METRICS ON DOUBLES AS AN INVERSE SEMIGROUP II

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ABSTRACT. We have shown recently that, given a metric space X, the coarse equivalence classes of metrics on the two copies of X form an inverse semigroup $M(X)$. Here we give several descriptions of the set $E(M(X))$ of idempotents of this inverse semigroup and its Stone dual space $\widehat{X}$. We also construct σ-additive measures on $\widehat{X}$ from finitely additive probability measures on X that vanish on bounded subsets. When X has a Følner sequence of special type then we show that the inner action of $M(X)$ on $E(M(X))$ gives rize to an action on the space of square-integrable functions on $\widehat{X}$.

INTRODUCTION

Given metric spaces X and Y, a metric d on $X \sqcup Y$ that extends the metrics on X and Y, depends only on the values of $d(x, y)$, $x \in X$, $y \in Y$, but it may be not easy to check which functions $d : X \times Y \to (0, \infty)$ determine a metric on $X \sqcup Y$: one has to check the triangle inequality too many times. The problem of description of all such extended metrics is difficult due to the lack of a nice algebraic structure on the set of metrics. Passing to quasi-equivalence (or coarse equivalence) of metrics, we can define a composition: if d is a metric on $X \sqcup Y$, and if ρ is a metric on $Y \sqcup Z$ then the formula $(\rho \circ d)(x, z) = \inf_{y\in Y}[d(x, y) + \rho(y, z)]$ defines a metric ρd on $X \sqcup Z$. The idea to consider metrics on the disjoint union of two spaces as morphisms from one space to another was suggested in [7].

It was a surprise for us to discover that in the case $Y = X$, there is a nice algebraic structure on the set $M^q(X)$ (resp., $M^c(X)$) of quasi-equivalence (resp., of coarse equivalence) classes of extended metrics on the double $X \sqcup X$: they form an *inverse* semigroup with respect to this composition [8].

Recall that a semigroup S is an inverse semigroup if for any $u \in S$ there exists a unique $v \in S$ such that $u = uvu$ and $v = vuv$ [5]. Philosophically, inverse semigroups describe local symmetries in a similar way as groups describe global symmetries, and technically, the construction of the (reduced) group C^*-algebra of a group generalizes to that of the (reduced) inverse semigroup C^*-algebra [10].

Any two projections of an inverse semigroup S commute, and the semilattice $E(S)$ of all projections of S generates a commutative C^*-algebra. Our aim is to get a better understanding of the Stone dual spaces $\widehat{X}^q$ of $E(M^q(X))$, and $\widehat{X}^c$ of $E(M^c(X))$.

It turns out that the coarse equivalence is better suited for study of the inverse semigroup from metrics on doubles, so we can give more detailed results for the semigroup $M^c(X)$.

We give several descriptions of $E(M^q(X))$ and $E(M^c(X))$, provide a description of the spaces $\widehat{X}^q$ and $\widehat{X}^c$ dual to $E(M^q(X))$ and to $E(M^c(X))$, respectively, and, under certain restrictions, we describe a dense set of $\widehat{X}^c$ in terms of free ultrafilters on X. We construct σ-additive measures on $\widehat{X}^c$ from finitely additive probability measures on X that vanish on bounded subsets. When X has a Følner sequence of special type then we show that

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