

# Algebraic Topology

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# Chapter 1

## Homotopy and Fundamental Group

### §1 Homotopy

**Definition 1.1** (Homotopy).  $f, g \in C(X, Y)$ . If  $\exists H \in C(X \times [0, 1], Y)$  s.t.  $H(x, 0) = f(x)$ ,  $H(x, 1) = g(x)$ , then we say  $f$  and  $g$  are **homotopic**, denoted by  $f \simeq g: X \rightarrow Y$  or just  $X \rightarrow Y$ .  $H$  is called a **homotopy** between  $f$  and  $g$ , denoted by  $H: f \simeq g$  or  $f \simeq_H g$ .

For  $t \in [0, 1]$ ,  $h_t: X \rightarrow Y; x \mapsto H(x, t)$  is called a ***t-slice***.

If  $f$  is homotopic to a constant mapping, we say that  $f$  is **null-homotopic**.

A **linear homotopy** is a homotopy between two functions to  $Y \subseteq \mathbb{R}^n$  that change linearly, i.e.

$$H(x, t) = (1 - t)f(x) + tg(x).$$

**Theorem 1.1** (Maps to convex set are homotopic).  $f, g \in C(X, Y)$ . If  $Y$  is a convex set in  $\mathbb{R}^n$ , then  $f \simeq g$ .

**Proof.** Consider linear homotopy. □

**Theorem 1.2.** *Homotopic relation is an equivalence relation.*

**Proof.** *reflexivity.*  $f \simeq f$ , just take  $H(x, t) = f(x)$  for any  $t$  (Such homotopy is called a **constant homotopy**).

*Symmetry.*  $f \simeq g$  then  $g \simeq f$ . Just take  $\bar{H}(x, t) = H(x, 1 - t)$  (Here  $\bar{H}$  is called the inverse of  $H$ ).

*Transitivity.*  $f \simeq g \wedge g \simeq h \rightarrow f \simeq h$ . Let

$$H_1 H_2(x, 2t) = \begin{cases} H_1(x, 2t) & t \in [0, 1/2], \\ H_2(x, 2t - 1) & t \in [1/2, 1]. \end{cases}$$

We can see that  $H_1 H_2$  is also a homotopy (see Theorem 11.6 in Point Set Topology) □

Hence, we can define **homotopy classes** on  $C(X, Y)$ , denoted by  $[X, Y]$ .

As you might expect after reading the proof of Theorem 1.2, the homotopies between mappings within a homotopy class form a group.

**Theorem 1.3** (Composition of homotopies).  $f_1 \simeq f_2: X \rightarrow Y$ ,  $g_1 \simeq g_2: Y \rightarrow Z$ , then  $g_1 \circ f_1 \simeq g_2 \circ f_2: X \rightarrow Z$ .

**Proof i.** Let  $F: f_1 \simeq f_2$ ,  $G: g_1 \simeq g_2$ . Define:

$$\mathbf{F}: X \times [0, 1] \rightarrow Y \times [0, 1]; (x, t) \mapsto (F(x, t), t).$$

It can be verified tht  $G \circ \mathbf{F}: g_1 \circ f_1 \simeq g_2 \circ f_2: X \rightarrow Z$ . □

**Proof ii.** Let  $F: f_1 \simeq f_2$ ,  $G: g_1 \simeq g_2$ .

We can verify that  $H_1: (x, t) \mapsto g_1 \circ F(x, t)$  is a homotopy between  $g_1 \circ f_1$  and  $g_1 \circ f_2$ ; Similarly  $H_2: g_1 \circ f_2 \simeq g_2 \circ f_2$  can be defined.

Now consider  $H = H_1 H_2$ , or in detailed,

$$H(x, t) = \begin{cases} g_1 \circ F(x, 2t) & (x, t) \in X \times [0, 1/2] \\ G(f_2(x), 2t - 1). & (x, t) \in X \times [1/2, 1] \end{cases}$$

□

**Lemma 1** (Identity map in convex space is null-homotopic).  $X \subseteq \mathbb{R}^n$  is a convex space.  $\forall x_0 \in X$ ,  $\text{id}_X \simeq (x \mapsto x_0)$ .

**Proof.** The linear homotopy can be constructed as:

$$H_{x_0}(x, t) = tx + (1 - t)x_0.$$

□

**Theorem 1.4** (Continuous mappings from a convex set are null-homotopic).  $X \subseteq \mathbb{R}^n$  is a convex set.  $\forall f \in C(X, Y)$ ,  $f$  is null-homotopic.

