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Part I

Basic Geometry

Chapter 1

Geometry in Regions of a Space

§1 Co-odinate and its transformation

Definition 1.1 (Jacobian)**.** A transformation of co-odinate from $\boldsymbol{x}$ to $\boldsymbol{y}$

$$\boldsymbol{y}(\boldsymbol{x}) = y^i(x^j)\hat{\boldsymbol{e}}_i = y(x^j).$$

Its ***Jacobian***:

$$\boldsymbol{J} = \left(\frac{\partial y^i}{\partial x^j}\right) \tag{1-1}$$

A ***vector*** $\boldsymbol{u}$ at point $\boldsymbol{x}_0$ under such transformation would follow:

$$v^i = \left.\frac{\partial y^i}{\partial x^j}\right|_{\boldsymbol{x}_0} u^i \tag{1-2}$$

$$\text{i.e.} \qquad \boldsymbol{v} = \boldsymbol{J}_0\boldsymbol{u}$$

A linear form (***covector***) $\ell\colon \boldsymbol{x} \mapsto \ell(\boldsymbol{x}) = l_i x^i$ under such transformation would follow:

$$l'_i\,\mathrm{d}y^i = l_j\,\mathrm{d}x^j \qquad \Rightarrow \qquad l'_i = \left.\frac{\partial x^j}{\partial y_i}\right|_{\boldsymbol{x}_0} l_j \tag{1-3}$$

$$\text{i.e.} \qquad \boldsymbol{l}' = \boldsymbol{l}\boldsymbol{J}_0^{-1}$$

A linear transformation $\mathscr{L}\colon \boldsymbol{x} \mapsto \boldsymbol{L}\boldsymbol{x}$ where $\boldsymbol{L} = (L^i{}_j)_{i,j\in n}$ under such transformation would follow:

$$\begin{aligned}\mathrm{d}y\left(\mathscr{L}^i(\boldsymbol{x})\right) &= (L')^i{}_j\,\mathrm{d}y^j \\ &= \left.\frac{\partial y^i}{\partial x^k}\right|_{\boldsymbol{x}_0}\mathrm{d}\mathscr{L}^k(\boldsymbol{x}) = \left.\frac{\partial y^i}{\partial x^k}\right|_{\boldsymbol{x}_0} L^k{}_h\,\mathrm{d}x^h = \left.\frac{\partial y^i}{\partial x^k}\right|_{\boldsymbol{x}_0} L^k{}_h\left.\frac{\partial x^h}{\partial y_j}\right|_{\boldsymbol{x}_0}\mathrm{d}y^j\end{aligned}$$

$$(L')^i{}_j = \left.\frac{\partial y^i}{\partial x^k}\right|_{\boldsymbol{x}_0} L^k{}_h \left.\frac{\partial x^h}{\partial y_j}\right|_{\boldsymbol{x}_0} \qquad \text{or} \qquad \boldsymbol{L}' = \boldsymbol{J}_0 \boldsymbol{L} \boldsymbol{J}_0^{-1} \tag{1-4}$$

A bilinear form $\mathscr{B}\colon \boldsymbol{x} \mapsto \boldsymbol{x}^{\mathrm{T}} \boldsymbol{b} \boldsymbol{x} = x^{\,i} b_{ij} x^j$:

$$b'_{ij}\,\mathrm{d}y^{\,i}\,\mathrm{d}y^j = b'_{ij} \left.\frac{\partial y^{\,i}}{\partial x^{\,k}}\right|_{\boldsymbol{x}_0} \left.\frac{\partial y^j}{\partial x^h}\right|_{\boldsymbol{x}_0} \mathrm{d}x^{\,k}\,\mathrm{d}x^h = b_{kh}\,\mathrm{d}x^{\,k}\,\mathrm{d}x^h \quad \Rightarrow \quad b'_{ij} = \left.\frac{\partial x^{\,k}}{\partial y^{\,h}}\right|_{\boldsymbol{x}_0} b_{kh} \left.\frac{\partial x^h}{\partial y^j}\right|_{\boldsymbol{x}_0}$$

$$\text{i.e.} \qquad \boldsymbol{b}' = (\boldsymbol{J}_0^{-1})^{\mathrm{T}} \boldsymbol{b} \boldsymbol{J}_0^{-1} \tag{1-5}$$

§2 Riemannian and Pseudo-Riemannnian Spaces

Definition 2.1 (Riemannian metric)**.** A ***Riemannian metric*** $\boldsymbol{G}$ is a smooth, positive-definite quadrutic form defined on a finite-dimensional vector space over $\mathbb{R}$.

