

where  $\sum_{S \in \text{ALL}} \text{mean}$  that we sum over all partitions  $S$  such that  $\boxed{I} \subseteq \boxed{S_1} \cup \boxed{S_2}$  or vice versa.

Calculate the degree of all summations using (2.1). We get the following fundamental system of quadratic relations among codimension zero classes:

$$\left( \sum_{\substack{(I, J) \\ (I \neq \emptyset)}} \sum_{\substack{a, b \\ a+b=0}} c(S) I_{0, (I_1+1, J)}^a (\otimes_{e(T_1)} \gamma_1) \otimes_{e(\Delta_1)} g^{ab} I_{0, (J_1+1, I_2)}^b (\Delta_2) \otimes (\otimes_{e(T_2)} \gamma_2) \right) =$$

$$\left( \sum_{\substack{(I, J) \\ (I \neq \emptyset)}} \sum_{\substack{a, b \\ a+b=0}} c(I) I_{0, (I_1+1, J)}^a (\otimes_{e(T_1)} \gamma_1) \otimes_{e(\Delta_1)} g^{ab} I_{0, (J_1+1, I_2)}^b (\Delta_2) \otimes (\otimes_{e(T_2)} \gamma_2) \right). \quad (3.3)$$

Now, define a partial order on pairs  $(\boxed{I}, \boxed{J})$ :  $\boxed{I} \leq \boxed{J}$ , by setting  $(\boxed{I}, \boxed{J}) \geq (\boxed{I}', \boxed{J}')$  if either  $(\boxed{I} \geq \boxed{I}' \wedge \boxed{J} \geq \boxed{J}')$  or  $(\boxed{I} = \boxed{I}' \wedge \boxed{J} > \boxed{J}')$ .

Observe that the highest order terms in (3.3) linearly. In fact, for these terms we have either  $(\boxed{I} = \boxed{J})$  or  $(\boxed{I} \leq \boxed{J})$ . The complementary class, with  $(\boxed{I} = \boxed{J})$  (resp.  $\boxed{I} \leq \boxed{J}$ ), can be zero if  $(\boxed{I}_1 = \boxed{J}_1)$  or  $(\boxed{I}_1 \leq \boxed{J}_1)$  (resp.  $(\boxed{I}_1) = \boxed{J}_1$  or  $(\boxed{I}_1, \boxed{I}_2) \geq (\boxed{J}_1, \boxed{J}_2)$ ), see (2.8). Hence there are four possibilities:  $\boxed{S}_1 = (\boxed{I}_1, \boxed{J}_1)$ ;  $\boxed{S}_2 = (\boxed{I}_1, \boxed{J}_2)$ ;  $\boxed{S}_3 = (\boxed{I}_2, \boxed{J}_1)$ ;  $\boxed{S}_4 = (\boxed{I}_2, \boxed{J}_2)$ .

Let us look, say, at the first group of highest terms:

$$c(S) \sum_{a,b} I_{0, (I_1+1, J)}^a (\gamma_1 \otimes \gamma_2 \otimes \Delta_1) g^{ab} I_{0, (J_1+1, I_2)}^b (\Delta_2) \otimes (\otimes_{e(T_2)} \gamma_2). \quad (3.4)$$

We have by (2.8):

$$I_{0, (I_1+1, J)}^a (\gamma_1 \otimes \gamma_2 \otimes \Delta_1) = \int_I \gamma_1 \wedge \gamma_2 \wedge \Delta_1.$$

Since  $I_{0, (I_1+1, J)}$  is (poly)linear, we can rewrite (3.4) as

$$c(S) \sum_{a,b} I_{0, (I_1+1, J)}^a (\sum_{i,k} \int_I \gamma_i \wedge \gamma_j \wedge \Delta_1) g^{ab} \Delta_2 \otimes (\otimes_{e(T_2)} \gamma_2) =$$

$$c(S) (I_{0, (I_1+1, J)}^a (\gamma_1 \otimes \gamma_2 \otimes (\otimes_{e(T_2)} \gamma_2)). \quad (3.5)$$

Using analogs of (3.5) for all four groups of highest order terms we can finally write (3.3) as

$$\begin{aligned} y &= N(a_1, b_1) N(a_2, b_2) \left( \frac{2a + 2b - 3}{2a_1 + 2b_1 - 1} \right)^{\frac{1}{2}} \\ &\quad \frac{1}{2a_1(a_2 + b_2 - 1)(b_1 a_2 + a_1 b_2) - (2a_1 + 2b_1 - 1)a_2 b_2}. \end{aligned} \quad (5.23)$$

Question: Can one deduce (5.20)–(5.23) directly from (5.19)?

