MAT137 Notes

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This document contains all notes starting from the second half of the limits unit, approximately 20% into the course. Images are not present in this document.

Contents

1 Proving limits with arbitrary parameters

Given L as a constant, suppose

$$\lim_{x\to 0} f(x) = L$$

Prove

$$\lim_{x\to 0} f(2x) = L$$

This means we suppose

$$\forall \varepsilon_2 > 0,\ \exists \delta_2 > 0,\ 0 < |x_2 - 0| < \delta_2 \Rightarrow |f(x_2) - L| < \varepsilon_2$$

We need to prove

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ 0 < |x - 0| < \delta \Rightarrow |f(2x) - L| < \varepsilon$$

1.1 Proving the above statement

Proof. Let $\varepsilon > 0$

Show the existence of delta to make the statement hold. Unfortunately, you can't find the delta. However, we can pick a particular value of the quantified variable labeled under for every. Choose one that is advantageous to the proof.

Pick $\varepsilon_2 = \varepsilon$. Then

$$\exists \delta_2 > 0,\ \forall x_2 \text{ if } 0 < |x_2 - 0| < \delta_2 \Rightarrow\ |f(x_2) - L| < \varepsilon_2$$

Now, ε_2 must always be picked prior to delta because the existence of delta depends on epsilon.

Pick $\delta =$ ___ (leave that blank for now).

Assume $0 < |x - 0| < \delta$

Prove $|f(2x) - L| < \varepsilon$

Consider (pick) $x_2 = 2x$. Why are we allowed to do this? Because we allowed the blue statement to be true, where $\forall x_2$ appears somewhere in the blue statement. Why did we choose x_2 to be that? Because it is advantageous to solving the problem. Now, the statement revisited:

$$\exists \delta_2 > 0,\ \forall x_2 \text{ if } 0 < |x_2 - 0| < \delta_2 \Rightarrow\ |f(x_2) - L| < \varepsilon_2$$

Because we picked $x_2 = 2x$, the red statement above becomes the statement we need to prove, as $|f(x_2) - L| = |f(2x) - L|$. How do we prove it? We need to prove the green statement.

When you want to prove an implication, you assume the statement after the "if" and prove the statement after the then.

However, **when you know an implication is true,** and you want to use it, you actually prove the statement after the "if" and then conclude the statement after the then. You do not prove the statement after the "then" because we already know the implication is true. This means *if the green statement is true, the red statement is true. Proving that the green statement is true verifies the red statement.*

Ask: $0 < |x_2 - 0| < \delta_2$. Can we get this to hold (what to solve, given $|x - 0| < \delta$)?

Since (scroll back up to our assumption), we can end up with:

$$\begin{aligned} |x - 0| = |x| &< \delta \\ 0 < |2x| &< 2\delta \\ 0 < |x_2 - 0| &< 2\delta \end{aligned}$$

However, we want to end up with $|x_2 - 0| < \delta_2$, not 2δ. This is where we can take advantage of our "pick $\delta =$ ___." Remember that we are free to pick δ to be whatever we want, and if we pick it to be $\delta = \frac{\delta_2}{2}$, then $\delta_2 = 2\delta$. Now we can substitute:

$$0 < |x_2 - 0| < \delta_2$$

Once we know that this (above) is true, we can also conclude (based on our assumption) that $|f(x_2) - L| < \varepsilon_2$ is true.

Because of our choices that $\varepsilon_2 = \varepsilon$ and $x_2 = 2x$:

$$|f(x_2) - L| < \varepsilon_2 \Leftrightarrow |f(2x) - L| < \varepsilon$$

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1.1.1 Shortened

Suppose

$$\forall \varepsilon_2 > 0,\ \exists \delta_2 > 0,\ 0 < |x_2 - 0| < \delta_2 \Rightarrow |f(x_2) - L| < \varepsilon_2$$

We need to prove

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ 0 < |x - 0| < \delta \Rightarrow |f(2x) - L| < \varepsilon$$

Proof. Let $\varepsilon > 0$. Pick $\varepsilon_2 = \varepsilon$, pick $\delta = \frac{\delta_2}{2}$.

Assume $|x - 0| < \delta$. Pick $x_2 = 2x$. This means

$$\begin{aligned} 2|x-0| &< 2\delta \\ |2x-0| &< 2\delta \\ |x_2-0| &< 2\delta \\ |x_2-0| &< \delta_2 \end{aligned}$$

This implies that $|f(x_2) - L| < \varepsilon_2$ and also that $|f(2x) - L| < \varepsilon$.

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1.2 Proving the sum of two limits

$|f(x) + g(x) - (L + M)| < \varepsilon$

Triangle inequality: While you can split absolute values over multiplications, they can't exactly be split when doing addition. $|x + y| \leq |x| + |y|$

Proving the triangle inequality Assume your inequality to be true (but don't write it) (start with the inequality you want to prove and do some algebra, simplify it, and end up with something that is necessarily true).

What is $|a|^2$? It's a^2. If a was negative and you square it, the negative would go away anyways. Use the square operator to remove the absolute value.

$$
\begin{aligned}
|x+y| &\leq |x|+|y| \\
|x+y|^2 &\leq (|x|+|y|)^2 \\
(x+y)^2 &\leq (|x|+|y|)^2 \\
x^2+2xy+y^2 &\leq |x|^2+2|x||y|+|y|^2 \\
x^2+2xy+y^2 &\leq x^2+2|x||y|+y^2 \\
2xy &\leq 2|x||y| \\
xy &\leq |xy|
\end{aligned}
$$

Is the absolute value of a number always greater than or equal to that number? Yes. $xy \leq |xy|$ is true beyond reasonable doubt.

But why does using algebra help prove the statement? Because after each algebraic step you are making a double-sided implication, where each step implies and are equivalent to each other.

1.2.1 Proof

We assumed the following

$$
\begin{aligned}
&\forall \varepsilon_2 > 0,\ \exists \delta_2 > 0,\ 0 < |x_2 - 0| < \delta \Rightarrow |f(x_2) - L| < \varepsilon_2 \\
&\forall \varepsilon_3 > 0,\ \exists \delta_3 > 0,\ 0 < |x_3 - 0| < \delta \Rightarrow |g(x_3) - M| < \varepsilon_3
\end{aligned}
$$

Prove $\lim_{x\to 0} f(x) + g(x) = L + M$

Prove $\forall \varepsilon > 0$ if $0 < |x-a| < \delta \Rightarrow |f(x)+g(x)-(L+M)| < \varepsilon$

Proof. Let $\varepsilon > 0$ be arbitrary.

Pick $\varepsilon_2 = \frac{\varepsilon}{2}$, (we wouldn't know what ε_2 is, but it is whatever we want any time in the future) and assume the bottom statement to be true:

$$\exists \delta_2 > 0\ if\ 0 < |x_2 - a| < \delta_2 \Rightarrow |f(x_2) - L| < \varepsilon_2$$

Pick $\varepsilon_3 = \frac{\varepsilon}{2}$, and assume the statement below to be true:

$$\exists \delta_3 > 0\ if\ 0 < |x_3 - a| < \delta_3 \Rightarrow |g(x_3) - M| < \varepsilon_3$$

Pick $\delta = \min\{\delta_2,\ d_3\} \Rightarrow \delta \leq \delta_2,\ \delta \leq d_3$

Prove what we need to solve: assume the “if” and prove the “then.”

Assume $0 < |x-a| < \delta$

Prove $|f(x)+g(x)-(L+M)| < \varepsilon$

Consider $x_2 = x,\ x_3 = x$.

Since $0 < |x-a| < \delta$ (we’re using the picked blue statement to justify our substitution):

$$
\begin{aligned}
0 \leq |x-a| = |x_2-a| < \delta \leq \delta_2 \\
0 < |x-a| = |x_3-a| < \delta \leq \delta_3
\end{aligned}
$$

Which helps up conclude:

$$
\begin{aligned}
&\Rightarrow |f(x_2)-L| < \varepsilon_2 \\
&\Rightarrow |g(x_3)-M| < \varepsilon_3 \\
&\quad |f(x)+g(x)-(L+M)| \\
&\quad = |(f(x)-L)+(g(x)-M)| \\
&\quad \leq |f(x)-L|+|g(x)-M| \\
&< \varepsilon_2+\varepsilon_3 = \varepsilon
\end{aligned}
$$

In the process, we picked $\varepsilon_2 = \frac{\varepsilon}{2}$ and $\varepsilon_3 = \frac{\varepsilon}{2}$, as we wanted to end our proof with epsilon.

We thus conclude $|f(x)+g(x)-(L+M)| < \varepsilon$.

■

2 Using existing statements to prove other statements

($\wedge$ means “and”; $\vee$ means “or”.)

A statement is **necessarily true** means we can prove that statement given using the assumption.

A statement is **not necessarily true** (considered false, usually the result of not all $\forall$ checks passing despite some) when there is a counterexample.

2.0.1 Format

Knowing

$$A \Rightarrow B$$

Prove that

$$C \Rightarrow D$$

1. Assume C is true.
2. Try to prove A is true, then conclude B is true for this context.
3. Use B to prove D or find a counterexample.

The end goal is to prove D is true and C is false. Remember that $C \Rightarrow D$ is logically equivalent to $\neg C \vee D$.

For example, if we know this is true (statement 1 a.k.a. first equation, from now on):

$$|x-2| < 1 \Rightarrow |f(x)-5| < 2$$

2.0.2 Example 1

We can prove the following:

$$|x-2| < 1 \Rightarrow |f(x)-5| < 3$$

Here's the rationale: If we assume $|x-2| < 1$, we know that $|f(x)-5| < 2$. Because $2 < 3$, $|f(x)-5| < 2 \Rightarrow |f(x)-5| < 3$ and therefore what we wanted to prove at the start is proven.

2.0.3 Example 2

Another statement to prove using the fact that statement 1 is true:

$$|x-2| < 0.5 \Rightarrow |f(x)-5| < 2$$

By assuming $|x-2| < 0.5$, we can conclude $|x-2| < 1$, which immediately implies $|x-2| < 0.5$ and $|f(x)-5| < 2$ (from the first statement, which turns out to be what we needed to prove true for the statement we needed to prove.

2.0.4 Example 3

$$|x-2| < 1 \Rightarrow |f(x)-5| < 1$$

Assume $|x-2| < 1$. This means $|f(x)-5| < 2$, but can we conclude $|f(x)-5| < 1$? No. Think about it. For example, if $3 < f(x) < 7$, then $|f(x)-5| < 1 \wedge |f(x)-5| < 2$, but if we let $f(x) = 7.5$, $|7.5-5| = 1.5 < 2$, but $1,5 \not< 1$. This is the reasoning why the statement we needed to prove is not necessarily true. In other words, the steps to prove/disprove a statement like this is:

1. Assume "if" of what to solve
2. If the "if" of what to solve implies the "if" of the first equation, then we can guarantee the "then" of the first equation is true.
3. The "then" of what to solve must be implied by the "then" of the first equation. You can also find a counterexample to disprove a statement.

A negation of an implication is where the "if" is true and the "then" is false, always.

Consider $x = 2$, and $f(x) = 6.5$. ($f(x)$ can be anything, as we did not put any constraints on it.)

$$|x-2| = 0 < 1 \Rightarrow |f(x)-5| = |6.5-5| \not< 1$$

(For what we know that is true must be followed when constructing the above work.)

2.0.5 Example 4

Knowing that:

$$|x-2| < 1 \Rightarrow |f(x)-5| < 2$$

Prove:

$$|x-2| < 0.5 \Rightarrow |f(x)-5| < 1$$

Proof. Assume $|x-2| < 0.5$; this implies $|x-2| < 1$, and thus implies $|f(x)-5| < 2$. If you scroll up, you can find why $|f(x)-5| < 1$ is false (not always true). Do this by giving an example where $|x-2| < 0.5$ is true and $|f(x)-5| < 1$ is false. Scroll up to question 3 for when that is used.

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2.0.6 Example 5

Knowing that

$$|x-2| < 1 \Rightarrow |f(x)-5| < 2$$

Prove

$$\exists \delta > 0,\ |x-2| < \delta \Rightarrow |f(x)-5| < 1$$

To show WTS is false, show

$$\neg(\exists \delta > 0,\ |x-2| < \delta \Rightarrow |f(x)-5| < 1)$$

$$\forall \delta > 0,\ |x-2| < \delta \wedge |f(x)-5| \not< 1$$

(A very small delta will imply that x can only be 2; any greater than it will exceed δ)

Let $\delta > 0$ be arbitrary. Consider $x = 2$. This means $|x-2| < \delta$. (Why did we pick $x = 2$? Because it is the only x that makes $|x-2| < \delta$ where delta can be as close to 0 as it wants but never negative)

Consider $f(x) = 6.5$

It satisfies $|x-2| = 0 < 1 \Rightarrow |f(x)-5| < 2$

With that checked off, the value we assigned to $f(x)$ is legal. Then $|6.5-5| = 1.5 \not< 1$.

This is satisfied, ultimately proving that

$$\forall \delta > 0,\ |x-2| < \delta \wedge |f(x)-5| \not< 1$$

Is true.

3 Proving that a limit doesn't exist

We still have the assumption

$$|x-2| < 1 \Rightarrow |f(x)-5| < 2$$

We know something about f, but we don't know what the function truly is. It is just the function that satisfies the assumption.

Now we suppose that

$$\lim_{x \to 2} f(x) = L$$

This means, we know the following is true:

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ 0 < |x-2| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

3.1 Deciding whether to prove the limit exists or not

Can $L = 2$? Give an example of a function with that limit being two. If you think the limit cannot be two, then prove it using the assumption.

If you say so, it means $L - \varepsilon < f(x) < L + \varepsilon \Rightarrow f(x) = 2$

Now what do we know about our first assumption? Then $|f(x) - 5| < 2 \Leftrightarrow 3 < f(x) < 7$

The two above statements contradict. L cannot be 2. A.K.A. because $f(x) \in (3,\ 7)$ and we wanted $f(x)$ to output 2, but we know that can't be the case because $f(x)$ is restricted.

To prove an implication is false: A is true, and B is false.

3.2 The "Proving the limit doesn't exist"

Prove $\lim_{x \to 2} f(x) \neq 2$. To prove a limit is not something, negate the limit definition. So, the limit definition, negated is:

$$\exists \varepsilon > 0,\ \forall \delta > 0,\ \exists x,\ 0 < |x-2| < \delta \wedge |f(x) - L| \geq \varepsilon$$

Pick $\varepsilon = 0.5$ (Initially blank and to be written down later. Could be small as we want.)

