2. Description of projections in $M(X)$ in terms of expanding sequences

For a subset $A \subset X$ we denote by $N_r(A)$ the r-neighborhood of A, i.e.

$$N_r(A) = \{x \in X : d_X(x, A) \leq r\}.$$

The sequence $\mathcal{A} = \{A_n\}_{n\in\mathbb{N}}$, where $A_n \subset X$, is *expanding* if it satisfies

(e1) A_n is not empty for some $n \in \mathbb{N}$;

(e2) $N_{1/2}(A_n) \subset A_{n+1}$ for any $n \in \mathbb{N}$, for which A_n is non-empty.

Note that $\cup_{n\in\mathbb{N}} A_n = X$.

A special class of expanding sequences is given by subsets of X. For $A \subset X$, set $\mathcal{E}_A = \{A_n\}_{n\in\mathbb{N}}$, where $A_n = N_{n/2}(A)$.

Let $\delta : X \to [1, \infty)$ be any function (not necessarily continuous) on X. We shall write $\delta(u, u')$ instead of $\delta(u)$ for $u \in X$ to show that this function measures distance in some sense.

Define a metric on the double of X by

$$d(x, y') = \inf_{u\in X}[d_X(x, u) + \delta(u, u') + d_X(u, y)].$$

Obviously, $d^* = d$.

Lemma 2.1. *d is a metric for any function δ.*

Proof. It suffices to check the two triangle inequalities.

1. Let $x_1, x_2, y \in X$. For any $u_1, u_2 \in X$ we have

$$\begin{aligned} d_X(x_1, x_2) &\leq d_X(x_1, u_1) + d_X(u_1, y) + d_X(y, u_2) + d_X(u_2, x_2) \\ &\leq [d_X(x_1, u_1) + \delta(u_1, u_1') + d_X(u_1, y)] + [d_X(y, u_2) + \delta(u_2, u_2') + d_X(u_2, x_2)], \end{aligned}$$

hence, passing to the infimum over u_1 and u_2, we obtain

$$d_X(x_1, x_2) \leq d(x_1, y') + d(x_2, y').$$

2. Take $\varepsilon > 0$, and let $\bar{u}_2$ satisfy

$$d(x_2, y') \geq [d_X(x_2, \bar{u}_2) + \delta(\bar{u}_2, \bar{u}_2') + d_X(\bar{u}_2, y)] - \varepsilon.$$

Then

$$\begin{aligned} d(x_1, y') &\leq [d_X(x_1, \bar{u}_2) + \delta(\bar{u}_2, \bar{u}_2') + d_X(\bar{u}_2, y)] \\ &\leq d_X(x_1, x_2) + d_X(x_2, \bar{u}_2) + \delta(\bar{u}_2, \bar{u}_2') + d_X(\bar{u}_2, y) \\ &\leq d_X(x_1, x_2) + d(x_2, y') + \varepsilon. \end{aligned}$$

As ε is arbitrary, we conclude that

$$d(x_1, y') \leq d_X(x_1, x_2) + d(x_2, y').$$

□

Lemma 2.2. *$[d]$ is a projection in $M(X)$ for any function δ.*

Proof. It follows from Theorem 1.2 and from the estimate

$$\begin{aligned} 2d(x, X') &= 2 \inf_{y,u\in X}[d_X(x, u) + \delta(u, u') + d_X(u, y)] \\ &= 2 \inf_{u\in X}[d_X(x, u) + \delta(u, u')] \\ &\geq \inf_{u\in X}[2d_X(x, u) + \delta(u, u')] = d(x, x') \end{aligned}$$

that $[d]_q$ is a projection in $M^q(X)$. As the canonical map $M^q(X) \to M^c(X)$ is a homomorphism, $[d]_c$ is a projection in $M^c(X)$ as well.

□