Magnetization from magnetic field

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1 Background

Gubbins et al. (2011) present a way to calculate vertically integrated magnetization from a crustal magnetic field model. Here I am re-deriving his work in our own notation to make sure that (1) everything is correct and (2) to be able to calculate and interpret my own vertically integrated magnetizations.

2 Units

The notation in this text is the same as by Plattner and Simons (2017). To get the physical units right, I here introduce the symbol Y_{lm}^*, which mathematically is the same as Y_{lm}, but has the physical unit m^{-2}. This allows for the right hand side of

$$\int_\Omega Y_{lm} Y_{lm}^* \, d\Omega = 1 \tag{1}$$

to be unitless. Without the definition of Y_{lm}^*, the equation

$$\int_\Omega Y_{lm} Y_{lm} \, d\Omega = 1 \tag{2}$$

would still hold, but the right hand side would have unit m^2. In a similar fashion, I introduce $\boldsymbol{F}_{lm}^*$, which is mathematically the same as $\boldsymbol{F}_{lm}$, but also with unit m^{-2}. Again, this leads to the right hand side of

$$\int_\Omega \boldsymbol{F}_{lm} \cdot \boldsymbol{F}_{lm}^* \, d\Omega = 1 \tag{3}$$

to be unitless. I think this is somewhat related to the bra-ket notation in physics.

3 Derivation

From Blakely (1995), his eq. (5.2), we have the following relationship between the spatial magnetization $\boldsymbol{M}(\hat{\boldsymbol{r}})$, defined between $r_p - d$ and r_p, and the resulting potential field

$$V(r_p \hat{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \int_{\Omega'} \int_{r_p - d}^{r_p} \boldsymbol{M}(r' \hat{\boldsymbol{r}}') \cdot \nabla' \left[\frac{1}{\left| r_p \hat{\boldsymbol{r}} - r' \hat{\boldsymbol{r}}' \right|} \right] dr' \, d\Omega' \tag{4}$$

From Plattner and Simons (2017), their eq. (4), we can describe the internal-source magnetic potential field via its scalar spherical-harmonic coefficients on the planet's surface at radial position r_p

$$v_{lm} = \int_{\Omega} V(r_p\hat{\boldsymbol{r}})Y^*_{lm}(\hat{\boldsymbol{r}})\, d\Omega. \tag{5}$$

The unit of v_{lm} is the same as the unit of $V(r_p\hat{\boldsymbol{r}})$, which is $\mathrm{T\,m}$.

We now put the description of $V(r_p\hat{\boldsymbol{r}})$ from eq. (4) into eq. (5) and obtain

$$v_{lm} = \int_{\Omega} \frac{\mu_0}{4\pi} \int_{\Omega'} \int_{r_p-d}^{r_p} \boldsymbol{M}(r'\hat{\boldsymbol{r}}') \cdot \nabla' \left[\frac{1}{\left| r_p\hat{\boldsymbol{r}} - r'\hat{\boldsymbol{r}}' \right|} \right] dr'\, d\Omega' Y^*_{lm}(\hat{\boldsymbol{r}})\, d\Omega. \tag{6}$$

By rearranging the integrals and functions we obtain

$$v_{lm} = \frac{\mu_0}{4\pi} \int_{\Omega'} \int_{r_p-d}^{r_p} \boldsymbol{M}(r'\hat{\boldsymbol{r}}') \cdot \nabla' \left[\int_{\Omega} \frac{Y^*_{lm}(\hat{\boldsymbol{r}})}{\left| r_p\hat{\boldsymbol{r}} - r'\hat{\boldsymbol{r}}' \right|}\, d\Omega \right] dr'\, d\Omega'. \tag{7}$$

From Freeden and Schreiner (2009), their eqs (10.13–10.15) we have the following expansion

$$\begin{aligned} \frac{1}{\left| r_p\hat{\boldsymbol{r}} - r'\hat{\boldsymbol{r}}' \right|} &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi r_p}{(2l+1)} \frac{1}{{r_p}^2} Y_{lm}(\hat{\boldsymbol{r}}) \left(\frac{r'}{r_p} \right)^l Y_{lm}(\hat{\boldsymbol{r}}') \\ &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{(2l+1){r_p}^{l+1}} (r')^l Y_{lm}(\hat{\boldsymbol{r}}) Y_{lm}(\hat{\boldsymbol{r}}') \end{aligned} \tag{8}$$

Because our spherical harmonics $Y_{lm}(\hat{\boldsymbol{r}})$ are orthogonal on the sphere and eq. (1) assures the correct physical units, we get

$$\int_{\Omega} \frac{Y^*_{lm}(\hat{\boldsymbol{r}})}{\left| r_p\hat{\boldsymbol{r}} - r'\hat{\boldsymbol{r}}' \right|}\, d\Omega = \frac{4\pi}{(2l+1){r_p}^{l+1}} (r')^l Y_{lm}(\hat{\boldsymbol{r}}') \tag{9}$$

