Harmonic Analysis: Fourier and Spherical

Notes on a document titled "Geos 419/519 Notes: Harmonic Analysis."

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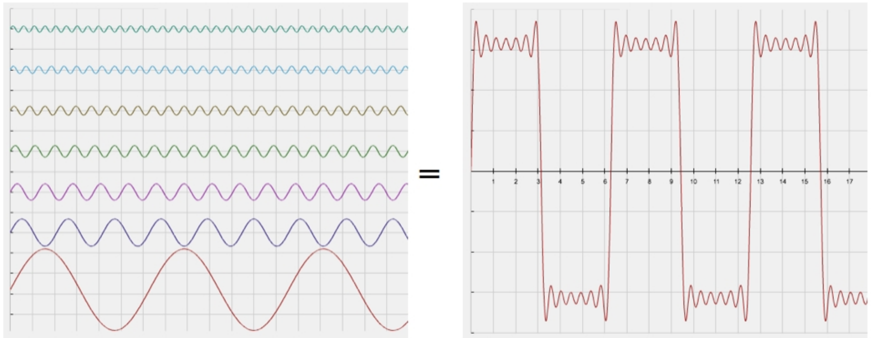
1. Fourier Transform/Analysis

1.1. General Definition

- Conceptually, Fourier Analysis takes a function g from the "time" domain (t) to a "frequency" domain $(a_n, b_n \vee A_n, \phi_n)$.
- To do this, we decompose f into a "Fourier Series" (sum of "Fourier terms") by finding coefficients a_n and b_n such that:

$$g(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{2\pi}{T} nx\right) + b_n \sin\left(\frac{2\pi}{T} nx\right), \tag{1}$$

seen graphically as:



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1.2. Finding Coefficients

We can perform a Fourier Transform on a function by:

1. Multiplying both sides of (1) by an element from a specific family of functions (trigonometric for Fourier, Legendre polynomials for Spherical), then
2. integrating term-by-term over an interval ($[0, 2\pi]$ for trig), and using the orthogonality † of the family of functions to cancel out terms and solve for coefficients a_n and b_n.
   - †Two functions are orthogonal if $\displaystyle\int_a^b dx\, f(x)g(x) = 0$.

In general, the relationships that give us coefficients for $n > 0$ are

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\, dx \tag{2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\, dx \tag{3}$$

1.3. Non-Periodic Functions

If the "signal" (g) we seek to run a Fourier Analysis on is non-periodic (as is most real-world data), we simply repeat the signal such that $g\left(\frac{-T}{2}\right) = g\left(\frac{T}{2}\right) = 0$.

1.4. Angular Paramters

Rather than using coefficients a_n, b_n, we can also describe the frequency domain with:

↳ $A_n = \sqrt{a_n^2 + b_n^2}$

: Amplitude of cosine wave with frequency $2\pi n f_f$ (?)

↳ $\phi_n = \tan^{-1}\left({}^{b_n}\!/_{a_n}\right)$

: phase (rightwards) shift of cosine waveform

Graphing n vs A_n yields an "amplitude spectrum".
Graphing n vs ϕ_n yields a "phase spectrum".

We can also do Fourier Analysis on multivariable functions like $f(x, y)$, which represents *planar* surfaces. However, we must use Spherical Harmonics when working with *spherical* surfaces.

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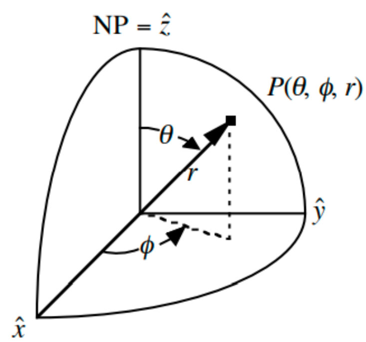
2. Spherical Harmonics

