Differential Geometry

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Contents

Part I

Domestic Differential Geometry

Chapter 1

Manifolds

Chapter 2

Scalar and Vector Fields

§1 Scalar Fields

Definition 1.1 (Scalar Field)**.** Let M be a smooth manifold, $f \in C^{(\infty)}(M)$ is called a ***scalar field***.

The scalar field over a manifold, form an algebra.

§2 Vector Fields

Definition 2.1 (vector field)**.** A ***vector field*** v over manifold M is a $C^{(\infty)}(M) \to C^{(\infty)}(M)$ map that satisfies

(a) $\forall f, g \in C^{(\infty)}(M)$, $\forall \lambda, \mu \in \mathbb{R}$, $v(\lambda f + \mu g) = \lambda v(f) + \mu v(g)$ (*linearity*).

(b) $\forall f, g \in C^{(\infty)}(M)$, $v(fg) = v(f)g + fv(g)$

The space of all vector fields on M is denoted by $\mathrm{Vect}(M)$

Definition 2.2 (tangent vector)**.** Let v be a vector field over M, p be a point on M. The tangent vector v_p at p is defined as a $C^{(\infty)}(M) \to C^{(\infty)}(M)$ map that satisfies

$$v_p(f) = v(f)(p). \tag{2-1}$$

The collection of tangent vectors at p is called the ***tangent space*** at p, denoted by T_pM.

The derivative of a path $\gamma\colon [0,1] \to M$ (or $\mathbb{R} \to M$) in a smooth manifold is defined as:

$$\begin{aligned} \gamma'(t)\colon C^{(\infty)}(M) &\to \mathbb{R}; \\ \gamma'(t)(f) &= \frac{\mathrm{d}}{\mathrm{d}t} f \circ \gamma(t) \end{aligned} \tag{2-2}$$

We can see that $\gamma'(t) \in T_{\gamma(t)}M$.

Let a path $\gamma\colon \mathbb{R}$ follows a vector field (a velocity field), that is

$$\gamma'(t) = v_{\gamma(t)}, \tag{2-3}$$

then we call γ the ***integral curve*** through $p := \gamma(0)$ of the vector field v.

Definition 2.3. Suppose v is an integrable vector field. Let $\varphi_t(p)$ be the point at time t on the integral curve through p.

$$\varphi_t\colon M \to M \tag{2-4}$$

is then called a ***flow*** generated by v.

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(p) = v_{\varphi_t(p)}. \tag{2-5}$$

§3 Covariant and Contravariant

Definition 3.1 (pullback)**.** Let f be a scalar field over N, $\varphi \in C^{(\infty)}(M, N)$. Then the ***pullback*** of f by φ

$$\varphi^*\colon C^{(\infty)}(N) \to C^{(\infty)}(M), \tag{3-1}$$

is defined as

$$\varphi^* f = f \circ \varphi \in C^{(\infty)}(M). \tag{3-2}$$

Fields that are pullbacked are ***covariant*** fields.

Definition 3.2 (pushforward)**.** Let v_p be a tangent vector of M at p, $\varphi \in C^{(\infty)}(M, N)$, $q = \varphi(p)$. Then the ***pushforward*** of v_p by φ

$$\varphi_* \colon T_p M \to T_q N, \tag{3-3}$$

is defined as

$$(\varphi_* v)_q(f) = v_p(\varphi^* f). \tag{3-4}$$

Note that the pushforward of a vector field can only be obtained when φ is a diffeomorphism.

Fields that are pushforwarded are ***contravariant*** fields.

Mathematicians and physicists might have disagreement on whether a tangent vector is covariant or contravariant. This is because of that physicists might consider the coordinates (v^μ) of a tangent vector as a vector field, instead of linear combination of bases ∂_μ.

§4 Components of Vector Fields

Let $\varphi\colon U \to \mathbb{R}^n$ be a chart of M ($U \subset M$).

Let $p \in U$, $\varphi(p) = x = (x^\mu)$ ($\mu = 0, \ldots, n-1$). Locally, a function $f \in C^{(\infty)}(M)$ can be written as

$$(\varphi^{-1})^* f = f \circ \varphi^{-1} \colon \mathbb{R}^n \to \mathbb{R}, \tag{4-1}$$

and a vector field $v \in \mathrm{Vect}(M)$ can be written as

$$(\varphi_* v)_x = \varphi_* v_p \colon C^{(\infty)}(\mathbb{R}^n) \to \mathbb{R}, \tag{4-2}$$

or

$$\varphi_* v \in \mathrm{Vect}(\mathbb{R}^n) \tag{4-3}$$

Since $T_x\mathbb{R}^n \cong \mathbb{R}^n$ is a linear space, one can find a basis for $T_x\mathbb{R}^n$ as

$$\partial_\mu \colon C^{(\infty)}(\mathbb{R}^n) \to C^{(\infty)}(\mathbb{R}^n), \tag{4-4}$$

and $(\varphi_* v)_x = v^\mu(x)\partial_\mu$.

