Category Theory

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Preface

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Chapter 1

Categories

§1 Categories

Definition 1.1 (Category)**.** A ***category*** $\mathcal{C}$ consists of three ingredients:

1. A *class* $\mathrm{obj}(\mathcal{C})$, called the ***objects***;
2. For any $A, B \in \mathrm{obj}(\mathcal{C})$, a set of ***morphisms*** $\mathrm{Hom}(A, B)$;
3. A function $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$, called the ***composition***, for any $A, B, C \in \mathrm{obj}(\mathcal{C})$, denoted as $(f, g) \mapsto gf$,

and they follow the following axioms:

(i) If $(A, B) \neq (A', B')$, then $\mathrm{Hom}(A, B) \cap \mathrm{Hom}(A', B') = \varnothing$;

(ii) *Associativity*: the composition is associative, i.e. $h(gf) = (hg)f$;

(iii) *Identity*: For any $A \in \mathrm{obj}(\mathcal{C})$, there is an identity morphism $\mathrm{id}_A \in \mathrm{Hom}(A, A)$, such that $f \,\mathrm{id}_A = f = \mathrm{id}_A f$, for any $B \in \mathrm{obj}(\mathcal{C})$ and $f \in \mathrm{Hom}(A, B)$.

A morphism can be shown by:

$$A \xrightarrow{\quad f \quad} B$$

Examples of categories: Set, Grp, Ab, Top, Ord, Ring, Mod, ...
If $\mathrm{obj}(\mathcal{C})$ is a set, then $\mathcal{C}$ is called a ***small category***.
If $(X, \le)$ is a preorder set, then $\forall x, y \in X$,

$$\mathrm{Hom}(x,y) = \begin{cases} \varnothing & x > y, \\ \{(x,y)\} & x \le y, \end{cases} \tag{1-1}$$

and $(y,z)(x,y) = (x,z)$. With this we can say that X is a category. The morphism (x,y) is also denoted by i_y^x.

Definition 1.2 (Isomorphism)**.** Let $\mathcal{C}$ be a category and $A, B \in \mathrm{obj}(\mathcal{C})$, $f \in \mathrm{Hom}(A,B)$. If $\exists g \in \mathrm{Hom}(B,A)$ s.t. $gf = \mathrm{id}_B$ and $fg = \mathrm{id}_A$, then f is called an ***isomorphism*** from A to B. g is called the ***inverse*** of f.

Definition 1.3 (Subcategory)**.** We say $\mathcal{S}$ a ***subcategory*** of $\mathcal{C}$, if

(i) $\mathrm{obj}(\mathcal{S}) \subseteq \mathrm{obj}(\mathcal{C})$;

(ii) $\forall A, B \in \mathrm{obj}(\mathcal{S})$, $\mathrm{Hom}_{\mathcal{S}}(A,B) \subseteq \mathrm{Hom}_{\mathcal{C}}(A,B)$;

(iii) $\forall A, B, C \in \mathrm{obj}(\mathcal{S})$,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

then gf is the same in both $\mathrm{Hom}_{\mathcal{S}}(A,C)$ and $\mathrm{Hom}_{\mathcal{C}}(A,C)$,

(iv) $\forall A \in \mathrm{obj}(\mathcal{S})$, $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{S}}(A,A)$ is the same in $\mathrm{Hom}_{\mathcal{C}}(A,A)$.

Definition 1.4 (Full subcategory)**.** Let $\mathcal{S}$ be a subcategory of $\mathcal{C}$. If $\forall A, B \in \mathrm{obj}(\mathcal{S})$, $\mathrm{Hom}_{\mathcal{S}}(A,B) = \mathrm{Hom}_{\mathcal{C}}(A,B)$, then $\mathcal{S}$ is called a ***full subcategory*** of $\mathcal{C}$.

Definition 1.5 (Generated full subcategory)**.** For any subclass $S \subseteq \mathrm{obj}(\mathcal{C})$, one can find a full subcategory $\mathcal{S}$ of $\mathcal{C}$ s.t. $\mathrm{obj}(\mathcal{S}) = S$, which is called the full subcategory generated by S.

