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Part I

Basic Geometry

# Chapter 1

## Geometry in Regions of a Space

### §1 Co-ordinate and its transformation

**Definition 1.1** (Jacobian). A transformation of co-ordinate from  $\mathbf{x}$  to  $\mathbf{y}$

$$\mathbf{y}(\mathbf{x}) = y^i(x^j)\hat{\mathbf{e}}_i = y(x^j).$$

Its *Jacobian*:

$$\mathbf{J} = \left( \frac{\partial y^i}{\partial x^j} \right) \quad (1-1)$$

A *vector*  $\mathbf{u}$  at point  $\mathbf{x}_0$  under such transformation would follow:

$$v^i = \left. \frac{\partial y^i}{\partial x^j} \right|_{\mathbf{x}_0} u^j \quad (1-2)$$

$$\text{i.e.} \quad \mathbf{v} = \mathbf{J}_0 \mathbf{u}$$

A linear form (*covector*)  $\ell: \mathbf{x} \mapsto \ell(\mathbf{x}) = l_i x^i$  under such transformation would follow:

$$l'_i dy^i = l_j dx^j \quad \Rightarrow \quad l'_i = \left. \frac{\partial x^j}{\partial y^i} \right|_{\mathbf{x}_0} l_j \quad (1-3)$$

$$\text{i.e.} \quad \mathbf{l}' = \mathbf{l} \mathbf{J}_0^{-1}$$

A linear transformation  $\mathcal{L}: \mathbf{x} \mapsto \mathbf{L}\mathbf{x}$  where  $\mathbf{L} = (L^i_j)_{i,j \in n}$  under such transformation would follow:

$$\begin{aligned} dy(\mathcal{L}^i(\mathbf{x})) &= (L^i_j)^i dy^j \\ &= \left. \frac{\partial y^i}{\partial x^k} \right|_{\mathbf{x}_0} d\mathcal{L}^k(\mathbf{x}) = \left. \frac{\partial y^i}{\partial x^k} \right|_{\mathbf{x}_0} L^k_h dx^h = \left. \frac{\partial y^i}{\partial x^k} \right|_{\mathbf{x}_0} L^k_h \left. \frac{\partial x^h}{\partial y^j} \right|_{\mathbf{x}_0} dy^j \end{aligned}$$

$$(L')^i_j = \frac{\partial y^i}{\partial x^k} \bigg|_{\mathbf{x}_0} L^k_h \frac{\partial x^h}{\partial y^j} \bigg|_{\mathbf{x}_0} \quad \text{or} \quad \mathbf{L}' = \mathbf{J}_0 \mathbf{L} \mathbf{J}_0^{-1} \quad (1-4)$$

A bilinear form  $\mathcal{B}: \mathbf{x} \mapsto \mathbf{x}^T \mathbf{b} \mathbf{x} = x^i b_{ij} x^j$ :

$$b'_{ij} dy^i dy^j = b'_{ij} \frac{\partial y^i}{\partial x^k} \bigg|_{\mathbf{x}_0} \frac{\partial y^j}{\partial x^h} \bigg|_{\mathbf{x}_0} dx^k dx^h = b_{kh} dx^k dx^h \quad \Rightarrow \quad b'_{ij} = \frac{\partial x^k}{\partial y^i} \bigg|_{\mathbf{x}_0} b_{kh} \frac{\partial x^h}{\partial y^j} \bigg|_{\mathbf{x}_0}$$

i.e.  $\mathbf{b}' = (\mathbf{J}_0^{-1})^T \mathbf{b} \mathbf{J}_0^{-1}$  (1-5)

## §2 Riemannian and Pseudo-Riemannian Spaces

**Definition 2.1** (Riemannian metric). A **Riemannian metric**  $\mathbf{G}$  is a smooth, positive-definite quadratic form defined on a finite-dimensional vector space over  $\mathbb{R}$ .

Given a basis, we usually denote the Riemannian metric by  $g_{ij}(\mathbf{x})$ .