Let $\delta > 0$ be randomly chosen.

Pick $x = 2 + \min\left\{\frac{\delta}{2}, \frac{1}{2}\right\}$ (Initially blank. You technically can pick $2 + \frac{\delta}{69696969}$ but please don't.)

Prove $0 < |x-2| < \delta \wedge |f(x) - L| \geq \varepsilon$

Firstly, prove that the left side is true. Pick a particular x that makes the blue statement is true. Remember: our assumption still holds:

$$
\begin{aligned}
&|x-2|<1 \Rightarrow |f(x)-5|<2\\
&\quad \Rightarrow 1<x<3\\
&\quad \Rightarrow 3<f(x)<7
\end{aligned}
$$

Okay, back to $0<|x-2|<\delta$. Notice that $x \neq 2$. 2 is the most obvious answer, but there is a slight technical detail that needs to be fixed. The statement $0<|x-2|<\delta \Rightarrow 2-\delta<x<2+\delta$. We know δ can be really small, but not at zero. Therefore, make x be slightly larger than 2. This is why we picked $x=2+\frac{\delta}{2}$, which was picked just now.

Okay, how would you know that $1<2+\frac{\delta}{2}<3$? δ could be anything. The way to solve this is to use the minimum function. This means $x=2+\min\left\{\frac{\delta}{2},\frac{1}{2}\right\}$.

Now since we chose x to be that:

$$
\begin{aligned}
&x \leq 2+\frac{\delta}{2}<2+\delta\\
&x \leq 2+\frac{1}{2}<3 \Leftrightarrow x \leq 2.5<3
\end{aligned}
$$

Compress this, and we have $x \leq 2+\frac{\delta}{2}$ and $x \leq \frac{5}{2}$.

Since $\frac{\delta}{2}>0$ and $\frac{1}{2}>0$, $x>2$ if we pick either of them. (x is designed to be ever so slightly greater than two.)

$$\Rightarrow 2-\delta<x<2+\delta$$

$$\Leftrightarrow 0<|x-2|<\delta$$

What we know: This limit exists. $\lim_{x\to 2} f(x)=L$. Can the limit be equal to two? Prove that it can't, which is what we want to prove.

Back to what we needed to prove: $0<|x-2|<\delta \wedge |f(x)-L| \geq \varepsilon$. We proved $0<|x-2|<\delta$, and now we need to prove $|f(x)-L| \geq \varepsilon$.

These are the assumptions we have in our disposal:

$$1<x<3 \Rightarrow 3<f(x)<7$$

We can confirm that we did declare x to be slightly greater than two and we know that it is less than 3 (scroll up), so we can confirm $2<x<3 \Rightarrow 1<x<3$

Now we finally know $3 < f(x) < 7$ is true. Knowing this is the fact, we need to prove $|f(x) - L| \geq \varepsilon$, which is **equivalent** to $f(x) \geq L + \varepsilon,\ f(x) \leq L - \varepsilon$ **or** $f(x) \notin (L - \varepsilon,\ L + \varepsilon)$ – if this statement meets for all intervals of $f(x)$, **then we proved what we needed to solve:** $|f(x) - L| \geq \varepsilon$ (Recall $L = 2$)

We didn't choose what epsilon is yet, so can we choose an epsilon in a way that the interval of $3 < f(x) < 7$ lies outside of the diagram above? Yes. Just let $\varepsilon = 0.5$. We know L is 2, and that guarantees the interval of $f(x)$ is outside the interval of $(L - \varepsilon,\ L + \varepsilon)$.

Therefore $f(x)$ does not lie in the interval $(1.5,\ 2.5)$

A diagram that visualizes it: $f(x)$ cannot be within the blue zone.

□

3.2.1 Formal Proof, shortened

Suppose the following are true:

$$|x - 2| < 1 \Rightarrow |f(x) - 5| < 2$$
$$\lim_{x \to 2} f(x) = L$$

Prove $L \neq 2$.

From the limit presented, we get this, which we can assume holds.

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ 0 < |x - 2| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

This is an implication that proves a limit exists. We want to prove that the limit doesn't exist, so we reverse the epsilon-delta definition of the limit $\lim_{x \to 2} f(x) = L$:

$$\exists \varepsilon > 0,\ \forall \delta > 0,\ 0 < |x - 2| < \delta \wedge |f(x) - L| \geq \varepsilon$$

Our task is to prove this with $L = 2$.

Pick $\varepsilon = 0.5$

Let $\delta > 0$ be picked arbitrarily / randomly chosen.

Pick $x = \min\left\{2 + \frac{\delta}{2},\ 2 + \frac{1}{2}\right\}$

Prove $0 < |x - 2| < \delta \wedge |f(x) - L| \geq \varepsilon$

We want x to be a value slightly greater than 2. So, we pick $x = \min\left\{2 + \frac{\delta}{2},\ 2 + \frac{1}{2}\right\}$ (The right side of the minimum function prevents x from being 3 or greater as it breaks one of our assumptions.

This means $x \leq 2 + \frac{\delta}{2}$ and $x \leq 2 + \frac{1}{2}$. The minimum function also tells us $x > 2$ as $\frac{\delta}{2}, \frac{1}{2} > 0$. We can thus write: (also $2 + \frac{\delta}{2} < 2 + \delta$)

$$
\begin{gathered}
2 < x < 2 + \delta \\
\Leftrightarrow 0 < |x - 2| < \delta
\end{gathered}
$$

We proved the left side of $0 < |x-2| < \delta \wedge |f(x) - L| \geq \varepsilon$. Now to prove $|f(x) - L| \geq \varepsilon$

Remember our assumptions: $|x - 2| < 1 \Rightarrow |f(x) - 5| < 2$

$$(|x-2| < 1 \Leftrightarrow 1 < x < 3) \Rightarrow (|f(x) - 5| < 2 \Leftrightarrow 3 < f(x) < 7)$$

It means $f(x)$ must be between 3 and 7, non-inclusively. Now, we let $L = 2$, remember? This means:

$$
\begin{aligned}
|f(x) - 2| &\geq \varepsilon \\
&f(x) \geq 2 + \varepsilon \vee f(x) \leq 2 - \varepsilon
\end{aligned}
$$

We said we could let ε be anything, so let's make it 0.5; this is what happens:

$$f(x) \geq 2.5 \vee f(x) \leq 1.5$$

$$2.5 \leq 3 < f(x) < 7$$

(This says $f(x)$ must be greater that 2.5 or something else. But $f(x)$ has to be between 3 and 7. This means $f(x)$ must be greater than 2.5; statement proved – we proved the other side of the "and".)

3.2.2 Shortened, again

$$
\begin{gathered}
|x - 2| < 1 \Rightarrow |f(x) - 5| < 2 \\
\lim_{x \to 2} f(x) = L
\end{gathered}
$$

$$\exists \varepsilon > 0,\ \forall \delta > 0,\ 0 < |x-2| < \delta \wedge |f(x) - L| \geq \varepsilon$$

Pick $\varepsilon = 0.5$, let $\delta > 0$ be picked arbitrarily / randomly chosen, and pick $x = \min\left\{2 + \frac{\delta}{2},\ 2 + \frac{1}{2}\right\}$

Prove $0 < |x-2| < \delta$. This statement is the same as $2 < x < 2 + \delta$. Because $x = \min\left\{2 + \frac{\delta}{2},\ 2 + \frac{1}{2}\right\}$, then $2 < x$ and $x < 2 + \delta$. We can thus verify the statement.

Prove $|f(x) - L| \geq \varepsilon$. Choose $\varepsilon = \frac{1}{2}$. Then we need to prove that $f(x)$ is not in $(L - \varepsilon,\ L + \varepsilon)$. This means $f(x)$ is not in $(2 - 0.5,\ 2 + 2.5)$. We can prove that is the case because $f(x)$ is assumed to be in the interval $(3,\ 7)$. Therefore, proven.

3.3 Preliminary conclusions

We know $0 < |x-2| < \delta$ is true, as L does not affect it.

$L = 2.5$? Choose $\varepsilon = \frac{1}{4}$. Then we need to prove that $f(x)$ is not in $(L - \varepsilon,\ L + \varepsilon)$. This means $f(x)$ is not in $(2.25,\ 2 + 2.75)$. We can prove that is the case because $f(x)$ is assumed to be in the interval $(3,\ 7)$. Therefore, proven.

$L = 8$? Choose $\varepsilon = \frac{1}{2}$. Then we need to prove that $f(x)$ is not in $(L - \varepsilon,\ L + \varepsilon)$. This means $f(x)$ is not in $(8 - 0.5,\ 8 + 0.5)$. We can prove that is the case because $f(x)$ is assumed to be in the interval $(3,\ 7)$. Therefore, proven.

We can conclude preliminary (by realizing) that $L \notin (3,\ 7) \Rightarrow$ limit does not exist. However, we have only checked the values of values out of $[3,\ 7]$.

$L = 5$? This means $3 < f(x) < 7$ has to hold. Therefore, we think that it is possible, but it is not a proper answer, as we need to give an example $f(x)$ with $L = 5$, so we can have $f(x) = 5$: it satisfies $3 < f(x) < 7$.

Suppose $|x-2| < 1$, prove $|f(x) - 5| < 2$. $|f(x) - 5| \Rightarrow |f(x) - 5| = |5 - 5| < 2$; therefore, we can be confident that $L = 5$ is a possibility, but not completely.

Consider $L = 4$. Suppose $|x-2| < 1$, prove $|f(x) - 4| < 2$. $|f(x) - 5| \Rightarrow |4 - 5| = |4 - 5| < 2$; therefore, we can be confident that $L = 4$ is a possibility, but not completely.

3.4 It is not always what it seems

$$|x-2| < 1 \Rightarrow |f(x) - 5| < 2$$
$$\lim_{x \to 2} f(x) = L$$

Remember the definition of a limit at $x = 2$:

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ 0 < |x-2| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Can $L = 3$? We cannot make the conclusion that $L = 3$ is impossible, even though making a preliminary conclusion makes it look like $L = 3$ is impossible. Suppose $|x-2| < 1$, prove $|f(x)-5| < 2$. $|f(x)-5| \Rightarrow |3-5| = |3-5| < 2 \Rightarrow 2 < 2 \ldots$ however, that does not rule out the possibility of $L = 3$. If we let $f(x)$ be a function represented by the black line with a hole at $x = 2$, the graph $y = f(x)$ approaches 3 from above but doesn't hit it.

Proof. Consider $f(x) = \begin{cases} (x-2)^2 + 3,\ x \neq 2 \\ 5,\ x = 2 \end{cases}$. The limit of $f(x)$ at 2 would be 3, but $f(2) = 5$. That point (in red) is irrelevant when we consider limits. We have a specific condition that $x \neq 2$. So:

Suppose $|x-2| < 1$ (an arbitrary value between 1 and 3). Prove $|f(x)-5| < 2$.

Case 1: $x = 2$. Then $f(x) = 5$, and $|f(x)-5| = |5-5| < 2$, sastifying that $|f(x)-5| < 2$

Case 2: $x \neq 2$. Since $|x-2| < 1$

$$\begin{aligned}
0 &< |x-2|^2 < 1^2 \\
0 &< (x-2)^2 < 1 \\
3 &< (x-2)^2 + 3 < 4 \\
3 &< f(x) < 4 \\
-2 &< f(x) - 5 < -1 \\
1 &< |f(x)-5| < 2
\end{aligned}$$

This proves satisfied that this function works. This means L can be 3, and we can also come up that $3 \leq f(x) \leq 7$ (though not proven yet).

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4 Limits and Continuity

Sided-limits are only meant for 2-dimensional visualizations and can be used for computations.

4.1 Two-sided limits

The **limit** at a point a of the function $f(x)$ is equal to the number L.

1. When you let x approach a from the left side **and** the right side of a, the graph of $f(x)$ gets close to number L.
2. The actual value of $f(a)$ does **NOT** matter at all.

$$\lim_{x \to a} f(x) = L$$

4.2 One-sided limits

A limit from the left side is $\lim_{x \to a^-} f(x) = L_1$. A limit from the right side is $\lim_{x \to a^+} f(x) = L_2$. If the two values match, the limit of $f(x)$ at a from both sides exist.

The formal definition of a left side limit is if $0 < a - x < \delta$, and the right limit is $a < x < a + \delta$. These two cannot be generalized for higher dimensions.

4.3 Continuity

Intuitively, $f(x)$ is continuous in an interval if the graph if the graph is in one piece. $f(x)$ is continuous at the point a if the limit a is equal to $f(x)$:

$$\lim_{x \to a} f(x) = f(a)$$

We can only define that a function is continuous on a point, but using that we can:

$f(x)$ is **continuous** in an interval means $f(x)$ is continuous at every point in the interval. Most functions are continuous. Transforming a continuous function using any means (such as +, ×, and composition) does not change whether the transformed interval is continuous (except division by zero.) Every standard function we know are continuous in their domains.

E.g., $\lim_{x \to 1} \left(x^2 + 2^x\right) = f(1) = 1^2 + 2^1 = 3$

Find conditions on A and B so that

$$f(x) = \begin{cases} Ax - B & x \leq 1 \\ 3x & 1 < x < 2 \\ Bx^2 - A & 2 \leq x \end{cases}$$

Is continuous at $x = 1$ and discontinuous at $x = 2$.

Continuous: This means $\lim_{x\to 1} f(x) = f(1)$ and $\lim_{x\to 2} f(x) \neq f(2)$. Fortunately, all of the pieces of the piecewise functions are continuous. To evaluate this question, we need to use left and right sided limits.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} Ax - B = A - B$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 3x = 3x = 3$$

$$A - B = 3 = \lim_{x\to 1} f(x) = f(1)$$

As long $A - B = 3$, then $f(x)$ is continuous at 1.

Discontinuous: This means $\lim_{x\to 2} f(x) \neq 2$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 3x = 6$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (Bx^2 - A) = 4B - A$$

Now, we need to make sure $\lim_{x\to 2} f(x) \neq f(2)$. If this two-sided limit were to exist, the left limit and the right limit would match, meaning $6 = 4B - A = \lim_{x\to 2} f(x) = f(2)$. When would the limit not exist? $6 \neq 4B - A$. Solve it.

$$\begin{cases} 6 \neq 4B - A \\ 3 = A - B \end{cases}$$

4.4 Limits approaching infinity

We allow a to be infinite as well. This is understood intuitively as:

As x becomes larger and largher, $f(x)$ gets closer to a number L.

$$\lim_{x\to\infty} f(x) = L$$

We can also have a being minus infinity:

$\lim_{x\to -\infty} f(x) = L$

4.5 Limit Properties

First, we **need** to know the following limits to exist:

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

If we are not certain these limits exist, we cannot use the properties below.

