Contents

1 Metric Space and Continuous Map

1.1 Metric Space

Definition 1.1. function

$$d : X^2 \to \mathbb{R} \tag{1-1}$$

$\forall x_1, x_2, x_2 \in X$ satisfied:

a) $d(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$;

b) $d(x_1, x_2) = d(x_2, x_1)$ (symmetry);

c) $d(x_1, x_3) \leqslant d(x_1, x_2) + d(x_2, x_3)$(Triangle inequality),

is called a **metric** or **distance** in X. Such X is said to be equiped with metric d, $(X; d)$ is called a **metric space**.

Some examples:

- $(\mathbb{R}^n; d_p)$, where $d_p(x_1, x_2) = \left(\sum_{i=1}^n \left|x_1^i - x_2^i\right|^p\right)^{1/p}$, while $d_\infty(x_1, x_2) = \max\limits_{1\leqslant i\leqslant n} \left|x_1^i - x_2^i\right|$.
- Similarly we can define metric spaces as $(C[a,b]; d_p)$ or $C_p[a,b]$. $d_p(f,g) = \left(\int_a^b |f-g|^p \,\mathrm{d}x\right)^{\frac{1}{p}}$. C_∞ is called a **Chebyshev metric**.
- On class $\tilde{\mathfrak{R}}[a,b]$ over $\mathfrak{R}[a,b]$ similar metric can be defined. Functions are considered of one same class if they are equivalent expect on a set not larger than null set.

Lemma 1. If $(X; d)$ is a metric space, then $\forall a, b, u, v$, $|d(a,b) - d(u,v)| \leq d(a,u) + d(b,v)$.

Proof. Without loss of generality, we assume that $d(a,b) > d(u,v)$. According to the triangle inequality (see def. 1-1), $d(a,b) \leq d(a,u) + d(u,v) + d(v,b)$, which is to proof. □

Definition 1.2. $\delta \in \mathbb{R}_+$, $a \in X$. Set

$$B(a;\delta) = \{x \in X | d(a,x) < \delta\}$$

is then called a **ball** with centre $a \in X$, and a radius of δ, or a **δ-ball** of point a.

Definition 1.3. A **open set** $G \subset X$ in metric space $(X; d)$ satisfies: $\forall x \in G$, $\exists B(x;\delta)$, s.t. $B(x;\delta) \subset G$.

Definition 1.4. A **closed set** F in metric space $(X; d)$ satisfies: $X - F$ is a open set in $(X; d)$.

$\tilde{B}(x;\delta) = \{x \in X | d(a,x) \leq r\}$ is an example of closed sets in $(X;d)$.

Proposition 1. a) An infinite union of open sets is an open set.

b) A definite intersection of open sets is an open set.

c) A definite union of closed sets is a closed set.

d) An infinite intersection of closed sets is a closed set.

Proof. a) If open sets $G_\alpha \subset X, \forall \alpha \in A$, $\forall a \in \bigcap\limits_{\alpha \in A} G_\alpha$, $\exists \alpha_0 \in A$, $a \in G_{\alpha_0}$, $\exists B(a;\delta) \subset G_{\alpha_0} \subset \bigcap\limits_{\alpha \in A} G_\alpha$.

b) Open sets $G_1 \cup G_2 \subset X$, $a \in G_1 \cap G_2$, therefore $\exists \delta_1, \delta_2 \in \mathbb{R}_+$, $B(a;\delta_1) \subset G_1, B(a;\delta_2) \subset G_2$, without loss of generality, let $\delta_1 \geq \delta_2$, $\quad a \in B(a;\delta_1) \cap B(a;\delta_2) = B(a;\delta_2) \subset G_1 \cap G_2$.

c) Just consider $\complement_X \left(\bigcap_{\alpha \in A} F_\alpha\right) = \bigcup_{\alpha \in A} \complement_X(F_\alpha)$ and a).

d) Similarly, $\complement_X (F_1 \cup F_2) = \complement_X(F_1) \cap \complement_X(F_2)$.

$\square$

Definition 1.5. If $x \in X$ is an element of an open set, then such open set is called a **neighbourhood** of point x in X, denoted by $U(x)$.

