MAT235 Master Document

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NOTE: I cannot guarantee the quality of my notes for this course. On the Github Repo hosting these notes, the courses with a star symbol tend to be higher quality.

1 Parametric equations

A function is something that takes in an input and returns an output. The set of acceptable inputs is the domain, and all the possible outputs based on the domain is the image. We can write a function as:

$$f\colon \mathbb{R} \to \mathbb{R}$$

To plot a single-variable, we need the y-axis, and we need the x-axis. We can use the vertical line test on a plot to test if something is a function or not.

Multi-dimensional maps include:

$$\begin{aligned}&f : \mathbb{R}^2 \to \mathbb{R}\\&(x,\ y) \mapsto f(x,\ y) \in \mathbb{R}\end{aligned}$$

For some reason, you can slice through 3D shapes, and you can figure out the formula.

1.1 Mapping curves

Do we have a curve? Then, we can map it as:

$$x = f(t),\ y = g(t)$$

Where $f\colon \mathbb{R} \mapsto \mathbb{R}$, $g\colon \mathbb{R} \mapsto \mathbb{R}$. t is a parameter.

Example 1: sketch the curve defined by the following equations:

$$\begin{aligned} x &= t^2 - 3 \\ y &= t + 2 \end{aligned}$$

From $-3 \leq t \leq 3$.

How do we draw this curve? Draw a table.

t	x	y
-3	6	-1
-2	1	0
-1	-2	1
0	-3	2
1	-2	3
2	1	4
3	6	5

Feel free to extend or make this table contain more information whatsoever. Now, we want to draw the curve for these points, and connect the dots. How do we check the answer? Rewrite the formula you have, but in cartesian form.

$$\begin{aligned} &y = t + 2 \Rightarrow t = y - 2 \\ &x = t^2 - 3 = (y-2)^2 - 3 = y^2 - 4y + 4 - 3 \\ &\quad = y^2 - 4y + 1 = x \end{aligned}$$

Use this with Desmos to check your answer. This is how you move from cartesian form to parametric. Maybe cartesian form is more convenient to imagine the curve for an equation.

Example 2 (harder): What curve is represented by the following parametric equations?

$$x = \cos(t),\ y = \sin(t),\ 0 \le t \le 2\pi$$

This makes a circle, as we know that $\cos^2(t) + \sin^2(t) = 1$:

$$x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1$$

But if we ever want to be precise:

t	$x = \cos(t)$	$y = \sin(t)$
0	1	0
$\frac{1}{2}\pi$	0	1
π	-1	0
$\frac{3}{2}\pi$	0	-1

Let's switch up the equation a bit: $0 \le t \le 2\pi$

$$x = \cos(2t),\ y = \sin\_2t$$

So:

$$x^2 + y^2 = \cos^2(2t) + \sin^2(2t) = 1$$

Do we get the same curve? The difference between this one and the previous one is that the circle is stacked twice.

What happens when we transform the circle by $(x_0,\ y_0)$ and make it have a radius of r? What is the parametric formula?

$$x = x_0 + r\cos(t)$$
$$y = y_0 + r\sin(t)$$

Let's bring up the formula for a circle on a cartesian plane, with center $(\alpha,\ \beta)$ and radius r:

$$(x - \alpha)^2 + (y - \beta)^2 = r^2$$

Do some substitution with the parametric equations, where we say $(\alpha,\ \beta) = (x_0,\ y_0)$:

$$
\begin{aligned}
(x_0 + r\cos(t) - \alpha)^2 + (y_0 + r\sin(t) - \beta)^2 &= r^2 \\
(r\cos(t))^2 + (r\sin(t))^2 &= r^2 \\
r^2\cos^2(t) + r^2\sin^2(t) &= r^2 \\
r^2\left(\cos^2(t) + \sin^2(t)\right) &= r^2 \\
r^2 &= r^2
\end{aligned}
$$

Left side is the right side.

1.2 Calculus with parametric

To write the tangent line, you need the point, and the slope. The formula for a tangent line is $y = m(x - x_0) + y_0$, which is $\frac{dy}{dx}\,|_{x_0,\, y_0}$.

Suppose that we have two functions $f\colon \mathbb{R} \to \mathbb{R}$, $g\colon \mathbb{R} \to \mathbb{R}$, and that they are differentiable. Suppose that we have a parametric curve:

$$x = f(t),\ y = g(t)$$

$$
\begin{gathered}
\frac{dx}{dt} \neq 0 \text{ and} \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \Rightarrow \\
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}
\end{gathered}
$$

The advantage of this is that we can easily compute $\frac{dy}{dt}$, because y is a function of t. We do not need to eliminate t. The same thing applies to $\frac{dx}{dt}$.

- A curve has a horizontal tangent when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$.
- A curve has a vertical tangent line when $\frac{dx}{dt} = 0$, given that $\frac{dy}{dt} \neq 0$.

The second derivative is denoted by

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

(Grab the equation for the first derivative – that is how this was derived.)

Suppose we have:

$$x = t^2,\ y = t^3 - 3t$$

What is the tangent line for $t = 2$? Use the first derivative formula.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t}$$

Substitute $t = 2$:

$$\frac{dy}{dx} = \frac{3(2)^2 - 3}{2(2)} = \frac{3}{2}$$

1.3 Areas

You have a graph $y = f(x)$, and you want the area underneath the curve from a to b. That is, $A = \int_a^b y dx$ (and let us assign $g(t) = y$ and $f(t) = x$ for no reason).

So:

$$\begin{aligned} A = \int_a^b y dx &= \int g(t) d\left(f(t)\right) \\ &= \int_\alpha^\beta g(t) f'(t) dt \\ \int_a^b y(t) x'(t) dt \end{aligned}$$

Where $a = f(\alpha),\ b = f(\beta)$. Good luck finding function inverses, or at least base the locations off t.

Example: Find the area of the region enclosed by the loop of the following curve (I swear the curve makes a loop):

$$
\begin{aligned}
x &= 1-t^2\\
y &= t-t^3\\
x &= 0\\
\Rightarrow 1-t^2 &= 0\\
\Rightarrow t &= \pm 1\\
\Rightarrow y &= 0
\end{aligned}
$$

At $t=1$, then $(x,y)=(0,\ 0)$. For $t=-1$, $(x,\ y)=(0,\ 0)$. Unfortunately, the best way to find out about this is by finding two distinct t-values: $t_1,\ t_2$ such that $f(t_1)=f(t_2)$. Good thing is that this function happens to be symmetrical with respect to x.

$$A=\int_{-1}^{1} g(t)f'(t)dt$$

1.4 Arc length

Where S is the arc length, and α is the value of t where the arc starts, and β is t where the parametric point is where the arc ends.

$$S=\int_{\alpha}^{\beta}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt$$

How it is derived: Length of curve from a function, from $(a,\ f(a))$ to $(b,\ f(b))$:

$$S=\int_{a}^{b}\sqrt{1+f'(x)}dx$$

To derive:

$$
\begin{aligned}
&\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\
&= \int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\
&= \int_\alpha^\beta \sqrt{1+\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\left(\frac{dx}{dt}\right)dt \\
&= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt
\end{aligned}
$$

Example. Find the length of the following parametric curves:

$$
\begin{aligned}
&x = 1+3t^2 \\
&y = 4+2t^3 \\
&0 \le t \le 1
\end{aligned}
$$

Solution:

$$
\begin{aligned}
S &= \int_0^1 \sqrt{(6t)^2+(6t^2)^2}\,dt \\
&= \int_0^1 \sqrt{36t^2+36t^4}\,dt
\end{aligned}
$$

2 Polar coordinates

A coordinate in the polar coordinate system can be described as

$$P(r,\ \theta)$$

This means “the polar coordinates of P.” Omit the P if you want.

2.1 Conventions

- Positive angles are measured counterclockwise

- Negative angles are measured in the clockwise direction
- If the point is placed at the origin, then $r = 0$ and $(0,\ \theta)$ represents the pole for any value of θ. In this case, what value θ takes will not matter.
- The points $(-r,\ \theta)$ and $(r,\ \theta)$ lie on the same line through the origin and at the same distance $|r|$ from the origin, but on the opposite sides of O (maybe a π angle difference). Moreover, $(r,\theta) = (-r,\ \theta \pm \pi)$

Polar to cartesian:

$$(r,\ \theta) = (r\cos(\theta),\ r\sin(\theta))$$

Cartesian to polar:

$$(x,\ y) = \left(\sqrt{x^2+y^2},\ \theta\right)$$

Where $\tan(\theta) = \frac{y}{x}$

When deciding the angle, it is better to visualize what quadrant $(x,\ y)$ is in, as the ratio $\frac{y}{x}$ itself may not be able to tell which quadrant is in unless you look at x and y as their own components.

2.2 Graphing polars

The graph of a polar equation $r = f(\theta)$ consists of all points that have at least once polar representations whose coordinates satisfy the equation.

Example: Polar equation $r = 2$? We need to find all points $(r,\ \theta)$ such that $r = 2$ – which happens to be a full 2-radius circle. Its equation is $x^2 + y^2 = r^2 = 4$.

Polar curve $\theta = \frac{\pi}{6}$. Solution:

$$\left\{(r,\theta) \mid \theta = \frac{\pi}{6}\right\}$$

Looks like $y = \arctan(\theta)$.

For a full equation, $r\cos(\theta)$: Make a table $\theta,\ r$ and connect the dots.

2.3 Calculus with polars – area of a slice of a circle

If you have a circle with the area $A = \pi r^2$, then the area of a section of the circle is $A = \frac{1}{2}r^2\theta$, where θ is the angle of the section of the circle you're taking.

If your circle isn't that much of a circle but something that is done through a polar curve, the area from angle a to b is:

$$
\begin{aligned}
A &= \int_a^b \frac{1}{2}r^2 d\theta \\
A &= \int_a^b \frac{1}{2}\left(f(\theta)\right)^2 d\theta \\
A &\approx \lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{2}\left(f\left(\theta_j^*\right)\right)^2\Delta\theta
\end{aligned}
$$

(You don't need to know the bottom one.)

Example: Find the area region enclosed by one loop (from one instance of $r = 0$ to the next instance of $r = 0$) of the curve $r = 4\cos(3\theta)$.

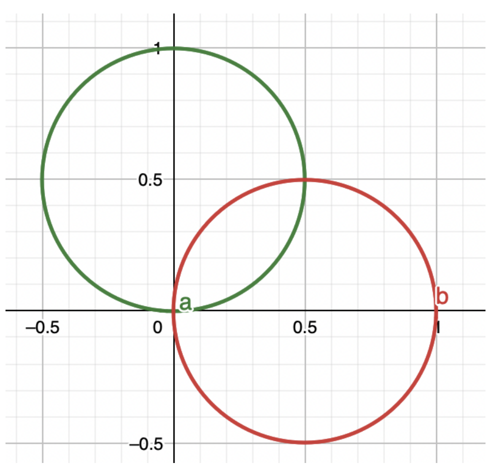
Here, $r = 0 \Rightarrow \cos 3\theta = 0 \Rightarrow 3\theta = \frac{\pi}{2} + n\pi,\ n \in \mathbb{Z} \Rightarrow \theta = \frac{\pi}{6} + \frac{n}{3}\pi$

We can choose θ to be $-\frac{\pi}{6}, \frac{\pi}{6}$

This means:

$$
\begin{aligned}
A &= \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (4\cos(3\theta))^2\, d\theta \\
&= 8\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2(3\theta) d\theta \\
&= 8\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1+\cos(6\theta)}{2} d\theta \\
&= 8\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}+\frac{\cos(6\theta)}{2} d\theta \\
&= 8\left(\frac{1}{2}\theta+\frac{1}{12}\cdot\sin(6\theta)\right)\,|_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \\
&= 8\left(\frac{1}{2}\cdot\frac{\pi}{6}+\frac{1}{12}\sin(\pi)+\frac{1}{2}\cdot\frac{\pi}{6}-\frac{1}{12}\sin(-\pi)\right) \\
&= 8\left(\frac{1}{2}\cdot\frac{\pi}{6}+\frac{1}{12}\sin(\pi)+\frac{1}{2}\cdot\frac{\pi}{6}+\frac{1}{12}\sin(\pi)\right) \\
&= 8\left(\frac{\pi}{6}+\frac{1}{6}\sin(\theta)\right) \\
&= \frac{4}{3}\pi+\frac{1}{6}
\end{aligned}
$$

2.3.1 Inside two circles



Find the area of the region that lies inside the curves:

$$A = \int_a^b \frac{1}{2} f(\theta)^2 d\theta$$

$r = \sin(\theta)$ and $r = \cos(\theta)$, $\theta = \frac{\pi}{4}$. This means the area is:

$$\frac{1}{2}\int_{0}^{\frac{\pi}{4}} \sin^2(\theta)d\theta + \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2(\theta)d\theta$$

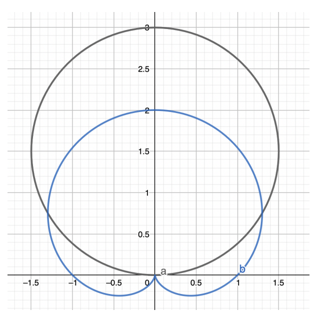
Visually, they look symmetrical, so we could say the answer is:

$$\int \sin^2(\theta)d\theta = \left(\frac{1}{2} - \frac{\cos(2\theta)}{4}\right)\Big|_0^{\frac{\pi}{4}}$$
$$= \left(\frac{1}{2} - \frac{\cos\left(\frac{\pi}{2}\right)}{4}\right) - \left(\frac{1}{2} - \frac{\cos(0)}{4}\right) = \frac{1}{2} - \frac{1}{2} + \frac{1}{4} = \frac{1}{4}$$

That seems to be the area – for now.

Another example:

Find the area of the region that lies **inside** the circle $r = 3\sin(\theta)$ ad **outside** the cardioid $r = 1 + \sin(\theta)$ (that's the shape)!!!



Area of the entire circle, from $\frac{\pi}{6}$ to $\frac{5\pi}{6}$:

$$\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 9\sin^2(\theta)d\theta$$

Area of the cardioid, from $\frac{\pi}{6}$ to $\frac{5\pi}{6}$:

$$\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1 + \sin(\theta))^2 d\theta$$

Subtract the top from the bottom, and you have the answer. The final answer is π, and it's up to you to figure it out.

In general, when you want to find the area between two curves, here's the formula:

$$\forall a \leq \theta \leq b,\ g(\theta) \geq f(\theta) \Rightarrow \text{Area between} = \frac{1}{2}\int_a^b (g(\theta))^2 - (f(\theta))^2\, d\theta$$

2.4 Tangents on a polar curve

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

To find the vertical/horizontal lines, we can use the same method that we use for parametric curves. This is when $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$ to find any vertical tangents; swap the $=$ and $\neq$ if you want vertical tangent lines.

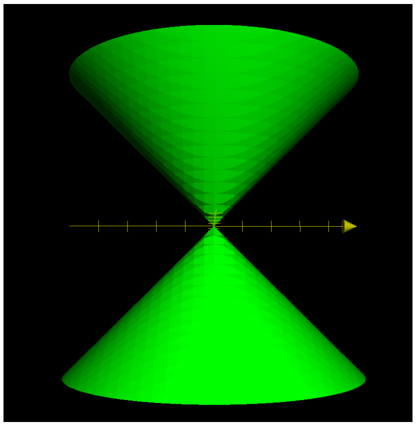
Alternatively, use the fact that

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

2.5 Arc length of a parametric curve

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

3 3D shapes and Conic sections



If I take a slice from it, what will I get? Here is the list of conic sections

- Circle
- Ellipse
- Hyperbola
- Parabola

3.1 Parabola

Definition 3.1 (Parabola)**.** A parabola is the set of points in a plane that are equidistant from a fixed-point F, called the focus, and a fixed line, called the direction.

The path an object travels when thrown creates a parabola. This type of curve has received a lot of attention.

The generic formula for a parabola with the vertex at the origin:

$$
\begin{aligned}
|y+p| = \left|\overline{pF}\right| &= \sqrt{x^2+(y-p)^2} \\
\Rightarrow (y+p)^2 &= x^2+(y-p)^2 \\
\Rightarrow y^2+2py+p^2 &= x^2+y^2-2py+p^2 \\
\Rightarrow x^2 &= 4py
\end{aligned}
$$

A parabola with a focus of $(0,\ p)$ is represented by this equation: $x^2 = 4py$, upwards or downwards. Swap x and y if the parabola rotates by 90°.

Instead of using $4p$, we can instead use a. There are four kinds of parabolas:

1. $y = ax^2,\ a > 0.$ U-shaped parabola.
2. $y = ax^2,\ a < 0.$ ∩-shaped parabola.
3. $x = ay^2,\ a > 0.$ C-shaped parabola.
4. $x = ay^2,\ a < 0.$ ⊃-shaped parabola.

Here, the vertex is always the origin.

A parabola with a vertex at $(x_0,\ y_0)$:

$$(x-x_0)^2 = 4p\,(y-y_0)$$

Would be the formula of the parabola.

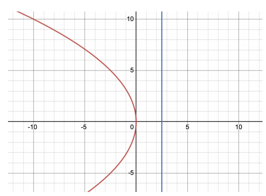
Example 3.1. Find the focus and directrix of this parabola. The focus is $(x_0,\ y_0+p)$ and the directrix is $y=-p$:

$$y^2+10x=0$$

The solution is as follows:

$$\begin{aligned} y^2 &= -10x \\ 4p &= -10 \\ p &= -\frac{5}{2} \\ x &= \frac{5}{2} \end{aligned}$$

The parabola is $\supset$-shaped.



Example 3.2. Find the focus and directrix of this parabola: $x^2-6x+3y=18$

The solution is as follows:

$$\begin{aligned} x^2-6x+3y+18 &= 0 \\ (x-3)^2-9+3y+18 &= 0 \\ (x-3)^2+3y &= -9 \\ (x-3)^2 &= -3y-9 \\ (x-3)^2 &= -3(y+3) \end{aligned}$$

The vertex of this parabola is $(3,\ -3)$.

$$\begin{aligned} -3 &= 4p \\ p &= -\frac{3}{4} \end{aligned}$$

The directrix is at $y=\frac{3}{4}$. The focus is at $\left(3,\ -3-\frac{3}{4}\right)=\left(3,\ -\frac{15}{4}\right)$.

3.2 Ellipse

An ellipse is the set of points in a plane that the sum of whose distances from two fixed points $f_1,\ f_2$ is a constant.

Instead of one focus, ellipses have two foci.

$$\begin{aligned}
\left|\overline{pf_1}\right| + \left|\overline{pf_2}\right| &= 2a > 0 \\
\Rightarrow \sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} &= 2a \\
\Rightarrow \sqrt{(x+c)^2+y^2} &= 2a - \sqrt{(x-c)^2+y^2} \\
\Rightarrow x^2+2cx+c^2+y^2 &= 4a^2+(x-c)^2+y^2-4a\sqrt{(n-c)^2+y^2} \\
&\Rightarrow \dots \\
\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2} &= 1
\end{aligned}$$

A horizontal ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1,\ a \geq b > 0$$

Has foci $(\pm c,\ 0)$ where $c^2 = a^2 - b^2$, and vertices $(\pm a,\ 0)$.

A vertical ellipse:

$$\frac{x^2}{b^2}+\frac{y^2}{a^2} = 1,\ a \geq b > 0$$

Note that $a,\ b$ are swapped.

There is a simpler way to figure out an ellipse is vertical or horizontal: take $x,\ y = 0$ and figure out if $a,\ b$ has a higher value.

If $a = b$, then you have a circle. It has one focus.

The standard form of an equation of an ellipse with center $(x_0,\ y_0)$ is:

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2} = 1$$

Example: sketch the graph of the following ellipses and locate the foci.

First:

$$
\begin{aligned}
9x^2 + 16y^2 &= 144 \\
&\frac{9x^2}{144} + \frac{16y^2}{144} + \frac{144}{144} \\
\frac{x^2}{16} + \frac{y^2}{9} &= 1
\end{aligned}
$$

This is the standard form of an ellipse.

