Algebraic Topology

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Chapter 1

Homotopy and Fundamental Group

§1 Homotopy

Definition 1.1 (Homotopy)**.** $f, g \in C(X, Y)$. If $\exists H \in C(X \times [0,1], Y)$ s.t. $H(x,0) = f(x)$, $H(x,1) = g(x)$, then we say f and g are ***homotopic***, denoted by $f \simeq g \colon X \to Y$ or just $X \to Y$. H is called a ***homotopy*** between f and g, denoted by $H \colon f \simeq g$ or $f \simeq_H g$.

For $t \in [0,1]$, $h_t \colon X \to Y; x \mapsto H(x,t)$ is called a t***-slice***.

If f is homotopic to a constant mapping, we say that f is ***null-homotopic***.

A ***linear homotopy*** is a homotopy between two functions to $Y \subseteq \mathbb{R}^n$ that change linearly, i.e.

$$H(x,t) = (1-t)f(x) + tg(x).$$

Theorem 1.1 (Maps to convex set are homotopic)**.** $f, g \in C(X, Y)$. *If Y is a convex set in $\mathbb{R}^n$, then $f \simeq g$.*

Proof. Consider linear homotopy. □

Theorem 1.2. *Homotopic relation is an equivalence relation.*

Proof. *reflexity.* $f \simeq f$, just take $H(x,t) = f(x)$ for any t (Such homotopy is called a ***constant homotopy***).

Symmetry. $f \simeq g$ then $g \simeq f$. Just take $\bar{H}(x,t) = H(x, 1-t)$ (Here $\bar{H}$ is called the inverse of H).

Transivity. $f \simeq g \wedge g \simeq h \to f \simeq h$. Let

$$H_1H_2(x, 2t) = \begin{cases} H_1(x, 2t) & t \in [0, 1/2], \\ H_2(x, 2t-1) & t \in [1/2, 1]. \end{cases}$$

We can see that H_1H_2 is also a homotopy (see Theorem 11.6 in Point Set Topology) □

Hence, we can define ***homotopy classes*** on $C(X,Y)$, denoted by $[X,Y]$.

As you might expect after reading the proof of Theorem 1.2, the homotopies between mappings within a homotopy class form a group.

Theorem 1.3 (Composition of homotopies)**.** *$f_1 \simeq f_2 \colon X \to Y$, $g_1 \simeq g_2 \colon Y \to Z$, then $g_1 \circ f_1 \simeq g_2 \circ f_2 \colon X \to Z$.*

Proof i. Let $F \colon f_1 \simeq f_2$, $G \colon g_1 \simeq g_2$. Define:

$$\boldsymbol{F} \colon X \times [0,1] \to Y \times [0,1]; (x,t) \mapsto (F(x,t),t).$$

It can be verified tht $G \circ \boldsymbol{F} \colon g_1 \circ f_1 \simeq g_2 \circ g_2 \colon X \to Z$. ☐

Proof ii. Let $F \colon f_1 \simeq f_2$, $G \colon g_1 \simeq g_2$.

We can verify that $H_1 \colon (x,t) \mapsto g_1 \circ F(x,t)$ is a homotopy between $g_1 \circ f_1$ and $g_1 \circ f_2$; Similarly $H_2 \colon g_1 \circ f_2 \simeq g_2 \circ f_2$ can be defined.

Now consider $H = H_1 H_2$, or in detailed,

$$H(x,t) = \begin{cases} g_1 \circ F(x,2t) & (x,t) \in X \times [0,1/2] \\ G(f_2(x), 2t-1). & (x,t) \in X \times [1/2,1] \end{cases}$$

☐

Lemma 1 (Identity map in convex space is null-homotopic)**.** *$X \subset \mathbb{R}^n$ is a convex space. $\forall x_0 \in X$, $\mathrm{id}_X \simeq (x \mapsto x_0)$.*

