CSC373 Notes

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1 Proofs

Why do particular algorithms run quickly? Why is it that traveling salesman takes forever, but something like sorting is quick? This course gives you the tools to help you design algorithms that are fast for solving these problems, and to give you what the features are that leads to something being fast or not.

We're trying to figure out whether an algorithm is efficient. There are two different types of efficiency:

- Weakly polynomial (the number)
- Strongly polynomial (no. of bits)

When can we find an example that runs in polynomial time and the structures of algorithms that end up running in polynomial time?

We have:

- Greedy
- Divide and conquer
- Network flow
- Dynamic programming
- Linear programming

These approaches are some of the most common approaches used to find examples for these algorithms that run in polynomial time.

We're also going to show example of problems we strongly suspect that do not run in polynomial time (NP hard or NP complete) – often by reducing them to instances of other NP-hard problems. For example, Super Mario Brothers is NP complete (you can build levels, and the level is beatable IFF the corresponding decision problem is correct). We'll see more fun examples along the way.

One of the things we're going to see is that some variants of problems are extremely easy, but when we change it a bit, it becomes extremely hard. Why?

Linear programming can be done in polynomial time, but if you change it into integer linear programming, all of a sudden, it becomes almost impossible to solve.

Randomized algorithms give us strategies where in case we get an NP problem, we argue how bad a random guess at a solution would be relative to the best case. In some cases, we can prove that the random solution isn't that bad.

This course is theoretical in nature. We'll drill down into the simplest problems.

1.1 How do I Structure Proofs?

Reduce everything down to a well-established proof structure. Putting the logic in a well-structured way makes errors easier to catch and makes it easier to be confident that your ideas are right. Use these structures:

- Induction, all forms
- Contradictions
  - Assume your claim isn't true, come up with an impossible statement
- Not of the two above
  - Beware, this is very error prone

How do I set up an inductive proof? Of the two common proof techniques, inductive proofs are easier to set up.

- Break the problem into a number of steps, $s(i)$
- Show the induction hypothesis holds for the base case $s(0)$, which is extremely easy to show
  - Show an array that contains one item is sorted
- Show that $s(i) \Rightarrow s(i+1)$

When you're proving something with induction, you should begin by drawing a picture. Figure out the structure of the problem before trying to prove it. When you see problems, you won't know how to solve them right away, but drawing some figures and

going through small examples makes the structure of the algorithm clearer. It makes it more obvious to set up the induction hypothesis, but it's rare to see it right away. **Don't try to do the prof without understanding the structure of the problem.**

1.2 Bubble sort Inductive Proof

Prove that one iteration of bubble sort puts the lowest element in the right.

How does this end up working? Induction on the array length

BASE CASE: $n = 1$. Our list is already sorted.

INDUCTION STEP:

Assume bubble sort will put the lowest number on the right for an array with $n - 1$ items.

Say we have a list A with n items and run our first iteration of bubblesort on $A[0 \ldots n-1]$. We'll end up with $\left[- \quad - \quad x\right][y]$

There are two cases:

Case 1: $x \leq y$: swap x and y, our last element would be x which is the smallest

Case 2: Do nothing, nothing happens, the right most element is the smallest

As we've covered all possible cases, we know that the induction step holds.

So by induction, our argument here works.

1.3 Bubble Sort Contradiction Proof

Assume that the opposite of the hypothesis was true.

Prove that one iteration of bubble sort puts the lowest element in the right.

Show that if the opposite were true then the assumptions of the problem would be violated.

The opposite is: (One iteration of bubble sort ran AND the lowest element isn't on the right)

Proof goes like this:

- Assume bubble sort fails and position i is first from right with $a[i] \leq a[i+1]$ (where $i = len(A) - 2$)
  - Here, we **assume a case of failure, that this was the result after the swap**
- If $a[i] \leq a[i+1]$, a swap should've happened but because we said that bubble sort failed, we don't swap. But we assumed that bubble sort must have run

I could really rephrase that as:

- Lowest element on the right $\Rightarrow$ One iteration of bubble sort couldn't have run

2 Divide and Conquer

2.1 Merge Sort

You know how it works. At least. Some preliminaries:

- Merging two sorted lists is $\mathcal{O}(n)$

Task: sort a list of integers $[1, 3, 4, 2]$

Merge sort works like this:

- Split A into L and R
- Call Mergesort on L and R
- Merge the two back together

I'm taking a problem, breaking them up into two, then I combine them back together.

2.1.1 Proving the Mergesort Algorithm

With induction: $P(n)$ means that Mergesort works for a list L of that size.

Base case: $P(2)$, there is no recursion, and we can very easily deduce that the list would be sorted.

Inductive step: For k that is a power of 2, assume $\forall i \in \{2, 4, \ldots, k\}, P(i)$. Prove $P(2k)$.

The list of size $2k$ will be split into two, so we will get two sorted lists coming in. Now, what we need to do is argue that merging two sublists works.

So really, here's our proof for the merge algorithm:

$$\text{Merge ran} \Rightarrow \text{Merged list is sorted}$$

Let's prove the contrapositive:

$$\neg\text{Merged list is sorted} \Rightarrow \neg\text{Merge ran}$$

Now, let's look at the merging algorithm again, and for this one concrete example: suppose that `merge` does this.

$$[4, 5][6, 7] \downarrow$$
$$[4, 6, 5, 7] = S$$

`Merge` couldn't have done this. Let's look at the first location on the list that isn't correctly sorted: at index 1, 6.

Because the list isn't sorted, $\exists i,\ S[i] > S[j],\ \ j > i$. If we assume this, we end up going through the list element by element and compare the two results. When we're breaking up the two list, we sort them and iterate over their positions.

This argument goes: if we actually run merge, we would get $[4, 5, 6, 7]$. Moreover, for $[4, 6, 5, 7]$ to be put as the output, $S[j] = 5 > S[i] = 6$, which is impossible according to the merge algorithm. This means that the merge algorithm couldn't have ran.

2.2 Master Theorem

I've seen it before. It's an amazingly powerful result. What it is, us that it gives you the asymptotic scaling for the solution of a recurrence relation. This is hugely important for divide and conquer algorithms, as they always have a recursive structure to it.

The cost of Mergesort is:

$$T(n) \leq 2\,T\left(\frac{n}{2}\right) + \mathcal{O}(n)$$

It goes by this:

Let $a \geq 1,\ b > 1$ be constants, $T(n)$ be defined on $\mathbb{Z}^{\geq 0}$, and $\frac{n}{b}$ can be $\lceil \frac{n}{b} \rceil$, and let $T(n)$ be:

$$T(n) \leq aT\left(\frac{n}{b}\right) + f(n)$$

Let $d = \log_b(a)$.

1. If $f(n) \in \mathcal{O}\left(n^{d-\varepsilon}\right)$ for some $\varepsilon > 0,\ T(n) = \Theta\left(n^d\right)$
2. If $f(n) \in \Theta\left(n^d \log^k(n)\right)$ for some $k \geq 0$, $T(n) = \Theta\left(n^d \log^{k+1} n\right)$
3. If $f(n) \in \Omega\left(n^{d+\varepsilon}\right)$ for some $\varepsilon > 0$, $T(n) = \Theta\left(f(n)\right)$

For case 2: Ideally, you want k as close to 0; 0 gives us the least upper bound. Picking a higher k gives a looser upper bound which we probably don't want.

For Mergesort, we have $a = 2,\ b = 2 \Rightarrow d = \log_2(2) = 1$.

In this case, we know that $f(n) \in \mathcal{O}(n)$. Case 1 can't be true: $f(n) \notin \mathcal{O}\left(n^{1-\varepsilon}\right)$. Yet, case 2 work.

Case 2: $f(n) \in \Theta\left(n^1 \log^0 n\right) = \Theta(n) \Rightarrow T(n) \in \Theta\left(n^d \log^{k+1} n\right)$

If you're using a specific master theorem, let the TAs know which one you're using.

2.3 Counting Inversions

Given an array a of length m, count the number of pairs (i, j) such that $i < j$ but $a[i] > a[j]$.

An inversion is a pair of elements opposite in the sorted order: e.g. [1, **3**, **2**, 6, **9**, **8**]. A list that is sorted in the reverse order would have around n^2 inversions.

The brute-force algorithm would take $\mathcal{O}\left(n^2\right)$ time.

Let's divide and conquer count inversions on the left, count inversions on the right, and then count the inversions that are between the left and the right.

Sorting kills all information about the inversions, so we want to store the output. That's why this algorithm has two outputs.

Here's how it works, SORT-AND-COUNT which returns (result, sorted list):

- Take a list L
- In the case of a base case, return $(0, L)$
- Break it in half. A, B
- Apply SORT-AND-COUNT, getting $\left(\underset{\text{NUM INVERSIONS}}{r_A}, A\right)$, (r_B, B) back
- Merge it back using the steps below

How do we count inversions (a, b) such that $a \in A, b \in B$? And how do we not merge sort over and over again?

We start with two halves, and we combine our inversion counting and sorting:

- Scan A and B LTR
- Compare a_i, b_j
- If $a_i \leq b_j$, do your merge step like usual
- Otherwise, if $a_i > b_j$, add 1 to the inversion count and do the merge step like usual

Meaning our runtime formula:

$$T(n) = 2T\left(\frac{n}{2}\right) + \mathcal{O}(n)$$
$$\Rightarrow \mathcal{O}\left(n\log(n)\right)$$

If we just relied on Mergesort and done this iteratively, this cost would be $n\log(n)$ so the second bit of the master theorem, we couldn't do $k = 0$ and we would have a squared algorithm.