If we take $y = 0 \Rightarrow \frac{x^2}{16} = 1 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4,\ (4,\ 0),\ (-4, 0)$

If we take $x = 0 \Rightarrow \ldots \Rightarrow \frac{y^2}{9} = 1 \Rightarrow y = \pm 3,\ (0,\ 3),\ (0,\ -3)$

Therefore, our ellipse can be drawn by getting the points $(\pm 4,\ 0),\ (0,\ \pm 3)$ and tracing through the dots. We can find the focus f_1 and f_2 in some way.

Example 2: The standard form of this ellipse:

$$
\begin{aligned}
3x^2 - 18x + 4y^2 + 16y &= -31 \\
3\left(x^2 - 6x\right) + 4\left(y^2 + 4y\right) &= -31 \\
3\left(x^2 - 6x + 9 - 9\right) + 4\left(y^2 + 4y + 4 - 4\right) &= -31 \\
3(x-3)^2 - 27 + 4(y+2)^2 - 16 &= -31 \\
3(x-3)^2 + 4(y+2)^2 &= -31 + 16 + 27 = 12 \\
\frac{(x-3)^2}{4} + \frac{(y+2)^2}{3} &= 1
\end{aligned}
$$

Tip: take $x = 3$ and find all values of y in the ellipse such that $x = 0$, and so on.

3.3 To sketch an ellipse

You need four points. If you have the equation for an ellipse:

$$
\frac{(x-2)^2}{16} + \frac{(y-1)^2}{9} = 1
$$

Take $x = 2$ and find y, and take $y = 1$ and find x. The reason why we took x and y to be these values because it effectively eliminates one side of the sum.

To find the center of an ellipse, it is $(x_0,\ y_0)$. Each component from each point is subtracted from x and y, respectively.

3.4 Hyperbola

A hyperbola is the set of all points in a plane the difference of whose distances from two fixed points f_1 and f_2 is any fixed constant.

$$\left|\overline{pf_1}\right| - \left|\overline{pf_2}\right| = \pm 2a$$

The standard formula for a hyperbola is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Has foci $(\pm c,\ 0)$, where $c^2 = a^2 + b^2$, vertices $(\pm a,\ 0)$ and asymptotes $y = \pm\left(\frac{b}{a}\right)x$.

You **only** need to know how to draw this hyperbola in a very simple way. Remember that what you need is the vertices and the asymptotes.

3.4.1 Vertical hyperbola

The hyperbola stated above is horizontal. For vertical, the equation is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$. Has foci $(0,\ \pm c)$ where $c^2 = a^2 + b^2$ and asymptotes $y = \pm\frac{a}{b}x$. **Note that the denominator of the positive part of the hyperbola equation (that that doesn't subtract) is always a.**

3.4.2 Hyperbola sketching

$$\begin{aligned} 9x^2 - 16y^2 &= 144 \\ \Rightarrow \frac{9x^2}{144} - \frac{16y^2}{144} &= 1 \\ \Rightarrow \frac{x^2}{16} - \frac{y^2}{9} &= 1 \\ a^2 &= 16,\ b^2 = 9 \\ \Rightarrow a &= \pm 4,\ b = \pm 3 \end{aligned}$$

The lines forming the hyperbola is $\pm\frac{b}{a}x$ and the vertices are at $(\pm a,\ 0)$.

4 3D coordinate system

Coordinate directions:

- x axis points to us
- y axis points to the right
- z axis points to the up.

The distance between two points is:

$$
\begin{aligned}
&|\overline{p_1 p_2}| \\
&= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}
\end{aligned}
$$

This is known as the norm: $||p_1 p_2||$.

4.1 Of a sphere

An equation of a sphere with center $(x_0,\ y_0,\ z_0)$ and radius r is:

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

Example: Find the center and the radius of the following sphere:

$$
\begin{aligned}
x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0 \\
\Rightarrow x^2 + 4x + y^2 - 6y + z^2 + 2z = -6 \\
\Rightarrow \left(x^2 + 4x + 4\right) - 4 + \left(y^2 - 6y + 9\right) - 9 + \left(z^2 + 2z + 1\right) - 1 = -6 \\
\Rightarrow (x + 2)^2 + (y - 3)^2 + (z + 1)^2 = 8
\end{aligned}
$$

Another example: What region in $\mathbb{R}^3$ is represented by the following:

$$1 \le x^2 + y^2 + z^2 \le 9,\ z \ge 0$$

The results end up being:

$$\left\{(x,\ y,\ z) \in \mathbb{R}^3 : 1 \le x^2 + y^2 + z^2 \le 9 \land z \ge 0\right\}$$

A hollow circle sliced across the XY plane, and we only keep the thing in the positive z-axis.

4.1.1 Of an ellipsoid

Just an extra component.

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}+\frac{(z-z_0)^2}{c^2}=1$$

Where $a,\ b,\ c \in \mathbb{R} \setminus \{0\}$.

If $a=b=c$, it's a sphere.

4.2 Vector

A vector is often represented by an arrow or a directed line segment. Goes from A, an initial point, to B, a terminal point. Then, we have $\overrightarrow{AB}$.

Add two vectors using the triangle law or the parallelogram law. Vector addition is commutative.

4.3 Hollow, not hollow

Suppose we have these two equations in $\mathbb{R}^3$:

$$\begin{aligned} x^2+y^2 &= 1 \\ x^2+y^2 &\leq 1 \\ x^2+y^2 &< 1 \end{aligned}$$

The top one is a hollow cylinder. The middle one is a solid cylinder. The bottom one only allows you to consider the points inside the cylinder.

4.4 Vector bracket notation

- $(a_1, a_2,\ a_3)$ is a point.
- $\langle a_1,\ a_2,\ a_3 \rangle$ is a vector, pointing from the origin.

4.5 Basis vectors

In $\mathbb{R}^2$, $\overrightarrow{i}=\langle 1,\ 0\rangle$ and $\overrightarrow{j}=\langle 0,\ 1\rangle$. In 3D space, the remaining one is $\overrightarrow{k}=\langle 0,\ 0,\ 1\rangle$.

If I have components of two vectors, I can consider vector addition and scalar multiplication.

Any vector can be expressed by the sum of the basis vectors:

$$\overrightarrow{a} = a_1 \overrightarrow{i} + a_2 \overrightarrow{j} + a_3 \overrightarrow{k}$$

I can express the magnitude of a vector by:

$$||\overrightarrow{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

A unit vector is a vector whose length is 1, i.e., $||\overrightarrow{a}|| = 1$.

4.5.1 Unit vectors based on another vector

From this one: $\overrightarrow{a} = \langle 2,\ -1,\ -2 \rangle$ and $||\overrightarrow{a}|| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$

So, divide each component of $\overrightarrow{a}$ by 3. This means $\frac{1}{3}\overrightarrow{a} = \left\langle \frac{2}{3},\ -\frac{1}{3},\ -\frac{2}{3} \right\rangle$, which is the unit vector that points in the same direction of $\overrightarrow{a}$. In other words, it is produced by y

$$\frac{\overrightarrow{a}}{||\overrightarrow{a}||}$$

4.6 Properties of vectors

- Commutativity $\overrightarrow{a} + \overrightarrow{b} = \overrightarrow{b} + \overrightarrow{a}$
- Associativity $\left(\overrightarrow{a} + \overrightarrow{b}\right) + \overrightarrow{c} = \overrightarrow{a} + \left(\overrightarrow{b} + \overrightarrow{c}\right)$
- Identity $\overrightarrow{a} + \overrightarrow{0} = \overrightarrow{a}$
- And so on.

4.7 Dot product

For dot product:

$$\langle a_1,\ a_2 \rangle \cdot \langle b_1,\ b_2 \rangle = a_1 b_1 + a_2 b_2$$

In general, in $\mathbb{R}^n$:

$$\overrightarrow{a} + \overrightarrow{b} = \sum_{i=1}^{n} a_i b_i$$

Dot products are also called inner products. They always return $\mathbb{R}$.

θ is the angle between the two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$. Using magnitudes and angles,

$$\begin{aligned}\overrightarrow{a}\cdot\overrightarrow{b} &= ||\overrightarrow{a}||\left|\left|\overrightarrow{b}\right|\right|\cos(\theta)\\ \Rightarrow \theta &= \arccos\left(\frac{\overrightarrow{a}\cdot\overrightarrow{b}}{||\overrightarrow{a}||\left|\left|\overrightarrow{b}\right|\right|}\right)\end{aligned}$$

Therefore, if you have two vectors such that they're perpendicular ($\theta = \frac{\pi}{2},\ \ldots$), then the dot product between them is 0. In other words

$$\overrightarrow{a}\cdot\overrightarrow{b} = 0 \Rightarrow \theta = 0$$

4.7.1 Properties of the dot product

- Itself $\overrightarrow{a}\cdot\overrightarrow{a} = ||\overrightarrow{a}||^2$
- Commutative
- Associative
- Closed under scalar multiplication
- Null element $\overrightarrow{0}\cdot\overrightarrow{a} = 0$

Corollary 1. *Two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ are orthogonal if and only if $\overrightarrow{a}\cdot\overrightarrow{b} = 0$.*

Exercise: Find a unit vector that is orthogonal to both $\overrightarrow{i} + \overrightarrow{j}$ and $\overrightarrow{i} + \overrightarrow{k}$, in other words $\langle 1,\ 1,\ 0\rangle$ and $\langle 1,\ 0,\ 1\rangle$.

This means we must find a vector $\overrightarrow{a}$ such that:

$$\sqrt{a_1^2 + a_2^2 + a_3^3} = 1$$

So:

$$\begin{cases}\overrightarrow{a}\cdot\left(\overrightarrow{i} + \overrightarrow{j}\right) = 0\\ \overrightarrow{a}\cdot\left(\overrightarrow{i} + \overrightarrow{k}\right) = 0\end{cases} \Rightarrow a_1 = -a_2 = -a_3 \Rightarrow \langle 1,\ -1,\ -1\rangle$$

The answer can be found there.

4.8 Cross product

For cross product, it returns a vector with the same dimensions of the two vectors that were used to compute the product. It is always perpendicular to both vectors. It is denoted as:

$$\overrightarrow{a} \times \overrightarrow{b}$$

It is:

$$\overrightarrow{a} \times \overrightarrow{b} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

If we only have the magnitudes and direction:

$$\left|\overrightarrow{a} \times \overrightarrow{b}\right| = ||\overrightarrow{a}|| \left|\left|\overrightarrow{b}\right|\right| \sin(\theta)$$

Properties of the cross product

$$\begin{aligned} \overrightarrow{a} \times \overrightarrow{b} &= -\overrightarrow{b} \times \overrightarrow{a} \\ \overrightarrow{a} \times \left(\overrightarrow{b} + \overrightarrow{c}\right) &= \overrightarrow{a} \times \overrightarrow{b} + \overrightarrow{a} \times \overrightarrow{c} \\ \overrightarrow{a} \cdot \left(\overrightarrow{b} \times \overrightarrow{c}\right) &= \left(\overrightarrow{a} \times \overrightarrow{b}\right) \cdot \overrightarrow{c} \\ \overrightarrow{a} \times \left(\overrightarrow{b} \times \overrightarrow{c}\right) &= (\overrightarrow{a} \cdot \overrightarrow{c})\, \overrightarrow{b} - \left(\overrightarrow{a} \cdot \overrightarrow{b}\right) \overrightarrow{c} \end{aligned}$$

4.8.1 The right hand rules

Grab your right hand out flat with four of your fingers pointing in the direction of $\overrightarrow{a}$. Then, make sure your hand is angled in such a way that you can close your hand to reach $\overrightarrow{b}$. The direction of your thumb is the direction of the cross product.

If two vectors are parallel, the cross product between them is 0.

4.9 Determinant

Order 2:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Order 3, using cofactors:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

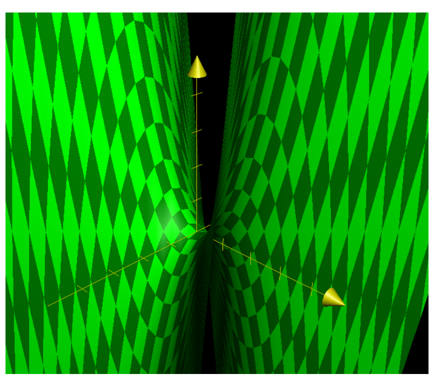
Example:

$$\begin{aligned} \begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix} &= \begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} - 1 \begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix} \\ &= (0 \cdot 2 - 1 \cdot 4) - 2(3 \cdot 2 + 1 \cdot 5) - 1(3 \cdot 4) \\ &= -38 \end{aligned}$$

Using determinant to find cross product: $\overrightarrow{a} = \langle 1,3,4 \rangle$, $\overrightarrow{b} = \langle 2,7,-5 \rangle$:

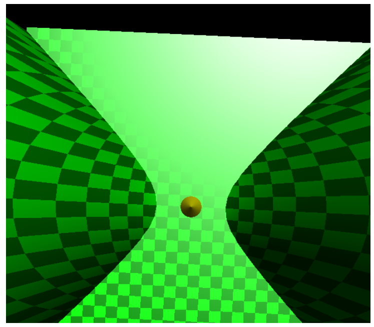
$$\begin{aligned} \overrightarrow{a} \times \overrightarrow{b} &= \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{j} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = \overrightarrow{i} \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} - \overrightarrow{j} \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} + \overrightarrow{k} \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \\ &= \overrightarrow{i}\,((-5) \cdot 3 - 4 \cdot 7) - \overrightarrow{j}\,((-5) \cdot 1 - 4 \cdot 2) + \overrightarrow{k}\,(1 \cdot 7 - 3 \cdot 2) \\ &= -43\overrightarrow{i} + 13\overrightarrow{j} + \overrightarrow{k} \end{aligned}$$

4.10 Different solids



Intersect this shape with two planes: one on the XY-plane and one on the ZX plane. What do we get?

XY: Hyperbola



4.11 Intersecting two planes

How can we obtain equations for planes intersecting planes?

4.12 Shapes

- $x^2 - y^2 - z^2 = 1$ produces a hyperbola
- $x^2 + y^2 - z^2 = 1$ produces an hourglass
- $z = x^2 - y^2$ produces the pringle
- $z = x^2 + y^2$ produces a paraboloid
- $z^2 = x^2 + y^2$ produces a double cone

5 Lines and planes in 3D spaces

In 2D, to write down the formula of a line, write down a point and a slope. The slope is really the direction of the line. When we move to 3D space. In 2D,

- Two lines may intersect
- Or be parallel or on top of each other.

In 3D spaces, what are the intersection cases for lines in 3D space?

- Parallel or at the same place
- Cross
- Not parallel, no cross (askew)

For planes:

- Parallel or at the same place
- Intersects in a line at one place

This means the standard formula for lines and planes are important.

5.1 Equations of lines and planes in space

Parametric equations for a line through the point $(x_0,\ y_0,\ z_0)$ and parallel to the direction vector $\langle a,\ b,\ c\rangle$ is:

$$\begin{bmatrix} x_0+at \\ y_0+bt \\ z_0+ct \end{bmatrix},\ t\in\mathbb{R}$$

For a plane, we need a point in the plane and one vector that is orthogonal to the plane (called the *normal vector*). A scalar equation of a plane through point $(x_0,\ y_0,\ z_0)$ with normal vector $\overrightarrow{n}=\langle a,\ b,\ c\rangle$:

$$a\,(x-x_0)+b\,(y-y_0)+c\,(z-z_0)=0$$

Example 5.1. Find parametric equations for the line that passes through the point $(5,\ 1,\ 3)$ and is parallel to the vector $\overrightarrow{i}+4\overrightarrow{j}-2\overrightarrow{k}$.

Well, $(x_0,\ y_0,\ z_0)=(5,\ 1,\ 3)$ and $\overrightarrow{d}=(1,\ 4,\ -2)$. So, the equation of the line i:

$$\begin{bmatrix} 5+t \\ 1+4t \\ 3-2t \end{bmatrix}$$

Therefore, $x=5+t$, $y=1+4t$, $z=3-2t$

If we need to find some other points in this line, take t to be any value we want and plug it in for as many times you want given it is in $\mathbb{R}$.

- If we take $t=1\Rightarrow(6,\ 5,\ 1)$
- If we take $t=-1\Rightarrow(4,\ -3,\ 5)$

That is the parametric formula for a line. Then, I can compute t by using any of the three formulas:

$$t = \tfrac{x-x_0}{a} \quad t = \tfrac{y-y_0}{b} \quad t = \tfrac{z-z_0}{c}$$

Thus, we have:

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

This is called the symmetric equations of a line.

If one of $a,\ b,\ c$ is 0, then we can still eliminate t. We just let $x = x_0$.

Example 5.2. Find parametric equations and symmetric equations of the line that passes through the points $p_0(2,\ 4,\ -3)$ and $p_1(3,\ -1,\ 1)$. At what point does this line intersect the xy plane?

We can find the direction vector by getting it from $p_1 - p_0$.

$$\overrightarrow{p_1} - \overrightarrow{p_0} = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$$

Then, grab one of p_0 and p_1, and the equation of the line is:

$$\begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$$

5.2 To show that two lines are skew

Not parallel and does not cross. To show that they aren't parallel, it is enough to show that their direction vectors are not parallel.

Example 5.3. Show that L_1 and L_2 do not cross. Then:

$$L_1 = \begin{bmatrix} 1+0t \\ -2+3t \\ 4-t \end{bmatrix},\ L_2 = \begin{bmatrix} 0+2s \\ 3+s \\ -3+4s \end{bmatrix}$$

$$\overrightarrow{a} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix},\ \overrightarrow{b} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

To find whether the two vectors are parallel or not, take the cross product of them. If the cross product is 0, then they are parallel; otherwise, they're not.

To show that the two lines are askew, attempt to take a system of equations and conclude that there are no answers.

5.3 On planes

How can we get the standard formula of a plane?

Firstly, let $\overrightarrow{p_1}$ be any vector that is parallel on the plan (if we project it on the plane the length of it doesn't change). Then:

$$\overrightarrow{n} \cdot \overrightarrow{p_1} = 0$$

As they are orthogonal.

THE STANDARD FORMULA OF A PLANE IS PRESENTED HERE:

If $n = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, then:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0$$

"NOT AS A VECTOR" FORM:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Where $x_0,\ y_0$, and z_0 form this point $(x_0,\ y_0,\ z_0)$ that lies on the plane – hint: take one of the points of the plane and sub them in $x_0,\ y_0$, and z_0.

Example 5.4. Find the plane passing through the following points: $(0,\ 1,\ 1)$, $(1,\ 0,\ 1)$, $(1,\ 1,\ 0)$.

What I need:

- One vector from one point to another
- A normal vector

Consider the two vectors that lie on the plane:

$$\overrightarrow{AB},\ \overrightarrow{AC}$$

Take the cross product, and you'll get the normal vector:

$$\overrightarrow{n} = \overrightarrow{AB} \times \overrightarrow{AC}$$

We have $\overrightarrow{AB} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\overrightarrow{AC} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Computing

$$\begin{aligned} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} &= \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \overrightarrow{i} \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} - \overrightarrow{j} \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} + k \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \end{aligned}$$

This means the equation of the plane is:

$$(x-0) + (y-1) + (z-1) = 0$$

Example 5.5. Find the angle between the following planes: $x+y+z=1$ and $x-2y+3z=1$. Hint: find the angle between the normal vectors.

The two planes can be represented as:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

And

$$\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Finding the angles between the normal vectors:

$$\cos(\theta)\left|\left|\begin{bmatrix}1\\-2\\3\end{bmatrix}\right|\right|\left|\left|\begin{bmatrix}1\\1\\1\end{bmatrix}\right|\right| = \begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\-2\\3\end{bmatrix}$$

$$\theta = \arccos\left(\frac{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\-2\\3\end{bmatrix}}{\left|\left|\begin{bmatrix}1\\-2\\3\end{bmatrix}\right|\right|\left|\left|\begin{bmatrix}1\\1\\1\end{bmatrix}\right|\right|}\right)$$

Whatever you get out of this is the angle between the two planes.

