

Algebraic Topology

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Chapter 1

Homotopy and Fundamental Group

§1 Homotopy

Definition 1.1 (Homotopy). $f, g \in C(X, Y)$. If $\exists H \in C(X \times [0, 1], Y)$ s.t. $H(x, 0) = f(x)$, $H(x, 1) = g(x)$, then we say f and g are **homotopic**, denoted by $f \simeq g: X \rightarrow Y$ or just $X \rightarrow Y$. H is called a **homotopy** between f and g , denoted by $H: f \simeq g$ or $f \simeq_H g$.

For $t \in [0, 1]$, $h_t: X \rightarrow Y; x \mapsto H(x, t)$ is called a ***t-slice***.

If f is homotopic to a constant mapping, we say that f is **null-homotopic**.

A **linear homotopy** is a homotopy between two functions to $Y \subseteq \mathbb{R}^n$ that change linearly, i.e.

$$H(x, t) = (1 - t)f(x) + tg(x).$$

Theorem 1.1 (Maps to convex set are homotopic). $f, g \in C(X, Y)$. If Y is a convex set in \mathbb{R}^n , then $f \simeq g$.

Proof. Consider linear homotopy. □

Theorem 1.2. *Homotopic relation is an equivalence relation.*

Proof. *reflexivity.* $f \simeq f$, just take $H(x, t) = f(x)$ for any t (Such homotopy is called a **constant homotopy**).

Symmetry. $f \simeq g$ then $g \simeq f$. Just take $\bar{H}(x, t) = H(x, 1 - t)$ (Here \bar{H} is called the inverse of H).

Transitivity. $f \simeq g \wedge g \simeq h \rightarrow f \simeq h$. Let

$$H_1 H_2(x, 2t) = \begin{cases} H_1(x, 2t) & t \in [0, 1/2], \\ H_2(x, 2t - 1) & t \in [1/2, 1]. \end{cases}$$

We can see that $H_1 H_2$ is also a homotopy (see Theorem ?? in Point Set Topology) □

Hence, we can define **homotopy classes** on $C(X, Y)$, denoted by $[X, Y]$.

As you might expect after reading the proof of Theorem 1.2, the homotopies between mappings within a homotopy class form a group.

Theorem 1.3 (Composition of homotopies). $f_1 \simeq f_2: X \rightarrow Y$, $g_1 \simeq g_2: Y \rightarrow Z$, then $g_1 \circ f_1 \simeq g_2 \circ f_2: X \rightarrow Z$.

Proof i. Let $F: f_1 \simeq f_2$, $G: g_1 \simeq g_2$. Define:

$$\mathbf{F}: X \times [0, 1] \rightarrow Y \times [0, 1]; (x, t) \mapsto (F(x, t), t).$$

It can be verified tht $G \circ \mathbf{F}: g_1 \circ f_1 \simeq g_2 \circ f_2: X \rightarrow Z$. □

Proof ii. Let $F: f_1 \simeq f_2$, $G: g_1 \simeq g_2$.

We can verify that $H_1: (x, t) \mapsto g_1 \circ F(x, t)$ is a homotopy between $g_1 \circ f_1$ and $g_1 \circ f_2$; Similarly $H_2: g_1 \circ f_2 \simeq g_2 \circ f_2$ can be defined.

Now consider $H = H_1 H_2$, or in detailed,

$$H(x, t) = \begin{cases} g_1 \circ F(x, 2t) & (x, t) \in X \times [0, 1/2] \\ G(f_2(x), 2t - 1). & (x, t) \in X \times [1/2, 1] \end{cases}$$

□

Lemma 1 (Identity map in convex space is null-homotopic). $X \subseteq \mathbb{R}^n$ is a convex space. $\forall x_0 \in X$, $\text{id}_X \simeq (x \mapsto x_0)$.

Proof. The linear homotopy can be constructed as:

$$H_{x_0}(x, t) = tx + (1 - t)x_0.$$

□

Theorem 1.4 (Continuous mappings from a convex set are null-homotopic). $X \subseteq \mathbb{R}^n$ is a convex set. $\forall f \in C(X, Y)$, f is null-homotopic.