Definition 1.6. $x \in X$, $E \subset X$.

a) If $\exists U(x) \subset E$, x is called an **interior point** of E.

b) If $\exists U(x) \subset X - E$, x is called an **exterior point** of E.

c) If x isn't an interior point nor exterior point of E, it is called a **boundary point** of E. The set of boundary points is called **boundary**, denoted by ∂E.

Definition 1.7. $a \in X$, $E \subset X$. If $\forall U(a)$, $|E \cap U(a)| = \infty$, a is called a **limit point** of E.

Definition 1.8. The intersections of $E \subset X$ and set of all its limit points is called the **closure** of E, denoted by $\overline{E}$.

Theorem 1.1. $F \subset X$ is a closed set in $X \Leftrightarrow \overline{F} = F$.

Proof. $\Rightarrow$: $\complement_X(F)$ is open, hence its elements are all its interior points. Therefore $\overline{F} - F = \overline{F} \cup \complement_X(F) = \varnothing$, $F \subset \overline{F} \Rightarrow F = \overline{F}$.

$\Leftarrow$: $F = \overline{F}$ means that $\forall x \in \complement_X(F)$, x is not a boundary of F, which indicates that x is an interior point of $X - F$. Therefore $F - X$ is open while F is closed. $\square$

Theorem 1.2. $\overline{E}$ is always closed.

Proof. $\forall x \in X - \overline{E}$, since it is not a element of the set E or its limit points, $\exists U(x)$ s.t. $U(x) \cap \overline{E} \varnothing$, which implies that x is an extorior point of E, therefore $\overline{E}$ is closed. □

Theorem 1.3. $\overline{E} = \overline{\overline{E}}$.

Proof. Since $\overline{E}$ is closed, its complement is open, which implies that its elements are all exterior point of $\overline{E}$, therefore $\overline{E}$ has contained all of its limit points. □

Definition 1.9. We called $(X'; d')$ a **subspace** of $(X; d)$ when $X' \subset X$ and $\forall x, y \in X', d'(x, y) = d(x, y)$.

1.2 Topological Space

Definition 1.10. We say X is equiped with a **topological space** or epuiped with **topology** if we assigned a $\mathscr{T} \subset 2^X$, which has got the following propoties:

a) $\varnothing \in \mathscr{T}; X \in \mathscr{T}$.

b) $(\forall \alpha \in A, \mathscr{T}_\alpha \in \mathscr{T}) \Rightarrow \bigcup\limits_{\alpha \in A} \mathscr{T}_\alpha \in \mathscr{T}$.

c) $(\mathscr{T}_1, \mathscr{T}_2 \in \mathscr{T}) \Rightarrow \mathscr{T}_1 \cap \mathscr{T}_2$.

Then we call $(X; \mathscr{T})$ a **topological space**.

These are correspondence of propoties of open sets (See proposition 1). Topology made of all open sets defined in metric space $(\mathbb{R}; d_2)$ is called the **standard topology** of n-dimension Euclidean space.

Definition 1.11. Topology $\mathscr{T}$'s elements are called **open sets**, and their complements are called **closed sets**.

Definition 1.12. $(X; \mathscr{T})$ is a topological space, $\mathfrak{B} \subset 2^X$. If $\forall G \in \mathscr{T}$, $\exists B_\alpha \in \mathfrak{B}$ $(\alpha \in A)$ s.t. $\bigcup\limits_{\alpha \in A} B_\alpha = G$, it is called a (topological or open) **base**.

Definition 1.13. The smallest possible cardinity of base is called the **weight** of the topological space.

Definition 1.14. If $x \in G$ and $G \in \mathscr{T}$, then G is a **neighbourhood** of x in topological space $(X; \mathscr{T})$.

For example, we define an equivalence relation $\sim$ in $C(\mathbb{R}; \mathbb{R})$. If $f, g \in C(\mathbb{R}; \mathbb{R})$, at point $a \in \mathbb{R}$:

$$f \sim_a g \Leftrightarrow (\exists U(a)\, (\forall x \in U(a), f(x) = g(x)))\,. \tag{1-2}$$

Then we call f and g define a **germ** at point a, denoted by f_a. If $f \in C(\mathbb{R}; \mathbb{R})$ is defined in $U(a)$, then we can call $f_x := \{f_x | x \in U(a)\}$ a neighbourhood of germ f_a. Class of neighbourhoods of each f_x constructs a base of topological space $(C(\mathbb{R}; \mathbb{R}); \mathscr{T})$, where $\mathscr{T}$ is made of the sets of germs of continuous function in $C(\mathbb{R}; \mathbb{R})$.