Then we can attain

$$\lim_{x \to a} [f(x) + g(x)] = L + M$$

$$\lim_{x \to a} [f(x) - g(x)] = L - M$$

$$\lim_{x \to a} c \cdot f(x) = c \cdot L$$

$$\lim_{x \to a} [f(x) \cdot g(x)] = L \cdot M$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M},\ M \neq 0$$

Third property onward, proving these become extremely hard.

If $f(x)$ is continuous, we can move the limit **inside** f:

$$\lim_{x \to a} f(g(x)) = f\left[\lim_{x \to a} g(x)\right]$$

4.5.1 Limit properties proofs

If we know two limits is true

Firstly, we have this statement brought from the previous subsection:

$$\lim(f)\ and \lim(g)\ exists \Rightarrow \lim(f+g) = \lim(f) + \lim(g) \tag{1}$$

Prove the following statement: Suppose $\lim_{x \to a} [f(x) + g(x)]$ exists and $\lim_{x \to a} f(x)$ exists, then $\lim_{x \to a} g(x)$ also exists. Unfortunately, it is not correct to use (1) at first. You need to prove (1) first, so we know that it holds, and for (1) to be true we need to know that the limit at f and g exists. However, we do not know for sure that $\lim(g)$ exists, meaning we **cannot assume that (1) is true.**

What would happen if we considered $\lim_{x\to a}(f+g) - \lim_{x\to a} f$. Are we allowed to consider this? This requires us to use this property:

$$\lim(f)\ and\ \lim(g)\ exists \Rightarrow \lim(f+g)\lim(f-g) = \lim(f) - \lim(g) \tag{2}$$

This doesn't hold. We do not know that $\lim_{x\to a} g(x)$ exists.

What if we consider the function $f+g$ and the function f? We know that $\lim(f+g)$ exists and so is $\lim(f)$, so we can deduce that $\lim((f+g)-f)$.

The key point is to use limit laws, you need to know that the two limits you would use the laws exist in the first place. We considered $\lim(f+g)$ to be one function, and $\lim(f)$ to be the other, which is why it is perfectly acceptable to write $\lim(f+g-f)$. Is this confusing? Think as if I made the substitution $\lim(f+g) = \lim(h)$.

We have $\lim(f+g-f) = \lim(g)$, **which we know exists** as it was derived from a limit that exists. It is precise to say:

Since $\lim(f+g), \lim(f)$ *exists, then we can conclude that* $\lim(f+g) - \lim(f)$ *exists, and so on.*

Part 2 **T/F?** Suppose $\lim_{x\to a}[f(x)+g(x)]$ exists but $\lim_{x\to a} f(x)$ does not exist, then $\lim_{x\to a} g(x)$ also does not exist.

Proof. Suppose $\lim_{x\to a} g(x)$ exists. We know $\lim_{x\to a}[f(x)+g(x)]$ exists.

Consider $\lim(f+g) - \lim(g) \Rightarrow \lim(f+g-g) \Rightarrow \lim(f)$ (we can do this because the limit of $f+g$ and g are assumed to exist), meaning we proved that $\lim(f)$ exists. However, we have reached a statement which is contradictory to something we assumed initially. This means it is not possible for $\lim(g)$ to exist.

When you reach a contradiction, your "supposed" assumption will automatically falsify itself (or turn true if supposed false). For regular proof questions, we cannot assume what we want to prove. However, when working with true/false questions, take a guess and go both paths.

Part 3 Suppose $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a}[f(x)+g(x)]$ does not exist. This statement is true. $\lim_{x\to a}[f(x)+g(x)]$ cannot exist if either limit of f or g do not exist – the limit property $\lim_{x\to a}[f(x)+g(x)]$ only works if the two functions as individual limits work in the first place. Here, we know the opposite.

■

4.6 Continuity law for composition

Prove the *continuity law for composition:*

If $f(x)$ is continuous at $\lim_{x\to a} g(x)$, then we can move the limit inside:

$$\lim_{x\to a} f(g(x)) = f\left[\lim_{x\to a} g(x)\right] \tag{1}$$

Deducing this implies that composition of continuous functions is continuous.

We know that if you put a continuous function inside another continuous function, you get a composite function. However, we think of continuous as a graph being in one piece. However, you have no idea.

> What does it mean for $f(x)$ to be continuous? It means $\lim_{x\to a} f(x) = f(a)$. How would we write this statement?
>
> $$\forall \varepsilon > 0,\ \exists \delta > 0,\ 0 < |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

Because $L = f(a)$, we modified our epsilon-delta definition accordingly. Our assumption is that $f(x)$ is continuous at $\lim_{x\to a} g(x)$, but that should be treated just like a number. This means, which we know is true:

$$\lim_{x\to \lim_{x\to a} g(x)} f(x) = f\left[\lim_{x\to a} g(x)\right] \tag{2}$$

This is the true definition of if $f(x)$ is continuous at $\lim_{x\to a} g(x)$ (we know $\lim_{x\to a} g(x)$ exists, because we must assume everything in the assumption to be true). This is what the question is asking.

Now, what does (1) mean? **WTS:**

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ 0 < |x-a| < \delta \Rightarrow \left| f(g(x)) - f\left[\lim_{x\to a} g(x)\right]\right| < \varepsilon$$

This is a modified $\varepsilon - \delta$ definition presented in the box if you scroll up. This statement is what (1) is trying to say. All we need to do is prove that (3) is true. This is where we start using our assumptions. Meaning we know

$$\forall \varepsilon_2 > 0, \exists \delta_2 > 0, 0 < \left| x_2 - \lim_{x \to a} g(x) \right| < \delta_2 \Rightarrow \left| f(x_2) - f\left(\lim_{x \to a} g(x)\right) \right| < \varepsilon_2$$

We used assumption (2) to help us construct the limit definition (4) that we can assume true when proving (3). Our (4) is something that we use.

Proof. Let $\varepsilon > 0$. Pick $\varepsilon_2 =$ __, which makes the statement below true.

$$\exists \delta_2 > 0,\ 0 < \left| x_2 - \lim_{x \to a} g(x) \right| < \delta_2 \Rightarrow \left| f(x_2) - f\left(\lim_{x \to a} g(x)\right) \right| < \varepsilon_2$$

Pick $\delta =$ __. It may depend on δ_2.

From (4), suppose $0 < |x - a| < \delta$

Prove $|f(g(x)) - f[\lim_{x \to a} g(x)]| < \varepsilon$.

We haven't started the proof yet. We have set up the proof, though.

So, we pick $\varepsilon_2 = \varepsilon$ and $x_2 = g(x)$. Then $|f(x_2) - f(\lim_{x \to a} g(x))| < \varepsilon_2$ from (4) becomes $|f(g(x)) - f(\lim_{x \to a} g(x))| < \varepsilon_2$ from (5.0). This means we know this is true:

$$\forall \varepsilon_2 > 0, \exists \delta_2 > 0, 0 < \left| g(x) - \lim_{x \to a} g(x) \right| < \delta_2 \Rightarrow \left| f(g(x)) - f\left(\lim_{x \to a} g(x)\right) \right| < \varepsilon$$

(We are so close! Verify $0 < |g(x) - \lim_{x \to a} g(x)| < \delta_2$)

Connection: If $a \to g(a) \approx \lim_{x \to a} g(x) \to f(g(a))$

If I pick an x that is very close to a, then $x \to g(x) \to f(g(x))$. We are asking if x was near a, can we prove $f(g(x))$ would be very close to $f(g(a))$? We then pick an $\varepsilon > 0$ which gives us an error range around $f(g(a))$, and we would like to ask: can we find a δ range on the left. Then $f(g(x))$ will be in the epsilon range within $f(g(a))$? Then notice that if you start with x, if you want to get to the right side, it needs to get to $g(x)$, then $g(x)$ would be inputted to f to obtain $f(g(x))$. Now, $\exists \delta_2$ around $g(a)$, then $f(g(x))$ would be very close to $f(g(a))$. If we take a look at how δ_2 is set up, the delta we are looking for is all the way on the left. They are not even in the same place.

There is an implicit assumption that $\lim_{x \to a} g(x) = M$ exists. Then, we know that

$$\forall \varepsilon_3 > 0,\ \exists \delta_3 > 0,\ 0 < |x_3 - a| < \delta_3 \Rightarrow \left| g(x_3) - \lim_{x \to a} g(x) \right| < \varepsilon_3$$

Pick $\varepsilon_3 = \delta_2$. Then

$$\exists \delta_3 > 0,\ 0 < |x_3 - a| < \delta_3 \Rightarrow \left| g(x_3) - \lim_{x \to a} g(x) \right| < \varepsilon_3$$

Pick $\delta = \delta_3$.

Now how can I make sure $g(x)$ lands in the δ_2 range?

Using the red statement: Give me any ε_3 range in the middle centered around the red point, I will make sure $g(x)$ lands in the ε_3 error range in the middle by giving a δ_3 control on the left side. This is why δ_2 is the control of the f function, which is the target of the g function. This justifies why we picked those variables. These guarantees $g(x)$ will be sent to $f(g(x))$. It is not that these choices are intuitive.

1. Set a target on the right
2. Want a control on the left
3. Two-leg journey broken up in two pieces
4. Have $g(x)$ land in $\delta_2 = \varepsilon_3$ which grants a δ_3 control range on the left
5. That will guarantee $f(g(x))$ will land on ε.

Considering $x_2 = g(x),\ x_3 = x$

Since $0 < |x - a| < \delta = \delta_3$

Then $|g(x_3) - \lim_{x \to a} g(x)| < \varepsilon_3 = \delta_2 \Rightarrow |g(x_3) - \lim_{x \to a} g(x)| < \varepsilon_3 = \delta_2$

Referring back to equation 6, we can then get: $|f(g(x)) - f(\lim_{x \to a} g(x))| < \varepsilon$

■

5 Limits approaching infinity

We do allow a to be infinite. This can be understood as: As x becomes larger and larger, $f(x)$ becomes closer to L.

Vertical Asymptotes:

$$f(x) = \frac{1}{x-1}$$

$$\lim_{x \to 1} = \pm\infty$$

Horizontal Asymptotes:

As x approaches $\pm\infty$, sometimes $f(x)$ will attain a limit. This is the horizontal asymptote. The graph is allowed to cross the horizontal asymptote. For example, $\arctan(x)$ gives us horizontal asymptotes of $\pm\frac{\pi}{2}$.

Anything divided by infinity, other than infinity, gives zero.

If a limit ends up evaluating to infinity, the limit does not exist. Also, the sum of a limit that exists and a limit that does not will exist.

5.1 Limit Evaluation Techniques

ALL INVOLVE GETTING AROUND HOLES. DON'T BREAK THE LIMIT UNTIL YOU'RE FINISHED! THESE ARE THE EXCEPTIONS TO THE REGULAR RULES.

Rational Functions

Rational functions is a polynomial dividing a polynomial. Factor the top and bottom and cancel out the factors. Any cases that would result in a 'hole' as you did in Grade 11 will not apply as you are taking limits.

Rational functions with $\mathbf{x}$ approaching infinity

Only the term with the highest exponent in the numerator and denominator matters when computing limits to infinity.

$$\lim_{x\to\infty} f(x) = \frac{24x^6+1}{2x^6+30x^5+1} \approx \frac{24x^6}{2x^6} = 12$$

To be precise, we divide by the highest exponent on the numerator and the denominator. In this example, x^6 is the highest exponent.

$$\frac{\frac{24x^6}{x^6}+\frac{1}{x^6}}{\frac{2x^6}{x^6}+\frac{30x^5}{x^6}+\frac{1}{x^6}} = \frac{24+0}{2+0+0} = \frac{24}{2} = 12$$

$$\frac{ax^m}{bx^n}$$

$m < n$ indicates the limit is zero.

$m = n$ indicates the limit is a horizontal asymptote.

$m > n$ limit is $\pm\infty$

Square Roots

Multiply the fraction by its conjugate. Treat the square root as an entire package. For example:

$$f(x) = \frac{\sqrt{4x+1}-3}{x-2}$$

The square root is $\sqrt{4x+1}$. Treat that as one term when conjugating.

Use this to get around a "hole".

Small angle approximation

If the limit is given as $x \to 0$ then we can make this substitution.

$$\sin(x) \approx x \approx \tan(x)$$

This means $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.

This works for $\arcsin(x)$ and $\arctan(x)$.

Example: $\lim_{x\to 0} \sin(2x) = 2$. $\lim_{x\to 0} \frac{\sin^2(5x)}{x^2} = \frac{(\sin(5x))^2}{x^2} = \frac{(5x)^2}{x^2} = \frac{25x^2}{x^2} = 25$

Look out for $\lim_{x\to 3} \frac{\sin(x-3)}{x}$.

Absolute Values

Approach from the left and right side separately. The moment you determine that the expression inside the absolute value is positive or negative, do what is necessary and remove the absolute value sign. If the limit exists, they should evaluate to the same values.

For example, approaching from the positive side guarantees x is positive, meaning $|x| = x$; from the negative side guarantees x is negative meaning $|x| = -x$.

It is better to write the actual values of the limits even if it does not exist, using two single-sided limits.

Squeeze Theorem

The only technique that can be extended to higher dimensions.

To attain $\lim_{x\to a} f(x)$, you can try to find $g(x)$ and $h(x)$ so that:

1. $g(x) \le f(x) \le h(x)$ near the point a
2. $\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x)$

Then we conclude $\lim_{x\to a} f(x) = L$

In practice, we want to show $\lim_{x\to a} f(x) = 0$. Start at $|f(x)|$, create a chian of inequality, simplify $|f(x)|$ to attain $|g(x)|$ which has limit 0:

$$|f(x)| \leq \ldots \leq |g(x)| \to 0 \text{ i.e. } \lim_{x\to a} |g(x)| = 0$$

The only possible way for $|f(x)|$ to be $\leq$ and $\geq$ to 0 is for $\lim_{x\to a} f(x) = 0$.