Proof. Let $H_{x_0}(x, t) = tx + (1 - t)x_0$. Then, any $f: X \rightarrow Y$ can be written as $f = f \circ \text{id}_X$, hence $f \simeq f \circ H_{x_0}(x, 1) = (x \mapsto f(x_0))$, which means f is null-homotopic. □

Theorem 1.5 (Constant mappings to a path-connected space belong to one homotopy class). If Y is a path-connected space, $y_0 \in Y$, then $[X, Y] = [x \mapsto y_0]$ (i.e. homotopy class of constant mapping to $\{y_0\}$)

Proof. Let $f_1(x) = y_1$, $f_2(x) = y_2$ be two constant mappings, a is a path from y_1 to y_2 . Then the homotopy between f_1 and f_2 can be defined as:

$$H(x, t) = a(t).$$

□

Definition 1.2 (Homotopy relative to a set). Let $A \subseteq X$, $H: f \simeq g$. If $\forall a \in A$, $\forall t \in [0, 1]$, $f(a) = g(a) = H(a, t)$, we say that f and g are **homotopic relative to** A , denoted by $H: f \simeq g \text{ rel } A$.

We can have parallel results as Theorem 1.2 and Theorem 1.3:

Theorem 1.6. Given $A \subseteq X$, $\simeq \text{rel } A$ is an equivalence relation in $C(X, Y)$.

Theorem 1.7 (Composition of relative homotopies). $f_1 \simeq f_2: X \rightarrow Y \text{ rel } A$, $g_1 \simeq g_2: Y \rightarrow Z \text{ rel } B$, and $f_1(A) \subset B$, then $g_1 \circ f_1 \simeq g_2 \circ f_2: X \rightarrow Z$.

Definition 1.3 (Fixed-endpoint Homotopy). Let a, b be two paths in X . If $a \simeq b \text{ rel } \{0, 1\}$, we say that a and b are **fixed-endpoint homotopic**. The paths in X modulus fixed-point homotopy is denoted by $[X]$, called the **path classes**. The path class which a belongs to is denoted by $\langle a \rangle$.

§2 Fundamental Group

Fundamental group of a topological space at a point is the path classes at this point. We need to introduce the multiplicative structure of path classes.

Theorem 2.1. Let a, b, c, d be four paths in X .

$$\begin{aligned} a \simeq b \text{ rel } \{0, 1\} &\leftrightarrow \bar{a} \simeq \bar{b} \text{ rel } \{0, 1\}, \\ a \simeq b \text{ rel } \{0, 1\} \wedge c \simeq d \text{ rel } \{0, 1\} &\rightarrow ac \simeq bd \text{ rel } \{0, 1\}. \end{aligned}$$

Definition 2.1 (Inverse and product of path classes). $\alpha, \beta \in [X]$, $a \in \alpha$, $b \in \beta$. $b(0) = a(1)$. We define $\alpha^{-1} := \langle \bar{a} \rangle$ to be the **inverse** of the path class α , and $\alpha\beta := \langle ab \rangle$ to be the **product** of the two path classes α and β .

While the product of paths does not obey associativity, we have:

Theorem 2.2 (Associativity of product of path classes). $\alpha, \beta, \gamma \in [X]$. $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ (if they are productible).

Proof. Consider $\forall a \in \alpha, \forall b \in \beta, \forall c \in \gamma$.

Let

$$\begin{aligned} \tilde{a}(t) &= t/3, \\ \tilde{b}(t) &= t/3 + 1/3, \\ \tilde{c}(t) &= t/3 + 2/3. \end{aligned}$$

\tilde{a}, \tilde{b} and \tilde{c} are three paths in $[0, 1]$, and $\tilde{a}(\tilde{b}\tilde{c}) \simeq (\tilde{a}\tilde{b})\tilde{c} \text{ rel } \{0, 1\}$, since $[0, 1]$ is convex, therefore there is a linear homotopy between the two product paths.

Now Let $f: [0, 1] \rightarrow X$ be

$$f(t) = \begin{cases} a(3t), & t \in [0, 1/3]; \\ b(3t - 1), & t \in [1/3, 2/3]; \\ c(3t - 2), & t \in [2/3, 1]. \end{cases}$$

$a(bc) = f \circ \tilde{a}(\tilde{b}\tilde{c}) \simeq f \circ (\tilde{a}\tilde{b})\tilde{c} = (ab)c \text{ rel } \{0, 1\}$, by Theorem 1.3. \square

Theorem 2.3 (Identity-like properties of point path). $\alpha \in [X]$. Let the initial and the terminal point of α be x_0 and x_1 . (i) $\alpha^{-1}\alpha = \langle t \mapsto x_1 \rangle$, $\alpha\alpha^{-1} = \langle t \mapsto x_0 \rangle$; (ii) $\alpha\langle t \mapsto x_0 \rangle = \alpha = \langle t \mapsto x_1 \rangle\alpha$.

