C^*-Algebras

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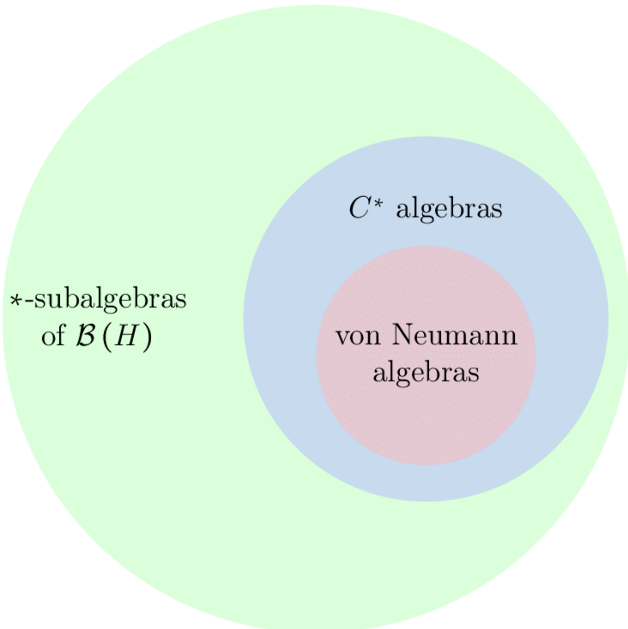

Preface

$\mathcal{H}$ means a Hilbert space by default. If not specified, the base field is $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Contents

Chapter 1

Banach Algebras

§1 Banach Algebras & Invertible Group

Definition 1.1: Banach algebra

A ***Banach algebra*** is a unital algebra $\mathscr{B}$ together with a norm $\| - \|$ s.t.

1. $\|1_{\mathscr{B}}\| = 1$;
2. $\forall a, b \in \mathscr{B},\ \|ab\| \leq \|a\|\|b\|$.

The most important example of Banach algebras may be the algebra $B(\mathcal{H})$ of bounded linear operators on a Banach space $\mathcal{H}$ with the operator norm:

$$\|L\|_{B(\mathcal{H})} = \sup_{\|v\|=1} \|Lv\|. \tag{1-1}$$

Chapter 2

C^*-Algebras

§2 C^*-Algebras

Definition 2.1: C^*-algebra
A C^****-algebra*** is a Banach algebra $\mathscr{A}$ together with an ***involution*** $*\colon \mathscr{A} \to \mathscr{A}$ s.t.

1. $\forall a \in \mathscr{A}$, $a^{**} = a$;
2. $\forall a, b \in \mathscr{A}$, $(ab)^* = b^* a^*$;
3. $\forall a, b \in \mathscr{A}$, $\forall \alpha, \beta \in \mathbb{K}$, $(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*$[a];
4. $\forall a \in \mathscr{A}$, $\|a^* a\| = \|a\|^2$.

The element a^* is called the ***adjoint*** of a.

[a] If $\mathbb{K} = \mathbb{R}$, then $\overline{\alpha} := \alpha$.

Definition 2.2: Projection
An element $p \in \mathscr{A}$ is called a ***projection*** if $p^2 = p = p^*$.

§3 Commutative C^*-Algebras

Appendix A

Appendix

Bibliography

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Here listed the important symbols used in this notes.