where

$$\sigma_K(t) := \sigma(Kt) \qquad \text{and} \qquad \lambda_K(t) := \frac{\lambda(Kt)}{K}.$$

It is straightforward to check that (**??**) satisfies A**??**-A**??**. Hence, by choosing

$$K := \frac{\|\lambda\|_{L^\infty(U)}}{\delta},$$

we fall into the required smallness regime.

We then have

$$\begin{aligned}\sup_{B_\rho}\left|\theta(x)-h(0)-Dh(0)\cdot x-\frac{x^TD^2h(0)x}{2}\right| &\leq \sup_{B_\rho}|\theta(x)-h(x)| \\ &\quad+\sup_{B_\rho}\left|h(x)-h(0)-Dh(0)\cdot x-\frac{x^TD^2h(0)x}{2}\right| \\ &\leq \varepsilon + C\rho^{2+\alpha},\end{aligned}$$

for some $\alpha \in (0,1)$, where $C>0$ is a universal constant (here the harmonicity of h plays a crucial role). Define

$$\rho := \left(\frac{1}{2C}\right)^{\frac{1}{\alpha}} \qquad \text{and} \qquad \varepsilon := \frac{\rho^2}{2}.$$

By setting $a := h(0)$, $b := Dh(0)$ and $M := D^2h(0)$, we conclude the proof.

□

Observe that the constant matrix $M = D^2h(0)$ in the proof of Proposition **??** satisfies $\mathrm{Tr}(M) = 0$. The next result is a discrete counterpart of Proposition **??**, at the scale ρ^n, for $n \in \mathbb{N}$.

Proposition 5.3. *Suppose A**??**-A**??** are in force and let*

$$(u,\theta) \in W_0^{1,p(\cdot)}(U) \times \mathcal{C}^{0,\alpha}(\overline{U})$$

*be a solution to (**??**).Then, there exists a sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ of the form*

$$P_n(x) := a_n + b_n\cdot x + \frac{x^TM_nx}{2},$$

satisfying

$$\mathrm{Tr}\,(M_n) = 0, \tag{5.2}$$

$$\sup_{B_{\rho^n}} |\theta(x) - P_n(x)| \leq \rho^{2n} \tag{5.3}$$

and

$$|a_n - a_{n-1}| + \rho^{n-1}|b_n - b_{n-1}| + \rho^{2(n-1)}|M_n - M_{n-1}| \leq C\rho^{2(n-1)}, \tag{5.4}$$