**Proof.** Let  $H_{x_0}(x, t) = tx + (1 - t)x_0$ . Then, any  $f: X \rightarrow Y$  can be written as  $f = f \circ \text{id}_X$ , hence  $f \simeq f \circ H_{x_0}(x, 1) = (x \mapsto f(x_0))$ , which means  $f$  is null-homotopic. □

**Theorem 1.5** (Constant mappings to a path-connected space belong to one homotopy class). If  $Y$  is a path-connected space,  $y_0 \in Y$ , then  $[X, Y] = [x \mapsto y_0]$  (i.e. homotopy class of constant mapping to  $\{y_0\}$ )

**Proof.** Let  $f_1(x) = y_1$ ,  $f_2(x) = y_2$  be two constant mappings,  $a$  is a path from  $y_1$  to  $y_2$ . Then the homotopy between  $f_1$  and  $f_2$  can be defined as:

$$H(x, t) = a(t).$$

□

**Definition 1.2** (Homotopy relative to a set). Let  $A \subseteq X$ ,  $H: f \simeq g$ . If  $\forall a \in A$ ,  $\forall t \in [0, 1]$ ,  $f(a) = g(a) = H(a, t)$ , we say that  $f$  and  $g$  are **homotopic relative to**  $A$ , denoted by  $H: f \simeq g \text{ rel } A$ .

We can have parallel results as Theorem 1.2 and Theorem 1.3:

**Theorem 1.6.** Given  $A \subseteq X$ ,  $\simeq \text{rel } A$  is an equivalence relation in  $C(X, Y)$ .

**Theorem 1.7** (Composition of relative homotopies).  $f_1 \simeq f_2: X \rightarrow Y \text{ rel } A$ ,  $g_1 \simeq g_2: Y \rightarrow Z \text{ rel } B$ , and  $f_1(A) \subset B$ , then  $g_1 \circ f_1 \simeq g_2 \circ f_2: X \rightarrow Z$ .

**Definition 1.3** (Fixed-endpoint Homotopy). Let  $a, b$  be two paths in  $X$ . If  $a \simeq b \text{ rel } \{0, 1\}$ , we say that  $a$  and  $b$  are **fixed-endpoint homotopic**. The paths in  $X$  modulus fixed-point homotopy is denoted by  $[X]$ , called the **path classes**. The path class which  $a$  belongs to is denoted by  $\langle a \rangle$ .

## §2 Fundamental Group

**Theorem 2.1.** Let  $a, b, c, d$  be four paths in  $X$ .

$$\begin{aligned} a \simeq b \text{ rel } \{0, 1\} &\leftrightarrow \bar{a} \simeq \bar{b} \text{ rel } \{0, 1\}, \\ a \simeq b \text{ rel } \{0, 1\} \wedge c \simeq d \text{ rel } \{0, 1\} \wedge a(1) = c(0) &\rightarrow ac \simeq bd \text{ rel } \{0, 1\}. \end{aligned}$$

**Definition 2.1** (Inverse and product of path classes).  $\alpha, \beta \in [X]$ ,  $a \in \alpha$ ,  $b \in \beta$ .  $b(0) = a(1)$ . We define  $\alpha^{-1} := \langle \bar{a} \rangle$  to be the **inverse** of the path class  $\alpha$ , and  $\alpha\beta := \langle ab \rangle$  to be the **product** of the two path classes  $\alpha$  and  $\beta$ .

While the product of paths does not obey associativity, we have:

**Theorem 2.2** (Associativity of product of path classes).  $\alpha, \beta, \gamma \in [X]$ .  $(\alpha\beta)\gamma = \alpha(\beta\gamma)$  (if they are productible).

**Proof.** Consider  $\forall a \in \alpha, \forall b \in \beta, \forall c \in \gamma$ .

Let  $\tilde{a}(t) = t/3$ ,  $\tilde{b}(t) = t/3 + 1/3$ ,  $\tilde{c}(t) = t/3 + 2/3$ .  $\tilde{a}, \tilde{b}$  and  $\tilde{c}$  are three paths in  $[0, 1]$ , and  $\tilde{a}(\tilde{b}\tilde{c}) \simeq (\tilde{a}\tilde{b})\tilde{c} \text{ rel } \{0, 1\}$ , since  $[0, 1]$  is convex, therefore there is a linear homotopy between the two product paths.

