von Neumann Algebras

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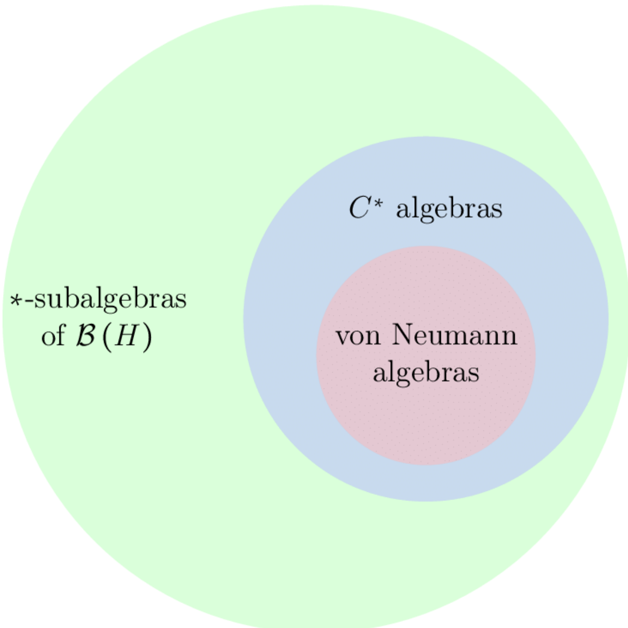

Preface

$\mathcal{H}$ means a Hilbert space by default. If not specified, the base field is $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Contents

Chapter 1

Operators on Hilbert Spaces

§1 Topologies on Spaces of Operators

Definition 1.1: Topology generated by semi-norms
Let V be a vector space over $\mathbb{K}$. If $\{\| - \|_i\}_{i\in I}$ is a family of seperated seminorms on V, where ``seperated'' means that

$$\forall v \in V,\ \exists i_0 \in I,\ v \neq 0 \to \|v\|_{i_0} \neq 0, \tag{1-1}$$

then the ***topology generated by*** $\{\| - \|_i\}_{i\in I}$ is the unique Hausdorff topology on V s.t.

$$\forall \langle v_n \rangle \in V^{\mathbb{N}}, \quad v_n \to v \in V \ \leftrightarrow\ \forall i \in I,\ \|v_n - v\|_i \to 0. \tag{1-2}$$

The locally convexity is given by the the balanced local base

$$\left\{ \left\{ v \in V \,\middle|\, \bigwedge_{k\in n} \|v\|_{i_k} < \varepsilon_k \right\} \,\middle|\, n \in \mathbb{N}_+,\ \forall k \in n,\ \varepsilon_k \in \mathbb{R}_+ \right\}. \tag{1-3}$$

Theorem 1.1 (Continuous linear functionals on a locally convex space)**.** *Let V be locally convex, with topology generated by $\{\| - \|_i \mid i \in I\}$. A linear functional $f \in V^*$ is continuous* iff

$$\begin{aligned} &\exists C \in \mathbb{R}_+,\ \exists n \in \mathbb{N}_+,\ \exists \{i_k \mid k \in n\} \subset I,\ \forall v \in V, \\ &|f(v)| \leq C \max_{k \in n} \|v\|_{i_k}. \end{aligned} \tag{1-4}$$

Definition 1.2: Strong-operator topology
Let $\mathcal{H}$ be a Hilbert space. The ***strong-operator topology*** (***SO***) on the space of all bounded operators $B(\mathcal{H})$ is the locally convex topology generated by the seminorms $\| - \|_x$ ($x \in \mathcal{H}$), defined as

$$\begin{aligned} \|\|_x \colon B(\mathcal{H}) &\to \mathbb{R} \\ T &\mapsto \|Tx\|. \end{aligned} \tag{1-5}$$

In SO topology, $T_n \to T$ iff $\forall x \in \mathcal{H}$, $\|T_n x - Tx\| \to 0$.

Definition 1.3: Weak-operator topology
The ***weak-operator topology*** (***WO***) on the space $B(\mathcal{H})$ is the locally convex topology generated by the seminorms $\|-\|_{x,y}$ ($x, y \in \mathcal{H}$), defined as

$$\begin{aligned} \|\|_{x,y} \colon B(\mathcal{H}) &\to \mathbb{R} \\ T &\mapsto |\langle Tx, y\rangle|. \end{aligned} \tag{1-6}$$

In WO topology, $T_n \to T$ iff $\forall x, y \in H$, $\langle T_n x, y\rangle \to \langle Tx, y\rangle$. If $T_n \to T$ in SO topology, then it is true also in WO topology.

Theorem 1.2. *Let $\mathscr{S}$ be a convex subset of $B(\mathcal{H})$. The WO closure of $\mathscr{S}$ coincides with the SO closure of $\mathscr{S}$.*

Chapter 2

Von Neumann Algebras

§2 Von Neumann Algebras

Definition 2.1: Von Neumann algebra
A C^*-subalgebra $\mathscr{A}$ of $B(\mathcal{H})$ is called a ***von Neumann algebra*** if $\mathscr{A}$ is closed in the SO topology.

By Theorem 1.2, a C^*-subalgebra $\mathscr{A}$ is a von Neumann algebra iff $\mathscr{A}$ is closed in the WO topology.

§3 Existence of Projections

Recall that a projection is an element $p \in \mathscr{A}$ s.t. $p^2 = p = p^*$, where $\mathscr{A}$ is a C^*-algebra.

Appendix A

Appendix

Bibliography

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Here listed the important symbols used in this notes.