we obtain

$$\theta \in W^{2,q}(U) \hookrightarrow \mathcal{C}^{1,\gamma}(\overline{U}), \tag{4.2}$$

for every $\gamma < \frac{1+\alpha}{2}$ so, in particular, for $\gamma = \alpha$. We define the operator

$$\begin{array}{rccc} \mathcal{T}: & \mathcal{C}^{1,\alpha}(\overline{U}) & \longrightarrow & \mathcal{C}^{1,\alpha}(\overline{U}) \\ \\ & \theta^* & \longmapsto & \mathcal{T}(\theta^*) := \theta. \end{array}$$

It is obvious that a fixed point θ^* of $\mathcal{T}$ provides the local weak solution (u, θ^*) of (**??**).

Step 2. The continuity of $\mathcal{T}$ is assured by continuous dependence results for the $p(x)$-Laplace equation and for Poisson's equation. In fact, if we take a sequence $\theta_n \to \theta_\infty$ in $\mathcal{C}^{1,\alpha}(\overline{U})$, then the corresponding variable exponents $\sigma(\theta_n(x))$ converge uniformly. Hence, we can apply the results in [**?**] (see also [**?**]) to show that $u_n \to u_\infty$, first in the variable exponent Sobolev space and then, using the regularity result in Lemma **??**, also in $\mathcal{C}^{1,\beta}(\overline{U})$, for a certain β. Finally, standard continuous dependence results for Poisson's equation give the continuity of $\mathcal{T}$.

Since the compactness follows from (**??**), it remains to show that $\mathcal{T}$ takes the unit ball in $\mathcal{C}^{1,\alpha}(\overline{U})$ into itself. Let $\|\theta^*\|_{\mathcal{C}^{1,\alpha}(\overline{U})} \leq 1$. Then

$$|\sigma \circ \theta^*(x) - \sigma \circ \theta^*(y)| \leq C_\sigma \, |\theta^*(x) - \theta^*(y)| \leq C_\sigma \, |x - y|$$

and thus

$$\|\sigma \circ \theta^*\|_{\mathcal{C}^{0,1}(\overline{U})} \leq C_\sigma.$$

Then, by Lemma **??**, we obtain

$$\|u\|_{\mathcal{C}^{1,\beta}(\overline{U})} \leq C_{data},$$

which gives a uniform control on the L^∞-norm of the gradient Du. We then have, by classical elliptic regularity theory,

$$\begin{aligned} \|\theta\|_{\mathcal{C}^{1,\alpha}(\overline{U})} &\leq C_{emb} \, \|\theta\|_{W^{2,\frac{2d}{1-\alpha}}(U)} \\ &\leq C_{emb} \left\| \lambda\left(\theta^*(x)\right) |Du|^{\sigma(\theta^*(x))} \right\|_{L^\infty(U)} \\ &\leq C_{emb} \, \|\lambda\|_{L^\infty(U)} \, C_{data} \\ &\leq 1, \end{aligned}$$

since $\|\lambda\|_{L^\infty(U)} \leq \frac{1}{C_{emb} C_{data}} =: \lambda^+$ due to A**??**.

$\square$