For example: $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$

Squeeze theorem is used very commonly for $\sin\left(\frac{1}{x}\right)$. Remember that $\sin(\theta) \in [-1,\ 1]$. This means $|\sin(\theta)| \in [0,\ 1]$. To use squeeze theorem, take the entire function in absolute value:

$$0 \leq \left| x^2 \sin\left(\frac{1}{x}\right) \right| \leq \left|x^2\right| \cdot 1 \to 0$$

Thus $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$ by squeeze theorem. Refer to the template: $|f(x)| = \left|x^2 \sin\left(\frac{1}{x}\right)\right|$ and $|g(x)| = \left|x^2\right| \cdot 1$

Note that $\left|\sin\left(\frac{1}{x}\right)\right| \leq 1$ because of $\sin(\theta)$'s domain.

Squeeze theorem is if you want to show that $\lim_{x\to a} f(x) = 2$, you can turn it into $\lim_{x\to a} f(x) - 2 = 0$. Regardless of what a is it should work regardless of what value it is unless some other problems independent of squeeze theorem arise.

Discussion: $y = \sin\left(\frac{1}{x}\right)$. Take a look at $y = x^2 \sin\left(\frac{1}{x}\right)$. Closer to zero, the function will never be above x^2 or $-x^2$. This means despite the infinite oscillation of $\sin\left(\frac{1}{x}\right)$, x^2 squeezes the function to zero.

x^2 can only be multiplied by a number in the interval $[-1,\ 1]$ in $y = x^2 \sin\left(\frac{1}{x}\right)$. This means it can never be multiplied by anything greater than 1. This is why x^2 is the max value, and $-x^2$ is the minimum value.

5.2 Types of discontinuity

Continuous at a point means the limit is the same as the point itself.

$$\lim_{x\to a} f(x) = f(a)$$

To say $f(x)$ is not continuous, it could fail either case:

Removable Discontinuity (HOLE)

$\lim_{x\to a} f(x) = L$ exists, but $f(a) \neq L$ or $f(a)$ DNE

In any case, we can redefine the function at a, by setting $f(a) = L$.

Removable discontinuity typically arises when a cancellation occurs in a fraction (rational function):

$$f(x) = \frac{(x-1)(x+1)}{x-1},\ x \neq 1$$

But since the function can be $x+1$ if you get rid of the hole, you can redefine the function as:

$$f_{redefined}(x) = \begin{cases} \frac{(x-1)(x+1)}{x-1},\ x \neq 1 \\ 2,\ x = 1 \end{cases}$$

What we have done is that despite the hole in $f(x)$, we can redefine it to fill in the hole of the function. That is the intuitive thing.

$f_{redefined}(x)$ is now continuous at $a = 1$. Note that $f(X) = x+1$ everywhere.

Jump Discontinuity

$$\lim_{x\to a} f(x) \text{ DNE as a 2-sided limit}$$

However, both sided limits exist but are not equal. For example:

$$f(x) = \frac{|x|}{x} = \begin{cases} 1,\ x > 0 \\ -1,\ x < 0 \end{cases}$$

This is a true discontinuity as you cannot fix it. We have $\lim_{x\to 0^-} f(x) = -1$ and $\lim_{x\to 0^+} f(x) = 1$

Infinite Discontinuity

When one or both of the 1-sided limits are **infinite.**

$$\lim_{x\to a^-} f(x) = \pm\infty \text{ or } \lim_{x\to a^+} f(x) = \pm\infty \text{ or both}$$

Infinite discontinuity typically arises when a fraction is dividing by 0. However, we require that the numerator to be nonzero. You need to deduce whether it is $+\infty$ or $-\infty$. This can change depending on what side you approach from.

Note that $\frac{0}{0}$ is not allowed. $\frac{0}{0}$ is called an indeterminant form, which requires further manipulations. Dividing by zero and computing limits when doing so can lead to untrue statements, a case of mathematical fallacy. Sometimes, cancellations may not be obvious.

Essential Discontinuity

Does not fit the description above. It only comes up in very specific functions:

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$

As $x \to 0,\ \theta = \frac{1}{x} \to \infty$. The oscillations become faster as x approaches zero. There is no way to describe the discontinuity. You cannot approach infinite numbers at the same time, as limits are required to be unique. It cannot be captured by $\varepsilon - \delta$ limit definitions. The same applies to $\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$.

5.3 The idea of a Derivative

The derivative of $f(x)$ from first principles is:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

You must do some manipulation to the function to solve this.

Note that $f'(a)$ is typically different with a different point a. This can be understood as a function:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Where x is now thought of as the variable to the function f'.

The three ideas can be presented very differently. They don't seem to be related at all, but they do end up being the same definition: the first principles equation. The three definitions are just a coincidence. In higher dimensions, the three different ideas, simply because they are very different, they end up being different flavors of derivatives, hence why they need to be presented.

5.3.1 Graphically (slope)

The slope of the line tangent to the graph of the function $y = f(x)$ at point a. If you pick a point a and pick another point, you can connect them together and that creates a straight line, which you can compute the slope for it. That new point would be called $a+h$, and the corresponding y values would be $f(a)$ and $f(a+h)$. To find the slope, it would be $\frac{\Delta y}{\Delta x}$. This is called the difference quotient which matches the slope of the secant line. We would like to shift $a+h$ closer to a, which gives you a better approximation. The closer you get it, the better the approximation of the slope would be to the slope of the function at a. If we approximate that difference to be zero, we get the tangent line. The tangent line at a would be defined as the derivative first-principles equation.

The **tangent line** at $x = a$ is defined to be the line with the **slope** as the derivative $f'(a)$ passing through the point $(a,\ f(a))$.

The point-slope form of a line is $y - y_0 = m(x - x_0)$. Here, the point-slope form of the line tangent to a point a of any function would be:

$$y - f(a) = f'(a)(x - a)$$

Where $y_0 = f(a),\ m = f'(a),\ x_0 = a$

5.3.2 Sensitivity to Input changes

The word "rate of change" actually has a more complicated meaning. It means *the ratio of the changes.* Rate of change is the sensitivity to input changes. For example, consider a function $f(x) = 2x$. If you start with the input $x = 1$, we have $f(1) = 2$. If we increase the input by 1 unit, with $x = 2$, we have $f(2) = 4$. The output changed by two units. If we were to change the input by h units, the output is changed by $2h$ units.

This has nothing to do with the slope. The reason why is because when we are talking about the slope of a graph, there needs to be a graph in the first place. However, we usually don't draw graphs all the time. In one dimension, a graph is a curve on the $x - y$ axis. In higher dimensions, it is a surface that floats in mid-air. We don't think of visual examples here.

If we change the input by a lot, the output will change by a lot. The meaningful question is not how much the output has change, but rather the ratio of the changes between the output and the input. It is the ratio of the change of the output of the function divided by the change in the input of a function.

$$\frac{\text{change in output}}{\text{change in input}}$$

Which is

$$\frac{f(a+h)-f(a)}{h} = \text{Average Rate of Change}$$

It is just a coincidence in one dimension that this idea algebraically becomes the same expression as the slope of the tangent of the graph, but the idea is completely different.

If we let $h \to 0$, we get the derivative. It measures the ratio when **very small** change is done to the input. However, the ratio of the changes between the output and the input may in fact be small or large.

(For example: effort becomes less correlated with reception the more effort you put in.)

The function $f(x) = x^2$, close to zero, changing inputs near it affects the output a lot less.

Price elasticity of demand has lots of applications here.

The derivative here would be the instantaneous rate of change. The derivative is a local phenomenon. $f'(a)$ does not give any information of $f(x)$ at other points.

5.3.3 Velocity

$f(t) = s(t)$ is the position of a particle at time t. The average velocity between times $t = a$ and $t = b$ is

$$v_{avg} = \frac{s(b)-s(a)}{b-a} = \frac{\text{difference in position}}{\text{time used}}$$

This corresponds to the slope of the secant line defined by two points $(a,\ s(a))$ and $(b,\ s(b))$ on the graph of $s(t)$.

The instantaneous velocity, or just simply velocity, at time $t = a$ is the derivative:

$$v_{inst} = v(a) = s'(a) = \lim_{h\to 0}\frac{s(a+h)-s(a)}{h} = \lim_{t\to a}\frac{s(t)-s(a)}{t-a}$$

The velocity can again be thought as a function of time:

$$v(t) = s'(t)$$

Velocity also generalizes to higher dimensions. It can be generalized to be the position of something, having multi-dimensional aspects. This would be completely different from the rate of change interpretation and no relation to the slope of the tangent.

5.4 Calculating derivatives through first principles

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$f(x) = x^3$$

$$\begin{aligned}
&\frac{(a+h)^3 - a^3}{h} \\
&\frac{a^3 + 3a^2h + 3ah^2 + h^3 - a^3}{h} \\
&\frac{3a^2h + 3ah^2 + h^3}{h} \\
&3a^2 + 3ah + h^2 \\
&3a^2
\end{aligned}$$

Find $f(x) = x^3$ at (1, 1)

$$\begin{aligned}
y - y_0 &= m\,(x - x_0) \\
y - 1 &= m(x-1) \\
f'(a) &= 3a^2 = 3a = 3 \\
y - 1 &= 3(x-1)
\end{aligned}$$

Let $f'(x) = 0 \forall x$ and $f(x) = 0$. What about $f(2)$? This means the function is not sensitive to input changes meaning $f(2) = 0$

The derivative is a local phenomenon. Let $f'(x) = 0$ and $f(x) = 0$. What about $f(2)$? We do not know for sure. For example, $f(x) = x^2,\ f'(x) = 2x.\ f'(0) = 0$. However, derivative increases away from zero. At 2, the function becomes more sensitive to input changes. However, for $f'(0) = 0$ we only know what happens to the function at zero and no where else. It Is not about the numerical difference. As long as x is not at zero, we do not know what $f(x)$ or $f'(x)$ is. There are infinitely as many values between 0 and 2. It is just as far from zero as 0.2 is, in terms of the number gap, because all of them are some finite distance away. Knowing something about the behavior of the function at zero tells us nothing about the behavior of the function elsewhere.

We however need a more power theorem to prove this statement more rigorously.

Next:

Let $s(t) = t^2 - 6t$. Find the average velocity for t between the interval $[4,\ 8]$ and the instantaneous velocity at $t = 8$.

$$v_{avg} = \frac{s(8) - s(4)}{8 - 4} = \frac{(64 - 48) - (16 - 24)}{4} = \frac{16 + 8}{4} = 6$$

Instantaneous velocity at $t = 8$? $v(t) = s'(t) = 2t - 6$ so, if you plug in $v(8) = 16 - 6 = 10$. The velocity is increasing as time increases. When you take the average of the velocity between 4 and 8, the number is somewhere in the middle.

6 Three Important Theorems

These theorems can't be proven with standard axioms of the real numbers.

The axiom of the least upper bound: TBA

6.1 Intermediate Value Theorem

If $f(x)$ is continuous on $[a,\ b]$ and $f(a) < 0,\ f(b) > 0 \Rightarrow \exists c \in (a,\ b)$ with $f(c) = 0$ (we mentioned an interval here). *To draw a continuous curve from below the x-axis to above the x-axis, the curve must cross the x-axis at least once.*

IVT Generalized

If $f(x)$ is continuous on $[a,\ b]$ then for any K between $f(a)$, $f(b)$, there exists $c \in (a,\ b)$ with $f(c) = K$

6.2 Extreme Value Theorem (EVT)

If $f(x)$ is continuous on $[a,\ b]$, then $f(x)$ has a maximum and minimum on the interval [a, b]:

1. $\exists x_M$, with $f(x_M) \geq f(x)$ for all $x \in [a,\ b]$ as a maximum.
2. $\exists x_m$, with $f(x_m) \leq f(x)$ for all $x \in [a,\ b]$ as a minimum.

(On a closed interval a continuous curve would have a highest and lowest point.)

6.3 Mean value theorem

If $f(x)$ is continuous on $[a,\ b]$ and is differentiable on $(a,\ b)$, then $\exists c \in (a,\ b)$ with

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

(In terms of velocity, there is a c where $f'(c) =$ average velocity)

Intuition: if the average speed of a car is 100kmh^{-1}, will there be a point where the car travels at exactly 100kmh^{-1}?

These theorems do not help find the exact values – sometimes, you cannot find exact values; you must use approximations. It just guarantees the existence of them.

(Note: differentiable on $[a,\ b] \Rightarrow$ differentiable on $(a,\ b)$ – we use the weakest assumption. Rare situation is when cusps exist on the edge of the interval.)

Jumps and cusps are not allowed. Jumps would be considered teleportation, and cups are sudden increases in speed with no acceleration for some time, and that cannot happen in real life.

When the assumption for the theorems is not satisfied, the theorem may not work. Even if the assumptions are not satisfied, the theorem may still work, but not always.

6.3.1 Q

Show that there are at least 3 solutions in $\left(\frac{x}{\pi}\right)^3 - 2\sin(x) + \frac{1}{2} = 0$

x is not solvable. Knowing you can't solve it, use IVT.

x	$f(x)$
0	$\frac{1}{2}$
π	$\frac{3}{2}$

Interval checking

7 Squeeze theorem

For every $x \neq 0$:

$$-1 \leq \sin\frac{1}{x} \leq 1$$

$$-x^2 \leq x^2 \sin\frac{1}{x} \leq x^2$$

A more general picture:

If you have two functions which have the same limit L at a, and I have a third function that is squeezed in between them, no matter how it behaves, it has the same limit.

This means:

Let $a,\ L \in \mathbb{R}$. Let $f,\ g,\ h$ be functions defined near a, except possibly at a

If:

- $h(x) \leq g(x) \leq f(x)$
- $\lim_{x\to a} f(x) = L$
- $\lim_{x\to a} h(x) = L$

Then

- $\lim_{x\to a} g(x) = L$

Compute $\lim_{x\to 0} x^2 \sin\frac{1}{x}$.

- For every $x \neq 0:\ -x^2 < x^2 \sin\frac{1}{x} \leq x^2$
- $\lim_{x\to 0} x^2 = 0$ and $\lim_{x\to 0} (-x^2) = 0$
- By the squeeze theorem: $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$

8 Derivative

The function is differentiable at a point if the limit exists as the derivative at the point.

