5. Improved regularity

Proposition **??** establishes the existence of a weak solution, which further belongs to the regularity class

$$(u,\theta) \in \mathcal{C}^{1,\beta}(\overline{U}) \times \mathcal{C}^{1,\alpha}(\overline{U}).$$

We notice that, were we given a solution in $W_0^{1,p(\cdot)}(U) \times \mathcal{C}^{0,\alpha}(\overline{U})$, elliptic regularity theory and Lemma **??** would lead to the same levels of regularity. In this section, we make this assumption to improve the regularity for θ.

The tangential analysis methods we will use operate in two distinct layers: they first finely combine stability and compactness for solutions and then localize the analysis through a scaling argument, unveiling the geometric properties of the problem. Accordingly, we proceed by producing an approximation result for the solutions to (**??**).

Proposition 5.1 (Approximation Lemma)**.** *Suppose A**??**-A**??** are in force and let*

$$(u,\theta) \in W_0^{1,p(\cdot)}(U) \times \mathcal{C}^{0,\alpha}(\overline{U})$$

be a solution to (**??**)*. Given $\varepsilon > 0$, there exists $\delta > 0$ such that, if*

$$\|\lambda\|_{L^\infty(U)} < \delta,$$

then there exists a harmonic function $h \in \mathcal{C}^\infty(U)$ satisfying

$$\|\theta - h\|_{L^\infty(U)} < \varepsilon.$$

Proof. We argue by a contradiction argument; suppose the statement of the Proposition is false. Then there exist $\varepsilon_0 > 0$ and sequences of functions $(u_n)_{n\in\mathbb{N}}$, $(\theta_n)_{n\in\mathbb{N}}$ and $(\lambda_n)_{n\in\mathbb{N}}$ such that

$$\|\lambda_n\|_{L^\infty(\mathbb{R})} \longrightarrow 0$$

and

$$\begin{cases} -\operatorname{div}\left(|Du_n|^{\sigma(\theta_n(x))-2} Du_n\right) = f & \text{in} \quad U \\ \\ -\Delta\theta_n = \lambda_n(\theta_n)\,|Du_n|^{\sigma(\theta_n)} & \text{in} \quad U, \end{cases}$$

but

$$\|\theta_n - h\|_{L^\infty(U)} > \varepsilon_0,$$

for every $n \in \mathbb{N}$ and $h \in \mathcal{C}^\infty(U)$ harmonic.