Point Set Topology

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Chapter 1

Topological Spaces and Continuous Mappings

§1 Metric Space

Definition 1.1. function

$$d\colon X^2 \to \mathbb{R} \tag{1-1}$$

$\forall x_1, x_2, x_2 \in X$ satisfied:
a) $d(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$;
b) $d(x_1, x_2) = d(x_2, x_1)$ (symmetry);
c) $d(x_1, x_3) \leqslant d(x_1, x_2) + d(x_2, x_3)$ (Triangle inequality),
is called a ***metric*** or ***distance*** in X. Such X is said to be equiped with metric d, (X, d) is called a ***metric space***.

Some examples:

- $(\mathbb{R}^n; d_p)$, where $d_p(x_1, x_2) = \left(\sum_{i=1}^n \left|x_1^i - x_2^i\right|^p\right)^{1/p}$, while $d_\infty(x_1, x_2) = \max\limits_{1 \leqslant i \leqslant n} \left|x_1^i - x_2^i\right|$.
- Similarly we can define metric spaces as $(C[a, b]; d_p)$ or $C_p[a, b]$. $d_p(f, g) = \left(\int_a^b |f - g|^p \, \mathrm{d}x\right)^{\frac{1}{p}}$. C_∞ is called a **Chebyshev metric**.
- On class $\tilde{\mathfrak{R}}[a, b]$ over $\mathfrak{R}[a, b]$ similar metric can be defined. Functions are considered of one same class if they are equivalent expect on a set not larger than null set.

Hilbert space denoted by $(\mathbb{H}; d)$ is defined as:

$$\mathbb{H} = \left\{ x = (x_1, x_2, \cdots) \mid \forall i \in \mathbb{Z}_+ \left(\forall x_i \in \mathbb{R} \wedge \sum_{i=1}^{\infty} x_1^2 < \infty \right) \right\} \tag{1-2}$$

equiped with a metirc d:

$$d\colon \mathbb{H}^2 \to \mathbb{R};\ x, y \mapsto \sqrt{\sum_{i=1}^{\infty}(x_i - y_i)^2}. \tag{1-3}$$

To justify this definition, we need to introduce a lemma:

Lemma 1.

$$\forall n \in \mathbb{Z} \forall \boldsymbol{u} \in \mathbb{R}^n \forall \boldsymbol{v} \in \mathbb{R}^n \left(\sum_{i=1}^{n} u_i v_i \le \sqrt{\sum_{i=1}^{n} u_i^2} \sqrt{\sum_{i=1}^{n} v_i^2} \right) \tag{1-4}$$

This is called **Schwarz inequality**.

Proof. If $\forall i = 1, \cdots, n(v_i = 0)$ the equivalence has already been satisfied, therefore the following considers the situation that $\exists i \in \{1, \cdots n\}(v_i \ne 0)$. $\forall \lambda \in \mathbb{R}$, the quadratic polynomial of λ

$$\sum_{i=1}^{n}(u_i + \lambda v_i)^2 = \sum_{i=1}^{n} u_i^2 + 2\lambda \sum_{i=1}^{n} u_i v_i + \lambda^2 \sum_{i=1}^{n} v_i^2 \ge 0$$

has at most one root. Hence $\Delta \le 0$ will lead to the inequality 1-4. $\square$

Apply this inequality to $\sum_{i=1}^{n}(u_i + v_i)^2 = \sum_{i=1}^{n} u_i^2 + \sum_{i=1}^{n} v_i^2 + 2\sum_{i=1}^{n} v_i \sum_{i=1}^{n} u_i$ we can get

$$\sum_{i=1}^{n}(u_i + v_i)^2 \le \sum_{i=1}^{n} u_i^2 + \sum_{i=1}^{n} v_i^2 + 2\sqrt{\sum_{i=1}^{n} u_i^2}\sqrt{\sum_{i=1}^{n} v_i^2} = \left(\sqrt{\sum_{i=1}^{n} u_i^2} + \sqrt{\sum_{i=1}^{n} v_i^2}\right)^2,$$

in which substitude u_i, v_i by $x_i - y_i, x_i + y_i$ will result in triangle inequality. The inequality holds as the n limits to $+\infty$.

Definition 1.2. Let (X, d) be a metric space, the ***distant*** between a non-empty set $\varnothing \ne A \in \mathscr{P}(X)$ and a point x is defined as:

$$d(A, x) = \inf\{d(x, y) \mid y \in A\},$$

and we let $d(x, A) = d(A, x)$. Also, the ***distant*** between two non-empty sets A, B is defined as:

$$d(A, B) = \inf\{d(x, y) \mid x \in A \land y \in B\}.$$

A metric space (X, d) is called ***discrete*** if

$$\forall x \in X \left(\exists \delta_x \in \mathbb{R}_+ \big(\forall y \in X (y \ne x \to d(x, y) > \delta_x)\big)\right).$$

Lemma 2. *If* (X, d) *is a metric space, then* $\forall a, b, u, v,\ |d(a, b) - d(u, v)| \le d(a, u) + d(b, v)$.

Proof. Without loss of generality, we assume that $d(a, b) > d(u, v)$. According to the triangle inequality (see def. 1.1), $d(a, b) \le d(a, u) + d(u, v) + d(v, b)$, which is to prove. $\square$

Definition 1.3. $\delta \in \mathbb{R}_+$, $a \in X$. Set

$$B(a;\delta) = \{x \in X \mid d(a,x) < \delta\}$$

is then called a ***ball*** with centre $a \in X$, and a radius of δ, or a ***δ-ball*** of point a.

Definition 1.4. An ***open set*** $G \subset X$ in metric space (X,d) satisfies: $\forall x \in G$, $\exists B(x;\delta)$, s.t. $B(x;\delta) \subset G$.

Definition 1.5. A set $F \subset X$ in metric space (X,d) is said to be a ***closed set*** if its complement $\complement_X(F)$ is open.

It can be proved that $\varnothing$ and X itself is both open and closed.

Proposition 1. a) An infinite union of open sets is an open set.

b) A finite intersection of open sets is an open set.

c) A finite union of closed sets is a closed set.

d) An infinite intersection of closed sets is a closed set.

Proof. a) If open sets $G_\alpha \subset X, \forall \alpha \in A$, $\forall a \in \bigcap\limits_{\alpha\in A} G_\alpha$, $\exists \alpha_0 \in A$, $a \in G_{\alpha_0}$, $\exists B(a;\delta) \subset G_{\alpha_0} \subset \bigcap\limits_{\alpha\in A} G_\alpha$.

b) Open sets $G_1 \cup G_2 \subset X$, $a \in G_1 \cap G_2$, therefore $\exists \delta_1, \delta_2 \in \mathbb{R}_+$, $B(a;\delta_1) \subset G_1, B(a;\delta_2) \subset G_2$, without loss of generality, let $\delta_1 \geq \delta_2$, 那么 $a \in B(a;\delta_1) \cap B(a;\delta_2) = B(a;\delta_2) \subset G_1 \cap G_2$.

c) Just consider $\complement_X\left(\bigcap_{\alpha\in A} F_\alpha\right) = \bigcup_{\alpha\in A} \complement_X(F_\alpha)$ and a).

d) Similarly, $\complement_X\left(F_1 \cup F_2\right) = \complement_X(F_1) \cap \complement_X(F_2)$.

□

§2 Topological Space

Definition 2.1. We say X is equiped with a ***topological space*** or equiped with ***topology*** if we assigned a $\mathscr{T} \subset 2^X$, which has got the following propoties:
a) $\varnothing \in \mathscr{T}; X \in \mathscr{T}$.
b) $\forall \alpha \in A(G_\alpha \in \mathscr{T}) \to \bigcup\limits_{\alpha\in A} G_\alpha \in \mathscr{T}$.
c) $G_1 \in \mathscr{T} \wedge G_2 \in \mathscr{T} \to G_1 \cap G_2 \in \mathscr{T}$.
Then we call $(X,\mathscr{T})$ a ***topological space***. Every $G \in \mathscr{T}$ is called an ***open set***.

Definition 2.2. A topology $\mathscr{T}_d$ insisting of the open sets in a metric space (X,d) is called a ***topology induced by metric*** d.

A trivial example of topological space is ***trivial topology***, which consists only of empty set and the space itself, i.e. $\mathscr{T} = \{\varnothing, X\}$. Another trivial example of topological space is ***discrete topology***, which consists of all the subsets of the space i.e. $\mathscr{T} = 2^X$.