Definition 1.15. We call a topological space $(X; \mathscr{T})$ a **Hausdorff space**, **separated space** or T_2 **space**, if $\forall x, y \in X$, $\exists U(x), U(y)$ s.t. $U(x) \cap U(y) = \varnothing$ (**Hausdorff axiom** or **separation axiom**).

Definition 1.16. $E \subset X$ is a **dense set** in topological space $(X; \mathscr{T})$, if $\forall x \in X$, $\forall U(x)$, $U(x) \cap E \neq \varnothing$.

Definition 1.17. If there is a countable dense set in topological space $(X; \mathscr{T})$, then $(X; \mathscr{T})$ is **separable**.

We can also define interior points, exterior points, boundary points, limit points in topological space as in metric space.

Definition 1.18. Each subset Y of X equiped with topology $\mathscr{T}$ can be given a **subspace topology** $\mathscr{T}_Y$ whose elements G_Y are intersections of the subset with an open set G in $(X; \mathscr{T})$ i.e. $\forall G_Y \in \mathscr{T}_Y$, $\exists G \in \mathscr{T}$ s.t. $G_Y = G \cap Y$. Subset equiped with such topology construct a **topological subspace** $(Y; \mathscr{T}_Y)$.

If two topology $\mathscr{T}_1, \mathscr{T}_2$ are defined on the same X, $\mathscr{T}_1$ is said to be **stronger** than $\mathscr{T}_2$ if $\mathscr{T}_1 \subsetneqq \mathscr{T}_2$.

1.3 Compact Set

Definition 1.19. Set K in topological space $(X; \mathscr{T})$ is called a **compact set** if each of its **open covers** has a finite **subcover**. Class Ω is called a open cover of K if $K \subset \cup\Omega$ and for all sets in Ω are open sets.

Specially, $\varnothing$ is compact.

Theorem 1.4. Set $K \subset X$ is compact in $(X; \mathscr{T})$ **iff** K is compact in $(K; \mathscr{T}_K)$ itself.

This theorem tells a truth that whether K is compact or not isn't dependent on the topological space it's in, it can be easily proofed: just need to notice that every open set G_K in $(K; \mathscr{T}_K)$ is an intersection of an open set G in $(X; \mathscr{T})$ and K.

Theorem 1.5. If K is compact in a Hausdorff space $(X; \mathscr{T})$ (See definition 1.15), then K is a closed set in $(X; \mathscr{T})$.

Proof. If x_0 is a limit point of K, which means $\forall U(x_0)$,

$$|U(x_0) \cap K| \notin \mathbb{N}.$$

Assume that $x_0 \notin K$. In a Hausdorff space, $\forall x \in K$, $\exists U(x)$ s.t. $U(x) \cap U(x_0) = \varnothing$. Such $U(x)$ construct a open cover $\Omega = \{U(x) | x \in K\} \subset 2^K$. Since K is compact, $\exists \Omega' \subset \Omega$ s,t. $|\Omega| \in \mathbb{N}$.

$$(\cup\Omega') \cap U(x_0) = \left(\bigcup_{k=1}^{n} U_k\right) \cap U(x_0) = \bigcup_{k=1}^{n} (U_k \cap U(x_0)) = \varnothing$$

Since $K \subset \cup\Omega'$, x_0 is an exterior point of K, which leads to a contradiction. Hence $x_0 \in K$. $\overline{K} = K$. $\square$

Theorem 1.6. Each decreasing **nested sequences** of non-empty compact sets has a non-empty limit, i.e. $\forall\{K_n\}$ s.t. $\forall n \in \mathbb{N}_+$, $K_n \supset K_{n+1} \wedge K_n \neq \varnothing \wedge (K_n$ is compact), $K_n \downarrow K \neq \varnothing$.