**5.2.6. Nonsingular rational curves.** Consider an effective class  $\boxed{S}$  with  $\boxed{p}_{\boxed{S}}(\boxed{I}) := (3, 3 + K_V, 2) \pm 1$ , i.e.,

$$-K \cdot \boxed{I} \cdot \boxed{J} \geq 2 \quad \text{and} \quad \boxed{J} \cdot \boxed{I} \geq -2.$$

An irreducible curve in this class must be nonsingular rational, so that passing through points imposes only linear conditions. Thus we may expect that  $\boxed{N(\boxed{I})} = \boxed{1}$  for each a class. This was observed numerically on cubic surfaces  $\boxed{E}$  by C. Itzykson.

Question: Can one deduce directly from (5.19) that  $\boxed{N(\boxed{I})} = \boxed{1}$  whenever  $\boxed{p}_{\boxed{S}}(\boxed{I}) = 0$ ?

Notice that there are infinitely many such classes on any  $\boxed{E}$  with  $\boxed{E} \leq \boxed{I}$ . The simplest family is:  $\boxed{I} = \boxed{I}_1 \otimes \boxed{I}_2 = n \boxed{I} - (n-1)\boxed{I}'$  projecting into rational curves of degree  $n$  with one point at  $(n-1)\boxed{I}'$  and one at  $\boxed{I}$ .

**5.3. Quantum multiplication in  $\boxed{H}(\boxed{P})$  at  $\boxed{E} = \boxed{I}$ .** Choose as above  $\boxed{N(\boxed{I})} = \boxed{1}$  ( $\boxed{I} = \boxed{I}_1 \otimes \boxed{I}_2$ ,  $\boxed{I}_1 = \boxed{I}_2 = \boxed{I}$ ). Calculate  $\boxed{A}_{\boxed{I}}$  with the help of (4.35). (We now drop the restriction  $\boxed{E} \leq \boxed{I}$ . Equivalently, we can say that we calculate the quantum multiplication with  $\boxed{E} = \boxed{I}$  but at all points of the subspace  $\boxed{H}^{\otimes 2} \subset \boxed{H}^2$ .)

The  $\boxed{I} \otimes \boxed{I}$  terms in (4.35) do not vanish only for  $\boxed{I} = \boxed{I}'$  of a line,  $\boxed{I} \leq \boxed{a}, \boxed{b} \leq \boxed{I}$  ( $\boxed{I} \otimes \boxed{I}$ ) or  $\boxed{I} = \boxed{I}'$  ( $\boxed{I} \otimes \boxed{I}$ ) for  $\boxed{I} \leq \boxed{a}, \boxed{b}$  ( $\boxed{I} \otimes \boxed{I}$ ), see (2.8). This range agrees also with geometric interpretation. Finally,  $\boxed{p}_{\boxed{S}}(\boxed{A}_{\boxed{I}}) = \boxed{1}$ . Putting this all together, we obtain:

**5.3.1. Proposition.**  $\boxed{A}_{\boxed{I}} = \boxed{A}_{\boxed{I}}$  for  $\boxed{I} \leq \boxed{a}, \boxed{b}$  and  $\boxed{A}_{\boxed{I}} = \boxed{0}$  for  $\boxed{I} \leq \boxed{n-1}$ .

Since  $\boxed{A}_{\boxed{I}}$  is the identity with respect to quantum multiplication, we see that

$$\boxed{H}_{\text{total}}(\boxed{P})|_{\boxed{I} = \boxed{I}} \cong \boxed{C}[t]/(t^{n+1} - q).$$

This formula was heuristically obtained for  $\boxed{P}$  in many papers, and was generalized by Batyrev for toric varieties, and by Givental and Kim for flag spaces.

We now see however, that (5.24) and these generalizations describe only a subspace of quantum deformations parameterized by  $\boxed{H}^2$ .