Tangent line

$$y - y_0 = m\,(x - x_0)$$

Tangent line at $(x_0,\ y_0) = (a,\ f(a))$ has slope $m = f'(a)$

$$e^{\ln(x)} = \ln\left(e^x\right)$$

(This works because an inverse function acts as an argument to the function)

Note: $\tan'(x) = \sec^2(x)$ and $(\sec(x))' = \sec(x)\tan(x)$

8.1 Inverse Function Theorem

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

$$\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$
$$acos(x) = -\frac{1}{\sqrt{1-x^2}}$$

$$atan'(x) = \frac{1}{1+x^2}$$

Square roots: $\sqrt{x} \geq 0$

- $x = 4$
- $\sqrt{x} = a$ (Square root is defined to simply represent the positive side.)
- $a^2 = 4 \Rightarrow a = \pm 2$

8.2 Higher order derivatives

$$\sin^2(x) = \sin^2(x) + \sin^2 x^2$$

Derivative is

$$2\sin(x)\cos(x) + \cos\left(x^2\right)2x + 2\sin\left(x^2\right)\cos\left(x^2\right)2x$$

Derivative of e^{x^e} is $e^{x^e}ex^{e-1}$

x^x (when both the base of the exponent and the exponent is the same, you cannot take the derivative.) **The power rule may not be applied if the exponent is variable**

$$x = e^{\ln(x)} \Rightarrow x^x = e^{(\ln(x))^x} = \left(e^{\ln(x)}\right)^x = e^{x\cdot\ln(x)}$$

Using $x = e^{\ln(x)}$ is workable. The derivative of $e^{x\cdot\ln(x)}$ is $e^{x\cdot\ln(x)} \cdot \left(1 \cdot \ln(x) + x\left(\frac{1}{x}\right)\right)$ (then simplify it using the product rule)

8.3 Nth order derivatives

$$f(x) = \ln(x)$$

$$\begin{aligned}
f'(x) &= x^{-1} \\
f''(x) &= -x^{-2} \\
f'''(x) &= (-1)(-2)x^{-3} \\
f^4(x) &= (-1)(-2)(-3)x^{-4} \\
f^n(x) &= (-1)^{n-1}\left(1 \times 2 \times \cdots \times (n-1)\right)x^{-4} = -1 \cdot (n-1)! \cdot x^{-4}
\end{aligned}$$

The $(-1)^{n-1}$ is how we compress signs. For example, the second derivative ends up. . . .

Firstly, derivative: $3x^2 = A$ at $x = 1$, meaning $3 = A$. This is not enough to guarantee that f is differentiable at 1.

Second, continuous: $x^3 = 3x + b$ at $x = 1$ means $1 = 3 + b \Rightarrow b = -2$

8.3.1 Qs

8.4 Exponents and logs

$$(e^x)' = e^x$$

What if base e are not used?

We will use the fact that e^x and $\ln(x)$ are inverse functions.

Use the relation: $a^x = \left(e^{\ln(a)}\right)^x = e^{x \cdot \ln(a)}$

$$(a^x)' = e^{x \cdot \ln(a)} \cdot \ln(a) = a^x \cdot \ln(a)$$

8.4.1 Natural log:

$$(\ln(x))' = \frac{1}{x}$$

(We define $\ln(x)$ for which its derivative is exactly $\frac{1}{x}$. It does not come from base 10 log or any other base. $\ln(x)$ is defined to be the integral of $\frac{1}{x}$)

8.4.2 Derivative of logs:

$$y = \log_a(x) \Leftrightarrow x = a^y$$

Using the relation $a = e^{\ln(a)}$:

$$x = a^y = \left(e^{\ln(a)}\right)^y = y \cdot \ln(a)$$

Take the ln of both sides:

$$\ln(x) = \ln\left(e^y \cdot \ln(a)\right) = y \cdot \ln(a)$$

Divide by the constant $\ln(a)$

$$y = \log_a(x) = \frac{\ln(x)}{\ln(a)}$$

(This is simply log change of base. $\ln(a)$ is simply a vertical stretch.)

Taking the derivative: $y' = (\log_a(x))' = \frac{1}{\ln(a)} \cdot \frac{1}{x} = \frac{1}{x\ln(a)}$

I mean:

$$f(x) = e^x$$
$$f(x \cdot \ln(a)) = e^{x \cdot \ln(a)}$$

8.4.3 Logarithmic differentiation:

When rolling a dice and flipping a coin, the probability of rolling a 1 and getting a head on the coin is the product of the probabilities of either. With the logical connection AND, the probabilities multiply together. When you have a lot of things connected with AND, it is a bunch of things multiplied by each other. The problem is, the derivative works very well with addition but not very well with multiplication.

$$(fg)' = f'g + fg'$$
$$(fgh)' = f'gh + fg'h + fgh'$$

(The prime carries from the left to the right.)

We can use ln to change multiplication to addition, division to subtraction, and exponentiation to multiplication. We have the following properties:

$$
\begin{aligned}
\ln(xy) &= \ln(x) + \ln(y) \\
\ln\left(\frac{x}{y}\right) &= \ln(x) - \ln(y) \\
\ln\left(x^y\right) &= y \ln(x)
\end{aligned}
$$

To use it:

$$y(x) = \frac{f(x)^2 g(x)}{h(x)}$$

$$\ln\left(y(x)\right) = \ln\left(\frac{f(x)^2 g(x)}{h(x)}\right)$$

Applying the log rules, we get (note that the square in $f(x)$ is part of the function, as $\ln\left(f(x)^2\right) = 2\ln f(x)$:

$$\ln\left(y(x)\right) = 2\ln\left(f(x)\right) + \ln\left(g(x)\right) - \ln\left(h(x)\right)$$

Take the derivative against x on both sides using the chain rule, we get:

$$\frac{1}{y(x)} \cdot y'(x) = \frac{2}{f(x)} \cdot f'(x) + \frac{1}{g(x)} \cdot g'(x) - \frac{1}{h(x)} \cdot h'(x)$$

Multiply both sides by $y(x)$

$$
\begin{aligned}
\frac{y(x)}{y(x)} \cdot y'(x) &= 2\frac{y(x)}{f(x)} \cdot f'(x) + \frac{y(x)}{g(x)} \cdot g'(x) - \frac{y(x)}{h(x)} \cdot h'(x) \\
y'(x) &= 2\frac{y(x)}{f(x)} \cdot f'(x) + \frac{y(x)}{g(x)} \cdot g'(x) - \frac{y(x)}{h(x)} \cdot h'(x)
\end{aligned}
$$

Okay good luck stuffing $y(x)$ into this mess

$$y'(x) = 2\frac{\frac{f(x)^2 g(x)}{h(x)}}{f(x)} \cdot f'(x) + \frac{\frac{f(x)^2 g(x)}{h(x)}}{g(x)} \cdot g'(x) - \frac{\frac{f(x)^2 g(x)}{h(x)}}{h(x)} \cdot h'(x)$$

An example

$$y(x) = \left(\frac{\sqrt{x^2+3}}{\sqrt[3]{x^2+2x}} \right)$$

What is $\ln(y(x))$? It is

$$\ln(y(x)) = \frac{1}{2}\ln\left(x^2+3\right) - \frac{1}{3}\ln\left(x^2+3x\right)$$

$$\frac{1}{y(x)}y'(x) = \frac{1}{2\left(x^2+3\right)} - \frac{1}{3\left(x^2+3x\right)}$$

$$y'(x) = \frac{y(x)}{2\left(x^2+3\right)} - \frac{y(x)}{3\left(x^2+3x\right)}$$

$$y'(x) = \frac{\frac{\sqrt{x^2+3}}{\sqrt[3]{x^2+2x}}}{2\left(x^2+3\right)} - \frac{\frac{\sqrt{x^2+3}}{\sqrt[3]{x^2+2x}}}{3\left(x^2+3x\right)}$$

9 Inverse functions (computationally difficult)

A function starts with a domain and a codomain.

$$f : A \to B$$

9.1 One-to-one

A function is **one-to-one** when any two different points in the domain will get sent (by *f*) to two different points in the codomain. Equivalently:

There cannot exist 2 different points that get sent by f to the same point.

The simplest logical statement for one-to-one is:

f is one-to-one $\Leftrightarrow (f(a) = f(b) \Rightarrow a = b)$

Why is this the case? If it seems as though we have 2 different points sent to the same point, then these two points have to be the same. What you are concluding is that it is **not** two different points.

$$f(a) = f(b) \Rightarrow a = b$$

We feel as though a and b are different, but we never said that. The only time where it seems that two different points are getting sent to the same point, you didn't start with two different points.

Contrapositive of $a \neq b \Rightarrow f(a) \neq f(b)$ *There cannot exist two points that get sent to the same point.*

Example:

$f(x) = x+3$. Suppose $f(a) = f(b)$, $a+3 = b+3$, so $a = b \Rightarrow f$ is one-to-one

9.2 Onto

The **image** is all possible values that f can take (based on the domain).

f is **onto** if: **image = codomain**

Consider this function, which we can assume to be one-to-one for now:

We have defined our function to be $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = \arctan(x)$

Consider $A = \mathbb{R}$ (on the left) and $B = \mathbb{R}$ (on the right). However, as x can be any real number in $A = \mathbb{R}, \arctan(x)$ can take on every value in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The function is not **onto,** as we typically define the codomain for 1D calculus to be $\mathbb{R}$, but the image does not cover $\mathbb{R}$.

However, we can redefine the codomain. If we set $f\colon \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then we have made the function onto. Then f is **one to one and onto.**

This means we can remove elements from the codomain by redefining what the function is supposed to map to in order to make our function onto.

(e.g. in linear algebra, $f : \mathbb{R}^2 \to \mathbb{R}^3$, codomain is $\mathbb{R}^3$. Yes, your linear transformation can be redundant as $[x_1 + x_2 + x_3,\ 0,\ 0]$ – if you redefine the codomain to be smaller you are going to have to get rid of dimensions of that linear transformation)

DOMAIN

If the function's domain is $\mathbb{R}$, only consider the point where the function is defined at that point.

9.3 Horizontal line test (one-to-one)

Given a function f : f is one-to-one if and only if every possible horizontal line in the x-y plane intersect the curve at most once.

9.4 One-to-one and onto

A function is **one-to-one and onto**

$$f : A \to B$$

if and only if there exists an inverse function $\mathbf{f^{-1}}$:

$$f^{-1} : B \to A$$

Suddenly, B is the domain.

One-to-one rationale – if it wasn't, our inverse wouldn't be a function

If it isn't onto, there will be missing correspondences. Better remove things from the codomain first, because if something is on the codomain that can't be associated with a point on the domain, then you have a missing correspondence.

The inverse comes in pairs. The function f takes $a \in A$ as input and gibves $b \in B$ as output. The function f^{-1} takes $b \in B$ and spits out $a \in A$ as output of f^{-1}.

The domain of f^{-1} is B and the image of f^{-1} is A.

The inverse of f^{-1} is f, written as $\left(f^{-1}\right)^{-1} = f$

Meaning

$$f^{-1}\left(f(1)\right) = 1; f\left(f^{-1}(4)\right) = 4$$

The inverse function and the regular function should cancel each other out.

There is a one-to-one correspondence between the points in A and B.

$$\begin{aligned} y &= f(x) \\ f^{-1}(y) &= f^{-1}\left(f(x)\right) \\ f^{-1}(y) &= x \end{aligned}$$

This means if $y = f(x)$, $y = \arctan(x) \Rightarrow f^{-1}(y) = \tan(y) \Rightarrow x = \tan(y)$

However: $f \colon \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$f^{-1} : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$$

The domain is restricted. The domain of the inverse function does not expand infinitely out. This is how inverse trigonometric functions are defined.

Although $f^{-1}(y)$ is defined to be a formula to $\tan(y)$, it has a different domain than the standard $\tan(y)$ function.

The inverse function only works is because we have restricted the domain of the input for the inverse. For $y \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have $\arctan(\tan(y)) \neq y$. To avoid confusion, instead of using y, we will be using θ for angle and x for the real number part.

9.5 Inverse trig function

Consider $f(\theta) = \sin(\theta)$. Let $f : A = \mathbb{R} \to B = \mathbb{R}$

We have $image(f) = [-1,\ 1]$ so we set

$$f : A = \mathbb{R} \to B[-1,\ 1]$$

We see that for a sine function, it is not 1:1 – the image is $[-1,\ 1]$ so the function is onto. We can fix that by redefining the codomain is $[-1,\ 1]$, so $\sin(\theta)$ is onto. But $\sin(\theta)$ is not one-to-one, as it fails the horizontal line test.

Restrict the domain by removing 2π and 4π if the circle on the left is our domain. At least something smaller than all real numbers. While the original $f(\theta)$ is the entire sine functiuon, we will restrict the domain to be the green portion of the curve.

The most common way for $f(\theta) = \sin(\theta)$ is to choose $A = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then f is **one-to-one and onto,** and $f^{-1}(x) = \arcsin(x)$

$$f(\theta) = \sin(\theta) : \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \to [-1,\ 1]$$

$$f^{-1}(x) = \arcsin(x) : [-1,\ 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Sin takes an angle and returns a ratio; arcsin takes a ratio and returns an angle.

We restrict around zero for the sake of simplicity.

These restrictions are universally agreed with to give the functions one-to-one properties. The confusing part about $\arcsin(\sin(6)) \neq 6$ – how do we deal with angles that are outside the domain that would make $\sin(\theta)$ bijective?

9.6 Working with inverse trigonometric functions

Following values:

$$x = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$$

$$y = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$\frac{2\pi}{3}$ is in the restricted domain $[0,\ \pi]$. This means $\arccos(x) = \arccos\left(\cos\left(\frac{2\pi}{3}\right)\right) = \frac{2\pi}{3}$

The two parts, $\arccos\left(\cos\left(\frac{2\pi}{3}\right)\right)$ are inverses of each other, giving exactly $\frac{2\pi}{3}$ back, meaning $\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$. We can only cancel out inverse functions if the input is in the restriction.

Okay. Now, consider $y = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$. Now what is $\arcsin\left(\sin\left(\frac{2\pi}{3}\right)\right) = \arcsin(y)$?

Unfortunately, $\arcsin\left(\sin\left(\frac{2\pi}{3}\right)\right) \neq \frac{2\pi}{3}$. However, if you look at how arcsine is defined, it will only take values between $[-1,\ 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

- We said $y = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$. Why is the representation of $\frac{\sqrt{3}}{2} = \sin\left(\frac{2\pi}{3}\right)$? Is there another way to represent $\frac{\sqrt{3}}{2}$? Consider the angle $\frac{\pi}{3}$. Note that $\frac{\sqrt{3}}{2} = \sin\left(\frac{\pi}{3}\right)$

$$\arcsin\left(\sin\left(\frac{2\pi}{3}\right)\right) = \arcsin(y) = \arcsin\left(\sin\left(\frac{\pi}{3}\right)\right)$$

Because $\frac{\pi}{3}$ falls in the region for how arcsine is defined, arcsine and sine here act as inverse functions to each other and give out the same thing. Meaning $\arcsin\left(\sin\left(\frac{2\pi}{3}\right)\right) = \arcsin\left(\sin\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$.