A ***cofinite space*** is a base set X equiped with a topology $\mathscr{T}$ defined as follows:

$$\mathscr{T} = \{U \in 2^X \mid U = \varnothing \vee \complement_X U \text{ is finite}\} \tag{2-1}$$

Proposition 2. The set $\mathscr{T}$ under definition 2-1 is a topology.

Proof. a) $\varnothing \in \mathscr{T}$, $X \in \mathscr{T}$.

b) $\forall i \in I\left(\left|\complement_X A_i\right| \in \mathbb{N}\right) \to \forall i_0 \in I\left(\left|\bigcap_{i\in I} \complement_X A_i\right| \leq \left|\complement_X A_{i_0}\right|\right)$, therefore $\bigcup_{i\in I} A_i \in \mathscr{T}$.

c) $\forall A \in \mathscr{T} \forall B \in \mathscr{T}(A \cap B = \varnothing \in \mathscr{T} \vee \complement_X(A \cap B) = \complement_X A \cup \complement_X B$ is finite), therefore $\forall A \in \mathscr{T} \forall B \in \mathscr{T}(A \cap B \in \mathscr{T})$.

□

Similarly, ***countable complement space*** can be defined.

Let X be equiped with two topology $\mathscr{T}_1, \mathscr{T}_2$. $\mathscr{T}_1 \cup \mathscr{T}_2$ is possibly not a topology of X. For example, $\mathscr{T}_1 = \{(x, +\infty) \mid x \in \mathbb{R}\} \cup \{\varnothing, \mathbb{R}\}$ and $\mathscr{T}_2 = \{(-\infty, y) \mid y \in \mathbb{R}\} \cup \{\varnothing, \mathbb{R}\}$ are both topologies of $\mathbb{R}$, but there union $T_1 \cup T_2$ is not.

Theorem 2.1. *Let X be equiped with two topology $\mathscr{T}_1, \mathscr{T}_2$. Their intersection $\mathscr{T}_1 \cap \mathscr{T}_2$ is also a topology on X.*

Proof. a) $\{\varnothing, X\} \subseteq \mathscr{T}_1 \wedge \{\varnothing, X\} \subseteq \mathscr{T}_2 \to \{\varnothing, X\} \subseteq \mathscr{T}_1 \cap \mathscr{T}_2$.

b) $\forall \alpha \in A(G_\alpha \in \mathscr{T}_1 \cap \mathscr{T}_2) \to \bigcup\limits_{\alpha\in A} G_\alpha \in \mathscr{T}_1 \wedge \bigcup\limits_{\alpha\in A} G_\alpha \in \mathscr{T}_2$.

c) $\forall G_1 \in \mathscr{T}_1 \cap \mathscr{T}_2 \forall G_2 \in \mathscr{T}_1 \cap \mathscr{T}_2 \big(G_1 \cap G_2 \in \mathscr{T}_1 \wedge G_1 \cap G_2 \in \mathscr{T}_2\big)$

□

Definition 2.3. Let $(X, \mathscr{T})$ be a topological space. If there exists a metric $d\colon X^2 \to \mathbb{R}$ s.t. $(X, \mathscr{T})$ is induced by d then call $(X, \mathscr{T})$ a ***metrizable space***, (X, d) is its ***metrization***.

§3 Neighbourhood

Definition 3.1. Let $(X, \mathscr{T})$ be a topological space. A set $U(x)$ is said to be a ***neighbourhood*** of a point $x \in X$ if $\exists G \in \mathscr{T}(G \subseteq U(x) \wedge x \in G)$. If $U(x) \in \mathscr{T}$, it is called an ***open neighbourhood***. Subset class $\{U(x) \subseteq X \mid U(x)$ is a neighbourhood of $x\}$ is called the ***neighbourhood system*** of point x, denoted by $\mathscr{U}_x$

Theorem 3.1. *Let $(X, \mathscr{T})$ be a topological space, U is a subset of X. U is an open set* iff $\forall x \in U$, *U is a neighbourhood of x.*

Proof. The necessity is trivial. $\forall x \in U \exists V(x)$ s.t. $V(x)$, being a subset of U, is an open neighbourhood of x. By definition of topology, $\bigcup\limits_{x\in U} V(x) \in \mathscr{T}$. $\forall x \in U(x \in \bigcup\limits_{x\in U} V(x)) \to U \subseteq V$, while $\forall x \in U(V(x) \subseteq U) \to \bigcup\limits_{x\in U} V(x) \subseteq U$, therefore $U = \bigcup\limits_{x\in U} V(x) \in \mathscr{T}$. □

Theorem 3.2. *Let $(X, \mathscr{T})$ be a topological space, $\mathscr{U}_x$ is a neighbourhood system of point $x \in X$.*

$$\forall U \in \mathscr{U}_x \forall V \in \mathscr{U}_x (U \cap V \in \mathscr{U}_x)$$

Proof. $\forall U \in \mathscr{U}_x \forall V \in \mathscr{U}_x \exists U_0 \in \mathscr{T} \exists V_0 \in \mathscr{T}(U_0 \subseteq U \wedge V_0 \subseteq V \wedge x \in U_0 \cap V_0)$, By definition of topology, $\mathscr{T} \ni U_0 \cap V_0 \subseteq U \cap V$. □

In history topologies were once built on neighbourhood systems. The following theorem shows the way.

Theorem 3.3. *Let X be a set and $\forall x \in X$ a collection of subsets $\mathscr{U}_x \in \mathscr{P}(X)$ is appointed, satisfying:*
(1) $\forall x \in X\big(\mathscr{U}_x \neq \varnothing \wedge \forall U \in \mathscr{U}_x(x \in U)\big)$;
(2) $\forall x \in X \forall U \in \mathscr{U}_x \forall V \in \mathscr{U}_x(U \cap V \in \mathscr{U}_x)$;
(3) $\forall x \in X \forall U \in \mathscr{U}_x \forall V \in \mathscr{P}(X)(U \subseteq V \to V \in \mathscr{U}_x)$;
(4) $\forall x \in X \forall U \in \mathscr{U}_x \exists V \in \mathscr{U}_x\big(V \subseteq U \wedge \forall y \in V(V \in \mathscr{U}_y)\big)$,
then there exists only one topology $\mathscr{T}$ on X s.t. $\forall x \in X$, $\mathscr{U}_x$ is the neighbourhood system of x in $(X, \mathscr{T})$.

Proof. Let $\mathscr{T} = \{G \in \mathscr{P}(X) \mid \forall x \in G(G \in \mathscr{U}_x)\}$.

a) Obviously $\varnothing \in \mathscr{T}$. Since the condition (1) and the condition (3) in theorem 3.3, $X \in \mathscr{T}$.

b) Let $A, B \in \mathscr{T}$. Consider the condition (2) in theorem 3.3 applied to $x \in A \cap B$.

c) Let $\forall i \in I(G_i \in \mathscr{T})$. $\forall x \in \bigcup_{i\in I} G_i$, there must exists a $i \in I$ s.t. $x \in G_i$ and $G_i \in \mathscr{U}_x$. Since the condition (3) in theorem 3.3, $G_i \subseteq \bigcup_{i\in I} G_i$ has implied $\bigcup_{i\in I} G_i \in \mathscr{U}_x$.

These tells that $\mathscr{T}$ is a topology on X.

The condition (4) in theorem 3.3 tells that there always exists a $G \subset U$ for all $x \in X$ and $U \in \mathscr{U}_x$ s.t. $G \in \mathscr{T}$. Therefore $\mathscr{U}_x$ must be a subset of the neighbourhood system of x.

For all neighbourhood U of $x \in X$, there must be a open neighbourhood subset $G \subseteq U$, which is also a member of $\mathscr{U}_x$. Since the condition (3) in theorem 3.3, $U \in \mathscr{U}_x$. Therefore the neighbourhood system of x must be a subset of $\mathscr{U}_x$.

Therefore, $\mathscr{U}_x$ is the neighbourhood system of x.

Now prove the uniqueness. Let there be another topology $\mathscr{T}'$. Since theorem 3.1, $\forall U\big(G \in \mathscr{T}' \leftrightarrow \forall x \in G(G \in \mathscr{U}_x)\big)$. Therefore $\mathscr{T}' = \mathscr{T}$. □

§4 Continuous Mappings

Definition 4.1. A mapping $f\colon X \to Y$, where X,Y is respectively equiped with topology $\mathscr{T}_X$,$\mathscr{T}_Y$, is said to be ***continuous*** at $x_0 \in X$ (let $y_0 = f(x_0) \in Y$), if $\forall U(y_0)$, $\exists U(x_0)$ s.t. $f(U(x_0)) \subset U(y_0)$. It is ***continuous*** in X if it is continuous at each point $x \in X$.