2.4 Closest Pair in R2

Given n points of the form (x_i, y_i) in the plane, find the closest pair of points. Assume that if I extract all x and y from all points I won't have duplicates.

Brute force: $\Theta\left(n^2\right)$. We'll have to look at every possible pair in the data. $\binom{n}{2} \in \mathcal{O}\left(n^2\right)$

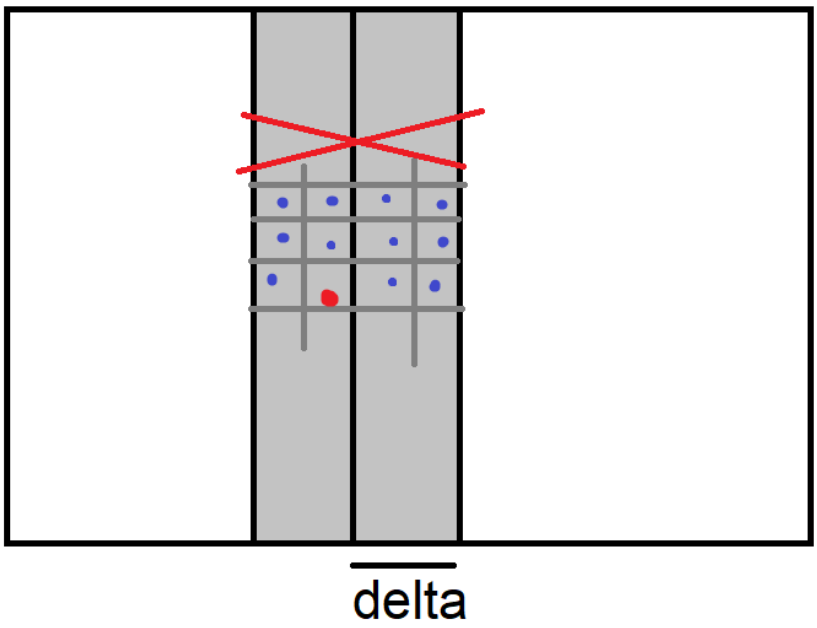
Divide and conquer what we do, is draw a line partway through the data such that half of the data is on the left, and half of the data is on the right. What I'll do, is I'll find the nearest pair on both sides then merge.

However, there's a big complication: merging is very hard.

What we need to do for the combine step, is to go through and check all points in the cross step. However:

Actually, we shouldn't think about doing our comparisons for everything. Reason why, is that:

δ is the lower of the shortest distance between two points on the left, on the right



We can save ourselves a bit of trouble and restrict our attention to our values in the δ region. Yet, we could have a situation where the large fraction of our points could lie inside this region.

Actually, something cool ends up happening. When you look at the sorted list of points, the maximum no. of points you need to compare on the opposite side is a constant. We only need to look at **11** points (I could skip work if the points are on the same side – the POINT of this is that it's a constant no. of points).

Let's break everything up.

Claim: If two points are at least 12 positions apart in the sorted list, their distance is at least δ

Proof: No points lie in the same $\frac{\delta}{2} \times \frac{\delta}{2}$ box

- If two of them were in the same cell, the maximum distance between the two corners is $\frac{\delta}{2}\sqrt{2} = \frac{\delta}{\sqrt{2}} < \delta$
- If two of them happen to show up in the same cell, the distance must be less than δ, but δ is the smallest distance between the two

So, what are the total comparisons that we need to do?

Merging takes $\mathcal{O}(n)$.

So, running time analysis.

- Finding points on strip: $\mathcal{O}(n)$
- Sorting by y coord: $\mathcal{O}\left(n\log(n)\right)$
- Testing against 11 points: $\mathcal{O}(n)$

Runtime:

$$T(n) \leq 2T\left(\frac{n}{2}\right) + \mathcal{O}\left(n\log(n)\right)$$

Master theorem tells us that the cost is $\mathcal{O}\left(n\log^2(n)\right)$

By using divide-and-conquer, we can speed up our algorithm by a lot.

We can improve this to $\mathcal{O}\left(n\log(n)\right)$ by doing a simple sort by y-coordinates at the start.

2.5 Karatsuba's Algorithm

Divide each integer into two parts:

$$x = x_1 \cdot 10^{\frac{n}{2}} + x_2,\ y = y_1 \times 10^{\frac{n}{2}} + y_2$$

$$xy = (x_1y_1)\,10^n + (x_1y_2 + x_2y_1)\,10^{\frac{n}{2}} + (x_2y_2)$$

For $\frac{n}{2}$-digit multiplications can be replaced by three:

$$x_1y_2 + x_2y_1 = (x_1 + x_2)\,(y_1 + y_2) - x_1y_1 - x_2y_2$$

We can reuse multiplications that we've already done.

Runtime:

$$T(n) \leq 3T\left(\frac{n}{2}\right) + O(n) \Rightarrow T(n) = \mathcal{O}\left(n^{\log_2(3)}\right)$$

Most of these multiplication algorithms have great scaling but they have horrible constant factors. Karatsuba is an exception.

2.6 Strassen's Algorithm

Instead of multiplying numbers, let's look at multiplying matrices.

Imagine we have two "block" matrices:

$$C = A \cdot B = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

Naively, this requires two multiplications of size $\frac{n}{2}$.

However, some of these multiplications, we can memorize and replicate after doing a clever substitution. By doing that, you would replace the 8 multiplications you would normally do with 7:

$$T(n) \leq 7T\left(\frac{n}{2}\right) + \mathcal{O}\left(n^2\right) \Rightarrow T(n) = \mathcal{O}\left(n^{\log_2(7)}\right)$$

The lower bound for matrix multiplication is $\Omega\left(n^2\right)$, but in practice it's $\mathcal{O}\left(n^{2.37}\right)$. It's interesting because the gap between n^2 and what we have right now isn't known. The problem with finding an optimum is really theoretically interesting.

2.7 Kth Smallest / Quick Select

Sorting it, find kth element, or using a heap isn't that quick. It won't be better than $n \log(n)$. Good news – selection is easier than sorting.

`QuickSelect` goes like follows:

- Find a pivot p
- Divide A into two sub-arrays: A_{less} = elements $\leq p$, A_{more} = elements $> p$
- If $|A_{\text{less}}| \geq k$, return kth smallest in A_{less}, else find $(k - |A_{\text{less}}|)$-th smallest element in A_{more}

Problem? If the pivot is close to the min or the max, we would get $T(n) \in \mathcal{O}\left(n^2\right)$. Best to find a pivot that evenly distributes the two elements.

So, are we going to select a pivot randomly? We want to do things deterministically. Select the average? That could be subject to outliers. Best solution?

2.7.1 Median of Medians

- Divide n elements into $\frac{n}{5}$ groups of 5 each
- Find the median of each group
  - Takes $\mathcal{O}(n)$ to here
- Find the median of $\frac{n}{5}$ medians = p^*
  - Takes $T\left(\frac{n}{5}\right)$, **this has to be done recursively**
- Create A_{less} and A_{more} according to p^*
  - Takes $\mathcal{O}(n)$
  - $\frac{n}{10}$ of the $\frac{n}{5}$ medians are $\leq p^*$
  - For each such median, there are *at least* 3 elements $\leq p^*$
  - $\Rightarrow$ There can be at most $\frac{7n}{10}$ elements that can be $\geq p^*$
  - In the other direction, there can be at most $\frac{7n}{10}$ elements that can be $\leq p^*$
- **Our work here is done. But, reiterating the last point from Quick Select:** Run selection on one of A_{less} or A_{more}, however you did it before
  - Takes $T\left(\frac{7n}{10}\right)$

We get that

$$T(n) \leq T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + \mathcal{O}(n)$$

Note that $\frac{n}{5} + \frac{7n}{10} = \frac{9n}{10}$. Only a fraction of n, using a similar analysis to the one in the master theorem, $T(n) = \mathcal{O}(n)$

By the way, you don't need to memorize the specific algorithms – you just need to know how to use these basic ideas. It may be easier or harder than the memorization challenges.

You will be tested on the ability to use these tools.

3 Greedy Algorithms

Find a solution x maximizing or minimizing an objective function f. The challenge is when the space of possible solutions x is too large. Does it find the best solution possible, and how much time does it take to find the best solution possible?

Here's our approach:

- **Compute it one part at a time**
  - For each part, they will always pick the best solution right now **without regard for the future**
- Select the next part greedily to get the most immediate benefit (this needs to be defined carefully for each problem)
- Guarantees polynomial time
- Need to prove that this will always return an optimal solution despite having no foresight

Greedy algorithms don't always give the optimal solution! We'll play with some heuristics that we can prove its optimal, and then we can prove that the heuristic is in fact optimal.

Beware:

- Most of the time, greedy algorithms, will find one of many equivalently optimal solutions. Greedy algorithms will return one of the optimal solutions.

3.1 When should I use it?

When **all of the** following hold:

- Optimal substructure
  - Solving one problem contains optimal solutions to all other subproblems
  - An optimal solution at one level can be thought of adding onto an optimal solution to a smaller level.
  - Otherwise, greedy algorithms won't give you the best option.
- Greedy choice property
  - Among all choices, the greedy solution performs best

3.2 Interval Scheduling

Given some intervals (job j starts as s_j and finishes at f_j), two jobs are compatible in the same way how UofT timetables don't conflict. Find the maximum-size subset of mutually compatible jobs.

How do we do it?

- Always pick the task that finishes first.