Proof. The linear homotopy can be constructed as:

$$H_{x_0}(x,t) = tx + (1-t)x_0.$$

☐

Theorem 1.4 (Continuous mappings from a convex set are null-homotopic)**.** *$X \subseteq \mathbb{R}^n$ is a convex set. $\forall f \in C(X,Y)$, f is null-homotopic.*

Proof. Let $H_{x_0}(x,t) = tx + (1-t)x_0$. Then, any $f \colon X \to Y$ can be written as $f = f \circ \mathrm{id}_X$, hence $f \simeq f \circ H_{x_0}(x,1) = (x \mapsto f(x_0))$, which means f is null-homotopic. ☐

Theorem 1.5 (Constant mappings to a path-connected space belong to one homotopy class)**.** *If Y is a path-connected space, $y_0 \in Y$, then $[X,Y] = [x \mapsto y_0]$ (i.e. homotopy class of constant mapping to $\{y_0\}$)*

Proof. Let $f_1(x) = y_1$, $f_2(x) = y_2$ be two constant mappings, a is a path from y_1 to y_2. Then the homotopy between f_1 and f_2 can be defined as:

$$H(x,t) = a(t).$$

☐

Definition 1.2 (Homotopy relative to a set)**.** Let $A \subseteq X$, $H \colon f \simeq g$. If $\forall a \in A$, $\forall t \in [0,1]$, $f(a) = g(a) = H(a,t)$, we say that f and g are ***homotopic relative to*** A, denoted by $H \colon f \simeq g \mathrm{rel} A$.

We can have parallel results as Theorem 1.2 and Theorem 1.3:

Theorem 1.6. *Given $A \subseteq X$, $\simeq \operatorname{rel} A$ is an equivalence relation in $C(X, Y)$.*

Theorem 1.7 (Composition of relative homotopies). *$f_1 \simeq f_2 \colon X \to Y \operatorname{rel} A$, $g_1 \simeq g_2 \colon Y \to Z \operatorname{rel} B$, and $f_1(A) \subset B$, then $g_1 \circ f_1 \simeq g_2 \circ f_2 \colon X \to Z$.*

Definition 1.3 (Fixed-endpoint Homotopy). Let a, b be two paths in X. If $a \simeq b \operatorname{rel} \{0, 1\}$, we say that a and b are ***fixed-endpoint homotopic***. The paths in X modulus fixed-point homotopy is denoted by $[X]$, called the ***path classes***. The path class which a belongs to is denoted by $\langle a \rangle$.

§2 Fundamental Group

Theorem 2.1. *Let a, b, c, d be four paths in X.*

$$a \simeq b \operatorname{rel} \{0, 1\} \leftrightarrow \bar{a} \simeq \bar{b} \operatorname{rel} \{0, 1\},$$
$$a \simeq b \operatorname{rel} \{0, 1\} \wedge c \simeq d \operatorname{rel} \{0, 1\} \wedge a(1) = c(0) \rightarrow ac \simeq bd \operatorname{rel} \{0, 1\}.$$

Definition 2.1 (Inverse and product of path classes). $\alpha, \beta \in [X]$, $a \in \alpha$, $b \in \beta$. $b(0) = a(1)$. We define $\alpha^{-1} := \langle \bar{a} \rangle$ to be the ***inverse*** of the path class α, and $\alpha\beta := \langle ab \rangle$ to be the ***product*** of the two path classes α and β.

While the product of paths does not obey associativity, we have:

Theorem 2.2 (Associativity of product of path classes). *$\alpha, \beta, \gamma \in [X]$. $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ (if they are productible).*

Proof. Consider $\forall a \in \alpha$, $\forall b \in \beta$, $\forall c \in \gamma$.

Let $\tilde{a}(t) = t/3$, $\tilde{b}(t) = t/3 + 1/3$, $\tilde{c}(t) = t/3 + 2/3$. $\tilde{a}, \tilde{b}$ and $\tilde{c}$ are three paths in $[0, 1]$, and $\tilde{a}(\tilde{b}\tilde{c}) \simeq (\tilde{a}\tilde{b})\tilde{c} \operatorname{rel} \{0, 1\}$, since $[0, 1]$ is convex, therefore there is a linear homotopy between the two product paths.