The set of continuous mappings from X into Y can be denoted by $C(X, Y)$ or $C(X)$ when Y is clear.

It can be easily proved that an identify function $e_X\colon X \to X$ where X is equiped with a topology $\mathscr{T}$ is a continuous function.

Theorem 4.1. *(***criterion of continuity***)*
Let $(X, \mathscr{T})$, $(Y; \mathscr{S})$ be two topological space. A mapping $f\colon X \to Y$ is continuous iff

$$\forall V \in \mathscr{S}\big(\exists U \in \mathscr{T}(U = f^{-1}(V))\big).$$

Proof. $\to$: It is obvious if $f^{-1}(G_Y) = \varnothing$. If $f^{-1}(G_Y) \neq \varnothing$ and $\forall x_0 \in f^{-1}(G_Y)$, since $f \in C(X, Y)$, for $G_Y \in \mathscr{S}$, $\exists U(x_0)$ s.t $f(U(x_0)) \subset G_Y$. Also notice that $f(U(x_0)) \subset G_Y \Rightarrow U(x_0) \subset f^{-1}(G_Y)$, therefore $f^{-1}(G_Y)$ is open (Theorem 3.1).

$\leftarrow$: $\forall x_0 \in X$, let $y_0 = f(x_0)$, $f^{-1}(U(y_0)) \in \mathscr{T}$ if $U(y_0) \in \mathscr{S} \cap \mathscr{U}_{y_0}$. Notice that $x_0 \in f^{-1}(U(y_0))$, $f^{-1}(U(y_0))$ is a neighbourhood of x_0, therefore $f \in C(X, Y)$. □

Theorem 4.2. *Let $(X, \mathscr{T}_X)$, $(Y, \mathscr{T}_Y)$, $(Z, \mathscr{T}_Z)$ be topological spaces. If $f: X \to Y$ and $g: Y \to Z$ are both continuous, $g \circ f: X \to Z$ is also continuous.*

Proof.

$$\forall W \in \mathscr{T}_Z \left(g^{-1}(W) \in \mathscr{T}_Y\right) \to \forall W \in \mathscr{T}_Z \left(f^{-1}\left(g^{-1}(W)\right)\right)$$

Since $f^{-1}\left(g^{-1}(W)\right) = (g \circ f)^{-1}(W)$, the theorem has been proved. □

Definition 4.2. $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ are both topological spaces. A bijective mapping $f: X \to Y$ is a ***homeomorphism*** if $f \in C(X, Y) \wedge f^{-1} \in C(Y, X)$.

Definition 4.3. Two topological spaces $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ are said to be ***homeomorphic*** if there exists a homeomorphism $f: X \to Y$.

Homeomorphic topological spaces are identical with respect to their topological propoties since the Theorem 4.1 has shown that their open sets correspond to each other. In fact homeomorphic relations are equivalent relations.

Lemma 3 (Gluing lemma)**.** *$\mathscr{F} = \{F_i \mid \complement_X F_i \in \mathscr{T}, i \in n\}$, $X \subseteq \cup\mathscr{F}$, $f: X \to Y$. If $\forall F \in \mathscr{F}$, $f|_F \in C(F, Y)$, then $f \in C(X, Y)$.*

Proof. $\forall F_Y (\complement_Y F_Y \in \mathscr{T})$,

$$f^{-1}(F_Y) = \bigcup_{i \in n} F_i \cap f^{-1}(F_Y) = \bigcup_{i \in n} f^{-1}|_{F_i}(F_Y).$$

The finite intersection of closed sets are also closed, while the pre-image of a closed set is also closed (when mapping is continuous). □

§5 Closure

Definition 5.1. Let X be a topological space and A be a subset of X. Let $x \in X$. If $\forall U \in \mathscr{U}_x\left(U \cap (A - \{x\}) \neq \varnothing\right)$, then x is called a ***accumulation point***, ***cluster point*** or ***limit point*** of A. The set $A' := \{x \in X \mid x$ is a accumulation point of $A\}$ is called the ***derived set*** of A. A point $a \in A$ is called a ***isolated point*** of A if $a \notin A'$.

Theorem 5.1. *Let X be a topological space and A, B be subsets of X. 1) $A \subseteq B \to A' \subseteq B'$; 2) $(A \cup B)' = A' \cup B'$; 3) $(A')' \subseteq A \cup A'$.*

Proof. 1) When $A \subseteq B$, $U \cap (A - \{x\}) \subseteq U \cap (B - \{x\})$.

2) $(A \cup B)' = \{x \in X \mid \forall U \in \mathscr{U}_x\left(U \cap (A \cup B - \{x\}) \neq \varnothing\right)\}$. Also $U \cap (A \cup B - \{x\}) = U \cap (X - \{x\}) \cap (A \cup B) = (U \cap A - \{x\}) \cup (U \cap B - \{x\})$.

3) If $x \notin A \cup A'$, then $\exists G \in \mathscr{U}_x \cap \mathscr{T}\big(G \cap (A - \{x\}) = G \cap A = \varnothing\big)$. $\forall y \in G$, G itself is a neighbourhood of y that $G \cap (A - \{y\}) = G \cap A = \varnothing$, therefore $y \notin A'$. This means that G is a neighbourhood of x that $G \cap (A' - \{x\}) = G \cap A' = \varnothing$, i.e. $x \notin (A')'$.

$\square$

Definition 5.2. Let $(X, \mathscr{T})$ be a topological space and F be a subset of X. F is said to be ***closed*** *iff* $\complement_X(F) \in \mathscr{T}$. The collection all closed sets is denoted by $\mathscr{F}$.

Theorem 5.2. *Let $(X, \mathscr{T})$ be a topological space and F be a subset of X. F is closed* iff $F' \subseteq F$.

Proof. $\to$: If $x \notin F$ then $x \in \complement_X(F)$, which is open in $(X, \mathscr{T})$. Then $\complement_X(F)$ is a neighbourhood that $\complement_X(F) \cap (F - \{x\}) = \complement_X(F) \cap F = \varnothing$, i.e. $x \notin F'$.

$\leftarrow$: $\forall x \notin F(x \notin F')$, then there exists an open neighbourhood U of x that $U \cap F = \varnothing$, then $\complement_X(F)$ is always a neighbourhood of its elements, since theorem 3.1, $\complement_X(F) \in \mathscr{T}$. $\square$

Definition 5.3. Let $(X, \mathscr{T})$ be a topological space and A be a subset of X. Set $\overline{A} := A \cup A'$ is called a ***closure*** of A.

Theorem 5.3. *Let $(X, \mathscr{T})$ be a topological space and A be a subset of X. Let $x \in X$.*

$$x \in \overline{A} \leftrightarrow \forall U \in \mathscr{U}(x)(U \cap A \neq \varnothing).$$

Proof. $\to$: If $x \in A$ then $\{x\} \subseteq U \cap A$, else if $x \in A'$ then $U \cap A \supset (U - \{x\}) \cap A \neq \varnothing$.

$\leftarrow$: If $\exists U \in \mathscr{U}(x)\big((U - \{x\}) \cap A = \varnothing\big) \wedge x \notin A$, then there exists a $U \in \mathscr{U}(x)$ s.t. $U \cap A = \varnothing$. $\square$

Theorem 5.4. *Let $(X, \mathscr{T})$ be a topological space and A be a subset of X. A is closed in $(X, \mathscr{T})$* iff $A = \overline{A}$.

