Point Set Topology

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1 Topological Spaces and Continuous Mappings

1.1 Metric Space

Definition 1.1. function

$$d\colon X^2 \to \mathbb{R} \tag{1-1}$$

$\forall x_1, x_2, x_2 \in X$ satisfied:

a) $d(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$;

b) $d(x_1, x_2) = d(x_2, x_1)$ (symmetry);

c) $d(x_1, x_3) \leqslant d(x_1, x_2) + d(x_2, x_3)$ (Triangle inequality),

is called a ***metric*** or ***distance*** in X. Such X is said to be equiped with metric d, (X, d) is called a ***metric space***.

Some examples:

- $(\mathbb{R}^n; d_p)$, where $d_p(x_1, x_2) = \left(\sum_{i=1}^n \left|x_1^i - x_2^i\right|^p\right)^{1/p}$, while $d_\infty(x_1, x_2) = \max\limits_{1\leqslant i\leqslant n} \left|x_1^i - x_2^i\right|$.
- Similarly we can define metric spaces as $(C[a, b]; d_p)$ or $C_p[a, b]$. $d_p(f, g) = \left(\int_a^b |f - g|^p \,\mathrm{d}x\right)^{\frac{1}{p}}$. C_∞ is called a **Chebyshev metric**.
- On class $\tilde{\mathfrak{R}}[a, b]$ over $\mathfrak{R}[a, b]$ similar metric can be defined. Functions are considered of one same class if they are equivalent expect on a set not larger than null set.

Hilbert space denoted by $(\mathbb{H}; d)$ is defined as:

$$\mathbb{H} = \left\{ x = (x_1, x_2, \cdots) \mid \forall i \in \mathbb{Z}_+ \left(\forall x_i \in \mathbb{R} \wedge \sum_{i=1}^{\infty} x_1^2 < \infty \right) \right\} \tag{1-2}$$

equiped with a metirc d:

$$d\colon \mathbb{H}^2 \to \mathbb{R};\ x, y \mapsto \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}. \tag{1-3}$$

To justify this definition, we need to introduce a lemma:

Lemma 1.

$$\forall n \in \mathbb{Z} \forall \boldsymbol{u} \in \mathbb{R}^n \forall \boldsymbol{v} \in \mathbb{R}^n \left(\sum_{i=1}^{n} u_i v_i \leq \sqrt{\sum_{i=1}^{n} u_i^2} \sqrt{\sum_{i=1}^{n} v_i^2} \right) \tag{1-4}$$

*This is called **Schwarz inequality**.*

Proof. If $\forall i = 1, \cdots, n(v_i = 0)$ the equivalence has already been satisfied, therefore the following consider the situation that $\exists i \in \{1, \cdots n\}(v_i \neq 0)$. $\forall \lambda \in \mathbb{R}$

$$\sum_{i=1}^{n}(u_i + \lambda v_i)^2 = \sum_{i=1}^{n} u_i^2 + 2\lambda \sum_{i=1}^{n} u_i v_i + \lambda^2 \sum_{i=1}^{n} v_i^2 \geq 0$$

has at most one root. Hence $\Delta \leq 0$ will lead to the inequality 1-4. □

Apply this inequality to $\sum_{i=1}^{n}(u_i+v_i)^2 = \sum_{i=1}^{n} u_i^2+\sum_{i=1}^{n} v_i^2+2\sum_{i=1}^{n} v_i \sum_{i=1}^{n} u_i$ we can get

$$\sum_{i=1}^{n}(u_i + v_i)^2 \leq \sum_{i=1}^{n} u_i^2 + \sum_{i=1}^{n} v_i^2 + 2\sqrt{\sum_{i=1}^{n} u_i^2}\sqrt{\sum_{i=1}^{n} v_i^2} = \left(\sqrt{\sum_{i=1}^{n} u_i^2} + \sqrt{\sum_{i=1}^{n} v_i^2}\right)^2,$$

in which substitude u_i, v_i by $x_i - y_i, x_i + y_i$ will result in triangle inequality. The inequality holds as the n limits to $+\infty$.