Proof. Note that $\text{id}_{[0,1]}$ is a path in the convex set $[0, 1]$. \square

For now path classes are not closed under production.

Definition 2.2 (Fundamental group). $x_0 \in X$. The path classes of loops at x_0 (paths that have both endpoints at x_0), equipped with production, is the **fundamental group** of X at x_0 , denoted by $\pi_1(X, x_0)$.

Definition 2.3 (Homomorphism induced by continuous function). $f \in C(X, Y)$, $x_0 \in X$. We define

$$f_\pi: [X] \rightarrow [Y], \quad \langle a \rangle \mapsto \langle f \circ a \rangle$$

where a is a path in X .

The limitation of f_π on $\pi_1(X, x_0)$ is said to be a **homomorphism induced by f** .

For simplicity, we would write such homomorphism by f_π (without explicitly referring limitation).

Theorem 2.4 (Isomorphism induced by homeomorphism). Let f be a homeomorphism from X to Y , then $\forall x_0 \in X$, f_π is an isomorphism from $\pi_1(X, x_0)$ to $\pi_1(Y, f(x_0))$.

Proof.

$$\begin{aligned} f^{-1} \circ f &= \text{id}_X \rightarrow (f^{-1})_\pi \circ f_\pi = \text{id}_{\pi_1(X, x_0)}; \\ f \circ f^{-1} &= \text{id}_Y \rightarrow f_\pi \circ (f^{-1})_\pi = \text{id}_{\pi_1(Y, f(x_0))}, \end{aligned}$$

therefore $(f^{-1})_\pi$ is the inverse of f_π . An invertible homomorphism is an isomorphism. \square

Theorem 2.5 (Fundamental group of product space). $x_0 \in X$, $y_0 \in Y$.

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Proof. Let $j_X \in C(X \times Y, X)$ and $j_Y \in C(X \times Y, Y)$ be projections ($j_X(x, y) = x$, $j_Y(x, y) = y$), and define a homomorphism

$$\begin{aligned} \varphi: \quad \pi_1(X \times Y, (x_0, y_0)) &\rightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0); \\ \gamma &\mapsto ((j_X)_\pi(\gamma), (j_Y)_\pi(\gamma)). \end{aligned}$$

φ is a *monomorphism*. Let $\langle c \rangle \in \ker \varphi$ i.e.

$$\varphi(\langle c \rangle) = (\langle t \mapsto x_0 \rangle, \langle t \mapsto y_0 \rangle).$$

Let

$$H_X: j_X \circ c \simeq t \mapsto x_0 \text{ rel } \{0\}, \quad H_Y: j_Y \circ c \simeq t \mapsto y_0 \text{ rel } \{0\}.$$

The homotopy between c and $t \mapsto (x_0, y_0)$ is defined as

$$F: [0, 1]^2 \rightarrow X \times Y; (t, s) \mapsto (H_X(t, s), H_Y(t, s)).$$

φ is an *epimorphism*. $\forall \langle a \rangle \in \pi_1(X, x_0)$ and $\forall \langle b \rangle \in \pi_1(Y, y_0)$. $c: t \mapsto (a(t), b(t)) \in C([0, 1], X \times Y)$. $\langle c \rangle \in \varphi^{-1}(\{(\langle a \rangle, \langle b \rangle)\})$. \square

Theorem 2.6 (Fundamental groups of path connected space at different points are isomorphic). X is path connected, $x_1, x_2 \in X$. $\pi_1(X, x_1) \cong \pi_1(X, x_2)$.

Proof. $\langle a \rangle \in \pi_1(X, x_1)$, $\langle b \rangle \in \pi_1(X, x_2)$, $\langle c \rangle$ is a path class with initial point x_1 and terminal point x_2 .