Proof. Assume that $K = \varnothing$. Compact subsets of K_1 are all colsed, while their complements are all open. An open cover Ω can be constructed as $\{K_1 - K_n | n \in \mathbb{N}_+\}$. Since K_1 is compact, there would be a finite subcover $\Omega' \subset \Omega$, notice that $\{X - K_n\}$ is also a nested sequence, there must be oone single $X - K_{n_0} \in \Omega'$ that covers K_1, which means $K_{n_0} = \varnothing$ contradicting that $\forall n \in \mathbb{N}_+$, K_n is non-empty. $\square$

Theorem 1.7. Closed subsets F of a compact set K are also compact.

Proof. If $\Omega_F \subset 2^K$ is an open cover of F. Notice that $K - F$ is open, $\Omega = (\cup\Omega_F) \cap \{K - F\}$ constructs an open cover over over K. Since K is compact there must be a finite cover $\Omega' \subset \Omega$ which obviously also covers over F. $\square$

The following propoties of compact sets are on the topological space induced from a metric space.

Definition 1.20. $(X; d)$ is a metric space, $E \subset X$. E is called an ε**-net** if $\forall x \in X, \exists e \in E$, $d(e, x) < \varepsilon$.

Theorem 1.8. If (K, d) is a compact metric space, then $\forall \varepsilon \in \mathbb{R}_+$, $\exists$ finite ε-net in $(K; d)$.

Proof. For each point $x \in K$, find it a $B(x, \varepsilon)$, of which an infinite cover Ω over K is made. Since K is compact, there exists a finite cover $\Omega' = \{B(x_1, \varepsilon), \cdots, B(x_n, \varepsilon)\}$ $(n \in \mathbb{N}_+)$. Therefore $\{x_1, \cdots, x_n\}$ is a finite ε-net in K. $\square$

Theorem 1.9. $(K; d)$ is compact **iff** it is **sequentially compact**, that is, $\forall\{x_n\}$ $(x_n \in K,\ n \in \mathbb{N}_+)$, it has convergent subsequence $\{x_{k_n}\}$ whose limit $a \in K$.

To proof it, we need to proof two lemmata first.

Lemma 2. If $(K; d)$ is sequentially compact, then $\forall \varepsilon \in \mathbb{R}_+$, $\exists$ finite ε-net in $(K; d)$.

Proof. Assume that there were no finite ε_0-net in $(K; d)$. Define such sequence : $\{x_n\}$ s.t. $\forall k, n \in \mathbb{N}_+$ $(1 \leq k < n)$, $d(x_n, x_k) \geq \varepsilon_0$ (There would always be the next one since there exists no ε_0-net). It has no convergent subsequence for it there were a $\{x_{k_n}\}$ convergent to $a \in K$, $\exists N, M \in \mathbb{N}_+$, $d(x_N, x_M) \leq d(x_N, a) + d(x_M, a) \leq \varepsilon_0$, which lead to a contradictary. $\square$

Lemma 3. If $(K; d)$ is sequentially compact then every nested sequence of closed non-empty sets $\{F_n\}$ in K have a non-empty intersection.

Proof. Let $\{x_{k_n}\}$ be a convergent subsequence of $\{x_n\}$, Let a be the limit of $\{x_{k_n}\}$ ($\forall n \in \mathbb{N}_+, x_n \in F_n$). Assume that $a \notin \bigcap_{n\in\mathbb{N}_+} F_n$, in metric space, $\exists U(a) \cap \left(\bigcap_{n\in\mathbb{N}_+} F_n\right) = \varnothing \Rightarrow U(a) \cap \left(\bigcap_{n\in\mathbb{N}_+} F_{k_n}\right) = \varnothing$. But this conflict the fact that $\exists N \in \mathbb{N}_+$, s.t. $n > N$, $x_{k_n} \in U(a)$ while $x_{k_n} \in F_{k_n}$. $\square$

Then get back to theorem 1.9.

Proof. $\Rightarrow$: If $|\{x_n\}| \in \mathbb{N}$, it is obvious; if $|\{x_n\}| = \infty$, make finite $\frac{1}{n}$-net (Theorem 1.8), $n \in \mathbb{N}_+$. For each n, there must be at least one $B(x_n; \frac{1}{n})$ that includes infinite elements in $\{x_n\}$. Select $x_{k_n} \in B(x_n; \frac{1}{n})$, and $\{\tilde{B}(x_n; \frac{1}{n})\}$ is a nested sequence of a closed non-empty sets in sequentially compact K, (Lemma 3) $\lim\limits_{n\to\infty} x_{k_n} \in K$.