Top_2 is the full subcategory of Top that is generated by the class of all Hausdorff spaces.

Definition 1.6 (Opposite category)**.** Let $\mathcal{C}$ be a category. Then $\mathcal{C}^{\mathrm{op}}$ is the category that:

1. $\mathrm{obj}(\mathcal{C}^{\mathrm{op}}) = \mathrm{obj}(\mathcal{C})$;
2. $\forall A, B \in \mathrm{obj}(\mathcal{C})$,

$$\mathrm{Hom}_{\mathcal{C}}(A, B) = \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(B, A). \tag{1-2}$$

§2 Definitions of Different Categories

Chapter 2

Functors

§3 Functors

Definition 3.1 (Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A ***functor*** $F : \mathcal{C} \to \mathcal{D}$ is a function that satisfies the following axioms:

(i) $\forall A \in \mathrm{obj}(\mathcal{C})$, $F(A) \in \mathrm{obj}(\mathcal{D})$;

(ii) $\forall A, B \in \mathrm{obj}(\mathcal{C}), \forall f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$;

(iii) $\forall A, B, C \in \mathrm{obj}(\mathcal{C})$,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

then $F(gf) = F(g)F(f)$.

(iv) $\forall A \in \mathrm{obj}(\mathcal{C})$, $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

We can restate some definition using functors

Theorem 3.1 (Subcategory, in language of functors)**.** *Let $\mathcal{C}$ and $\mathcal{S}$ be two categories, $\mathcal{S} \subseteq \mathcal{C}$. If the inclusion $I\colon \mathcal{S} \to \mathcal{C}$ is a functor, then $\mathcal{S}$ is a subcategory of $\mathcal{C}$.*

The ***identity functor*** from $\mathcal{C}$ to itself is $1_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ s.t. $\forall C, D \in \mathcal{C}$, $\forall f \in \mathrm{Hom}(C, D)$,

$$1_{\mathcal{C}}(C) = C, \quad 1_{\mathcal{C}}(f) = f. \tag{3-1}$$

Theorem 3.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. $F\colon \mathcal{C} \to \mathcal{D}$ is a functor. $\forall A, B \in \mathrm{obj}(\mathcal{C})$, if $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is an isomorphism, then $F(f)$ is an isomorphism.*

Definition 3.2 (Hom)**.** Let $\mathcal{C}$ be a category and $A \in \mathrm{obj}(\mathcal{C})$. The ***Hom functor*** $F_A\colon \mathcal{C} \to \mathsf{Set}$ is defined as

$$\begin{aligned} F_A(B) &= \mathrm{Hom}(A, B), \\ F_A(f)\colon\ &\mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C);\ h \mapsto fh. \end{aligned} \tag{3-2}$$

The Hom functor is also denoted by $\mathrm{Hom}(A, -)$. We call the $F_A(f) =: \mathrm{Hom}(A, f)$ the ***induced map***, and denote it by f_*

$$f_* h = fh. \tag{3-3}$$

Definition 3.3 (Faithful functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A ***faithful functor*** $F\colon \mathcal{C} \to \mathcal{D}$ is a functor that satisfies $\forall A, B \in \mathrm{obj}(\mathcal{C})$,

$$i\colon\ \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B));\ f \mapsto F(f) \tag{3-4}$$

is injective.

Definition 3.4 (Concrete category)**.** Let $\mathcal{C}$ be a category. $\mathcal{C}$ is called a ***concrete category*** if there exists a faithful functor $F\colon \mathcal{C} \to \mathsf{Set}$.