Given a basis, we usually denote the Riemannian metric by $g_{ij}(\boldsymbol{x})$.

We can define ***arc length*** ℓ and ***inner product*** $\langle , \rangle$ in a ***Riemannian space*** (i.e. a vector space equiped with a Riemannian metric):

$$\ell := \int_{t_1}^{t_2} \sqrt{g_{ij}[\boldsymbol{x}(t)] \frac{\mathrm{d}x^i}{\mathrm{d}t} \frac{\mathrm{d}x^j}{\mathrm{d}t}}\,\mathrm{d}t, \qquad \langle \boldsymbol{u}, \boldsymbol{v} \rangle := g_{ij} u^i v^j.$$

We can also introduce the following notation: $u_i := g_{ij} u^j$, which means the linear form $\boldsymbol{v} \mapsto u_i v^i$; and $\mathrm{d}\ell^2 = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$.

Definition 2.2 (Euclidean metric)**.** If a metric $\boldsymbol{G}(\boldsymbol{x})$ is said to be ***Euclidean*** if there exists a coodinates $\boldsymbol{y}(\boldsymbol{x})$ s.t.

$$g_{ij} = \delta_{k\ell} \frac{\partial y^k}{\partial x^i} \frac{\partial y^\ell}{\partial x^j}.$$

Such coodinates $\boldsymbol{y}(\boldsymbol{x})$ are said to be a ***Euclidean coodinates***.

Definition 2.3 (Pseudo-Riemannian metric)**.** A ***pseudo-Riemannian metric*** $\boldsymbol{G}$ is a smooth, indefinite quadrutic form defined on a finite-dimensional vector space over $\mathbb{R}$.

A pseudo-Riemannian metric shall have the following cannonical form at some coodinates:

$$\boldsymbol{G} = \mathrm{diag}(\eta_1^2, \ldots, \eta_p^2, -\eta_{p+1}^2, \ldots, -\eta_n^2)$$

where η

By Sylvester's law of inertia, the index of inertia i.e. the number of positive terms on the caninical form, shall conserve under any coodinate change.

Definition 2.4 (Pseudo-Euclidean metric)**.** If a metric $\boldsymbol{G}(\boldsymbol{x})$ is said to be ***pseudo-Euclidean*** if there exists a coodinates $\boldsymbol{y}(\boldsymbol{x})$ s.t.

$$g_{ij} = \sum_{k=1}^{p} \frac{\partial y^k}{\partial x^i} \frac{\partial y^k}{\partial x^j} - \sum_{\ell=p+1}^{n} \frac{\partial y^\ell}{\partial x^i} \frac{\partial y^\ell}{\partial x^j}.$$

Such coodinates $\boldsymbol{y}(\boldsymbol{x})$ are said to be a ***pseudo-Euclidean coodinates***.

We denote a pseudo-Euclidean space by $\mathbb{R}^n_{p,n-p}$, where p is the index of innertia. Especially, we call $\mathbb{R}^4_{1,3}$ the ***Minkowski space***, since its significance in relativistic mechanics.

Bibliography

[1] R.G. Burns et al. *Modern Geometry — Methods and Applications: Part I: The Geometry of Surfaces, Transformation Groups, and Fields*. Graduate Texts in Mathematics. Springer New York, 1991. ISBN: 9780387976631. URL: https://books.google.co.jp/books?id=FC0QFlx12pwC.

Symbol List

Here listed the important symbols used in this notes.

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