Definition 1.2. Let (X, d) be a metric space, the ***distant*** between a non-empty set $\varnothing \neq A \in \mathscr{P}(X)$ and a point x is defined as:

$$d(A, x) = \inf\{d(x, y) \mid y \in A\},$$

and we let $d(x, A) = d(A, x)$. Also, the ***distant*** between two non-empty sets A, B is defined as:

$$d(A, B) = \inf\{d(x, y) \mid x \in A \wedge y \in B\}.$$

A metric space (X, d) is called ***discrete*** if

$$\forall x \in X \left(\exists \delta_x \in \mathbb{R}_+ \left(\forall y \in X (y \neq x \to d(x, y) > \delta_x)\right)\right).$$

Lemma 2. *If (X, d) is a metric space, then $\forall a, b, u, v$, $|d(a, b) - d(u, v)| \leq d(a, u) + d(b, v)$.*

Proof. Without loss of generality, we assume that $d(a, b) > d(u, v)$. According to the triangle inequality (see def. 1.1), $d(a, b) \leq d(a, u)+d(u, v)+d(v, b)$, which is to prove. □

Definition 1.3. $\delta \in \mathbb{R}_+$, $a \in X$. Set

$$B(a; \delta) = \{x \in X \mid d(a, x) < \delta\}$$

is then called a ***ball*** with centre $a \in X$, and a radius of δ, or a δ-***ball*** of point a.

Definition 1.4. An ***open set*** $G \subset X$ in metric space (X, d) satisfies: $\forall x \in G$, $\exists B(x; \delta)$, s.t. $B(x; \delta) \subset G$.

Definition 1.5. A set $F \subset X$ in metric space (X, d) is said to be a ***closed set*** if its complement $\complement_X(F)$ is open.

It can be proved that $\varnothing$ and X itself is both open and closed.

Proposition 1. a) An infinite union of open sets is an open set.

b) A finite intersection of open sets is an open set.

c) A finite union of closed sets is a closed set.

d) An infinite intersection of closed sets is a closed set.

Proof. a) If open sets $G_\alpha \subset X, \forall \alpha \in A$, $\forall a \in \bigcap\limits_{\alpha \in A} G_\alpha$, $\exists \alpha_0 \in A$, $a \in G_{\alpha_0}$, $\exists B(a; \delta) \subset G_{\alpha_0} \subset \bigcap\limits_{\alpha \in A} G_\alpha$.

b) Open sets $G_1 \cup G_2 \subset X$, $a \in G_1 \cap G_2$, therefore $\exists \delta_1, \delta_2 \in \mathbb{R}_+$, $B(a; \delta_1) \subset G_1, B(a; \delta_2) \subset G_2$, without loss of generality, let $\delta_1 \geq \delta_2$, $\quad a \in B(a; \delta_1) \cap B(a; \delta_2) = B(a; \delta_2) \subset G_1 \cap G_2$.

c) Just consider $\complement_X \left(\bigcap_{\alpha \in A} F_\alpha \right) = \bigcup_{\alpha \in A} \complement_X(F_\alpha)$ and a).

d) Similarly, $\complement_X (F_1 \cup F_2) = \complement_X(F_1) \cap \complement_X(F_2)$.

$\square$

1.2 Topological Space

Definition 1.6. We say X is equiped with a ***topological space*** or equiped with ***topology*** if we assigned a $\mathscr{T} \subset 2^X$, which has got the following propoties:

a) $\varnothing \in \mathscr{T}; X \in \mathscr{T}$.

b) $\forall \alpha \in A (G_\alpha \in \mathscr{T}) \to \bigcup\limits_{\alpha \in A} G_\alpha \in \mathscr{T}$.

c) $G_1 \in \mathscr{T} \wedge G_2 \in \mathscr{T} \to G_1 \cap G_2 \in \mathscr{T}$.

Then we call $(X, \mathscr{T})$ a ***topological space***. Every $G \in \mathscr{T}$ is called an ***open set***.

Definition 1.7. A topology $\mathscr{T}_d$ insisting of the open sets in a metric space (X, d) is called a ***topology induced by metric*** d.

