Category Theory

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Preface

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Chapter 1

Categories

§1 Categories

Definition 1.1 (Category)**.** A ***category*** $\mathcal{C}$ consists of three ingredients:

1. A *class* $\mathrm{obj}(\mathcal{C})$, called the ***objects***;
2. For any $A, B \in \mathrm{obj}(\mathcal{C})$, a set of ***morphisms*** $\mathrm{Hom}(A, B)$;
3. A function $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$, called the ***composition***, for any $A, B, C \in \mathrm{obj}(\mathcal{C})$, denoted as $(f, g) \mapsto gf$,

and they follow the following axioms:

(i) If $(A, B) \neq (A', B')$, then $\mathrm{Hom}(A, B) \cap \mathrm{Hom}(A', B') = \varnothing$;

(ii) *Associativity*: the composition is associative, i.e. $h(gf) = (hg)f$;

(iii) *Identity*: For any $A \in \mathrm{obj}(\mathcal{C})$, there is an identity morphism $\mathrm{id}_A \in \mathrm{Hom}(A, A)$, such that $f\,\mathrm{id}_A = f = \mathrm{id}_A\, f$, for any $B \in \mathrm{obj}(\mathcal{C})$ and $f \in \mathrm{Hom}(A, B)$.

A morphism can be shown by:

$$A \xrightarrow{\quad f \quad} B$$

Examples of categories: Set, Grp, Ab, Top, Ord, Ring, Mod, …
If $\mathrm{obj}(\mathcal{C})$ is a set, then $\mathcal{C}$ is called a ***small category***.
If $(X, \leq)$ is a preorder set, then $\forall x, y \in X$,

$$\mathrm{Hom}(x,y) = \begin{cases} \varnothing & x > y, \\ \{(x,y)\} & x \leq y, \end{cases} \tag{1-1}$$

and $(y,z)(x,y) = (x,z)$. With this we can say that X is a category. The morphism (x,y) is also denoted by i_y^x.

Definition 1.2 (Isomorphism)**.** Let $\mathcal{C}$ be a category and $A, B \in \mathrm{obj}(\mathcal{C})$, $f \in \mathrm{Hom}(A,B)$. If $\exists g \in \mathrm{Hom}(B,A)$ s.t. $gf = \mathrm{id}_B$ and $fg = \mathrm{id}_A$, then f is called an ***isomorphism*** from A to B. g is called the ***inverse*** of f.

Definition 1.3 (Subcategory)**.** We say $\mathcal{S}$ a ***subcategory*** of $\mathcal{C}$, if

(i) $\mathrm{obj}(\mathcal{S}) \subseteq \mathrm{obj}(\mathcal{C})$;

(ii) $\forall A, B \in \mathrm{obj}(\mathcal{S})$, $\mathrm{Hom}_{\mathcal{S}}(A,B) \subseteq \mathrm{Hom}_{\mathcal{C}}(A,B)$;

(iii) $\forall A, B, C \in \mathrm{obj}(\mathcal{S})$,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

then gf is the same in both $\mathrm{Hom}_{\mathcal{S}}(A,C)$ and $\mathrm{Hom}_{\mathcal{C}}(A,C)$,

(iv) $\forall A \in \mathrm{obj}(\mathcal{S})$, $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{S}}(A,A)$ is the same in $\mathrm{Hom}_{\mathcal{C}}(A,A)$.

Definition 1.4 (Full subcategory)**.** Let $\mathcal{S}$ be a subcategory of $\mathcal{C}$. If $\forall A, B \in \mathrm{obj}(\mathcal{S})$, $\mathrm{Hom}_{\mathcal{S}}(A,B) = \mathrm{Hom}_{\mathcal{C}}(A,B)$, then $\mathcal{S}$ is called a ***full subcategory*** of $\mathcal{C}$.

Definition 1.5 (Generated full subcategory)**.** For any subclass $S \subseteq \mathrm{obj}(\mathcal{C})$, one can find a full subcategory $\mathcal{S}$ of $\mathcal{C}$ s.t. $\mathrm{obj}(\mathcal{S}) = S$, which is called the full subcategory generated by S.