Pushing forward $v^\mu(x)\partial_\mu$ by φ^{-1} one obtains v.

In an abuse of symbols, one may just omit the pullback and pushforward, and refer to the f and v by $(\varphi^{-1})^* f$ and $\varphi_* v$.

Consider another chart $\psi \colon U \to \mathbb{R}^n$ of M, and

$$y = \psi(p), \quad (\psi_* v)_x = u^\mu \partial_\mu, \tag{4-5}$$

where we have chosen the same basis in $T_y\mathbb{R}^n$ as in $T_x\mathbb{R}^n$.

We would like to know how to relate v^μ and u^μ i.e. we want to know how the components of v transforms under a coordinate transformation $\tau = \psi \circ \varphi^{-1}$.

Consider any $f \in C^{(\infty)}(M)$,

$$v(f) = \varphi_* v((\varphi^{-1})_* f) = \psi_* v((\psi^{-1})_* f) \tag{4-6}$$

$\Rightarrow$

$$u^\mu \partial_\mu (f \circ \psi^{-1}) = v^\mu \partial_\mu (f \circ \varphi^{-1}) = v^\mu \partial_\mu (f \circ \psi^{-1} \circ \tau) = v^\mu {\tau'}^\nu_{\ \mu} \partial_\nu (f \circ \psi^{-1}) \tag{4-7}$$

$\Rightarrow$

$$u^\mu = v^\nu {\tau'}^\mu_{\ \nu}, \tag{4-8}$$

where

$${\tau'}^\mu_{\ \nu} = \frac{\partial y^\mu}{\partial x^\nu}. \tag{4-9}$$

§5 Lie Bracket

Definition 5.1 (Lie bracket)**.** Let $v, w \in \mathrm{Vect}(M)$, then the ***Lie bracket*** of v and w is defined as

$$[v, w] \colon C^{(\infty)}(M) \to C^{(\infty)}(M);\ f \mapsto v \circ w(f) - w \circ v(f). \tag{5-1}$$

The Lie bracket is an antisymmetric bilinear map[1], and an important property of the Lie bracket is the Leibniz rule:

$$[v, w](fg) = [v, w](f)g + f[v, w](g). \tag{5-2}$$

Another important property of the Lie bracket is the Jacobi identity:

$$[v, [w, u]] + [w, [u, v]] + [u, [v, w]] = 0. \tag{5-3}$$

[1]Note that it is not $C^{(\infty)}$-linear

Chapter 3

Differential Forms

§6 1-forms

Definition 6.1 (1-form)**.** A ***1-form*** ω on M is a $\mathrm{Vect}(M) \to C^{(\infty)}(M)$ which satisfies that

(a) $\forall v, w \in \mathrm{Vect}(M)$, $\forall f, g \in C^{(\infty)}(M)$,

$$\omega(fv + gw) = f\omega(v) + g\omega(w). \tag{6-1}$$

The space of all 1-forms on M is denoted as $\Omega^1(M)$, which is a module over $C^{(\infty)}(M)$.

The operator d, when given a $C^{(\infty)}(M)$ function (which is called a ***0-form***), would give a 1-form:

$$(\mathrm{d}f)(v) = v(f). \tag{6-2}$$

This is called the ***exterior derivative*** or ***differential*** of f.

The ***cotangent vector*** or ***covector*** is similar as the tangent vector:

$$\omega_p(v_p) = \omega(v)(p). \tag{6-3}$$

The space of cotangent vectors at p on M is denoted by T_p^*M.

1-forms are covariant, that is, if $\varphi\colon M \to N$, then the pushforward of a 1-form ω by φ is

$$(\varphi^*\omega)_p(v_p) = \omega_q(\varphi_* v_p), \tag{6-4}$$

where $\varphi(p) = q$.

Theorem 6.1. *$f \in C^{(\infty)}(N)$, $\varphi\colon M \to N$ is differential, then*

$$\varphi^*(\mathrm{d}f) = \mathrm{d}(\varphi^* f). \tag{6-5}$$

§7 Components of 1-Forms

Let $\varphi\colon U \to \mathbb{R}^n$ be a chart of M ($U \subset M$).

