6 Further integrability conditions

It is convenient to take equation (**??**) in the form

$$(u_{xz}v_x - u_{xx}u_{xy})w_y = (u_{xy}v_x + u_{xx}u_{xz})w_z - \Delta[u](w + u_{xx})w_x. \tag{6.1}$$

The integrability conditions $(v_t)_y = (v_y)_t$ and $(v_t)_z = (v_z)_t$ with the use of (**??**) simplify to

$$\Delta[u_y] = \frac{u_{xy}}{w + u_{xx}}\Delta[u_x], \quad \Delta[u_z] = \frac{u_{xz}}{w + u_{xx}}\Delta[u_x] \tag{6.2}$$

with the integrability condition $(\Delta[u_y])_z - (\Delta[u_z])_y = 0$ resulting in

$$u_{xz}w_y - u_{xy}w_z = 0. \tag{6.3}$$

On account of (**??**) equation (**??**) becomes

$$u_{xx}(u_{xy}w_y + u_{xz}w_z) - \Delta[u](w + u_{xx})w_x = 0. \tag{6.4}$$

Equations (**??**) and (**??**) can be put in the form

$$w_y = \frac{u_{xy}}{u_{xx}}w_x, \quad w_z = \frac{u_{xz}}{u_{xx}}w_x \tag{6.5}$$

with the integrability condition $(w_y)_z = (w_z)_y$ identically satisfied. Differentiating the definition (**??**) of w with respect to t we obtain

$$w_t = \frac{v_x}{u_{xx}}w_x. \tag{6.6}$$

The integrability conditions $(w_t)_y = (w_y)_t$ and $(w_t)_z = (w_z)_t$ of equations (**??**) and (**??**) are identically satisfied.

Equations (**??**) are integrated by the method of characteristics with the result $w = w(u_x, t)$ whereas (**??**) further implies $w = w(u_x)$. The remaining equations (**??**) take the form of two first-order PDEs for $\Delta[u]$

$$\begin{aligned}(u_{xx} + w(u_x))(\Delta[u])_y - u_{xy}(\Delta[u])_x &= 0\\ (u_{xx} + w(u_x))(\Delta[u])_z - u_{xz}(\Delta[u])_x &= 0.\end{aligned} \tag{6.7}$$

We start with the combination of these equations

$$u_{xz}(\Delta[u])_y - u_{xy}(\Delta[u])_z = 0 \tag{6.8}$$