2.1. Spherical Coordinates



Colatitude: $\theta \in [0, \pi]$
Longitude: $\phi \in [0, 2\pi]$

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

2.2. $g(\phi)$ (Circle of Colatitude)

If we consider only one "layer"/"plane" of colatitude, θ_0 (i.e. fix z), then we can represent a function g with a Fourier series:

$$g(\theta_0, \phi) = \sum_{m=0}^{\infty} a_m \cos(m\phi) + b_m \sin(m\phi).$$

As we vary θ, we get different a_m and b_m. In other words, a_m and b_m are functions of θ:

$$g(\theta, \phi) = \sum_{m=0}^{\infty} a_m(\theta) \cos(m\phi) + b_m(\theta) \sin(m\phi).$$

Our goal now is to expand $a_m(\theta)$ and $b_m(\theta)$ (i.e. approximate them with a series). Due to singularities at the poles of the coordinate system, we can't use a Fourier Series. So we turn to Legendre polynomials...

2.2.1. Legendre Polynomials

An especially compact expression for the Legendre polynomials is given by Rodrigues' formula:

$$P_n(\cos\theta) = \frac{1}{2^n n!}\,\frac{\partial^n}{\partial(\cos\theta)^n}\left[\cos^2\theta - 1\right]^n. \tag{4}$$

Question: Why is it a function of cos?

We can notationally simplify this with the substitution $\mu = \cos\theta \rightarrow d\mu = -\sin\theta\, d\theta$:

$$P_n(\mu) = \frac{1}{2^n n!}\,\frac{\partial^n}{\partial\mu^n}\left[\mu^2 - 1\right]^n. \tag{5}$$

Here's a handy reference table for the first few Legendre polynomials:

n	$P_n(\mu)$
0	1
1	μ
2	$\frac{1}{2}(3\mu^2 - 1)$
3	$\frac{1}{2}(5\mu^3 - 3\mu)$

.

The orthogonality relationship is given by:

$$\int_0^\pi P_{n_1} P_{n_2} \sin\theta\, d\theta = \int_{-1}^1 P_{n_1}(\mu) P_{n_1}(\mu) = \begin{cases} 0 & , n_1 \neq n_2 \\ \frac{2}{2n_1+1} & , n_1 = n_2 \end{cases}.$$

Question: Is the first also a function of cos?

2.3. $g(\theta)$ (Axially Symmetric Surface)

The normal Legendre polynomials are able to expand expand $a_m(\theta)$ and $b_m(\theta)$ when the function g is independent of longitude ϕ (that is to say, axially symmetric). We represent $g(\theta, \phi) = g(\theta)$ as an infinite sum of Legendre polynomials:

$$g(\theta) = \sum_{n=0}^{\infty} a_n P_n(\theta) \quad \text{or} \quad g(\mu) = \sum_{n=0}^{\infty} a_n P_n(\mu).$$

We use the same general method outlined in section 1.2 to find the coefficent a_n. In general, we find that:

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} d\mu\ f(\mu)P_n(\mu).$$

2.4. $g(\theta, \phi)$ (Any Spherical Surface)

2.4.1. Associated Legendre Polynomials

The "associated Legendre polynomials" extend the Legendre polynomials to include longitudinal dependence, defined by:

$$P_n^m(\theta) = \left[2\frac{(n-m)!}{(n+m)!}\right]^{1/2} \sin(\theta)\ \frac{d^m P_n(\theta)}{d(\cos\theta)^m}. \tag{6}$$

where

- n : "order" $P_n^m(\theta)$.
- m : "degree" $P_n^m(\theta)$.
- $[]$: Gauss-Schmidt normalizatoin to keep high-degree polynomials for getting too big.

Some cool facts that make the associated Legendre Polynomials less intimidating are:

1. When $m > n$, $P_n^m(\theta) = 0$.
   (a) $P_n(\theta)$ contains $\cos^n(\theta)$, which equals zero when differentiationd $n+1$ times.
2. When $m = 0$, $P_n^m(\theta) = P_n(\theta)$.
3. $P_{n_1}^{m_1}(\theta)$ is orthogonal to $P_{n_2}^{m_2}(\theta)$ for $n_1 \neq n_2 \wedge m_1 \neq m_2$.

