4. Existence of weak solutions

In this section, we show that the system (**??**) admits at least one weak solution in the sense of Definition **??**. As is customary for this type of coupled system, we will use Schauder's Fixed Point Theorem. This strategy has been pursued before in the literature, *e.g.*, in [**?**].

We emphasize two aspects of our argument. First, it bypasses previous dimensional constraints; this advance is due to an improved regularity result produced in [**?**]. In fact, suppose $\theta \in \mathcal{C}^{0,\beta}(U)$ is given; under A**??** and A**??** the variable exponent $p := \sigma \circ \theta$ is Hölder continuous and bounded (even if σ is unbounded above). Together with the conditions on the source term, it frames the first equation in (**??**) within the scope of [**?**, Theorems 1.1-1.2] and unveils the improved regularity for u. This gain-of-regularity mechanism opens room for standard compact imbedding results; see, for instance, [**?**, Theorem 2.84].

Proposition 4.1 (Existence of weak solutions)**.** *Suppose A**??**-A**??** are in force. Then, there exists a weak solution of* (**??**) *in the sense of Definition* **??**.

Proof. The proof is performed in two steps: we first define an operator and then show it fits the requirements of Schauder's Fixed Point Theorem.

Step 1. We are going to apply Schauder's Fixed Point Theorem in $\mathcal{C}^{1,\alpha}(\overline{U})$, for some $0 < \alpha < 1$. Take $\theta^* \in \mathcal{C}^{1,\alpha}(\overline{U})$ and, using the results in [**?**], solve

$$\begin{cases} -\operatorname{div}\left(|Du|^{\sigma(\theta^*(x))-2}\, Du\right) = f & \text{in } U \\ \\ u = 0 & \text{on } \partial U, \end{cases} \tag{4.1}$$

obtaining $u \in W_0^{1,\sigma\circ\theta^*(\cdot)}(U)$. By Lemma **??**, we have that $u \in \mathcal{C}^{1,\beta}(\overline{U})$, with a uniform estimate, and thus $|Du|$ is globally bounded and so is the right-hand side of

$$-\Delta\theta = \lambda\left(\theta^*(x)\right)|Du|^{\sigma(\theta^*(x))}.$$

We then solve this Poisson equation, with the Dirichlet condition $\theta = 0$ on ∂U, obtaining a solution

$$\theta \in W_0^{2,q}(U), \quad \forall\, 1 < q < \infty.$$

In particular, by taking

$$q := \frac{2d}{1-\alpha},$$