Category Theory

Hoyan Mok[1]

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[1]hoyanmok@outlook.com

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Preface

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Chapter 1

Category

§1 Category

Definition 1.1 (Category)**.** A ***category*** $\mathcal{C}$ consists of three ingredients:

1. A *class* $\mathrm{obj}(\mathcal{C})$, called the ***objects***;
2. For any $A, B \in \mathrm{obj}(\mathcal{C})$, a set of ***morphisms*** $\mathrm{Hom}(A, B)$;
3. A function $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$, called the ***composition***, for any $A, B, C \in \mathrm{obj}(\mathcal{C})$, denoted as $(f, g) \mapsto gf$,

and they follow the following axioms:

(i) If $(A, B) \neq (A', B')$, then $\mathrm{Hom}(A, B) \cap \mathrm{Hom}(A', B') = \varnothing$;

(ii) *Associativity*: the composition is associative, i.e. $h(gf) = (hg)f$;

(iii) *Identity*: For any $A \in \mathrm{obj}(\mathcal{C})$, there is an identity morphism $\mathrm{id}_A \in \mathrm{Hom}(A, A)$, such that $f \,\mathrm{id}_A = f = \mathrm{id}_A f$, for any $B \in \mathrm{obj}(\mathcal{C})$ and $f \in \mathrm{Hom}(A, B)$.

A morphism can be shown by:

$$A \xrightarrow{f} B$$

Examples of categories: Set, Grp, Ab, Top, Ord, Ring, Mod, ...
If $\mathrm{obj}(\mathcal{C})$ is a set, then $\mathcal{C}$ is called a ***small category***.
If $(X, \leq)$ is a preorder set, then $\forall x, y \in X$,

$$\mathrm{Hom}(x,y) = \begin{cases} \varnothing & x > y, \\ \{(x,y)\} & x \leq y, \end{cases} \tag{1-1}$$

and $(y,z)(x,y) = (x,z)$. With this we can say that X is a category. The morphism (x,y) is also denoted by i_y^x.

Definition 1.2 (Subcategory)**.** We say $\mathcal{S}$ a ***subcategory*** of $\mathcal{C}$, if

(i) $\mathrm{obj}(\mathcal{S}) \subseteq \mathrm{obj}(\mathcal{C})$;

(ii) $\forall A, B \in \mathrm{obj}(\mathcal{S})$, $\mathrm{Hom}_{\mathcal{S}}(A,B) \subseteq \mathrm{Hom}_{\mathcal{C}}(A,B)$;

(iii) $\forall A, B, C \in \mathrm{obj}(\mathcal{S})$,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

then gf is the same in both $\mathrm{Hom}_{\mathcal{S}}(A,C)$ and $\mathrm{Hom}_{\mathcal{C}}(A,C)$,

(iv) $\forall A \in \mathrm{obj}(\mathcal{S})$, $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{S}}(A,A)$ is the same in $\mathrm{Hom}_{\mathcal{C}}(A,A)$.

Definition 1.3 (Full subcategory)**.** Let $\mathcal{S}$ be a subcategory of $\mathcal{C}$. If $\forall A, B \in \mathrm{obj}(\mathcal{S})$, $\mathrm{Hom}_{\mathcal{S}}(A,B) = \mathrm{Hom}_{\mathcal{C}}(A,B)$, then $\mathcal{S}$ is called a ***full subcategory*** of $\mathcal{C}$.

Definition 1.4 (Generated full subcategory)**.** For any subclass $S \subseteq \mathrm{obj}(\mathcal{C})$, one can find a full subcategory $\mathcal{S}$ of $\mathcal{C}$ s.t. $\mathrm{obj}(\mathcal{S}) = S$, which is called the full subcategory generated by S.

Top_2 is the full subcategory of Top that is generated by the class of all Hausdorff spaces.

Chapter 2

Functors

§2 Functors

Definition 2.1 (Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A ***functor*** $F : \mathcal{C} \to \mathcal{D}$ is a function that satisfies the following axioms:

(i) $\forall A \in \mathrm{obj}(\mathcal{C}),\ F(A) \in \mathrm{obj}(\mathcal{D})$;

(ii) $\forall A, B \in \mathrm{obj}(\mathcal{C}), \forall f \in \mathrm{Hom}_{\mathcal{C}}(A, B),\ F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$;

(iii) $\forall A, B, C \in \mathrm{obj}(\mathcal{C})$,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

then $F(gf) = F(g)F(f)$.

(iv) $\forall A \in \mathrm{obj}(\mathcal{C}),\ F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

Appendix A

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