Observe that, from (**??**) and (**??**), we have

$$a_n \to \theta(0), \qquad b_n \to D\theta(0),$$

with

$$|a_n - \theta(0)| \leq \rho^{2n}, \qquad |b_n - D\theta(0)| \leq C\rho^n. \tag{5.6}$$

Although we can not conclude about the convergence of the sequence of matrices $(M_n)_n$, we still obtain, again from (**??**), the estimate

$$|M_n| \leq Cn. \tag{5.7}$$

We conclude the paper with the proof of Theorem **??**, which amounts to producing the continuous version of Proposition **??**.

Proof of Theorem **??**. Let $0 < r \leq \rho \ll 1$ be given. Take $n \in \mathbb{N}$ such that $\rho^{n+1} < r \leq \rho^n$. Observe that then

$$n \leq \frac{\ln r}{\ln \rho}.$$

From Proposition **??** and estimates (**??**) and (**??**), we obtain

$$\begin{aligned}
\sup_{B_r} |\theta(x) - (\theta(0) + D\theta(0)\cdot x)| &\leq \sup_{B_{\rho^n}} |\theta(x) - (\theta(0) + D\theta(0)\cdot x)| \\
&\leq \sup_{B_{\rho^n}} \left| (\theta - P_n) + a_n - \theta(0) + b_n \cdot x - D\theta(0)\cdot x + \frac{x^t M_n x}{2} \right| \\
&\leq \rho^{2n} + \rho^{2n} + C\rho^{2n} + \frac{C}{2} n\rho^{2n} \\
&\leq C\left(\rho^{2n} + n\rho^{2n}\right) \\
&\leq \frac{C}{\rho^2}\left(\rho^{2(n+1)} + n\rho^{2(n+1)}\right) \\
&\leq \frac{C}{\rho^2}\left(r^2 + \frac{\ln r}{\ln \rho} r^2\right) \\
&\leq \frac{C}{\rho^2 |\ln \rho|}\left(|\ln \rho| + \ln \frac{1}{r}\right) r^2 \\
&\leq \frac{2C}{\rho^2 |\ln \rho|} r^2 \ln \frac{1}{r} \\
&\leq Cr^2 \ln \frac{1}{r},
\end{aligned}$$

for a universal constant $C > 0$. The conclusion that, locally, θ has continuous first order derivatives, with a Log-Lipschitz modulus of continuity, follows from standard arguments in regularity theory.

□