$\Leftarrow$: Assume that there were a open cover Ω over K having no finite subcover, $\forall n \in \mathbb{N}_+$, $\exists$ finite $\frac{1}{n}$-net (Lemma 3), in which there would be at least one x_n whose $\tilde{B}(x_n; \frac{1}{n})$ can't be covered finitely. Then $\tilde{B}(x_n; \frac{1}{n}) \downarrow B = \{a\}$ (Theorem 1.6) can't be finitely covered by any subcover of Ω which means Ω can't cover the whole K, leading to the contradiction. $\square$

1.4 Connected Set

Definition 1.21. Topological space $(X; \mathscr{T})$ is called **connected** if there is no **open-closed set** (i.e. both open and closed) besides $\varnothing$ and X itself.

Notice that if $A \subset X$ is open-closed, its complement $X - A$ is also open-closed, which means a topological space is connected **iff** it is not a union of its two open subsets.

Definition 1.22. $(X; \mathscr{T})$ is a topological space. Subset C is said to be **connected** if subspace $(C; \mathscr{T}_C)$ is connected.

Theorem 1.10. $(X; \mathscr{T})$ is a topological space. $\forall \alpha \in A$, C_α are connected subsets of X. If $\bigcap\limits_{\alpha\in A} C_\alpha \neq \varnothing$, then $\bigcup\limits_{\alpha\in A} C_\alpha$ is also connected.

Proof. If $C = \bigcup\limits_{\alpha\in A} C_\alpha$ were not connected, $\exists E \subset C$ s.t. $E \neq \varnothing \wedge E \neq C \wedge E, C - E \in \mathscr{T}_C$. For E is not empty there exists a $\beta \in A$ s.t. $E \cap C_\beta \neq \varnothing$. It can be proofed that $C_\beta \subset E$.

Suppose that $C_\beta \not\subseteq E$, which implies that $(C-E)\cap C_\beta \neq \varnothing$. $E, C-E, C_\beta \in \mathscr{T}_C \Rightarrow E\cap C_\beta, (C-E)\cap C_\beta \in \mathscr{T}_C$. This conflicts to the fact that C_β is connected. Therefore $C_\beta \subset E$.

Hence there exists a $B \subsetneqq A$, $\bigcup\limits_{\beta\in B} C_\beta = A$. Since C_γ, $\gamma \in A - B$ would have a empty intersection with E, which contradicts $\bigcap\limits_{\alpha\in A} C_\alpha \neq \varnothing$. $\square$

Theorem 1.11. Connected sets have connected closure.

Proof. $\square$

Theorem 1.12. $E \subset \mathbb{R}$ is connected **iff** that if $\forall x, z \in E$, $y \in \mathbb{R}$ s.t. $x < y < z$, then $y \in C$.

Proof. $\Rightarrow$: Assume that there were such $y \in \mathbb{R}$ that $\exists x, z \in C$, $x < y < z$ but $y \notin C$. $\{x \in C | x < y\}$ and $\{x \in C | x > y\}$ are open in $C$for they are intersection of open sets in $\mathbb{R}$ and C. Since they're each other's complement, they are both open-closed, which conflict to the definition of connected set.

$\Leftarrow$: It can be proofed that $(\inf C, \sup C) \subset C$. Assume that there were an open-closed proper subset $E \neq \varnothing$ contained in C. Find two points $x \in E$, $z \in C - E$. Without loss of generality, let $x < z$. Since E and $C - E$ are closed, $c_1 = \inf\{E \cap [a, b]\} \in E$ while $c_2 = \inf\{(C - E) \cap [a, b]\} \in C - E$. However $E \cap (C - E) = \varnothing \Rightarrow c_1 < c_2$, which means $(c_1, c_2) \cap E = \varnothing$. Here's the contradiction. $\square$

Definition 1.23. A topological space $(X; \mathscr{T})$ is said to be **locally connected** if $\forall x \in X$, $\exists U(x)$ s.t. $U(x)$ is connected.

