omega'_X| \in \mathbb{N}_+ \wedge K \subset \cup\Omega'_X)$, $f(K) \subset f(\cup\Omega'_X)$. $f(G'_X) \in \Omega_Y$, hence $\Omega'_Y = \{f(G'_X) \mid G'_X \in \Omega'_X\}$ is a finite subcover over $f(K)$. □

Theorem 6.3. *$(K; \mathscr{T}_K)$ is a compact space and $(Y, \mathscr{T}_Y)$ is a Hausdorff space. If a bijective $f : K \to Y \in C(K, Y)$, then it is a homeomorphism.*

Proof. $\forall F = K - G$ s.t. $G \in \mathscr{T}_K$ is compact (Theorem **??**). Hence $f(F)$ is compact (Theorem 6.2), then it is also closed (Theorem **??**). This fact shows that f^{-1} is continuous (Theorem 6.1). □

Theorem 6.4. *$(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ are both topological spaces. $E \subset X$ is a connected set. If $f : X \to Y \in C(X, Y)$, then $f(E)$ is also connected.*

Proof. Only to notice that the open-closed sets in $(f(E); \mathscr{T}_{f(E)})$ have concurrently open-closed pre-images in $(E; \mathscr{T}_E)$. □

§7 Contraction

Definition 7.1. A point $a \in X$ is a ***fixed point*** of a mapping $f : X \to X$ if $f(a) = a$.

Definition 7.2. Let $(X; d)$ be a metric space. A mapping $f : X \to X$ is called a ***contraction*** if $\exists q \in (0, 1) \subset \mathbb{R}$ s.t. $\forall x_1, x_2 \in X$,

$$d(f(x_1), f(x_2)) \leq qd(x_1, x_2). \tag{7-1}$$

Lemma 4. *A contraction $f : X \to X$ is always continuous.*

Proof. $\forall x \in X$, $\forall\varepsilon \in \mathbb{R}_+$, $\exists\delta < \varepsilon/q$, according to inequality 7-1:

$$f(B(x; \delta)) \subset B(f(x); \varepsilon).$$

□

Theorem 7.1 (**Picard-Banach fixed-point principle** or **contraction mapping principle**). *Let $(X; d)$ be a complete metric space. Each contraction $f : X \to X$ has a unique fixed point a. Also, $\forall\{x_n\} \subset X$ s.t. $\forall n \in \mathbb{N}\, (f(x_n) = x_{n+1})$ then $\lim_{n\to\infty} x_n = a$, and*

$$d(x_n, a) \leq \frac{q^n}{1 - q} d(x_1, x_0). \tag{7-2}$$

Proof. By the inequality 7-1:

$$d(x_{n+1}, x_n) \leq qd(x_n, x_{n-1}) \leq \cdots \leq q^n d(x_1, x_0)$$

Therefore, $\forall n, k \in \mathbb{N}$,

$$d(x_{n+k}, x_n) \leq \sum_{i=0}^{k-1} d(x_{n+i+1}, x_{n+i}) \leq \sum_{i=0}^{k-1} q^{n+i} d(x_1, x_0) \leq \frac{q^n}{1-q} d(x_1, x_0), \tag{7-3}$$

which implies that x_n is a Cauchy sequence in a complete space $(X; d)$, hence it converges to a point $a \in X$.

To proof that a is a fixed point of f, since f is continuous (Lemma 4), just notice that

$$a = \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x_n).$$

If there were a second fixed point $a' \in X$ of f, then:

$$0 \leq d(a, a') = d(f(a), f(a')) \leq qd(a, a')$$

which can't be true unless $a = a'$.

By passing to the limit as $k \to \infty$ in the inequality 7-3, we have the inequality 7-2. □

Chapter 2

Normed Linear Space and Differential Calculus

§8 Normed Linear Space

Definition 8.1. Let V be a linear space over $\mathbb{R}$ or $\mathbb{C}$. A function $\| \ \| : X \to \mathbb{R}$ assigning to each vector $\boldsymbol{x} \in X$ a real number $\|x\|$ is called a ***norm*** in the linear space X if:

a) $\|\boldsymbol{x}\| = 0 \Leftrightarrow \boldsymbol{x} = \mathbf{0}$ (nondegeneracy);

b) $\|\lambda \boldsymbol{x}\| = |\lambda| \|\boldsymbol{x}\|$ (homogeneity);

c) $\|\boldsymbol{x}_1 + \boldsymbol{x}_2\| \leq \|\boldsymbol{x}_1\| + \|\boldsymbol{x}_2\|$ (the triangle inequality).

A linear space with a norm defined on it is called ***normed***.

Bibliography

[1] Vladimir A Zorich. *Mathematical analysis II; 2nd ed.* Universitext. Berlin: Springer, 2016. DOI: 10.1007/978-3-662-48993-2. URL: https://cds.cern.ch/record/2137923.

Symbol List

Here listed the important symbols used in this notes.

Index