Here comes the earliest finishing time heuristic.

For scheduling-type problems, there is a set of typical heuristics that could end up working well. Go through each heuristic and come up with a counterexample. If you come up with one, abandon strategy and try another.

Look for the one you can't prove is suboptimal, then prove that it is optimal.

What are the heuristics? Choose in the order of

- Earliest start time
  - Not – if the earliest task takes the entire day, blocking everything else out
- **Earliest finishing time (the best)**
- Shortest interval
  - Which happens to rule everything else out: two other tasks that one ends at noon, one starts at noon, yet the shortest job overlaps noon
- Fewest conflicts
  - Which happens to block everything else out; we can cram a bunch of other conflicts

You would struggle looking for a contradiction with earliest finishing time. You would switch over and take a look at the finishing time as the heuristic.

Before we improve its optimality, let's look at how long it will take for it to run.

To do earliest finishing time, we need to compute the earliest finishing time. The easiest way to do that is by sorting. To find the one that has the earliest initial starting time, just sort the list. Sorting takes $\mathcal{O}\left(n \log(n)\right)$. Tiebreaks do not matter.

We then start picking tasks according to that list. Look at all tasks that end later than our current task in the same order and choose the earliest compatible one. You might think this will take $\mathcal{O}\left(n^2\right)$ time, but no, because we only need to compare all future-ending intervals with our previous interval, it's constant time in total.

3.2.1 Interval Scheduling Proof of Optimality By Contradiction

- Assume what we're proving is false
- Derive an inconsistency

For a contradiction, **assume that greedy is optimal**

Suppose that greedy selects jobs $i_1, \ldots, i_k$ sorted by finish time

Consider an optimal solution $j_1, j_2, \ldots, j_m$ which matches greedy for as many indices as possible from the start (we want $j_1 = i_1, \ldots, j_r = i_r$ for the greatest possible r)

Both i_{r+1} and j_{r+1} must be compatible with the previous selection (the Greedy algorithm can never choose something invalid; so is the optimal solution)

What are we going to do with this?

Consider a new solution $i_1, i_2, \ldots, i_r, \mathbf{i_{r+1}}, j_{r+2}, \ldots, j_m$

We have replaced j_{r+1} with i_{r+1} in our reference optimal solution.

- A conflict is impossible: $j_r = i_r$ and greedy cannot select anything that conflicts
- i_{r+1}.finish $\leq$ j_{r+1}.finish. If that wasn't the case, greedy would've chosen j_{r+1} instead.
- j_{r+1}.finish $\leq j_{r+2}$.start, so, the right side of i_{r+1} is still compatible.

This results in a contradiction: the new solution matches greedy for $r + 1$ intervals. This means that greedy is optimal, **as any step of it can be extended into an optimal strategy**.

Key fact:

$$\text{Can be extended into an optimal strategy in any step} \Rightarrow \text{Optimal}$$

3.2.2 Proof of Optimality by Induction

Induction gives you a lot of scaffolding and gives you a structure to work off from. The structure of inductive proof gives you guides of what you should do.

Let S_j be the **subset of jobs picked by greedy after considering the first j jobs in the increasing order of finish time.**

- Define $S_0 = \varnothing$.

We call a partial solution *promising* if there is a way to extend it to an optimal solution by picking some subject of jobs $j + 1, \ldots, n$ (indices of jobs sorted by finish time)

- $\exists T \subseteq \{j+1, \ldots, n\}$ such that $O_j = S_j \cup T$ is optimal

Our inductive claim: $\forall t \in \{0, 1, \ldots, n\}$, S_t is promising.

If S_n is promising, then it must be optimal as there are no more jobs to consider.

Base case:

For $t = 0$, $S_0 = \emptyset$ is promising, because any optimal solution can extend this set.

Induction hypothesis: Suppose claim holds for $t = j - 1$ and optimal solution O_{j-1} extends S_{j-1}. WTP: O_j extends S_{j-1} with no issues

Induction step: At $t = j$, we have two possibilities.

On considering the jth job:

If greedy **did not** select job j:

- j must conflict with some job in S_{j-1}
- Since $S_{j-1} \subseteq O_{j-1}$, O_{j-1} cannot include job j either
- $O_j = O_{j-1}$ also extends $S_j = S_{j-1}$ (transitivity of the = sign)

If greedy **did** select job j:

- $S_j = S_{j-1} \cup \{j\}$
- Consider the earliest job r in $O_{j-1} \setminus S_{j-1}$
- Consider O_j obtained by replacing r with j in O_{j-1}
- What happens if $r \neq j$? We still need to show that there are no conflicts.
  - We know greedy can't select anything with conflicts, so there are no conflicts
  - Because greedy selects jobs in ascending order of finish time, job j's finish time must be **before** job r's finish time
- We know that O_j extends S_j with no issues (really obvious just read the above again).

This means that $\forall j \in \{0, \ldots, n\}$, S_j is promising.

3.3 What are the Arguments Saying?

Both proof methods make the same claim:

- The greedy solution after j iterations can be extended into an optimal solution $\forall j$

The same key argument is used:

- If the greedy solution after j iterations can be extended to an optimal solution, then the greedy solution after $j + 1$ iterations can be extended to an optimal solution as well

The difference:

- Induction: this is the induction step
- Contradiction: we take the greatest j for which the greedy solution can be extended to an optimal solution (where anything ahead **we assumed for a contradiction cannot**), and we derive a contradiction by saying that it can (by extending the greedy solution after $j + 1$ iterations).

3.3.1 Another Flavor of Greedy Proofs

Greedy stays ahead

Let $i_1, \ldots, i_k$ be the greedy solution sorted by finish time

Let $j_1, \ldots, j_m$ be an optimal solution sorted by finish time

Claim: $f_{i_r} \leq f_{j_r}$ $\forall r$

Proof:

Base case: $f_{i_1} \leq f_{j_1}$ because greedy sorts in finish time and will always initially choose the lecture that finishes the earliest.

Inductive step: Assume $f_{i_l} \leq f_{j_l}$ (inductive hypothesis). Show $f_{i_{l+1}} \leq f_{j_{l+1}}$.

By the greedy algorithm, i_{l+1} is the lecture that finishes the earliest that is compatible with f_{i_l}. Can $f_{i_{l+1}} > f_{j_{l+1}}$? No, as if it were, greedy would've selected $f_{j_{l+1}}$ (contradiction spotted). So, this means that our claim is proven.

Why does the claim imply Greedy is optimal?

Suppose greedy is not optimal ($k < m$).

By the claim, $f_{i_k} \leq f_{j_k}$.

However, $s_{j_{k+1}} \geq f_{j_k}$. But then, $s_{j_{k+1}} \geq f_{i_k}$, so $s_{j_{k+1}}$ must have been considered by the greedy algorithm. This is our contradiction.

3.4 Interval Partitioning / Scheduling Lectures In Rooms

Jobs j starts at time s_j and finishes at f_j. Two jobs are compatible if they don't overlap. Goal: group jobs into fewest partitions such that jobs in the same partition are compatible.

So, what heuristic should we use? Schedule, considering *this* ordering first:

- Finish time
- Shortest interval
- Fewest conflicts
- **Start time**
  - Start time is the one that works; everything else has counterexamples

So, the algorithm goes like this:

Input: a set of n lectures

- Sort lectures by start time
- $d = 0$ (number of allocated classrooms)
- For $j = 1$ to n:
  - If lecture j is compatible with some classroom (doesn't matter which one)
    - Schedule it in that classroom
  - Else:
    - Allocate a new classroom $d + 1$

        * Schedule lecture j there
        * $d = d + 1$
- Return the schedule

3.4.1 Runtime

Key step to check if a lecture is compatible with some classroom:

- Store classrooms in a priority queue, key = latest finish time of any lecture in the classroom

If lecture j compatible with some classroom?

- Same as, is $s_j \geq$ latest finish time of that classroom
  - Yes: add it, increase minimum key of that classroom to f_j
  - No: Create a new classroom, add lecture j, set key to f_j
- $\mathcal{O}(n)$ priority queue operations, $\mathcal{O}\left(n \log(n)\right)$ time

3.4.2 Proof of Optimality

This is a different style of proof than the one before. We're going to prove a lower bound: that it is not possible to do interval partitioning with fewer than some number of classrooms. Then, we need to show that our algorithm matches that number.

First, prove that there is no way I can do better than k (the depth). Then, I need to show that greedy will always give me k (the depth) classrooms. If greedy achieves that and there is no way to do better than that, then greedy is optimal.