Let $p \in U$, $\varphi(p) = x = (x^\mu)$ ($\mu = 0, \dots, n-1$). Locally a 1-form $\omega \in \Omega^1(M)$ can be written as

$$(\varphi^{-1})^*\omega \in T_x^*\mathbb{R}^n. \tag{7-1}$$

A natural way to impose a basis $\mathrm{d}x^\mu$ in $T_x^*\mathbb{R}^n$ is

$$\mathrm{d}x^\mu(\partial_\nu) = \delta^\mu_\nu, \tag{7-2}$$

and $(\varphi^{-1})^*\omega = \omega_\mu(x)\,\mathrm{d}x^\mu$.

Now by the definition of 1-form:

$$\omega_\mu\,\mathrm{d}x^\mu(v^\nu\partial_\nu) = v^\nu\omega_\mu\delta^\mu_\nu = v^\mu\omega_\mu. \tag{7-3}$$

By the transformation rule of components of a vector, one have

$$\tau'^{\nu}_{\ \mu}\alpha_\nu = \omega_\mu, \tag{7-4}$$

where $\psi\colon U \to \mathbb{R}^n$, $(\psi^{-1})_*\omega = \alpha_\mu\,\mathrm{d}x^\mu$, $\tau = \psi \circ \varphi^{-1}$.

§8 k-Forms

Definition 8.1. If we assign an antisymmetric multilinear k-form $\omega_p \in \bigotimes_{i\in k} T_p^* M$ to each point $p \in M$, we say we have a ***k-form*** on M.

The collection of all k-forms is denoted by $\Omega^k(M)$, and $\Omega(M) := \bigcup_{k\in\mathbb{N}} \Omega^k(M)$.

Theorem 8.1 (Dimension of forms)**.** *If M is an nD manifold, then the dimension of $\Omega^k(M)$ is $\frac{n!}{k!(n-k)!}$ ($k \leq n$), and 0 for $k > n$; The dimension of $\Omega(M)$ is 2^n.*

Definition 8.2 (Wedge product)**.** The ***wedge product*** $\wedge$ is defined as a binary operater that takes a k-form and ℓ-form and gives a $(k+\ell)$-forms, satisfying $\forall\alpha \in \Omega^k(M)$, $\forall\beta \in \Omega^\ell(M)$:

(a) (Associativity) $\forall\gamma \in \Omega^m(M)$,
$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma). \tag{8-1}$$

(b) (Supercommutativity)
$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha. \tag{8-2}$$

(c) (Distributiveness) $\forall\gamma \in \Omega^\ell(M)$,
$$\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma. \tag{8-3}$$

(d) (Bilinearity over $C^{(\infty)}(M)$) $\forall f \in C^{(\infty)}(M)$,
$$(f\alpha) \wedge \beta = f(\alpha \wedge \beta). \tag{8-4}$$

(e) (Naturality) If $\varphi\colon M \to N$ is a smooth map, then the pullback of a form by φ can be given by repeatingly applying ($\forall\gamma \in \Omega^\ell(M)$)
$$\begin{aligned} \varphi^*(\beta + \gamma) &= \varphi^*\alpha + \varphi^*\beta \\ \varphi^*(\alpha \wedge \beta) &= \varphi^*\alpha \wedge \varphi^*\beta, \end{aligned} \tag{8-5}$$
while the pullback of a 0-form and a 1-form agree with what we have already defined before.

By convention if $f \in C^{(\infty)}(M)$ then

$$f \wedge \omega =: f\omega. \tag{8-6}$$

It can be shown that any k-form ω can be written as

$$(\varphi^{-1})^* \omega = \frac{\omega_{\mu_1 \cdots \mu_k}}{n!} \bigwedge_{i=1}^{k} \mathrm{d}x^{\mu_i}, \tag{8-7}$$

where $\varphi\colon M \to \mathbb{R}^n$ is a chart.

§9 Exterior Derivative

Definition 9.1 (Exterior derivative)**.** The ***exterior derivative*** d is defined as a linear operator that takes a k-form and gives a $(k+1)$-form, satisfying $\forall \alpha \in \Omega^k(M)$, $\forall \beta \in \Omega^\ell(M)$:

(a) (Linearity) $\forall \lambda, \mu \in \mathbb{R}$, $\forall \gamma \in \Omega^\ell(M)$,

$$\mathrm{d}(\lambda\beta + \mu\gamma) = \lambda\,\mathrm{d}\alpha + \mu\,\mathrm{d}\beta. \tag{9-1}$$

(b) (Leibniz rule)

