

First, let $x \in U$ be such that $Du_\infty(x) \neq 0$. Then, there exists $N \in \mathbb{N}$ such that $Du_n(x) \neq 0$ for every $n \geq N$. Therefore,

$$\begin{aligned} & \left| |Du_\infty(x)|^{\sigma(\theta_\infty(x))-2} Du_\infty(x) - |Du_n(x)|^{\sigma_n(\theta_n(x))-2} Du_n(x) \right| \\ & \leq |Du_n(x)|^{\sigma_n(\theta_n(x))-2} |Du_n(x) - Du_\infty(x)| \\ & \quad + \left| |Du_n(x)|^{\sigma_n(\theta_n(x))-2} - |Du_\infty(x)|^{\sigma(\theta_\infty(x))-2} \right| |Du_\infty(x)| \\ & \leq o(1) + C \left| e^{(\sigma_n(\theta_n(x))-2) \ln |Du_n(x)|} - e^{(\sigma(\theta_\infty(x))-2) \ln |Du_\infty(x)|} \right| \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$.

Consider next the case of $x \in U$ so that $Du_\infty(x) = 0$. Now,

$$\begin{aligned} 0 & \leq \left| |Du_\infty(x)|^{\sigma(\theta_\infty(x))-2} Du_\infty(x) - |Du_n(x)|^{\sigma_n(\theta_n(x))-2} Du_n(x) \right| \\ & \leq |Du_n(x)|^{\sigma_n(\theta_n(x))-1} + |Du_\infty(x)|^{\sigma(\theta_\infty(x))-1} \\ & = |Du_n(x)|^{\sigma_n(\theta_n(x))-1}. \end{aligned}$$

Because $|Du_n|$ converges uniformly to $|Du_\infty|$, there exists $N \in \mathbb{N}$ such that $|Du_n(x)| < 1/2$ for every $n > N$. We conclude that

$$\begin{aligned} 0 & \leq \left| |Du_\infty(x)|^{\sigma(\theta_\infty(x))-2} Du_\infty(x) - |Du_n(x)|^{\sigma_n(\theta_n(x))-2} Du_n(x) \right| \\ & \leq |Du_n(x)|^{\sigma^- - 1} \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$.

To treat I_n , notice that

$$|f_n(x)\phi(x)| \leq C |\phi(x)| \in L^1(U).$$

In addition, by assumption, we conclude that $f_n(x)\phi(x) \rightarrow f(x)\phi(x)$ for every $x \in U$, as $n \rightarrow \infty$. Therefore, Lebesgue's Dominated Convergence Theorem ensures that (??) holds true.

In what regards (??), notice that $D\theta_n(x) \rightarrow D\theta_\infty(x)$, for every $x \in U$. Furthermore, the assumptions of the proposition yield

$$\left(\lambda_n(\theta_n(x)) |Du_n|^{\sigma_n(\theta_n(x))} \right) \psi(x) \rightarrow 0,$$

as $n \rightarrow \infty$. A further application of Lebesgue's Dominated Convergence Theorem produces (??). The proof of the proposition is then complete. \square

The stability ensured by Proposition ?? will play an instrumental role in our improved-regularity argument of section ??.