Proof of optimality (lower bound) – I CAN'T DO BETTER THAN DEPTH:

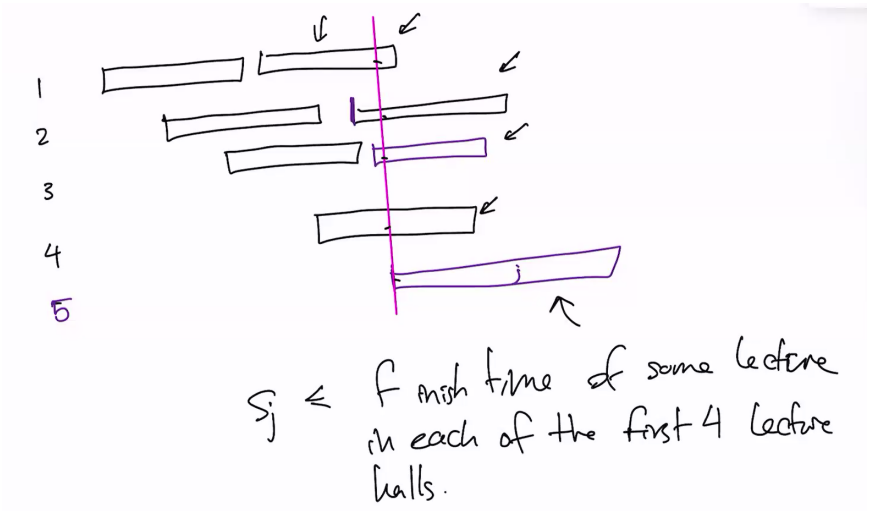
- WTS: d = Classrooms needed by greedy $\geq$ depth
  - Where depth = maximum no. of lectures running at any time
  - Job i runs in $[s_i, f_i)$

  - The rationale, is that if you have 4 classes running at the same time, you at least will have to pack them into four different classrooms

Now, we want to claim our greedy algorithm uses only these many classrooms

Proof of optimality (upper bound) – I CAN'T DO WORSE THAN DEPTH:

- Let d = no. classrooms used by greedy. Show that $d \leq$ depth.
- Classroom d was opened because there was a lecture j which was incompatible with some lecture already scheduled in each of $d - 1$ other classrooms.
  - All these d lectures end after s_j.
  - Since we sorted by the start time, they all start at or before s_j.
- $\Rightarrow$ So, at time s_j, we have d mutually overlapping (conflicting) lectures
  - **We're showing that greedy can also detect the minimum depth (max no. of concurrently conflicting lectures).** (With our lower bound, we can get rid of the word "minimum")
- Hence, depth $\geq d$ = no. classrooms used by greedy. In other words, **the depth (highest no. of concurrently conflicting lectures) is at most the no. of classrooms used by greedy.**
  - When I opened the dth classroom, the depth at that point was d. The depth cannot be smaller than d.
  - From the lower bound proof: the depth cannot be larger than d, as if it were so the classroom would've already been made for that.



Overall, we showed that depth $\geq d$ and $d \geq$ depth, then it can only be concluded that d = depth and greedy only uses as many classrooms as the depth.

3.5 Minimizing Maximum Lateness

Problem:

- We have a single machine
- Each job j requires t_j units of time and is due by time d_j
- If it's scheduled to start at s_j, it will finish at $f_j = s_j + t_j$
- Lateness: $l_j = \max\left(0,\ f_j - d_j\right)$
- Goal: figure out the ordering of the tasks that minimize the **maximum** lateness of any of my tasks: $L = \max_j l_j$. *This is not minimizing the sum of all task lateness!*

Contrast with interval scheduling:

- We decide the start time
- There are soft deadlines

Let's look at the greedy template:

- Shortest processing time: t_j ascending

  - Counter: $(t, d) : (1, 100), (10, 10)$ – opt: 0, this heuristic: 1

- Earliest deadline first: d_j ascending
  - It's this one
- Smallest slack first: $d_j - t_j$ ascending
  - Counter: $(t, d) : (1, 2, s = 1), (10, 10, s = 0)$ – opt: 1, this heuristic: 11

So, how do we prove the optimality of earliest deadline first? Let's make a couple observations:

1. All optimal solutions have no idle time. Solutions with idle time can be optimized to have no idle time.
2. Earliest deadline first has no idle time.
   a. An inversion is (i, j) such that $d_i < d_j$ but j is scheduled before i (later deadline scheduled earlier)
3. By definition, earliest deadline first has no inversions.
   a. Any schedule with no idle time and no inversions have the same maximum lateness. If two tasks have the same deadlines, it doesn't matter what order I pick them.
4. If a schedule with no idle time has at least one inversion, it has a pair of inverted jobs scheduled consecutively.
   a. Whenever we have an inversion, we can swap two tasks such that the maximum lateness doesn't increase, and we will have no inversions.
5. Swapping adjacently scheduled inverted jobs do not increase lateness but reduces inversion count by one
   a. Draw it out. This pushes overall due times back for both jobs.
   b. Proof: $d_j \geq d_i$ yet j was scheduled first. Check that swapping an adjacent inverted pair reduces the total no. of inversions by 1. Tasks not i, j: swapping does not change lateness. Task i is shifted earlier, so task i can only be less late. Task j is shifted later, but it cannot be later than how late task i was.

c. Lateness of j after swap is $f - d_j$

d. Lateness of i before swap is $f - d_i$

e. Since $d_j > d_i$, $f - d_j < f - d_i$

Here's how the proof of optimality by contradiction works **by showing that EDF is an optimal solution:**

Remember that optimal solutions can have inversions, for example if the only have due times as ∞.

We use this fact: *If an optimal solution did not fully match greedy (has inversions), we can swap an adjacent inverted pair and reduce the no. of inversions by 1.*

- Suppose that the greedy earliest deadline first solution isn't optimal
- Consider optimal schedule S^* with the **fewest but at least one inversion** (otherwise it would be the same as EDF). Suppose it has no idle time.
- Because EDF is not optimal, S^* has at least one inversion $\Rightarrow$ it has an adjacent inversion.
- Swapping the adjacent pair keeps the schedule optimal but reduces the no. of inversions by 1
- Contradiction

Here's the proof by reverse induction:

Claim: For each $r \in \left\{0, 1, \ldots, \binom{n}{2}\right\}$, there is an optimal schedule with at most r inversions.

Base case $r = \binom{n}{2}$: trivial, any optimal schedule satisfies this as it covers every single combination possible

Induction hypothesis: Suppose the claim holds for $r = t + 1$.

Induction step: Take an optimal schedule with at most $t + 1$ inversions.

- If it has at most t inversions, we're done
- If it has exactly $t + 1 \geq 1$ inversions

  - Assume no idle time without loss of generality
  - Find and swap an adjacent inverted pair, which removes the inversion
  - No. of inversion reduces by 1 to t (meaning there is an optimal schedule with at most t inversions)

Claim for $r = 0$ shows optimality of EDF

3.6 Lossless Compression

We have a document that is written using n distinct labels, represented in binary.

The naïve way is that if we have n distinct labels, we can encode them in $\lceil \log(n) \rceil$ bits. If the document has length m, this uses $m \log(n)$ bits. Thing is, in English, there are some characters that are way more frequent. The idea is, can we save space by assigning shorter codes to more frequent letters?

The problem:

- $a = 0,\ b = 1,\ c = 01$: what happens if we observe the encoding 01? Is it ab or is it c?

We need a prefix-free encoding. **No encoding may be the prefix to the other.**

The formal problem:

Given n symbols and their frequencies $(w_1, \ldots, w_n)$ find a prefix-free encoding with lengths $(l_1, \ldots, l_n)$ assigned to the symbols which minimizes $\mathbf{w}^T\mathbf{l}$.

A prefix-free encoding can be represented as a binary tree.

Here are some observations:

- A prefix-free encoding can be seen as a tree

The idea for Huffman is:

- Build a priority queue by adding $(x,\ w_x)$ for each symbol x
- While $|\text{queue}| \geq 2$

- Take two symbols with the lowest weight $(x,\ w_x)$ and $\left(y,\ w_y\right)$
- Merge them into one symbol with weight $w_x + w_y$

3.6.1 Proof of Huffman Encoding Optimality

Runtime is $\mathcal{O}\left(n \log(n)\right)$.

Proof of optimality:

- Induction on the number of symbols n, that the Huffman tree built is optimal

Base case: For $n = 2$, both encoding which assigns 1 bit to each symbol are optimal

Hypothesis: Assume it returns an optimal encoding with $n - 1$ symbols

Idea: Run Huffman for 1 step. Let $x,\ y$ be the characters selected by Huffman. x and y get put into a binary tree with priority $w_x + w_y$.

Treat this as a single symbol "$\left(w_x + w_y\right)$" with weight $\left(w_x + w_y\right)$ and now we have $n - 1$ symbols.

By IH, Huffman will produce the optimal tree where $\left(w_x + w_y\right)$ is treated as one symbol.

Let T be optimal (for n symbols). Transform T such that x and y are siblings with the lowest weights, argue that the length of $H \leq$ length of T.

3.6.1.1 Lemma 1 Consider the case of n symbols. If $w_x < w_y$, then $l_x \geq l_y$ in any optimal tree. "If y occurs more often than x, then it shouldn't have a longer encoding".

3.6.1.2 Proof of that lemma Suppose for contradiction that $w_x < w_y$ and $l_x < l_y$. Swapping x and y strictly reduces the overall length as $w_x l_y + w_y l_x < w_x l_x + w_y l_y$. This is the rearrangement inequality:

If $a_1 \ldots a_n$ are sorted and so are $b_1, \ldots b_n$, $\mathbf{a}^T \mathbf{b}$ maximizes the sum. Any rearrangement of the bs will only decrease the sum.

3.6.1.3 Lemma 2 Consider the two symbols x and y with lowest frequency which Huffman combines in the first step. $\exists$ optimal tree T in which x and y are siblings. In other words, for some p, they are assigned encodings $p0$ and $p1$.

3.6.1.4 Proof of that lemma:

1. Take any optimal tree
2. Let x be the label with the lowest frequency.
3. If x doesn't have the longest encoding, swap it with one that has (as swapping does not increase the score)
4. Due to optimality, x must have a sibling (if it didn't, then that prefix code would have to be used somewhere in the tree).
5. If x's sibling is not y (the "second lowest frequency"), then swap it with y
6. Check that we didn't increase the average compression rate $\mathbf{w}^T\mathbf{l}$
7. Initial encoding was optimal so the modified one was also

Why this insight? Let's look at how the Huffman encoding tree was constructed. Right from the start, we take our two lowest frequency symbols and split them. Every time we split them, two children get created. The first children that are created are the entries with the lowest frequency. This process will always end up generating trees where the outcomes will have the lowest frequency notes as siblings.