Proof. Since theorem 5.2, A is closed iff $A' \subseteq A$, which iff $A = A \cup A' = \overline{A}$. $\square$

Corollary 1. *Let $(X, \mathscr{T})$ be a topological space and A be a subset of X. $\overline{A}$ is always closed.*

Proof. Since (3) of theorem 5.1, $\overline{\overline{A}} = \overline{A}$. $\square$

Lemma 4. *Let $(X, \mathscr{T})$ be a topological space and A, B be subsets of X. $A \subseteq B \to \overline{A} \subseteq \overline{B}$.*

Proof. $A \subseteq B \to A' \subseteq B'$ ((1) of theorem 5.1), so $A \cup A' \subseteq B \cup B'$, i.e. $\overline{A} \subseteq \overline{B}$. $\square$

We can say that the closure of a set is the smallest closed set containing it, as long as we prove the following theorem:

Theorem 5.5. *Let $(X, \mathscr{T})$ be a topological space and A be a subset of X.*

$$\overline{A} = \bigcap_{F \in \mathscr{F} \wedge A \subseteq F} F.$$

Proof. Since $\overline{A}$ itself is closed (corollary 1), $\bigcap_{F \in \mathscr{F} \wedge A \subseteq F} F \subseteq \overline{A}$. On the other hand, $\bigcap_{F \in \mathscr{F} \wedge A \subseteq F} F$ is closed, so $\overline{\bigcap_{F \in \mathscr{F} \wedge A \subseteq F} F} = \bigcap_{F \in \mathscr{F} \wedge A \subseteq F} F$. Therefore, $A \subseteq \bigcap_{F \in \mathscr{F} \wedge A \subseteq F} F \to \overline{A} \subseteq \bigcap_{F \in \mathscr{F} \wedge A \subseteq F} F$ (Lemma 4). $\square$

Theorem 5.6. *Let (X, d) be a metric space and A be a non-empty subset of X.*

1) $\forall x \in X,\ x \in A' \leftrightarrow d(x, A - \{x\}) = 0.$

2) $\forall x \in X,\ x \in \overline{A} \leftrightarrow d(x, A) = 0.$

Proof. 1) We have $x \in A'$ *iff* $\forall \varepsilon \in \mathbb{R}_+ \big(B(x, \varepsilon) \cap (A - \{x\}) \neq \varnothing\big)$, which is established *iff* $\forall \varepsilon \in \mathbb{R}_+ \exists y \in A - \{x\} \big(d(x, y) < \varepsilon\big)$.

2) We only need to substitude $A - \{x\}$ with A in 1).

□

Theorem 5.7. *Let $(X, \mathscr{T})$ and $(Y, \mathscr{S})$ be two topological spaces, and $f\colon X \to Y$. Note the collections of closed sets in X and Y by $\mathscr{F}_X$, $\mathscr{F}_Y$. The statements below are equivalent:*

(1) $f \in C(X, Y)$.

(2) $\forall F \in \mathscr{F}_Y \big(f^{-1}(F) \in \mathscr{F}_X\big)$.

(3) $\forall A \in \mathscr{P}(X) \big(f(\overline{A}) \subseteq \overline{f(A)}\big)$.

(4) $\forall B \in \mathscr{P}(Y) \big(\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})\big)$.

Proof. (1) → (2): Only to notice that $f^{-1}(\complement_Y F) = \complement_X f^{-1}(F)$.

(2) → (3): $\forall A \in \mathscr{P}(X)$ we have $f(A) \subseteq \overline{f(A)}$, so $A \subseteq f^{-1}\big(\overline{f(A)}\big)$. By (2) we know that $f^{-1}\big(\overline{f(A)}\big)$ is closed, therefore $\overline{A} \subseteq \overline{f^{-1}\big(\overline{f(A)}\big)} = f^{-1}\big(\overline{f(A)}\big)$, so $f(\overline{A}) \subseteq \overline{f(A)}$.

(3) → (4): By (3) we know that $\forall B \in \mathscr{P}(Y)$, $f\big(\overline{f^{-1}(B)}\big) \subseteq \overline{f(f^{-1}(B))}$. Also $f(f^{-1}(B)) \subseteq B$ (equality satisfied when f is surjective), then $f\big(\overline{f^{-1}(B)}\big) \subseteq \overline{B}$, so $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.

(4) → (1): $\forall G \in \mathscr{S}$, $\complement_Y G \in \mathscr{F}_Y$, so by (3), we have

$$\overline{\complement_X f^{-1}(G)} = \overline{f^{-1}\big(\complement_Y G\big)} \subseteq f^{-1}\big(\overline{\complement_Y G}\big) = f^{-1}\big(\complement_Y G\big) = \complement_X f^{-1}(G).$$

However by the definition of closure $\complement_X f^{-1}(G) \subseteq \overline{\complement_X f^{-1}(G)}$. Therefore $\complement_X f^{-1}(G) = \overline{\complement_X f^{-1}(G)}$, which means $\complement_X f^{-1}(G)$ is closed (theorem 5.4), i.e. $f^{-1}(G)$ is open. □

§6 Interior Points and Boundary

Definition 6.1. Let $(X, \mathscr{T})$ be a topological space and A be a subset of X. We call x an ***interior point*** of A if A is a neighbourhood of x. We call x an ***exterior point*** of A if $\complement_X(A)$ is a neighbourhood of x. The sets of all interior points of A is the ***interior*** of A, noted by $\operatorname{int} A$.

Theorem 6.1. *Let $(X, \mathscr{T})$ be a topological space and A be a subset of X.* $\operatorname{int} A = \complement_X\big(\overline{\complement_X(A)}\big)$, $\overline{A} = \complement_X\big(\operatorname{int} \complement_X(A)\big)$.

Proof. $x \in \operatorname{int} A$ implies an open set G which is a subset of A and $x \in G$. The complement of G is closed, thus its closure is $\complement_X(G)$ itself. $\complement_X(A) \subseteq \complement_X(G) \to \overline{\complement_X(A)} \subseteq \complement_X(G)$ (Lemma 4), therefore $x \notin \overline{\complement_X(A)}$, which is the first equation to prove.

To prove the second one only need to replace the A with $\complement_X(A)$ in the first equation. □

Theorem 6.2. *Let $(X, \mathscr{T})$ be a topological space and G be a subset of X.*

$$G \in \mathscr{T} \leftrightarrow \operatorname{int} G = G$$

Proof. G is open *iff* the complement of G is closed. And $\complement_X(G) = \overline{\complement_X(G)}$. The complement of the both size of this equation and Theorem 6.1 give the proof of the theorem. $\square$

With the propoties of closure and Theorem 6.1 the folowing statements should be easy to prove:

Theorem 6.3. *Let $(X, \mathscr{T})$ be a topological space and A, B be subsets of X.* $\operatorname{int}(A \cap B) = \operatorname{int} A \cap \operatorname{int} B$, $\operatorname{int}(\operatorname{int} A) = \operatorname{int} A$,

$$\operatorname{int} A = \bigcup_{G \in \mathscr{T} \wedge G \subseteq A} G.$$

Therefore we can say that the interior of A is the largest open set contianed in A.

Definition 6.2. Let $(X, \mathscr{T})$ be a topological space and A be a subset of X. A point x is said to be a ***boundary point*** of A if $\forall U \in \mathscr{U}(x)\big(U \cap A \neq \varnothing \wedge U \cap \complement_X(A) \neq \varnothing\big)$. The set of all boundary points of A is called the ***boundary*** of A, noted by ∂A.

Theorem 6.4. *Let $(X, \mathscr{T})$ be a topological space and A be a subset of X. (1) $\partial A = \overline{A} \cap \overline{\complement_X(A)}$; (2) $\operatorname{int} A = \overline{A} - \partial A$; (3) $\overline{A} = \operatorname{int} A \cup \partial A$.*

Proof. (1): Apply Theorem 5.3 to both A and $\complement_X(A)$.

(2): $\operatorname{int} A \cup \partial A = A \cup (\overline{A} \cap \overline{\complement_X(A)}) = \overline{A} \cap \big(\operatorname{int} A \cup \complement_X(\operatorname{int} A)\big) = \overline{A}$.

(3) $\overline{A} - \partial A = \overline{A} - (\overline{A} \cap \overline{\complement_X(A)}) = \overline{A} \cap \complement_X\big(\overline{\complement_X(A)}\big) = \overline{A} \cap \operatorname{int} A = \operatorname{int} A$. $\square$

§7 Basis

Definition 7.1 (Basis)**.** Let $(X, \mathscr{T})$ be a topological space and $\mathscr{B}$ be a subset of $\mathscr{T}$. If $\forall G \in \mathscr{T} \exists \mathscr{B}_G \in \mathscr{P}(\mathscr{B})\big(G = \cup \mathscr{B}_G\big)$, then we call $\mathscr{B}$ a ***basis*** or a ***base*** of the topology $\mathscr{T}$. We call the minimum topology of which $\mathscr{B}$ is a basis the ***closure*** of basis $\mathscr{B}$.

Theorem 7.1. *Let $(X, \mathscr{T})$ be a topological space and $\mathscr{B}$ be a subset of $\mathscr{T}$. $\mathscr{B}$ is a basis of $\mathscr{T}$* iff *$\forall x \in X \forall U \in \mathscr{U}_x \exists B \in \mathscr{B}(x \in B \wedge B \subseteq U)$.*

Proof. $\to$: $U \in \mathscr{U}_x$ implies an open set $G \in \mathscr{P}(U)$ that contians x, which is the union of elements in $\mathscr{B}$. Therefore there exists a $B \in \mathscr{B}$, which is the subset of U and it contians x.

