Contents

Chapter 1

Geometry in Regions of a Space

§1 Co-odinate and its transformation

Definition 1.1. A transformation of co-odinate from $\boldsymbol{x}$ to $\boldsymbol{y}$

$$\boldsymbol{y}(\boldsymbol{x}) = y^i(x^j)\hat{\boldsymbol{e}}_i = y(x^j).$$

Its ***Jacobian***:

$$\boldsymbol{J} = \left(\frac{\partial y^i}{\partial x^j}\right) \tag{1-1}$$

A vector $\boldsymbol{u}$ at point $\boldsymbol{x}_0$ under such transformation would follow:

$$v^i = \left.\frac{\partial y^i}{\partial x^j}\right|_{\boldsymbol{x}_0} u^i \tag{1-2}$$

$$\text{i.e.} \qquad \boldsymbol{v} = \boldsymbol{J}_0\boldsymbol{u}$$

A linear form $\ell : \boldsymbol{x} \mapsto \ell(\boldsymbol{x}) = l_i x^i$ under such transformation would follow:

$$l'_i \,\mathrm{d}y^i = l_j \,\mathrm{d}x^j \qquad \Rightarrow \qquad l'_i = \left.\frac{\partial x^j}{\partial y_i}\right|_{\boldsymbol{x}_0} l_j \tag{1-3}$$

$$\text{i.e.} \qquad \boldsymbol{l}' = \boldsymbol{l}\boldsymbol{J}_0^{-1}$$

A linear transformation $\mathscr{L} : \boldsymbol{x} \mapsto \boldsymbol{L}\boldsymbol{x}$ where $\boldsymbol{L} = \left(L^i{}_j\right)$ under such transformation would follow:

$$\begin{aligned}
\mathrm{d}y\left(\mathscr{L}^i(\boldsymbol{x})\right) &= (L')^i{}_j \,\mathrm{d}y^j \\
&= \left.\frac{\partial y^i}{\partial x^k}\right|_{\boldsymbol{x}_0} \mathrm{d}\mathscr{L}^k(\boldsymbol{x}) = \left.\frac{\partial y^i}{\partial x^k}\right|_{\boldsymbol{x}_0} L^k{}_h \,\mathrm{d}x^h = \left.\frac{\partial y^i}{\partial x^k}\right|_{\boldsymbol{x}_0} L^k{}_h \left.\frac{\partial x^h}{\partial y_j}\right|_{\boldsymbol{x}_0} \mathrm{d}y^j
\end{aligned}$$

$$(L')^i{}_j = \left.\frac{\partial y^i}{\partial x^k}\right|_{\boldsymbol{x}_0} L^k{}_h \left.\frac{\partial x^h}{\partial y_j}\right|_{\boldsymbol{x}_0} \qquad \text{or} \qquad \boldsymbol{L}' = \boldsymbol{J}_0 \boldsymbol{L} \boldsymbol{J}_0^{-1} \tag{1-4}$$

A bilinear form $\mathscr{B} : \boldsymbol{x} \mapsto \boldsymbol{x}^{\mathrm{T}} \boldsymbol{b} \boldsymbol{x} = x^{\,i} b_{ij} x^j$:

$$b'_{ij}\,\mathrm{d}y^{\,i}\,\mathrm{d}y^j = b'_{ij} \left.\frac{\partial y^{\,i}}{\partial x^{\,k}}\right|_{\boldsymbol{x}_0} \left.\frac{\partial y^j}{\partial x^h}\right|_{\boldsymbol{x}_0} \mathrm{d}x^{\,k}\,\mathrm{d}x^h = b_{kh}\,\mathrm{d}x^{\,k}\,\mathrm{d}x^h \qquad \Rightarrow \qquad b'_{ij} = \left.\frac{\partial x^{\,k}}{\partial y^{\,h}}\right|_{\boldsymbol{x}_0} b_{kh} \left.\frac{\partial x^h}{\partial y^j}\right|_{\boldsymbol{x}_0}$$

$$\text{i.e.} \qquad \boldsymbol{b}' = (\boldsymbol{J}_0^{-1})^{\mathrm{T}} \boldsymbol{b} \boldsymbol{J}_0^{-1} \tag{1-5}$$

Bibliography

[1] R.G. Burns et al. *Modern Geometry — Methods and Applications: Part I: The Geometry of Surfaces, Transformation Groups, and Fields*. Graduate Texts in Mathematics. Springer New York, 1991. ISBN: 9780387976631. URL: https://books.google.co.jp/books?id=FC0QFlx12pwC.

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