Algebraic Topology

Hoyan Mok

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Chapter 1

Homotopy and Fundamental Group

§1 Homotopy

Definition 1.1 (Homotopy)**.** $f, g \in C(X, Y)$. If $\exists H \in C(X \times [0,1], Y)$ s.t. $H(x, 0) = f(x)$, $H(x, 1) = g(x)$, then we say f and g are ***homotopic***, denoted by $f \simeq g \colon X \to Y$ or just $X \to Y$. H is called a ***homotopy*** between f and g, denoted by $H \colon f \simeq g$ or $f \simeq_H g$.

For $t \in [0,1]$, $h_t \colon X \to Y; x \mapsto H(x,t)$ is called a ***t-slice***.

If f is homotopic to a constant mapping, we say that f is ***null-homotopic***.

A ***linear homotopy*** is a homotopy between two functions to $Y \subseteq \mathbb{R}^n$ that change linearly, i.e.

$$H(x,t) = (1-t)f(x) + tg(x).$$

Theorem 1.1 (Maps to convex set are homotopic)**.** $f, g \in C(X, Y)$. *If Y is a convex set in $\mathbb{R}^n$, then $f \simeq g$.*

Proof. Consider linear homotopy. □

Theorem 1.2. *Homotopic relation is an equivalence relation.*

Proof. *reflexity.* $f \simeq f$, just take $H(x,t) = f(x)$ for any t (Such homotopy is called a ***constant homotopy***).

Symmetry. $f \simeq g$ then $g \simeq f$. Just take $\bar{H}(x,t) = H(x, 1-t)$ (Here $\bar{H}$ is called the inverse of H).

Transivity. $f \simeq g \wedge g \simeq h \to f \simeq h$. Let

$$H_1H_2(x, 2t) = \begin{cases} H_1(x, 2t) & t \in [0, 1/2], \\ H_2(x, 2t-1) & t \in [1/2, 1]. \end{cases}$$

We can see that H_1H_2 is also a homotopy (see Theorem 11.6 in Point Set Topology) □

Hence, we can define ***homotopy classes*** on $C(X, Y)$, denoted by $[X, Y]$.

As you might expect after reading the proof of Theorem 1.2, the homotopies between mappings within a homotopy class form a group.

Theorem 1.3 (Composition of homotopies)**.** *$f_1 \simeq f_2 \colon X \to Y$, $g_1 \simeq g_2 \colon Y \to Z$, then $g_1 \circ f_1 \simeq g_2 \circ f_2 \colon X \to Z$.*

Proof i. Let $F \colon f_1 \simeq f_2$, $G \colon g_1 \simeq g_2$. Define:

$$\boldsymbol{F} \colon X \times [0,1] \to Y \times [0,1]; (x,t) \mapsto (F(x,t), t).$$

It can be verified tht $G \circ \boldsymbol{F} \colon g_1 \circ f_1 \simeq g_2 \circ g_2 \colon X \to Z$. □

Proof ii. Let $F \colon f_1 \simeq f_2$, $G \colon g_1 \simeq g_2$.

We can verify that $H_1 \colon (x,t) \mapsto g_1 \circ F(x,t)$ is a homotopy between $g_1 \circ f_1$ and $g_1 \circ f_2$; Similarly $H_2 \colon g_1 \circ f_2 \simeq g_2 \circ f_2$ can be defined.

Now consider $H = H_1 H_2$, or in detailed,

$$H(x,t) = \begin{cases} g_1 \circ F(x, 2t) & (x,t) \in X \times [0, 1/2] \\ G(f_2(x), 2t-1). & (x,t) \in X \times [1/2, 1] \end{cases}$$

□

Theorem 1.4 (Identity map in convex space is null-homotopic)**.**

Theorem 1.5 (All mappings from a convex set to a path-connected space are null-homotopic)**.** *If $X \subseteq \mathbb{R}^n$ is a convex set, Y is path-connected, then any $f \colon X \to Y$ is null-homotopic.*

Proof. First we verify that the identity id_X is null-homotopic. The linear homotopy can be constructed as:

$$H_{x_0}(x,t) = tx + (1-t)x_0.$$

Then, any $f \colon X \to Y$ can be written as $f = f \circ \mathrm{id}_X$, hence $f \simeq f \circ H_{x_0}(x, 1) = (x \mapsto f(x_0))$, which means f is null-homotopic. □

Theorem 1.6 (All constant mappings to a path-connected space belong to one homotopy class)**.** *If Y is a path-connected space, $y_0 \in Y$, then $[X, Y] = [x \mapsto y_0]$ (i.e. homotopy class of constant mapping to $\{y_0\}$)*

Proof. Let $f_1(x) = y_1$, $f_2(x) = y_2$ be two constant mappings, a is a path from y_1 to y_2. Then the homotopy between f_1 and f_2 can be defined as:

$$H(x,t) = a(t).$$

□

Definition 1.2 (Homotopy relative to a set)**.** Let $A \subseteq X$, $H \colon f \simeq g$. If $\forall a \in A$, $\forall t \in [0,1]$, $f(a) = g(a) = H(a,t)$, we say that f and g are ***homotopic relative to*** A, denoted by $H \colon f \simeq g \operatorname{rel} A$.

We can have parallel results as Theorem 1.2 and Theorem 1.3:

Theorem 1.7. *Given $A \subseteq X$, $\simeq \operatorname{rel} A$ is an equivalence relation in $C(X, Y)$.*

Theorem 1.8 (Composition of relative homotopies)**.** *$f_1 \simeq f_2 \colon X \to Y \operatorname{rel} A$, $g_1 \simeq g_2 \colon Y \to Z \operatorname{rel} B$, and $f_1(A) \subset B$, then $g_1 \circ f_1 \simeq g_2 \circ f_2 \colon X \to Z$.*

Definition 1.3 (Fixed-endpoint Homotopy)**.** Let a, b be two paths in X. If $a \simeq b \operatorname{rel}\{0, 1\}$, we say that a and b are ***fixed-endpoint homotopic***. The paths in X modulus fixed-point homotopy is denoted by $[X]$, called the ***path classes***. The path class which a belongs to is denoted by $\langle a \rangle$.

bibliography

[1] 尤承业. 基础拓扑学讲义. 北京: 北京大学出版社, 1997. ISBN: 9787301031032.

[2] 熊金诚, ed. 点集拓扑讲义. 2nd ed. 北京: 高等教育出版社, 1998. ISBN: 9787040062823.

Symbol List

Here listed the important symbols used in this notes

$\langle a \rangle$, 3

$f \simeq g$, 1
$f \simeq_H g$, 1

$\bar{H}$, 1

$H\colon f \simeq g$, 1
$H\colon f \simeq g \operatorname{rel} A$, 2

$[X]$, 3
$[X, Y]$, 2

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