Hence why there will always exist an optimal tree with the lowest frequency symbols as siblings.

3.6.1.5 Back to the Proof Proof of Huffman encoding optimality (that minimizes $\mathbf{w}^T\mathbf{l}$)

WTP: All Huffman trees of size $\leq n-1$ are optimal $\Rightarrow$ all Huffman trees of size n are optimal

Let x and y be the two least frequent symbols that Huffman combines in the first step into xy

- Let H be the Huffman tree produced
- Let T be an optimal tree in which x and y are siblings (Lemma 2)
- Let H' and T' be obtained from H and T respectively by treating xy as one symbol with frequency $w_x + w_y$
  - This case, when encoding x and y would be treated as the same character
- Induction hypothesis: $\text{Length}\left(H'\right) \leq \text{Length}\left(T'\right)$ (expected length of an arbitrary symbol)
- Induction step:
  - $\text{Length}(H) = \text{Length}\left(H'\right) + \left(w_x + w_y\right) \cdot 1$
    - By disambiguating "xy" we need 1 more bit per xy to represent the whole string
  - $\text{Length}(T) = \text{Length}\left(T'\right) + \left(w_x + w_y\right) \cdot 1$

4 Dynamic Programming

Where and when should we use greedy algorithms? In practice, greedy algorithms will almost never give you the optimal solution for anything. The cases covered in class before were just very specific examples where you can prove that greedy algorithms will give you the best answer. That doesn't usually happen.

Going forward, let's look at algorithms that give you advantages that aren't greedy algorithms.

Dynamic programming is just:

- Breaking the problem down into simpler subproblems, solve each subproblem once, and store their solutions. This is called "memoization." It's just the art of keeping the best solutions in a type of data structure to store the optimal solutions to all the subproblems we end up getting.

4.1 Weighted Interval Scheduling

This is very similar to the interval scheduling problem. Find a set of mutually compatible jobs with the highest total weights/

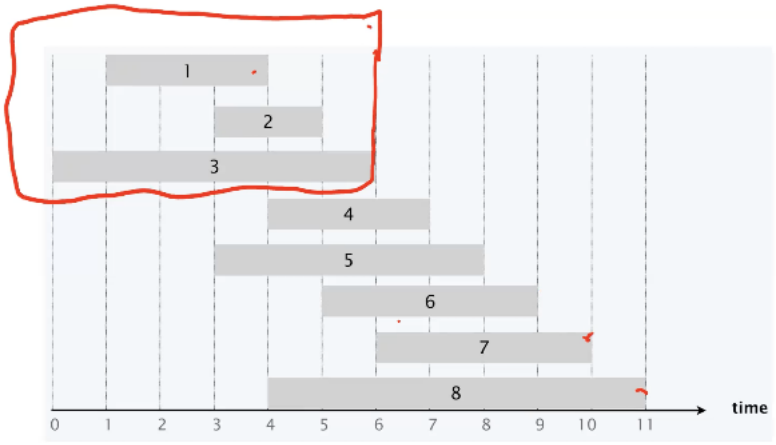
- Job j starts at s_j and finishes at f_j
- Each job has a weight w_j
- Two jobs are compatible if they don't overlap
- Goal: Find a set of S mutually compatible jobs with the highest total weight: $\sum_{j \in S} w_j$

This problem will fail horribly if there is a massive imbalance of the weights if we tried to use our algorithms for interval scheduling. So is picking the largest weight. You can actually prove that any greedy algorithm can't guarantee an optimal solution.

So, how do we solve this?

- Jobs are sorted by finish time: $f_1 \leq f_2 \leq \cdots \leq f_n$
- $p[j]$ = **largest i** $< j$ such that job i is compatible with job j ($f_i < s_j$). This takes $\mathcal{O}\big(\log(n)\big)$ time.

Why do we want to sort them by finish time? By sorting them like that, it's easier to see if a particular job is compatible. It's a convenient thing to do, as we'll be able to say, if we have job 1 that has a finishing time that is before job 2's start time, by sorting them according to the finishing time, we can binary search to figure out which jobs are compatible with the other.



- $p[8] = 1$
- $p[7] = 3$
- $p[2] = 0$
- $Opt[7] = \max(Opt[6],\ w_7 + Opt[3])$

If our optimal strategy has job 7, then our optimum has the weight of 7 plus the optimal value of $p[7] = 3$ which is $Opt[3]$

So really, (Opt: optimal choice given you start at that job)

$$Opt[N] = \max\left(Opt[N-1],\ w_N + Opt\left[p[N]\right]\right)$$

The bellman equation is:

$$Opt[j] = \left(0 \text{ if } j == 0 \text{ else } \max\left(Opt[j-1] + w_j + Opt\left(P[j]\right)\right)\right)$$

We won't solve this algebraically. The bellman equation tells us how to write our code.

What's the cost of executing this?

No dynamic programming? It's going to take 2^n times. We end up creating a binary tree of depth n for our execution, so the leaf count is going to be $\mathcal{O}(2^n)$. At least we're getting an optimal solution – the optimum value is going to be the maximum of the optimal

value we are going to get if we take or don't take the action. Just by construction, the value you get at the next level up has to be optimal.

How do we do better? We store the values we were computing along the way. How many different values do we end up getting. We only have n items. So, there are only n distinct calls we end up making to each of these.

This gets us down to tie $\mathcal{O}\left(n \log(n)\right)$.

The idea is to use the optimal substructure property.

4.2 Top-Down vs. Bottom-up Approach

Assume our problem looks like computing the nth Fibonacci number.

- Top down uses recursive calls, from n.
  - Translates right from the bellman equation.
  - Looks like: $\max\left(Opt(j-1),\ w_j + Opt\left(p[j]\right)\right)$
- Bottom up uses an iterative loop, building the look-up table. A loop would go from 1 to n, where a value i calculated will always depend on values less than i.
  - However, it can waste memory or time. You may not need to fill up the entire look-up table.
  - However, it does not waste stack space. Recursion is slow.
  - It does not require a global array, unlike top-down.
  - No waste in the worst-case scenario.
  - Looks like: For $j = 1$ to n: $M[j] = \max\left(M[j-1],\ w_j + M\left[p[j]\right]\right)$

Neither one of the two approaches is inherently better. Pay attention to the use cases.

In this course, do what you want. Figure out which way of thinking works the best for you.

4.3 Optimal Value vs. Optimal Solution

For the weighted scheduling problem, the optimal value is:

$$OPT(j) = \begin{cases} 0 & j = 0 \\ \max\left(OPT(j-1),\ w_j + OPT\left(p[j]\right)\right) & j > 0 \end{cases}$$

Yet, the optimal solution is, as we need to know what `max` selects:

$$S(j) = \begin{cases} \varnothing & j = 0 \\ S(j-1) & (j > 0) \wedge OPT(j-1) \geq w_j + OPT\left(p[j]\right) \\ \{j\} \cup S\left(p[j]\right) & \text{else} \end{cases}$$

You're going to have to run both. This works for top-down and bottom-up.

4.4 Optimal Substructure Property

Where our optimal values depend on the optimal values for a sliced version of our input.

4.5 Knapsack Problem

Problem:

- n items: item i provides value $v_i > 0$ and weight $w_i > 0$. Items are not sorted.
- Knapsack has weight cap W
- Assume all values are integers
- Maximize $\sum v$ subject to $\sum w \leq W$

Solution:

$OPT(i,\ w)$ = max. value we can pack using only items 1 to i in backpack of capacity w. Compute $OPT(n,\ W)$.

Consider item i:

- If $w_i > w$, can't choose it. Pointless; use $OPT(i-1,\ w)$
- If $w_i \leq w$ (otherwise):
  - If we choose i, the best value is $v_i + OPT\ (i-1,\ w-w_i)$
  - If we didn't, best value is $OPT(i-1,\ w)$

This gives us the Bellman equation:

$$OPT(i,\ w) = \begin{cases} 0 & i = 0 \\ OPT(i-1,\ w) & w_i > w \\ \max\left(OPT(i-1,\ w),\ v_i + OPT\ (i-1,\ w-w_i)\right) & w_i \leq w \end{cases}$$

So, our look-up table is 2D.

4.5.1 Running Time

Considering for $OPT(i,\ w)$, $i \in \{1, \ldots,\ n\}$ and $w \in \{1 \ldots W\}$, there are $\mathcal{O}(nW)$ possible evaluations of OPT. However, each is evaluated at most once using memorization. The total running time is $\mathcal{O}(nW)$. The algorithm scales with the maximum weight, not the no. of bits needed to represent this algorithm. If we assume $W = \text{poly}(n)$, then the algorithm would run in polynomial time.

Is this polynomial? No, it's pseudo-polynomial. The inputs, W, $\mathbf{v}$, $\mathbf{w}$ – the time should be polynomial in $\log W + \sum_{i=1}^{n} \left(\log v_i + \log w_i\right)$

4.6 Single Source Shortest Paths

This is a really important in general for navigation. Let's imagine that we have a **connected** graph that has a series of nodes on it. We would like to get between 2 nodes, one labeled s and t. We'd like to do it using the shortest possible paths. Each path has a distance (weight). We have a directed graph $G = (V,\ E)$.