Now Let $f \colon [0, 1] \to X$ be

$$f(t) = \begin{cases} a(3t), & t \in [0, 1/3]; \\ b(3t - 1), & t \in [1/3, 2/3]; \\ c(3t - 2), & t \in [2/3, 1]. \end{cases}$$

$a(bc) = f \circ \tilde{a}(\tilde{b}\tilde{c}) \simeq f \circ (\tilde{a}\tilde{b})\tilde{c} = (ab)c \operatorname{rel} \{0, 1\}$, by Theorem 1.3. □

Theorem 2.3 (Identity-like properties of point path). *$\alpha \in [X]$. Let the initial and the terminal point of α be x_0 and x_1.* (i) *$\alpha^{-1}\alpha = \langle t \mapsto x_1 \rangle$, $\alpha\alpha^{-1} = \langle t \mapsto x_0 \rangle$;* (ii) *$\alpha\langle t \mapsto x_0 \rangle = \alpha = \langle t \mapsto x_1 \rangle\alpha$.*

Proof. Note that $\mathrm{id}_{[0,1]}$ is a path in the convex set $[0, 1]$. □

For now path classes are not closed under production.

Definition 2.2 (Fundamental group). $x_0 \in X$. The path classes of loops at x_0 (paths that have both endpoints at x_0), equiped with production, is the ***fundamental group*** of X at x_0, denoted by $\pi_1(X, x_0)$.

Definition 2.3 (Homomorphism induced by continuous function)**.** $f \in C(X, Y)$, $x_0 \in X$. We define $f_\pi \colon [X] \to [Y]$ as $f_\pi(\langle a \rangle) = \langle f \circ a \rangle$, where a is a path in X.

The limitation of f_π on $\pi_1(X, x_0)$ is said to be a ***homomorphism induced by*** f.

For simplicity, we would write such homomorphism by f_π (without explicitly referring limitation).

Theorem 2.4 (Isomorphism induced by homeomrphism)**.** *Let f be a homeomorphism from X to Y, then $\forall x_0 \in X$, f_π is an isomorphism from $\pi_1(X, x_0)$ to $\pi_1(Y, f(x_0))$.*

Proof. $f^{-1} \circ f = \mathrm{id}_X \;\to\; (f^{-1})_\pi \circ f_\pi = \mathrm{id}_{\pi_1(X,x_0)}$, $f \circ f^{-1} = \mathrm{id}_Y \;\to\; f_\pi \circ (f^{-1})_\pi = \mathrm{id}_{\pi_1(Y,f(x_0))}$, therefore $(f^{-1})_\pi$ is the inverse of f_π. An invertible homomorphism is an isomorphism. □

Theorem 2.5 (Fundamental groups of path connected space at different points are isomorphic)**.** *X is path connected, $x_1, x_2 \in X$. $\pi_1(X, x_1) \cong \pi_1(X, x_2)$.*

Proof. $\langle a \rangle \in \pi_1(X, x_1)$, $\langle b \rangle \in \pi_1(X, x_2)$, $\langle c \rangle$ is a path class with initial point x_1 and terminal point x_2.

It can be verify that

$$g \colon \pi_1(X, x_1) \to \pi_2(X, x_2); \langle a \rangle \mapsto \langle \bar{c} a c \rangle$$

is a homomorphism. Same as $g'(\langle b \rangle) = c b \bar{c}$.

$$g \circ g'(\langle b \rangle) = \langle \bar{c} c b \bar{c} c \rangle = \mathrm{id}_{\pi_1(X,x_2)}; \quad g' \circ g(\langle a \rangle) = \langle c \bar{c} a c \bar{c} \rangle = \mathrm{id}_{\pi_1(X,x_2)},$$

therefore g is an isomorphism. □

With Theorem 2.5, we can write the fundamental group of a path-connected space X by $\pi_1(X)$.

