A 4 (Integrability of the source). *The source term $f : U \to \mathbb{R}$ is in $L^\infty(U)$ and there exists $C_f > 0$ such that*

$$\|f\|_{L^\infty(U)} \le C_f.$$

We recall that a function is Log-Lipschitz continuous if it has a modulus of continuity of the type $\omega(\sigma) = \sigma \ln(1/\sigma)$. Since, for $0 < \gamma < 1$, we have

$$\sigma \le \sigma \ln(1/\sigma) \le \frac{1}{(1-\gamma)e}\, \sigma^\gamma, \quad \forall\, 0 < \sigma \le \frac{1}{e},$$

this is weaker than Lipschitz continuity but implies the $C^{0,\gamma}$ Hölder continuity, for every $0 < \gamma < 1$. Observe that the constant on the right-hand side of the second inequality (which, in fact, holds for every $\sigma > 0$) blows up as $\gamma \to 1^-$.

We now state the main theorem of our paper, the $\mathcal{C}^{1,\text{Log-Lip}}_{loc}(U)$-regularity for the temperature θ.

Theorem 2.1. *Suppose A**??**-A**??** are in force and let (u, θ) be a weak solution to (**??**) such that*

$$(u, \theta) \in W_0^{1,p(\cdot)}(U) \times \mathcal{C}^{0,\alpha}(\overline{U}).$$

Then, $\theta \in \mathcal{C}^{1,\text{Log-Lip}}_{loc}(U)$ and, for every compact $K \subset U$, there exists $C > 0$, depending only on the data and K, such that

$$\|\theta\|_{\mathcal{C}^{1,\text{Log-Lip}}(K)} \le C.$$

The theorem can be interpreted as follows: a slight refinement in the existence class resonates through the highly nonlinear coupling of the system to yield a substantial gain in regularity.

Remark 2.1 (Fully nonlinear variant). *The tangential analysis techniques we will be using find application in a variety of distinct contexts. In particular, we believe that our arguments are flexible enough to accommodate a fully nonlinear variation of our problem, for example,*

$$\begin{cases} |Du|^{\sigma(\theta(x))}\, F(D^2 u, x) \;=\; f \\ \\ G(D^2\theta, x) \;=\; \lambda(\theta(x))\, |Du|^{\sigma(\theta(x))+2}, \end{cases}$$

where F, $G : \mathcal{S}(d) \times U \to \mathbb{R}$ are fully nonlinear elliptic operators. By designing an appropriate limiting profile, we would expect to establish improved regularity of the solutions in Hölder spaces.