Example 5.6. Find the equation of the plane passing through $(2,\ 1,\ 0)$ and is parallel to the plane $x+4y-3z=1$.

The normal vector of the plane stated is:

$$\begin{bmatrix}1\\4\\-3\end{bmatrix}$$

The point is given, so the equation of the plane that is asked by the question is:

$$\begin{bmatrix}1\\4\\-3\end{bmatrix}\cdot\begin{bmatrix}x-2\\y-1\\z\end{bmatrix}=0$$
$$\Rightarrow (x-2)+4(y-1)-3z=0$$

How to solve any problem involving finding the plane given one line, one point:

- Let $\overrightarrow{v}$ be any vector that points in the same direction as the line
- Let p be the point that we are given
- Let q be any point on the line
- Then the normal vector of the plane is $\overrightarrow{v}\times\overrightarrow{pq}$
- And the equation of the line is $(\overrightarrow{v}\times\overrightarrow{pq})\cdot\begin{bmatrix}x-p_1\\y-p_2\\z-p_3\end{bmatrix}=0$

6 Vector-valued functions

A function takes an input and returns an output. It maps from a set to a set. In the context of single-variable calculus, functions we've seen are $f\colon \mathbb{R} \to \mathbb{R}$. This means if I input x to f, I won't get a vector.

A vector valued function is when I have a function that looks like $f\colon \mathbb{R} \to \mathbb{R}^n$. Here's an example:

$$r(t) = \begin{bmatrix} t^2 \\ \sin(t) \\ \cos(t) \end{bmatrix}$$

Then $r\colon \mathbb{R} \to \mathbb{R}^3$, because I input one $\mathbb{R}$ that is t and I get a vector that is $\begin{bmatrix} t^2 \\ \sin(t) \\ \cos(t) \end{bmatrix}$, which has 3 components.

6.1 Parametric equations

Remember parametric functions?

$$x^2 + y^2 = 1 \to \begin{matrix} x = \cos(t) \\ y = \sin(t) \end{matrix}$$

We can represent it as:

$$\overrightarrow{r}(t) = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}$$

We can use the same method, but in 3D space.

6.2 Points of intersections of two 3D functions

Let $f,\ g,\ h$ be real-valued functions. For every t in the domain of $\overrightarrow{r}$, we have a unique vector $\overrightarrow{r}(t) = \begin{bmatrix} f(t) \\ g(t) \\ h(t) \end{bmatrix}$ ...?

Example 6.1. Describe the vector function given by $\overrightarrow{r}(t) = \begin{bmatrix} 1+2t \\ 1-5t \\ -2+6t \end{bmatrix}$

So, we have

$$\begin{aligned} x &= 1+2t \\ y &= 1-5t \quad \Rightarrow \text{parametric eq of a line} \\ z &= -2+6t \end{aligned}$$

Where the line's direction vector is $\begin{bmatrix} 2 \\ -5 \\ 6 \end{bmatrix}$ that passes through $(1,\ 1,\ -2)$.

Example 6.2. Sketch the curve whose vector equation is:

$$\begin{aligned} \vec{r}(t) &= 2\cos(t)\,\vec{i} + 2\sin(t)\,\vec{j} + t\,\vec{k} \\ &= \begin{bmatrix} 2\cos(t) \\ 2\sin(t) \\ t \end{bmatrix} \end{aligned}$$

If we project this onto the xy plane, we get $\begin{bmatrix} 2\cos(t) \\ 2\sin(t) \end{bmatrix}$, which looks like a radius-2 circle centered at the origin.

A point is $(x_0,\ y_0,\ z_0)$. A vector is $\langle x_0,\ y_0,\ z_0 \rangle$ or $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$.

We had this vector function: $\begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix}$

When projected on the xy plane, we have $\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$.

Example 6.3. Find a vector function that represents the curve of intersection of the cylinder $x^2+z^2=1$ (going through the y-axis) and $z+y=1$ (a plane).

As a parametric equation:

$$\begin{aligned} x &= \cos(t) \\ z &= \sin(t) \end{aligned}$$

And $z+y=1$ so $y=1-z=1-\sin(t)$

This means the vector function that represents the intersection of the curve, and the cylinder is

$$\begin{bmatrix} \cos(t) \\ 1-\sin(t) \\ \sin(t) \end{bmatrix},\ 0 \le t \le 2\pi$$

Example 6.4. The upper part of the sphere $x^2+y^2+z^2=1$ and the upper half of the cone $z^2=x^2+y^2$

We have

$$\begin{cases} z^2 = x^2+y^2 \\ x^2+y^2+z^2=1 \end{cases}$$

$$\begin{aligned} \Rightarrow z^2+z^2 &= 1 \\ 2z^2 &= 1 \\ z &= \frac{1}{\sqrt{2}} \end{aligned}$$

So:

$$\begin{aligned} x^2+y^2+\frac{1}{2} &= 1 \\ x^2+y^2 &= \frac{1}{2} \\ x &= \frac{1}{\sqrt{2}}\cos(t) \\ y &= \frac{1}{\sqrt{2}}\sin(t) \end{aligned}$$

We thus have this parametric equation:

$$\begin{bmatrix} \frac{1}{\sqrt{2}}\cos(t) \\ \frac{1}{\sqrt{2}}\sin(t) \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

6.3 Limits of vector functions

If I have a function $f\colon \mathbb{R} \to \mathbb{R}$, then

$$\lim_{x \to x_0} f(x) \in \mathbb{R}$$

If the limit exists. If I swap $\mathbb{R}$ with $\mathbb{C}$, it should work the same. At least, if $\forall \varepsilon > 0,\ \exists \delta > 0,\ |x-a| < \delta \Rightarrow |f(x)-L| < \varepsilon$. I will never see that equation again.

If I have $f\colon \mathbb{R} \to \mathbb{R}^3$, then the limit is also in $\mathbb{R}^3$.

$$\overrightarrow{r}\colon \mathbb{R} \to \mathbb{R}^3$$

We can define $\lim_{t \to t_0} \overrightarrow{r}(t) \in \mathbb{R}^3$ if the limit exists.

Definition 6.1. Let $f,\ g,\ h \in \mathbb{R} \to \mathbb{R}$ be functions and let $\overrightarrow{r}(t) = \begin{bmatrix} f(t) \\ g(t) \\ r(t) \end{bmatrix}$

Then $\lim_{t \to a} \overrightarrow{r}(t) = \begin{bmatrix} \lim_{t \to a} f(t) \\ \lim_{t \to a} g(t) \\ \lim_{t \to a} h(t) \end{bmatrix}$, provided the limits of the component functions exist.

$\overrightarrow{r}$ is **continuous** at a if $\lim_{t \to a} \overrightarrow{r}(t) = \overrightarrow{r}(a)$.

Example 6.5. Find the $\lim_{t \to 0} \overrightarrow{r}(t)$, were

$$\begin{aligned} &\overrightarrow{r}(x) \\ &= \left(\frac{1}{\ln\left(x + \sqrt{x^2+1}\right)} - \frac{1}{\ln(x+1)} \right) \overrightarrow{i} \\ &+ \left((1+x)^{\frac{1}{x}} \right) \overrightarrow{j} + \left(\frac{\sin(5x)}{3x} \right) \overrightarrow{k} \end{aligned}$$

The answer is:

$$\left(-\frac{1}{2}\right) \overrightarrow{i} + e \overrightarrow{j} + \frac{5}{3} \overrightarrow{k}$$

You *may* have to use Hospital's rule for this.

Solution for the $\overrightarrow{k}$ component: TBA

6.4 Derivative

$$f\colon \mathbb{R} \to \mathbb{R}$$

Then $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

For a vector function, you have the same thing, but instead of real numbers, you must deal with vectors.

What is the appropriate notion of the derivative if $f\colon \mathbb{R} \to\mathbb{R}^3$? Or $f : \mathbb{R}^3 \to \mathbb{R}^3$?

If we have a vector-valued-function in 3D:

$$\frac{d\overrightarrow{r}}{dt} = \overrightarrow{r}'(t) = \lim_{h\to 0} \frac{\overrightarrow{r}(t+h) - \overrightarrow{r}(t)}{h}$$

Theorem 6.1. *The derivative of a vector function is the derivative of its components.*

$$\overrightarrow{r}(t) = \begin{bmatrix} f(t) \\ g(t) \\ h(t) \end{bmatrix} \Rightarrow \overrightarrow{r}'(t) = \begin{bmatrix} f'(t) \\ g'(t) \\ h'(t) \end{bmatrix}$$
$$\overrightarrow{r}''(t) = \begin{bmatrix} f''(t) \\ g''(t) \\ h''(t) \end{bmatrix}$$

The **unit tangent vector,** denoted $\overrightarrow{T}$, is defined by:

$$\overrightarrow{T}(t) = \frac{\overrightarrow{r}(t)}{\left|\overrightarrow{r}(t)\right|}$$

6.5 The TNB frame

The normalized tangent vector, where the length is always 1.

The normal vector, $\overrightarrow{N}$, is defined by: *The derivative of the normal vector, then normalized again for some reason*

$$\vec{N}(t) = \frac{\vec{T'}(t)}{\left|\vec{T'}(t)\right|}$$

The binominal vector, defined by $\vec{B}$, is defined by *the cross product of the previous two, $\vec{T}$ being first*

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

Visually, suppose you have a curve. For each point of the curve, you want an orthogonal frame to the curve. Only $\vec{T}(t)$ will be asked on the exam, but it is best to know all three.

Example 6.6. Let $\vec{r}(t) = \begin{bmatrix} 1+t^3 \\ te^{-t} \\ \sin(2t) \end{bmatrix}$

Then $\vec{r'}(t) = \begin{bmatrix} 3t^2 \\ e^{-t} - te^{-t} \\ 2\cos(2t) \end{bmatrix}$, and $\vec{r''}(t) = \begin{bmatrix} 6t \\ -2e^{-t} + te^{-t} \\ -4\sin(2t) \end{bmatrix}$.

The normalized tangent vector is $\frac{\vec{r'}(t)}{\left|\vec{r'}(t)\right|} = \frac{\begin{bmatrix} 3t^2 \\ e^{-t} - te^{-t} \\ 2\cos(2t) \end{bmatrix}}{\sqrt{9t^4 + (e^{-t} - te^{-t})^2 + 4\cos^2(2t)}}$

The normal is

$$\frac{\left(\frac{\begin{bmatrix} 6t \\ -2e^{-t}+te^{-t} \\ -4\sin(2t) \end{bmatrix}}{\sqrt{9t^4+(e^{-t}-te^{-t})^2+4\cos^2(2t)}}\right)}{\frac{\left\|\begin{bmatrix} 6t \\ -2e^{-t}+te^{-t} \\ -4\sin(2t) \end{bmatrix}\right\|}{\sqrt{9t^4+(e^{-t}-te^{-t})^2+4\cos^2(2t)}}}$$

$$= \frac{\begin{bmatrix} 6t \\ -2e^{-t}+te^{-t} \\ -4\sin(2t) \end{bmatrix}}{\left\|\begin{bmatrix} 6t \\ -2e^{-t}+te^{-t} \\ -4\sin(2t) \end{bmatrix}\right\|}$$

$$= \frac{\begin{bmatrix} 6t \\ -2e^{-t}+te^{-t} \\ -4\sin(2t) \end{bmatrix}}{\sqrt{(6t)^2+(-2e^{-t}+te^{-t})^2+(-4\sin(2t))^2}}$$

And

$$\vec{B}(t) = \frac{\begin{bmatrix} 3t^2 \\ e^{-t}-te^{-t} \\ 2\cos(2t) \end{bmatrix}}{\sqrt{9t^4+(e^{-t}-te^{-t})^2+4\cos^2(2t)}} \times \frac{\begin{bmatrix} 6t \\ -2e^{-t}+te^{-t} \\ -4\sin(2t) \end{bmatrix}}{\sqrt{(6t)^2+(-2e^{-t}+te^{-t})^2+(-4\sin(2t))^2}}$$

You see, this isn't the cleanest thing to exist.

6.6 Parametric to cartesian

With $x = \sin(t),\ y = \cos(t)$, the cartesian form is $x^2 + y^2 = 1$.

With $x = \cos(t),\ y = \sin(t)$, the cartesian form is $x^2 + y^2 = 1$.

No, cartesian equations may not always be parameterized uniquely. Cartesian equations only have paths, whilst parametric equations have directions. It doesn't matter if we take $x = \sin(t)$ or $x = \cos(t)$ given y isn't the same as x.

In the future, because we want to look at green's theorem, we usually consider the direction of the parametric equation as **counterclockwise.** Hence, it is strongly recommended to consider that everything is supposed to go in the counterclockwise direction.

To find the direction of a curve, good luck tracing.

6.7 Derivative rules

Most of the rules from single-variable calculus apply.

1) Sum: $(\overrightarrow{u}(t) + \overrightarrow{v}(t))' = \overrightarrow{u}'(t) + \overrightarrow{v}'(t)$
2) Closed under multiplication
3) Multiplication with a scalar and vector
4) Dot product: $(\overrightarrow{u}(t) \cdot \overrightarrow{v}(t))' = \overrightarrow{u'}(t) \cdot \overrightarrow{v}(t) + \overrightarrow{u}(t) \cdot \overrightarrow{v'}(t)$
5) Cross product: $(\overrightarrow{u}(t) \cdot \times \overrightarrow{v}(t))' = \overrightarrow{u'}(t) \times \overrightarrow{v}(t) + \overrightarrow{u}(t) \times \overrightarrow{v'}(t)$
6) Chain rule: $(\overrightarrow{u}\,(f(t)))' = \overrightarrow{u'}\,(f(t)) \cdot f'(t)$

Theorem 6.2. $\forall t \in \mathbb{R}$*, If* $|\overrightarrow{r}(t)| = c$ *is a constant, then* $\overrightarrow{r'}(t)$ *is orthogonal to* $\overrightarrow{r}(t)$*.*

Proof. $\overrightarrow{r}(t) \cdot \overrightarrow{r}(t) = ||\overrightarrow{r}(t)||^2 = c^2$

Take the derivative on both sides:

$$\begin{aligned} &\overrightarrow{r'}(t)\overrightarrow{r}(t) + \overrightarrow{r}(t)\overrightarrow{r'}(t) \\ &= 2\overrightarrow{r'}(t) \cdot \overrightarrow{r}(t) = 0 \\ \Rightarrow \overrightarrow{r'}(t) \cdot \overrightarrow{r}(t) = 0& \\ &\Rightarrow \overrightarrow{r'}(t) \perp \overrightarrow{r}(t) \end{aligned}$$

$\blacksquare$

Remember circular motion?

7 Functions of two variables

Motivation. Suppose that there's a sphere. Let T be the temperature of a point on the sphere depending on the longitude x and the latitude y. Then:

$$T = f(x,\ y)$$

We can think of T as functions of two variables.

In a mathematical example, suppose that you have a cylinder. If you want to compute the volume of this cylinder, if you know the radius of the base and the height of the edge, the volume is $\pi r^2 h$. If we substitute a different cylinder with different radius and height, the function is still the same: $\pi r^2 h$. We can consider volume as a function of h and r which is $\pi r^2 h$. This is a function of two variables. $f(r,\ h) = \pi r^2 h$.

Definition 7.1. A function of n variables is a rule that assigns to $x_1,\ x_2,\ \ldots x_n \in \mathbb{R}^n$ in a set $D \subseteq \mathbb{R}^j$, denoted by $f\,(x_1,\ \ldots,\ x_n)$. The set D is the domain of f and its image (range) is $\{f\,(x_1,\ \ldots,\ x_n) \mid (x_1, x_2,\ \ldots,\ x_n) \in D\}$. In this course, we'll only focus on 2-3 variables.

If we have a function of two variables, we often write $z = f(x,\ y)$ to make explicit the value taken by f of the general point $(x,\ y)$. The domain is which point you can pick on the XY plane that returns something.

Example 7.1. For this function, evaluate $f(3,\ 2)$ and find and sketch the domain.

$$f(x,\ y) = \frac{\sqrt{x+y+1}}{x-1}$$

$$f(3,\ 2) = \frac{\sqrt{3+2+1}}{2-1} = \frac{\sqrt{5}}{1} = \sqrt{5}$$

The domain of this function is $\left\{(x,\ y) \in \mathbb{R}^2 \mid x+y+1 \geq 0 \wedge x \neq 1\right\}$

And

$$f(x,\ y) = \frac{\ln\left(x^2-y\right)}{(x-1)\sqrt{1-x^2}}$$

The domain of this function is $\left\{(x,\ y) \in \mathbb{R}^2 \mid y < x^2 \wedge x \neq 0 \wedge -1 < x < 1\right\}$. The domain is an area in the XY plane that should be precisely shown.

7.1 Level curves

Definition 7.2. The level curves of a function f of two variables are the curves with equations $f(x,\ y) = c$, where c is a constant number in the image of f.

$$f^{-1}(c) = \{(x_1,\ x_2,\ \dots) \mid f(x_1,\ x_2,\ \dots) = c\}$$

A collection of level curves is called a contour map. These help us visualize functions of two or more variables.

The definition of a graph is $\{(x,\ f(x)) \mid x \in D_f\}$

For a 3D graph, it is:

$$\{(x,\ y,\ z) \mid z = f(x,\ y)\}$$

So, the result will be in a 3D curve.

Note that it is very difficult to visualize a function f of three variables by its graph.

Example 7.2. Sketch the graph of $f(x,\ y) = 6-3x-2y$ and sketch the level curves for $k = -6,\ 0,\ 6$.

Solution 7.1.

$$\begin{aligned}
k = -6 &\Rightarrow f(x,\ y) = -6 \\
\Rightarrow 6x-3x-2y &= -6 \\
\Rightarrow 3x+2y &= 12 \\
\Rightarrow y &= -\frac{3}{2}x+6 \\
k = 0 &\Rightarrow f(x,\ y) = 0 \\
\Rightarrow 6-3x-2y &= 0 \\
\Rightarrow 3x+2y &= 6 \\
\Rightarrow y &= -\frac{3x}{2}+3 \\
k = 6 &\Rightarrow f(x,\ y) = 6 \\
\Rightarrow 6-3x-2y &= 6 \\
\Rightarrow -3x-2y &= 0 \\
\Rightarrow 3x+2y &= 0 \\
\Rightarrow y &= -\frac{3x}{2}
\end{aligned}$$

The question asks for me to sketch the level curves. These are just lines. This can be as simple as sketching the lines. A collection of these lines forms a contour map. The graph of this function is a plane, as it is $z = 6 - 3x - 2y$.

To sketch the 3D graph, we need three points. Choose two variables, find the last one, and that's a point on the graph.

You can approximate how a graph looks like using its contour maps if you have enough information.

7.1.1 Not a plane

Example 7.3. Let $f(x,\ y) = 4x^2 + y^2$.

Find the domain and range of f. There are no restrictions from the domain, but the image is $[0,\ \infty)$.

Sketch level curves. Plot $(0,\ 0)$, $1 = x^2 + \frac{y^2}{4}$, and $1 = \frac{x^2}{4} + \frac{y^2}{16}$.

Sketch graph. Looks like a stretched paraboloid, the longer side being on the y-axis.

Example 7.4. Sketch the contour map of $f(x,\ y) = ye^x$.

7.2 Limits in 1D space

Recall the epsilon delta:

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

That applies.

Theorem 7.1 (Squeeze theorem)**.** *Also known as the sandwich theorem. Let* $f,\ g,\ h : \mathbb{R} \to \mathbb{R}$ *be functions and let* $a \in \mathbb{R}$*. If*

$$f(x) \le g(x) \le h(x)$$

Where x *is near but not at* a*, and*

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

Then

$$\lim_{x \to a} g(x) = L$$

Example 7.5. Show that $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

7.3 Limits in 2D and 3D space

Definition 7.3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ and let $(a,\ b) \in D_f \subseteq \mathbb{R}^2$. We say that the limit of $f(x,\ y)$ as $(x,\ y)$ approaches $(a,\ b)$ is L, denoted $\lim_{(x,\ y)\to(a,\ b)} f(x,\ y) = L$:

$$\forall (x,\ y) \in D_f,\ 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x,\ y) - L| < \varepsilon$$

There are two ways to show the limit exists:

- Use the epsilon-delta method
- Use the squeeze theorem

To show that a limit doesn't exist at a point, show that the limit for two different paths don't match (like from the left to the right, but now you have 3 dimensions).