$\frac{\sqrt{3}}{2}$ has multiple ways of being represented. In other words, if I were to apply arcsine on both sides of the equation, we get $\arcsin(y) = \arcsin\left(\sin\left(\frac{2\pi}{3}\right)\right)$ which is the natural thing to do, but then if you were to do this, sine and arcsine cannot act as inverse functions as each other, as $\frac{2\pi}{3}$ is not in the correct angle domain for sine and arcsine that would make them inverse functions of each other.

So instead, $\frac{\sqrt{3}}{2}$ not only can be represented as $\sin\left(\frac{2\pi}{3}\right)$, but I could also represent it as $\sin\left(\frac{\pi}{3}\right)$.

Now, other cases: $x = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$; $\arccos\left(\cos\left(\frac{2\pi}{3}\right)\right) = \frac{2\pi}{3}$

Now $\arctan\left(\tan\left(\frac{2\pi}{3}\right)\right)$

Note that $z = \tan\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{-1}$

Use the same technique: what between $(-\frac{\pi}{2}, \frac{\pi}{2})$ if inputted into tan gives us $-\sqrt{3}$?

Use the CAST rule. $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$. Using this insight, $-\frac{\pi}{3} = \arctan\left(\tan\left(-\frac{\pi}{3}\right)\right)$. Notice that $-\frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The only matching angle in the interval where the inverse trig functions would be defined.

9.6.1 Inverse on the inside

What if we do the reverse?

$$\cos(\arccos(x))$$

If we take a look at the arccosine function, it takes in a ratio between $[-1,\ 1]$ and gives us an angle in $[0,\ \pi]$. Then cosine takes an angle in its correct angle domain according to how arccosine is defined. This means $x = \cos(\arccos(x))$, ONLY IF $x \in [-1,\ 1]$. $\arccos(2) =$ DNE

Consider $\sin(\arcsin(x)) = x$, as $\arcsin(x) = \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, ONLY IF $x \in [-1,\ 1]$. $\arcsin(2) =$ DNE; don't try.

For tangent, we get the same conclusion. $\tan(\arctan(x)) = x$, which works for $x \in \mathbb{R}$. It is true always.

Therefore, having the inverse trigonometric function in the inside, you will always get x as long as x is in the domain of the inverse function.

We can conclude that:

$\arccos(\cos(\theta)) = \theta$ for $\theta \in [0,\ \pi]$

$\arcsin(\sin(\theta)) = \theta$ for $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Some questions:

$$\arccos\left(\cos\left(-\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$$

(Work: $\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right)$; $\arccos\left(\cos\left(-\frac{\pi}{3}\right)\right) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3} \neq -\frac{\pi}{3}$. $\frac{\pi}{3} \in [0,\ \pi]$, required interval for $\arccos(\dots)$ to be defined).

Now, how do we do this for any angle that is not in the special triangles?

9.7 Trig properties

To easily calculate $\arccos(\cos(\theta))$ for any θ, use the following properties:

- $\cos(\theta) = \cos(-\theta)$ as cosine is an even function.
- $\cos(\theta \pm 2k\pi) = \cos(\theta)\ \forall k \in \mathbb{Z}$ as cosine is a periodic function. Note $2\pi \approx 6.28$

Sine is more complicated because the same properties do not hold.

- $\sin(-\theta) = -\sin(\theta)$ as sine is an odd function – you can move the negative sign in or out
- $\sin(\theta \pm \pi) = -\sin(\theta)$
  - As $\sin(\theta)\cos(\pi) + \cos(\theta)\sin(\pi) = -\sin(\theta) + 0$ from the addition property.
  - We can imply $\sin(\theta \pm (2k+1)\pi) = -\sin(\theta)\ \forall k \in \mathbb{Z}$
- $\sin(\theta + 2k\pi) = \sin(\theta)\ \forall k \in \mathbb{Z}$ as sine is a periodic function. Note $2\pi \approx 6.28$

Example:

$$\begin{aligned}\arcsin\left(\sin\left(\frac{5\pi}{6}\right)\right) &= -\arcsin\left(\sin\left(\frac{5\pi}{6} - \pi\right)\right) = \arcsin\left(-\sin\left(-\frac{\pi}{6}\right)\right) \\ &= \arcsin\left(\sin\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}\end{aligned}$$

9.8 Graphing inverse of itself for trig functions

9.8.1 Sine

Sine is a periodic function. The arcsine of sine is also a periodic function because sine is periodic. To figure out how $\arcsin(\sin(\theta))$ will look like, you only need to deduce what it will look like for **one period** of the function; the function then repeats from there infinitely. By the way, the period for $\sin(\theta)$ is 2π.

Firstly, we know $y = \arcsin(\sin(\theta)) = \theta$ if $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. This gives us how the function behaves between that region, which covers $\frac{1}{2}$ of the period we need.

Expanding regions

Now, what is y when

$$\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$

We can make the following deductions (REMEMBER: θ STAYS FAITHFUL TO ITS DEFINED INTERVAL, $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$):

$$f(\theta) = \arcsin(\sin(\theta)) \neq 0$$

Use the property $\sin(\theta \pm \pi) = -\sin(\theta)$, so subtract π and add the negative sign to sine.

$$f(\theta) = \arcsin(-\sin(\theta - \pi))$$

Then

$$\theta - \pi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Unfortunately, arcsine and sine cannot cancel each other out. But since we know

$$\theta - \pi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Leftrightarrow \pi - \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Then

$$f(\theta) = \arcsin(-\sin(\theta - \pi)) = \arcsin(\sin(\pi - \theta)) = \pi - \theta$$

Because $-\theta + \pi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the above statement is true. This means

$$f(\theta) = -\theta + \pi,\ \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$

By this point, we have found $f(\theta)$ for $\theta \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

Looping

Note that $\sin(\theta) = \sin(\theta + 2\theta)$. Therefore, we can conclude our graph looks like this:

In other words: loop the following function:

$$f(\theta) = \begin{cases} \theta,\ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ -\theta + \pi,\ \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \end{cases}$$

9.9 Converting between sine and cosine

Sine is just cosine transformed.

$$\sin(\theta) = \cos\left(\theta - \frac{\pi}{2}\right)$$

Trying to solve $\arccos(\sin(\theta))$, $\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

$$\begin{aligned}\arccos(\sin\theta) &= \arccos\left(\cos\left(\theta - \frac{\pi}{2}\right)\right),\ \theta - \frac{\pi}{2} \in [0,\ \pi] \\ &= \theta - \pi/2\end{aligned}$$

As long as x is between the domain that lets cosine be injective, $\arccos(\cos(x)) = x$

Now consider θ being in a different region, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Then $\arccos\left(\cos\left(\theta - \frac{\pi}{2}\right)\right)$, $\theta - \frac{\pi}{2} \in [-\pi,\ 0]$. Not in the same region where cos and arccosine would cancel each other out. If you add 2π, it will overshoot, so don't. Consider negating the input to cosine (taking advantage of the fact that cosine is an even function).

$$= \arccos\left(\cos\left(-\theta + \frac{\pi}{2}\right)\right) = -\theta + \frac{\pi}{2} \in [0,\ \pi] \text{ (same as negating inequalities)}$$

Recall: placing the arc function inside the regular function will cancel out, except the domain will be restricted. If you put the trig function on the inside, any angle can be placed in.

$$\sec(\arccos(x)),\ x \in (0,\ 1]$$

$$\arccos(x) = \theta \in [0,\ \pi]$$

$$\cos(\arccos(x)) = \cos(\theta) = x$$

By limiting x to be positive, it requires that θ be positive.

$$\frac{A}{H} = \frac{x}{1} = x = \cos(\theta)$$

$$\sec(\arccos(x))$$

But $\arccos(x) = \theta$ so

$$\frac{1}{\cos(\arccos(x))} = \frac{1}{x}$$

Now what about $\sin(\arccos(x))$ and knowing that x is positive, then

$$\arccos(x) = \theta,\ \theta \in \left[0, \frac{\pi}{2}\right] \text{ (theta restriction from x forced positive)}$$

$$\sin(\theta) = \frac{l}{1} = \sqrt{1-x^2}$$

This diagram: say that $x = \cos(\theta)$. Now, calculate the height of the opposite of θ. Using the Pythagoras theorem, $l = \sqrt{1-x^2}$

10 Solving inverse functions

Solving for an inverse function:

1. Set $y = f(x)$
2. Isolate x on one side, $x = g(y)$, where $g(y)$ is the right side with only y
3. Write $f^{-1}(y) = g(y)$, then switch all y to x, so $f^{-1}(x) = g(x)$. Note that switching the letter back to x is optional, but the moment you switch the x is NOT the same as the x of the original function.

10.1 Inverse Function Theorem

Why do we use f^{-1} to indicate inverse functions? Note: $\det\left(A^{-1}\right) = (\det(A))^{-1} = \frac{1}{\det(A)}$. The inverse of any number is its reciprocal. Now, get a function, and let's say we have an inverse function: f and f^{-1}. Now, what is $(f^{-1})'$? Note that you can interchange derivative operators with inverse operators. This means:

$$\left(f^{-1}\right)' = \left(f'\right)^{-1}$$

$$f(x) = y,\ x = f^{-1}(y)$$
$$\left(f^{-1}\right)'(y) = \left(f'(x)\right)^{-1} = \frac{1}{f'(x)} = \frac{1}{f'\left(f^{-1}(y)\right)}$$

Note: input changes when you swap a function's inverse and derivative operator. You have a function in terms of y, so substitute. The input value y must show up on the right. x and y are related by their inverse functions.

10.2 Usage

$$f(x) = \sin(x),\ f^{-1}(x) = \arcsin(x)$$

Here, we are using x as both of the variables but we aren't supposed to do that. If you are interested in taking the derivative of arcsine:

$$\begin{aligned}
\left(f^{-1}\right)'(x) &= \frac{1}{\cos\left(\arcsin(x)\right)} \\
\arcsin(x) &= \theta \\
x &= \sin(\theta) \\
l^2 + x^2 &= 1^2 \\
l &= \sqrt{1-x^2} = \cos(\theta)
\end{aligned}$$

11 Implicit differentiation

Not functions

1. Replace all y with $y(x)$
2. Take derivative against x on both sides.
3. Isolate $y'(x) = \frac{\mathrm{d}y}{\mathrm{d}x}$

Remember to use the product and chain rule when taking derivatives.

For example:

$$\begin{aligned}
\cos(xy) &= 1 + \sin(y) \\
\cos\left(x \cdot y(x)\right) &= 1 + \sin\left(y(x)\right) \\
\frac{d}{dx}\cos\left(x \cdot y(x)\right) &= \frac{d}{dx}\left[1 + \sin\left(y(x)\right)\right] \\
-\sin\left(x \cdot y(x)\right) \cdot \left[1 \cdot y(x) + x \cdot \frac{dy}{dx}\right] &= \cos\left(y(x)\right) \cdot \frac{dy}{dx}
\end{aligned}$$

Isolate $\frac{dy}{dx}$:

$$\begin{aligned}
-\sin(x \cdot y(x)) \cdot y(x) - x \cdot \sin(x \cdot y(x)) \cdot \frac{dy}{dx} - \cos(y(x)) \cdot \frac{dy}{dx} &= 0 \\
[-x \cdot \sin(x \cdot y(x)) - \cos(y(x))] \cdot \frac{dy}{dx} &= \sin(x \cdot y(x)) \cdot y(x) \\
\frac{dy}{dx} &= \frac{\sin(x \cdot y(x)) \cdot y(x)}{-x \cdot \sin(x \cdot y(x)) - \cos(y(x))}
\end{aligned}$$

Notation stuff:

$$\begin{aligned}
y &= y(x) \\
y'(x) &= \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} y(x) = y'
\end{aligned}$$

12 Related rates

TAKE DERIVATIVE FIRST BEFORE PLUGGING IN VALUES

1. Assign symbols to all variables in the problem
2. Writes an eqn that relates all the variables
3. Identify which variables change with time (If some quantity x changes as time passes, replace x with $x(t)$).
4. Take derivative against t on both sides using implicit differentiation
5. Plug in the values given for the variables in the problem.

Example: vol of cone increases at 30m^3 per minute. Diameter and height are always equal. How fast is the height increasing when it is 10 meters tall?

$$v = \text{vol},\ r = \text{radius},\ h = \text{ht}$$

$$v = \frac{1}{3}\pi r^2 h$$

Using assumption that $d = h$, $d = 2r = h \Rightarrow r = \frac{h}{2}$

$$v=\frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h=\frac{1}{12}\pi h^3$$

Because both change with time

$$\begin{aligned}
v(t) &= \frac{1}{12}\pi h(t)^3 \\
\frac{d}{dt}v(t) &= \frac{1}{12}\pi\cdot\frac{d}{dt}\left(h(t)\right)^3 \\
\frac{dv}{dt} &= \frac{1}{12}\pi\cdot 3\cdot h(t)^2\cdot\frac{dh}{dt} \\
30 &= \frac{1}{12}\cdot 3\cdot\left(10^2\right)\cdot\frac{dh}{dt}=25\pi\cdot\frac{dh}{dt}\Rightarrow\frac{dh}{dt}=\frac{6}{5\pi}
\end{aligned}$$

(Objective: find $\frac{dh}{dt}$ or change in height over time.)