$\leftarrow$: $\forall G \in \mathscr{T}$, it is a neighbourhood of all the points in G. For all x in G assign a $B_x \in \mathscr{B}$ which contians x and is the subset of G, so that G is the Union of all the B_x. $\square$

Theorem 7.2. *Let $\mathscr{B}$ be a basis of a topological space $(X, \mathscr{T})$.*

$$\forall B_1 \in \mathscr{B} \forall B_2 \in \mathscr{B} \forall x \in B_1 \cap B_2 \exists B \in \mathscr{B} (x \in B \subseteq B_1 \cap B_2).$$

Proof. By definition of topology and basis, B_1 and B_2 are both open and their intersection $B_1 \cap B_2$ is open as well. Then there exists a collections of sets in $\mathscr{B}$ whose union is $B_1 \cap B_2$, there must be at least a set B that contians x. $\square$

The topology $\mathscr{T}$ on X is determinded if the basis $\mathscr{B}$ is given, that is, if the union of $\mathscr{B}$ is X and it satisfies Theorem 7.2, then $\mathscr{T}$, which is defined by the collection of the unions of Bs in $\mathscr{B}$, is the *only* topology on X such that $\mathscr{B}$ is a basis of it.

For example, ***lower limit topology*** $\mathscr{T}_\ell$ on $\mathbb{R}$ is defined by giving a basis:

$$\mathscr{B}_\ell = \{[a,b) \mid a,b \in \mathbb{R} \wedge a < b\},$$

and $(\mathbb{R}, \mathscr{T}_\ell)$ is called the ***lower limit topological space***, the ***Sorgenfrey line*** or the ***arrow***, denoted by $\mathbb{R}_\ell$.

It is obvious that $\mathscr{T} \subsetneqq \mathscr{T}_\ell$, where $\mathscr{T}$ is the standard topology on $\mathbb{R}$.

Definition 7.2. Let $(X, \mathscr{T})$ be a topological space and $\mathscr{S}$ be a subset of $\mathscr{T}$. If the collection $\mathscr{B}$ of the finite intersections of the non-empty sets in $\mathscr{S}$ is a basis of $\mathscr{T}$, i.e.

$$\mathscr{B} = \left\{\bigcap_{i=1}^{n} S_i \mid S_i \in \mathscr{S},\ i = 1, \cdots, n,\ n \in \mathbb{N}_+\right\},$$

then we call $\mathscr{S}$ a ***subbasis*** or a ***subbase*** of $\mathscr{T}$.

A set X, given a collection $\mathscr{S}$ of subsets whose union is X itself, can be equiped with only one topology $\mathscr{T}$ so that $\mathscr{S}$ is a subbasis of the topology.

Theorem 7.3. *Let $(X, \mathscr{T}_X)$,$(Y, \mathscr{T}_Y)$ be two topological space and $f\colon X \to Y$. The following statements are equivalent:*

(1) $f \in C(X,Y)$;

(2) There exists a basis $\mathscr{B}_Y$ of Y that $\forall B \in \mathscr{B}_Y \big(f^{-1}(B) \in \mathscr{T}_X\big)$;

(3) There exists a subbasis $\mathscr{S}_Y$ of Y that $\forall S \in \mathscr{S}_Y \big(f^{-1}(S) \in \mathscr{T}_X\big)$.

Proof. It is almost obvious that (1) $\to$ (2) and (1) $\to$ (3). Since

$$f^{-1}\left(\bigcup_{B \in \mathscr{B}_Y} B\right) = \bigcup_{B \in \mathscr{B}_Y} f^{-1}(B)$$
$$f^{-1}\left(\bigcap_{k=1, S_k \in \mathscr{S}}^{n} S_k\right) = \bigcap_{k=1, S_k \in \mathscr{S}}^{n} f^{-1}(S_k)$$

, (2) $\to$ (1) and (3) $\to$ (2) can be proved. □

Definition 7.3. Let X be a topological space, $x \in X$ and $\mathscr{U}_x$ be a neighbourhood system of x. If $\mathscr{V}_x \subseteq U_x$, and $\forall U \in \mathscr{U}_x \exists V \in \mathscr{V}_x (V \subseteq U)$, then we call $\mathscr{V}_x$ a basis of $\mathscr{U}_x$ or a basis at point x. If $\mathscr{W}_x \subseteq U_x$, If the collection $\mathscr{V}_x$ of the finite intersections of the non-empty sets in $\mathscr{W}_x$ is a ***basis*** of $\mathscr{U}_x$, i.e.

$$\mathscr{V}_x = \left\{\bigcap_{i=1}^{n} W_i \mid W_i \in \mathscr{W}_x,\ i = 1, \cdots, n,\ n \in \mathbb{N}_+\right\},$$

then we call $\mathscr{W}_x$ a ***subbasis*** of $\mathscr{U}_x$ or of the point x.

There is also a theorem that is similar with the Theorem 7.3.

Theorem 7.4. *Let $(X, \mathscr{T}_X)$,$(Y, \mathscr{T}_Y)$ be two topological space and $f\colon X \to Y$. $x \in X$, $y = f(x) \in Y$. The following statements are equivalent:*

(1) f is continuous at point x;
(2) There exists a basis $\mathscr{V}_y$ at y that $\forall V \in \mathscr{V}_y\big(f^{-1}(V) \in \mathscr{U}_x\big)$;
(3) There exists a subbasis $\mathscr{W}_y$ at y that $\forall W \in \mathscr{W}_y\big(f^{-1}(W) \in \mathscr{U}_x\big)$.

The proof of this theorem is similar to the proof of the Theorem 7.3

Theorem 7.5. *Let X be a topological space and $x \in X$.*

(1) If $\mathscr{B}$ is a basis of X, then

$$\mathscr{B}_x = \{B \in \mathscr{B} \mid x \in B\}$$

is a basis at point x.

(2) If $\mathscr{S}$ is a subbasis of X, then

$$\mathscr{S}_x = \{S \in \mathscr{S} \mid x \in S\}$$

is a subbasis at point x.

Proof. From Theorem 7.1 (1) can be easily derived.
Let $\mathscr{B}_x$ be $\{B \in \mathscr{B} \mid x \in B\}$ where

$$\mathscr{B} = \left\{\bigcap_{i=1}^{n} S_i \mid S_i \in \mathscr{S},\ i = 1, \cdots, n,\ n \in \mathbb{N}_+\right\}.$$

Let

$$\tilde{\mathscr{B}}_x = \left\{\bigcap_{i=1}^{n} S_i \mid S_i \in \mathscr{S}_x,\ i = 1, \cdots, n,\ n \in \mathbb{N}_+\right\}.$$

We need to prove that $\tilde{\mathscr{B}}_x = \mathscr{B}_x$. If $U \in \mathscr{B}_x$ then $x \in U$ and $\exists \mathscr{S}_U \subseteq \mathscr{S}$ and $\cup \mathscr{S}_U = U$. Since $\forall S_U \in \mathscr{S}_U (x \in S_U)$ hence $\mathscr{S}_U \subseteq \mathscr{S}_x$. Therefore $\cup \mathscr{S}_U = U \in \tilde{\mathscr{B}}_x$. This is the proof of $\mathscr{B}_x \subseteq \tilde{\mathscr{B}}_x$, and $\tilde{\mathscr{B}}_x \subseteq \mathscr{B}_x$ can be proved similarly. $\square$

Chapter 2

Basic Properties of Topological Spaces

§8 Seperability

Seperabilities are some properties that describe how to can distinguish points in a topological space with their neighbourhoods. Since they are some additional properties to our definition, they are often call "axioms".

Definition 8.1 (T_0 *or* Kolmogorov)**.** Let $(X, \mathscr{T})$ be a topological space. If $\forall x \in X\, \forall y \in X(x \neq y)$, $\exists U \in \mathscr{U}(x)\, \exists V \in \mathscr{U}(y)$ s.t. $x \notin V \vee y \notin U$, we say that $(X, \mathscr{T})$ is a T_0 ***space*** or ***Kolmogorov space***.

Definition 8.2 (T_1 *or* Fréchet)**.** Let $(X, \mathscr{T})$ be a topological space. If $\forall x \in X\, \forall y \in X(x \neq y)$, $\exists U \in \mathscr{U}(x)\, \exists V \in \mathscr{U}(y)$ s.t. $x \notin V \wedge y \notin U$, we say that $(X, \mathscr{T})$ is a T_1 ***space***or ***Fréchet space***.