7.4 Checking whether a limit exists

It exists

- Plug it in and it's not indeterminate
- $\varepsilon - \delta$
- Squeeze
- Polar cords

It doesn't

- Approaches $(x,\ y)$ from different directions and obtain two different values for the limit
- Negation of $\varepsilon - \delta$

Limit properties:

- Limit of a constant, wherever it may approach is the constant

- You can sum and subtract limits
- You can multiply limits
- You can divide limits if it's not division by zero
- The continuity law of composition holds

7.4.1 Examples

$$\lim_{(x,y)\to(1,0)} \sin\left(\frac{1+x^2}{x^2+xy+1}\right) = \sin\left(\frac{1+1}{1+1}\right) = 1$$

$$\begin{aligned}
&\lim_{(x,y)\to(3,3)} \frac{\sqrt{x+1}-\sqrt{y-1}}{x-y} \\
&= \lim_{(x,y)\to(3,3)} \frac{\sqrt{x+1}-\sqrt{y-1}}{x-y} \cdot \frac{\sqrt{x+1}+\sqrt{y+1}}{\sqrt{x+1}+\sqrt{y+1}} \\
&= \lim_{(x,y)\to(3,3)} \frac{(x+1)-(y+1)=x-y}{(x-y)\left(\sqrt{x+1}+\sqrt{y+1}\right)} \\
&= \frac{1}{\sqrt{x+1}+\sqrt{y+1}} = \frac{1}{\sqrt{4}+\sqrt{4}} \\
&= \frac{1}{4}
\end{aligned}$$

7.4.2 To show that a limit at a point doesn't exist

Show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$$

In two directions:

$$\begin{aligned}
f(x,0) &= \frac{x^2}{x^2} = 1,\ x \neq 0 \\
f(0,\ y) &= \frac{0}{y^2} = 0,\ y \neq 0
\end{aligned}$$

Therefore, the limit does not exist.

7.4.3 Using polar coordinates

$$x = r\cos(\theta) \quad y = r\sin(\theta)$$

This means:

$$x^2 + y^2 = r^2, \\ (x,\ y) \to (0,\ 0) \Rightarrow r \to 0^+$$

$$\lim_{r\to 0^+} \frac{x^2}{x^2+y^2} = \lim_{r\to 0^+} \frac{r^2\cos^2(\theta)}{r^2} = \cos^2(\theta)$$

The answer depends on θ, so the limit doesn't exist. It depends on θ if a different value of θ can change the limit.

7.4.4 It doesn't exist

Example 7.6. This:

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+3y^2}$$

$$\lim_{(x,x)\to(0,0)} \frac{2x^2}{x^2+3x^2} = \lim_{(x,x)\to(0,0)} \frac{2x^2}{4x^2} = \frac{1}{2}$$

$$\lim_{(x,3x=y)\to(0,0)} \frac{2x\cdot 3x}{x^2+3(3x)^2} = \lim_{(x,3x)\to(0,0)} \frac{6x^2}{28x^2} = \frac{3}{14}$$

Nope, not the same. Limit doesn't exist.

There's nothing stopping us from taking $y = mx,\ m \in \mathbb{R}$. Then:

$$\lim_{(x,mx)\to(0,0)} \frac{2mx^2}{x^2+3(mx)^2} = \lim_{(x,\ mx)\to(0,0)} \frac{2mx^2}{x^2+3m^2x^2}$$
$$= \lim_{(x,\ mx)\to(0,0)} \frac{2mx^2}{(1+3m^2)x^2} = \lim_{(x,\ mx)\to(0,0)} \frac{2m}{(1+3m^2)}$$

Because the answer of this limit depends on m, the limit doesn't exist.

Example 7.7.

$$\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2(x)}{x^4+y^4}$$

$$f(x,y) = f(x,x)$$
$$\lim_{(x,x)\to(0,0)} \frac{x^2\sin^2(x)}{x^4+x^4} = \lim_{(x,x)\to(0,0)} \frac{x^2\sin^2(x)}{2x^4}$$
$$\lim_{(x,x)\to(0,0)} \frac{\sin^2(x)}{2x^2} = \lim_{(x,x)\to(0,0)} \frac{x^2}{2x^2} = \frac{1}{2}$$

Take $y = 2x$

$$f(x,y) = f(x,2x)$$
$$\lim_{(x,x)\to(0,0)} \frac{(2x)^2\sin^2(x)}{x^4+(2x)^4} = \lim_{(x,x)\to(0,0)} \frac{4x^2\sin^2(x)}{17x^4}$$
$$\lim_{(x,x)\to(0,0)} \frac{4}{17x^2}\sin^2(x) = \frac{4}{17}$$

Nope, limit doesn't exist.

Example 7.8. Presented below:

$$\lim_{(x,y)\to(0,0)} \frac{2xy^2}{3x^2+y^4}$$

$$\lim_{(x,mx)\to(0,0)} \frac{2x(mx)^2}{3x^2+(mx)^4} = \lim_{(x,mx)\to(0,0)} \frac{2x^3m^2}{3x^2+m^4x^4}$$
$$= \lim_{(x,mx)\to(0,0)} \frac{2xm^2}{3+m^4x^2} = \frac{0}{3} = 0$$

Well, look for another point:

$$x = y^2$$

$$\lim_{\left(y^2,y\right)\to(0,0)} \frac{2y^4}{3y^4+y^4} = \frac{2y^4}{4y^4} = \frac{1}{2}$$

Because we have two different limits when we take different paths, the limit doesn't exist.

Example 7.9. This requires a completely different method.

$$\lim_{(x,y)\to(1,1)} \frac{y-x}{1-y+\ln(x)}$$

You can expand this into three dimensions.

$y = x$:

$$f(x,\ y) = f(x,x) = \frac{0}{1-x+\ln(x)} = 0,\ x \neq 1$$

Beware of pitfalls: there must exist a value where the limit to the point agrees. Try $x = e^{1-y}$:

$$\begin{aligned} f(x,\ y) = f\left(x,\ x^2\right) &= \frac{y-e^{1-y}}{1-y+\ln\left(e^{1-y}\right)} \\ &= \frac{y-e^{1-y}}{1-y+(1-y)}, \lim_{y\to 1} \frac{y-e^{1-y}}{2(1-y)} = \lim_{y\to 1} \frac{1+e^{1-y}}{-2} = -1 \end{aligned}$$

Try any other expression, that when graphed, there's a point on the curve that holds $(1,\ 1)$. It's not always easy to find different paths, and many examples must be done.

7.5 Squeeze Theorem 2.0

Theorem 7.2. *Let $a,\ b,\ L \in \mathbb{R}$ and let $f,\ g\ : \mathbb{R}^2 \to \mathbb{R}$ be functions. IF:*

$$|f(x,\ y) - L| \leq g(x,\ y)$$

Then $\forall (x,\ y) \in \mathbb{R}^2$ inside a disk centered at $(a,\ b)$, maybe except $(a,\ b)$:

$$\left(\lim_{(x,y)\to(a,b)} g(x,y) = 0\right) \Rightarrow \left(\lim_{(x,y)\to(a,b)} f(x,\ y) = L\right)$$

And like always

$$\lim_{x\to x_0} |f(x)| \text{ exists} \not\Rightarrow \lim_{x\to x_0} f(x)$$

So, proceed with caution.

Example 7.10. Compute this:

$$\lim_{(x,y)\to(0,0)} xy\sin\left(\frac{1}{x^2+y^2}\right)$$

Well, we know that $0 \le \left|\sin\left(\frac{1}{x^2+y^2}\right)\right| \le 1$, so:

$$\begin{aligned} 0 \le \left|xy\sin\left(\frac{1}{x^2+y^2}\right)\right| &\le xy \\ \lim_{(x,y)\to(0,0)} |xy| &= 0 \\ \Rightarrow \lim_{(x,y)\to(0,0)} xy\sin\left(\frac{1}{x^2+y^2}\right) &= 0 \end{aligned}$$

With another version of the squeeze theorem, I can say that this limit exists:

$$\begin{aligned} \left|xy\sin\left(\frac{1}{x^2+y^2}\right) - 0\right| \le |xy|\left|\sin\left(\frac{1}{x^2+y^2}\right)\right| &= 0 \\ \Rightarrow \lim_{(x,y)\to(0,0)} xy\sin\left(\frac{1}{x^2+y^2}\right) &= 0 \end{aligned}$$

7.6 Continuity

No hole.

Definition 7.4. A function of two variables is continuous at $(a,\ b)$ if:

$$\lim_{(x,y)\to(a,b)} f(x,\ y) = f(a,\ b)$$

We say that f is continuous on D_f (domain of f) if f is continuous at every point $(a,\ b)$ in D.

Example 7.11. Where is $f(x,\ y) = \begin{cases} \frac{e^{-x^2-y^2}-1}{x^2+y^2} & (x,\ y) \ne (0,\ 0) \\ -1 & (x,\ y) = (0,\ 0) \end{cases}$

Firstly, find $\lim_{(x,\ y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2}$, which is:

$$\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2}$$

Use polar coordinates: $x = r\cos(\theta),\ y = r\sin(\theta)$:

$$\lim_{r\to 0^+} \frac{e^{-r^2}-1}{r^2} = \frac{e^{-r^2}}{r^2} - \frac{1}{r^2}$$

$$\begin{aligned} &r^2\cos^2(\theta) + r^2\sin^2(\theta) \\ &= r^2\left(\cos^2(\theta)+\sin^2(\theta)\right) = r^2 \end{aligned}$$

Take Hospital's rule twice:

$$\ldots = -1$$

Then, the function is continuous at $(0,\ 0)$ and it is thus continuous everywhere.

Note that r goes to 0 from the right because $r = x^2+y^2 \geq 0$.

$$\lim_{r\to 0^+} r\cos(\theta)\sin(\theta) = 0$$

Dependence on theta. Now, it doesn't

$$\lim_{r\to 0^+} \sin(\theta)\cos(\theta) \Rightarrow \text{The limit DNE}$$

If the limit when turned to something that r approaches doesn't depend on, r then the limit does not exist.

No Hospital's rule? Use Taylor series of e^{-r^2} and factor r out.

8 Partial derivative

The difference between derivatives and differential? Suppose we have

$$f\colon \mathbb{R} \to \mathbb{R}$$

The difference is that:

- To differentiate is an operator: $\frac{d}{dx}$
- The derivative is the function: f'.

$$f \to \frac{d}{dx} \to f'$$

For $f : \mathbb{R}^2 \to \mathbb{R}$, it is not clear which flavor of derivative we can have. Suppose I have this function:

$$f(x,\ y) = y + x$$

Take $f(1,\ y) = 1 + y$. This is a function based on y. If we take $y = 2$, then we would have $f(x,\ 2) = x + 2$. Therefore, this function $f(x,\ c)$ is a single-variable function based on x. If you take x or y (one of them in a certain value) then we will come up with a certain function based on x or y. Now, we can talk about $f'(1,\ y)$ and $f'(x,\ c)$ (where c is a constant).

What does that mean? We have $\frac{df}{dy}$. We can also talk about $f'(x,\ 2) = \frac{df(x,\ 2)}{dx}$.

For partial derivatives, we use this notation:

$$\frac{\partial f}{\partial x}$$

If I want a partial derivative with respect to x, I want y to be a certain number $(y = b)$, and I make a slice: now I see a parabola. Now I can consider the partial derivative.

If I want the partial derivative with respect to y, I slice something on the x-axis: $x = a$. Then I end up with another single variable function.

To make it clear:

$$\frac{\partial f}{\partial x}(a,\ b) = \frac{df(x,\ b)}{dx}\ |_{x=a}$$

This is the partial derivative, with respect to x , at $a,\ b$ is the differentiation of the function $f(x,\ b)$ computed at $x = a$.

Official definition for partials:

Let f: $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function of two variables. Let $(a,\ b) \in D_f$.

The partial derivative of f with respect to x at $(a,\ b)$, denoted vy $f_x(a,b)$ or $\frac{\partial f}{\partial x}(a,\ b)$ is:

$$\frac{\partial f}{\partial x}(a,\ b) = \lim_{h \to 0} \frac{f(a+h,\ b) - f(a,\ b)}{h}$$

The partial derivative of f with respect to x at $(a,\ b)$ denoted by $f_x(a,\ b)$ denoted by $\frac{\partial f}{\partial x}\ (a,b)$ s:

$$f_y(a,b) = \lim_{h \to 0} \frac{f(a,\ b+h) - f(a,b)}{h}$$

8.1 Notation for partial derivatives

If $z = f(x,\ y)$, we write:

$$\begin{aligned} f_x(x,\ y) &= \frac{\partial f}{\partial x}\ (x,\ y) = \frac{dz}{dx} \\ f_y(x,\ y) &= \frac{\partial f}{\partial y}\ (x,\ y) = \frac{dz}{dy} \end{aligned}$$

Example 8.1. Let $f(x,\ y) = \ln\left(2x^2 + y^3 + 1\right)$. Find $f_x(2,\ 1)$ and $f_y(2,\ 1)$.

$$\begin{aligned} \frac{\partial f}{\partial x}(x,\ y) &= \frac{4x}{2x^2 + y^3 + 1} \\ f_x(2,\ 1) &= \frac{8}{10} = \frac{4}{5} \\ f_y(x,\ y) &= \frac{3y^2}{2x^2 + y^3 + 1} \\ f_y(2,\ 1) &= \frac{3}{8+1+1} = \frac{3}{10} \end{aligned}$$

Example 8.2. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined as:

$$e^z = xyz$$

First one: use implicit differentiation. take the derivative of both sides, treating y as a constant and keeping in mind that z is a function of x: $\frac{\partial z}{\partial x}$

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(e^z\right) &= \frac{\partial}{\partial x}(xyz) \\
\frac{\partial z}{\partial x}e^z &= yz + xy\left(\frac{\partial z}{\partial x}\right) \\
\frac{\partial z}{\partial x}e^z - xy\left(\frac{\partial z}{\partial x}\right) &= yz \\
\frac{\partial z}{\partial x}\left(e^z - xy\right) &= yz \\
\frac{\partial z}{\partial x} &= \frac{yz}{e^z - xy}
\end{aligned}
$$

Computing $\frac{\partial z}{\partial y}$:

$$
\begin{aligned}
\frac{\partial}{\partial y}\left(e^z\right) &= \frac{\partial}{\partial y}(xyz) \\
\frac{\partial z}{\partial y}e^z &= xz + xy\frac{\partial z}{\partial y} \\
\frac{\partial z}{\partial y}\left(e^z - xy\right) &= xz \\
\frac{\partial z}{\partial y} &= \frac{xz}{e^z - xy}
\end{aligned}
$$

8.2 With a curve

Suppose we have $z = f(x,\ y) = x^2 + 2y^2$.

Then:

$$
\begin{aligned}
f(1,\ y) &= 1 + 2y^2 = g(y) \\
g'(y) &= 4y
\end{aligned}
$$

This means

$$\frac{df}{dy} = 4y$$

This means:

$$\frac{\partial f}{\partial y}(1,\ 1) = f_y(1,1) = 4y\ |_{y=1} = 4$$

And:

$$g'(1) = \lim_{h \to 0} \frac{g(1+h) - g(1)}{h}$$
$$\frac{\partial f}{\partial y}(1,\ 1) = \lim_{h \to 0} \frac{f(1,\ 1+h) - f(1,\ 1)}{h}$$

8.3 Four dimensional

Example 8.3. Find $f_x,\ f_y,\ f_z$ if $f(x,\ y,\ z) = e^{xy}\ln(z)$.

Then, treat uninvolved parameters as constants:

$$f_x = ye^{xy}\ln(z)$$
$$f_y = xe^{xy}\ln(z)$$
$$f_z = \frac{e^{xy}}{z}$$

9 Higher derivatives

Let $f \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function of two variables. Then f_x and f_y are given functions of two variables. So, we consider their partial derivatives:

$$(f_x)_x,\ (f_x)_y,\ (f_y)_x,\ (f_y)_y$$

We call them higher derivatives. These are just different uses of notation.

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

Beware of notations: they involve nesting.

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\ \left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

Note that the y is placed on the left.

Example 9.1. Find $f_{xy} - 2f_{yx}$ if $f(x,\ y) = e^{xy}\sin(y)$.

Solution 9.1. $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

$$\begin{aligned}
&= \frac{\partial}{\partial y}\left(ye^{xy}\sin(y)\right) \\
&= \left(e^{xy} + xe^{xy}\right)\sin(y) + \left(ye^{xy}\cos(y)\right)
\end{aligned}$$

And $f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

$$\begin{aligned}
&= \frac{\partial}{\partial x}\left(xe^{xy}\sin(y) + e^{xy}\cos(y)\right) \\
&= e^{xy}\sin(y) + xe^{xy}\sin(y) + ye^{xy}\cos(y)
\end{aligned}$$

Theorem 9.1 (Clairaut's theorem)**.** *Suppose f is defined on a disk D that contains the point $(a,\ b)$. If f_{xy} and f_{yx} are both continuous, then:*

$$f_{xy}(a,\ b) = f_{yx}(a,\ b)$$

To find partial derivatives for functions with potential discontinuities, like piecewise functions, you must use the definition of the derivative.

Example 9.2. Let

$$f(x,\ y) = \begin{cases} \frac{x^3y - xy^3}{x^2+y^2} & (x,\ y) \neq (0,\ 0) \\ 0 & (x,\ y) = (0,\ 0) \end{cases}$$

Find $f_x(x,\ y)$ and $f_y(x,\ y)$.

For $(x,\ y) \neq 0$:

$$\begin{aligned}
f_x &= \frac{\left(3x^2y - y^3\right)\left(x^2 + y^2\right) - \left(x^3y - xy^3\right)\left(2x + y^2\right)}{\left(x^2+y^2\right)^2} \\
f_y &= \frac{\left(x^3 - 3xy^2\right)\left(x^2 + y^2\right) - \left(x^3y - xy^3\right)\left(x^2 + 2y\right)}{\left(x^2+y^2\right)^2}
\end{aligned}$$

If $(x,\ y) = 0$:

$$\begin{aligned}
f_x(0,\ 0) &= \lim_{h\to 0} \frac{f(0+h,\ 0) - f(0,\ 0)}{h} \\
&= \lim_{h\to 0} \frac{f(h,\ 0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0 \\
f_y(0,\ 0) &= 0
\end{aligned}$$

9.1 For potential discontinuities

Say we have:

$$f(x,\ y) = \begin{cases} \frac{x^3y - xy^3}{x^2+y^2} & (x,\ y) \neq (0,\ 0) \\ 0 & (x,\ y) = (0,\ 0) \end{cases}$$

For $(x,\ y) \neq (0,\ 0)$:

$$f_x(x,\ y) = \frac{\left(3x^2y - y^3\right)\left(x^2+y^2\right) - \left(x^3y - xy^3\right)(2x)}{(x^2+y^2)}$$

For $(0,\ 0)$:

$$f_x(0,\ 0) = \lim_{h\to 0} \frac{f(h,\ 0) - f(0,\ 0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

This means:

$$\begin{aligned}
&f_x(x,\ y) \\
&= \begin{cases} \frac{\left(3x^2y-y^3\right)\left(x^2+y^2\right)-\left(x^3y-xy^3\right)(2x)}{\left(x^2+y^2\right)} & (x,\ y) \neq (0,\ 0) \\ 0 & (x,\ y) = 0 \end{cases}
\end{aligned}$$

By the same method, you can compute f_y. Consider two cases, $(x,\ y) = 0$ and $(x,\ y) \neq 0$.

$$\begin{aligned}
f_{xy}(0,\ 0) &= \lim_{h\to 0} \frac{f_x(0,\ h) - f_x(0,\ 0)}{h} \\
&= \lim_{h\to 0} \frac{\frac{\left(-h^3\right)}{\left(h^2\right)}}{h} = \lim_{h\to 0} -\frac{h}{h} = -1
\end{aligned}$$

10 Partial Differential Equations

Laplace's equation:

For $u : D \subseteq \mathbb{R}^2 \mapsto \mathbb{R}$, the Laplace's equation is:

$$u_{xx} + u_{yy} = 0$$

Solution of these equations are called harmonic equations. If u can satisfy this equation, then it is a solution of Laplace's equation and u is harmonic.

