which is integrated by the method of characteristics to give $\Delta[u] = f(u_x, x, t)$. Then equations (**??**) are satisfied by the solution

$$\Delta[u] = f(\zeta, t), \quad \text{where} \quad \zeta = \omega(u_x) + x \quad \text{and} \quad \omega(u_x) = \int \frac{d\,u_x}{w(u_x)}. \tag{6.9}$$

Equation (**??**) becomes

$$u_{xy}^2 + u_{xz}^2 = A^2, \quad \text{where} \quad A = \sqrt{f\left(u_{xx} + \frac{1}{\omega'}\right)}. \tag{6.10}$$

We rewrite (**??**) in the form

$$u_{xy} = A\sin\theta, \quad u_{xz} = A\cos\theta \tag{6.11}$$

where we have introduced the new unknown $\theta = \theta(x, y, z, t)$. The definition of $A$ in (**??**) implies

$$\begin{aligned} A_y &= a\sin\theta + \frac{f}{2}\cos\theta\cdot\theta_x, \quad A_z = a\cos\theta - \frac{f}{2}\sin\theta\cdot\theta_x \\ A_x &= \frac{f_\zeta A^3}{2f^2}\omega' - \frac{A}{2}\frac{\omega''}{\omega'^2} + \frac{f}{2A}\left(\frac{\omega''}{\omega'^3} + u_{xxx}\right) \end{aligned} \tag{6.12}$$

where

$$a = \frac{3}{4}\left(\frac{f_\zeta}{f}A^2\omega' - f\frac{\omega''}{\omega'^2}\right) + \frac{f^2}{4A^2}\left(\frac{\omega''}{\omega'^3} + u_{xxx}\right). \tag{6.13}$$

The integrability condition $(u_{xy})_z - (u_{xz})_y = 0$ of the system (**??**) has the form

$$\sin\theta\cdot\theta_y + \cos\theta\cdot\theta_z = \frac{f}{2A}\theta_x. \tag{6.14}$$

The first equation $\Delta[u] = f(\zeta, t)$ in (**??**) implies

$$\Delta[u_x] = (u_{xy})_y + (u_{xz})_z = (A\sin\theta)_y + (A\cos\theta)_z = f_\zeta(\omega' u_{xx} + 1) \tag{6.15}$$

which finally results in the equation

$$\cos\theta\cdot\theta_y - \sin\theta\cdot\theta_z = \frac{f_\zeta A}{f}\omega' - \frac{a}{A}. \tag{6.16}$$

We solve algebraically the system of equations (**??**) and (**??**) to obtain

$$\theta_y = \frac{f}{2A}\sin\theta\cdot\theta_x + b\cos\theta, \quad \theta_z = \frac{f}{2A}\cos\theta\cdot\theta_x - b\sin\theta \tag{6.17}$$