Point Set Topology

Hoyan Mok

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Contents

1 Topological Spaces and Continuous Mappings

1.1 Metric Space

Definition 1.1. function

$$d\colon X^2 \to \mathbb{R} \tag{1-1}$$

$\forall x_1, x_2, x_2 \in X$ satisfied:

a) $d(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$;

b) $d(x_1, x_2) = d(x_2, x_1)$ (symmetry);

c) $d(x_1, x_3) \leqslant d(x_1, x_2) + d(x_2, x_3)$ (Triangle inequality),

is called a ***metric*** or ***distance*** in X. Such X is said to be equiped with metric d, $(X; d)$ is called a ***metric space***.

Some examples:

- $(\mathbb{R}^n; d_p)$, where $d_p(x_1, x_2) = \left(\sum_{i=1}^n \left|x_1^i - x_2^i\right|^p\right)^{1/p}$, while $d_\infty(x_1, x_2) = \max\limits_{1\leqslant i\leqslant n} \left|x_1^i - x_2^i\right|$.
- Similarly we can define metric spaces as $(C[a,b]; d_p)$ or $C_p[a,b]$. $d_p(f,g) = \left(\int_a^b |f-g|^p \,\mathrm{d}x\right)^{\frac{1}{p}}$. C_∞ is called a **Chebyshev metric**.
- On class $\tilde{\mathfrak{R}}[a,b]$ over $\mathfrak{R}[a,b]$ similar metric can be defined. Functions are considered of one same class if they are equivalent expect on a set not larger than null set.

Hilbert space denoted by $(\mathbb{H}; d)$ is defined as:

$$\mathbb{H} = \left\{ x = (x_1, x_2, \cdots) \mid \forall i \in \mathbb{Z}_+ \left(\forall x_i \in \mathbb{R} \wedge \sum_{i=1}^{\infty} x_1^2 < \infty \right) \right\} \tag{1-2}$$

equiped with a metirc d:

$$d\colon \mathbb{H}^2 \to \mathbb{R};\ x, y \mapsto \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}. \tag{1-3}$$

To justify this definition, we need to introduce a lemma:

Lemma 1.

$$\forall n \in \mathbb{Z} \forall \boldsymbol{u} \in \mathbb{R}^n \forall \boldsymbol{v} \in \mathbb{R}^n \left(\sum_{i=1}^n u_i v_i \leq \sqrt{\sum_{i=1}^n u_i^2} \sqrt{\sum_{i=1}^n v_i^2} \right) \tag{1-4}$$

*This is called **Schwarz inequality**.*

Proof. If $\forall i = 1, \cdots, n(v_i = 0)$ the equivalence has already been satisfied, therefore the following consider the situation that $\exists i \in \{1, \cdots n\}(v_i \neq 0)$. $\forall \lambda \in \mathbb{R}$

$$\sum_{i=1}^{n}(u_i + \lambda v_i)^2 = \sum_{i=1}^{n} u_i^2 + 2\lambda \sum_{i=1}^{n} u_i v_i + \lambda^2 \sum_{i=1}^{n} v_i^2 \geq 0$$

has at most one root. Hence $\Delta \leq 0$ will lead to the inequality 1-4. $\square$

Apply this inequality to $\sum_{i=1}^{n}(u_i+v_i)^2 = \sum_{i=1}^{n} u_i^2+\sum_{i=1}^{n} v_i^2+2\sum_{i=1}^{n} v_i \sum_{i=1}^{n} u_i$ we can get

$$\sum_{i=1}^{n}(u_i + v_i)^2 \leq \sum_{i=1}^{n} u_i^2 + \sum_{i=1}^{n} v_i^2 + 2\sqrt{\sum_{i=1}^{n} u_i^2}\sqrt{\sum_{i=1}^{n} v_i^2} = \left(\sqrt{\sum_{i=1}^{n} u_i^2} + \sqrt{\sum_{i=1}^{n} v_i^2}\right)^2,$$

in which substitude u_i, v_i by $x_i - y_i, x_i + y_i$ will result in triangle inequality. The inequality holds as the n limits to $+\infty$.

A metric space $(X; d)$ is called ***discrete*** if

$$\forall x \in X \left(\exists \delta_x \in \mathbb{R}_+ \big(\forall y \in X(y \neq x \to d(x,y) > \delta_x)\big)\right).$$

Lemma 2. *If $(X; d)$ is a metric space, then $\forall a, b, u, v,\ |d(a,b) - d(u,v)| \leq d(a,u) + d(b,v)$.*

Proof. Without loss of generality, we assume that $d(a,b) > d(u,v)$. According to the triangle inequality (see def. 1.1), $d(a,b) \leq d(a,u)+d(u,v)+d(v,b)$, which is to prove. $\square$

Definition 1.2. $\delta \in \mathbb{R}_+$, $a \in X$. Set

$$B(a;\delta) = \{x \in X \mid d(a,x) < \delta\}$$

is then called a ***ball*** with centre $a \in X$, and a radius of δ, or a δ***-ball*** of point a.