A trivial example of topological space is ***trivial topology***, which consists only of empty set and the space itself, i.e. $\mathscr{T} = \{\varnothing, X\}$. Another trivial example of topological space is ***discrete topology***, which consists of all the subsets of the space i.e. $\mathscr{T} = 2^X$.

A ***cofinite space*** is a base set X equiped with a topology $\mathscr{T}$ defined as follows:

$$\mathscr{T} = \{U \in 2^X \mid U = \varnothing \vee \complement_X U \text{ is finite}\} \tag{1-5}$$

Proposition 2. The set $\mathscr{T}$ under definition 1-5 is a topology.

Proof. a) $\varnothing \in \mathscr{T}$, $X \in \mathscr{T}$.

b) $\forall i \in I \left(\left|\complement_X A_i\right| \in \mathbb{N}\right) \to \forall i_0 \in I \left(\left|\bigcap_{i\in I} \complement_X A_i\right| \leq \left|\complement_X A_{i_0}\right|\right)$, therefore $\bigcup_{i\in I} A_i \in \mathscr{T}$.

c) $\forall A \in \mathscr{T} \forall B \in \mathscr{T} (A \cap B = \varnothing \in \mathscr{T} \vee \complement_X (A \cap B) = \complement_X A \cup \complement_X B$ is finite), therefore $\forall A \in \mathscr{T} \forall B \in \mathscr{T} (A \cap B \in \mathscr{T})$.

□

Similarly, ***countable complement space*** can be defined.

Let X be equiped with two topology $\mathscr{T}_1, \mathscr{T}_2$. $\mathscr{T}_1 \cup \mathscr{T}_2$ is possibly not a topology of X. For example, $\mathscr{T}_1 = \{(x, +\infty) \mid x \in \mathbb{R}\} \cup \{\varnothing, \mathbb{R}\}$ and $\mathscr{T}_2 = \{(-\infty, y) \mid y \in \mathbb{R}\} \cup \{\varnothing, \mathbb{R}\}$ are both topologies of $\mathbb{R}$, but there union $T_1 \cup T_2$ is not.

Theorem 1.1. *Let X be equiped with two topology $\mathscr{T}_1, \mathscr{T}_2$. Their intersection $\mathscr{T}_1 \cap \mathscr{T}_2$ is also a topology on X.*

Proof. a) $\{\varnothing, X\} \subseteq \mathscr{T}_1 \wedge \{\varnothing, X\} \subseteq \mathscr{T}_2 \to \{\varnothing, X\} \subseteq \mathscr{T}_1 \cap \mathscr{T}_2$.

b) $\forall \alpha \in A (G_\alpha \in \mathscr{T}_1 \cap \mathscr{T}_2) \to \bigcup\limits_{\alpha \in A} G_\alpha \in \mathscr{T}_1 \wedge \bigcup\limits_{\alpha \in A} G_\alpha \in \mathscr{T}_2$.

c) $\forall G_1 \in \mathscr{T}_1 \cap \mathscr{T}_2 \forall G_2 \in \mathscr{T}_1 \cap \mathscr{T}_2 \big(G_1 \cap G_2 \in \mathscr{T}_1 \wedge G_1 \cap G_2 \in \mathscr{T}_2\big)$

□

Definition 1.8. Let $(X, \mathscr{T})$ be a topological space. If there exists a metric $d\colon X^2 \to \mathbb{R}$ s.t. $(X, \mathscr{T})$ is induced by d then call $(X, \mathscr{T})$ a ***metrizable space***, (X, d) is its ***metrization***.