Theorem 8.1 (Finite subspaces are closed iff T_1)**.** *Topological space $(X, \mathscr{T})$ is T_1 iff $\forall F \in 2^X$, $\operatorname{card} F \in \mathbb{N} \to \complement_X F \in \mathscr{T}$.*

Corollary 2. *Let X be T_1, and $A \in 2^X$, $a \in A$. If a is a accumulation point of A, then $\forall U \in \mathscr{U}(a)$, $\operatorname{card} U \cap A \geq \omega$.*

Definition 8.3 (T_2 *or* Hausdorff)**.** Let $(X, \mathscr{T})$ be a topological space. If $\forall x \in X\, \forall y \in X(x \neq y)$, $\exists U \in \mathscr{U}(x)\, \exists V \in \mathscr{U}(y)$ s.t. $U \cap V = \varnothing$, we say that $(X, \mathscr{T})$ is a T_2 ***space***or ***Hausdorff space***.

Theorem 8.2 (The uniqueness of limit in T_2)**.** *Let X be T_2, $\langle x_n \rangle \in X^{\mathbb{N}}$. If a, b are both limits of $\langle x_n \rangle$, then $a = b$.*

§9 Countability

§10 Compactness

Definition 10.1 (Open cover)**.** Let $(X, \mathscr{T})$ be a topological space, $K \in 2^X$ and $\mathit{\Omega} \in 2^{\mathscr{T}}$. We call $\mathit{\Omega}$ to be an ***open cover*** over K, if $K \subset \cup\mathit{\Omega}$. If there are two open covers $\mathit{\Omega}$, $\mathit{\Omega}'$ over K, and $\mathit{\Omega}' \subset \mathit{\Omega}$, we say that $\mathit{\Omega}'$ is a ***subcover*** of $\mathit{\Omega}$.

Definition 10.2 (Compact set)**.** A set $K \in 2^X$ in topological space $(X, \mathscr{T})$ is called a ***compact set*** if each of its open covers has a *finite* subcover.

Specially, $\varnothing$ is compact.

Theorem 10.1. *A set $K \subset X$ is compact in $(X, \mathscr{T})$* **iff** *K is compact in $(K, \mathscr{T}_K)$ itself.*

This theorem tells a truth that whether K is compact or not doesn't dependent on the topological space it's in. This fact can be easily proved: we just need to notice that every open set G_K in $(K, \mathscr{T}_K)$ is an intersection of an open set G in $(X, \mathscr{T})$ and K.

Theorem 10.2 (Compact $\to$ closed (Hausdorff))**.** *If K is compact in a Hausdorff space $(X, \mathscr{T})$, then K is a closed set in $(X, \mathscr{T})$.*

Proof. Let x_0 be a limit point of K, which means $\forall U(x_0)$,

$$\operatorname{card} U(x_0) \cap K \notin \mathbb{N}.$$

Assume that $x_0 \notin K$. In a Hausdorff space, $\forall x \in K - \{x_0\}$, $\exists U(x)$ s.t. $U(x) \cap U(x_0) = \varnothing$. Such $U(x)$ construct an open cover $\mathit{\Omega} = \{U(x) | x \in K\} \subset 2^K$. Since K is compact, $\exists \mathit{\Omega}' \subset \mathit{\Omega}$ s.t. $\operatorname{card} \Omega \in \mathbb{N}$.

$$(\cup \Omega') \cap U(x_0) = \left(\bigcup_{k=1}^{n} U_k \right) \cap U(x_0) = \bigcup_{k=1}^{n} (U_k \cap U(x_0)) = \varnothing\,.$$

Since $K \subset \cup \Omega'$, x_0 is an exterior point of K, which leads to a contradiction.

Hence $x_0 \in K$. $\overline{K} = K$. ☐

Theorem 10.3. *Each decreasing* **nested sequences** *of non-empty compact sets has a non-empty limit, i.e. $\forall (K_n)_{n \in \mathbb{N}} \in \mathscr{P}(X)^{\mathbb{N}}$ s.t. $\forall n \in \mathbb{N}_+$, $K_n \supset K_{n+1} \wedge K_n \neq \varnothing \wedge$ (K_n is compact): $K_n \downarrow K \neq \varnothing$.*

Proof. Assume that $K = \varnothing$. Compact subsets of K_1 are all colsed, while their complements are all open. An open cover $\mathit{\Omega}$ can be constructed as $\{K_1 - K_n \mid n \in \mathbb{N}_+\}$. Since K_1 is compact, there would be a finite subcover $\mathit{\Omega}' \subset \mathit{\Omega}$, notice that $(X - K_n)_{n \in \mathbb{N}}$ is also a nested sequence, there must be one single $X - K_{n_0} \in \mathit{\Omega}'$ that covers K_1, which means $K_{n_0} = \varnothing$ contradicting that $\forall n \in \mathbb{N}_+$, K_n is non-empty. ☐

Theorem 10.4. *A Closed subset F of a compact set K is also compact.*

Proof. If $\mathit{\Omega}_F \subset 2^K$ is an open cover of F. Notice that $K - F$ is open, $\mathit{\Omega} = (\cup \mathit{\Omega}_F) \cap \{K - F\}$ constructs an open cover over over K. Since K is compact there must be a finite cover $\mathit{\Omega}' \subset \mathit{\Omega}$ which obviously also covers over F. ☐

The following propoties of compact sets are about topological spaces induced from metric spaces.

Definition 10.3 (net)**.** (X, d) is a metric space, $E \in 2^X$. E is called an ***ε-net*** if $\forall x \in X$,$\exists e \in E$, $d(e, x) < \varepsilon$.

Theorem 10.5 (Finite ε-net exists)**.** *If (K, d) is a compact metric space, then $\forall \varepsilon \in \mathbb{R}_+$,* $\exists$ finite *ε-net in (K, d).*

Proof. For each point $x \in K$, find it a $B(x, \varepsilon)$, of which an infinite cover Ω over K is made. Since K is compact, there exists a finite subcover $\mathit{\Omega}' = \{B(x_i, \varepsilon)\}_{i \in n}$ $(n \in \mathbb{N}_+)$. Therefore $\{x_i\}_{i \in n}$ is a finite ε-net in K. ☐

Theorem 10.6 (Sequentially compact)**.** *A metric space (K, d) is compact* **iff** *it is* **sequentially compact***, that is, $\forall (x_n)_{n \in \mathbb{N}} \in K^{\mathbb{N}}$, it has a convergent subsequence $(x_{k_n})_{n \in \mathbb{N}}$ ($k_n \in \mathbb{N}$; $k_{n+1} > k_n$) whose limit $a \in K$.*

To prove Theorem 10.6, we need to prove two lemmata first.

Lemma 5. *If (K, d) is sequentially compact, then $\forall \varepsilon \in \mathbb{R}_+$, $\exists$ finite ε-net in (K, d).*

Proof. Assume that $\exists \varepsilon_0 \in \mathbb{R}_+$, there were no finite ε_0-net in (K, d). Define such sequence: $(x_n)_{n \in \mathbb{N}}$ s.t. $\forall n \in \mathbb{N}$ $\forall k \in n$, $d(x_n, x_k) \geq \varepsilon_0$ (There would always be a next one since there exists no finite ε_0-net or $\{B(x_n; \varepsilon_0)\}_{n \in N}$ gives such). It has no convergent subsequence: if there were a $(x_{k_n})_{n \in \mathbb{N}}$ convergent to $a \in K$, $\exists N, M \in \mathbb{N}_+$, $d(x_N, x_M) \leq d(x_N, a) + d(x_M, a) \leq \varepsilon_0$, which lead to a contradictary. ☐

Lemma 6. *If (K, d) is sequentially compact then every nested sequence of closed non-empty sets $\{F_n\}_{n \in \mathbb{N}}$ in K have a non-empty intersection.*

Proof. Let $(x_{k_n})_{n \in \mathbb{N}}$ be a convergent subsequence of $(x_n)_{n \in \mathbb{N}}$, where $\forall n \in \mathbb{N}$, $x_n \in F_n$. Let a be the limit of $(x_{k_n})_{n \in \mathbb{N}}$.