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta. \tag{9-2}$$

(c)

$$\mathrm{d}^2\omega = 0. \tag{9-3}$$

(d) (Naturality) If $\varphi\colon M \to N$ is a smooth map, then

$$\varphi^*\,\mathrm{d}\omega = \mathrm{d}\varphi^*\omega. \tag{9-4}$$

Chapter 4

Metric

§10 Pseudo-Riemannian Metric

Definition 10.1 (Psedudo-Riemannian metric)**.** Let M be a manifold. A ***pseudo-Riemannian metric*** or simply ***metric*** g on a manifold M is a field ($g \in \Gamma(T^*M \otimes T^*M)$) that $\forall p \in M$,

$$g_p \colon T_p^*M \times T_p^*M \to \mathbb{R}, \tag{10-1}$$

is a bilinear form satisfying the following properties:

(a) (Symmetry) $\forall u, v \in T_pM$,

$$g_p(u, v) = g_p(v, u). \tag{10-2}$$

(b) (Non-degenerate)

$$u \mapsto g_p(u, -) \colon T_pM \to T_p^*M \tag{10-3}$$

is an isomorphism.

(c) (Bilinearity) $\forall p \in M$, $\forall u, v \in T_pM$, $\forall \lambda, \mu \in \mathbb{R}$,

$$g_p(\lambda u + \mu v, w) = \lambda g_p(u, w) + \mu g_p(v, w). \tag{10-4}$$

(d) (Smoothness) If $v, u \in \mathrm{Vect}(M)$, then

$$p \mapsto g_p(v_p, u_p) \in C^{(\infty)}(M). \tag{10-5}$$

Given a metric, $\forall p \in M$, we can always find an orthonormal basis $\{e_\mu\}$ of T_pM such that

$$g_p(e_\mu, e_\nu) = \mathrm{sign}(\mu)\delta_{\mu\nu}, \tag{10-6}$$

where $\mathrm{sign}(\mu) = \pm 1$. Conventionally we order the basis such that $\mathrm{sign}(\mu) = 1$ for $\mu \in s$ and $\mathrm{sign}(\mu) = -1$ for $\mu - s \in n - s$, and say that the metric has ***signature*** $(s, n-s)$.

If $\gamma\colon [0,1] \to M$ is a smooth path and $\forall t, s \in [0,1]$,

$$g(\gamma'(t), \gamma'(t))g(\gamma'(s), \gamma'(s)) \geq 0, \tag{10-7}$$

then we can define the arclength of γ as

$$\int_0^1 \sqrt{|g(\gamma'(t), \gamma'(t))|}\,\mathrm{d}t \tag{10-8}$$

if the integral converges.

The metric gives an ***inner product*** on $\mathrm{Vect}(M)$:

$$\langle u, v\rangle := g(u, v). \tag{10-9}$$

The metric also gives a way to relate a vector field v to a 1-form ω. If v and ω satisfies: $\forall u \in \mathrm{Vect}(M)$,

$$g(v, u) = \omega(u), \tag{10-10}$$

then we say that v is the corresponding vector field of ω, and ω is the corresponding 1-form of v.

We can also define the ***inner product*** on $\Omega^1(M)$ by

$$\langle \alpha, \beta\rangle = \langle a, b\rangle, \tag{10-11}$$

where a and b is the corresponding vector fields of α and β.

The ***inner product*** on $\Omega^k(M)$ is defined by induction with

$$\langle \bigwedge_{i\in k} \alpha_i, \bigwedge_{i\in k} \beta_i \rangle = \det(\langle \alpha_i, \beta_j \rangle)_{i,j\in k}. \tag{10-12}$$

Hence, if $\{e_\mu\}$ is an orthonormal basis (field) of T_pM, while the corresponding covectors are $\{f^\mu\}$ ($f^\mu(e_\nu) = \delta^\mu_\nu$) then

$$\langle \bigwedge_{i\in k} f^{\mu_i}, \bigwedge_{i\in k} f^{\mu_i} \rangle = \prod_{i\in k} \operatorname{sign}(\mu_i). \tag{10-13}$$

Specially, when $f, g \in \Omega^0(M) = C^{(\infty)}(M)$,

$$\langle f, g \rangle = fg. \tag{10-14}$$

§11 Volume Form

Notice that if M is an nD manifold, $\dim \Omega^n(M) = 1$, meaning at $p \in M$, $\{\omega_p \mid \omega \in \Omega^n(M)\}$ can be labelled by a parametre $\lambda_p \in \mathbb{R}$. If we have a basis $\{f^\mu\}$ of T^*_pM (or corresponding vectors $\{e_\mu\}$), then

$$\{\omega_p \mid \omega \in \Omega^n(M)\} = \lambda_p \bigwedge_{\mu\in n} f^\mu. \tag{11-1}$$

If there were another basis $\{g^\mu\}$ of T^*_pM (or corresponding vectors $\{h_\mu\}$), and the transformation between the two bases is given by

$$Pe^\mu = f^\mu, \tag{11-2}$$

where $P \in \operatorname{Aut}(T^*_pM)$. When $\det P > 0$, we say that $\{f^\mu\}$ and $\{g^\mu\}$ have the same ***orientation***.