It can be verified that

$$g_c: \pi_1(X, x_1) \rightarrow \pi_1(X, x_2); \langle a \rangle \mapsto \langle \bar{c}ac \rangle \quad (2-1)$$

is a homomorphism. Same as $g_{\bar{c}}(\langle b \rangle) = cb\bar{c}$.

$$\begin{aligned} g_c \circ g_{\bar{c}}(\langle b \rangle) &= \langle \bar{c}cb\bar{c} \rangle = \text{id}_{\pi_1(X, x_2)}; \\ g_{\bar{c}} \circ g_c(\langle a \rangle) &= \langle c\bar{c}ac\bar{c} \rangle = \text{id}_{\pi_1(X, x_1)}, \end{aligned}$$

therefore g_c is an isomorphism. \square

With Theorem 2.6, we can write the fundamental group of a path-connected space X by $\pi_1(X)$.

For different path-connected branches, a topological space can have different fundamental groups, while they are isomorphic within one branch.

Theorem 2.7. Let X and Y be two topological spaces, $f_1 \simeq f_2: X \rightarrow Y$, $x_0 \in X$, $f_1(x_0) = y_1$, $f_2(x_0) = y_2$; If $\exists c \in C([0, 1], Y)$ s.t. $c(0) = y_1$, $c(1) = y_2$, and g_c is defined as in Eq. (2-1), then $g_c \circ f_{1,\pi} = f_{2,\pi}$.

$$\begin{array}{ccc} & & \pi_1(Y, y_1) \\ & \nearrow f_{1,\pi} & \downarrow g_c \\ \pi_1(X, x_0) & & \pi_1(Y, y_2) \\ & \searrow f_{2,\pi} & \end{array}$$

Proof. \square

Definition 2.4 (Simply connected). If the fundamental group of a path connected space X is trivial i.e. $\pi_1(X) \cong \{1\}$, we say that X is *simply connected*.

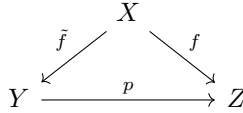
Theorem 2.8 (Convex set is simply connected). *If $X \subset \mathbb{R}^n$ is convex, then X is simply connected.*

Proof. $x_0 \in X$, $a \in C([0, 1], X)$ s.t. $a(0) = a(1) = x_0$. $H_{a, x_0}(s, t) = (1 - t)a(s) + tx_0$. \square

§3 Examples of Fundamental Groups

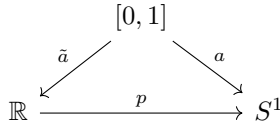
3.1 S^1

Definition 3.1 (Lift). Let X, Y, Z be three topological spaces, and $f \in C(X, Z)$, $p \in C(Y, Z)$. If $\tilde{f} \in C(X, Y)$, s.t. $f = p \circ \tilde{f}$, we say that \tilde{f} is a **lift** of f .



In some case, given f and p , \tilde{f} might do not exist.

Lemma 2 (Lift of path). $a \in C([0, 1], S^1)$, $p: \mathbb{R} \rightarrow S^1; x \mapsto e^{2\pi xi}$. Let $t_0 \in \mathbb{R}$ s.t. $p(x_0) = a(0)$. There exists a unique lift $\tilde{a} \in C([0, 1], \mathbb{R})$ of a s.t. $\tilde{a}(0) = x_0$.



Proof. *Existence.* The collection of open sets that the images under a do not cover S^1 , $\{(\alpha_i, \beta_i) \cap [0, 1] \mid a_i, b_i \in \mathbb{R}^I \wedge S^1 \subsetneq a((\alpha_i, \beta_i))\}$, is a cover of $[0, 1]$ by the definition of continuity. Since $[0, 1]$ is compact, there exists a finite subcover $\{(\alpha_i, \beta_i) \cap [0, 1] \mid a_i, b_i \in \mathbb{R}^n \wedge S^1 \subsetneq a((\alpha_i, \beta_i))\}$, where $n \in \mathbb{N}$. By dividing these open intervals

into closed intervals that has no inner points intersecting, we can get $\Omega = \{I_k := [t_i, t_{i+1}] \mid k \in m\}$ (This can be done by sorting α_i and β_i).

The mapping p is locally homeomorphic i.e. there exists $[x_i, x'_i] \subset \mathbb{R}$ s.t. $p_i := p|_{[x_i, x'_i]}: [x_i, x'_i] \rightarrow a(I_i)$ is a homeomorphism (and $p_i(x_i) = a(t_i)$), therefore $\tilde{a}_i := p_i^{-1} \circ a$ is a lift of $a_i := a|_{I_i}$.

