for every $n \in \mathbb{N}$.

Proof. The result follows from an induction argument. The statement of Proposition **??** accounts for the case $n = 1$. Suppose the case $n = k$ has already been verified. We consider the case $n = k + 1$.

Consider the auxiliary function

$$v_k(x) \,:=\, \frac{\theta(\rho^k x) \,-\, P_k(\rho^k x)}{\rho^{2k}}.$$

Notice that v_k solves

$$-\Delta v_k \,=\, \lambda\left(\rho^{2k} v_k(x) + P_k(\rho^k x)\right)\left|Du(\rho^k x)\right|^{\sigma\left(\rho^{2k} v_k(x) + P_k(\rho^k x)\right)}.$$

Hence, by imposing a suitable smallness regime on the L^∞-norm of $\lambda(x)$, v_k falls within the scope of Proposition **??**. Therefore, there exists $\overline{h} \in \mathcal{C}^\infty(U)$ such that

$$\sup_{B_\rho}\left|v_k(x) \,-\, \overline{h}(0) \,-\, D\overline{h}(0)\cdot x \,-\, \frac{x^T D^2\overline{h}(0) x}{2}\right| \,\leq\, \rho^2. \tag{5.5}$$

Set $P_{k+1}(x)$ as

$$P_{k+1}(x) \,:=\, a_k \,+\, \rho^{2k}\overline{h}(0) \,+\, \left(b_k \,+\, \rho^k D\overline{h}(0)\right)\cdot x \,+\, \frac{x^T\left(M_k \,+\, D^2\overline{h}(0)\right)x}{2}.$$

It follows from (**??**) that

$$\sup_{B_{\rho^{k+1}}} \left|\theta(x) \,-\, P_{k+1}(x)\right| \,\leq\, \rho^{2(k+1)}$$

and, in addition,

$$\operatorname{Tr}\left(M_k \,+\, D^2\overline{h}(0)\right) = 0.$$

Finally, we also have

$$|a_{k+1} \,-\, a_k| \,=\, \rho^{2k}\left|\overline{h}(0)\right|,$$

$$|b_{k+1} \,-\, b_k| \,=\, \rho^{k}\left|D\overline{h}(0)\right|$$

and

$$|M_{k+1} \,-\, M_k| \,=\, \left|D^2\overline{h}(0)\right|.$$

Hence,

$$|a_{k+1} - a_k| + \rho^k\,|b_{k+1} - b_k| + \rho^{2k}\,|M_{k+1} - M_k| \leq C\rho^{2k},$$

where $C > 0$ is a universal constant.

□