Now Let  $f: [0, 1] \rightarrow X$  be

$$f(t) = \begin{cases} a(3t), & t \in [0, 1/3]; \\ b(3t - 1), & t \in [1/3, 2/3]; \\ c(3t - 2), & t \in [2/3, 1]. \end{cases}$$

$a(bc) = f \circ \tilde{a}(\tilde{b}\tilde{c}) \simeq f \circ (\tilde{a}\tilde{b})\tilde{c} = (ab) \text{crel}\{0, 1\}$ , by Theorem 1.3.  $\square$

**Theorem 2.3** (Identity-like properties of point path).  $\alpha \in [X]$ . Let the initial and the terminal point of  $\alpha$  be  $x_0$  and  $x_1$ . (i)  $\alpha^{-1}\alpha = \langle t \mapsto x_1 \rangle$ ,  $\alpha\alpha^{-1} = \langle t \mapsto x_0 \rangle$ ; (ii)  $\alpha\langle t \mapsto x_0 \rangle = \alpha = \langle t \mapsto x_1 \rangle\alpha$ .

**Proof.** Note that  $\text{id}_{[0,1]}$  is a path in the convex set  $[0, 1]$ .  $\square$

For now path classes are not closed under production.

**Definition 2.2** (Fundamental group).  $x_0 \in X$ . The path classes of loops at  $x_0$  (paths that have both endpoints at  $x_0$ ), equipped with production, is the **fundamental group** of  $X$  at  $x_0$ , denoted by  $\pi_1(X, x_0)$ .

**Definition 2.3** (Homomorphism induced by continuous function).  $f \in C(X, Y)$ ,  $x_0 \in X$ . We define  $f_\pi: [X] \rightarrow [Y]$  as  $f_\pi(\langle a \rangle) = \langle f \circ a \rangle$ , where  $a$  is a path in  $X$ .

The limitation of  $f_\pi$  on  $\pi_1(X, x_0)$  is said to be a **homomorphism induced by  $f$** .

For simplicity, we would write such homomorphism by  $f_\pi$  (without explicitly referring limitation).

**Theorem 2.4** (Isomorphism induced by homeomorphism). Let  $f$  be a homeomorphism from  $X$  to  $Y$ , then  $\forall x_0 \in X$ ,  $f_\pi$  is an isomorphism from  $\pi_1(X, x_0)$  to  $\pi_1(Y, f(x_0))$ .

**Proof.**  $f^{-1} \circ f = \text{id}_X \rightarrow (f^{-1})_\pi \circ f_\pi = \text{id}_{\pi_1(X, x_0)}$ ,  $f \circ f^{-1} = \text{id}_Y \rightarrow f_\pi \circ (f^{-1})_\pi = \text{id}_{\pi_1(Y, f(x_0))}$ , therefore  $(f^{-1})_\pi$  is the inverse of  $f_\pi$ . An invertible homomorphism is an isomorphism.  $\square$

**Theorem 2.5** (Fundamental group of product space).  $x_0 \in X$ ,  $y_0 \in Y$ .

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

**Proof.** Let  $j_X \in C(X \times Y, X)$  and  $j_Y \in C(X \times Y, Y)$  be projections ( $j_X(x, y) = x$ ,  $j_Y(x, y) = y$ ), and define a homomorphism

$$\begin{aligned} \varphi: \quad \pi_1(X \times Y, (x_0, y_0)) &\rightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0); \\ \gamma &\mapsto ((j_X)_\pi(\gamma), (j_Y)_\pi(\gamma)). \end{aligned}$$

$\varphi$  is a *monomorphism*. Let  $\langle c \rangle \in \ker \varphi$  i.e.

$$\varphi(\langle c \rangle) = (\langle t \mapsto x_0 \rangle, \langle t \mapsto y_0 \rangle).$$

Let

$$H_X: j_X \circ c \simeq t \mapsto x_0 \text{ rel } \{0\}, \quad H_Y: j_Y \circ c \simeq t \mapsto y_0 \text{ rel } \{0\}.$$

The homotopy between  $c$  and  $t \mapsto (x_0, y_0)$  is defined as

$$F: [0, 1]^2 \rightarrow X \times Y; (t, s) \mapsto (H_X(t, s), H_Y(t, s)).$$

$\varphi$  is an *epimorphism*.  $\forall \langle a \rangle \in \pi_1(X, x_0)$  and  $\forall \langle b \rangle \in \pi_1(Y, y_0)$ .  
 $c: t \mapsto (a(t), b(t)) \in C([0, 1], X \times Y)$ .  $\langle c \rangle \in \varphi^{-1}(\{(\langle a \rangle, \langle b \rangle)\})$ .  $\square$