Definition 11.1 (Volume form)**.** Let M be an orientable manifold. If $\forall p \in M$, we find an oriented orthonormal basis $\{f_\mu\}$ of T^*_pM at point p, then the ***volume form*** vol is defined by

$$\bigwedge_{\mu\in n} f_\mu = \operatorname{vol}_p. \tag{11-3}$$

§12 Hodge Star Operator

Definition 12.1 (Hodge Star Operator)**.** Let M be an orientable manifold. The ***Hodge star operator*** $\star$ is defined by the linear map

$$\star\colon \Omega^k(M) \to \Omega^{n-k}(M), \tag{12-1}$$

$\forall \alpha, \beta \in \Omega^k(M)$,

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \operatorname{vol}. \tag{12-2}$$

We call $\star\omega$ the ***dual*** of ω.

The special case is when $k = 0$,

$$\star f = f \operatorname{vol}, \tag{12-3}$$

and $k = n$,

$$\star(f \operatorname{vol}) = f \prod_{\mu \in n} \operatorname{sign}(\mu) = (-1)^{n-s} f \tag{12-4}$$

if the signature of the metric is $(s, n-s)$.

§13 Metric and Coordinates

Chapter 5

De Rham Theory

Chapter 6

Bundles and Connections

§17 Fibre Bundles

Definition 17.1 (Bundle)**.** A ***bundle*** is a triple (E, π, B), where $\pi \colon E \to B$ is a surjective map. E is called the ***total space***, π is called the ***projection map***, and B is called the ***base space***.

A bundle (E, π, B) can be denoted as $\pi \colon E \to B$ or $E \xrightarrow{\pi} B$.

Definition 17.2 (Fibre)**.** For $p \in B$, $\pi^{-1}(\{p\})$ is the ***fibre*** over b.

Definition 17.3 (Subbundle)**.** Let $\pi \colon E \to B$ be a bundle. $F \subset E$, $C \subset B$, $\rho \colon F \to C$. If $\pi|_C = \rho$, then $\rho \colon F \to C$ is called a ***subbundle*** of $\pi \colon E \to B$.

Definition 17.4 (Section)**.** A ***section*** is a map $s \colon B \to E$ such that

$$p \circ s = \mathrm{id}_B\,. \tag{17-1}$$

All sections of a bundle $\pi \colon E \to B$ is denoted as $\Gamma(E)$.

Definition 17.5 (Fibre bundle)**.** A ***fibre bundle*** (E, π, B, F) is a bundle $\pi\colon E \to B$, where E, B, F are topology spaces, and π is a continuous map, and $\forall p \in B$, $\exists U \in \mathscr{U}(p)$ s.t.

$$\varphi\colon \pi^{-1}(U) \to U \times F, \tag{17-2}$$

is a homeomorphism and $\pi_1 \circ \varphi = \pi$. π_1 is defined as $\pi_1(p, q) = p$.

A fibre bundle can be denoted as the exact sequence

$$F \longrightarrow E \xrightarrow{\ \pi\ } B \tag{17-3}$$

The last condition is called the ***local triviality condition***. F is called the ***standard fibre***

If $E = B \times F$, then (E, π, B, F) is called a ***trivial fibre bundle***.

Definition 17.6 (Morphism)**.** Let $\pi\colon E \to B$, $\rho\colon F \to C$ be two fibre bundles. A ***morphism*** (φ, ψ) is a pair of two continuous maps such that

$$\begin{tikzcd} E \arrow[r, "\psi"] \arrow[d, "\pi"] & F \arrow[d, "\rho"] \\ B \arrow[r, "\varphi"] & C \end{tikzcd} \tag{17-4}$$

commutes.

§18 Vector Bundles

Definition 18.1 (Vector bundle)**.** A ***vector bundle*** is a fibre bundle (E, π, B, F), where F is a vector space, and the local trivialisation $\varphi\colon \pi^{-1}(U) \to U \times F$ (U is a neibourhood of $p \in B$) satisfies that $\forall x \in U$, $\forall v \in F$,

$$\begin{aligned} F &\to \pi^{-1}(\{x\}) \\ v &\mapsto \varphi^{-1}(x, v) \end{aligned} \tag{18-1}$$

is a linear isomorphism (***fibrewise linear***).