We can define **arc length**  $\ell$  and **inner product**  $\langle, \rangle$  in a **Riemannian space** (i.e. a vector space equipped with a Riemannian metric):

$$\ell := \int_{t_1}^{t_2} \sqrt{g_{ij}[\mathbf{x}(t)] \frac{dx^i}{dt} \frac{dx^j}{dt}} dt, \quad \langle \mathbf{u}, \mathbf{v} \rangle := g_{ij} u^i v^j.$$

We can also introduce the following notation:  $u_i := g_{ij} u^j$ , which means the linear form  $\mathbf{v} \mapsto u_i v^i$ ; and  $d\ell^2 = g_{ij} dx^i dx^j$ .

**Definition 2.2** (Euclidean metric). If a metric  $\mathbf{G}(\mathbf{x})$  is said to be **Euclidean** if there exists a coordinates  $\mathbf{y}(\mathbf{x})$  s.t.

$$g_{ij} = \delta_{k\ell} \frac{\partial y^k}{\partial x^i} \frac{\partial y^\ell}{\partial x^j}.$$

Such coordinates  $\mathbf{y}(\mathbf{x})$  are said to be a **Euclidean coordinates**.

**Definition 2.3** (Pseudo-Riemannian metric). A **pseudo-Riemannian metric**  $\mathbf{G}$  is a smooth, indefinite quadratic form defined on a finite-dimensional vector space over  $\mathbb{R}$ .

A pseudo-Riemannian metric shall have the following canonical form at some coordinates:

$$\mathbf{G} = \text{diag}(\eta_1^2, \dots, \eta_p^2, -\eta_{p+1}^2, \dots, -\eta_n^2)$$

where  $\eta$

By Sylvester's law of inertia, the index of inertia i.e. the number of positive terms on the canonical form, shall conserve under any coordinate change.

**Definition 2.4** (Pseudo-Euclidean metric). If a metric  $\mathbf{G}(\mathbf{x})$  is said to be **pseudo-Euclidean** if there exists a coordinates  $\mathbf{y}(\mathbf{x})$  s.t.

$$g_{ij} = \sum_{k=1}^p \frac{\partial y^k}{\partial x^i} \frac{\partial y^k}{\partial x^j} - \sum_{\ell=p+1}^n \frac{\partial y^\ell}{\partial x^i} \frac{\partial y^\ell}{\partial x^j}.$$

Such coordinates  $\mathbf{y}(\mathbf{x})$  are said to be a **pseudo-Euclidean coordinates**.

We denote a pseudo-Euclidean space by  $\mathbb{R}_{p,n-p}^n$ , where  $p$  is the index of inertia. Especially, we call  $\mathbb{R}_{1,3}^4$  the **Minkowski space**, since its significance in relativistic mechanics.

### §3 The Simplest Groups of Transformations of a Euclidean Space

When we say a **transformation** from  $\Omega$  to  $\Omega'$ , we refer to a bijective  $\varphi$  s.t. both  $\varphi$  and  $\varphi^{-1}$  are smooth, i.e. a **diffeomorphism**. If  $\Omega = \Omega'$ , we would call it the transformation of  $\Omega$ .

The transformations form a group, we might call it the **transformation group**.

**Definition 3.1** (Isometry). If a transformation  $\varphi$  of  $\Omega$ , with a metric  $\mathbf{G}$ , satisfies that  $\forall \mathbf{x} \in \Omega$ ,

$$g_{ij} = g_{k\ell} \frac{\partial \varphi^k}{\partial x^i} \frac{\partial \varphi^\ell}{\partial x^j},$$

we shall call it a **isometry**, or a **motion** of the given metric.

**Theorem 3.1** (Isometry group). *The isometries of a region form a group.*

We might call this group the **isometries** or the **group of motions**. If an isometry preserve the orientation of  $\mathbb{R}^n$ , we might call it a **proper isometry**. The proper isometries is a subgroup of the isometries of  $\mathbb{R}^n$ .

**Lemma 1** (Proper isometries of Euclidean space). *Every proper isometry of  $\mathbb{R}^n$  is either a translation along a vector, or a rotation about a point.*

The proper isometries can be parameterised by a special orthogonal matrix  $\mathbf{A} \in \text{SO}(n)$  (2D special orthogonal group), and a vector  $\mathbf{v}$ :

$$\mathbf{x} \mapsto \mathbf{Ax} + \mathbf{v}.$$

We can find the group a representation by a matrix:

$$\begin{pmatrix} \mathbf{A} & \mathbf{v} \\ \mathbf{0}^T & 1 \end{pmatrix} \quad (3-1)$$

If  $\det \mathbf{A} = \pm 1$ , then we have the parametrisation of the isometry of the Euclidean space  $\mathbb{R}^n$ .