Since $p_0(t_0) = a(t_0)$, $p_{i+1}(t_i) = p_i(t_i)$, we can define piecewisely the lift of a by $\tilde{a} = \cup\{\tilde{a}_i \mid i \in m\}$.

Uniqueness. Let \tilde{a}' be another lift of a , $p(\tilde{a}'(t) - \tilde{a}(t)) = p \circ \tilde{a}'(t)/p \circ \tilde{a}(t) = a(t)/a(t) = 1$, therefore $\tilde{a}'(t) - \tilde{a}(t) \in \mathbb{Z}$. Since $[0, 1]$ is connected, the image of $t \mapsto \tilde{a}'(t) - \tilde{a}(t)$ must be connected, which is possible only if it is constant. $\tilde{a}'(0) = \tilde{a}(0) = x_0$, therefore $\tilde{a} = \tilde{a}'$. \square

Notice that we have the freedom to set $\tilde{a}(0) \in \mathbb{Z}$ (the lift is unique after setting that), what really matter is the difference $\tilde{a}(1) - \tilde{a}(0)$. One can proof that $q(a) := \tilde{a}(1) - \tilde{a}(0)$ does not depend on the chose of $\tilde{a}(0) \in \mathbb{Z}$. We call $q(a)$ the **loop number** of path a .

Lemma 3 (Two loops that are never antipodal have the same loop number). *Let a, b be two loops at z_0 in S^1 . If $\forall t \in [0, 1]$, $a(t) \neq -b(t)$, then $q(a) = q(b)$.*

Proof. Choose $\tilde{a}(0) = \tilde{b}(0) = 0$ (if not so, just translate the lift by an integer). In this case, $q(a) = \tilde{a}(1)$, $q(b) = \tilde{b}(1)$.

If $q(a) \neq q(b)$, without loss of generality, $q(a) > q(b)$, then $f := t \mapsto \tilde{a}(t) - \tilde{b}(t)$ is a continuous function from a compact space $[0, 1]$ to \mathbb{R} , therefore by the connectedness of $[0, 1]$, $\exists t_0 \in [0, 1]$ s.t. $f(t_0) = 1/2 \in [0, q(a) - q(b)]$, when

$$p \circ \tilde{a}(t_0) + p \circ \tilde{b}(t_0) = e^{2\pi i(\tilde{b}(t_0)+1/2)} + e^{2\pi i\tilde{b}(t_0)} = 0.$$

\square

Lemma 4 (Same loop number iff homotopic relative to endpoint). *Let a, b be two loops at z_0 in S^1 . $a \simeq b \text{ rel } \{0\}$ iff $q(a) = q(b)$.*

Proof. \rightarrow : Let $H: a \simeq \text{brel}\{0\}$, $h_s = t \mapsto H(t, s)$, $f_t = s \mapsto H(t, s)$.
 $\forall(t, s) \in [0, 1]^2$, $U := \{H(t, s)e^{i\theta} \mid \theta \in (-\pi, \pi)\} \in \mathcal{U}_{S^1}(H(t, s))$.
 Since $f_t \in C([0, 1], S^1)$, $\exists V(s) \in \mathcal{U}_{[0, 1]}(s)$ s.t. $H(V(s)) \subset U$. Which means, $\forall t \in [0, 1]$, $\forall s_1, s_2 \in V(s)$, $f_t(s_1) \neq -f_t(s_2)$ or $h_{s_1}(t) \neq -h_{s_2}(t)$. By Lemma 3, $q(h_s) = q(h_{s'})$.

$\Omega = \{V(s) \mid s \in [0, 1]\}$ is an open cover of the compact space $[0, 1]$, therefore has a finite subcover $\Omega = \{V_i \in \Omega \mid i \in n\}$. In each $V(s_i)$, h_s has the same loop numbers.

We therefore have $q(a) = q(h_0) = q(h_1) = q(b)$.

\leftarrow : $H: [0, 1]^2 \rightarrow S^1; (t, s) \mapsto p((1-s)\tilde{a}(t) - s\tilde{b}(t))$. □

Theorem 3.1. $\pi_1(S^1) \cong \mathbb{Z}$.

Proof. $z_0 \in S^1$. Let $Q: \pi_1(S^1, z_0) \rightarrow \mathbb{Z}; \langle a \rangle \mapsto q(a)$.