1.3 Neighbourhood

Definition 1.9. Let $(X, \mathscr{T})$ be a topological space. A set $U(x)$ is said to be a ***neighbourhood*** of a point $x \in X$ if $\exists G \in \mathscr{T} (G \subseteq U(x) \wedge x \in G)$. If $U(x) \in \mathscr{T}$, it is called a ***open neighbourhood***. Subset class $\{U(x) \subseteq X \mid U(x)$ is a neighbourhood of $x\}$ is called the ***neighbourhood system*** of point x, denoted by $\mathscr{U}_x$

Theorem 1.2. *Let $(X, \mathscr{T})$ be a topological space, U is a subset of X. U is an open set* iff *$\forall x \in U$, U is a neighbourhood of x.*

Proof. The necessity is trivial. $\forall x \in U \exists V(x)$ s.t. $V(x)$, being a subset of U, is a open neighbourhood of x. By definition of topology, $\bigcup\limits_{x \in U} V(x) \in \mathscr{T}$. $\forall x \in U (x \in \bigcup\limits_{x \in U} V(x)) \to U \subseteq V$, while $\forall x \in U (V(x) \subseteq U) \to \bigcup\limits_{x \in U} V(x) \subseteq U$, therefore $U = \bigcup\limits_{x \in U} V(x) \in \mathscr{T}$. □

Theorem 1.3. *Let $(X, \mathscr{T})$ be a topological space, $\mathscr{U}_x$ is a neighbourhood system of point $x \in X$.*

$$\forall U \in \mathscr{U}_x \forall V \in \mathscr{U}_x (U \cap V \in \mathscr{U}_x)$$

Proof. $\forall U \in \mathscr{U}_x \forall V \in \mathscr{U}_x \exists U_0 \in \mathscr{T} \exists V_0 \in \mathscr{T}(U_0 \subseteq U \wedge V_0 \subseteq V \wedge x \in U_0 \cap V_0)$, By definition of topology, $\mathscr{T} \ni U_0 \cap V_0 \subseteq U \cap V$. □

In history topologies were once built on neighbourhood systems. The following theorem shows the way.

Theorem 1.4. *Let X be a set and $\forall x \in X$ a collection of subsets $\mathscr{U}_x \in \mathscr{P}(X)$ is appointed, satisfying:*

(1) $\forall x \in X \big(\mathscr{U}_x \neq \varnothing \wedge \forall U \in \mathscr{U}_x (x \in U)\big)$;

(2) $\forall x \in X \forall U \in \mathscr{U}_x \forall V \in \mathscr{U}_x (U \cap V \in \mathscr{U}_x)$;

(3) $\forall x \in X \forall U \in \mathscr{U}_x \forall V \in \mathscr{P}(X)(U \subseteq V \to V \in \mathscr{U}_x)$;

(4) $\forall x \in X \forall U \in \mathscr{U}_x \exists V \in \mathscr{U}_x \big(V \subseteq U \wedge \forall y \in V (V \in \mathscr{U}_y)\big)$,

then there exists only one topology $\mathscr{T}$ on X s.t. $\forall x \in X$, $\mathscr{U}_x$ is the neighbourhood system of x in $(X, \mathscr{T})$.

Proof. Let $\mathscr{T} = \{G \in \mathscr{P}(X) \mid \forall x \in G(G \in \mathscr{U}_x)\}$.

a) Obviously $\varnothing \in \mathscr{T}$. Since the condition (1) and the condition (3) in theorem 1.4, $X \in \mathscr{T}$.

b) Let $A, B \in \mathscr{T}$. Consider the condition (2) in theorem 1.4 applied to $x \in A \cap B$.

c) Let $\forall i \in I (G_i \in \mathscr{T})$. $\forall x \in \bigcup_{i \in I} G_i$, there must exists a $i \in I$ s.t. $x \in G_i$ and $G_i \in \mathscr{U}_x$. Since the condition (3) in theorem 1.4, $G_i \subseteq \bigcup_{i \in I} G_i$ has implied $\bigcup_{i \in I} G_i \in \mathscr{U}_x$.

These tells that $\mathscr{T}$ is a topology on X.

The condition (4) in theorem 1.4 tells that there always exists a $G \subset U$ for all $x \in X$ and $U \in \mathscr{U}_x$ s.t. $G \in \mathscr{T}$. Therefore $\mathscr{U}_x$ must be a subset of the neighbourhood system of x.