Negative path weights don't really make sense in this context. If there is a negative path cycle, shortest paths are not even well-defined – you can traverse the cycle arbitrarily many times to get arbitrarily short paths. We'll be removing these cases. So, assume all our weights are positive.

Claim: with no negative cycles, there is always a shortest path from any vertex to any other vertex that is **simple (does not loop).**

- Consider the shortest $s \to t$ path with the fewest edges among all the shortest $s \to t$ paths
- If it has a cycle, removing the cycle creates a path with fewer edges that is no longer the original path.

We're trying to attack the optimal substructure property. I'm trying to argue that the optimum path can be always chosen to have a particular form – simple and no loop. Why are we doing this? When we're coming up with a dynamic programming problem, we want to take the optimum of the score as a function of the problem size. The problem size is now the no. of edges our path ends up including.

The idea is, let's take a look at the optimum solution that we can find to the problem that uses paths up to a fixed length.

Consider a simple shortest $s \to t$ path P: it could be just a simple edge. But if P has more than one edge, consider u with immediately preceeds t in the path.

- If $s \to t$ is shortest, $s \to u$ must be shortest as well and it must use one fewer edge than the $s \to t$.

Let $OPT(t,\ i)$ = length of the shortest path from s to t using at most i edges.

Then:

- Either this path uses at most $i - 1$ edges $\Rightarrow OPT(t,\ i - 1)$
- It uses exactly i edges $\Rightarrow \min_u OPT(u,\ i - 1) + l_{u \to t}$ where u and t has an edge (specifically, $u \to t$).

So, the full bellman equation takes up the following form:

$$OPT(t,\ i) = \begin{cases} 0 & i = 0 \vee t = s \\ \infty & i = 0 \wedge t \neq s \\ \min\{OPT(t,\ i-1), \min_u OPT(u,\ i-1) + l_{u \to t}\} & \text{otherwise} \end{cases}$$

Running time: The node count is n. $O\left(n^2\right)$ calls, each takes $O(n)$ times, so the TOTAL runtime is $O\left(n^3\right)$ where n is the edge count. Pay attention to the bellman equation. Now, we have to take the minimum value over all possible stopovers. Because it's the minimum value, if we have a series of nodes we're interested in, is that we have to look at all nodes.

No, it's not $O(|E|)$ as there are no cycles. The longest path length is n.

This is a polynomial time algorithm. It is not a terribly dumb algorithm. Trying to do a brute-force algorithm can take $n!$ Time.

For your interest, $\widetilde{O}$ is big O but neglecting subdominant polylog functions. Some authors disagree.

4.7 Maximum Length Paths

Can we use a similar DP to compute maximum length paths from s to all other vertices? (If there are positive cycles, we want SIMPLE paths).

The previous algorithm doesn't work for this. Why? If we use a path $s \to t$? The path $s \to u$ might in turn go through t, making the path no longer simple causing a cycle at the end. The maximum-length simple path is NP-hard. This means:

- We have good reason to suspect that a small variant on these existing problems lead to situations where no polynomial time graph exists.

The problems that we can solve in polynomial time is fragile. A small change can make them difficult is a lot. So, how can we tell if a problem is hard, and when will the techniques like dynamic programming won't work.

4.8 Chain Matrix Product

Need to matrix multiply? You better do it.

- Input: Matrices $M_{1\ldots n}$, where M_i is $d_{i-1} \times d_i$ (the matrix multiplication chain is valid)
- Goal: Compute $\prod_{i=1}^{n} M_i$

Matrix multiplication is associative. While $A(BC) = (AB)C$, computing either may take very different time. So, it's not just doing exactly $n - 1$ multiplications in any order. **We have to decide the order of the bracketing.**

How expensive is matrix multiplication? If I need to multiply a $P \times Q$ and $Q \times R$ matrix, the complexity is $\mathcal{O}(PQR)$. For the purposes of this discussion, we'll just consider the $\mathcal{O}\left(n^3\right)$ matrix multiplication for now.

In the case where we do a matrix-vector multiplication, we're multiplying an $N \times N$ by a $N \times 1$ is $\mathcal{O}\left(N^2\right)$. Yet, for a matrix-matrix multiplication, it's $\mathcal{O}\left(N^3\right)$. So, what's our insight:

- $(MM)\mathbf{v}$ has complexity $\mathcal{O}\left(N^3 + N^2\right)$
- $M(M\mathbf{v})$ has complexity $\mathcal{O}\left(2N^2\right)$

Bracketing our matrix multiplication so that our matrix multiplication always happens on the vector gives substantial savings. Can we generalize this? In this case, it was dead simple.

But if all the matrices have different shapes, it's going to be harder. What we would like to do, is take a specification of a bunch of matrices being fed to us, and we'd like to output an algorithm that finds the cheapest way to multiply all of them together. Hence, this acts as a multiplier compiler, which figures out the cost and the optimal way to multiply.

4.8.1 A Small Example

- M_1 is 5×10

- M_2 is 10×100
- M_3 is 100×50

You will notice that

$$(M_1 M_2) M_3 \longrightarrow 5 \cdot 10 \cdot 100 + 5 \cdot 100 \cdot 50 = 30000$$
$$M_1 (M_2 M_3) \longrightarrow 10 \cdot 100 \cdot 50 + 5 \cdot 10 \cdot 50 = 52500 \text{ operations}$$

4.8.2 Why use DP?

DP requires an optimal substructure problem. We want to argue that an optimal solution can be expressed as a series of smaller optimal sub-solutions. At the end of the day, you can collapse matrix multiplications, but what's the best way to collapse them?

$$[X_1 X_2]\,[X_3, X_4, X_5]\,B$$

The way that dynamic programming comes into play, is that if we figure out the optimal cost of a matrix, it will lower the cost of the entire algorithm.

So, how are we doing to do it? We're just going to brute force ourselves every way we can decompose a series of matrix multiplications into two. For instance: from $ABCDE$ to

- $A(BCDE)$
- $(AB)(CDE)$
- $(ABC)(DE)$
- $(ABCD)E$

That's it. Because those are all the options we have for doing the final multiplication, by definition we are choosing the cheapest, and then choosing the cheapest implementation inside the outer one that is the cheapest.

This is how you should approach the DP problem. How can we think about the optimal solution being a function of smaller optimal solutions.

Now, let's formalize it.

4.8.3 Formalizing It + Runtime

$OPT(i,j)$ = min operations required to compute $M_i \cdot \ldots \cdot M_j$, where $1 \leq i \leq j \leq n$.

All that we need to specify is the first and the last index.

Then, we'll try to write an optimal solution for $M_1 \cdot M_2 \cdot M_3 \cdot M_4 \cdot M_5$. Then, $OPT(1,\ n)$ would be the whole optimum expression.

$$OPT(i,\ j) = \begin{cases} 0 & i = j \\ \min_{k\ :k\in[i,j)} OPT(i,k) + OPT(k+1,j) + d_{i-1}d_k d_j & i < j \end{cases}$$

To outline the expression better:

$$\underset{OPT(i,\ k)}{[X_i \ldots X_k]} \underset{OPT(k+1,j)}{\left[X_{k+1} \ldots X_j\right]}$$

The cost of then multiplying the two together is $d_{i-1}d_k d_j$ (where d_{i-1} is the height of X_i)

So, what's the runtime?

- Running over all decompositions, `min` takes $\mathcal{O}(n)$. Assuming recursive runtimes are constant, the non-recursive runtime is just $\mathcal{O}(n)$.
- We have up to $\mathcal{O}\left(n^2\right)$, which is the size of the memorization array we're making.
- Our total runtime is $\mathcal{O}\left(n^2\right) \times \mathcal{O}(n) = \mathcal{O}\left(n^3\right)$.

4.9 Edit Distance

List indices are inclusive here.

Early spellcheckers use this exclusively. The edit distance problem is: how similar are the two strings?

$$X = x_1, \ldots, x_m \quad Y = y_1, \ldots, y_m$$

Where **similarity** is how many symbols I need to **delete** or **replace** to have one match the other.

For example, two strings:

```
ocurrance
occurrence
```

This would take 6 replacements and 1 deletion.

Formalizing the problem:

- Strings X of length m and Y of length n
- $d(a)$: cost of deleting symbol a
- $r(a,\ b)$: cost of replacing symbol with a with b
  - Assume $r(a,b) = r(b,a)$ and $r(a,a) = 0\ \forall a, b$

Goal: compute the minimum total cost for matching the two strings. What's the optimal structure?

- **Want to delete/replace at one end and recurse**
- Figure out distance of $X[:m-1]$ and $Y[:n-1]$ and then determine cost of $X[m]$ and $Y[m]$ (last index)

How are we going to deal with this? So, the bellman equation is, where $E[i,j]$ is the edit distance between $x_1, \ldots, x_i$ and $y_1 \ldots y_i$:

$$E[i,j] = \begin{cases} 0 & i = j = 0 \\ B & i = 0 \land j > 0 \\ A & i > 0 \land j = 0 \\ \min(A,B,C) & \text{otherwise} \end{cases}$$

Where:

$$A = d(x_i) + E[i-1, j]$$
$$B = d(y_j) + E[i, j-1]$$
$$C = r(x_i, y_j) + E[i-1, j-1]$$

What does each variable mean?

- A: delete from $X[:i]$
- B: delete last from $Y[:j]$
- C: replace last of $X[:i]$ with last of $Y[:j]$, or the other way around. It doesn't matter.

So, what's the complexity?

- $\mathcal{O}(nm)$ time
- $\mathcal{O}(nm)$ space

There is no idea of insertion as we can pad the string with "invisible" characters.