$\forall \langle a \rangle, \langle b \rangle \in \pi_1(S^1, z_0)$, choose $\tilde{a}(1) = \tilde{b}(0)$,

$$\begin{aligned} Q(\langle a \rangle \langle b \rangle) &= Q(\langle ab \rangle) = q(ab) \\ &= \tilde{b}(1) - \tilde{a}(0) = \tilde{b}(1) - \tilde{b}(0) + \tilde{a}(1) - \tilde{a}(0) = q(a) + q(b) \\ &= Q(\langle a \rangle) + Q(\langle b \rangle), \end{aligned}$$

which means Q is a homomorphism.

By Lemma 4, Q is a monomorphism. $\forall n \in \mathbb{Z}$, $Q(\langle t \mapsto e^{2\pi n t i} \rangle) = n$, therefore Q is also an epimorphism. □

3.2 S^n , $n > 2$

The situation for S^n is much simpler:

Theorem 3.2. $\forall n \in \mathbb{N}$, if $n \geq 2$, then S^n is simply connected.

Proof. Let $x_0 \in S^n$, and a be a loop at x_0 in S^n . $x \in S^n$ and $x \neq x_0$. Embed S^n into \mathbb{R}^{n+1} and let $B(x; \delta)$ be a $(n+1)$ -D ball with radius δ around x that $x_0 \notin B(x; \delta)$.

$a^{-1}(B(x; \delta) \cap S^n)$ is a collection of open, disjoint intervals in $[0, 1]$, which can be considered as an open cover of $a^{-1}(\{x\})$, which

is compact. Let the finite subcover of $a^{-1}(\{x\})$ be $\{(\alpha_i, \beta_i) \cap [0, 1] \mid \alpha_i, \beta_i \in \mathbb{R}, i \in m\}$, where $m \in \mathbb{N}$.

Let $P: \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R} \rightarrow \mathbb{R}^{n+1}$ be:

$$P(y, y_0, r) = \frac{y - y_0}{\|y - y_0\|} r + y_0,$$

which means project y to the sphere with radius r around y_0 .

Now we define the loop b that go as:

$$b(t) = \begin{cases} a(t), & t \notin (\alpha_i, \beta_i), \forall i \in m; \\ P[P(a(t), x, \delta), 0, 1], & t \in (\alpha_i, \beta_i) - a^{-1}(\{x\}), \exists i \in m; \\ \lim_{t' \rightarrow t} b(t'), & t \in a^{-1}(\{x\}) \cap (\alpha_i, \beta_i), \exists i \in m, \\ & t' \in (\alpha_i, \beta_i) - a^{-1}(\{x\}) \end{cases}$$

and the homotopy between a and b can be written as

$$H: [0, 1]^2 \rightarrow S^n; (t, s) \mapsto P[(1 - s)a(t) + sb(t), 0, 1].$$

Since b is a loop in $S^n - \{x\}$, while $S^n - \{x\} \cong \mathbb{R}^n$ (by stereographic projection), which is simply connected, we know that b is homotopic to $t \mapsto x_0$ i.e. null-homotopic. \square

By Theorem 2.5, the fundamenal group of $T^2 := S^1 \times S^1$ is $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$, which is not isomorphic to S^2 , therefore $T^2 \not\cong S^2$.

§4 Homotopy Types

Definition 4.1 (Homotopy type). If $\exists f \in C(X, Y)$, $\exists g \in C(Y, X)$ s.t.

$$g \circ f \simeq \text{id}_X, \quad f \circ g \simeq \text{id}_Y,$$

then we say X and Y are **homotopy equivalent**, or they are of the same **homotopy type**, denoted by $X \simeq Y$. f is called a **homotopy map** or a **homotopy equivalence** from X to Y , and g is called a **homotopy inverse** of f .

An inverse of a homotopy map is not unique.

Some examples of spaces having same homotopy types:

- $\mathbb{R} \simeq \mathbb{R}^n$ ($n \in \mathbb{N}_+$).
- $X \times [0, 1] \simeq X$.

Theorem 4.1. *If $X \simeq Y$ and f is a homotopy map from X to Y , $f(x_0) = y_0$, then f_π is an isomorphism from $\pi_1(X, x_0)$ to $\pi_1(Y, y_0)$.*

Proof.

□

Spaces with the simplest homotopy type are contractable spaces.

Definition 4.2 (Contractable space). If $X \simeq \{x\}$, we call X a **contractable space**.