For all neighbourhood U of $x \in X$, there must be a open neighbourhood subset $G \subseteq U$, which is also a member of $\mathscr{U}_x$. Since the condition (3) in theorem 1.4, $U \in \mathscr{U}_x$. Therefore the neighbourhood system of x must be a subset of $\mathscr{U}_x$.

Therefore, $\mathscr{U}_x$ is the neighbourhood system of x.

Now prove the uniqueness. Let there be another topology $\mathscr{T}'$. Since theorem 1.2, $\forall U \big(G \in \mathscr{T}' \leftrightarrow \forall x \in G(G \in \mathscr{U}_x)\big)$. Therefore $\mathscr{T}' = \mathscr{T}$. □

1.4 Continuous Mappings

Definition 1.10. A mapping $f\colon X \to Y$, where X,Y is respectively equiped with topology $\mathscr{T}_X$,$\mathscr{T}_Y$, is said to be ***continuous*** at $x_0 \in X$ (let $y_0 = f(x_0) \in Y$), if $\forall U(y_0)$, $\exists U(x_0)$ s.t. $f(U(x_0)) \subset U(y_0)$. It is ***continuous*** in X if it is continuous at each point $x \in X$.

The set of continuous mappings from X into Y can be denoted by $C(X, Y)$ or $C(X)$ when Y is clear.

It can be easily proved that an identify function $e_X\colon X \to X$ where X is equiped with a topology $\mathscr{T}$ is a continuous function.

Theorem 1.5. ***(criterion of continuity)***

Let $(X, \mathscr{T})$, $(Y; \mathscr{S})$ *be two topological space. A mapping* $f\colon X \to Y$ *is continuous* iff

$$\forall V \in \mathscr{S}\big(\exists U \in \mathscr{T}(U = f^{-1}(V))\big).$$

Proof. $\to$: It is obvious if $f^{-1}(G_Y) = \varnothing$. If $f^{-1}(G_Y) \neq \varnothing$ and $x_0 \in X$, since $f \in C(X, Y)$, for G_Y, $\exists U(x_0)$ s.t $f(U(x_0)) \subset G_Y$. Also notice that $f(U(x_0)) \subset G_Y \Rightarrow U(x_0) \subset f^{-1}(G_Y)$, therefore $f^{-1}(G_Y)$ is open.

$\leftarrow$: $\forall x_0 \in X$, let $y_0 = f(x_0)$, $f^{-1}(U(y_0)) \in \mathscr{T}_X$. Notice that $x_0 \in f^{-1}(U(y_0))$, therefore $f \in C(X, Y)$. ☐

Theorem 1.6. *Let* $(X, \mathscr{T}_X)$, $(Y, \mathscr{T}_Y)$, $(Z, \mathscr{T}_Z)$ *be topological spaces. If* $f\colon X \to Y$ *and* $g\colon Y \to Z$ *are both continuous,* $g \circ f\colon X \to Z$ *is also continuous.*

Proof.

$$\forall W \in \mathscr{T}_Z\left(g^{-1}(W) \in \mathscr{T}_Y\right) \to \forall W \in \mathscr{T}_Z\left(f^{-1}\big(g^{-1}(W)\big)\right)$$

Since $f^{-1}\big(g^{-1}(W)\big) = (g \circ f)^{-1}(W)$, the theorem has been proved. ☐

Definition 1.11. $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ are both topological spaces. A bijective mapping $f\colon X \to Y$ is a ***homeomorphism*** if $f \in C(X, Y) \wedge f^{-1} \in C(Y, X)$.

Definition 1.12. Two topological spaces $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ are said to be ***homeomorphic*** if there exists a homeomorphism $f\colon X \to Y$.

Homeomorphic topological spaces are identical with respect to their topological propoties since the theorem 1.5 has shown that their open sets correspond to each other. In fact homeomorphic relations are equivalent relations.

1.5 Closure

Definition 1.13. Let X be a topological space and A be a subset of X. Let $x \in X$. If $\forall U \in \mathscr{U}_x\big(U \cap (A - \{x\}) \neq \varnothing\big)$, then x is called a ***accumulation point***, ***cluster point*** or ***limit point*** of A. The set $A' := \{x \in X \mid x \text{ is a accumulation point of } A\}$ is called the ***derived set*** of A. A point $a \in A$ is called a ***isolated point*** of A if $a \notin A'$.