4.10 The Traveling Salesman Problem

Given a network of locations, figure out the way I can visit each of them with the shortest path.

The general problem:

- Given a graph in n vertices such that each edge has positive weight w_{ij}, find a path P such that:
  - First vertex is 1
  - Last vertex is 1 (this is a cycle)
  - Each vertex except 1 appears once in the path
  - $\sum_{i=1}^{n+1} w_{P_i P_{i+1}}$ is minimal

How efficient do we expect this problem to be? The brute force method would take $\mathcal{O}(n!)$. The question, is can we improve this with DP? If we're assuming that everything is a cycle, start with our start node, jump to the rest of them, and consider the fastest way we can travel among them.

How are we going to do it?

Input:

- **Complete** (all vertices are connected to each other by one edge) directed graph $G = (V, E)$
- $d_{i,j}$ = distance from node i to node j

Output:

- Min. distance which needs to be travelled to the start from some node v, visit every other node exactly once, and come back to v. That is, the minimum cost of a Hamiltonian cycle.

The correct starting point doesn't matter.

Approach:

- Let's start at node $v_1 = 1$.
- We want to visit the other nodes in some order, say $v_2, \ldots v_n$
- Total distance is $\sum_{i=1}^{n} v_i v_{(i+1)}$, which we want to minimize

With DP, we can do a little bit better than $n!$ Solutions needed. Consider v_n (the last node before returning to $v_1 = 1$.

If $v_n = c$, find the optimal order of visiting nodes $\{2, \ldots, n\}$ that ends at c. We need to keep track of the subset of nodes to be visited and the end node.

$OPT[S, c]$ = minimum total distance when starting at 1, visiting each node in S exactly once, and ending at $c \in S$

Answer to the original problem, which has to go back to 1:

$$\min_{c \in S} OPT[S,\ c] + d_{c,1} \qquad S = \{2, \ldots,\ n\}$$

To compute $OPT[S, c]$, we can condition over the vertex visited right before c in the optimal trip. So, the bellman equation is:

$$OPT[S,\ c] = \min_{m \in S \setminus \{c\}} \left(OPT\left[S \setminus \{c\},\ m\right] + d_{m,c} \right)$$

And the final solution is just:

$$\min_{c \in \{2, \ldots, n\}} \left(OPT\left[\{2, \ldots, n\},\ c\right] + d_{c,1} \right)$$

We'll have to do a total of $\mathcal{O}\left(n \cdot 2^n\right)$ calls. It takes $\mathcal{O}(n)$ time per call, so the total runtime is $\mathcal{O}\left(n^2 \cdot 2^n\right)$. This is much better than the naïve solution which has $\left(\frac{n}{e}\right)^n$ runtime.

This problem is NP-complete. Some problems have fundamental limits to how good algorithms can go.

5 Network Flow

Network flow deals with the following type of problem:

Input:

- A directed graph $G = (V, E)$
- Edge capacities $c : E \longrightarrow \mathbb{R}_{\geq 0}$
- Source node s, target node t

Output:

- Maximum flow from s to t

Sorry, greedy does not work well for this. What's a flow?

5.1 What's a Flow?

An assignment of weights to individual edges of a graph, which denotes how much material can flow through the pipe – that is $f(e)$, on edge e.

Here are the flow axioms:

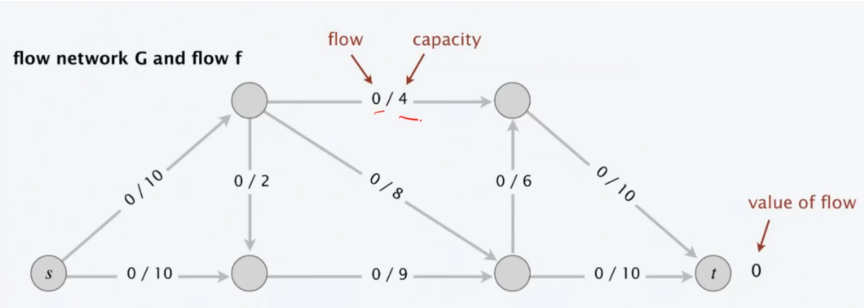
- Actual flow through a pipe can't exceed its capacity
- Flow is conserved: for each node: $\sum$ what goes in = $\sum$ what comes out

5.2 Why Doesn't Greedy Work?

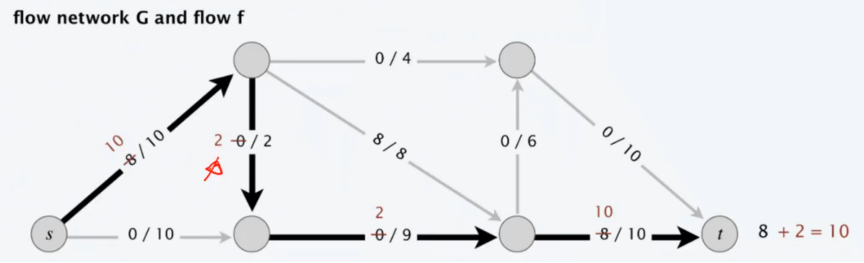
Greedy doesn't let us backtrack from bad decision. What else works? We'll need to be able to reverse bad decisions here.

Anyway, here's the greedy approach:

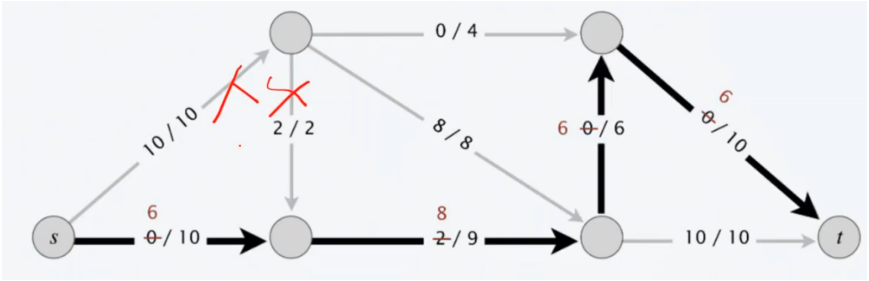
Let's imagine we label our graph with flow and capacity.



A natural greedy way is to pick up any path and figure out the maximum matter we can shove through there. In this case, the **bottleneck** is the maximum matter we can shove through. We can no longer shove through the bottleneck.



And finally:



This is unfortunately not the optimal solution.

Greedy just keeps making paths.

To recap, this is our greedy algorithm:

- Start from zero flow ($f(e) = 0$ for each e)
- While there exists an $s - t$ path P in G such that $f(e) < c(e)$ for each $e \in P$
  - Find any path P
  - Compute $\Delta = \min_{e \in P} (c(e) - f(e))$ (the bottleneck)
  - Increase the flow on each edge $e \in P$ by Δ

We made a sub-optimal allocation mid-way. This is a problem with greedy strategies. We don't get the ability to move back or correct our problems.

5.3 Residual Graph

Suppose the current flow is f. Define the residual graph G_f of flow f. G_f has the same vertices as G, but:

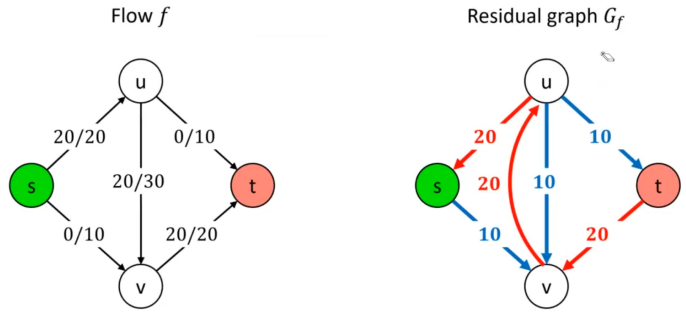
- For each edge $e = (u, v)$ in G, G_f has at most two edges
  - Forward edge: $e = (u, v)$ with capacity $c(e) - f(e)$
    - We can send this much additional flow on e
    - $c(e)$ is the original capacity of that edge
    - $c(e) - f(e)$ is the residual capacity of the pipe

- Reverse edge $e^{rev} = (v, u)$ with capacity $f(e)$
  - The maximum reverse flow we can send is the maximum amount by which we can reduce flow on e, which is $f(e)$
- We only really add edges of capacity > 0

All the information about the flow and the capacity are stored. Now, we have the flow left and the flow in total we can push through.

For any edge in the following diagrams that have 0 capacity, we don't draw that edge.

Here's an example:



The forward edge in the residual graph is the residual capacity. It is $c(s, v) - f(s, v) =$ leftover that can flow through.

The backwards edge represents how much flow is going through it.

5.4 Augmenting Paths

An augmentation involves finding a path from s to t and adding the bottleneck to each forward edge. Then, recompute the backward edges. I don't care which path I find as of right now, just find **a** path.

Let P be an $s - t$ path in the residual graph G_f. In a residual path, you wouldn't care whether you're walking the forward and reverse direction. The new graph just generates itself.

This lets us undo our mess-ups. Can you figure out why? When you flow stuff through a backwards edge, what you're doing is cancelling what flows into the tip of the backwards edge.

How does augmenting work? We augment flow f by sending bottleneck(P, f) units of flow along path P. To send x units of flow along P, would mean:

- For each forward edge $e \in P$, increase flow on e by x
- For each reverse edge $e^{rev} \in P$, decrease the flow on e by x

For each edge, the sum of the forwards and backwards edge will always equal to the same value.