If we put eq. (9) into eq. (7) we get

$$v_{lm} = \frac{\mu_0}{4\pi} \int_{\Omega'} \int_{r_p-d}^{r_p} \boldsymbol{M}(r'\hat{\boldsymbol{r}}') \cdot \nabla' \left[\frac{4\pi}{(2l+1){r_p}^{l+1}} (r')^l Y_{lm}(\hat{\boldsymbol{r}}') \right] dr'\, d\Omega', \tag{10}$$

which simplifies to

$$v_{lm} = \frac{\mu_0}{(2l+1){r_p}^{l+1}} \int_{\Omega'} \int_{r_p-d}^{r_p} \boldsymbol{M}(r'\hat{\boldsymbol{r}}') \cdot \nabla' \left[(r')^l Y_{lm}(\hat{\boldsymbol{r}}') \right] dr'\, d\Omega'. \tag{11}$$

We can expand, using the definition of ∇ by Plattner and Simons (2017) just above their eq. (13),

$$\begin{aligned} \nabla' \left[(r')^l Y_{lm}(\hat{\boldsymbol{r}}') \right] &= \hat{\boldsymbol{r}}' \partial_{r'} \left[(r')^l Y_{lm}(\hat{\boldsymbol{r}}) \right] + \frac{1}{r'} \nabla'_1 \left[(r')^l Y_{lm}(\hat{\boldsymbol{r}}') \right] \\ &= (r')^{l-1} \left[\hat{\boldsymbol{r}} l Y_{lm}(\hat{\boldsymbol{r}}') + \nabla'_1 Y_{lm}(\hat{\boldsymbol{r}}') \right] \\ &= (r')^{l-1} \sqrt{l(2l+1)} \boldsymbol{F}_{lm}(\hat{\boldsymbol{r}}'). \end{aligned} \tag{12}$$

In the last step we used the definition of the $\boldsymbol{F}_{lm}$ by Plattner and Simons (2017) in their eq. (16). Here, the function $\boldsymbol{F}_{lm}$ arose from Y_{lm}, hence no "$*$".

By including eq. (12) into eq. (11), we get

$$v_{lm} = \frac{\mu_0}{{r_p}^2}\sqrt{\frac{l}{2l+1}}\int_{\Omega'}\int_{r_p-d}^{r_p}\left(\frac{r'}{r_p}\right)^{l-1}\boldsymbol{M}(r'\hat{\boldsymbol{r}}')\cdot\boldsymbol{F}_{lm}(\hat{\boldsymbol{r}}')\,dr'\,d\Omega'. \tag{13}$$

3.1 Gubbins' approach

Gubbins et al. (2011) defined the radially integrated magnetization for degree l

$$\bar{\boldsymbol{M}}_{l-1} = \int_{r_p-d}^{r_p}\left(\frac{r'}{r_p}\right)^{l-1}\boldsymbol{M}(r'\hat{\boldsymbol{r}}')\,dr' \tag{14}$$

and hence get

$$v_{lm} = \frac{\mu_0}{{r_p}^2}\sqrt{\frac{l}{2l+1}}\int_{\Omega'}\bar{\boldsymbol{M}}_{l-1}\cdot\boldsymbol{F}_{lm}(\hat{\boldsymbol{r}}')\,d\Omega'. \tag{15}$$

Gubbins et al. (2011) propose to approximate $\bar{\boldsymbol{M}}_i$ for $i>0$ by $\bar{\boldsymbol{M}}_0$, which tends to overestimate the magnetization, because $\left(\frac{r'}{r_p}\right) < 1$ in the integral in eq. (14). However, doing this and defining $\bar{\boldsymbol{M}} := \bar{\boldsymbol{M}}_0$, allows us to directly expand the vertically integrated magnetization in vector spherical harmonics. Set

$$\bar{m}_{lm} := \int_{\Omega}\bar{\boldsymbol{M}}\cdot\boldsymbol{F}_{lm}(\hat{\boldsymbol{r}})\,d\Omega \tag{16}$$

for $l \geq 1$. This yields

$$v_{lm} = \frac{\mu_0}{{r_p}^2}\sqrt{\frac{l}{2l+1}}\bar{m}_{lm}, \tag{17}$$

and finally

$$\bar{m}_{lm} = v_{lm}\frac{{r_p}^2}{\mu_0}\sqrt{\frac{2l+1}{l}}, \tag{18}$$

allowing us to expand the vertically integrated magnetization as

$$\bar{\boldsymbol{M}}(\hat{\boldsymbol{r}}) \approx \sum_{l=1}^{L_o}\sum_{m=-l}^{l} v_{lm}\frac{{r_p}^2}{\mu_0}\sqrt{\frac{2l+1}{l}}\boldsymbol{F}^*_{lm}(\hat{\boldsymbol{r}}). \tag{19}$$

The physical units of the $\bar{m}_{lm}$ is $\mathrm{A\,m^2}$ because they arose, in eq. (16), from an integral with $\boldsymbol{F}_{lm}$ and not $\boldsymbol{F}^*_{lm}$. Therefore, the coefficients "live" in the dual-space (or $*$-space) and have to be expanded using the $\boldsymbol{F}^*_{lm}$ functions, hence eq. (19).