§5 Retractability

Definition 5.1 (Retractability). $A \subset X$, $i: A \rightarrow X$ is an **inclusion** from A to X , meaning $\forall a \in A$, $i(a) = a$. If $\exists r \in C(X, A)$ s.t. $r \circ i = \text{id}_A$, then A is called a **retract** of X , r is a **retraction**, and X is said to be **retractable**.

Definition 5.2 (Deformation retractability). $A \subset X$, $i: A \rightarrow X$ is an inclusion. If $\exists r \in C(X, A)$ s.t. $r \circ i = \text{id}_A \wedge H: i \circ r \simeq \text{id}_X$, then A is called a **deformation retract** of X , H is a **deformation retraction** of X , and X is said to be **deformation retractable**.

Theorem 5.1 (Spaces are homotopically equivalent to their deformation retracts). *If $X \simeq Y$ and X is a deformation retract of Y , then $X \simeq Y$. And, the retraction $r: Y \rightarrow X$ and the inclusion $i: X \rightarrow Y$ are homotopy inverse to each other.*

Theorem 5.2 (Contractable space can be deformationally retracted to all its points). *If X is a contractable space, $\forall x \in X$, $\{x\}$ is a deformation retract of X .*

Definition 5.3 (Strong deformation retractability). $A \subset X, i: A \rightarrow X$ is an inclusion. If $\exists r \in C(X, A)$ s.t. $r \circ i = \text{id}_A \wedge H: i \circ r \simeq \text{id}_X \text{ rel } A$, then A is called a **strong deformation retract** of X , H is a **strong deformation retraction** of X , and X is said to be **strongly deformation retractable**.

Some examples:

- $X \times [0, 1]$ has strong deformation retracts $X \times \{t\}$ for each $t \in [0, 1]$.
- S^{n-1} is a strong deformation retract of $\mathbb{R}^n \setminus \{0\}$.
- Topological cone $CX = X \times [0, 1]/X \times \{1\}$ has a strong deformation retract at the tip of the cone i.e. $X \times \{1\}$.
- Möbius belt can be strong-deformatively retracted to the circle which is the centre line of the belt.

Chapter 2

Van-Kampen Theorem

§6 Free Abelian Group and Finitely Generated Group

In this section, we only talk about Abelian groups, and their multiplications are called “addition”, i.e. $(G, +)$.

Definition 6.1 (Free Abelian group). Let $(F, +)$ be an Abelian group. If $\exists A \subset F$ s.t. $\forall f \in F, \exists! n_f: A \rightarrow \mathbb{Z}$ s.t.

$$f = \sum_{a \in A} n_f(a)a, \quad \text{card}\{a \in A \mid n_f(a) \neq 0\} \in \mathbb{N},$$

then we call F a **free Abelian group**, A is a **basis** of F .

In plain words, all elements in F can be uniquely decided by finite integer-linear combinations of the elements in A . Notice that A can be infinite.

Typical free Abelian groups are integer vectors groups \mathbb{Z}^n ($n \in \mathbb{N}_+$), while \mathbb{Z}

Theorem 6.1 (Homomorphism induced by any function of basis to a group). *Let F be a free Abelian group, A be a basis of F , G is another Abelian group. $\forall f: A \rightarrow G$, $\exists! \varphi \in \text{Hom}(F, G)$ s.t. $\forall a \in A$, $\varphi(a) = f(a)$.*

Proof. If $x = \sum_{i \in m} n_i a_i \in F$ ($n_i \in \mathbb{Z}$, $m \in \mathbb{N}$, $a_i \in A$), then

$$\varphi(x) = \sum_{i \in m} n_i f(a_i).$$

□

Definition 6.2 (Finitely generated Abelian group). $(F, +)$ is an Abelian group. If $\exists A \subset F$ s.t. $\text{card } A \in \mathbb{Z}$ and $\forall f \in F$, $\exists n_f: A \rightarrow \mathbb{Z}$ s.t.

$$f = \sum_{a \in A} n_f(a) a,$$

then F is called a **finitely generated Abelian group**, A **generates** F . A is a **generating set** of F .

