From the regularity theory available for (u_n, θ_n), we infer the existence of a pair $(u_\infty, \theta_\infty)$ such that

$$\|u_n - u_\infty\|_{\mathcal{C}^{1,\beta}(U)} + \|\theta_n - \theta_\infty\|_{\mathcal{C}^{1,\beta}(U)} \longrightarrow 0,$$

as $n \to \infty$. By the stability of weak solutions, Proposition **??**, we conclude that θ_∞ satisfies

$$\Delta\theta_\infty = 0 \quad \text{in} \quad U$$

and hence, $\theta_\infty \in \mathcal{C}^\infty(U)$ is harmonic. Therefore, by setting $h := \theta_\infty$, we reach a contradiction and the proof is complete.

□

In the next proposition, we produce an oscillation control for the difference of θ and a paraboloid $P(x)$.

Proposition 5.2. *Suppose A**??**-A**??** are in force and let*

$$(u, \theta) \in W_0^{1,p(\cdot)}(U) \times \mathcal{C}^{0,\alpha}(\overline{U})$$

be a solution to (**??**). *Then, there exists a universal constant* $0 < \rho \ll 1$ *such that*

$$\sup_{B_\rho} \left| \theta(x) - \left(a + b \cdot x + \frac{1}{2} x^T M x \right) \right| \leq \rho^2,$$

for some $a \in \mathbb{R}$, $b \in \mathbb{R}^d$ *and* $M \in \mathcal{S}(d)$.

Proof. Take $0 < \varepsilon < 1$, to be determined further down, and use Proposition **??** to obtain a harmonic function $h \in \mathcal{C}^\infty(U)$ such that

$$\|\theta - h\|_{L^\infty(U)} < \varepsilon.$$

Note that we have a universal control on the L^∞-norm of h since

$$\|h\|_{L^\infty(U)} \leq \|\theta - h\|_{L^\infty(U)} + \|\theta\|_{L^\infty(U)} < \varepsilon + M \leq C.$$

We point out that a standard scaling argument (see, for example [**?**, **?**]) puts us in the (then unrestrictive) smallness regime required by Proposition **??**. Indeed, given a solution (u, θ) to (**??**), define

$$\theta_K(x) := \frac{\theta(x)}{K},$$

for some $K > 0$ to be fixed. Notice the pair (u, θ_K) solves

$$\begin{cases} -\operatorname{div}\left(|Du|^{\sigma_K(\theta_K(x))-2} Du\right) = f \\ \\ -\Delta\theta_K = \lambda_K(\theta_K(x))\,|Du|^{\sigma_K(\theta_K(x))}, \end{cases} \tag{5.1}$$