Definition 18.2 (Morphism (vector bundle))**.** The morphism between two vector bundles (E, π, B, F) and (E', π', B', F') is a morphism (φ, ψ) such that $\forall x \in B$,

$$\psi_* \colon \pi^{-1}(\{x\}) \to (\pi')^{-1}(\{\varphi(x)\}) \tag{18-2}$$

is a linear homomorphism.

Definition 18.3 (Smooth vector bundle)**.** A ***smooth vector bundle*** is a vector bundle (E, π, B, F), where the projection $\pi \colon E \to B$ and the local trivialisation $\varphi \colon \pi^{-1}(U) \to U \times F$ are smooth.

Definition 18.4 (Tangent bundle)**.** The ***tangent bundle*** TM is the smooth vector bundle over an nD smooth manifold M with the standard fibre $T_pM = \mathbb{R}^n$.

A vector field $v \in \mathrm{Vect}(M)$ is the smooth section of the tangent bundle $\Gamma(TM)$.

Definition 18.5 (Cotangent bundle)**.** The ***cotangent bundle*** of an nD manifold M, denoted by T^*M, is the smooth vector bundle over with the standard fibre $T_p^*M = (\mathbb{R}^n)^*$.

A 1-form $\omega \in \Omega^1(M)$ is the smooth section of the cotangent bundle $\Gamma(T^*M)$.

§19 Constructions of Vector Bundles

Definition 19.1 (Duality)**.**

§20 Connections

Definition 20.1 (Connection)**.** A ***connection*** on a smooth vector bundle (E, π, M, F) is map

$$D \colon \Gamma(TM) \times \Gamma(E) \to \Gamma(E), \tag{20-1}$$

that satisfies the following conditions: $\forall v, w \in \Gamma(TM)$, $\forall s, t \in \Gamma(E)$, $\forall f \in C^{(\infty)}(M)$,

(a) $D_v(s+t) = D_v s + D_v t$;

(b) $D_v(fs) = v(f)s + fD_v s$;

(c) $D_{v+w}s = D_v s + D_w s$;

(d) $D_{fv}s = fD_v s$.

When a vector field $v \in \Gamma(TM)$ is given to the connection D, the map $D_v : \Gamma(E) \to \Gamma(E)$ is called the ***covariant derivative*** with respect to v.

Definition 20.2 (Vector potential)**.** A ***vector potential*** A is an $\mathrm{End}(E)$-valued 1-form, that is

$$A \in \Gamma(\mathrm{End}(E) \otimes T^*M), \tag{20-2}$$

where $\mathrm{End}(E) \cong E \otimes E^*$ can be considered as a vector bundle over M with the standard fibre $\mathrm{End}(E_p) \cong E_p \otimes E_p^*$ ($p \in E$).

Locally if $s \in \Gamma(E)$ we can have a trivialisation $\varphi\colon E|_U \to U \times F$ ($U \subset M$). If we assign a basis $\{f_i\}_{i\in m}$ for the mD standard fibre F, then

$$s = s^i e_i := s^i \varphi^{-1}(f_i), \quad s^i \in C^{(\infty)}(U), \tag{20-3}$$

where we can call $\{s^i\}_{i\in m}$ the ***components of the section*** s. With this specific normalisation, one can define that

$$D_v^0 s = v(s^i)e_i \tag{20-4}$$

where D^0 is called the ***standard flat connection*** (which depends on trivialisation).

Theorem 20.1. *Let (E, π, M, F) be a smooth vector bundle. If D is a connection on E, $A \in \Gamma(\mathrm{End}(E)) \otimes T^*M$, then the $D+A$, which defined as*

$$D + A\colon (v, s) \mapsto D_v s + A(v)s, \tag{20-5}$$

is also a connection.

Theorem 20.2. *Let (E, π, M, F) be a smooth vector bundle, and D^0 is the standard flat connection on $U \subset E$ with the trivialisation $\varphi\colon E|_U \to U \times F$. If D is a connection on a (E, π, M, F), then $\exists A \in \Gamma(\mathrm{End}(E)) \otimes T^*M$ s.t.*

$$D = D^0 + A. \tag{20-6}$$

§21 Parallel Transport

Definition 21.1 (Parallel transport)**.** Let (E, π, M, F) be a smooth vector bundle, and D is a connection on E. A ***paralell transport*** of $s_0 \in \pi^{-1}(\{p\})$ $(p \in M)$along a curve $\gamma\colon [0,1] \to M$ is a section $s \in \Gamma(E|_{\gamma([0,1])})$ such that

$$\forall t \in [0,1], \quad D_{\gamma'(t)} s(t) = 0, \quad s(0) = s_0, \tag{21-1}$$

where $s(t) := s_{\gamma(t)}$.