Theorem 6.2 (Finitely generated iff quotient of a free Abelian group). *F is an Abelian group. F is finitely generated \leftrightarrow there exists a free Abelian group H , $\exists j: H \rightarrow F$ s.t. j is an epimorphism.*

Definition 6.3 (Direct sum of Abelian group). Let H_i ($i \in n$) be Abelian subgroups of H . If $\forall h \in H$, $\exists! h_i$ ($i \in n$) s.t.

$$h = \sum_{i \in n} h_i,$$

then we say that H is a **direct sum** of H_i ($i \in n$), denoted as:

$$H = \bigoplus_{i \in n} H_i.$$

If H_i are not subgroup of H and $H \cong \bigoplus_{i \in n} H_i$, H is also called a direct sum of H_i ($i \in n$). In order to avoid confusion, this is called an outer direct sum.

The following theorem is very useful to construct a direct sum:

Theorem 6.3. H_1, H_2, H are Abelian groups. If $H = H_1 + H_2$ (this is the Abelian group version of $H = H_1 H_2$) and $H_1 \cap H_2 = \{0\}$, then $H = H_1 \oplus H_2$.

Theorem 6.4. Let $j: H \rightarrow F$ be an epimorphism, F is a free Abelian group.

$$H \cong \ker j \oplus F.$$

We define some concepts that are very familiar in vector spaces:

Definition 6.4 (Independence and basis). Let H be an Abelian group, A is a subset of H . If $\forall n: A \rightarrow \mathbb{Z}$,

$$\sum_{a \in A} n(a)a = 0 \rightarrow \forall a \in A, n(a) = 0,$$

then A is an *independent* set. And if A generates H , we call it a *basis* of H .

Theorem 6.5. Let H be an Abelian group, and there exists a basis of H . All bases of H have same cardinality.

§7 Free Product of Groups

§8 Van-Kampen Theorem

Chapter 3

Covering Space

§9 Covering space

Definition 9.1 (Even cover). Let $p: E \rightarrow B$ be a continuous surjective, \mathcal{T}_E is the topology of E , U be an open subset of B , I be an index set. If $\exists \langle V_i \rangle_{i \in I} \in \mathcal{T}_E^I$ s.t.

$$p^{-1}(U) = \coprod_{i \in I} V_i,$$

and $p|_{U_i}: U_i \rightarrow U$ is a homeomorphism from U_i to U , then U is said to be evenly covered by p . Each U_i is called a **sheet** or a **slice**.

Definition 9.2 (Covering space). Let $p: E \rightarrow B$ be a continuous surjective. If $\forall b \in B$, $\exists U \in \mathcal{U}(b)$ (**evenly covered neighbourhood**) s.t. U is evenly covered by p , then we call (E, p) a **covering space**, p is a **covering map**, B is the **base space**.

Many authors ([4]) impose path connectivity and local path connectivity onto E and B .

Theorem 9.1. Let (E, p) be a covering space. p is an open mapping.

Proof. Let G be an open set in E . $\forall b \in p(G)$, $\exists U \in \mathcal{U}(b)$ s.t. U is evenly covered by p .

Choose $e \in p^{-1}(b)$, which should be contained in a sheet $V \subset p^{-1}(U)$. $V \cap G$ is open, and since $p|_V$ is a homeomorphism, $p(V \cap G) \subset U \cap p(G) \subset p(G)$ is an open set contained in G which b belongs to.

Therefore, $p(G)$ is open. □

Theorem 9.2 (Restriction of a covering map). *Let (E, p) be a covering space onto B , $B_0 \subset B$, $E_0 = p^{-1}(B_0)$. $(E_0, p|_{E_0})$ is a covering space onto B_0 .*

Theorem 9.3 (Product of covering maps). *Let (E, p) and (F, q) be covering spaces onto B and C . The **product** of (E, p) and (F, q) (E and F , or p and q)*

$$\begin{aligned} p \times q: E \times F &\rightarrow B \times C; \\ (e, f) &\mapsto (p(e), q(f)), \end{aligned}$$

is also a covering map (onto $B \times C$).

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Symbol List

Here listed the important symbols used in these notes

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(E, p) , 17

\tilde{f} , 8

f_π , 5

$f \simeq g$, 1

$f \simeq_H g$, 1

\bar{H} , 2

$H \colon f \simeq g$, 1

$H \colon f \simeq g \text{ rel } A$, 3

$\pi_1(X)$, 7

$\pi_1(X, x_0)$, 5

$q(a)$, 9

$[X]$, 4

$X \simeq Y$, 11

$[X, Y]$, 2

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