1.5 Complete Metric Spaces

We now take a closer look at one of the most important sorts of metric spaces: complete spaces.

Definition 1.24. A sequence $\{x_n \mid n \in \mathbb{N}\}$ of points of a metric space $(X; d)$ is called a **fundamental** or **Cauchy sequence** if $\forall \varepsilon \in \mathbb{R}_+$, $\exists N \in \mathbb{N}$ s.t. as long as $m, n > N$, $d(x_n, x_m) < \varepsilon$.

Definition 1.25. A metric space $(X; d)$ is **complete** if every Cauchy sequence of its points is convergent.

For example, metric space $C_\infty[a, b]$ is complete while $C_1[a, b]$ isn't. Proof see p22, Zorich. Consider incomplete space $\mathbb{Q}_1$, which is a subspace of the complete space $\mathbb{R}_1$. If $\mathbb{R}_1$ is the smallest complete space containing $\mathbb{Q}_1$, we can say that we have achieved a **completion** of $\mathbb{Q}_1$. However, the definition of "completion" hasn't been defined yet.

Definition 1.26. If a metric space $(X; d)$ is a subspace of a complete metric space $(Y; d)$ and everywhere dense in it, we call the latter one the **completion** of $(X; d)$.

We need to confirm that such completion is the smallest and unique. So we introduce:

Definition 1.27. If there exists a **isometry** $f : X_1 \to X_2$ when $(X_1; d_1)$ and $(X_2; d_2)$ are both metric space, i.e. f is a bijective and for each $a, b \in X_1$, $d_2\left(f(a), f(b)\right) = d_1\left(a, b\right)$, then these two metric space is **isometric**.

This relation is reflexive (e), symmetric (f^{-1}), and transitive $(f \circ g)$, so it is a equivalence relation, noted by $\sim$. We shall consider isometric spaces are identical.

Theorem 1.13. If metirc spaces $(Y_1; d_1)$ and $(Y_2; d_2)$ are both completions of $(X; d)$, then they are isometric.

Proof. Such isometry $f : Y_1 \to Y_2$ can be defined: if $x_1, x_2 \in X$,

$$d_2(f(x_1), f(x_2)) = d(f(x_1), f(x_2)) = d(x_1, x_2) = d_1(x_1, x_2).$$

For each $y_1 \in Y_1 - X_1$, a Cauchy sequence $\{x_n\}$ can be found in the nested sequence of balls centered in y_1. It is obvious that $\{x_n\}$ is also fundamental in Y_2, limitting to $y_2 \in Y_2$. Different sequences of points $\{x'_n\}$ selected won't result in a diffrent y'_2, or $d(x_n, x'_n)$ wouldn't converge to 0, which violate the fact that the radii of balls converge to 0. Let $f(y_1) = y_2$.

a) For each $y_2 \in Y_2 - X$, there always exists a Cauchy sequence converging to it, which implies that f is a surjection.

b) Also notice that $\forall y'_1, y''_1 \in Y_1 - X$,

$$d_1(y'_1, y''_1) = \lim_{n\to\infty} d(x'_n, x''_n) = d_2(y'_2, y''_2)$$

while $\{x'_n\}$ and $\{x''_n\}$ are both Cauchy sequence. This equality also proofed that f is a injection. □

Theorem 1.14. There always exists a completion for every metric space.

Proof. A isometric space $(S_X; d)$ to the metric space $(X; d_X)$ can be constructed, which consists of constant sequence of points in X. Its completion $(S; d)$ can be defined as Cauchy sequences whose mutual distances' limits are not 0. □

1.6 Continuous Mapping

Let's recall the definition of the limitation.

Definition 1.28. A set $\mathscr{B} \subset 2^X$ is called a **(filter) base** in X if the following conditions hold:

a) $\varnothing \notin \mathscr{B}$.

b) $\forall B_1, B_2 \in \mathscr{B}$, $\exists B \in \mathscr{B}$ s.t. $B \subset B_1 \cap B_2 \subset B_2$.

Introduction of the limits in a topological space is as follows.