1D wave equation:

For $u : D \subseteq \mathbb{R}^2 \mapsto \mathbb{R}$, the 1D wave equation is:

$$u_{yy} = a^2 u_{xx}$$

Where a is a constant. If your function u can satisfy this equation, then it is a 1D wave equation.

Example 10.1. Let $f(x,\ y) = e^{-x}\cos(y) - e^{-y}\cos(x)$. Show that f is a solution of Laplace's equation. In other words: (note that $\frac{\partial f}{\partial x} = f_x$)

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Well:

$$\begin{aligned}
f_x &= -e^{-x}\cos(y) + e^{-y}\sin(x) \\
f_{xx} &= e^{x}\cos(y) + e^{-y}\cos(x) \\
f_y &= -e^{-x}\sin(y) + e^{-y}\cos(x) \\
f_{yy} &= -e^{-x}\cos(y) - e^{-y}\sin(x)
\end{aligned}$$

If we add them, do they cancel?

$$\begin{aligned}
u_{xx} + u_{yy} &= e^{x}\cos(y) + e^{-y}\cos(x) - e^{-x}\cos(y) - e^{-y}\sin(x) \\
&= 0
\end{aligned}$$

Now we know. f is a solution of Laplace's equation.

Example 10.2. Verify that $u(x,\ t)=\sin(x-at)+\ln(x+at)$ where a is a constant is a solution of the wave equation.

$$\begin{aligned}
u_x &= \cos(x-at)+\frac{1}{x+at}\\
u_{xx} &= -\sin(x-at)-\frac{1}{(x+at)^2}\\
u_t &= -\cos(x-at)\cdot a+\frac{a}{x+at}\\
u_{tt} &= -\sin(x-at)\cdot a-\frac{a^2}{(x+at)^2}
\end{aligned}$$

Putting them together:

$$-\sin(x-at)-\frac{1}{(x+at)^2}=\left(-\sin(x-at)\cdot a-\frac{a^2}{(x+at)^2}\right)k$$

I probably messed up. Maybe not. What do I do.

11 Tangent Planes

1. Let S be a surface whose equation is $z=f(x,\ y)$
2. Let first partial derivatives of f be continuous.
3. Let $p\,(x_0,\ y_0,\ z_0)$ be a point on S.
4. Let C_1 be the curve obtained by intersecting the vertical plane $y=y_0$. Let T_1 be the tangent line to the curve C_1 at the point p.
5. Let C_2 be the curve obtained by the curve obtained by intersecting the vertical plane $x=x_0$. Let T_2 be the tangent line ot the curve C_2 at the point p. What is the slope of T_2?

Equation of a tangent plane

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z=f(x,\ y)$ at the point $p\,(x_0,\ y_0,\ z_0)$ is:

$$
\begin{aligned}
& z - z_0 \\
& = \overset{\text{normal}}{f_x}(x_0,\ y_0)\,(x - x_0) \\
& + \overset{\text{normal}}{f_y}(x_0,\ y_0)\,(y - y_0)
\end{aligned}
$$

The gradient of a function is:

$$
\begin{gathered}
f : \mathbb{R}^2 \to \mathbb{R} \\
\nabla f(x,\ y) = (f_x(x,\ y),\ f_y(x,\ y))
\end{gathered}
$$

∇f is always orthogonal to the surface.

If $f : \mathbb{R}^3 \to \mathbb{R}$, then $\nabla f(x,\ y,\ z) = \begin{bmatrix} f_x(x,\ y,\ z) \\ f_y(x,\ y,\ z) \\ f_z(x,\ y,\ z) \end{bmatrix}$

Example 11.1. Find the equation of the tangent plane for $z = y^2 e^x$ at the point $(0,\ 3,\ 9)$.

Solution 11.1. I can say that

$$
\begin{gathered}
z = f(x,\ y) = y^2 e^x \\
f_x(x,\ y) = y^2 e^x \Rightarrow f_x(0,\ 3) = 9 \\
f_y(x,\ y) = 2y e^x \Rightarrow f_y(0,\ 3) = 6
\end{gathered}
$$

We have

$$
\begin{aligned}
z - z_0 &= f_x(0,\ 3)\,(x - x_0) + f_y(0,\ 3)\,(y - y_0) \\
\Rightarrow z - 9 &= 9x + 6(y - 3) \\
9 &= 9x + 6y - z
\end{aligned}
$$

You may also wish to consider:

$$
\begin{aligned}
z &= y^2 e^x \\
z - y^2 e^x &= 0 \quad z = F(x,\ y,\ z) \\
\nabla F(0,\ 3,\ 9) &= \begin{bmatrix} f_x(0,\ 3,\ 9) \\ f_y(0,\ 3,\ 9) \\ f_z(0,\ 3,\ 9) \end{bmatrix} \\
F_x(x,\ y) &= -y^2 e^x \\
F_y(x,\ y) &= -2ye^x \\
F_z(x,\ y) &= 1 \\
\overrightarrow{N} &= \begin{bmatrix} -9 \\ -6 \\ 1 \end{bmatrix} \quad p_0 = (0,\ 3,\ 9)
\end{aligned}
$$

$$
\begin{bmatrix} -9 \\ -6 \\ 1 \end{bmatrix} \begin{bmatrix} x \\ y-3 \\ z-9 \end{bmatrix} = 0
$$

Example 11.2. Determine the equation of the tangent plane to the surface $z = x^3 - 3xy + y^3$ at point $(1,\ 2,\ 3)$. If $z = f(x,\ y)$, then

$$
\begin{aligned}
f_x &= 3x^2 - 3y \\
f_y &= 3x + 3y^2 \\
z - z_0 &= f_x(x_0,\ y_0)(x - x_0) + f_y(x_0,\ y_0)(y - y_0) \\
\ldots \Rightarrow 3x - 9y + z &= -12
\end{aligned}
$$

12 Linear Approximations

A linear function $f\colon \mathbb{R} \to \mathbb{R}$ can be represented as $f(x) = ax + b$, where a and b are constants. It looks like a map when graphed.

Perhaps you have a function $f\colon \mathbb{R} \to \mathbb{R}$ and the graph looks like some curve that isn't a line.

Say we have a point $(a,\ f(a))$. If the function is differentiable at a, then consider the tangent line at $(a,\ f(x))$, which we call $L(x)$. What is the formula for this tangent line? It is:

$$
y - f(a) = f'(a)(x - a)
$$

If you consider some point b that is close to a, then:

- A point on the function is $(b,\ f(b))$
- Consider $L(b)$: $f(b)$ is close to $L(b)$. $L(b)$ is linear, so I can compute it easily.

If you consider a point c that is far from a, we can still approximate $f(c)$ with $L(c)$, but it may not be a good approximation. But if it is close, you can compute it.

The problem is, how can we extend what we said already if we have a function from $\mathbb{R}^2 \to \mathbb{R}$? What does linear mean in 3D space>

$f : \mathbb{R}^2 \to \mathbb{R}$ is a linear function if we can write f as the following:

$$f(x,\ y) = ax + by + c$$

Where $a,\ b,\ c$ are constants. This is the standard equation for a plane. If I have a surface that I don't know the formula for, but I want to consider the tangent plane at some point on the surface, if the visual gap is small then I have made a good approximation.

The linear function whose graph is the tangent plane is:

$$L(x,\ y) = f_x(a,\ b)(x-a) + f_y(a,\ b)(y-b) + f(a,\ b)$$

This is called the linearization of f at $(a,\ b)$.

The linear approximation of f at $(a,\ b)$ is:

$$f(x,y) \approx L(x,y) = f_x(a,\ b)(x-a) + f_y(a,\ b)(y-b) + f(a,\ b)$$

Example 12.1. Let $f(x,\ y) = xe^{xy}$. Find the linearization of f at $(1,\ 0)$. Approximate $f(1.1,\ -0.1)$

Solution 12.1. Firstly,

$$f_x(x,\ y) = e^{xy} + xe^{xy}y$$
$$f_y(x,\ y) = x^2e^{xy}$$

$$\begin{aligned} L(x,\ y) &= f_x(1,\ 0)(x-1) + f_y(1,\ 0)(y-0) + f(1,\ 0) \\ &= (1+0)(x-1) + (1)(y-0) + f(1,\ 0) \\ &= x - 1 + y + 1 \\ &= x + y \end{aligned}$$

To approximate $f(1.1,\ -0.1)$, just plug it in the linear approximation: $1.1-0.1=1$. There we go.

Comparisons: $f(1.1,\ -0.1)=(1.1)e^{(1.1)y}=0.985$. It's close to 1.

If $f:\mathbb{R}^2\to\mathbb{R}$ does not have continuous first partal derivatives, the linearization for the function f fails.

12.1 The Nabla Operator

If you have an $\mathbb{R}^n$ cartesian system $(x_1,x_2,\dots,\ x_n)$, we define nabla (∇) as:

$$\nabla=\begin{bmatrix}\frac{\partial}{\partial x_1}\\ \frac{\partial}{\partial x_2}\\ \vdots\\ \frac{\partial}{\partial x_n}\end{bmatrix}$$

This means, in $\mathbb{R}^2$:

$$\nabla=\begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{bmatrix}$$

In $\mathbb{R}^3$:

$$\nabla=\begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{bmatrix}$$

If we have $f:\mathbb{R}^2\to\mathbb{R}$, we can define

$$\nabla f=\left\langle\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right\rangle$$

If we have $f:\mathbb{R}^3\to\mathbb{R}$, we can define:

$$\nabla f=\left\langle\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right\rangle$$

We call this the *gradient* of f.

12.2 Upcoming

If we have $f : \mathbb{R}^n \rightarrow \mathbb{R}$, we may have:

$$div(f) = \nabla \cdot f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$

We also have $curl(f) = \nabla \times f$

For Laplace's equation, this satisfies it: $\nabla^2 f = 0$.

13 Multivariable derivatives

In f: $\mathbb{R} \rightarrow \mathbb{R}$, $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$,

$$\Leftrightarrow \lim_{h \rightarrow 0} \frac{f(x+h) - f(x) - f'(x)h}{h} = 0$$

If the limit exists. Now, we can say that f' is a 1-by-1 matrix (a number). We can also say that it is some sort of operator:

$$A(h) := f'(x) \cdot h$$

$A(h)$ is a linear operator (a linear transformation). A **linear** transformation is, if you have a mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $T\left(\overrightarrow{x} + \lambda \overrightarrow{y}\right) = T\left(\overrightarrow{x}\right) + \lambda T\left(\overrightarrow{y}\right)$. This means linear transformations always fix the origin.

If we want to extend this notion for functions $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, what should be done? How should I define the notion of derivative?

$$\lim_{h \rightarrow (0,\ 0)} \frac{\left| f\left(\overrightarrow{x} + \overrightarrow{h}\right) - f\left(\overrightarrow{x}\right) - A\left(\overrightarrow{h}\right) \right|}{\left|\overrightarrow{h}\right|}$$

Then, A is a 1×2 matrix (column vector) and is a transformation from $\mathbb{R}^2 \rightarrow \mathbb{R}$. **If** I can find A:

Then:

$$f'(x) = A$$

If $f_x,\ f_y$ are continuous, then

$$A = \begin{bmatrix} f_x & f_y \end{bmatrix} = \nabla f$$

If we have $\mathbb{R}^3$, the same formula applies, and if $f_x,\ f_y,\ f_z$ are continuous, then $A = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix}$

In a much more general setting (which we won't focus on), if you have a function $\mathbb{R}^m \to \mathbb{R}^n$, then A is an $n \times m$ matrix and is composed of a matrix I would like to avoid seeing in this course.

To show that something is differentiable at a point, I just need to show that $\langle f_x,\ f_y \rangle$ is continuous.

Definition 13.1 (Differentiability at a point)**.** Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function and let $x \in E$. If there exists a 1×2 matrix A such that

$$\lim_{h \to 0} \frac{|f(x+h) - f(x) - A(h)|}{|h|} = 0 \quad h = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \in \mathbb{R}^2$$

$$\lim_{h \to 0} \frac{|f(x+h) - f(x) - A(h)|}{|h|} = 0 \quad h = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \in \mathbb{R}^2$$

Then we say that f is differentiable at x and we can write $f'(x) = A$, which is a 1×2 matrix.

Theorem 13.1 (I can tell that it is differentiable there)**.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. If f_x and f_y exist near the point $(a,\ b)$, and $f_x,\ f_y$ are continuous at $(a,\ b)$, then f is differentiable at $(a,\ b)$ and $f'(a,\ b) = \langle f_x(a,\ b),\ f_y(a,\ b) \rangle$ (column vector).*

Let me say this again:

- If $f_x,\ f_y$ exist near $(a,\ b)$
- And $f_x,\ f_y$ are continuous at $(a,\ b)$
- Then f is differentiable at $(a,\ b)$ and $f'(a,\ b) = \begin{bmatrix} f_x(a,\ b) & f_y(a,b) \end{bmatrix}$.

The **converse** of this theorem is **NOT TRUE.** Simple counterexample:

$$f(x,\ y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & (x,\ y) \neq (0,\ 0) \\ 0 & (x,\ y) = (0,\ 0) \end{cases}$$

We can see that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ is not continuous.

Compared with $\nabla f(a,\ b)$, it means that f is in the class of C_1.

Definition 13.2 (Smoothness)**.** Let $n,\ k \in \mathbb{Z}_{>0}$. Let $E \subseteq \mathbb{R}^n$ be an open set and let $f:\mathbb{R}^n \to \mathbb{R}$ be a function. We say that f is of the class C^k on E if all kth partial derivatives if f is continuous on E. We cay that f is of the class C^∞ on E if all partial derivatives of all orders of f are continuous on E.

Example 13.1. Let $f(x,\ y) = 1 + x\ln(xy-5)$.

1. Is f differentiable at the point $(2,\ 3)$?
2. What is the linear approximation of f at $(2,\ 3)$?
3. Find an appropriate approximate for $f(2.1,\ 3.1)$

Solution 13.1. Checking differentiability:

$$
\begin{aligned}
f_x &= \ln(xy-5) + x \cdot \frac{y}{xy-5} \\
f_y &= x \cdot \frac{x}{xy-5}
\end{aligned}
$$

Continuous at the point $(2,\ 3)$? It is – substitute!! Then f is differentiable at $(2,\ 3)$ and it is $\langle f_x(2,\ 3),\ f_y(2,\ 3)\rangle = \langle 6,\ \ 4\rangle$

14 Differential

Motivation: Let $y = f(x)$ be a function.

For f: $\mathbb{R} \to \mathbb{R}$, $f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. Then $dy = f'(x)dx$. If you change x by a bit, then can you quantify the value of y, which may have also changed?

1. Change of f as x changes form a to $a + \Delta x$: $\Delta f = f(a+\Delta x) - f(a)$.

If you have a function from $\mathbb{R}^2 \to \mathbb{R}$, the definition may have changed by a bit:

$$
\begin{aligned}
z &= f(x,\ y) \\
\Rightarrow dz &= \nabla f(x,\ y) \cdot \langle dx,\ dy\rangle \\
&= f_x dx + f_y dy
\end{aligned}
$$

Example 14.1. Let $z = f(x,\ y) = x^2 - 3xy - y^2$. Find the total differential of f. If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

Then: $f_x = 2x+3y,\ f_y = 3x-2y,\ df = (2x+3y)dx+(3x-2y)dy.$

Then:

$$
\begin{aligned}
(x,\ y) = (2,3),\ dx = 0.5,\ dy = -0.4 \\
dz &= f_x(2,\ 3)\cdot 0.5 + f_y(2,3)\cdot(-0.4) \\
&= \frac{13}{2} + 13\cdot(-0.4)
\end{aligned}
$$

In other words:

$$
\Delta z = f(2.05,\ 2.96) - f(2,\ 3) = TBA
$$

15 Chain Rule

If $g,\ f\colon \mathbb{R}\to\mathbb{R}$

Then, $f\left(g(t)\right)' = f'\left(g(t)\right)g'(t)$

$$
\begin{aligned}
y = f(x) = f\left(g(t)\right),\ x = g(t) \\
\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}
\end{aligned}
$$

Now suppose you have a function $\mathbb{R} \xrightarrow{g} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$ and $f(x,\ y) = f\left(x(t),\ y(t)\right)$. $g(t) = (x(t),\ y(t))$

$$
t \mapsto \left(x(t),\ y(t)\right),\ (x,\ y)\mapsto f(x,\ y)
$$

Then:

$$
f\circ g(t) = f\left(x(t),\ y(t)\right)
$$

Now, I want to compute $(f\circ g)'(t)$. Then:

$$f'(g(t))\,g'(t) = f'(g(t))\,g'(t) = \begin{bmatrix} f_x(g(t)) & f_y(g(t)) \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \end{bmatrix}$$
$$= f_x g(t) \cdot \frac{dx}{dt} + f_y g(t) \cdot \frac{dy}{dt}$$

If I have $z = f(x,\ y)$, then:

z is a function of x and y; $x = x(t)$ and $y = y(t)$. $x,\ y$ are the intermediate variables and t is the independent variable (t does not depend on anything).

If I want to define $\frac{dz}{dt}$:

- $z \to \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$
- Each leading to $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

Then:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Because $z = f(x,\ y)$, then $\frac{\partial z}{\partial x} = f_x$. If you change t, both x and y will be changed. Because z is the result of a function of x and y, z will be changed. This means you'll have to consider the changes in both x and y should t change by just a bit.

Example 15.1. Let $z = x^2y^2 + 3x^2y^4$ and let $x = \sin(2t)$ and $y = \cos(2t)$. Find $\frac{dz}{dt}$.

We note that z is a function of $x,\ y$ and $x,\ y$ are functions of t. All you need to know to compute the partial derivatives is to draw this diagram:

$$\begin{array}{ccc} & z & \\ \swarrow & & \searrow \\ \frac{\partial z}{dx} \to x & & \frac{\partial z}{dy} \to y \\ | & & | \\ \frac{dx}{dt} \to t & & \frac{dy}{dt} \to t \end{array}$$

So:

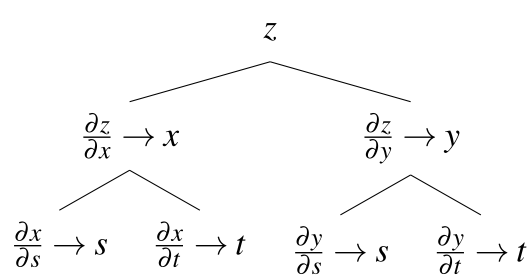
$$
\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} \\
&= \left(2xy^2 + 6xy^4\right)(2\cos(2t)) + \left(2x^2y + 4x^2y^3\right)(-2\sin(2t)) \\
&\left(2\sin(2t)\cos^2(2t) + 6\sin(2t)\cos^4(2t)\right)(2\cos(2t)) \\
&+ \left(2\sin^2(2t)\cos(t) + 4\sin^2(2t)\cos^3(2t)\right)(-2\sin(2t))
\end{aligned}
$$

15.1 Chain Rule Case 2

Suppose that $z = f(x,\ y)$ is differentiable, where $x = g(s,\ t)$ and $y = h(s,\ t)$ are differentiable functions of $s,\ t$. Then:

$$
\begin{aligned}
\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} \\
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}
\end{aligned}
$$

The tree diagram goes like the following:



So, I want to compute $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$ and $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$.

If not writing dot products hurt my hand too much, $\frac{\partial z}{\partial s} = \begin{bmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial s} \end{bmatrix}$.