Chapter 7

Curvature

Definition 21.2 (Curvature)**.** A ***curvature*** of a connection D on a smooth vector bundle (E, π, M, F) is a section $F \in \Gamma(\mathrm{End}(E) \otimes \Omega^2(M))$ (a End(E)-*valued* 2-*form*) defined as

$$F(v, w)s = D_v D_w s - D_w D_v s - D_{[v,w]} s, \quad v, w \in \Gamma(TM), \quad s \in \Gamma(E). \tag{21-1}$$

If $\forall v, w \in \Gamma(TM)$, $\forall s \in \Gamma(E)$, $F(v, w)s = 0$, then D is called a ***flat connection***.

Consider a local trivialisation $\varphi \colon E|_U \to U \times F$ $(U \subset M)$ s.t.

$$s = s^i e_i := s^i \varphi^{-1}(f_i), \tag{21-2}$$

where $s \in \Gamma(E|_U)$, $s^i \in C^{(\infty)}(U)$ and $\{f_i\}_{i \in m}$ is a set of bases of F, and $\sigma \colon U \to \mathbb{R}^n$ is a chart of M, $\sigma_* d_\mu := \partial_\mu$. Notice that

$[\partial_\mu, \partial_\nu] = 0,$

$$\begin{aligned}
&F(v,u)(s^i e_i) = v^\mu u^\nu F(d_\mu, d_\nu)(s^i e_i) \\
&= v^\mu u^\nu [D_\mu(d_\nu(s^i)e_i + s^i A^j_{\nu i} e_j) - D_\nu(d_\mu(s^i)e_i + s^i A^j_{\mu i} e_j)] \\
&= v^\mu u^\nu \big[d_\nu d_\mu(s^i)e_i + d_\nu(s^i)A^j_{\mu i}e_j + d_\mu(s^i A^j_{\nu i})e_j + s^i A^j_{\nu i} A^k_{\mu j} e_k \\
&\quad - d_\mu d_\nu(s^i)e_i - d_\mu(s^i)A^j_{\nu i}e_j - d_\nu(s^i A^j_{\mu i})e_j - s^i A^j_{\mu i} A^k_{\nu j} e_k\big] \\
&= v^\mu u^\nu s^i \big[d_\mu(A^k_{\nu i}) + A^j_{\nu i} A^k_{\mu j} - d_\nu(A^k_{\mu i}) - A^j_{\mu i} A^k_{\nu j}\big] e_k
\end{aligned} \tag{21-3}$$

§22 Bianchi Identity

$$[D_u, [D_v, D_w]] + [D_v, [D_w, D_u]] + [D_w, [D_u, D_v]] = 0 \tag{22-1}$$

$$[D_\mu, F_{\nu\lambda}] + [D_\nu, F_{\lambda\mu}] + [D_\lambda, F_{\mu\nu}] = 0 \tag{22-2}$$

Chapter 8

Pseudo-Riemannian Geometry

§23 Tensors

Definition 23.1 (Tensor)**.** Let M be a smooth manifold. A (r, s)-***tensor*** is a smooth section of the tensor product of rth tensor power of TM and sth tensor power of T^*M:

$$t \in \Gamma(TM^{\otimes r} \otimes T^*M^{\otimes s}). \tag{23-1}$$

In local coordinates:

$$t^{\mu_1 \cdots \mu_r}_{\nu_1 \cdots \nu_s} \bigotimes_{k=1}^{r} \partial_{\mu_k} \otimes \bigotimes_{k=1}^{s} \partial_{\nu_k}. \tag{23-2}$$

It is conventional to use the local coordinates form in pseudo-Riemannian geometry, and do not distinguish between a tensor and its components, written in forms of ***abstract indices***, where indices are written just to indicates types and operations on tensors.

And since we can raise and lower indices of a tensor, it is sometimes important to distinguish the orders between covariant and contravariant indices. e.g. ${T^\mu}_\nu \neq {T^\nu}_\mu$.