(page 18 gives the first few associated Legendre polynomials)

The associated Legendre polynomials also have a series form better suited for computation:

$$P_n^m(\theta) = \left[2\frac{(n-m)!}{(n+m)!}\right]^{1/2} \frac{\sin^m\theta}{2^n n!} \sum_{t=0}^{\kappa} \frac{(-1)^t(2n-2t)!}{t!(n-t)!(n-m-t)!}(\cos\theta)^{n-m-2t}, \tag{7}$$

where κ is the largest integer $\leq \frac{1}{2}(n-m)$.

2.4.2. Surface Harmonic Functions

We can finally combine the associated Legendry polynomials with the Fourier series to define a set of functions (called "surface harmonics" and denoted as $S_n^m(\theta)$) that can be used to represent any arbitrary function $g(\theta, \phi)$ at every point on a spherical surface with any radius:

$$g(\theta,\phi) = S_n^m(\theta,\phi) = P_n^m(\theta)\begin{bmatrix} \cos m\phi \\ \sin m\phi \end{bmatrix}. \tag{8}$$

The general surface harmonic series representation of a function is given by:

$$g(\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} P_n^m(\theta)\,\left(A_n^m \cos m\phi + B_n^m \sin m\phi\right), \tag{9}$$

where:

$$\begin{bmatrix} A_n^m \\ B_n^m \end{bmatrix} = \frac{2n+1}{4\pi}\int_0^{\pi}\int_0^{2\pi} d\phi\, d\theta\;\; g(\theta,\phi)\, P_n^m(\theta)\begin{bmatrix} \cos m\phi \\ \sin m\phi \end{bmatrix}\sin\theta \tag{10}$$

Note that the properties of the associated Legendre polynomials listed in $4.2.1$ extend to these as well.

2.4.3. Completely Normalized Harmonics

(See page 22 for equations, not sure what they mean though or if they're relevant to magnetic applications.)

2.5. Graphical Intuition

We can draw the zero crossings of some $S_n^m(\theta,\phi)$ using these properties:

1. $P_n^m(\theta)$ divides the sphere in $n-m+1$ bands (bounded by parallels of latitude, $\theta \in [0,\pi]$). In other words, there are $n-m$ "zero crossings".
2. There are $2m$ bands (same with zero crossings) around circles of colatidude, $\phi \in [0,2\pi]$.
3. Zonal harmonics (i.e. $m=0$, or pure Legendre polynomial) have values:
$$\begin{cases} +1 & \theta = 0 \\ (-1)^n & \theta = \pi \end{cases}.$$

(page 20 gives some graphical examples; page 21 gives a general chart with +/- values)

2.6. Amplitude Spectum for Surface Harmonics

The amplitude spectrum or "degree variance" is given by:

$$\sigma_n = \left[\sum_{m=0}^{n} \frac{\left(A_n^m\right)^2 + \left(B_n^m\right)^2}{2n+1} \right]^{1/2}$$

Because of the complicated form of the terms with $m > 0$, a phase spectrum wouldn't be particularly meaningful.

2.7. Application

If we restrict to small values of n, surface harmonics can serve as a smoothing/interpolation device for noisy data sets.

2.7.1. Radial Dependence

Some stuff about Laplace's Equations that I'm not sure if I need to know.

The general form of the spherical harmonic representation of a potential function (where the sources are within the sphere of radius a) is:

$$f(\theta, \phi, r) = \frac{1}{a} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^{n+1} P_n^m(\theta) \left(A_n^m \cos m\phi + B_n^m \cos m\phi\right)$$

2.7.2. Planetary Magnetic Potential (Earth)

(page 28 explains specific equations)