Example 15.2. Let $z = x^2 + 2y^2$ and let $x = \frac{r}{s}$ and $y = r^2 + \ln(s)$. Find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial s}$. Note that

$$
\begin{aligned}
\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r} \\
&= \left(2\frac{r}{s}\right)\left(\frac{1}{s}\right) + \left(4(r^2 + \ln(s)\right)\left(2r^2\right)
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}\\ &= (2x)\left(-\frac{r}{s^2}\right)+(4y)\left(\frac{1}{s}\right)\end{aligned}$$

Example 15.3. Suppose that $z = f(x,\ y)$ has continuous second-order partial derivatives and $x = r^2+s^2$ and $y = 2rs$. Find $\frac{dz}{dr}$ and $\frac{d^2z}{dr^2}$. It is:

$$\begin{aligned}\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\cdot 2r+\frac{\partial z}{\partial y}\cdot 2s\\ \frac{\partial^2 z}{\partial r^2} &= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\cdot 2r+\frac{\partial z}{\partial y}\cdot 2s\right) = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\cdot 2r\right)+\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\cdot 2s\right)\\ &= 2\cdot\frac{\partial z}{\partial x}+2r\cdot\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right)+0+2s\cdot\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)\\ &= 2\cdot\frac{\partial z}{\partial x}+2r\cdot\left(2r\cdot\frac{\partial^2 z}{\partial r^2}+2s\cdot\frac{\partial^2 z}{\partial y\partial x}\right)+\\ &2s\cdot\left(\frac{\partial^2 z}{\partial x\partial y}\left(\frac{\partial x}{\partial r}\right)\cdot\frac{\partial x}{\partial r}+\frac{\partial^2 z}{\partial y^2}\cdot\frac{\partial y}{\partial r}\right)\\ &= 2\cdot\frac{\partial z}{\partial x}+4r^2\cdot\frac{\partial^2 z}{\partial x^2}+8rs\cdot\frac{\partial^2 z}{\partial x\partial y}+4s^2\cdot\partial^2 z/\partial y^2\end{aligned}$$

We know that

$$\begin{aligned}\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) &= \frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial x}{\partial r}+\frac{\partial^2 z}{\partial y\partial x}\cdot\frac{\partial y}{\partial r}\\ &= 2r\cdot\frac{\partial^2 z}{\partial r^2}+2s\cdot\frac{\partial^2 z}{\partial y\partial x}\end{aligned}$$

And

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x\partial y}\left(\frac{\partial x}{\partial r}\right)\cdot\frac{\partial x}{\partial r}+\frac{\partial^2 z}{\partial y^2}\cdot\frac{\partial y}{\partial r}$$

Based on this tree diagram:

$$\begin{tikzcd}
 & & & \frac{\partial z}{\partial x} \arrow[dll, dash] \arrow[drr, dash] & & & \\
 & \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \to x \arrow[dl, dash] \arrow[dr, dash] & & & & \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) \arrow[dl, dash] \arrow[dr, dash] & \\
\frac{\partial x}{\partial r} \to r & & \frac{\partial x}{\partial s} \to s & & \frac{\partial y}{\partial r} \to r & & \frac{\partial y}{\partial s} \to s
\end{tikzcd}$$

Example 15.4. The radius of a right circular clone is increasing at a rate of 1.8 while height decreases at rate of 2.5. What are is the volume of the cone changing when radius is 120 and height is 140?

$$\begin{aligned}
v(r,\ h) &= \frac{1}{3}(r \cdot h) \\
\frac{dv}{dt} &= \frac{\partial v}{\partial r} \cdot \frac{dr}{rt} + \frac{\partial v}{\partial h} \cdot \frac{dh}{dt} \\
\frac{dv}{dt} &= \frac{1}{3}h \cdot 1.8 - \frac{1}{3}r \cdot 2.5
\end{aligned}$$

Plug in $h = 140$ and $r = 120$ to find the answer. (Note that if I get a negative answer, the question is illegal)

Suppose we have

$$\begin{aligned}
& f(x,\ y) \\
& = x^2 e^{\sin(4x)} + \left(x^2 + y^2\right) e^{\cos(5x)} \\
& + \sin(10x) e^{(\sin(x)+\cos(y))^2}
\end{aligned}$$

Find

$$(f_{xy} + f_{yx})^2 - 4f_{xy}$$

f is continuous everywhere, so Clairaut's theorem works. So, the answer is 0.

16 Implicit Function Theorem

Theorem 16.1. *Suppose that $F(x,\ y) = 0$ is differentiable and the equation $F(x,\ y) = 0$ defines y as a differentiable function of x. Then, at any point where $F_y \neq 0$:*

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Proof. $0 = F(x,\ y) \overset{\text{differentiate both sides}}{\Rightarrow} 0 = F_x\left(\frac{dx}{dx}\right) + F_y\left(\frac{dy}{dx}\right)$

$$\Rightarrow 0 = F_x + F_y\left(\frac{dy}{dx}\right)$$
$$\overset{F_y \neq 0}{\Rightarrow} \frac{dy}{dx} = -\frac{F_x}{F_y}$$

■

For functions of three variables:

1. Suppose that $z = f(x,\ y)$.
2. z can implicitly be given as $F(x,\ y,\ z) = 0$. This means $F(x,\ y,\ f(x,\ y)) = 0$. Note that F is a different function than f.
3. Assume that F and f are differentiable.
4. Using the chain rule for $F(x,\ y,\ 0)$, we get:

$$F(x,\ y,\ z) = 0 \overset{\text{diff w/ x}}{\Rightarrow} 0 = F_x\left(\frac{dx}{dx}\right) + F_z\left(\frac{\partial z}{\partial x}\right)$$
$$\overset{\text{if } F_z \neq 0}{\Rightarrow} \frac{\partial z}{\partial y} = -\frac{F_y(x,\ y,\ z)}{F_z(x,\ y,\ z)}$$

Example 16.1. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(0,\ 0,\ 0)$ if $x^3 + z^2 + ye^{xz} + z\cos(y) = 0$.

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + zye^{xz}}{2z + xye^{xz} + \cos(y)}$$
$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{e^{xz}}{2z + xye^{xz} + \cos(y)}$$

To calculate it without the implicit function theorem:

$$\begin{aligned}
0 &= \frac{\partial}{\partial x}\left(x^3 + z^2 + ye^{xz} + z\cos(y)\right) \\
&= 3x^2 + 0 + \frac{\partial}{\partial x}\left(ye^{xz}\right) + 0 \\
\frac{\partial}{\partial x}\left(e^{xz}\right) &= ze^{zx} + x\frac{\partial z}{\partial x}e^{xz} \\
&\Rightarrow \\
&= 3x^2 + yze^{zx} + \frac{\partial z}{\partial x}\left(2z + yxe^{xz} + \cos(y)\right) = 0 \\
\Rightarrow \frac{\partial z}{\partial x} &= -\frac{yze^{xz} + 3x^2}{2z + yxe^{xz} + \cos(y)} = -\frac{F_x}{F_z}
\end{aligned}$$

16.1 Implicit function theorem with a single variable function

Suppose we have a function $f(x) = x^2$. Then, $f'(x) = 2x = \frac{dy}{dx}$. We can see that x is the independent variable, and y depends on x. Yet, this implies $\underbrace{y - x^2}_{F(x,\ y)} = 0$. Now, consider $F(x,\ y) = y - x^2 = 0$. In this context for capital F, x and y are both independent variables. Applying the **implicit function theorem,** we get:

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-2x}{1} = 2x$$

17 The Gradient Vector

Definition 17.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. The gradient vector at a point $(x_0,\ y_0)$ is the vector:

$$\nabla f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0,\ y_0)\,\hat{i} + \frac{\partial f}{\partial y}(x_0,\ y_0)\,\hat{j}$$

$$\begin{aligned}\nabla f\left(x_0,\ y_0,\ z_0\right) \\ = \frac{\partial f}{\partial x}\left(x_0,\ y_0,\ z_0\right)\widehat{i} \\ + \frac{\partial f}{\partial y}\left(x_0,\ y_0,\ z_0\right)\widehat{j} \\ + \frac{\partial f}{\partial z}\left(x_0,\ y_0, z_0\right)\widehat{j}\end{aligned}$$

$$\nabla f(x,\ y,\ z,\ w) = \begin{bmatrix} f_x \\ f_y \\ f_z \\ f_w \end{bmatrix}$$

17.1 Algebraic rules for gradients

It's linear, and most properties from derivatives cover over. Instead of prime, we use ∇.

1. $\nabla(f \pm g) = \nabla f \pm \nabla g$
2. $\nabla(\lambda f) = \lambda \nabla f$
3. Product rule is $\nabla(fg) = \nabla f \cdot g + f \cdot \nabla g$
4. Quotient rule also applies

17.2 The direction of the Gradient, and Why It's Normal

Theorem 17.1. *Let $k \in \mathbb{R}$ be a constant. Let S be a surface with an equation $f(x,\ y,\ z) = k$, and that f is a differentiable function. Let $p\left(x_0, y_0,\ z_0\right)$ be a point on S. Let C be any curve that lies on S and passes p (the path of C must crawl on S). Let $\overrightarrow{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$ be a continuous vector function describing C. Then:*

$$\underbrace{\nabla F\left(x_0,\ y_0,\ z_0\right)}_{\textit{normal}} \cdot \overrightarrow{r'}(t) = 0$$

Recall that $\overrightarrow{r'}(t)$ is always tangent to the surface S that it is on, meaning it is always perpendicular to ∇F. Also, this theorem holds regardless of k, as ∇F is not impacted by k.

∇F is orthogonal to the tangent plane. We can consider the gradient as the normal vector of the plane, at the point.

Proof. $\overrightarrow{r'}(t) = \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix}$. Do a bit of substituting:

$$f(x,\ y,\ z) = f\left(x(t),\ y(t),\ z(t)\right) = k$$

$$\begin{aligned} 0 &= \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt} \\ 0 &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle \\ &= \nabla f\left(x_0,\ y_0,\ z_0\right) \cdot \mathbf{r}'(t) \end{aligned}$$

■

Remark 1*.* Let $\nabla F\left(x_0,\ y_0,\ z_0\right) \neq 0$. Then, ∇F at the point $p\left(x_0,\ y_0,\ z_0\right)$ is orthogonal to the tangent plane to the level surface $F(x,\ y,\ z) = k$ at $p\left(x_0,\ y_0,\ z_0\right)$.

$$\begin{aligned} &\frac{\partial F}{\partial x}\left(x_0,\ y_0,\ z_0\right)\left(x - x_0\right) \\ &+ \frac{\partial F}{\partial y}\left(x_0,\ y_0, z_0\right)\left(y - y_0\right) \\ &+ \frac{\partial F}{\partial z}\left(x_0,\ y_0,\ z_0\right)\left(z - z_0\right) \\ &= 0 \end{aligned}$$

Definition 17.2 (Level surface)**.** $\{(x,\ y,\ z)\ : f(x,\ y,\ z) = c\}$, for $c \in \mathbb{R}$.

Compare it with the case that $z = f(x,\ y)$. Rearrange things. What do we have?

$$\begin{aligned} F(x,\ y,\ z) &= z - f(x,\ y) = 0 \\ &\Rightarrow \nabla F = \left(-f_x,\ -f_y,\ 1\right) \end{aligned}$$

Note: $\frac{\partial}{\partial x}\left(z - f(x,\ y)\right) = \frac{\partial}{\partial x}\left(-f(x,\ y)\right) = f_x(x,\ y)$. That should clarify things.

Definition 17.3. The normal line to S at P is the line passing through P and orthogonal to the tangent plane at P.

Remark 2. The direction of the normal line is given by the gradient $\nabla F(x_0,\ y_0,\ z_0)$.

$$\begin{aligned}&\frac{x-x_0}{F_x(x_0,\ y_0,\ z_0)}\\ &=\frac{y-y_0}{F_y(x_0,\ y_0,\ z_0)}\\ &=\frac{z-z_0}{F_z(x_0,\ y_0,\ z_0)}\end{aligned}$$

Where $F_{x\ldots y\ldots z}(x_0,\ y_0,\ z_0)\neq 0$.

This comes from this form of the equation of a line:

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

Where $a=F_x(x_0,\ y_0,\ z_0),\ b=F_y(x_0,\ y_0,\ z_0),\ c=F_z(x_0,\ y_0,\ z_0)$

Parametric is:

$$\overrightarrow{r}(t)=\begin{bmatrix}x_0\\ y_0\\ z_0\end{bmatrix}+\begin{bmatrix}F_x(x_0,\ y_0,\ z_0)\\ F_y(x_0,\ y_0,\ z_0)\\ F_z(x_0,\ y_0,\ z_0)\end{bmatrix}t$$

17.3 The VERY Long Example

Let $f(x,\ y)=x^2+y^2$ and $F(x,\ y,\ z)=x^2+y^2-z$.

1. What is the difference between f and F?

   a. f is $\mathbb{R}^2\to\mathbb{R}$ and F is $\mathbb{R}^3\to\mathbb{R}$.

2. Find $\nabla f(x,\ y)$ and $\nabla F(x,\ y,\ z)$. Difference?

   a. $\nabla f(x,\ y)=\begin{bmatrix}2x\\ 2y\end{bmatrix}$ and $\nabla F(x,\ y,\ z)=\begin{bmatrix}2x\\ 2y\\ -1\end{bmatrix}$.

b. Note that $\nabla F(x,\ y,\ z) = \begin{bmatrix} \frac{d\mathbf{F}}{dx}(x,\ y,\ \mathbf{z}) \\ \frac{d\mathbf{F}}{dy}(x,\ y,\ \mathbf{z}) \\ \frac{\mathbf{dF}}{\mathbf{dz}}(\mathbf{x},\ \mathbf{y},\ \mathbf{z}) \end{bmatrix}$ (the boldface is what is separate from $\nabla f(x,\ y)$).

3. Find $\nabla f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $\nabla F\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2},\ 1\right)$.

   a. $\nabla f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$ and $\nabla F\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2},\ 1\right) = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ -\mathbf{1} \end{bmatrix}$

4. Find the level **curve** $f(x,\ y) = 1$ and the graph of f.

   a. The **level curve** is $\left\{(x,\ y) \in \mathbb{R}^2\ : x^2 + y^2 = 1\right\}$. Hence, it's a 1-radius circle centered at the origin. If we imagine $z = f(x,\ y)$, the level curve is that function projected at $z = f(x,\ y) = 1$.
   b. The **graph** of f: form a contour map or attempt to sketch the 3D surface.
      i. $z = 1 \Rightarrow x^2 + y^2 = 1$
      ii. $z = 2 \Rightarrow x^2 + y^2 = 2$ (Radius $\sqrt{2}$)
      iii. The surface forms a paraboloid.

5. Find the level surface of $F(x,\ y,\ z) = 0$ and plot the graph.

   a. What is the level surface? **The same as the level curve of** $f(x,\ y)$.
   b. $F(x,\ y,\ z) = 0 \Leftrightarrow z = x^2 + y^2$. The level surface is $\left\{(x,\ y,\ z) \in \mathbb{R}^3\ : z = x^2 + y^2\right\}$, where I could replace $z = x^2 + y^2$ with $F(x,\ y,\ z) = 0$ and the level surface will still be the same.

6. Is there any relationship between the graph of f and the level surface $F(x,\ y,\ z) = 0$?

   a. The **level surface** of F and the **graph** of f look the same.

7. Is $\nabla f(x,\ y)$ orthogonal to any level **curve** $f(x,\ y) = c$ where c is constant?

   a. $\nabla f(x,\ y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$. Plot this as a **vector field**, and they all point away from any level curve $f(x,\ y) = c$ (if it forms a circle, and if it does it would be centered at the origin).

b. Or I could use the theorem $\nabla f(x,\ y) \cdot \overrightarrow{r'}(t) = 0$. I could use the chain rule. If I have a parametric equation $\begin{matrix} x = x(t) \\ y = y(t) \end{matrix}$, then $f\left(x(t),\ y(t)\right) = 1$ implies $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot x'(t) + \frac{\partial f}{\partial y} y'(t) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \nabla f(x,\ y) \cdot \overrightarrow{r'}(t) \Rightarrow \nabla f(x,\ y)$ is orthogonal to any level curve.

8. Is $\nabla F(x,\ y,\ z)$ orthogonal to the level **surface** of $F(x,\ y,\ z) = 1$?

   a. The level surface is $\left\{(x,\ y,\ z) \in \mathbb{R}^3\ : F(x,\ y,\ z) = 1\right\}$. Which is, $x^2 + y^2 - z = 1$. I mean, **this surface is the same as the graph of** $f(x,\ y)$, so $\nabla F(x,\ y,\ z)$ anywhere should be orthogonal to the surface.

9. Find the tangent plane to the graph of f at the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2},\ 1\right)$.

   a. $z - 1 = \sqrt{2}\left(x - \frac{\sqrt{2}}{2}\right) + \sqrt{2}\left(y - \frac{\sqrt{2}}{2}\right)$

   b. By the way, when we set $G(x,\ y,\ z) = x^2 + y^2 - z = 0$, then $\nabla G\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2},\ 1\right) = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ -1 \end{bmatrix} = \overrightarrow{N}$. Which is a normal vector to the tangent plane at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2},\ 1\right)$.

   So, the equation of the tangent plane is $\begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x - \frac{\sqrt{2}}{2} \\ y - \frac{\sqrt{2}}{2} \\ 1 \end{bmatrix} = 0$.

10. Find the points $(a,\ b,\ c)$ in which the tangent plane to the graph of f is orthogonal to the xz-plane (which is $y = 0$).

   a. Two planes are orthogonal if and only if their normal vectors are perpendicular.

   b. A normal vector for the xz plane might be $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

   c. The normal vector for the tangent plane at any point is $\nabla G(x,\ y,\ z) = \begin{bmatrix} 2x \\ 2y \\ -1 \end{bmatrix}$.

   Now, all we need to do is figure out what values of $x,\ y,\ z$ make $\begin{bmatrix} 2x \\ 2y \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$. Which is $2y = 0 \Rightarrow y = 0$. This means the answer is $\{(x,\ y,\ z) \in \mathbb{R}^3\ : y = 0\}$.

11. *Instead of orthogonal, where we use parallel, then use proportions – one is a scalar multiple of the other, a.k.a.* $\overrightarrow{N_1} = \lambda \overrightarrow{N_2}$ *.*

18 Directional Derivative

A derivative in a certain direction.

Definition 18.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function and let $\alpha = (\alpha_1,\ \alpha_2),\ x = (x_1,\ x_2) \in \mathbb{R}^2$. Let $\overrightarrow{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ be a unit vector. Then, the directional derivative of f at $x = (x_1,\ x_2)$ in the direction of $\overrightarrow{u}$ is denoted by $D_{\overrightarrow{u}} f(x)$, is:

$$
\begin{aligned}
D_{\overrightarrow{u}} f(x) &= \lim_{t \to 0} \frac{f\left((x_1,\ x_2) + t\,(u_1,\ u_2)\right) - f\left(x_1, x_2\right)}{t} \\
&= \lim_{t \to 0} \frac{f\left(x_1 + tu_1,\ x_2 + tu_2\right) - f\left(x_1, x_2\right)}{t}
\end{aligned}
$$

Assume $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then:

$$
\lim_{t \to 0} \frac{f\left(x_1 + t,\ x_2\right) - f\left(x_1,\ x_2\right)}{t}
$$

Assume $u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then:

$$
\lim_{t \to 0} \frac{f\left(x_1,\ x_2 + t\right) - f\left(x_1,\ x_2\right)}{t}
$$

So, partial derivatives are a special case of the directional derivative. $\overrightarrow{u}$ tells us the direction.

Definition 18.2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and let $a = (\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n)$ and $x = (x_1,\ x_2,\ \ldots,\ x_n) \in \mathbb{R}^n$. Let $\overrightarrow{u}$ be a unit vector in $\mathbb{R}^n$. The directional derivatives of f at $\overrightarrow{x}$ in the direction of u ... (TBA)

18.1 Interpretation of directional derivatives

$\overrightarrow{u}$ is always a unit vector. If I slice a surface by a plane (a contains subspace of $\mathbb{R}^3$), the plane **should** include $\overrightarrow{u}$, the tangent line to the curve is called the directional derivative in direction $\overrightarrow{u}$.