**Theorem 3.2** (Mazur-Ulam Theorem). *The isometry group of  $\mathbb{R}^n$  is the semiproduct of translations ( $\simeq \mathbb{R}^n$ ) and the orthogonal group  $\text{O}(n)$  (rotations and rotatory reflections).*

The isometry group can be represented by Eq. 3-1.

**Theorem 3.3.**  *$\text{SO}(n)$ , as a Lie group, is continuous.*

**Proof.** find the rotation a basis so that it take canonical form

$$\begin{pmatrix} \pm 1 & & & & & & & \\ & \ddots & & & & & & \\ & & \pm 1 & & & & & \\ & & & \cos \theta_1 & -\sin \theta_1 & & & \\ & & & \sin \theta_1 & \cos \theta_1 & & & \\ & & & & & \ddots & & \\ & & & & & & \cos \theta_m & -\sin \theta_m \\ & & & & & & \sin \theta_m & \cos \theta_m \end{pmatrix}$$

Hence by replacing  $\theta_i$  by  $t\theta_i$ , we can get the rotation from identity continuously by changing  $t$  from 0 to 1.  $\square$

The **Galilean group** is the group of **Galilean transformation**, characterising by a rotatory reflection  $\mathbf{A} \in \text{O}(3)$ , a translation  $\mathbf{x}_0 \in \mathbb{R}^3$  a velocity  $\mathbf{v} \in \mathbb{R}^3$ :

$$x := (ct, \mathbf{x}) \mapsto (ct, \mathbf{Ax} + \mathbf{x}_0 - \mathbf{vt}).$$

The Galilean group can be represented as:

$$\begin{pmatrix} \mathbf{A} & \mathbf{x}_0 & \mathbf{v} \\ \mathbf{0}^T & 1 & 0 \\ \mathbf{0}^T & 0 & 1 \end{pmatrix} \quad (3-2)$$

## §4 Curvature and Torsion

Let  $\gamma: [t_1, t_2] \rightarrow \mathbb{R}^3; t \mapsto \mathbf{x}(t)$  be a diffeomorphism to a smooth curve in  $\mathbb{R}^3$ , the velocity and the acceleration are defined as:

$$\mathbf{v} := \dot{\mathbf{x}} := \frac{d\mathbf{x}}{dt} = \gamma'(t), \quad \mathbf{a} := \ddot{\mathbf{x}} := \frac{d^2\mathbf{x}}{dt^2} = \gamma''(t).$$

**Definition 4.1** (Natural parametre).  $s(t)$  is the **natural parametre** of the curve defined as:

$$s(t) = \int_0^t |\mathbf{v}| dt.$$

We shall denote the velocity and the acceleration by  $\mathbf{v}_s$  and  $\mathbf{a}_s$ , with natural parametre. To change the parametre, we only need:

$$ds = |\mathbf{v}| dt, \quad \frac{d}{ds} = \frac{1}{|\mathbf{v}|} \frac{d}{dt}.$$

By differentiating  $\mathbf{v}_s \cdot \mathbf{v}_s = 1$  we know the orthogonality of  $\mathbf{a}_s$  and  $\mathbf{v}_s$ .

Define three unit vectors for a parameterised curve  $t \mapsto \mathbf{x}(t)$ : the tangent vector  $\hat{\mathbf{t}} := \mathbf{v}/|\mathbf{v}|$ , the (primary) normal  $\hat{\mathbf{n}} := \frac{d\hat{\mathbf{t}}}{ds}/|\frac{d\hat{\mathbf{t}}}{ds}|$  and  $\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$ .

$$\mathbf{v}_s = \hat{\mathbf{t}}, \quad \hat{\mathbf{n}} = \hat{\mathbf{a}}_s/|\hat{\mathbf{a}}_s|.$$

**Definition 4.2** (Curvature and torsion). The **curvature**  $\kappa$  is the norm of  $\mathbf{a}_s$ , and the torsion is defined as  $\tau = \frac{d\hat{\mathbf{n}}}{ds}$ .