**Theorem 2.6** (Fundamental groups of path connected space at different points are isomorphic).  *$X$  is path connected,  $x_1, x_2 \in X$ .  $\pi_1(X, x_1) \cong \pi_1(X, x_2)$ .*

**Proof.**  $\langle a \rangle \in \pi_1(X, x_1)$ ,  $\langle b \rangle \in \pi_1(X, x_2)$ ,  $\langle c \rangle$  is a path class with initial point  $x_1$  and terminal point  $x_2$ .

It can be verified that

$$g: \pi_1(X, x_1) \rightarrow \pi_2(X, x_2); \langle a \rangle \mapsto \langle \bar{c}ac \rangle$$

is a homomorphism. Same as  $g'(\langle b \rangle) = cb\bar{c}$ .

$$g \circ g'(\langle b \rangle) = \langle \bar{c}cb\bar{c} \rangle = \text{id}_{\pi_1(X, x_2)}; \quad g' \circ g(\langle a \rangle) = \langle \bar{c}cac\bar{c} \rangle = \text{id}_{\pi_1(X, x_2)},$$

therefore  $g$  is an isomorphism.  $\square$



With Theorem 2.6, we can write the fundamental group of a path-connected space  $X$  by  $\pi_1(X)$ .

For different path-connected branches, a topological space can have different fundamental groups, while they are isomorphic within one branch.

**Definition 2.4** (Simply connected). If the fundamental group of a path connected space  $X$  is trivial i.e.  $\pi_1(X) \cong \{1\}$ , we say that  $X$  is *simply connected*.

**Theorem 2.7** (Convex set is simply connected). *If  $X \subset \mathbb{R}^n$  is convex, then  $X$  is simply connected.*

**Proof.**  $x_0 \in X$ ,  $a \in C([0, 1], X)$  s.t.  $a(0) = a(1) = x_0$ .  $H_{a, x_0}(s, t) = (1 - t)a(s) + tx_0$ .  $\square$

## §3 Examples of Fundamental Groups

### 3.1 $S^1$

**Definition 3.1** (Lift). Let  $X, Y, Z$  be three topological spaces, and  $f \in C(X, Z)$ ,  $p \in C(Y, Z)$ . If  $\tilde{f} \in C(X, Y)$ , s.t.  $f = p \circ \tilde{f}$ , we say that  $\tilde{f}$  is a *lift* of  $f$ .

In some case, given  $f$  and  $p$ ,  $\tilde{f}$  might do not exist.

**Lemma 2** (Lift of path).  $a \in C([0, 1], S^1)$ ,  $p: \mathbb{R} \rightarrow S^1; x \mapsto e^{2\pi xi}$ . Let  $t_0 \in \mathbb{R}$  s.t.  $p(x_0) = a(0)$ . There exists a unique lift  $\tilde{a} \in C(\mathbb{R}, S^1)$  of  $a$  s.t.  $\tilde{a}(0) = x_0$ .

**Proof.** *Existence.* The collection of open sets that the images under  $a$  do not cover  $S^1$ ,  $\{(\alpha_i, \beta_i) \cap [0, 1] \mid a_i, b_i \in \mathbb{R}^I \wedge S^1 \subsetneq a((\alpha_i, \beta_i))\}$ , is a cover of  $S^1$  by the definition of continuity. Since  $S^1$  is compact, there exists a finite subcover  $\{(\alpha_i, \beta_i) \cap [0, 1] \mid a_i, b_i \in \mathbb{R}^n \wedge S^1 \subsetneq a((\alpha_i, \beta_i))\}$ , where  $n \in \mathbb{N}$ . By dividing these open intervals into closed intervals that has no inner points intersecting, we can get  $\Omega = \{I_k := [t_i, t_{i+1}] \mid k \in m\}$  (This can be done by sorting  $\alpha_i$  and  $\beta_i$ ).

The mapping  $p$  is locally homeomorphic i.e. there exists  $[x_i, x'_i] \subset \mathbb{R}$  s.t.  $p_i := p|_{[x_i, x'_i]}: [x_i, x'_i] \rightarrow a(I_i)$  is a homeomorphism (and  $p_i(x_i) = a(t_i)$ ), therefore  $\tilde{a}_i := p_i^{-1} \circ a$  is a lift of  $a_i := a|_{I_i}$ .