**5.4.  $\boxed{\mathfrak{B}}$ -module  $\boxed{\mathfrak{B}}$  at  $\boxed{E} = \boxed{I}$ .** We can now easily write for  $\boxed{I} \otimes \boxed{P}$  the equation (4.34) at  $\boxed{E} = \boxed{I}$ . In fact, the matrix  $\boxed{B}(I)$  describes the quantum multiplication by  $\boxed{A}_{\boxed{I}}$  ( $\boxed{I} = (n-1)\boxed{I}$ ) in the basis  $\boxed{\Delta}_{\boxed{I}}$ :

$$\boxed{B}(I) = \begin{pmatrix} -\mu & 0 & 0 & \dots & 0 & (n-1)n \\ 0 & -\mu & 0 & \dots & 0 & (n-1)n \\ 0 & 0 & -\mu & \dots & 0 & (n-1)n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\mu & (n-1)n \\ 0 & 0 & 0 & \dots & 0 & (n-1)n \end{pmatrix} \begin{pmatrix} \partial^2/\partial p \\ \partial^2/\partial p \\ \vdots \\ \partial^2/\partial p \\ \vdots \\ \partial^2/\partial p \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix}.$$

The finite singular points are  $(n-1) \sqrt{n}/2$ .

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the differential equation (1.1) for  $\boxed{E} \leq \boxed{I}$  and the boundary conditions (1.2). By a change of variable, the existence of a positive solution of (1.1), (1.2) may be shown by the existence of a positive radial solution of the semilinear elliptic equation  $\boxed{-K} u + \boxed{f}(x) u = \boxed{0}$  in the annulus  $\boxed{R} \leq \boxed{x} \leq \boxed{R}'$  subject to the certain boundary conditions for  $\boxed{x} = \boxed{R}$  and  $\boxed{x} = \boxed{R}'$ . (Here  $\boxed{\|\cdot\|}$  denotes the Euclidean norm.) We refer to [11] for some additional details.

## 2. EXISTENCE RESULTS

The main result of this paper is:

**Theorem 1.** Assume (A.1)–(A.3) hold. Then the BVP (1.1)–(1.2) has at least one positive solution in the case

- (i)  $\boxed{E} = \boxed{0}$  and  $\boxed{f}(x) \equiv 0$  (superlinear),
- (ii)  $\boxed{f}(x) \equiv 0$  and  $\boxed{f}'(x) = 0$  (sublinear).

It will be seen in the proof that Theorem 1 is also valid for the more general equation

$$(1.1') \quad \boxed{K} u + \boxed{f}(x, u) = \boxed{0},$$

with the same boundary conditions (1.2), provided we assume a certain uniformity with respect to the  $x$  variable. We state this more general result as

**Corollary 1.** Assume  $\boxed{f}$  is continuous,  $\boxed{f}(x, u) \geq 0$  for  $x \in \boxed{[0, 1]}$ , and  $K \geq 0$  with  $\boxed{f}(x, u) \not\equiv 0$  on any subinterval of  $\boxed{[0, 1]}$  for  $\boxed{K} \geq \boxed{0}$ ; and let condition (A.3) hold. Then the BVP (1.1)–(1.2) has at least one positive solution in the case

- (i')  $\lim_{u \rightarrow 0^+} \max_{x \in \boxed{[0, 1]}} \frac{\boxed{f}(x, u)}{u} = 0$  and  $\lim_{u \rightarrow \infty} \min_{x \in \boxed{[0, 1]}} \frac{\boxed{f}(x, u)}{u} = \infty$ , or
- (ii')  $\lim_{u \rightarrow 0^+} \min_{x \in \boxed{[0, 1]}} \frac{\boxed{f}(x, u)}{u} = 0$  and  $\lim_{u \rightarrow \infty} \max_{x \in \boxed{[0, 1]}} \frac{\boxed{f}(x, u)}{u} = 0$ .

The proof of Theorem 1 will be based on an application of the following Fixed Point Theorem due to Krasnoselskii [9]. The proof of Corollary 1 follows from the proof of Theorem 1 with obvious slight modifications that we shall omit.