Why is this so neat? When we cannot augment anymore, we can prove that what we have is optimal. That is, you cannot have anything more flow out of s, the start node.

5.5 Augmenting Always Satisfies Capacity Constraints

Argue that the new flow after augmenting is a valid flow. Well:

- Increasing flow on e, we can do by at most the capacity of the forward edge e in G_f, which is $c(e) - f(e)$, so the new flow can be at most $f(e) + (c(e) - f(e)) = c(e)$
- Decreasing flow on e, we can do most by the capacity of the reverse edge e^{rev} in G_f, which is $f(e)$, so new flow is at least $f(e) - f(e) = 0$
- Meaning flow will always be $\in [0, c(e)]$

An alternative argument is by saying that for each edge we pass through (directions have to make sense), we note that:

$$\text{bottleneck} \leq \text{minimum residual edge capacity}$$

So no edge can be subtracted to be below 0.

For backward edges, our argument changes to:

$$\text{bottleneck} \leq \text{min flow down a reverse edge}$$

5.6 Augmenting Always Conserves Flow

For each node, what goes in, must come out. How do we prove this?

Say we have a path that has a series of forward edges and backwards edges. The idea is that, let's assume that x is the bottleneck. If x is the bottleneck, what we want to do for all edges is add x units of flow each time we pass a forward edge and subtract x units of flow each time we pass a backward edge.

For instance:

- This assumes that all edges are forwards edges; backwards edges would look like $\overleftarrow{-A}$
- $\overrightarrow{A} \circ$ means net flow is $+A$
- $\circ \overrightarrow{A}$ means that net flow is $-A$

Each node on the path, except for s and t, has exactly **two** incident edges on the path.

- If both edges go into the node or leave the node, then the net flow for that node is increased on both or decreased on both, to that node.
  - $\overrightarrow{x} \circ \overleftarrow{-x} \quad x + (-x) = 0$
  - $\overleftarrow{-x} \circ \overrightarrow{x} \quad -(-x) - x = 0$
- If there is one forward and one reverse node, then flow is increased on the incoming node but that same flow has to be decreased on the other side
  - $\overrightarrow{x} \circ \overrightarrow{x} \quad x - x = 0$
  - $\overleftarrow{-x} \circ \overleftarrow{-x} \quad -(-x) - (-x) = 0$

Since net flow remains 0, the new flow is a valid flow.

5.7 Ford-Fulkerson Algorithm

Here it is:

```
def MaxFlow(G) -> Flow:
    set f(e) = 0 for all e in G
    # while there is an s-t path in G_f
    while P = FindPath(s, t, Residual(G, F)) != None:
        f = Augment(f, P)
        UpdateResidual(G, f)
    return f
```

Will this algorithm halt?

5.7.1 Runtime – Number of Augmentations

- At every step, flow and capacities remain integers
- For path P in G_f, bottleneck$(P, f) > 0 \Rightarrow$ bottleneck$(P, f) \geq 1$
- Each augmentation increases flow by at least 1
- Max flow, hence max no. of augmentations (they are equal) is at most $C = \sum_{e \text{ leaving } s} c(e)$
  - Sum of the capacities of all the edges leaving the sources

5.7.2 Runtime – Time to perform an augmentation

- G_f has n vertices and at most $2m$ edges (where m is the existing no. of edges in the non-residual non-augmented graph)
- Finding P and computing bottleneck(P, f), and updating G_f has $\mathcal{O}(m+n)$ time
  - It takes $\mathcal{O}(m)$ time to check and update all edges
    - By BFS/DFS
  - Updating all nodes, taking $\mathcal{O}(n)$

So, the total time is $\mathcal{O}((m+n)C)$

This is pseudo-polynomial time. The value of C can be exponentially large in the input length (the no. of bits required to write down the edge capacities). So, this can take an

extremely long time under some circumstances. With irrational capacities, you can show that this algorithm could loop forever (with rationals, we can use LCMs). Can we convert this to polynomial time?

5.8 Can we convert this to polynomial time?

Not if we choose an arbitrary path in G_f at each step. In the path below, we might end up repeatedly sending 1 unit of flow across $a \to b$ then reversing it. It takes X steps, which can be exponential in the input length.

5.9 Types of Polynomial Running Times

We have two quantities: number of integers provided as input and total number of bits provided as input

- Strongly polynomial
  - Running time polynomial in number of bits
  - Operation count polynomial in integers but not dependent on bits
- Weakly polynomial
  - Number of operations polynomial in number of bits
  - Example: scales logarithmically with an input integer
- Pseudo-polynomial
  - Number of operations is polynomial in no. of bits if input was written in unary (in other words, polynomial in the values of the input integers)
  - For example, polynomial to the value of type `int` you pass in

5.10 How to Achieve Polynomial Time

To achieve a weakly polynomial time, find the maximum bottleneck capacity augmenting path. This results in $\mathcal{O}\left(m^2 \log(C)\right)$ operations. This is weakly polynomial time.

We can use BFS, which is how the Edmonds-Karp algorithm goes. It runs in $\mathcal{O}\left(nm^2\right)$ operations.

Would you prefer a pseudo-polynomial with better scaling of m and n, or one with worser scaling but with no scaling on C?

5.11 Ford-Fulkerson Correctness

Notation:

For a node u, $f^{out}(u)$ and $f^{in}(u)$ is the total flow out of and into u respectively. This can apply for a set of nodes as well: $f^{out}(X)$ and $f^{in}(X)$.

$$v(f) = f^{out}(s) = f^{in}(t) = \text{value of the flow for the graph}$$

So, we can rewrite our constraints:

- Capacity: $0 \leq f(e) \leq c(e)$
- Flow conservation: $f^{out}(u) = f^{in}(u),\ \forall u \neq s, t$

5.11.1 Graph Cuts

$(A,\ B)$ is an $s-t$ cut if a partition of vertex set V (such as $A \cup B = V$, $A \cap B = \varnothing$ with $s \in A$ and $t \in B$)

It's capacity, denoted $cap(A,\ B)$, is the **sum of capacities of edges leaving** A

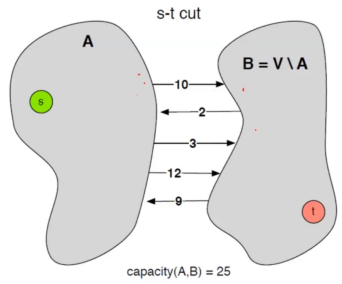


Image above: 10 + 3 + 12 = 25

Cuts aren't unique.

Why do we care about this notion?

Theorem: for any flow f and any $s-t$ cut $(A,\ B)$:

$$v(f) = f^{out}(A) - f^{in}(A)$$

Things drained to t is stuff coming in minus stuff coming out.

Theorem: For any flow f and any $s-t$ cut $(A,\ B)$, $v(f) \leq cap(A,\ B)$.

This seems quite obvious as t can't drain more than what I could *possibly* have entering the cut that has t, but I'll lay it out:

$$\begin{aligned} v(f) &= f^{out}(A) - f^{in}(A) \\ &\leq f^{out}(A) \\ &= \sum_{e \text{ leaving } A} f(e) \\ &\leq \sum_{e \text{ leaving } A} c(e) \\ &= cap(A,\ B) \end{aligned}$$

Why do we care? Now we know that $v(f) \leq cap(A,\ B)$, then

$$\max_f v(f) \leq \min_{A,\ B} cap(A,\ B)$$

Which in other words, means **max value of any flow ≤ min capacity of any s – t cut (which follows from previous)**

We will now prove that the value of the flow generated by Ford-Fulkerson = capacity of *some* cut.

Implications:

- Max flow = min. cut
- Ford-Fulkerson generates max flows

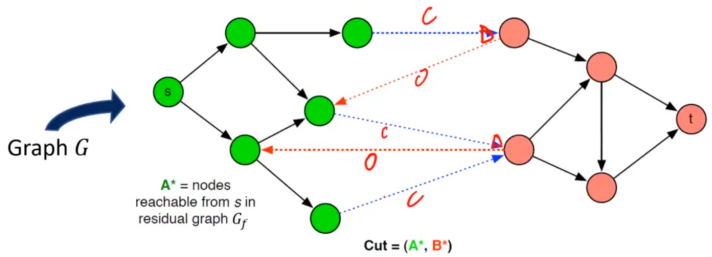
Theorem: Ford-Fulkerson finds maximum flow.

Proof:

- f: flow returned by FF
- A^* = nodes reachable from s in G_f
- $B^* = V \setminus A^*$

Claim: (A^*, B^*) is a valid cut. $s \in A^*$, by definition. $t \in B^*$, because when FF terminates, there are no s – t paths in G_f as nothing can come out of s, so $t \notin A^*$

- Each blue edge (u, v) must be saturated
  - Otherwise G_f would have its forward edge (u, v) and then $v \in A^*$
- Each red edge (v, u) must have zero flow
  - Otherwise G_f would have its reverse edge (u, v) and then $v \in A^*$



Saturated means edge at its cap

So $v(f) = f^{out}(A^*) - f^{in}(A^*) = cap(A^*, B^*)$

Max flow-min cut theorem: in any graph, the value of the max. flow is equal to the capacity of the minimum cut.

Our proof already gives an algorithm to find a min cut:

- Run FF to find a max flow f
- Construct tis residual graph G_f
- Let A^* = set of all nodes reachable from s in G_f
  - Easy to compute with BFS
- Then, $\left(A^*,\ V \setminus A^*\right)$ is a min cut (cut that minimizes $cap(A,\ B)$).