Example 18.1. Find the derivative of $f(x,\ y) = xy$ at $p_0(1,\ 2)$ in the direction of the vector of the vector $\begin{bmatrix}1\\1\end{bmatrix}$.

To solve this, you should always normalize the direction vector:

$$\frac{\begin{bmatrix}1\\1\end{bmatrix}}{\sqrt{2}} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$$

And now, we can compute the directional derivative:

$$\begin{aligned}
&\lim_{t\to 0} \frac{f\left(1+t\frac{1}{\sqrt{2}},\ 2+t\frac{1}{\sqrt{2}}\right) - f(1,2)}{t} \\
&= \lim_{t\to 0} \frac{\left(\left(1+t\frac{1}{\sqrt{2}}\right)\left(2+t\frac{1}{\sqrt{2}}\right)\right) - 1\cdot 2}{t} \\
&= \frac{\left(2+3\frac{t}{\sqrt{2}}+t^2\cdot\frac{1}{2}\right) - 2}{t} \\
&= \frac{t\left(\frac{3}{\sqrt{2}}+t\frac{1}{2}\right)}{t} = \frac{3}{\sqrt{2}}
\end{aligned}$$

Fortunately, we have a **theorem.**

Theorem 18.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a **differentiable** function. Let $\overrightarrow{u} \in \mathbb{R}^n$ be a unit vector and let $p \in \mathbb{R}^n$. Then:*

$$D_{\overrightarrow{u}} f(p) = \nabla f(p) \cdot \overrightarrow{u}$$

This theorem does not work if f isn't differentiable.

Example 18.2. Find the derivative of $f(x,\ y) = xy$ at $p_0(1,\ 2)$ in the direction of the vector of the vector $\begin{bmatrix}1\\1\end{bmatrix}$, using the above theorem.

Solution 18.1. $f(x,\ y) = xy$ is differentiable everywhere as $\frac{\partial f(x,\ y)}{\partial y}$ and $\frac{\partial f(x,\ y)}{\partial x}$ exist and are continuous everywhere. So, where $\overrightarrow{u} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$:

$$D_{\overrightarrow{u}} f(1,\ 2) = \begin{bmatrix} y \\ x \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{y}{\sqrt{2}} + \frac{x}{\sqrt{2}} = \frac{\sqrt{2}}{2}(x+y)$$
$$= \frac{\sqrt{2}}{2}(1+2) = \frac{\sqrt{2}}{2} \cdot 3 = \frac{3}{\sqrt{2}}$$

But why may we only use this theorem if the function is differentiable? There's a counterexample.

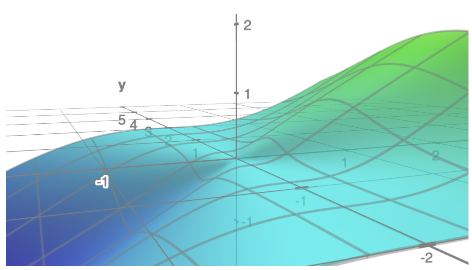
Proposition 1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. If all directional derivatives of f exist at $(a,\ b)$ then f is differentiable at $(a,\ b)$. This statement is false.*

Here's a counterexample: Say we have $f(x) = |x|$. It is **not** differentiable at 0, because there could be two tangent lines there. In $\mathbb{R}^3$, we would say there could be two or more possible tangent planes.

Example 18.3. Show that all the directional derivatives at $(0,\ 0)$ exist. Show that f is not differentiable at $(0,\ 0)$.

$$f(x,\ y) = \begin{cases} \frac{x^3}{x^2+y^2} & (x,\ y) \neq (0,\ 0) \\ 0 & (x,\ y) = (0,\ 0) \end{cases}$$

Graph it out. Look at the origin. I can't look for a distinct tangent plane I could give it.



Behaves weirdly near the origin.

$$D_{\overrightarrow{u}} f(0,\ 0) = u_1^3$$

Now, suppose that f is differentiable at $(0,\ 0)$, we can apply the theorem (somehow, we chose a random directional derivative):

$$D_{\vec{u}} f(0,\ 0) = \nabla f(0,\ 0) \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$
$$2 = 1$$

Contradiction. With the assumption that f is differentiable at $(0,\ 0)$.

Example 18.4. $f(x,\ y) = xe^y + \cos(xy)$. Find the directional derivative of f at the point $(2,\ 0)$ in the direction of $v = 3i - 4j$.

This function is continuous as it is composed of continuous functions. $\frac{\vec{v}}{|\vec{v}|} = \frac{\langle 3,\ -4 \rangle}{\sqrt{3^2+4^2}} = \frac{\langle 3,\ -4 \rangle}{\sqrt{25}} = \left\langle \frac{3}{5},\ -\frac{4}{5} \right\rangle$.

18.2 Finding the Direction of Maximum Change

Theorem 18.2. *If we have a function that is* ***differentiable****, the maximum value of the directional derivative $D_{\vec{u}} f(x)$ is $|\nabla f(x)|$ and it occurs when $\vec{u}$ has the same direction as the gradient vector $\nabla f(x)$. (Remember 3b1b)?*

The minimum value of the directional derivative is $-|\nabla f(x)|$ and occurs when $\vec{u}$ has the opposite direction as ∇f

The value of the direction derivative is zero if u is orthogonal to $\nabla f(x) \neq 0$.

Proof. Since f is differentiable, it follos that

$$D_{\vec{u}} f(x) = \nabla f(x) \cdot u = |\nabla f(x)|\,|u| \cos(\theta) = |\nabla f(x)| \cos(\theta)$$

Where θ is the angle between ∇f and u. Maximize it by making θ zero. Minimize it by making θ a right angle, π. ■

Example 18.5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function given by $f(x\ ,y) = \frac{x^2}{2} + \frac{y^2}{2}$. Find the directions in which f increases the most rapidly at $(1,\ 1)$.

Well, $\nabla f(x,\ y) = \begin{bmatrix} x \\ y \end{bmatrix}$. At $(1,\ 1)$, then in direction $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. That's the direction where it increases the most rapidly. And in the opposite direction $-\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ gives us the minimum value.

19 Minimum and Maximum Values

Motivation. $f : \mathbb{R} \to \mathbb{R}$

Definition 19.1. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a function and let D_f be the domain of f (a set):

1. f has an absolute (or global) maximum value on D_f at the point $c \in D_f$ if $f(x) < f(c)$, well, if it exists.
2. f has an absolute (or minimum) maximum value on D_f at the point $c \in D_f$ if $f(x) < f(c)$, well, if it exists.

Notation. Let $r \in (0, \infty)$ and let $a \in \mathbb{R}$. A neighborhood of a of radius r, denoted by $N_r(a)$, is:

$$N_r(a) = \{x \in \mathbb{R} \ : |x - a| < r\} = \{x \in \mathbb{R} \ : x \in (a - r, \ a + r)\}$$

(Some interval centered around a, on the number line, with radius r).

19.1 Local Min Max

Local minimum on D_f at $c \in D_f$ if $f(x) \leq f(c)$, for all $x \in N_r(c) \subseteq D_f$ for some $r \in (0, \infty)$.

19.2 Critical Point

- c is a critical point if $f'(c) = 0$ or $f'(c)$ is undefined
- c is a stationary point if $f'(c) = 0$

Definition 19.2. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a function and let D_f be the domain of f. Let $c \in D_f$. c is an inflection point if the graph of f has a tangent line and concavity changes. I mean critical point for f'.

- All local max points, local min points, all inflexion points $\subseteq$ stationary points (0 derivative) $\subseteq$ all critical points
- Any continuous function on a compact set can attain a minimum and maximum value.

19.3 Neighborhoods in $\mathbb{R}^2$

Open disk:

$$N_r(a,\ b) := \left\{(x,\ y) \in \mathbb{R}^2\ : (x-a)^2 + (y-b)^2 < r^2\right\}$$

Closed disk:

$$\overline{N_r(a,\ b)}\ := \left\{(x,\ y) \in \mathbb{R}^2\ : (x-a)^2 + (y-b)^2 \leq r^2\right\}$$

Theorem 19.1 (the first derivative theorem for local extreme values)**.** *Let $c \in D_f$ be a local max or local min. If f' is defined at c, then $f'(c) = 0$.*

Theorem 19.2 (EVT)**.** *If $f\colon \mathbb{R} \to \mathbb{R}$ is **continuous** in a compact subset of $\mathbb{R}$ like $[a,\ b]$ (any closed **and** bounded interval), then f attains both an absolute max and absolute min.*

19.4 Max and Mins in Two Dimensions

"Tip of a hill in all directions" vs. "pringle"

The "pringle" gives us saddle points: where it may be a min/max if we cut through either axis.

Definition 19.3 (critical points for multivariable functions)**.** A critical point at any $\mathbb{R}^2 \to \mathbb{R}$ function is all $(x,\ y)$ when $\nabla f(x,\ y) = \overrightarrow{0}$. This implies that $f_x(x,\ y) = 0$ and $f_y(x,\ y) = 0$. Also, if one partial doesn't exist so does the gradient (flat tangent plane or tangent plane DNE?).

The definition for $\mathbb{R}^2 \to \mathbb{R}$:

f has a local max at $(a,\ b)$ if $f(x,\ y) \geq f(a,\ b)$ for all $(x,\ y) \in N_r(a,\ b) \subseteq D_f$ for some $r \in (0,\ \infty)$ (can be epsilon-sized a.k.a. infinitely small)

f has a local min at $(a,\ b)$ if $f(x,\ y) \leq f(a,\ b)$ for all $(x,\ y) \in N_r(a,\ b) \subseteq D_f$ for some $r \in (0,\ \infty)$

19.5 CRITICAL POINTS IN R2 – Testing for local extremes

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function and let $(a,\ b) \in D_f$ be a local max or min point. If the first order partial derivatives of f exist, then $\nabla f(a,\ b) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Note the following equivalences:

$$\nabla f(a,\ b) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} f_x(a,\ b) \\ f_y(a,\ b) \end{bmatrix} \Leftrightarrow f_x(a,\ b) = 0,\ f_y(a,\ b) = 0$$

19.6 Hessian Matrix, Finding Critical Points, and Min/Max Values

We have some critical points. What do they mean?

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. The hessian matrix of f at $(a,\ b)$, denoted $Hf(a,\ b)$, is:

$$Hf(a,\ b) = \begin{bmatrix} f_{xx}(a,\ b) & f_{xy}(a,\ b) \\ f_{yx}(a,\ b) & f_{yy}(a,\ b) \end{bmatrix}$$

$Hf(a,\ b)$ happens to be exactly what the second derivative of f would be, which is formed by taking the derivative of ∇f.

By the way, an $\mathbb{R}^n \to \mathbb{R}^m$ function, the matrix would be $m \times n$.

Second partial derivatives are packed here and are arranged into a matrix.

If f is C^2 at $(a,\ b)$, then:

$$\det(Hf(a,\ b)) = f_{xx}(a,\ b) f_{yy}(a,\ b) - (f_{xy}(a,\ b))^2$$

Theorem 19.3 (Second derivative test for local extreme values)**.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function and let $(a,\ b) \in D_f$. Suppose first and second partial derivatives of f are continuous on $\overline{N_r(a,\ b)}$ (consider the boundary of the disk as well). Suppose that $f_x(a,\ b) = f_y(a,\ b) = 0$ (so $(a,\ b)$ is a critical point).* ***Then:***

- $\det(Hf(a,\ b)) > 0$
  - $f_{xx}(a,\ b) > 0 \Rightarrow (a,\ b)$ is local min $\cup$
  - $f_{xx}(a,\ b) < 0 \Rightarrow (a,\ b)$ is local max $\cap$
  - Neither otherwise
- $\det(Hf(a,\ b)) < 0$
  - $f(a,\ b)$ is a saddle point
- $\det(Hf(a,\ b)) = 0$

– The test is inconclusive.

Example 19.1. Obtain all the critical points at $xy - x^2 - y^2 - 2x - 2y + 4$.

This requires a system of equations. $\nabla f(x,\ y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ at when? $f_x(x,\ y) = y - 2x - 2 \quad f_y(x,\ y) = x - 2y - 2$

$$\begin{cases} y - 2x - 2 = 0 \\ x - 2y - 2 = 0 \end{cases} \Rightarrow \begin{cases} y = 2x + 2 \\ x = 2y + 2 \end{cases} \Rightarrow x = -2 \Rightarrow y = -2$$

$\Rightarrow (-2,\ -2)$ is a critical point. But what type? Compute the Hessian matrix. By the way, $f_{xy}(x,\ y) = 1$. So:

$$\begin{gathered} Hf(x,\ y) = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \\ \det(Hf(x,\ y)) = 4 - 1 = 3 \end{gathered}$$

The result from the hessian matrix is positive. $f_{xx}(-2,\ -2) = -2$, which is less than 0, so $(-2,\ -2)$ is a local maximum.

Example 19.2. Find and classify the critical points for this: $f(x,\ y) = 3y^2 - 2y^3 - 3x^2 + 6xy$

Firstly: $\nabla f(x,\ y) = \begin{bmatrix} -6x + 6y \\ 6y - 6y^2 + 6x \end{bmatrix}$

And the hessian matrix: $\begin{bmatrix} -6 & 6 \\ 6 & 6 - 12y \end{bmatrix}$

Match $\begin{bmatrix} -6x + 6y \\ 6y - 6y^2 + 6x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$:

$$\begin{cases} -6x+6y=0 \\ 6y-6y^2+6x=0 \end{cases}$$

$$\begin{aligned}
\Rightarrow 12y-6y^2 &= 0 \\
\Rightarrow 6\left(2y-y^2\right) &= 0 \\
\Rightarrow 2y-y^2 &= 0 \\
2y\left(1-\frac{1}{2}y\right) &= 0
\end{aligned}$$

$$\Rightarrow y=0 \text{ or } 1-\frac{1}{2}y=0$$

$$\Rightarrow -\frac{1}{2}y=-1 \Rightarrow y=2$$

If $y=0,\ -6x=0 \Rightarrow x=0$. If $y=2,\ -6x+12=0 \Rightarrow -6x=-12 \Rightarrow x=2$

So, our critical points are $(0,\ 0)$ and $(2,\ 2)$. I want to classify these points. Firstly, take the determinant of the hessian matrix:

$$\begin{aligned}
\begin{vmatrix} -6 & 6 \\ 6 & 6-12y \end{vmatrix} &= -6(6-12y)-36 \\
&= -36+72y-36 \\
&= -72+72y = 72(y-1)
\end{aligned}$$

When $(x,\ y)=(0,\ 0)$, the result of the determinant of the hessian matrix is $72(0-1)=-72$, so it is negative and $(0,\ 0,)$ is a **saddle point** of f **.**

When $(x,\ y)=(2,\ 2)$, the result of the determinant of the hessian matrix is $72(2-1)=72$. As $f_{xx}(2,\ 2)$ is negative, $(2,\ 2)$ is a **local maximum** of f **.**

19.7 To find the absolute min and max

For $\mathbb{R}^2 \rightarrow \mathbb{R}$ in a certain region (closed and bounded set D):

1. Find the values of f at the **critical points** of f in D
2. Find the extreme values of f at the **boundary** of D
3. The largest of the values from step 1 and step 2 is the absolute maximum value
4. The smallest of the values from step 1 and step 2 is the absolute minimum value

Example 19.3. Find the absolute max and min values of $f(x,\ y) = 2 + 2x + 4y - x^2 - y^2$ on the triangular region

$$D = \{(x,\ y)\ : x = 0,\ y = 0,\ y = 9 - x\}$$

Solution 19.1. Get the area. Find all critical points in the area. Then, inspect everything in the boundary.

Firstly, compute the critical points of f.

$$\nabla f(x,\ y) = \begin{bmatrix} 2 - 2x \\ 4 - 2y \end{bmatrix}$$

$$\nabla f(x,\ y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 2 - 2x = 0 \\ 4 - 2y = 0 \end{matrix} \Rightarrow \begin{matrix} x = 1 \\ y = 2 \end{matrix}$$

We have one critical point: $(1,\ 2)$. $f(1,\ 2) = 7$.

This is an interior point in this region.

Then, consider the boundaries. There's a theorem saying any continuous function on any compact region can attain maximum and minimum values. The function given is continuous, for sure, and we have a compact (closed) region.

The first boundary is $x = 0$. Because of this, I'll have to compute $f(0,\ y) = 4y - y^2$.
$f'(0,\ y) = 4 - 2y = 0 \Rightarrow y = 2$

Therefore, the next potentially max or min point: $(0,\ 2)$.

AND the boundary: $(0,\ 0)$, $(0,\ 2)$, $(0,\ 9)$.

Go over lines $y = 0$ and $y = 9 - x$ but make sure you're inside the set. After that, you'll get an entire graph:

$$f(0,\ 0) = 2 f(0,\ 9) = -43 \vdots$$

And the lowest one is the min; the highest one is the max.

Find the absolute min and max values of

$$f(x,\ y) = 2 + 2x + 4y - x^2 - y^2$$

On $D = \left\{(x,\ y) \in \mathbb{R}^2\ : x = 0,\ y = 0,\ y = 9 - x\right\}$ (the boundaries)

You're going to slice the graph of f three times. We've already found the critical points: $(1,\ 2)$. $f(1,\ 2) = 7$

Case 1. $x = 0$. Then, $f(x,\ y) = f(0,\ y) = 2 + 4y - y^2$. Max/min points in $[0,\ 9]$? $f'(0,\ y) = 0 \Rightarrow 4 - 2y = 0 \Rightarrow y = 2$. When $y = 2,\ x = 0,\ f(0,\ 2) = 6.\ f(0,\ 0) = 2,\ f(0,\ 9) = -43$.

Case 2. $y = 0$, inside $[0,\ 9]$. Then, $f(x,\ 0) = 2 + 2x - x^2$. $f'(x,\ 0) = 2 - 2x$. When $f'(x,\ 0) = 0 \Rightarrow 2 - 2x = 0 \Rightarrow x = 1$. Then, $f(1,\ 0) = 3,\ f(9,\ 0) = 2 + 18 - 81 = 20 - 81 = -61$. The lowest is at $(9,\ 0)$ with a value of -61.

Case 3. $y = 9 - x$. $f(x,\ y) = f(x,\ 9-x) = 2 + 2x + 4(9-x) - x^2 - (9-x)^2 = 2 + 2x - 36 - 4x - x^2 - 81 + 18x - x^2 = -43 + 16x - 2x^2$. Now, find the min/max on values such that $x \in [0,\ 9]$. $f'(x,\ 9-x) = 16 - 4x = 0 \Rightarrow x = 4$. When $x = 4,\ y = 9 - 4 = 5$, so $f(4,\ 5) = 2 + 8 + 20 - 16 - 25 = -11$.

The absolute minimum is -61 located at $(9,\ 0)$ and the absolute maximum is 7, at $(1,\ 2)$.

19.8 The Nabla Operator for Over Two Dimensions

$$\nabla g(x,\ y,\ z) = \begin{bmatrix} f_x & f_y & f_z \\ h_x & h_y & h_z \end{bmatrix}$$
$$\text{where } g(x,\ y,\ z) = (f(x,\ y,\ z),\ h(x,\ y,\ z))$$
$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$\nabla f(x,\ y) = \begin{bmatrix} f_x \\ f_y \end{bmatrix}$$

...

20 Lagrange Multipliers for Constrained Optimization

With a single constraint vs. with multiple constraints

Motivation. Sometimes, we must find the extreme. Values of a function whose domain is constrained to some **subset of the plane like a disk or along a curve.**

Find the max. of f subject to it being on a curve.