Theorem 1.7. *Let X be a topological space and A, B be subsets of X. 1) $A \subseteq B \to A' \subseteq B'$; 2) $(A \cup B)' = A' \cup B'$; 3) $(A')' \subseteq A \cup A'$.*

Proof. 1) When $A \subseteq B$, $U \cap (A - \{x\}) \subseteq U \cap (B - \{x\})$.

2) $(A \cup B)' = \{x \in X \mid \forall U \in \mathscr{U}_x \big(U \cap (A \cup B - \{x\}) \neq \varnothing\big)\}$. Also $U \cap (A \cup B - \{x\}) = U \cap (X - \{x\}) \cap (A \cup B) = (U \cap A - \{x\}) \cup (U \cap B - \{x\})$.

3) If $x \notin A \cup A'$, then $\exists G \in \mathscr{U}_x \cap \mathscr{T} \big(G \cap (A - \{x\}) = G \cap A = \varnothing\big)$. $\forall y \in G$, G itself is a neighbourhood of y that $G \cap (A - \{y\}) = G \cap A = \varnothing$, therefore $y \notin A'$. This means that G is a neibourhood of x that $G \cap (A' - \{x\}) = G \cap A' = \varnothing$, i.e. $x \notin (A')'$.

$\square$

Definition 1.14. Let $(X, \mathscr{T})$ be a topological space and F be a subset of X. F is said to be ***closed*** *iff* $\complement_X(F) \in \mathscr{T}$. The collection all closed sets is denoted by $\mathscr{F}$.

Theorem 1.8. *Let $(X, \mathscr{T})$ be a topological space and F be a subset of X. F is closed* iff $\forall x \in F'(x \in F)$.

Proof. $\to$: If $x \notin F$ then $x \in \complement_X(F)$, which is open in $(X, \mathscr{T})$. Then $\complement_X(F)$ is a neighbourhood that $\complement_X(F) \cap (F - \{x\}) = \complement_X(F) \cap F = \varnothing$, i.e. $x \notin F'$.

$\leftarrow$: $\forall x \notin F(x \notin F')$, then there exists a open neighbourhood U of x that $U \cap F = \varnothing$, then $\complement_X(F)$ is always a neighbourhood of its elements, since theorem 1.2, $\complement_X(F) \in \mathscr{T}$. $\square$

Definition 1.15. Let $(X, \mathscr{T})$ be a topological space and A be a subset of X. Set $\overline{A} := A \cup A'$ is called a ***closure*** of A.

Theorem 1.9. *Let $(X, \mathscr{T})$ be a topological space and A be a subset of X. A is closed in $(X, \mathscr{T})$* iff $A = \overline{A}$.

Proof. Since theorem 1.8, A is closed iff $A' \subseteq A$, which iff $A = A \cup A' = \overline{A}$. $\square$

Corollary 1. *Let $(X, \mathscr{T})$ be a topological space and A be a subset of X. $\overline{A}$ is always closed.*

Proof. Since (3) of theorem 1.7, $\overline{\overline{A}} = \overline{A}$. $\square$

Lemma 3. *Let $(X, \mathscr{T})$ be a topological space and A, B be subsets of X. $A \subseteq B \to \overline{A} \subseteq \overline{B}$.*

Proof. $A \subseteq B \to A' \subseteq B'$ ((1) of theorem 1.7), so $A \cup A' \subseteq B \cup B'$, i.e. $\overline{A} \subseteq \overline{B}$. $\square$

We can say that the closure of a set is the smallest closed set containing it, as long as we prove the following theorem:

Theorem 1.10. *Let $(X, \mathscr{T})$ be a topological space and A be a subset of X.*

$$\overline{A} = \bigcap_{F \in \mathscr{F} \wedge A \subseteq F} F.$$

Proof. Since $\overline{A}$ itself is closed (corollary 1), $\bigcap_{F \in \mathscr{F} \wedge A \subseteq F} F \subseteq \overline{A}$. On the other hand, $\bigcap_{F \in \mathscr{F} \wedge A \subseteq F} F$ is closed, so $\overline{\bigcap_{F \in \mathscr{F} \wedge A \subseteq F} F} = \bigcap_{F \in \mathscr{F} \wedge A \subseteq F} F$. Therefore, $A \subseteq \bigcap_{F \in \mathscr{F} \wedge A \subseteq F} F \to \overline{A} \subseteq \bigcap_{F \in \mathscr{F} \wedge A \subseteq F} F$ (Lemma 3). □

Theorem 1.11. *Let (X, d) be a metric space and A be a non-empty subset of X.*

1) $\forall x \in X$*,* $x \in A' \leftrightarrow d(x, A - \{x\}) = 0$*.*

2) $\forall x \in X$*,* $x \in \overline{A} \leftrightarrow d(x, A) = 0$*.*

Proof. 1) We have $x \in A'$ *iff* $\forall \varepsilon \in \mathbb{R}_+ \big(B(x, \varepsilon) \cap (A - \{x\}) \neq \varnothing\big)$, which is established *iff* $\forall \varepsilon \in \mathbb{R}_+ \exists y \in A - \{x\} \big(d(x, y) < \varepsilon\big)$.

2) We only need to substitude $A - \{x\}$ with A in 1).

□

Theorem 1.12. *Let $(X, \mathscr{T})$ and $(Y, \mathscr{S})$ be two topological spaces, and $f \colon X \to Y$. Note the collections of closed sets in X and Y by $\mathscr{F}_X$, $\mathscr{F}_Y$. The statements below are equivalent:*

(1) $f \in C(X, Y)$*.*

(2) $\forall F \in \mathscr{F}_Y \big(f^{-1}(F) \in \mathscr{F}_X\big)$*.*

(3) $\forall A \in \mathscr{P}(X) \big(f(\overline{A}) \subseteq \overline{f(A)}\big)$*.*

(4) $\forall B \in \mathscr{P}(Y) \big(\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})\big)$*.*

Proof. (1) $\to$ (2): Only to notice that $f^{-1}(\complement_Y F) = \complement_X f^{-1}(F)$.

(2) $\to$ (3): $\forall A \in \mathscr{P}(X)$ we have $f(A) \subseteq \overline{f(A)}$, so $A \subseteq f^{-1}\big(\overline{f(A)}\big)$. By (2) we know that $f^{-1}\big(\overline{f(A)}\big)$ is closed, therefore $\overline{A} \subseteq \overline{f^{-1}\big(\overline{f(A)}\big)} = f^{-1}\big(\overline{f(A)}\big)$, so $f(\overline{A}) \subseteq \overline{f(A)}$.

(3) $\to$ (4): By (3) we know that $\forall B \in \mathscr{P}(Y)$, $f\big(\overline{f^{-1}(B)}\big) \subseteq \overline{f(f^{-1}(B))}$. Also $f(f^{-1}(B)) \subseteq B$ (equality satisfied when f is surjective), then $f\big(\overline{f^{-1}(B)}\big) \subseteq \overline{B}$, so $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.

(4) $\to$ (1): $\forall G \in \mathscr{S}$, $\complement_Y G \in \mathscr{F}_Y$, so by (3), we have

$$\overline{\complement_X f^{-1}(G)} = \overline{f^{-1}\big(\complement_Y G\big)} \subseteq f^{-1}\big(\overline{\complement_Y G}\big) = f^{-1}\big(\complement_Y G\big) = \complement_X f^{-1}(G).$$

However by the definition of closure $\complement_X f^{-1}(G) \subseteq \overline{\complement_X f^{-1}(G)}$. Therefore $\complement_X f^{-1}(G) = \overline{\complement_X f^{-1}(G)}$, which means $\complement_X f^{-1}(G)$ is closed (theorem 1.9), i.e. $f^{-1}(G)$ is open.

□

1.6 Interior Points and Boundary

2 Subspace

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