Top_2 is the full subcategory of Top that is generated by the class of all Hausdorff spaces.

Definition 1.6 (Opposite category)**.** Let $\mathcal{C}$ be a category. Then $\mathcal{C}^{\mathrm{op}}$ is the category that:

1. $\mathrm{obj}(\mathcal{C}^{\mathrm{op}}) = \mathrm{obj}(\mathcal{C})$;
2. $\forall A, B \in \mathrm{obj}(\mathcal{C})$,

$$\mathrm{Hom}_{\mathcal{C}}(A, B) = \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(B, A). \tag{1-2}$$

§2 Definitions of Different Categories

Chapter 2

Functors

§3 Functors

Definition 3.1 (Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A ***functor*** $F : \mathcal{C} \to \mathcal{D}$ is a function that satisfies the following axioms:

(i) $\forall A \in \mathrm{obj}(\mathcal{C})$, $F(A) \in \mathrm{obj}(\mathcal{D})$;

(ii) $\forall A, B \in \mathrm{obj}(\mathcal{C}), \forall f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$;

(iii) $\forall A, B, C \in \mathrm{obj}(\mathcal{C})$,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

then $F(gf) = F(g)F(f)$.

(iv) $\forall A \in \mathrm{obj}(\mathcal{C})$, $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

We can restate some definition using functors

Theorem 3.1 (Subcategory, in language of functors)**.** *Let $\mathcal{C}$ and $\mathcal{S}$ be two categories, $\mathcal{S} \subseteq \mathcal{C}$. If the inclusion $I\colon \mathcal{S} \to \mathcal{C}$ is a functor, then $\mathcal{S}$ is a subcategory of $\mathcal{C}$.*

The ***identity functor*** from $\mathcal{C}$ to itself is $1_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ s.t. $\forall C, D \in \mathcal{C}$, $\forall f \in \mathrm{Hom}(C, D)$,

$$1_{\mathcal{C}}(C) = C, \quad 1_{\mathcal{C}}(f) = f. \tag{3-1}$$

Theorem 3.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. $F\colon \mathcal{C} \to \mathcal{D}$ is a functor. $\forall A, B \in \mathrm{obj}(\mathcal{C})$, if $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is an isomorphism, then $F(f)$ is an isomorphism.*

Definition 3.2 (Hom)**.** Let $\mathcal{C}$ be a category and $A \in \mathrm{obj}(\mathcal{C})$. The ***Hom functor*** $F_A\colon \mathcal{C} \to \mathsf{Set}$ is defined as

$$\begin{aligned} &F_A(B) = \mathrm{Hom}(A, B), \\ &F_A(f)\colon\ \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C);\ h \mapsto fh. \end{aligned} \tag{3-2}$$

The Hom functor is also denoted by $\mathrm{Hom}(A, -)$. We call the $F_A(f) =: \mathrm{Hom}(A, f)$ the ***induced map***, and denote it by f_*

$$f_* h = fh. \tag{3-3}$$

Definition 3.3 (Faithful functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A ***faithful functor*** $F\colon \mathcal{C} \to \mathcal{D}$ is a functor that satisfies $\forall A, B \in \mathrm{obj}(\mathcal{C})$,

$$i\colon\ \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B));\ f \mapsto F(f) \tag{3-4}$$

is injective.

Definition 3.4 (Concrete category)**.** Let $\mathcal{C}$ be a category. $\mathcal{C}$ is called a ***concrete category*** if there exists a faithful functor $F\colon \mathcal{C} \to \mathsf{Set}$.

§4 Contravariant Functors

Definition 4.1 (Contravariant functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A ***contravariant functor*** $F : \mathcal{C} \to \mathcal{D}$ is a function that satisfies the following axioms:

(i) $\forall A \in \mathrm{obj}(\mathcal{C})$, $F(A) \in \mathrm{obj}(\mathcal{D})$;

(ii) $\forall A, B \in \mathrm{obj}(\mathcal{C}), \forall f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(B), F(A))$;

(iii) $\forall A, B, C \in \mathrm{obj}(\mathcal{C})$,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

then $F(gf) = F(f)F(g)$.

