we restrict ourselves to the *special first nonlocal symmetry* setting $\Phi = 0$ and $\chi = 0$.

In Section **??**, we show in detail how the careful analysis of integrability conditions specifies various functional parameters in the invariance equations with no additional assumptions made. In Section **??**, we integrate completely all the obtained equations and end up with a noninvariant solution of CMA which is a general form of the solution invariant under the first nonlocal symmetry in the hierarchy. In Section **??**, we use this solution for constructing the corresponding (anti-)self-dual gravitational metrics with either Euclidean or neutral signature.

2 Real variables and 2-component form of CMA

In our earlier paper [**?**] we presented bi-Hamiltonian structure of the two-component version of (**??**), which by Magri's theorem [**?**] proves that it is a completely integrable system in four dimensions.

We impose additional reality condition for all the objects in the theory. The transformation from complex to real variables has the form

$$t = z^1 + \bar{z}^1, \quad x = i(\bar{z}^1 - z^1), \quad y = z^2 + \bar{z}^2, \quad z = i(\bar{z}^2 - z^2). \tag{2.1}$$

Introduce the notation

$$a = \Delta(u) = u_{yy} + u_{zz}, \quad b = u_{xy} - v_z, \quad c = v_y + u_{xz}, \quad Q = \frac{b^2 + c^2 + \varepsilon}{a} \tag{2.2}$$

where $v = u_t$ is the second component of the unknown and $\Delta = D_y^2 + D_z^2$ is the two-dimensional Laplace operator. The definitions (**??**) imply the relations

$$a_x = b_y + c_z, \quad c_y - b_z = \Delta(v). \tag{2.3}$$

The CMA equation (**??**) in the real variables becomes

$$(u_{tt} + u_{xx})\Delta(u) - b^2 - c^2 - \varepsilon = 0$$

or in the two-component form

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} v \\ Q - u_{xx} \end{pmatrix} \tag{2.4}$$

which we will call CMA system.