Definition 1.29. Let $a \in Y$ be the **limit** over the base $\mathscr{B} \subset 2^{\mathscr{D}(f)}$ of a mapping $f : \mathscr{D}(f) \to Y$, in which Y is epuiped with a topology $\mathscr{T}$.

$$\lim_{\mathscr{B}} f = a \quad := \quad \forall U(a) \subset Y \; \exists B \in \mathscr{B}(f(B) \subset U(a)).$$

Such definition is parallel to the definition we have introduced on the limits of real number, hence it basically holds the same propoties.

Definition 1.30. A mapping $f : X \to Y$, where X,Y is respectively equiped with topology $\mathscr{T}_X$,$\mathscr{T}_Y$, is said to be **continuous** at $x_0 \in X$ (let $y_0 = f(x_0) \in Y$), if $\forall U(y_0)$, $\exists U(x_0)$ s.t. $f(U(x_0)) \subset U(y_0)$. It is **continuous** in X if it is continuous at each point $x \in X$.

The set of continuous mappings from X into Y can be denoted by $C(X, Y)$ or $C(X)$ when Y is clear.

Theorem 1.15. (Criterion for continuity)
$(X; \mathscr{T}_X)$ and $(Y; \mathscr{T}_Y)$ are both topological spaces. A mapping $f : X \to Y$ is continuous **iff** $\forall G_Y \in \mathscr{T}_Y$, $f^{-1}(G_Y) \in \mathscr{T}_X$.

Proof. $\Rightarrow$: It is obvious if $f^{-1}(G_Y) = \varnothing$. If $f^{-1}(G_Y) \neq \varnothing$ and $x_0 \in X$, since $f \in C(X, Y)$, for G_Y, $\exists U(x_0)$ s.t $f(U(x_0)) \subset G_Y$. Also notice that $f(U(x_0)) \subset G_Y \Rightarrow U(x_0) \subset f^{-1}(G_Y)$, therefore $f^{-1}(G_Y)$ is open.

$\Leftarrow$: $\forall x_0 \in X$, let $y_0 = f(x_0)$, $f^{-1}(U(y_0)) \in \mathscr{T}_X$. Notice that $x_0 \in f^{-1}(U(y_0))$, therefore $f \in C(X, Y)$. □

Definition 1.31. $(X; \mathscr{T}_X)$ and $(Y; \mathscr{T}_Y)$ are both topological spaces. A bijective mapping $f : X \to Y$ is a **homeomorphism** if $f \in C(X, Y) \wedge f^{-1} \in C(Y, X)$.

Definition 1.32. Two topological spaces $(X; \mathscr{T}_X)$ and $(Y; \mathscr{T}_Y)$ are said to be **homeomorphic** if there exists a homeomorphism $f : X \to Y$.

Homeomorphic topological spaces are identical with respect to their topological propoties since the theorem 1.15 has shown that their open sets correspond to each other.

Theorem 1.16. $(X; \mathscr{T}_X)$ and $(Y; \mathscr{T}_Y)$ are both topological spaces. $K \subset X$ is a compact set. If $f : X \to Y \in C(X, Y)$, then $f(K)$ is compact.

Proof. For each open cover $\Omega_Y = \{G_Y \in \mathscr{T}_Y\} \subset \mathscr{T}_Y$ over $f(K)$, $f^{-1}(G_Y) \in \mathscr{T}_X$ (Therem 1.15). $f(K) \subset \cup\Omega_Y \Rightarrow K \subset f^{-1}(\cup\Omega_Y) = \cup\Omega_X$, where $\Omega_X = \{f^{-1}(G_Y) \mid G_Y \in \Omega_Y\}$ is an open cover over K. Since K is compact, $\exists\Omega'_X \subset \Omega_X$ $(|\Omega'_X| \in \mathbb{N}_+ \ \wedge \ K \subset \cup\Omega'_X)$, $f(K) \subset f(\cup\Omega'_X)$. $f(G'_X) \in \Omega_Y$, hence $\Omega'_Y = \{f(G'_X) \mid G'_X \in \Omega'_X\}$ is a finite subcover over $f(K)$. □

Theorem 1.17. $(K; \mathscr{T}_K)$ is a compact space and $(Y; \mathscr{T}_Y)$ is a Hausdorff space. If a bijective $f : K \to Y \in C(K, Y)$, then it is a homeomorphism.