Since  $p_0(t_0) = a(t_0)$ ,  $p_{i+1}(t_i) = p_i(t_i)$ , we can define piecewisely the lift of  $a$  by  $\tilde{a} = \cup \{\tilde{a}_i \mid i \in m\}$ .

*Uniqueness.* Let  $\tilde{a}'$  be another lift of  $a$ ,  $p(\tilde{a}'(t) - \tilde{a}(t)) = p \circ \tilde{a}'(t)/p \circ \tilde{a}(t) = a(t)/a(t) = 1$ , therefore  $\tilde{a}'(t) - \tilde{a}(t) \in \mathbb{Z}$ . Since  $[0, 1]$  is connected, the image of  $t \mapsto \tilde{a}'(t) - \tilde{a}(t)$  must be connected, which is possible only if it is constant.  $\tilde{a}'(0) = \tilde{a}(0) = x_0$ , therefore  $\tilde{a} = \tilde{a}'$ .  $\square$

Notice that we have the freedom to set  $\tilde{a}(0) \in \mathbb{Z}$  (the lift is unique after setting that), what really matter is the difference  $\tilde{a}(1) - \tilde{a}(0)$ . One can proof that  $q(a) := \tilde{a}(1) - \tilde{a}(0)$  does not depend on the chose of  $\tilde{a}(0) \in \mathbb{Z}$ . We call  $q(a)$  the **loop number** of path  $a$ .

**Lemma 3** (Two loops that are never antipodal have the same loop number). *Let  $a, b$  be two loops at  $z_0$  in  $S^1$ . If  $\forall t \in [0, 1]$ ,  $a(t) \neq -b(t)$ , then  $q(a) = q(b)$ .*

*Proof.* Choose  $\tilde{a}(0) = \tilde{b}(0) = 0$  (if not so, just translate the lift by an integer). In this case,  $q(a) = \tilde{a}(1)$ ,  $q(b) = \tilde{b}(1)$ .

If  $q(a) \neq q(b)$ , without loss of generality,  $q(a) > q(b)$ , then  $f := t \mapsto \tilde{a}(t) - \tilde{b}(t)$  is a continuous function from a compact space  $[0, 1]$  to  $\mathbb{R}$ , therefore by the connectedness of  $[0, 1]$ ,  $\exists t_0 \in [0, 1]$  s.t.  $f(t_0) = 1/2 \in [0, q(a) - q(b)]$ , when

$$p \circ \tilde{a}(t_0) + p \circ \tilde{b}(t_0) = e^{2\pi i(\tilde{b}(t_0)+1/2)} + e^{2\pi i\tilde{b}(t_0)} = 0.$$

$\square$

**Lemma 4** (Same loop number iff homotopic relative to endpoint). *Let  $a, b$  be two loops at  $z_0$  in  $S^1$ .  $a \simeq b$  rel  $\{0\}$  iff  $q(a) = q(b)$ .*

*Proof.*  $\rightarrow$ : Let  $H: a \simeq b$  rel  $\{0\}$ ,  $h_s = t \mapsto H(t, s)$ ,  $f_t = s \mapsto H(t, s)$ .

$\forall (t, s) \in [0, 1]^2$ ,  $U := \{H(t, s)e^{i\theta} \mid \theta \in (-\pi, \pi)\} \in \mathcal{U}_{S^1}(H(t, s))$ . Since  $f_t \in C([0, 1], S^1)$ ,  $\exists V(s) \in \mathcal{U}_{[0, 1]}(s)$  s.t.  $H(V(s)) \subset U$ . Which

means,  $\forall t \in [0, 1], \forall s_1, s_2 \in V(s), f_t(s_1) \neq -f_t(s_2)$  or  $h_{s_1}(t) \neq -h_{s_2}(t)$ . By Lemma 3,  $q(h_s) = q(h_{s'})$ .

$\Omega = \{V(s) \mid s \in [0, 1]\}$  is an open cover of the compact space  $[0, 1]$ , therefore has a finite subcover  $\Omega = \{V_i \in \Omega \mid i \in n\}$ . In each  $V(s_i)$ ,  $h_s$  has the same loop numbers.

We therefore have  $q(a) = q(h_0) = q(h_1) = q(b)$ .

