4. Description of projections in terms of functions on X

Let $\mathcal{C}_m(X)$ be the set of all continuous functions f on X taking values in $[0,\infty)$ such that

(f1) there exists $\varepsilon > 0$ such that $f(x) \geq \varepsilon$ for any $x \in X$;
(f2) $|f(x) - f(y)| \leq 2d_X(x,y)$ for any $x, y \in X$.

We have two kinds of equivalence on $\mathcal{C}_m(X)$, quasi-equivalence and coarse equivalence. Let $C_m^q(X)$ (resp., $C_m^c(X)$) be the set of quasi-equivalence (resp., of coarse equivalence) classes of $\mathcal{C}_m(X)$. If the kind of equivalence doesn't matter then we write just $C_m(X)$ for any of them.

We say that $[f]_q \preceq [g]_q$ if there exist $\alpha \geq 0$, $\beta \geq 1$ such that $g(x) \leq \beta f(x) + \alpha$ for any $x \in X$. This makes $C_m^q(X)$ a partially ordered set. Similarly, we say that $[f]_c \preceq [g]_c$ if there exists a continuous function ψ on $[0,\infty)$ with $\lim_{t\to\infty} \psi(t) = \infty$ such that $g(x) \leq \psi(f(x))$ for any $x \in X$. Clearly, $[f]_q \preceq [g]_q$ implies $[f]_c \preceq [g]_c$, and $C_m^c(X)$ is a partially ordered set as well.

Set $f \wedge g(x) = \max(f(x), g(x))$ and $f \vee g(x) = \min(f(x), g(x))$, respectively.

Lemma 4.1. *If $f, g \in \mathcal{C}_m(X)$ then $f \vee g, f \wedge g \in \mathcal{C}_m(X)$.*

Proof. The property (f1) for $f \vee g$ and for $f \wedge g$ is obvious. The property (f2) for them can be checked by direct calculation.

□

Lemma 4.2. *Let $f' \sim f$. Then $f' \vee g \sim f \wedge g$ and $f' \vee g \sim f \wedge g$.*

Proof. As the two statements are similar, we check only the first one, and only for quasi-equivalence. If $f'(x) \leq \beta f(x) + \alpha$ for any $x \in X$ then

$$\begin{aligned} \max(f'(x), g(x)) &\leq \max(\beta f(x) + \alpha, g(x)) \leq \max(\beta f(x) + \alpha, \beta g(x) + \alpha) \\ &= \beta \max(f(x), g(x)) + \alpha. \end{aligned}$$

□

Thus, $[f] \vee [g] = [f \vee g]$ and $[f] \wedge [g] = [f \wedge g]$ are well-defined in $C_m(X)$ (both for quasi-equivalence and for coarse equivalence).

It is clear that $[f] \wedge [g] \preceq [f], [g] \preceq [f] \vee [g]$.

Lemma 4.3. *If $h \in \mathcal{C}_m(X)$ satisfies $[f], [g] \preceq [h]$ and $[h] \preceq [f] \vee [g]$ then $[h] = [f] \vee [g]$. If $h \in \mathcal{C}_m(X)$ satisfies $[h] \preceq [f], [g]$ and $[f] \wedge [g] \preceq [h]$ then $[h] = [f] \wedge [g]$.*

Proof. Once again, it suffices to prove only the first statement and only for quasi-equivalence. By assumption, there exist $\alpha \geq 0$, $\beta \geq 1$ such that

$$h(x) \leq \beta f(x) + \alpha, \quad h(x) \leq \beta g(x) + \alpha, \tag{4.1}$$

$$\min(f(x), g(x)) \leq \beta h(x) + \alpha \tag{4.2}$$

For any $x \in X$. Then it follows from (4.1) that $h(x) \leq \beta \min(f(x), g(x)) + \alpha$, and this, together with (4.2), implies $h \sim_q f \vee g$.

□

Thus, $C_m^q(X)$ and $C_m^c(X)$ are lattices.

Let $d \in \mathcal{M}(X)$. Set $F(d) = f$, where $f(x) = d(x, x')$. Clearly, this determines maps $M^q(X) \to C_m^q(X)$ and $M^c(X) \to C_m^c(X)$. Consider the restriction of these maps to $E(M^q(X))$ and $E(M^c(X))$,

$$F^q : E(M^q(X)) \to C_m^q(X); \quad F^c : E(M^c(X)) \to C_m^c(X). \tag{4.3}$$

Theorem 4.4. *The maps F^q and F^c (4.3) are bijections.*