allows us to bypass the embedding related constraints surfacing in higher dimensions. We show in fact that a weak solution exists in the regularity class $(u, \theta) \in \mathcal{C}^{1,\beta} \times \mathcal{C}^{1,\alpha}$. We then improve the regularity for the temperature θ, producing a $\mathcal{C}^{1,\text{Log-Lip}}$ local regularity result, with appropriate estimates.

The paper is organised as follows. In addition to specifying what we mean by a weak solution, we state in section **??** the assumptions on the data of the problem and present our main result. In section **??**, we gather a few auxiliary results that will be instrumental in the sequel. The existence of weak solutions is established in section **??**, by means of Schauder's Fixed Point Theorem. The final section **??** brings the proof of the improved regularity.

2. Assumptions and main result

The system (**??**) holds in a given smooth domain $U \subset \mathbb{R}^d$, $d \geq 2$. We start with the formal definition of weak solution. To slightly assuage the notation, set

$$p := \sigma \circ \theta \qquad \text{and} \qquad a := \lambda \circ \theta.$$

Definition 2.1. *A weak solution of* (**??**)*, coupled with homogeneous Dirichlet boundary conditions*

$$u = 0 \qquad \text{and} \qquad \theta = 0 \qquad \text{on } \partial U,$$

is a pair

$$(u, \theta) \in W_0^{1,p(\cdot)}(U) \times H_0^1(U)$$

such that

$$\int_U |Du|^{p(x)-2} \, Du \cdot D\varphi = \int_U f\varphi, \quad \forall \varphi \in W_0^{1,p(\cdot)}(U) \tag{2.1}$$

and

$$\int_U D\theta \cdot D\psi = \int_U a(x) \, |Du|^{p(x)} \, \psi, \quad \forall \psi \in H_0^1(U) \cap L^\infty(U). \tag{2.2}$$

It is timely to comment again on the mathematical difficulties arising from this definition. First, we notice that $u \in W_0^{1,p(\cdot)}(U)$ leads to

$$|Du|^{p(\cdot)} \in L^1(U).$$

As a consequence, the structure of (**??**) falls short in producing further regularity for the temperature θ since the integrability of the right-hand side of Poisson's equation does not even ensure the continuity of solutions. To circumvent this structural difficulty of the system, we will resort to an improved regularity result for the first equation in (**??**).