3.2 New approach

Describe the magnetization as a product of vertical variation and horizontal variation (separation of variables)

$$\boldsymbol{M}(r'\hat{\boldsymbol{r}}') = M_r(r')\boldsymbol{M}_t(\hat{\boldsymbol{r}}'). \tag{20}$$

The function $M_r(r')$ is unitless, while the function $\boldsymbol{M}_t(\hat{\boldsymbol{r}}')$ carries the physical unit $\mathrm{A/m}$.

With this, eq. (13) turns into

$$v_{lm} = \frac{\mu_0}{{r_p}^2}\sqrt{\frac{l}{2l+1}}\int_{r_p-d}^{r_p}\left(\frac{r'}{r_p}\right)^{l-1} M_r(r')\,dr' \int_{\Omega'} \boldsymbol{M}_t(\hat{\boldsymbol{r}}')\cdot\boldsymbol{F}_{lm}(\hat{\boldsymbol{r}}')\,d\Omega'. \tag{21}$$

Eq. (21) follows from eq. (13), hence the $\boldsymbol{F}_{lm}$ and not $\boldsymbol{F}^*_{lm}$.

We can now use prior information about magnetization depth, or include assumptions. For example, if we expect the magnetization distribution to look like a boxcar function between $r_p - d_b$ and $r_p - d_t$

$$M_r(r') = \begin{cases} 1 & \text{for } r_p - d_b \le r' \le r_p - d_t, \\ 0 & \text{elsewhere.} \end{cases} \tag{22}$$

Note that any other form for $M_r(r')$ would work just as well. For example a function of $(r')^2$ with maximum value at some depth, or a linear function decreasing with depth.

We can now calculate the integral

$$\int_{r_p-d}^{r_p}\left(\frac{r'}{r_p}\right)^{l-1} M_r(r')\,dr' = \frac{(r_p-d_t)^l - (r_p-d_b)^l}{l{r_p}^{l-1}}, \tag{23}$$

for which both sides of the equation have the unit m.

Including this into eq. (21) yields

$$v_{lm} = \frac{\mu_0}{r_p}\frac{(r_p-d_t)^l - (r_p-d_b)^l}{{r_p}^l}\sqrt{\frac{1}{l(2l+1)}}\int_{\Omega'} \boldsymbol{M}_t(\hat{\boldsymbol{r}}')\cdot\boldsymbol{F}_{lm}(\hat{\boldsymbol{r}}')\,d\Omega'. \tag{24}$$

We define

$$\tilde{m}_{lm} = \int_{\Omega'} \boldsymbol{M}_t(\hat{\boldsymbol{r}}')\cdot\boldsymbol{F}_{lm}(\hat{\boldsymbol{r}}')\,d\Omega'. \tag{25}$$

The physical unit of the $\tilde{m}_{lm}$ is $\mathrm{A\,m}$ and they live in the $*$-space, because they arise from inner products with the $\boldsymbol{F}_{lm}(\hat{\boldsymbol{r}}')$. They therefore will need to be expanded using the $\boldsymbol{F}^*_{lm}$.

From eqs (24) and (25) we can calculate $\tilde{m}_{lm}$ directly from v_{lm} via

$$\tilde{m}_{lm} = v_{lm}\frac{{r_p}^{l+1}\sqrt{l(2l+1)}}{\mu_0\left[(r_p-d_t)^l - (r_p-d_b)^l\right]} \tag{26}$$

As explained below eq. (25), we need to expand the $\tilde{m}_{lm}$ with the $\boldsymbol{F}^*_{lm}(\hat{\boldsymbol{r}}')$

$$\boldsymbol{M}_t(\hat{\boldsymbol{r}}') = \sum_{l=1}^{L_o}\sum_{m=-l}^{l} v_{lm}\frac{{r_p}^{l+1}\sqrt{l(2l+1)}}{\mu_0\left[(r_p-d_t)^l - (r_p-d_b)^l\right]}\boldsymbol{F}^*_{lm}(\hat{\boldsymbol{r}}') \tag{27}$$

The vector-valued function $\boldsymbol{M}_t(\hat{\boldsymbol{r}}')$ describes the magnetization assuming that its vertical variation follows the description in eq. (22), i.e. it is homogeneous between depths d_b and d_t below the surface. The unit of $\boldsymbol{M}_t(\hat{\boldsymbol{r}}')$ is $\mathrm{A/m}$.

References

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