**Theorem 2** [4, 9]. Let  $\boxed{B}$  be a Banach space, and let  $\boxed{K} \subset \boxed{B}$  be a cone in  $\boxed{B}$ . Assume  $\boxed{\Omega}_1, \boxed{\Omega}_2$  are open subsets of  $\boxed{B}$  with  $\boxed{0} \in \boxed{\Omega}_1$ ,  $\boxed{\Omega}_1 \subset \boxed{\Omega}_2$ , and let

$$\boxed{A}: \boxed{K} \cap (\boxed{\Omega}_2 \setminus \boxed{\Omega}_1) \rightarrow \boxed{B}$$

be a completely continuous operator such that either

- (i)  $\boxed{A}u \leq \boxed{u}$  for  $\boxed{u} \in \boxed{K} \cap \partial \boxed{\Omega}_1$ , and  $\boxed{A}\boxed{u} \geq \boxed{u}$  for  $\boxed{u} \in \boxed{K} \cap \partial \boxed{\Omega}_2$ ,
- (ii)  $\boxed{A}u \geq \boxed{u}$  for  $\boxed{u} \in \boxed{K} \cap \partial \boxed{\Omega}_1$ , and  $\boxed{A}\boxed{u} \leq \boxed{u}$  for  $\boxed{u} \in \boxed{K} \cap \partial \boxed{\Omega}_2$ .

Then  $\boxed{A}$  has a fixed point in  $\boxed{K} \cap (\boxed{\Omega}_2 \setminus \boxed{\Omega}_1)$ .

We will apply the first and second parts of the above Fixed Point Theorem to the superlinear and sublinear cases, respectively.

*Proof of Theorem 1. Superlinear case.* Suppose then that  $\boxed{E} = \boxed{0}$  and  $\boxed{f}(x, u) \equiv 0$ .

We wish to show the existence of a positive solution of (1.1), (1.2). Now (1.1), (1.2) has a solution  $\boxed{u} = \boxed{u}(x)$  if and only if  $\boxed{u}$  solves the operator equation

$$\boxed{u}(x) = \int_0^1 \boxed{k}(x, s) \boxed{a}(s) \boxed{f}(u(s)) ds \geq M \|\boxed{u}\|,$$

$$\boxed{u} \in \boxed{C}[0, 1].$$

TABLE II: Results for TFD-ICDAR2019

	IOU $\geq 0.75$			IOU $\geq 0.5$		
	Precision	Recall	F-score	Precision	Recall	F-score
ScanSSD*	<b>0.781</b>	<b>0.690</b>	<b>0.733</b>	<b>0.848</b>	<b>0.749</b>	<b>0.796</b>
RIT 2†	0.753	0.625	0.683	0.831	0.670	0.754
RIT 1	0.632	0.582	0.606	0.744	0.685	0.713
Mitchiking	0.191	0.139	0.161	0.369	0.270	0.312
Samsung‡	0.941	0.927	0.934	0.944	0.929	0.936

\* Used TFD-ICDAR2019v2 dataset

† Earlier ScanSSD, placed 2<sup>nd</sup> in TFD-ICDAR 2019 competition [5]

‡ Used character information

## B. Quantitative Results

We used two evaluation methods, based on the ICDAR 2019 Typeset Formula Detection competition [5] (Table II), and the character-level detection metrics used by Ohyama et al. [4] (Table III).

<sup>7</sup>Details are available in [31]

<sup>8</sup><https://github.com/amdegoert/ssl.pytorch>

**Formula detection.** An earlier version of ScanSSD placed second in the ICDAR 2019 competition on Typeset Formula Detection (TFD) [5].<sup>9</sup> The new ScanSSD system outperforms the other systems from the competition that did not use character locations and labels from ground truth.

Figure 7 gives the document-level f-scores for each of the 10 testing documents, for matching constraints  $IOU \geq 0.5$  and  $IOU \geq 0.75$ . The highest and lowest f-scores for  $IOU \geq 0.75$  are 0.8518 for Erbe94, and 0.5898 for Emden76. We think this variance is due to document styles: we have more training documents with a style similar to Erbe94 than Emden76. With more diverse training data we expect better results.

Examining the effect of the IOU matching threshold on results demonstrates that the detection regions found by ScanSSD are highly precise: 70.9% of the ground-truth formulas are found at their exact location (i.e., IOU threshold of 1.0). Requiring this exact matching of detected and ground truth formulas also yields a precision of 62.67%, and an f-score of 66.5%. To obtain a more complete picture, we next look at the detection of math symbols.

<sup>9</sup>The first place system used provided character information.