Theorem 20.1. *Let $S \subseteq \mathbb{R}^n$ and let $f,\ g : S \to \mathbb{R}$ be functions of class C^1. Let $x \in \mathbb{R}^n$ be a local minimum point or local maximum point of f to the constraint $g(x) = 0$. If*

$\nabla g(x) \neq \overrightarrow{0}$, *then* $\exists\lambda \in \mathbb{R}$ *such that the following system of equation is satisfied by* x *and* λ*:*

$$\begin{cases} \nabla f(x) = \lambda \nabla g(x) \\ g(x) = 0 \end{cases}$$

Theorem 20.2. *Let* $S \subseteq \mathbb{R}^n$ *and let* $f,\ g,\ h : S \to \mathbb{R}$ *be* C^1*. Let* $x \in \mathbb{R}^n$ *be a local minimum point or local maximum point of* f *constraint to* $g(x) = 0$ *and* $h(x) = 0$*. Suppose that* ∇g *is not parallel to* ∇h*. Then,* $\exists\lambda,\ \mu \in \mathbb{R}$ *such that*

$$\begin{cases} \nabla f(x) = \lambda g(x) + \mu \nabla h(x) \\ g(x) = 0 \\ h(x) = 0 \end{cases}$$

Example 20.1. Find the extreme values of $f(x,\ y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

Set $g(x,\ y) = x^2 + y^2 + 1$. We'll apply the Lagrange theorem to solve this.

$$\begin{cases} \nabla f(x,\ y) = \lambda \nabla g(x,\ y) \\ g(x,\ y) = 1 \end{cases}$$
$$\Rightarrow \begin{cases} \begin{bmatrix} 2x \\ 4y \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix} \\ x^2 + y^2 = 1 \end{cases}$$
$$\Rightarrow \begin{cases} 2x = \lambda 2x \\ 4y = \lambda 2y \\ x^2 + y^2 = 1 \end{cases}$$

Solve this system. Number the equations on the matrix from 1 to 3 from top to bottom.

Suppose that $2x = 2\lambda x$. There are two possibilities:

- $x = 0$
  - Then, $0 = 0$ and any λ works… but that wouldn't work.
- $x \neq 0$
  - Then, $\lambda = 1$ by canceling stuff out.
  - Then by the second equation, we get $4y = 2y \Rightarrow y = 0$

  - Therefore, I have one candidate: $y = 0,\ \lambda = 1,\ x = TBA$
  - By the bottom-most equation, we have $x^2 + y^2 = 1 \Rightarrow x^2 = 1 - y^2 \Rightarrow x = \pm 1$
  - Okay, we have two candidates:
    * $A(1,\ 0) \quad B(-1,\ 0)$

- $y \neq 0$
  - $\Rightarrow 4y = \lambda 2y \Rightarrow \lambda = 2$
    * $\Rightarrow x = \pm 1$. It satisfies the first equation.
    * So, we have $C(0,\ 1) \quad D(0,\ -1)$

- $y = 0$
  - Then, $x = 0$, but this results in a contradiction.

Try plugging in points:

- $f(A) = 1$
- $f(B) = 1$
- $f(C) = 2$
- $f(D) = 2$

The maximum value of f is 2 and the function f attains the maximum values at points C and D. The minimum value is 1 and the function f attains the minimum value at A and B.

The solution above is a bit of a mess and you probably shouldn't use this to learn about LaGrange multipliers for the first time.

20.1 Taking a Ratio

$$\begin{cases} 2x = \lambda 2x \\ 4y = \lambda 2y \\ x^2 + y^2 = 1 \end{cases}$$

If $x \neq 0$, then there is a case where $\lambda = 1$

If $y \neq 0$, then there is a case where $\lambda = 2$

20.2 Another Example

Lagrange theorem can only find potential max/mins on the boundary. In the interior, you'll have to use $\nabla f(x,\ y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Example 20.2. Find the points on the hyperbolic cylinder

$$x^2 - z^2 - 1 = 0$$

That are the closet to $(0,\ 0,\ 0)$. Note that $d = \sqrt{x^2 + y^2 + z^2}$.

So, the function is $\sqrt{x^2 + y^2}$ and the constraint is $x^2 - z^2 - 1 = 0$.

$$\begin{aligned} f(x,\ y) &= \sqrt{x^2 + y^2 + z^2} \\ g(x,\ y) &= x^2 - z^2 - 1 = 0 \end{aligned}$$

Apply the LaGrange theorem. I can amend $f(x,\ y)$ to $f(x,\ y) = x^2 + y^2 + z^2$ because regardless, the maximum I get will at least be the maximum.

$$\begin{cases} \nabla f(x,\ y,\ z) = \lambda \nabla g(x,\ y,\ z) \\ g(x,\ y,\ z) = 0 \end{cases}$$

What I have is

$$\begin{cases} \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 0 \\ -2z \end{bmatrix} \\ x^2 - z^2 - 1 = 0 \end{cases}$$

So, we have:

$$\begin{cases} 2x = 2\lambda x \\ 2y = 0 \\ 2 = -2\lambda z \\ x^2 - z^2 - 1 = 0 \end{cases}$$

Immediately, we can obtain $y = 0$. Now, $2x = 2\lambda x$. We have two possibilities.

- $x = 0$

- Then $z^2 = -1$. Because we're only working with $\mathbb{R}^3 \to \mathbb{R}$, this is a contradiction.

- $x \neq 0$
  - $\Rightarrow \lambda = 1$
  - $2z = -2z \Rightarrow 4z = 0 \Rightarrow z = 0$
  - If $z = 0$, then, we have $x^2 - 1 = 0 \Rightarrow x = \pm 1$
  - So, our potential points are $(1,\ 0,\ 0)$ and $(-1,\ 0,\ 0)$

- For $2z = -2\lambda z$:
  - $z = 0$
    - TBA
  - $z \neq 0$
    - $\lambda = -1$
    - $x = 0$
    - Contradiction; does not satisfy bottom equation

Then: $f(A) = 1,\ f(B) = 1$. As $\sqrt{1} = 1$, the maximum value is 1.

20.3 When is the Gradient Is Orthogonal to the Curve

If we have $r(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, and we want $f\left(x(t),\ y(t)\right) = 1$, differentiate both sides:

$$0 = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$

We can write this as two vectors:

$$\begin{aligned} &\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \end{bmatrix} \\ &= \nabla f(x,\ y) \cdot \begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \end{bmatrix} \\ &= \nabla f(x,\ y) \cdot \overrightarrow{r}(t) = 0 \end{aligned}$$

So, $\overrightarrow{r}'(t)$ is always tangent to $\overrightarrow{r}$, and the gradient is always orthogonal to r'.

Now, if we have two vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x' \\ y' \end{bmatrix}$, what can we say about the relationship between them?

$$\exists \lambda \in \mathbb{R} \, : \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Suppose we have a function $f(x,\, y) = 1$ and suppose you have another function $g(x,\, y) = 1$. g and f at some point is tangent (they share the exact same tangent line, including the slope and a point of it).

If we disregard g, ∇f at the point where f and g share the same tangent line $x_0,\, y_0$ will always be perpendicular to the tangent line (at $x_0,\, y_0$).

If we disregard f, ∇g at the point where f and g share the same tangent line will always be perpendicular to the tangent line.

Hence:

$$\nabla f\,(x_0,\, y_0) = \lambda \nabla g\,(x_0,\, y_0)$$

In LaGrange multipliers, we have a function $z = f(x,\, y)$ and we have a level curve of a function $f(x,\, y) = 1,\, 2,\, 3,\, \ldots$ and so on, and we also have a constraint, $g(x,\, y)$. We want to maximize (know about the maximum value of the graph of f) subject to $g(x,\, y)$. As we see, if we move from one level to the other, the value of f moves from 1 to 2.

Look for the level curve where a level curve of $g(x,\, y) = c$ has a point that shares the same tangent line to somewhere in the level curve of $f(x,\, y)$.

And I end up with this system:

$$\begin{aligned} \nabla f\,(x_0,\, y_0) &= \lambda g\,(x_0,\, y_0) \\ g(x,\, y) &= 1 \end{aligned}$$

For g, consider a single level curve. For f, consider all level curves and look for one that works.

20.4 Gradient Descent

(Will not be tested)

Machine learning concepts are typically optimization problems. Input is something, and output is something else. For each example in the dataset, we compare the output of the neural network to the ideal output. Take the squared sum difference, we get the cost. Then, we get the overall cost function. How do we find the right input value that makes the cost unction as small as possible.

Here comes gradient descent. It feels like making Newton iterations. You want to make the function decrease in each iteration. It feels like dropping a ball over a surface and hoping that it gets to the lowest value.

20.5 Easiest Optimization Problem

The temperature at point $(x,\ y,\ z)$ on the unit sphere is given by $f(x,\ y,\ z) = 2xy + z^2 - z$ in the constraint $g(x) = x^2 + y^2 + z^2\ = 1$.

$$\nabla f(x,\ y,\ z) = \lambda \nabla g(x,\ y,\ z)$$
$$x^2 + y^2 + z^2 = 1$$
$$\begin{bmatrix} 2y \\ 2x \\ 2z-1 \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$$
$$\Rightarrow \begin{cases} 2y = \lambda 2x \\ 2x = \lambda 2y \\ 2z-1 = \lambda 2z \end{cases}$$

- $2y = \lambda \cdot \lambda 2y$
- $\Rightarrow 2y = \lambda^2 2y$
- $\Rightarrow y = \lambda^2 y$
- Case 0. $z = 0$.
  - $2z - 1 = 0$
  - $-1 = 0$
  - Not permissible.
- Case 1. $y = 0$
  - Then, $2x = 0 \Rightarrow x = 0$.
  - $z^2 = 1 \Rightarrow z = \pm 1$, so we get $(0,\ 0,\ 1)$ and $(0,\ 0,\ -1)$

- But we need to check for contradiction on the other equations.
  * $2-1=\lambda 2 \Rightarrow 1=\lambda 2$. There exists a λ, so no contradiction for $z=1$.
  * $-2-1=-2\lambda \Rightarrow -3=-2\lambda$. There exists a λ, so no contradiction for $z=-1$.

- Case 2. $y \neq 0$.
  - Then, $1=\lambda^2 \Rightarrow \lambda=\pm 1$
  - Sub-case 1: $\lambda=1$
    * $2y=2x \Rightarrow x=y$
    * $\begin{matrix} 2x^2+z^2=1 \\ 2z-1=2z \end{matrix}$
    * No solutions for z exist.
  - Sub-case 2: $\lambda=-1$
    * $2y=-2x \Rightarrow x=-y$. Then, $x \neq 0$.
    * $2z-1=-2z$
    * $4z=1 \Rightarrow z=\frac{1}{4}$
    * Then, $2x^2+\frac{1}{16}=1$
    * $\Rightarrow x^2=\frac{15}{32}$
    * $x=\pm\sqrt{\frac{15}{32}}$
    * Therefore, we get the points
    * $\left(\sqrt{\frac{15}{32}},\ -\sqrt{\frac{15}{32}}, \frac{1}{4}\right)$ and $\left(-\sqrt{\frac{15}{32}},\ \sqrt{\frac{15}{32}}, \frac{1}{4}\right)$

Plug and chug to figure out what is the max and min.

Question: If I don't end up coincidentally find a case where $x=0$, do I have to do make up a case? Or do I have to automatically do it all the time?

20.6 Closest To The Origin

Find the points closest to the origin on the curve of the intersection of the plane $2y+2z=5$ and the cone $z^2=4x^2+4y^2$.

We have these equations

$$\begin{aligned} \text{set } f(x,\ y,\ z) &= x^2+y^2+z^2 \\ g(x,\ y,\ z) &= 2y+2z-5=0 \\ h(x,\ y,\ z) &= 4x^2+4y^2-z^2=0 \end{aligned}$$

Using LaGrange multipliers, set this up:

$$\begin{cases} \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} 8x \\ 8y \\ -2z \end{bmatrix} \\ 2y + 2z - 5 = 0 \\ 4x^2 + 4y^2 - z^2 = 0 \end{cases}$$

So, we get:

$$\begin{cases} x = 4\mu x \\ y = \lambda + \mu 4y \\ z = \lambda - \mu z \\ 2y + 2z - 5 = 0 \\ 4x^2 + 4y^2 - z^2 = 0 \end{cases}$$

We need to solve this system:

- Case 1: $x = 0$
  - $\Rightarrow z^2 = 4y^2 \Rightarrow z = \pm 2y$
  - Case 1A: $z = 2y$
    - If $z = 2y$, then $2y + 4y = 5 \Rightarrow y = \frac{5}{6}$
    - And $\frac{5}{3} + 2z - 5 = 0 \Rightarrow 2z = 5 - \frac{5}{3} \Rightarrow z = \frac{5 - \frac{5}{3}}{2} \Rightarrow z = \frac{5}{3}$
    - Hence, we have a point: $A\left(0, \frac{5}{6}, \frac{5}{3}\right)$
  - Case 1B: $z = -2y$ and $-z = 2y$
    - Then, $-2y = 5 \Rightarrow y = -\frac{5}{2},\ z = 5$
    - We get $B\left(0,\ -\frac{5}{2}, 5\right)$
- Case 2: $x \neq 0$
  - $\mu = \frac{1}{4}$
  - $y = \lambda + y \Rightarrow \lambda = 0$
  - $\Rightarrow z = \frac{1}{4}z \Rightarrow z = 0$
  - $2y = 5 \Rightarrow y = \frac{5}{2}$
  - $4x^2 + 4y^2 = 0 \Rightarrow 4y^2 = -4x^2 \Rightarrow y^2 = -x^2 \Rightarrow$ the only working values that can satisfy this equation is when $x = 0$ and $y = 0$, which is a contradiction as we initially said $x \neq 0$ and $y = \frac{5}{2}$.

I'm not going to do any calculations, but I'm pretty sure $A\left(0, \frac{5}{6}, \frac{5}{3}\right)$ is the minimum and $B\left(0,\ -\frac{5}{2}, 5\right)$ is the maximum.

$$f\left(0, \frac{5}{6}, \frac{5}{3}\right) = \frac{25}{36} + \frac{25}{9}$$
$$f\left(0,\ -\frac{5}{2},\ 5\right) = \frac{25}{4} + 25$$

WARNING – if we are asked for the distance, keep in mind that $f(x,\ y,\ z) = x^2 + y^2 + z^2$. Hence, f actually gives us the distance squared, so we need to square root anything we get from f to call it distance.

20.7 Cuts

> The plane $x + y + z = 1$ cuts the cylinder $x^2 + y^2 = 1$ in an ellipse. Find the points on the ellipse that lie closest and farthest from the origin.

We have the distance from the origin squared:

$$f(x,\ y,\ z) = x^2 + y^2 + z^2$$

And we have the constraints:

$$g(x,\ y,\ z) = x^2 + y^2 = 1$$
$$h(x,\ y,\ z) = x + y + z = 1$$

Leaving us with this system:

$$\begin{cases} \nabla f(x,\ y,\ z) = \lambda \nabla g(x,\ y,\ z) + \mu \nabla h(x,\ y,\ z) \\ x^2 + y^2 = 1 \\ x + y + z = 1 \end{cases}$$

We may expand it and get:

$$\begin{cases} 2x = \lambda 2x + \mu & (1) \\ 2y = \lambda 2y + \mu & (2) \\ 2z = \mu & (3) \\ x^2 + y^2 = 1 & (4) \\ x + y + z = 1 & (5) \end{cases}$$

It is often in our best interests to solve for λ and μ first.

From (3), we can make our substitutions into (1) and (2):

$$\begin{cases} 2x = \lambda 2x + 2z & (1) \\ 2y = \lambda 2y + 2z & (2) \\ 2z = \mu & (3) \\ x^2 + y^2 = 1 & (4) \\ x + y + z = 1 & (5) \end{cases}$$

I then subtract (1) and (2):

$$2x - 2y = \lambda(2x - 2y)$$

Case 1: $x \neq y$. Then:

- $\frac{2x-2y}{2x-2y} = \lambda \Rightarrow \lambda = 1$
- We then get the relation $2x = 2x + 2z$ and thus $0 = 2z$. Hence, $z = 0$.
- Hence, $\begin{cases} x^2 + y^2 = 1 \\ x + y = 1 \Rightarrow y = 1 - x \end{cases}$ and we get the solutions $A(0,\ 1,\ 0)$ and $B(1,\ 0,\ 0)$.

Case 2: $x = y$. Then:

- $2x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$ and $y = x$. We also know that $z = 1 - x - y$.
  - Sub-case 1: $x,\ y = \frac{1}{\sqrt{2}}$. Then, $z = 1 - \frac{2}{\sqrt{2}}$, and we get the point $C\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}},\ 1 - \sqrt{2}\right)$
  - Sub-case 2: $x,\ y = -\frac{1}{\sqrt{2}}$. Then, $z = 1 + \frac{2}{\sqrt{2}}$, and we get the point $D\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}},\ 1 + \sqrt{2}\right)$.

Then, I plug in each point into $f(x,\ y,\ z) = x^2 + y^2 + z^2$:

- $f(0,\ 1,\ 0) = f(1,\ 0,\ 0) = 1$. The distance is 1.
- $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}},\ 1 - \sqrt{2}\right) = \left(\frac{1}{2} + \frac{1}{2} + \left(1 - 2\sqrt{2} + 2\right)\right) \approx\ 4 - 2.8 = 1.2$
- $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}},\ 1 + \sqrt{2}\right) = \left(\frac{1}{2} + \frac{1}{2} + 1 + 2\sqrt{2} + 2\right) = \left(5 + 2\sqrt{2}\right)$

Hence, the longest distance is $\sqrt{5 + 2\sqrt{2}}$ and the shortest distance is 1.

20.8 Extreme values

Find the extreme values of $f(x,\ y) = \frac{x+y}{1+x^2+y^2}$ subject to the constraint $x^2 + y^2 = R^2$ where $R \in \mathbb{R}_{\geq 0}$.

$$g(x) = x^2 + y^2 = R^2$$

Hence:

$$\begin{cases} \nabla f(x,\ y) = \lambda \nabla g(x) \\ x^2 + y^2 = R^2 \end{cases}$$

As an alternative way, we can consider $x = R\cos(\theta)$ and $y = R\sin(\theta)$. Then, find the extreme values of $f\left(R\cos(\theta),\ R\sin(\theta)\right)$. Then, we have:

$$f\left(R\cos(\theta),\ R\sin(\theta)\right) = \frac{R}{1+R^2} \cdot \left(\cos(\theta) + \sin(\theta)\right)$$

Take $\frac{df}{d\theta} =$ some single variable function. We should end up with $\sin(\theta) = \cos(\theta)$ which implies that $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$. Plug in those values in f, with $x = R\cos(\theta)$ and $y = R\sin(\theta)$.

20.9 The Last One

Let $f(x,\ y) = (x+y)^4 + y^4$. Find the minimum of f subject to $x^4 + y^4 = 1$.

Solve

$$\begin{cases} \nabla f(x,\ y) = \lambda \nabla g(x,\ y) \\ x^4 + y^4 = 1 \end{cases}$$

$$\begin{bmatrix} 4(x+y)^3 \\ 4(x+y)^3 + 4y^3 \end{bmatrix} = \lambda \begin{bmatrix} 4x^3 \\ 4y^3 \end{bmatrix}$$

$$\Rightarrow (x+y)^3 = \lambda x^3$$

$$(x+y)^3 + y^3 = \lambda y^3$$

We end up with this system:

$$\begin{cases} (x+y)^3 = \lambda x^3 \\ (x+y)^3 + y^3 = \lambda y^3 \\ x^4 + y^4 = 1 \end{cases}$$

Multiply both sides by:

$$\begin{cases} x(x+y)^3 = \lambda x^4 \\ y(x+y)^3 + y^4 = \lambda y^4 \\ x^4 + y^4 = 1 \end{cases}$$

As $x^4 = 1 - y^4$ and $y^4 = 1 - x^4$

...

We get:

$$\begin{aligned} y^3 &= \lambda y^3 - \lambda x^3 \\ \Rightarrow y^3 - \lambda y^3 &= -\lambda x^3 \\ \Rightarrow \lambda y^3 - y^3 &= \lambda x^3 \\ \Rightarrow y^3(\lambda - 1) &= \lambda x^3 \end{aligned}$$