Differential Geometry

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Chapter 1

Manifolds

Chapter 2

Scalar and Vector Fields

§1 Scalar Fields

Definition 1.1 (Scalar Field)**.** Let M be a smooth manifold, $f \in C^{(\infty)}(M)$ is called a ***scalar field***.

The scalar field over a manifold, form an algebra.

§2 Vector Fields

Definition 2.1 (vector field)**.** A ***vector field*** v over manifold M is a $C^{(\infty)}(M) \to C^{(\infty)}(M)$ map that satisfies

(a) $\forall f, g \in C^{(\infty)}(M)$, $\forall \lambda, \mu \in \mathbb{R}$, $v(\lambda f + \mu g) = \lambda v(f) + \mu v(g)$ (*linearity*).

(b) $\forall f, g \in C^{(\infty)}(M)$, $v(fg) = v(f)g + fv(g)$

Definition 2.2 (tangent vector)**.** Let v be a vector field over M, p be a point on M. The tangent vector v_p at p is defined as a $C^{(\infty)}(M) \to C^{(\infty)}(M)$ map that satisfies

$$v_p(f) = v(f)(p). \tag{2-1}$$

The collection of tangent vectors at p is called the ***tangent space*** at p, denoted by T_pM.

The derivative of a path $\gamma\colon [0,1] \to M$ (or $\mathbb{R} \to M$) in a smooth manifold is defined as:

$$\begin{aligned} \gamma'(t)\colon C^{(\infty)}(M) &\to \mathbb{R}; \\ \gamma'(t)(f) &= \frac{\mathrm{d}}{\mathrm{d}t} f \circ \gamma(t) \end{aligned} \tag{2-2}$$

We can see that $\gamma'(t) \in T_{\gamma(t)}M$.

§3 Covariant and Contravariant

Definition 3.1 (pullback)**.** Let f be a scalar field over M, $\varphi \in C^{(\infty)}(M, N)$. Then the ***pullback*** of f by φ is defined as

$$\varphi^* f = f \circ \varphi \in C^{(\infty)}(N). \tag{3-1}$$

Fields that are pullbacked are ***covariant*** fields.

Definition 3.2 (pushforward)**.** Let v_p be a tangent vector of M at p, $\varphi \in C^{(\infty)}(M, N)$, $q = \varphi(p)$. Then the ***pushforward*** of v_p by φ is defined as

$$(\varphi_* v)_q(f) = v_p(\varphi^* f). \tag{3-2}$$

Note that the pushforward of a vector field can only be obtained when φ is a diffeomorphism.

Fields that are pushforwarded are ***contravariant*** fields.

Mathematicians and physicists might have disagreement on whether a tangent vector is covariant or contravariant. This is because of that physicists might consider the coordinates (v^μ) of a tangent vector as a vector field, instead of linear combination of bases ∂_μ.

§4 Flows

Let a path $\gamma\colon \mathbb{R}$ follows a vector field (a velocity field), that is

$$\gamma'(t) = v_{\gamma(t)}, \tag{4-1}$$

then we call γ the ***integral curve*** through $p := gamma(0)$ of the vector field v.

Definition 4.1. Suppose v is an integrable vector field. Let $\varphi_t(p)$ be the point at time t on the integral curve through p.

$$\varphi_t\colon M \to M \tag{4-2}$$

is then called a ***flow*** generated by v.

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(p) = v_{\varphi_t(p)}. \tag{4-3}$$

Chapter 3

Differential Forms

§5 1-forms

Definition 5.1 (1-form)**.** A ***1-form*** $\mathrm{d}\omega$ on M is a $\mathrm{Vect}(M) \to C^{(\infty)}(M)$ which satisfies that

(a) $\forall v, w \in \mathrm{Vect}(M)$, $\forall f, g \in C^{(\infty)}(M)$,

$$\mathrm{d}\omega(fv + gw) = f\mathrm{d}\omega(v) + g\mathrm{d}\omega(w) \tag{5-1}$$

Symbol List

Here listed the important symbols used in these notes

T_pM, 3

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1-form, 5

contravariant, 3
covariant, 3

flow, 4

integral curve, 4

pullback, 3
pushforward, 3

scalar field, 2

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