Assume that $a \notin \bigcap_{n \in \mathbb{N}} F_n$, in a metric space, $\exists U(a) \in \mathscr{U}(a)$ s.t. $U(a) \cap \left(\bigcap_{n \in \mathbb{N}} F_n\right) = \varnothing$, therefore $U(a) \cap \left(\bigcap_{n \in \mathbb{N}} F_{k_n}\right) = \varnothing$. But this conflict the fact that $\exists N \in \mathbb{N}$, s.t. $n > N \to x_{k_n} \in U(a)$ while $x_{k_n} \in F_{k_n}$. ☐

Then we get back to the Theorem 10.6.

Proof. $\to$: If $\operatorname{card}\{x_n\}_{n \in \mathbb{N}} \in \mathbb{N}$, it is obvious; Now we let $\operatorname{card}\{x_n\}_{n \in \mathbb{N}} \notin \mathbb{N}$. We can always find finite $1/k$-net $\{B(a_{k,i}, 1/k)\}_{i \in m}$ (Theorem 10.5, $m \in \mathbb{N}$, $a_i \in K$), for all $k \in \mathbb{N}_+$. For each k, there must be at least one $B(a_{k,i_0}; 1/k)$ (for simplication, we denote a_{k,i_0} by a_k) that includes infinite elements in $(x_n)_{n \in \mathbb{N}}$. $\forall n \in \mathbb{N}_+$ (let $k_0 = 0$), select $x_{k_n} \in B(a_{n,0}; 1/n)$, and $\{\overline{B}(x_n; 1/k)\}$ is a nested sequence of a closed non-empty sets in sequentially compact K, (Lemma 6) $\lim_{n \to \infty} x_{k_n} \in K$.

$\leftarrow$: Assume that there were an open cover $\mathit{\Omega}$ over K having no finite subcover, $\forall n \in \mathbb{N}_+$, $\exists$ finite $1/n$-net (Lemma 6), in which there would be at least one x_n whose $\overline{B}(x_n; \frac{1}{n})$ can't be covered finitely. Then $\overline{B}(x_n; 1/n) \downarrow B = \{a\}$ (Theorem 10.3) can't be finitely covered by any subcover of $\mathit{\Omega}$, which means $\mathit{\Omega}$ can't cover the whole K, leading to the contradiction. ☐

We now prove a very useful special case for compact sets: compact sets in $\mathbb{R}$.

Lemma 7 (n-dimemsional cuboids are compact)**.** *Let I be a cuboid in $\mathbb{R}^n$ i.e.*

$$I := \{\boldsymbol{x} \in \mathbb{R}_n \mid a_i \leq x_i \leq b_i, \forall i \in n\}.$$

The cuboid I is compact.

Proof. We only need to prove that I is sequentially compact (Theorem 10.6). Let $(\boldsymbol{x}_i)_{i\in\mathbb{N}} \in I^{\mathbb{N}}$.

Denote $S_0 := I$. We divide S_m $(m \in \mathbb{N})$ into 2^n parts by equally dividing every $I_i := \{\boldsymbol{x} \in \mathbb{R}_n \mid a_i \le x_i \le b_i\}$ into two. Choose one that contains inifinite points of $(\boldsymbol{x}_i)_{i\in\mathbb{N}}$ as S_{m+1}. Then we get a closed nested sequence $S := (S_i)_{i\in\mathbb{N}}$. Notice that $\forall i \in \mathbb{N}$, S_i can be conceived as a product of n 1-dimension intervals. These intervals are also closed nested sequence, but in $\mathbb{R}$. We have leaned that $\exists! \boldsymbol{\xi} := (\xi_i)_{i\in n}$ s.t. $\{\boldsymbol{\xi}\} := \bigcap S$ from the theory of real numbers.

In every S_k we can find an $\boldsymbol{x}_{i_k}$, which is a convergent subsequence of the arbitrary sequence $(\boldsymbol{x}_i)_{i\in\mathbb{N}}$. □

Theorem 10.7 (Compact iff closed and bounded in $\mathbb{R}^n$)**.** *Let $K \in \mathscr{P}(\mathbb{R}^n)$, $n \in \mathbb{N}_+$. The set K is compact* iff *it is closed and bounded.*

Proof. $\rightarrow$: We have proved that compact sets are closed in a Hausdorff space (Theorem 10.2). Now we prove that K is also bounded. Let $\boldsymbol{x} \in \mathbb{R}^n$, and we could find an open covers of K:

$$\mathit{\Omega} := \{B(\boldsymbol{x}; n) \mid n \in \mathbb{N}_+\}\,.$$

Assume that we find a finite subcover $\mathit{\Omega}' := \{B(\boldsymbol{x}; n_k) \mid k \in m\}$, then $d(K) < n_m$.

$\leftarrow$: Since K is bounded, we can find it a n-dimension cuboid I, which we have proved to be compact (Lemma 7). The closed set K in the compact set I is compact (Theorem 10.4). □

Theorem 10.8 (Conservation of compactness)**.** *Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be two topological spaces. Let $K \subset X$ be a compact set. If $f\colon X \to Y \in C(X,Y)$, then $f(K)$ is compact.*

Proof. For each open cover $\Omega_Y = \{G_Y \in \mathscr{T}_Y\} \subset \mathscr{T}_Y$ over $f(K)$, $f^{-1}(G_Y) \in \mathscr{T}_X$ (Therem **??**). $f(K) \subset \cup\Omega_Y \Rightarrow K \subset f^{-1}(\cup\Omega_Y) = \cup\Omega_X$, where $\Omega_X = \{f^{-1}(G_Y) \mid G_Y \in \Omega_Y\}$ is an open cover over K. Since K is compact, $\exists \Omega'_X \subset \Omega_X$ $(|\Omega'_X| \in \mathbb{N}_+ \wedge K \subset \cup\Omega'_X)$, $f(K) \subset f(\cup\Omega'_X)$. $f(G'_X) \in \Omega_Y$, hence $\Omega'_Y = \{f(G'_X) \mid G'_X \in \Omega'_X\}$ is a finite subcover over $f(K)$. □

Theorem 10.9 (Weierstrass maximum-value theorem)**.** *Let K be a compact topological space, and $f \in C(K, \mathbb{R})$. $\exists x_m, x_M \in K$, s.t. $f(x_m) = m := \inf f(K)$, $f(x_M) = M := \sup f(K)$.*

Proof. By Theorem 10.8, $f(K)$ is also compact, and therefore closed and bounded (Theorem 10.7). If $M \notin f(K)$, then open covers $\{B(M; (M-m)/n) - \tilde{B}(M; (M-m)/(n+1)) \mid n \in \mathbb{N}_+\}$ would not have a finite subcover, which is a contradiction to the compactness of $f(K)$. □

Theorem 10.10 (Bijective from compact space to Hausdorff space is homeomorphism)**.** *Let $(K, \mathscr{T}_K)$ be a compact space and $(Y, \mathscr{T}_Y)$ be a Hausdorff space. Let $f \in Y^K$ be a bijective. If $f \in C(K, Y)$, then f is a homeomorphism.*

Proof. $\forall F = K - G$ s.t. $G \in \mathscr{T}_K$ is compact (Theorem 10.4). Hence $f(F)$ is compact (Theorem 10.8), then it is also closed (Theorem 10.2). This fact shows that f^{-1} is continuous (Theorem **??**). □

§11 Connectness

Definition 11.1 (Connectness)**.** If a topological space $(X, \mathscr{T})$ cannot be divided by two non-intersecting non-empty open sets, i.e.

$$\forall U \in \mathscr{T}(U \neq \varnothing \wedge U \neq X), \quad \complement_X U \notin \mathscr{T},$$

then we say it is a ***connected*** topological space. If X is a subspace of a topological space Y and it is connected, we say X is ***connected*** in Y or X is a connected set in Y.

If we write a non-connected space by the union of several non-intersecting connected subspaces, we call those connected subspaces ***connected branches***.

Theorem 11.1 (Conservation of connectedness)**.** *Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be two topological spaces, and $E \subset X$ be a connected set. If $f \in C(X, Y)$, then $f(E)$ is also connected.*

Proof. Only to notice that the open-closed sets in $(f(E), \mathscr{T}_{f(E)})$ have corresponding open-closed pre-images in $(E, \mathscr{T}_E)$. □

Theorem 11.2. *Let $(X, \mathscr{T})$ be a topological space, and $\{C_\alpha\}_{\alpha \in A}$ be connected subsets of X. If $\bigcap_{\alpha \in A} C_\alpha \neq \varnothing$, then $\bigcup\limits_{\alpha \in A} C_\alpha$ is also connected.*

Proof. Assume that $C = \bigcup_{\alpha \in A} C_\alpha$ were not connected, $\exists E \in 2^C$ s.t. $E \neq \varnothing$, $E \neq C$ and $E, C - E \in \mathscr{T}_C$. For E is not empty there exists a $\beta \in A$ s.t. $E \cap C_\beta \neq \varnothing$.