**Theorem 4.1** (Serret-Frenet formulae for space curve).

$$\frac{d}{ds} \begin{pmatrix} \hat{\mathbf{t}} \\ \hat{\mathbf{n}} \\ \hat{\mathbf{b}} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{t}} \\ \hat{\mathbf{n}} \\ \hat{\mathbf{b}} \end{pmatrix}. \quad (4-1)$$

## §5 Pseudo-Euclidean Spaces

$\mathbb{R}_{1,2}^4$  (or, more general case,  $\mathbb{R}_{1,n-1}^n$ ) is called the Minkowski space. The metric is always denoted by  $\eta_{\mu\nu}$ ,  $\eta_{00} = 1$ ,  $\eta_{ij} = -\delta_{ij}$ ,  $i, j \neq 0$ . (By convention, greek index take its value from 0 to  $n-1$ )

Considering the physics reality, we usually restrict our study to the region where  $x_\mu x^\mu = \eta_{\mu\nu} x^\mu x^\nu \geq 0$ , that is, the **time-like** region bounded by the **light cone** (or **isotropic cone**).

A **world-line** is the curve  $x(t) := (ct, \mathbf{x}(t))$ , where  $t := x^0/c$ . The natural parametre of a world-line is called the **proper time**  $\tau$  (not to be confused with the torsion).

The isomerics of  $\mathbb{R}_{1,3}^4$  is called the **Poincaré group**.

**Theorem 5.1.** *Any isometry of  $\mathbb{R}_{1,3}^4$  is an affine transformation.*

**Proof.** We need to prove that if there points are collinear, the image of them under the isometry is also collinear. After a translation, which is affine, we can set one of the point to be the origin, and the remaining two points are in the same branch (both  $t > 0$  or both  $t < 0$ ).

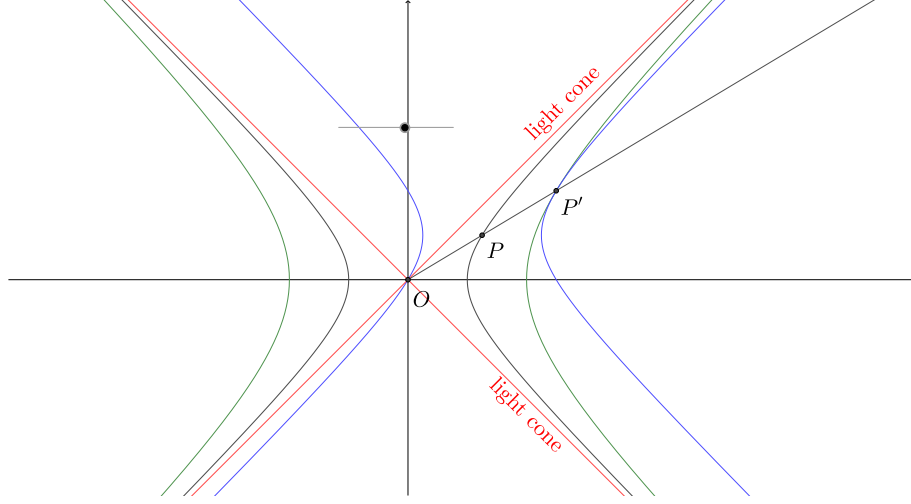


Figure 1.1: Pseudo-spheres in  $\mathbb{R}_{1,1}^2$ .

A pseudo-sphere centred at a point is a hyperboloid (of two sheets) centred at it and the principal axis is along the  $x^0$ -axis. Since the isometry preserve the distance, under which the points must move along the pseudo-sphere centred at origin. As illustrated in Fig. 1.1. If  $x_\mu x^\mu = \ell$ , and  $y^\mu = \lambda x^\mu$ , under an isometry,  $x \mapsto x'$ . The pseudo-sphere centred at  $x'$  of radius  $(\lambda - 1)\ell$  and the pseudo-sphere centred at origin of radius  $\lambda\ell$  are tangent at  $y'$ , which along with  $x'$  and the origin are collinear.  $\square$



# Bibliography

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# Symbol List

Here listed the important symbols used in this notes.

$g_{ij}(\boldsymbol{x})$ , 3

$\tau$ , 5, 6

$\kappa$ , 5

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