(iv) $\forall A \in \mathrm{obj}(\mathcal{C})$, $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

To distinguish functors from contravariant functors, we sometimes call the functors ***covariant functors***.

$-^{\mathrm{op}} : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ is a contravariant functor.

Definition 4.2 (Contravariant Hom)**.** Let $\mathcal{C}$ be a category and $A \in \mathrm{obj}(\mathcal{C})$. The ***contravariant Hom functor*** $F_A : \mathcal{C} \to \mathsf{Set}$ is defined as

$$\begin{aligned} F_A(B) &= \mathrm{Hom}(B, A), \\ F_A(f)&\colon\ \mathrm{Hom}(B, A) \to \mathrm{Hom}(C, A);\ h \mapsto hf. \end{aligned} \tag{4-1}$$

The contravariant Hom functor is also denoted by $\mathrm{Hom}(-, A)$. We call the $F_A(f) =: \mathrm{Hom}(f, A)$ the ***induced map***, and denote it by f^*

$$f^* h = hf. \tag{4-2}$$

§5 Diagrams

Definition 5.1 (Diagram)**.** A ***diagram*** in a category $\mathcal{C}$ is a functor $D\colon \mathcal{D} \to \mathcal{C}$ where $\mathcal{D}$ is a small category.

We have already seemed drawn diagrams like

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"] \arrow[dr, bend right, "h'"'] & B \arrow[d, "g"] \\ & C \end{tikzcd} \tag{5-1}$$

where $A, B, C \in D(\mathrm{obj}(\mathcal{D}))$, and each arrow from one to another is a morphism in the image of morphism in $\mathcal{D}$ under D e.g. $\exists D_A, D_B \in \mathrm{obj}(\mathcal{D})$ s.t.

$$f \in D(\mathrm{Hom}_{\mathcal{D}}(D_A, D_B)) \subseteq \mathrm{Hom}_{\mathcal{C}}(A, B). \tag{5-2}$$

Definition 5.2 (Path)**.** A ***path*** in a category $\mathcal{C}$ is a functor $P\colon n+1 \to \mathcal{C}$ where $n+1$ is considered as a preorder with morphism defined in Eq. (1-1).

Conventionally we denote a path as:

$$P_0 \xrightarrow{f_0} P_1 \xrightarrow{f_1} P_2 \longrightarrow \cdots \longrightarrow P_{n-1} \xrightarrow{f_{n-1}} P_n \tag{5-3}$$

A ***simple path*** is a path such that $\forall i, j \in n+1$, $P_i = P_j \to i = j$.

A diagram D ***commutes*** iff $A, B \in D(\mathrm{obj}\,\mathcal{D})$, the compositions of morphisms in any two simple paths from A to B are the same.

§6 Natural transformations

Definition 6.1 (Natural transformation)**.** Let $\mathcal{C}, \mathcal{D}$ be two categories and $F, G\colon \mathcal{D} \to \mathcal{C}$ be functors. A ***natural transformation*** $\alpha\colon F \to G$ is *one-parametre family of morphisms* in $\mathcal{D}$:

$$\alpha\colon\ \mathrm{obj}(\mathcal{C}) \to \{\mathrm{Hom}(F(A), G(A)) \mid A \in \mathrm{obj}(\mathcal{C})\};\ A \mapsto \alpha_A, \tag{6-1}$$

such that $\forall A, B \in \mathrm{obj}(\mathcal{C})$, $\forall f \in \mathrm{Hom}(A, B)$, the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ \big\downarrow{\scriptstyle F(f)} & & \big\downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow{\alpha_B} & G(B) \end{array} \tag{6-2}$$

or,

$$\alpha_B F(f) = G(f)\alpha_A. \tag{6-3}$$

Appendix A

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