The distance D is the distance between car W and car S. (W and S indicate direction of car). x is the distance W driven so far, and y is S driven so far. $x(t),\ y(t)$ current position of car, $x'(t)$ and so on is the speed of the car right now.

$$\begin{aligned}
x^2+y^2 &= D^2 \\
x(t)^2+y(t)^2 &= D(t)^2 \\
2x(t)\cdot x'(t)+2y(t)\cdot y'(t) &= 2D(t)\cdot D'(t) \\
x(t)\cdot x'(t)+2y(t)\cdot y'(t) &= D(t)\cdot D'(t)
\end{aligned}$$

At $t=2,\ x(t)=50,\ y(t)=120,\ D(t)=\sqrt{x^2+y^2}=\sqrt{50^2+120^2}=\sqrt{2500+14400}=\sqrt{100}+\sqrt{25+144}=10\sqrt{169}=10\cdot 13=130$

$$50\cdot 25+120\cdot 60=1300D'(t)$$

$$\begin{aligned}
x(t)^2+200^2 &= D(t)^2 \\
2x(t)x'(t) &= 2D(t)D'(t) \\
x(t)x'(t) &= D(t)D'(t)
\end{aligned}$$

When the police did the measurement, the radar indicates that the distance between the police and the car is increasing by 70 km/h. That is your $D'(t)$

Solve for $x'(t) \approx 100$

$$\begin{aligned}
200x'(t) &= 100\sqrt{8}\cdot 70 \\
x'(t) &= \frac{100\sqrt{8}\cdot 70}{200} = \frac{1}{2}\sqrt{8}\cdot 70 = 35\sqrt{8} \\
x(t)^2+8^2 &= D(t)^2 \\
x(t)^2+8^2 &= (4s(t))^2 = 16\,(s(t))^2 \\
2x(t)x'(t) &= 32s(t)\cdot s'(t) \\
2x(t)\frac{x'(t)}{32s(t)} &= s'(t)
\end{aligned}$$

Because $s(t) = \frac{D(t)}{4}$, and $D(t) = \sqrt{x(t)^2+64}$ then

$$x(t)\frac{x'(t)}{4\sqrt{x(t)^2+64}} = s'(t)$$

$$s'(t) = 6\cdot\left(\frac{2}{4\sqrt{4+64}}\right) = \frac{12}{4\sqrt{68}} = \frac{12}{8\sqrt{17}} = \frac{3}{2\sqrt{17}}$$

13 MVT revisited

Consequences of the MVT

$$f(b) = f(a) \Rightarrow \exists c \in (a,\ b),\ f'(c) = 0$$

Constant function

If $f'(x) = 0 \forall x$ in an interval, $\Rightarrow f(x)$ is a constant function.

If $f'(x) \neq 0 \forall x$ in an interval, then $f(x)$ is 1-1.

If $f'(x) > 0 \forall x$ in an interval, then $f(x)$ is increasing.

If $f'(x) < 0 \ldots\ f(x)$ is decreasing.

Local max: A point c is a local max if $\exists$ interval $(c-l,\ c+l)$ such that for all x in interval, $f(x) \leq f(c)$ when $f(x)$ is defined.

Local extrema: in the interior of interval $[a,\ b]$ then c is a critical point $f'(c) = 0$ or $f'(c)$ DNE.

$$
\begin{aligned}
f(x) &= \arcsin\left(\frac{1-x}{1+x}\right) + 2\arctan\left(\sqrt{x}\right) \\
&= \frac{1}{\sqrt{1-\left(\frac{1-x}{1+x}\right)^2}} \cdot \left(-\frac{2}{(1+x)^2}\right) \\
&= -\frac{2}{\sqrt{4x}(1+x)} + \frac{1}{(1+x)\sqrt{x}} = -\frac{1}{\sqrt{x}(1+x)} + \frac{1}{(1+x)\sqrt{x}} = 0
\end{aligned}
$$

Note:

$$f(0) = c = \arcsin\left(\frac{1}{1}\right) + 2\arctan(0) = \frac{\pi}{2} + 0$$

(Test a value somewhere in the interval.)

Deducing domain of the function:

$$\arcsin\left(\frac{1-x}{1+x}\right) + 2\arctan\left(\sqrt{x}\right)$$

True

$$
\begin{aligned}
1 - x &\leq 1 + x \\
1 &\leq 1 + 2x \\
0 &\leq 2x
\end{aligned}
$$

Another question

Find all functions defined on $\mathbb{R}$ that has $f'(x) = 3$ everywhere.

That is $f(x) = 3x + C$, where C is a constant. We can mention it in another way: there is an infinite number of functions that have $f'(x) = 3$ everywhere in the form $f(x) = 3x + C$.

Suppose $g(x) \neq 3x + C$ and $g'(x) = 3$ everywhere. How do we show that? The trick here is to consider:

$$f'(x) - g'(x) = 0, \text{ as } f'(3) - g'(3) = 0$$

This means

$$f - g = D, \text{ a constant}$$

Is a constant function.

However, if you solve for g: $f - g = D \Rightarrow g = f - D \Rightarrow g = 3x + C - D$

But $C - D$ is just another constant.

13.1 At most some zeroes

Suppose $f'(x)$ has only k zeroes on $[a, b]$. Then $f(x)$ can have at most $k+1$ zeroes. This can be proved by induction.

14 Maximum and minimum (second derivative test)

What to do at each critical point:

- $f''(a) > 0$, a is a local minimum.
- $f''(a) < 0$, then local max
- $f''(a) = 0$ then it is unclear
- We can also use the first derivative test for the local max or min.

Compare all values of $f(a)$ for all critical points and endpoints of an interval. The largest value is a global maximum. The smallest value is an absolute (global) minimum.

14.1 Absolute max and min

For $\lim_{x\to\infty} f(x) = +\infty = \lim_{x\to-\infty} f(x)$ then there exists an absolute minimum.

For $\lim_{x\to\infty} f(x) = -\infty = \lim_{x\to-\infty} f(x)$ then there exists an absolute max.

For $\lim_{x\to\infty} f(x) = \pm\infty,\ \lim_{x\to-\infty} f(x) = \mp\infty$ then no absolute max or min.

14.1.1 Optimization

1. Identify quantity to be maximized/minimized. Set as y.
2. Identify changeable quantity. Set as x.
3. Write an equation that relates all variables and draw a diagram if needed.
4. Calculate $\frac{dy}{dx}$ and set to 0.
5. Solve for x and the quantities needed.
6. Context helps you determine which x is correct.
7. -

15 Concavity and Point of Inflection

If $f'(x) > 0$ function increasing, negative then the function is decreasing.

For $f''(x)$, positive $\Rightarrow$ concave up; negative then concave down.

A point of inflexion is when $f''(x) = 0$ and the sign changes. If $f''(x)$ changes sign at x but $f''(x)$ DNE it can still be a point of inflexion. Test points around the point of inflexion.

16 Graph sketching

Draw a decent enough picture so calculus can be done on it. We need the following:

- Domain
- Vertical asymptotes: $\lim_{x \to a} f(x) = \pm\infty$
- Horizontal asymptotes and slant asymptotes (when the power on top is greater than the power on the bottom). Obtain slant asymptotes using long division.
- Points of intersection with axis (x and y intercept)
- Critical points. Set derivative equal to 0.
- Points of inflexion, but don't do that.
- Others.

16.1 Examples

$$\frac{x^3}{x^2+1}$$

No VA, as $x^2 \neq -1$

May have slant asymptotes. $\lim_{x\to\infty} f = +\infty$, $\lim_{x\to-\infty} f = -\infty$

Occurs when power on top of the rational function is higher than the bottom.

The remainder means if I were to remove the remainder from the beginning, then the division would go through. This means if you subtract the remainder from the numerator:

$$\frac{x^3-(-x)}{x^2+1} = x$$

You can have this:

$$\frac{x^3}{x^2+1} = x + \left(\frac{-x}{x^2+1}\right)$$

The function highlighted in orange will always have a lower degree on the top.

If we now take $\lim_{x\to\pm\infty}\left(-\frac{x}{x^2+1}\right)$, we will get 0, as the degree is larger on the bottom. This means, in the long run, you will end with x. This means

$$\lim_{x\to\pm\infty}\left(\frac{x^3}{x^2+1}\right) \approx x$$

This function, $y = x$ is power 1 is because the numerator x^3 and the denominator x^2+1 differ by one power. When the top power is 1 more than the bottom, the quotient will be a linear function. The function essentially becomes linear as x approaches infinity. In general, if you have $\frac{x^4}{x^2+1}$, you will get a quadratic as the quotient, so instead of having a straight line to what the function will conform to, you will get a parabola instead.

Point of intersect: $(0,\ 0)$ (just plug it in)

Take the derivative and find critical points:

$$
\begin{aligned}
f' = \frac{3x^2\left(x^2+1\right) - 2x \cdot x^3}{\left(x^2+1\right)^2} = 0 \\
3x^4 + 3x^2 - 2x^4 = 0 \\
x^2\left(x^2+3\right) = 0 \\
x^2 = 0,\ x = 0
\end{aligned}
$$

At $x = 0$, this is the only critical point.

Order of zeroes: The only zero is at $x = 0$, which is what we found. If we take a look at the function, $f = \frac{x^3}{x^2+1}$. This means that this is an order 3, as the power on x is 3. You only need to worry about the numerator, is that the idea is, when we are interested in order of 0, we only care how the function behaves near the zero, which turns out to be $x = 0$. When x is near 0, the denominator $x^2 + 1$ basically becomes a constant, approaching 1. Near $x = 0$, the function is essentially $\frac{x^3}{1}$. That is the reason why x^3 governs the behavior of the function.

16.2 Order of zeroes

Aside:

$$
g = \frac{(x-1)^2(x+2)^3}{x-2}
$$

Zero: $x = 1,\ x = -2$. Order of $x = 1$ is 2; order of $x = -2$ is 3. If the order is even, the graph will bounce off. If the order is odd, the function will make a cubic-like wiggle across the x-axis.

17 L'Hopital's rule

L'Hopital's rule makes no sense of why it works until you get to Taylor series, but not every function can be written as a Taylor series.

When

$$
\frac{\lim f(x)}{\lim g(x)} = \frac{0}{0} \text{ or} \pm \frac{\infty}{\infty}
$$

Then

$$\lim\left(\frac{f(x)}{g(x)}\right) = \lim\left(\frac{f'(x)}{g'(x)}\right)$$

17.1 Examples

$$\lim_{x\to 0}\frac{x^3}{x^2+x} = \frac{0}{0}$$

$$= \lim_{x\to 0}\frac{3x^2}{2x+1} = \frac{0}{1} = 0$$

This is done by taking the derivative of the numerator and the denominator.

17.2 Flip and multiply

Case of $0\cdot\infty$

$$\lim\left(f(x)\cdot g(x)\right) = 0\cdot\infty$$

Turn it into a fraction by forcing either f or g onto the denominator. Choose either, better the simpler one. Whatever gets flipped turns from $\pm\infty$ to 0 or the other way around.

$$\lim\left(\frac{f(x)}{\frac{1}{g(x)}}\right) = \frac{0}{0} \text{ or } \pm\frac{\infty}{\infty}$$
$$\lim\left(\frac{g(x)}{\frac{1}{f(x)}}\right) = \frac{0}{0} \text{ or } \pm\frac{\infty}{\infty}$$

Then proceed with L'Hopital's rule.

17.3 Non-indeterminate forms

$$\frac{0}{\infty} = 0$$

$$\frac{\infty}{0} = \infty$$

17.4 How would we have figured that out?

Suppose $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$ (an infinite polynomial) and $g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \dots$ and have

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{a_0 + a_1x + a_2x^2 + a_3x^3 + \dots}{b_0 + b_1x + b_2x^2 + b_3x^3 + \dots}$$

Since every term at the back contains an x, most of the time, we should end up with

$$\frac{a_0}{b_0}$$

L'Hopital's rule says that this answer is true most of the time, but if **both** $b_0 = 0$ and $a_0 = 0$, then we end up with an indeterminate form.

When you take L'Hopital's rule, only these have an impact:

$$\frac{a_0 + a_1x + a_2x^2 + a_3x^3 + \dots}{b_0 + b_1x + b_2x^2 + b_3x^3 + \dots}$$

With a_0 and b_1 disappearing, we should actually get

$$\frac{a_1 + a_2x + a_3x^2 + \dots}{b_1 + b_2x + b_3x^2 + \dots} = \frac{a_1}{b_1}$$

Repeat process if a_1 and b_1 are zero.

When you take the derivative of $f(x)$ and $g(x)$, you remove a_0 and b_0. Meaning:

$$\frac{f'(x)}{g'(x)} = \frac{a_1 + a_2x + a_3x^2 + \dots}{b_1 + b_2x + b_3x^2 + \dots}$$

Note that this fraction is completely different from $\frac{f}{g}$. This gets rid of a_0 and b_0.

Now we have $\frac{a_1}{b_1}$. This is a way to extract it.

Analytic functions can be written as polynomials, and nearly all functions aren't. Yet, the functions we work with the most often are analytic functions.

17.5 Exponent cases

When the base of an exponent **includes** x, then use the formula:

$$f(x)^{g(x)} = \left(e^{\ln(f(x))}\right)^{g(x)} = e^{g(x)\cdot\ln(f(x))}$$

$$\begin{aligned} f(x) &= e^{\ln(f(x))} \\ f(x)^{g(x)} &= \left(e^{\ln(f(x))}\right)^{g(x)} \\ x^x &= e^{\ln(x^x)} = e^{x\ln(x)} \end{aligned}$$

For example:

$$\lim_{x\to 0^+} (\tan(2x))^x = \lim_{x\to 0^+} e^{x\cdot\ln(\tan(2x))}$$

Using the property of limits, we can move the limit onto the exponent.

$$e^{\lim_{x\to 0^+} x\cdot\ln(\tan(2x))}$$

So we only need to compute:

Simplify stuff the moment you take the derivative.

You may have to take L'Hopital's again.

18 The idea of integration

Assume $f(x) > 0$

We want the area under the curve of $f(x)$ on $[a,\ b]$. Split the area underneath into many rectangles and add up their total area.

Partition: the way we pick the bases of the rectangles. There's nothing stopping you from splitting the partitions into equal length intervals, but you should. For example, Partition P: some number of rectangles between a and b. You can refine a partition by adding other points. The result would be called a **refinement.**

Choosing the heights of the rectangles: There are two methods we can do (WIDTH OF RECTANGLE IS CONTROLLED HERE):

- Upper sum: Highest point; maximum value of f (supremum) of f in the rectangle; called $U_f(P)$; area over-estimator $\geq$

- Lower sum: Lowest point; takes the infimum of f; called $L_f(P)$; area under-estimator

While these two are great for theoretical purposes they aren't computationally feasible. Here are the methods that are easy for computation:

- Right-point method

- Left-point method

There are drawbacks of these two methods (often due to inaccuracies), but the height is already determined for you the moment you choose where your rectangles go. For example:

$$h_1 = f(t_1),\ h_2 = f(t_2),\ h_3 = f(t_3)$$

18.1 Formulas for these methods

Given a partition of $[a,\ b]$ where $t_0 = a$ and $t_n = b$

$$P = t_0 < t_1 < \ldots < t_n$$

We may compute the total area explicitly. In the ith rectangle with base $[t_{i-1},\ t_i]$, the right most endpoint is t_i and left is t_{i-1}.

Total area by the right-point method:

$$A = \sum_{i=1}^{n} b_i \cdot h_i = \sum_{i=1}^{n} (t_i - t_{i-1}) \cdot f(t_i)$$

For the left-point method:

$$A = \sum_{i=1}^{i} b_i \cdot h_i = \sum_{i=1}^{n} (t_i - t_{i-1}) \cdot f(t_{i-1})$$

The right-point method is slightly easier.