Now we show that $C_\beta \subset E$. Suppose that $C_\beta \not\subseteq E$, which implies that $(C - E) \cap C_\beta \neq \varnothing$. $E, C - E, C_\beta \in \mathscr{T}_C$, by the definition of the topology, $E \cap C_\beta, (C - E) \cap C_\beta \in \mathscr{T}_C$. This conflicts to the fact that C_β is connected. Therefore $C_\beta \subset E$.

Hence, there exists a $B \subsetneqq A$, $\bigcup_{\beta \in B} C_\beta = A$. Since C_γ, $\gamma \in A - B$ would have a empty intersection with E, which contradicts $\bigcap_{\alpha \in A} C_\alpha \neq \varnothing$. □

Theorem 11.3. *Connected sets have connected closure.*

Proof. □

Theorem 11.4. *$C \subset \mathbb{R}$ is connected* **iff** $\forall x, z \in C \forall y \in \mathbb{R}(x < y < z) \to y \in C$.

Proof. $\to$: Assume that there were such $y \in \mathbb{R}$ that $\exists x, z \in C$, $x < y < z$ but $y \notin C$. $\{x \in C \mid x < y\}$ and $\{x \in C \mid x > y\}$ are open in C for they are intersection of open sets in $\mathbb{R}$ and C. Since they're each other's complement, they are both open-closed, which conflicts to the definition of a connected set.

$\leftarrow$: It can be proved that $(\inf C, \sup C) \subset C$. Assume that there were an open-closed proper subset $E \neq \varnothing$ contained in C. Find two points $x \in E$, $z \in C - E$. Without loss of generality, let $x < z$. Since E and $C - E$ are closed, $c_1 = \inf\big(E \cap [a, b]\big) \in E$ while $c_2 = \inf\big((C - E) \cap [a, b]\big) \in C - E$. However $E \cap (C - E) = \varnothing$, hence $c_1 < c_2$, which means $(c_1, c_2) \cap E = \varnothing$. Here's the contradiction. □

Theorem 11.5 (Intermediate-value theorem)**.** *Let $(X, \mathscr{T})$ be a connected topological space, and $f \in C(X, \mathbb{R})$, $f(a) = A$, $f(b) = B$, $A < B$. $\forall C \in [A, B]$, $\exists c \in X$, $f(c) = C$.*

Proof. by Theorem 11.1, $f(X)$ must be a connected set. Hence by Theorem 11.4, we know that $\forall C \in [A, B]$, $C \in f(X)$. □

Definition 11.2 (Locally connected)**.** A topological space $(X, \mathscr{T})$ is said to be ***locally connected*** if $\forall x \in X$, $\exists U(x)$ s.t. $U(x)$ is connected.

Definition 11.3 (Path)**.** A continuous map $a\colon [0,1] \to X$ is a ***path***. $a(0)$ is called the ***initial point*** and $a(1)$ is called the ***terminal point*** When $a(0) = a(1)$, a is a ***loop***.

In some texts, the domain of a path can be any closed interval.
We can define the ***composition*** of two paths a, b with

$$ab(t) = \begin{cases} a(2t) & t \in [0, 1/2], \\ b(2t-1) & t \in [1/2, 1], \end{cases}$$

if $a(1) = b(0)$.

Theorem 11.6. *The composition of two paths is also a path.*

Proof. See Lemma 3. □

The ***inverse*** of a path a is $\bar{a}(t) = a(1-t)$.

Definition 11.4 (Path-connected)**.** If $\forall x, y \in X$, there exists a path a in X s.t. $a(0) = x$ and $a(1) = y$, we say that X is ***path-connected***

Theorem 11.7 (Path-connected then connected)**.** *A path-connected space is also connected.*

Theorem 11.8 (Conservation of path-connectedness)**.** *The image of a connected mapping from a path-connected set is also path-connected.*

§12 Convexity

Definition 12.1 (Convexity)**.** A ***convex set*** is a set in Euclidean space that contains all points on the straight segment jointing any two points i.e. $\forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in C$, $[\boldsymbol{x}_1, \boldsymbol{x}_2] \subset C$.

Chapter 3

Product Spaces and Quotient Spaces

§13 Product Space

Definition 13.1 (Product space)**.** Let $(X_1, \mathscr{T}_1, (X_2, \mathscr{T}_2)$ be two topological spaces. The topology $\mathscr{T}$ of the ***product space*** $(X_1 \times X_2, \mathscr{T})$ is defined by the closure of basis $\mathscr{B} = \{U_1 \times U_2 \mid U_1 \in \mathscr{T}_1 \wedge U_2 \in \mathscr{T}_2\}$.

By definition, we can tell that the projection mappings $\pi_1 \colon X_1 \times X_2 \to X_1; (x_1, x_2) \mapsto x_1$ is continuous, so is π_2.

Theorem 13.1. *$f \colon Y \to X_1 \times X_2 \in C(Y, X_1 \times X_2) \iff \pi_1 \circ f \in C(Y, X_1) \wedge \pi_2 \circ f \in C(Y, X_2)$.*

Theorem 13.2 (Connectedness is productible)**.** *If X and Y are two connected space, so is $X \times Y$.*

Proof. □

Theorem 13.3 (Path-connectedness is productible)**.** *If X and Y are two path-connected space, so is $X \times Y$.*

Theorem 13.4 (Compactness is productible)**.** *If X and Y are two compact space, so is $X \times Y$.*

Proof. □

§14 Quotient Space

Definition 14.1 (Quotient space)**.** Let $(X, \mathscr{T})$ be a topological space, $\sim$ be a equivalence relation of X, $p \colon X \to X/\sim; x \mapsto [x]$. The topological space $(X/\sim, p(\mathscr{T}))$ is called the ***quotient space*** of X with respect to $\sim$. Here p is called a ***quotient map*** associated to $\sim$.

Theorem 14.1. *$f \in C(X/\sim, Y) \iff f \circ p \in C(X, Y)$.*

Definition 14.2 (Quotient map)**.** $f \colon X \to Y$ is called a ***quotient map*** if:

(1) $f \in C(X, Y)$;
(2) $f(X) = Y$;
(3) $f^{-1}(V) \in \mathscr{T}_X \to V \in \mathscr{T}_Y$.

We can consider fibres of f be equivalence classes of X, in that way, $X/\sim_f \simeq Y$ (Y and $X/\sim_f$ are homeomorphic).

Theorem 14.2 (Continuous surjective to Hausdorff space is quotient). *Let Y be a Hausdorff space. If $f \in C(X, Y)$ is surjective, then f is a quotient map.*

Theorem 14.3 (Composition of quotient maps is quotient). *If $f: X \to Y$ and $g: Y \to Z$ are both quotient, then $g \circ f$ is also quotient.*

Definition 14.3 (Quotient space divided by a subset). $A \subseteq X$. If we define $\sim$ by $x \sim y \iff (x \in A \wedge y \in A) \vee (x \notin A \wedge y \notin A)$, we denote the quotient space $X/\sim$ by X/A.

Definition 14.4 (Topological cone). The ***topological cone*** of X is defined as $\mathrm{C}X := X \times [0,1]/X \times \{1\}$.

Fig. 3.1 illustrates a topological cone.

Definition 14.5 (Gluing). X and Y are both subspaces of Z, f is a surjective map from X to Y ($f(X) = Y$). Define $\sim_f$ on Z as: $x \sim y \iff x = y \vee y = f(x)$. Then $Z_f := Z/\sim_f$ is called the ***gluing*** of X and Y along f.

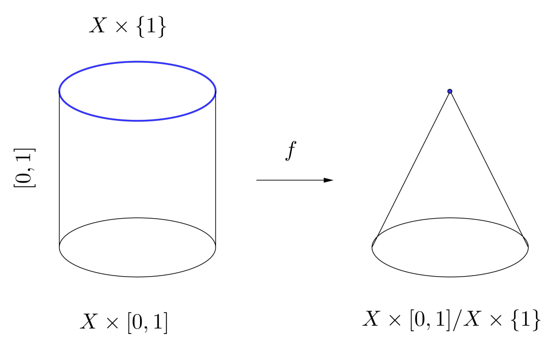

Figure 3.1: A topological cone

Chapter 4

Topological Manifolds

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Symbol List

Here listed the important symbols used in this notes

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