§4 Contravariant Functors

Definition 4.1 (Contravariant functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A ***contravariant functor*** $F : \mathcal{C} \to \mathcal{D}$ is a function that satisfies the following axioms:

(i) $\forall A \in \mathrm{obj}(\mathcal{C})$, $F(A) \in \mathrm{obj}(\mathcal{D})$;

(ii) $\forall A, B \in \mathrm{obj}(\mathcal{C}), \forall f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(B), F(A))$;

(iii) $\forall A, B, C \in \mathrm{obj}(\mathcal{C})$,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

then $F(gf) = F(f)F(g)$.

(iv) $\forall A \in \mathrm{obj}(\mathcal{C})$, $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

To distinguish functors from contravariant functors, we sometimes call the functors ***covariant functors***.

$-^{\mathrm{op}} \colon \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ is a contravariant functor.

Definition 4.2 (Contravariant Hom)**.** Let $\mathcal{C}$ be a category and $A \in \mathrm{obj}(\mathcal{C})$. The ***contravariant Hom functor*** $F_A \colon \mathcal{C} \to \mathsf{Set}$ is defined as

$$\begin{aligned} F_A(B) &= \mathrm{Hom}(B, A), \\ F_A(f) &\colon \mathrm{Hom}(B, A) \to \mathrm{Hom}(C, A);\ h \mapsto hf. \end{aligned} \tag{4-1}$$

The contravariant Hom functor is also denoted by $\mathrm{Hom}(-, A)$. We call the $F_A(f) =: \mathrm{Hom}(f, A)$ the ***induced map***, and denote it by f^*

$$f^* h = hf. \tag{4-2}$$

§5 Diagrams

Definition 5.1 (Diagram)**.** A ***diagram*** in a category $\mathcal{C}$ is a functor $D\colon \mathcal{D} \to \mathcal{C}$ where $\mathcal{D}$ is a small category.

We have already seemed drawn diagrams like

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"] \arrow[dr, bend right=60, "h'"'] & B \arrow[d, "g"] \\ & C \end{tikzcd} \tag{5-1}$$

where $A, B, C \in D(\mathrm{obj}(\mathcal{D}))$, and each arrow from one to another is a morphism in the image of morphism in $\mathcal{D}$ under D e.g. $\exists D_A, D_B \in \mathrm{obj}(\mathcal{D})$ s.t.

$$f \in D(\mathrm{Hom}_{\mathcal{D}}(D_A, D_B)) \subseteq \mathrm{Hom}_{\mathcal{C}}(A, B). \tag{5-2}$$

Definition 5.2 (Path)**.** A ***path*** in a category $\mathcal{C}$ is a functor $P\colon n+1 \to \mathcal{C}$ where $n+1$ is considered as a preorder with morphism defined in Eq. (1-1).

Conventionally we denote a path as:

$$P_0 \xrightarrow{f_0} P_1 \xrightarrow{f_1} P_2 \longrightarrow \cdots \longrightarrow P_{n-1} \xrightarrow{f_{n-1}} P_n \tag{5-3}$$

A ***simple path*** is a path such that $\forall i, j \in n+1$, $P_i = P_j \to i = j$.

A diagram D ***commutes*** iff $A, B \in D(\mathrm{obj}\,\mathcal{D})$, the compositions of morphisms in any two simple paths from A to B are the same.

§6 Natural transformations

Definition 6.1 (Natural transformation)**.** Let $\mathcal{C}, \mathcal{D}$ be two categories and $F, G\colon \mathcal{D} \to \mathcal{C}$ be functors. A ***natural transformation*** $\alpha\colon F \to G$ is *one-parametre family of morphisms* in $\mathcal{D}$:

$$\alpha\colon\ \mathrm{obj}(\mathcal{C}) \to \{\mathrm{Hom}(F(A), G(A)) \mid A \in \mathrm{obj}(\mathcal{C})\};\ A \mapsto \alpha_A, \tag{6-1}$$

such that $\forall A, B \in \text{obj}(\mathcal{C})$, $\forall f \in \text{Hom}(A, B)$, the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ \Big\downarrow{\scriptstyle F(f)} & & \Big\downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow{\alpha_B} & G(B) \end{array} \tag{6-2}$$

or,

$$\alpha_B F(f) = G(f)\alpha_A. \tag{6-3}$$

All natural transformations between two functors $F, G \colon \mathcal{C} \to \mathcal{D}$ is denoted as $\text{Nat}(F, G)$. However, $\text{Nat}(F, G)$ can only be considered as an object in our metalanguage, since it does not even have to be a class.