Tensor Product Let T_1, T_2 be (p_1, q_1) and (p_2, q_2) tensors, we can have their tensor product:

$$T_1 \otimes T_2 \in \Gamma(TM^{\otimes(p_1+p_2)} \otimes T^*M^{\otimes(q_1+q_2)}), \tag{23-3}$$

where at each point $p \in M$, the tensor product is but the tensor product of the corresponding multilinear functions.

In abstract indices, we have:

$$\begin{aligned} &{(T_1 \otimes T_2)^{\mu_1\cdots\mu_{p_1+p_2}}}_{\nu_1\cdots\nu_{q_1+q_2}} \\ &= {T_1^{\mu_0\cdots\mu_{p_1-1}}}_{\nu_0\cdots\nu_{q_0-1}} {T_2^{\mu_{p_1}\cdots\mu_{p_1+p_2-1}}}_{\nu_{q_1}\cdots\nu_{q_1+q_2-1}}. \end{aligned} \tag{23-4}$$

Contractions The contraction is a generalisation of the inner product of vectors. Let T is a $(p+1, q+1)$ tensor, we can define the $(i, p+j)$ contraction of T as

$$\begin{aligned} &\operatorname{tr}_{(i,p+j)} T \colon TM^p \times T^*M^q \to \mathbb{R} \\ &(v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_p, \omega_0, \ldots, \omega_{j-1}, \omega_{j+1}, \ldots, \omega_q) \mapsto \\ &\sum_{\mu\in N} T(v_0, \ldots, v_{i-1}, \partial_\mu, v_{i+1}, \ldots, v_p, \omega_0, \ldots, \omega_{j-1}, \mathrm{d}x^\mu, \omega_{j+1}, \ldots, \omega_q). \end{aligned} \tag{23-5}$$

The index-free notation can be found at [1].

§24 Levi-Civita Connection

Definition 24.1 (Levi-Civita connection)**.** Let $E \to M$ be a smooth vector bundle, where M is a Riemanian manifold with metric $g \in T^*M \otimes T^*M$. Let $\nabla \in \Gamma(\mathrm{End}(E) \otimes T^*M^{\otimes 2})$ be a connection on E. Then ∇ is called a ***Levi-Civita connection*** if

$$ug(v, w) = g(\nabla_u v, w) + g(v, \nabla_u w), \tag{24-1}$$

and

$$[v,w] = \nabla_v w - \nabla_w v, \tag{24-2}$$

where $u, v, w \in \Gamma(TM)$.

In local coordinates:

$$\nabla_\alpha \partial_\beta = \Gamma^\gamma_{\alpha\beta}\partial_\gamma, \tag{24-3}$$

where $\Gamma^\gamma_{\alpha\beta}$ is the ***Christoffel symbol***.

For any $T \in \Gamma(TM^{\otimes p} \otimes T^*M^{\otimes q})$, we have

$$\nabla T = {T^{\alpha_0\cdots\alpha_{p-1}}}_{\beta_0\cdots\beta_{q-1};\mu} \bigotimes_{k\in p} \partial_{\alpha_k} \otimes \bigotimes_{\ell \in q} \mathrm{d}x^{\beta_\ell} \otimes \mathrm{d}x^\mu \tag{24-4}$$

$$\begin{aligned} {T^{\alpha_0\cdots\alpha_{p-1}}}_{\beta_0\cdots\beta_{q-1};\mu} = {} & {T^{\alpha_0\cdots\alpha_{p-1}}}_{\beta_0\cdots\beta_{q-1},\mu} \\ & + \sum_{i\in p} \Gamma^{\alpha_i}_{\lambda\mu} {T^{\alpha_0\cdots\alpha_{i-1}\lambda\alpha_{i+1}\cdots\alpha_{p-1}}}_{\beta_0\cdots\beta_{q-1}} \\ & - \sum_{i\in q} \Gamma^{\lambda}_{\beta_i\mu} {T^{\alpha_0\cdots\alpha_{p-1}}}_{\beta_0\cdots\beta_{i-1}\lambda\beta_{i+1}\cdots\beta_{q-1}}. \end{aligned} \tag{24-5}$$

bibliography

[1] Yuri Vyatkin (https://math.stackexchange.com/users/2002/yuri-vyatkin). *Coordinate-free notation for tensor contraction?* Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/1804213 (version: 2016-05-29). eprint: https://math.stackexchange.com/q/1804213. URL: https://math.stackexchange.com/q/1804213.

[2] Javier P. Muniain John C. Baez. *Gauge Fields, Knots and Gravity (Series on Knots and Everything)*. Series on Knots and Everything. World Scientific Publishing Company, 1994. ISBN: 9789810217297,9810217293,9810220340.

Symbol List

Here listed the important symbols used in these notes

Index