Definition 1.3. An ***open set*** $G \subset X$ in metric space $(X; d)$ satisfies: $\forall x \in G$, $\exists B(x;\delta)$, s.t. $B(x;\delta) \subset G$.

Definition 1.4. A set $F \subset X$ in metric space $(X; d)$ is said to be a ***closed set*** if its complement $\complement_X(F)$ is open.

It can be proved that $\varnothing$ and X itself is both open and closed.

Proposition 1. a) An infinite union of open sets is an open set.

b) A finite intersection of open sets is an open set.

c) A finite union of closed sets is a closed set.

d) An infinite intersection of closed sets is a closed set.

Proof. a) If open sets $G_\alpha \subset X, \forall\alpha \in A$, $\forall a \in \bigcap\limits_{\alpha\in A} G_\alpha$, $\exists\alpha_0 \in A$, $a \in G_{\alpha_0}$, $\exists B(a;\delta) \subset G_{\alpha_0} \subset \bigcap\limits_{\alpha\in A} G_\alpha$.

b) Open sets $G_1 \cup G_2 \subset X$, $a \in G_1 \cap G_2$, therefore $\exists\delta_1, \delta_2 \in \mathbb{R}_+$, $B(a;\delta_1) \subset G_1, B(a;\delta_2) \subset G_2$, without loss of generality, let $\delta_1 \geq \delta_2$, $\quad a \in B(a;\delta_1) \cap B(a;\delta_2) = B(a;\delta_2) \subset G_1 \cap G_2$.

c) Just consider $\complement_X \left(\bigcap_{\alpha\in A} F_\alpha\right) = \bigcup_{\alpha\in A} \complement_X(F_\alpha)$ and a).

d) Similarly, $\complement_X \left(F_1 \cup F_2\right) = \complement_X(F_1) \cap \complement_X(F_2)$.

$\square$

1.2 Topological Space and Continuous Mapping

Definition 1.5. We say X is equiped with a ***topological space*** or epuiped with ***topology*** if we assigned a $\mathscr{T} \subset 2^X$, which has got the following propoties:

a) $\varnothing \in \mathscr{T}; X \in \mathscr{T}$.

b) $\forall\alpha \in A(G_\alpha \in \mathscr{T}) \to \bigcup\limits_{\alpha\in A} G_\alpha \in \mathscr{T}$.

c) $G_1 \in \mathscr{T} \wedge G_2 \in \mathscr{T} \to G_1 \cap G_2 \in \mathscr{T}$.

Then we call $(X;\mathscr{T})$ a ***topological space***. Every $G \in \mathscr{T}$ is called an ***open set***.

Definition 1.6. A topology $\mathscr{T}_d$ insisting of the open sets in a metric space $(X;d)$ is called a ***topology induced by metric*** d.

A trivial example of topological space is ***trivial topology***, which consists only of empty set and the space itself, i.e. $\mathscr{T} = \{\varnothing, X\}$. Another trivial example of topological space is ***discrete topology***, which consists of all the subsets of the space i.e. $\mathscr{T} = 2^X$.

A ***cofinite space*** is a base set X equiped with a topology $\mathscr{T}$ defined as follows:

$$\mathscr{T} = \{U \in 2^X \mid U = \varnothing \vee \complement_X U \text{ is finite}\} \tag{1-5}$$

Proposition 2. The set $\mathscr{T}$ under definition 1-5 is a topology.

Proof. a) $\varnothing \in \mathscr{T}$, $X \in \mathscr{T}$.

b) $\forall i \in I \left(\left|\complement_X A_i\right| \in \mathbb{N}\right) \to \forall i_0 \in I \left(\left|\bigcap_{i\in I} \complement_X A_i\right| \leq \left|\complement_X A_{i_0}\right|\right)$, therefore $\bigcup_{i\in I} A_i \in \mathscr{T}$.

c) $\forall A \in \mathscr{T} \forall B \in \mathscr{T}(A \cap B = \varnothing \in \mathscr{T} \vee \complement_X(A \cap B) = \complement_X A \cup \complement_X B$ is finite$)$, therefore $\forall A \in \mathscr{T} \forall B \in \mathscr{T}(A \cap B \in \mathscr{T})$.

$\square$

Similarly, ***countable complement space*** can be defined.

Definition 1.7. Let $(X;\mathscr{T})$ be a topological space. If there exists a metric $d\colon X^2 \to \mathbb{R}$ s.t. $(X;\mathscr{T})$ is induced by d then call $(X;\mathscr{T})$ a ***metrizable space***, $(X;d)$ is its ***metrization***.

Definition 1.8. Let $(X;\mathscr{T})$, $(Y;\mathscr{S})$ be two topological space. A mapping f is said to be continuous if

$$\forall V \in \mathscr{S}\big(\exists U \in \mathscr{T}(U = f^{-1}(V))\big).$$

Index