Functional Analysis

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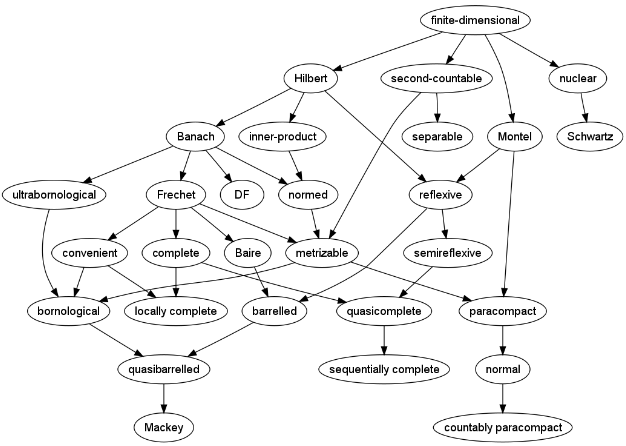

Preface

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Chapter 1

Topological Vector Spaces

§1 Topological Vector Spaces

If not specified, $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$, V is a vector space over $\mathbb{K}$.

Definition 1.1 (Topological Vector Space)**.** A ***topological vector space*** is a vector space V over field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R} \vee \mathbb{K} = \mathbb{C}$) s.t. the vector addition $+\colon V \times V \to V$ and the scalar multiplication $\cdot\colon \mathbb{K} \times V \to V$ are both continuous.

Definition 1.2 (Balanced set)**.** A subset C of V over $\mathbb{K}$ is ***balanced*** if $\forall \lambda \in \mathbb{K}$, $\forall x \in C$, if $|\lambda| \leq 1$, then $\lambda x \in C$.

It means that, if $x \in C$, the disk with x on its boundary and centered at 0 is also contained in C.

Definition 1.3 (Locally convexity)**.** A topological vector space V is ***locally convex*** if there exists a local base of balanced, convex sets at 0.

Chapter 2

Normed and Banach Spaces

Chapter 3

Inner Product and Hilbert Spaces

Chapter 4

Linear Operators

Chapter 5

Duality and Hahn-Banach Theorem

§2 Sublinear Functionals

Definition 2.1 (Sublinear functional)**.** Let V be a vector space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R} \vee \mathbb{K} = \mathbb{C}$). A ***sublinear functional*** on V is a function $p\colon V \to \mathbb{R}$ s.t.

1. $\forall v \in V$, $\forall \lambda \in \mathbb{R}$, if $\lambda \geq 0$, then $p(\lambda v) = \lambda p(v)$ (***non-negative homogeneity*** or ***positive homogeneity***);
2. $p(v + w) \leq p(v) + p(w)$ for all $v, w \in V$ (***subadditivity*** or ***triangle inequality***).

Definition 2.2 (Semi-norm)**.** A ***semi-norm*** on a vector space V over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R} \vee \mathbb{K} = \mathbb{C}$) is a function $p\colon V \to \mathbb{R}$ s.t.

1. $\forall v \in V, \forall \lambda \in \mathbb{K}, p(\lambda v) = |\lambda| p(v)$ (***absolute homogeneity***);
2. $p(v + w) \leq p(v) + p(w)$ for all $v, w \in V$ (***subadditivity***).

The definition implies that a semi-norm is also non-negative ($p(v) \geq 0$).

By comparing definition, we can tell that a semi-norm is a sublinear functional, and a norm is a semi-norm with $p(v) > 0$ for non-zero v.

§3 The Hahn-Banach Theorem

Definition 3.1 (Extension)**.** An ***extension*** of a linear functional $f_W: W \to \mathbb{K}$ on a subspace W of a vector space V over $\mathbb{K}$ is a linear functional $f: V \to \mathbb{K}$ s.t.

$$\forall w \in W,\ f(w) = f_W(w).$$

Theorem 3.1 (Hahn-Banach)**.** *Let V be a vector space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R} \vee \mathbb{K} = \mathbb{C}$). $p: V \to \mathbb{R}$ is a seminorm. $W \subset V$ is a subspace of V.*

If $f_W: W \to \mathbb{K}$ is a linear functional s.t.

$$\forall w \in W,\ |f_W(w)| \leq p(w),$$

then there exists an extension $f: V \to \mathbb{K}$ s.t.

$$\forall v \in V,\ |f(v)| \leq p(v).$$

Chapter 6

Linear Operators on Hilbert Spaces

Chapter 7

Compact Operators

Chapter 8

Integral and Differential Equations

Appendix A

Appendix

Bibliography

[1] Bryan P. Rynne and Martin A. Youngson. *Linear Functional Analysis*. Springer Undergraduate Mathematics Series. London: Springer, 2008. ISBN: 978-1-84800-004-9 978-1-84800-005-6. DOI: 10.1007/978-1-84800-005-6. URL: http://link.springer.com/10.1007/978-1-84800-005-6 (visited on 12/07/2023).

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Here listed the important symbols used in this notes.