For different path-connected branches, a topological space can have different fundamental groups, while they are isomorphic within one branch.

Definition 2.4 (Simply connected)**.** If the fundamental group of a path connected space X is trivial i.e. $\pi_1(X) \cong \{1\}$, we say that X is ***simply connected***.

Theorem 2.6 (Convex set is simply connected)**.** *If $X \subset \mathbb{R}^n$ is convex, then X is simply connected.*

Proof. $x_0 \in X$, $a \in C([0, 1], X)$ s.t. $a(0) = a(1) = x_0$. $H_{a,x_0}(s, t) = (1 - t)a(s) + t x_0$. □

Now we can calculate the fundamental group of S^n.

Definition 2.5 (Lift)**.** Let X, Y, Z be three topological spaces, and $f \in C(X, Z)$, $p \in C(Y, Z)$. If $\tilde{f} \in C(X, Y)$, s.t. $f = p \circ \tilde{f}$, we say that $\tilde{f}$ is a ***lift*** of f.

In some case, given f and p, $\tilde{f}$ might do not exist.

Lemma 2 (Lift of path)**.** *$a \in C([0, 1], S^1)$, $p \colon \mathbb{R} \to S^1; x \mapsto \mathrm{e}^{2\pi x \mathrm{i}}$. Let $t_0 \in \mathbb{R}$ s.t. $p(x_0) = a(0)$. There exists a unique lift $\tilde{a} \in C(\mathbb{R}, S^1)$ of a s.t. $\tilde{a}(0) = x_0$.*

Proof. *Existence.* The collection of open sets that the images under a do not cover S^1, $\{(\alpha_i, \beta_i) \cap [0, 1] \mid a_i, b_i \in \mathbb{R}^I \wedge S^1 \subsetneq a((\alpha_i, \beta_i))\}$, is a cover of S^1 by the definition of continuity. Since S^1 is compact, there exists a finite subcover $\{(\alpha_i, \beta_i) \cap [0, 1] \mid a_i, b_i \in \mathbb{R}^n \wedge S^1 \subsetneq a((\alpha_i, \beta_i))\}$, where $n \in \mathbb{N}$. By dividing these open intevals into closed intevals that has no inner points intersecting, we can get $\Omega = \{I_k := [t_i, t_{i+1}] \mid k \in m\}$ (This can be done by sorting α_i and β_i).

The mapping p is locally homeomorphic i.e. there exists $[x_i, x_i'] \subset \mathbb{R}$ s.t. $p_i := p|_{[x_i,x_i']} \colon [x_i, x_i'] \to a(I_i)$ is a homeomorphism (and $p_i(x_i) = a(t_i)$), therefore $\tilde{a}_i := p_i^{-1} \circ a$ is a lift of $a_i := a|_{I_i}$.

Since $p_0(t_0) = a(t_0)$, $p_{i+1}(t_i) = p_i(t_i)$, we can define piecewisely the lift of a by $\tilde{a} = \cup\{\tilde{a}_i \mid i \in m\}$.

Uniqueness. Let $\tilde{a}'$ be another lift of a, $p(\tilde{a}'(t) - \tilde{a}(t)) = p \circ \tilde{a}'(t)/p \circ \tilde{a}(t) = a(t)/a(t) = 1$, therefore $\tilde{a}'(t) - \tilde{a}(t) \in \mathbb{Z}$. Since $[0, 1]$ is connected, the image of $t \mapsto \tilde{a}'(t) - \tilde{a}(t)$ must be connected, which is possible only if it is constant. $\tilde{a}'(0) = \tilde{a}(0) = x_0$, therefore $\tilde{a} = \tilde{a}'$. $\square$

bibliography

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Symbol List

Here listed the important symbols used in these notes

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