Proof. $\forall F = K - G$ s.t. $G \in \mathscr{T}_K$ is compact (Theorem 1.7). Hence $f(F)$ is compact (Theorem 1.16), then it is also closed (Theorem 1.5). This fact shows that f^{-1} is continuous (Theorem 1.15). □

Theorem 1.18. $(X; \mathscr{T}_X)$ and $(Y; \mathscr{T}_Y)$ are both topological spaces. $E \subset X$ is a connected set. If $f : X \to Y \in C(X, Y)$, then $f(E)$ is also connected.

Proof. Only to notice that the open-closed sets in $(f(E); \mathscr{T}_{f(E)})$ have concurrently open-closed pre-images in $(E; \mathscr{T}_E)$. □

1.7 Contraction

Definition 1.33. A point $a \in X$ is a **fixed point** of a mapping $f : X \to X$ if $f(a) = a$.

Definition 1.34. Let $(X; d)$ be a metric space. A mapping $f : X \to X$ is called a **contraction** if $\exists q \in (0,1) \subset \mathbb{R}$ s.t. $\forall x_1, x_2 \in X$,

$$d(f(x_1), f(x_2)) \leq q d(x_1, x_2). \tag{1-3}$$

Lemma 4. A contraction $f : X \to X$ is always continuous.

Proof. $\forall x \in X$, $\forall \varepsilon \in \mathbb{R}_+$, $\exists \delta < \varepsilon / q$, according to inequality 1-3:

$$f\left(B(x;\delta)\right) \subset B\left(f(x);\varepsilon\right).$$

□

Theorem 1.19. (**Picard-Banach fixed-point principle** or **contraction mapping principle**) Let $(X; d)$ be a complete metric space. Each contraction $f : X \to X$ has a unique fixed point a. Also, $\forall \{x_n\} \subset X$ s.t. $\forall n \in \mathbb{N}\,(f(x_n) = x_{n+1})$ then $\lim\limits_{n\to\infty} x_n = a$, and

$$d(x_n, a) \leq \frac{q^n}{1-q} d(x_1, x_0). \tag{1-4}$$

Proof. By the inequality 1-3:

$$d(x_{n+1}, x_n) \leq q d(x_n, x_{n-1}) \leq \cdots \leq q^n d(x_1, x_0)$$

Therefore, $\forall n, k \in \mathbb{N}$,

$$d(x_{n+k}, x_n) \leq \sum_{i=0}^{k-1} d(x_{n+i+1}, x_{n+i}) \leq \sum_{i=0}^{k-1} q^{n+i} d(x_1, x_0) \leq \frac{q^n}{1-q} d(x_1, x_0), \tag{1-5}$$

which implies that x_n is a Cauchy sequence in a complete space $(X; d)$, hence it converges to a point $a \in X$.

To proof that a is a fixed point of f, since f is continuous (Lemma 4), just notice that

$$a = \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x_n).$$

If there were a second fixed point $a' \in X$ of f, then:

$$0 \leq d(a, a') = d(f(a), f(a')) \leq q d(a, a')$$

which can't be true unless $a = a'$.

By passing to the limit as $k \to \infty$ in the inequality 1-5, we have the inequality 1-4. □

2 Normed Linear Space and Differential Calculus

2.1 Normed Linear Space

Definition 2.1. Let V be a linear space over $\mathbb{R}$ or $\mathbb{C}$. A function $\| \| : X \to \mathbb{R}$ assigning to each vector $\boldsymbol{x} \in X$ a real number $\|x\|$ is called a **norm** in the linear space X if:

a) $\|\boldsymbol{x}\| = 0 \Leftrightarrow \boldsymbol{x} = \mathbf{0}$ (nondegeneracy);

b) $\|\lambda \boldsymbol{x}\| = |\lambda| \|\boldsymbol{x}\|$ (homogeneity);

c) $\|\boldsymbol{x}_1 + \boldsymbol{x}_2\| \leq \|\boldsymbol{x}_1\| + \|\boldsymbol{x}_2\|$ (the triangle inequality).

A linear space with a norm defined on it is called **normed**.

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