$\leftarrow: H: [0, 1]^2 \rightarrow S^1; (t, s) \mapsto p((1-s)\tilde{a}(t) - s\tilde{b}(t))$ .  $\square$

**Theorem 3.1.**  $\pi_1(S^1) \cong \mathbb{Z}$ .

**Proof.**  $z_0 \in S^1$ . Let  $Q: \pi_1(S^1, z_0) \rightarrow \mathbb{Z}; \langle a \rangle \mapsto q(a)$ .

$\forall \langle a \rangle, \langle b \rangle \in \pi_1(S^1, z_0)$ , choose  $\tilde{a}(1) = \tilde{b}(0)$ ,

$$\begin{aligned} Q(\langle a \rangle \langle b \rangle) &= Q(\langle ab \rangle) = q(ab) \\ &= \tilde{b}(1) - \tilde{a}(0) = \tilde{b}(1) - \tilde{b}(0) + \tilde{a}(1) - \tilde{a}(0) = q(a) + q(b) \\ &= Q(\langle a \rangle) + Q(\langle b \rangle), \end{aligned}$$

which means  $Q$  is a homomorphism.

By Lemma 4,  $Q$  is a monomorphism.  $\forall n \in \mathbb{Z}, Q(\langle t \mapsto e^{2\pi n t i} \rangle) = n$ , therefore  $Q$  is also an epimorphism.  $\square$

### 3.2 $S^n, n > 2$

The situation for  $S^n$  is much simpler:

**Theorem 3.2.**  $\forall n \in \mathbb{N}$ , if  $n \geq 2$ , then  $S^n$  is simply connected.

**Proof.** Let  $x_0 \in S^n$ , and  $a$  be a loop at  $x_0$  in  $S^n$ .  $x \in S^n$  and  $x \neq x_0$ . Embed  $S^n$  into  $\mathbb{R}^{n+1}$  and let  $B(x; \delta)$  be a  $(n+1)$ -D ball with radius  $\delta$  around  $x$  that  $x_0 \notin B(x; \delta)$ .

$a^{-1}(B(x; \delta) \cap S^n)$  is a collection of open, disjoint intervals in  $[0, 1]$ , which can be considered as an open cover of  $a^{-1}(\{x\})$ , which is compact. Let the finite subcover of  $a^{-1}(\{x\})$  be  $\{(\alpha_i, \beta_i) \cap [0, 1] \mid \alpha_i, \beta_i \in \mathbb{R}, i \in m\}$ , where  $m \in \mathbb{N}$ .

Let  $P: \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R} \rightarrow \mathbb{R}^{n+1}$  be:

$$P(y, y_0, r) = \frac{y - y_0}{\|y - y_0\|} r + y_0,$$

which means project  $y$  to the sphere with radius  $r$  around  $y_0$ .

Now we define the loop  $b$  that go as:

$$b(t) = \begin{cases} a(t), & t \notin (\alpha_i, \beta_i), \forall i \in m; \\ P[P(a(t), x, \delta), 0, 1], & t \in (\alpha_i, \beta_i) - a^{-1}(\{x\}), \exists i \in m; \\ \lim_{t' \rightarrow t} b(t'), & t \in a^{-1}(\{x\}) \cap (\alpha_i, \beta_i), \exists i \in m, \\ & t' \in (\alpha_i, \beta_i) - a^{-1}(\{x\}) \end{cases}$$

and the homotopy between  $a$  and  $b$  can be written as

$$H: [0, 1]^2 \rightarrow S^n; (t, s) \mapsto P[(1 - s)a(t) + sb(t), 0, 1].$$

Since  $b$  is a loop in  $S^n - \{x\}$ , while  $S^n - \{x\} \cong \mathbb{R}^n$  (by stereographic projection), which is simply connected, we know that  $b$  is homotopic to  $t \mapsto x_0$  i.e. null-homotopic.  $\square$

By Theorem 2.5, the fundamenal group of  $T^2 := S^1 \times S^1$  is  $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$ , which is not isomorphic to  $S^2$ , therefore  $T^2 \not\cong S^2$ .

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# Symbol List

Here listed the important symbols used in these notes

$\langle a \rangle$ , 4

$\tilde{f}$ , 7

$f_\pi$ , 5

$f \simeq g$ , 1

$f \simeq_H g$ , 1

$\bar{H}$ , 2

$H: f \simeq g$ , 1

$H: f \simeq g \text{ rel } A$ , 3

$\pi_1(X)$ , 7

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