

First, let  $x \in U$  be such that  $Du_\infty(x) \neq 0$ . Then, there exists  $N \in \mathbb{N}$  such that  $Du_n(x) \neq 0$  for every  $n \geq N$ . Therefore,

$$\begin{aligned} & \left| |Du_\infty(x)|^{\sigma(\theta_\infty(x))-2} Du_\infty(x) - |Du_n(x)|^{\sigma_n(\theta_n(x))-2} Du_n(x) \right| \\ & \leq |Du_n(x)|^{\sigma_n(\theta_n(x))-2} |Du_n(x) - Du_\infty(x)| \\ & \quad + \left| |Du_n(x)|^{\sigma_n(\theta_n(x))-2} - |Du_\infty(x)|^{\sigma(\theta_\infty(x))-2} \right| |Du_\infty(x)| \\ & \leq o(1) + C \left| e^{(\sigma_n(\theta_n(x))-2) \ln |Du_n(x)|} - e^{(\sigma(\theta_\infty(x))-2) \ln |Du_\infty(x)|} \right| \rightarrow 0, \end{aligned}$$

as  $n \rightarrow \infty$ .

Consider next the case of  $x \in U$  so that  $Du_\infty(x) = 0$ . Now,

$$\begin{aligned} 0 & \leq \left| |Du_\infty(x)|^{\sigma(\theta_\infty(x))-2} Du_\infty(x) - |Du_n(x)|^{\sigma_n(\theta_n(x))-2} Du_n(x) \right| \\ & \leq |Du_n(x)|^{\sigma_n(\theta_n(x))-1} + |Du_\infty(x)|^{\sigma(\theta_\infty(x))-1} \\ & = |Du_n(x)|^{\sigma_n(\theta_n(x))-1}. \end{aligned}$$

Because  $|Du_n|$  converges uniformly to  $|Du_\infty|$ , there exists  $N \in \mathbb{N}$  such that  $|Du_n(x)| < 1/2$  for every  $n > N$ . We conclude that

$$\begin{aligned} 0 & \leq \left| |Du_\infty(x)|^{\sigma(\theta_\infty(x))-2} Du_\infty(x) - |Du_n(x)|^{\sigma_n(\theta_n(x))-2} Du_n(x) \right| \\ & \leq |Du_n(x)|^{\sigma^- - 1} \rightarrow 0, \end{aligned}$$

as  $n \rightarrow \infty$ .

To treat  $I_n$ , notice that

$$|f_n(x)\phi(x)| \leq C |\phi(x)| \in L^1(U).$$

In addition, by assumption, we conclude that  $f_n(x)\phi(x) \rightarrow f(x)\phi(x)$  for every  $x \in U$ , as  $n \rightarrow \infty$ . Therefore, Lebesgue's Dominated Convergence Theorem ensures that (??) holds true.

In what regards (??), notice that  $D\theta_n(x) \rightarrow D\theta_\infty(x)$ , for every  $x \in U$ . Furthermore, the assumptions of the proposition yield

$$\left( \lambda_n(\theta_n(x)) |Du_n|^{\sigma_n(\theta_n(x))} \right) \psi(x) \rightarrow 0,$$

as  $n \rightarrow \infty$ . A further application of Lebesgue's Dominated Convergence Theorem produces (??). The proof of the proposition is then complete.  $\square$

The stability ensured by Proposition ?? will play an instrumental role in our improved-regularity argument of section ??.