A ***natural isomorphism*** is a natural transformation $\alpha \colon F \to G$ such that $\forall A \in \text{obj}(\mathcal{C})$, α_A is an isomorphism.

Natural transformations can compose, and for any functor, there exists an identity natural isomorphism.

You can define the contravariant version of natrual transformation too.

Theorem 6.1 (Yoneda lemma)**.** *Let $\mathcal{C}$ be a category and $F : \mathcal{C} \to \mathsf{Set}$ be a functor. Then, $\forall A \in \text{obj}(\mathcal{C})$, $\text{Nat}(\text{Hom}_{\mathcal{C}}(A, -), G)$ is a set*[1]*, and there exists a bijection*

$$y\colon \text{Nat}(\text{Hom}_{\mathcal{C}}(A, -), G) \to F(A) \tag{6-4}$$

s.t.

$$y(\tau) = \tau_A(1_A). \tag{6-5}$$

Proof. $\tau \in \text{Nat}(\text{Hom}_{\mathcal{C}}(A, -), G)$ means $\forall A, B \in \text{obj}(\mathcal{C})$, the diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(A, B) & \xrightarrow{\tau_B} & F(B) \\ \Big\downarrow{\scriptstyle \varphi_*} & & \Big\downarrow{\scriptstyle F(\varphi)} \\ \text{Hom}_{\mathcal{C}}(A, C) & \xrightarrow{\tau_C} & F(C) \end{array} \tag{6-6}$$

[1] Why?

commutes. Setting $B = A$ we have

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(A, A) \arrow[r, "\tau_A"] \arrow[d, "\varphi_*"] & F(A) \arrow[d, "F(\varphi)"] \\ \mathrm{Hom}_{\mathcal{C}}(A, C) \arrow[r, "\tau_C"] & F(C) \end{tikzcd} \tag{6-7}$$

which gives

$$F(\varphi)\tau_A(1_A) = \tau_C\varphi_*(1_A) = \tau_C(\varphi). \tag{6-8}$$

Injectivity Now assuming there exists another natural transformation $\sigma\colon \mathrm{Hom}_{\mathcal{C}}(A, -) \to G$ such that $\sigma_A(1_A) = \tau_A(1_A)$, we have $\forall C \in \mathrm{obj}(\mathcal{C})$,

$$\sigma_C(\varphi) = F(\varphi)\sigma_A(1_A) = F(\varphi)\tau_A(1_A) = \tau_C(\varphi), \tag{6-9}$$

i.e. $y(\tau) = y(\sigma) \to \tau = \sigma$, or in plain words, y is an injection.

Surjectivity $\forall a \in F(A)$, find a morphism $\tau_A\colon \mathrm{Hom}_{\mathcal{C}}(A, A) \to F(A)$ s.t. $\tau_A(1_A) = a$ (this is always possible e.g. we can set τ_A to be the constant function $\mathrm{Hom}_{\mathcal{C}}(A, A) \ni f \mapsto a$). Then, $\forall C \in \mathrm{obj}(\mathcal{C})$, $\forall \varphi \in \mathrm{Hom}_{\mathcal{C}}(A, C)$, define morphism as $\tau_C(\varphi) = F(\varphi)(a)$.

We have not yet proved that τ is natural, so we check if $\psi \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, $\forall \vartheta \in \mathrm{Hom}_{\mathcal{C}}(A, B)$:

$$\tau_C\psi_*(\vartheta) = F(\psi_*\vartheta)(a) = F(\psi\vartheta)(a) = F(\psi)F(\vartheta)(a) = F(\psi)\tau_B(\vartheta). \tag{6-10}$$

□

Appendix A

Appendix

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