18.2 Even splits

If we split an interval $[a,\ b]$ into n-equal sized pieces:

Each rectangle would have base length $b_i = \frac{b-a}{n}$

This means to determine where a split point t is, where $t_0 = a$:

$$t_1 = a + \frac{b-a}{n}$$
$$t_i = a + i\frac{(b-a)}{n}$$

Using equal-length intervals, using rectangle width and height (depends on right point or left point):

$$A_R = \sum_{i=1}^{n} b_i \cdot h_i = \sum_{i=1}^{n} \frac{b-a}{n} \cdot f\left(a + i\frac{(b-a)}{n}\right)$$
$$A_L = \sum_{i=1}^{n} b_i \cdot h_i = \sum_{i=1}^{n} \frac{b-a}{n} \cdot f\left(a + (i-1)\frac{(b-a)}{n}\right)$$

18.3 True area

As we use more rectangles, the better approximation to the **true area** under the curve we get of $f(x)$.

An informal definition of the true area (named integral) under the curve of $f(x)$ is approximated using more and more angles.

$$A_T = \int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} b_i \cdot h_i$$

(Remember how b_i and h_i are defined, but h_i represents some way we choose the height of the rectangles. As the number of rectangles reaches infinity, we are supposed to get the true answer every time.)

When n becomes large, b_i becomes verry narrow so it is centered around a point x, so we can change that notation from $b_i \to dx$. $f(x)$ comes from the fact that $h_i \to f(x)$ when b_i, the box length becomes very narrow.

18.4 Positive and negative area

About net area:

When $f(x) > 0$, we count the area above the x-axis and below $f(x)$ to be positive area.

When $f(x) < 0$, we count the area below the x-axis and below $f(x)$ to be negative area.

The net area is the sum of both positive area and negative area.

18.5 Integrals without integral rules

Compute $\int_0^1 2xdx$. Area is $A = \frac{1}{2} \cdot 1 \cdot 2 = 1$ (triangle).

Using integrals, we can do this instead, for $f(x) = 2x,\ [a = \ 0, b = 1]$:

$$
\begin{aligned}
A &= \sum_{i=1}^{n} b_i \cdot h_i = \sum_{i=1}^{n} \frac{b-a}{n} \cdot f\left(a + i\frac{(b-a)}{n}\right) \\
&= \sum_{i=1}^{n} \frac{1}{n} \cdot f\left(i\frac{1}{n}\right) = \sum_{i=1}^{n} \frac{1}{n} \cdot 2\left(i\frac{1}{n}\right) \\
&= 2\sum_{i=1}^{n} \frac{i}{n^2} = \frac{2}{n^2}\sum_{i=1}^{n} i = \frac{2}{n^2} \cdot \frac{n(n+1)}{2} = \frac{n+1}{n}
\end{aligned}
$$

Anything that does not involve i can be taken out of summations, so you can use this fact to your advantage.

Taking the limit of n to infinity:

$$\lim_{n\to\infty} \frac{n+1}{n} = \frac{n}{n} + \frac{1}{n} = 1$$

19 1D Integration

The upper and lower sums are the methods to theoretically find the integral. The true area is always sandwiched between the upper sum and the lower sum.

For any partitions P_1 and P_2:

$$L_f(P_1) \leq \text{True Area} \leq U_f(P_2)$$

This holds even if P_1 and P_2 are different.

By picking more rectangles, we can make the upper sum and lower sum closer and closer together.

If P_2 is a **refinement** of P_1 (P_2 has all the rectangles of P_1 and some more), then P_2 should provide a better estimate:

$$L_f(P_1) \leq L_f(P_2) \leq \text{True Area} \leq U_f(P_2) \leq U_f(P_1)$$

Here, there are 4 rectangles instead of the 3. The area of the green rectangles is larger than the area of the black rectangles, and the area of the red rectangles is less than the area of the blue rectangles.

The more rectangles, the closer to the true area we are, but we haven't defined true area yet.

19.1 Defining true area without defining true area

Approximate the true area up to some error ε we are comfortable with, such that $U_f(P) - L_f(P) < \varepsilon$. Pick an ε as small as possible.

Definition 19.1. A function is **integrable** if the upper sum and lower sum can be made as close as you want (by picking more rectangles).

$$\forall \varepsilon > 0, \ \exists P \text{ such that } U_f(P) - L_f(P) < \varepsilon$$

Note the same partition P is used here in the upper and lower sum. The theory here is that the error can be made as small as possible (you can always make your error smaller provided that it is positive), but there will always be some error.

The **true area** should be the minimum of all possible upper sums whilst also being equal to the maximum of all possible lower sums. There are however infinitely as many upper and lower sums, so a maximum and minimum wouldn't exist. We could believe that an intuitive max/min exists for them:

- The supremum of a set S is intuitively understood as the maximum of the set S.
- The infimum of a set S is intuitively understood as the minimum of the set S.
- This is from point 13 of the axioms of real numbers.

Definition 19.2. True area:

$$
\begin{aligned}
\text{True Area} &= \int_a^b f \\
&= \text{infimum of all possible } \{U_f(P)\} \\
&= \bar{I}_a^b \\
&= \text{supremum of all possible } \{L_f(P)\} \\
&= \underline{I}_a^b
\end{aligned}
$$

I'd rather use definition 1.

$U_f(P) - L_f(P)$ is liked the shaded area below:

Almost all functions are integrable. Even though definition 1 can be used, it doesn't define true area.

19.2 Proving that a function is integrable

Show whether the following functions are integrable.

$$f(x) = 2,\ x \in [0,\ 2]$$

Prove $\forall \varepsilon > 0,\ \exists P,\ U_f(P) - L_f(P) < \varepsilon$.

Proof. Let $\varepsilon > 0$. Pick P, which is the rectangle from 0 to 2. There is only one rectangle, the base being 2 and the height being 2 by looking at the function, so $U_f(P) = 2_{\text{height}} \cdot 2_{\text{base}} = 4$.

For $L_f(P)$, calculate it by $L_f(P) = 2_{\text{height}} \cdot 2_{\text{base}} = 4$.

Then:

$$U_f(P) - L_f(P) = 2 - 2 = 0 < \varepsilon$$

Here, the error is always less than epsilon. ■

We can try something more complicated:

$$f(x) = \begin{cases} 2 & x \neq 1 \\ 4 & x = 1 \end{cases},\ x \in [0,\ 2]$$

This is a jump discontinuity.

Proof. Let $\varepsilon > 0$. Pick partition $P : 0 \to 1 - \frac{l}{2} \to 1 + \frac{l}{2} \to 2$ (the partitions do not need to be equal length).

Effectively, you are creating three rectangles. This means:

$$\begin{aligned} U_f(P) = & \overbrace{b_1 \cdot 2}^{\text{left rectangle}} \\ & + \overbrace{l \cdot 4}^{\text{middle rectangle}} + \overbrace{b_3 \cdot 2}^{\text{right rectangle}} \\ L_f(P) = & \, b_1 \cdot 2 + l \cdot 2 + b_2 \cdot 2 \end{aligned}$$

Now:

$$\begin{aligned} U_f(P) - L_f(P) &< \varepsilon \\ (2b_1 + 4l + 2b_3) - (2b_1 + 2l + 2b_2) &< \varepsilon \end{aligned}$$

Choose $l = \frac{\varepsilon}{4}$. You can choose whatever you want, but you should be choosing l such that the end result does not look ugly.

$$\begin{aligned} 2b_1 + \varepsilon + 2b_3 - 2b_1 - \frac{\varepsilon}{2} - 2b_2 &< \varepsilon \\ \frac{\varepsilon}{2} &< \varepsilon \end{aligned}$$

■

What does this mean? Integrals are more versatile than the derivative. It can handle discontinuities, which can be done by picking small rectangles around discontinuities, which will control the discontinuity for you.

Another example:

$$f(x) = \begin{cases} 2 & x \neq 1 \\ 4 & x = 1 \end{cases}, \; x \in [0,\ 2]$$

Proof. Let $\varepsilon > 0$, and pick $P : 0 \to 1 - \frac{l}{2} \to 1 + \frac{l}{2} \to 2 - \frac{l}{2} \to 2$. This means

$$
\begin{aligned}
U_f(P) &= 2b_1 + 4l + 4b_3 + 6 \cdot \frac{l}{2} \\
L_f(P) &= 2b_1 + 2l + 4b_3 + 4 \cdot \frac{l}{2} \\
U_f(P) - L_f(P) &< \varepsilon \\
2b_1 + 4l + 4b_3 + 6 \cdot \frac{l}{2} - 2b_1 - 2l - 4b_3 - 4 \cdot \frac{l}{2} &< \varepsilon \\
(4-2)l + \frac{(6-4)l}{2} &= 3l < \varepsilon
\end{aligned}
$$

Pick $l = \frac{\varepsilon}{6}$

$$
\begin{aligned}
3 \cdot \frac{\varepsilon}{6} &< \varepsilon \\
\frac{\varepsilon}{2} &< \varepsilon
\end{aligned}
$$

■

This method handles all discontinuities.

For a function that isn't flat: $f(x) = x,\ x \in [0,\ 2]$

$$
\begin{aligned}
b_1 h_1 &= b_2 h_2 \\
U_f(P) &= 1 \cdot 1 + 1 \cdot 2 \\
L_f(P) &= 1 \cdot 2 + 1 \cdot 1
\end{aligned}
$$

Notice that f is increasing so if you pick the same base for each rectangle, then the adjacent rectangle's area of the upper sum and the lower sum would cancel.

Proof. Let $\varepsilon > 0$. Let P be n-equal spaced intervals, meaning $b = \frac{2}{n}$. Pick n such that $\frac{4}{n} < \varepsilon$.

Then:

$$
\begin{aligned}
U_f(P) &= b \cdot f(t_1) + b \cdot f(t_2) + \cdots + b \cdot f(t_n) \\
L_f(P) &= b \cdot f(t_0) + b \cdot f(t_1) + \cdots + b \cdot f(t_{n-1})
\end{aligned}
$$

$$U_f(P) - L_f(P) = b \cdot f(t_n) - b \cdot f(t_0) = b(f(t_n) - f(t_0)) = \frac{2}{n} \cdot 2 = \frac{4}{n} < \varepsilon$$

■

19.2.1 Finitely many discontinuities

Suppose $g(x)$ is integrable and bounded (by M) on $[a,\ b]$. Let $f(x) = g(x)$ except at finitely many points $\{c_1,\ \ldots,\ c_n\}$ where $f(c_i) \neq g(c_i)$.

The idea is to have small rectangles around each of the discontinuities. Make each of the errors $\frac{\varepsilon}{2n}$ and the total error is $n \cdot \frac{\varepsilon}{2n} = \frac{\varepsilon}{2} < \varepsilon$.

Key word: **finitely** as many discontinuities.

Basically, if you have a continuous function with finitely many holes.

19.2.2 Infinitely as many discontinuities

This will cause you to set something like $\frac{\varepsilon}{\infty} = 0$, which can't happen, as the error can't be zero. The techniques of splitting up partitions and creating rectangles on discontinuities wouldn't work.

Example, where $\mathbb{Q}$ (standing for quotient) means rational numbers, which can be represented as $\frac{p}{q},\ p,\ q \in \mathbb{Z}$:

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

There are a lot of rational numbers (more like infinitely as many) scattered between $[0,\ 1]$. It is an infinitely scattering of points in $[0,\ 1]$. It is not a solid line; irrational numbers like $\frac{\pi}{4}$ exist. This means $f\left(\frac{\pi}{4}\right) = 0$. But also $\frac{\pi}{8}$ is irrational, so $f\left(\frac{\pi}{8}\right) = 0$, and so is $\frac{\pi}{16},\ \frac{\pi}{5}$, and so on, and don't forget $\frac{e}{5},\ \sqrt{2}$, and so on. There is an infinite scattering of points at $y = 0$, as there are a lot of irrational numbers scattered at $y = 0$.

At a visual standpoint, there are points at $y = 1$ infinitely scattered and points at $y = 0$ infinitely scattered.

The trick of controlling errors can only work for finite discontinuities. If there are infinitely as many discontinuities, it feels like the function is not continuous everywhere, or even anywhere. At every single point, there's a discontinuity.

It does not mean that the function isn't integrable, but we run into this problem (note that $a < b$):

$$\forall[a,\ b],\ \exists x_1 \in \mathbb{Q}, x_1 \in [a,\ b]$$
$$\forall[a,\ b],\ \exists x_1 \notin \mathbb{Q}, x_2 \in [a,\ b]$$

We can see where the problem comes.

The upper sum for any interval (that isn't squished to a point) will always be 1, and 0 for the lower sum. There will always be an error of 1 overall, and $1 \not< \varepsilon$. Is it possible to make the error less than ε? Regardless of the partition you choose, the error will stay at 1.

What does it mean to be integrable? $\forall \varepsilon > 0,\ \exists P,\ U_f(P) - L_f(P) < \varepsilon$. I don't think this can be satisfied.

What is the negation of this definition?

$$\exists \varepsilon > 0,\ \forall P,\ U_f(P) - L_f(P) \geq \varepsilon$$

This is our proof that $f(x)$ here isn't integrable.

Proof. Pick $\varepsilon = \frac{1}{2}$. Let P be arbitrary. Regardless of how we choose our partition P, $U_f(P) - L_f(P)$ will always be greater or equal than ε:

$$U_f(P) = \sum b_i \cdot h_i = \sum b_i \cdot 1 = 1$$
$$L_f(P) = \sum b_i \cdot h_i = \sum b_i \cdot 0 = 0$$

This always happens. What can we conclude?

$$U_f(P) - L_f(P) = 1$$

...regardless of what P we choose. The error here is always going to be greater than 1, greater than or equal to the ε we set, $\frac{1}{2}$. We are unable to minimize the error to be under ε. ■

Ever so slightly harder example:

$$f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \text{ written as an irreducible fraction} \\ 0 & x \notin \mathbb{Q} \end{cases} \quad x \in [0,\ 1]$$

What does it mean? $f\left(\frac{2}{3}\right) = \frac{1}{3},\ f\left(\frac{4}{5}\right) = \frac{1}{5}, f\left(\frac{6}{10}\right) = \frac{1}{5}$

19.2.3 That for an increasing function

Suppose f is increasing on $[a,\ b]$. Prove f is integrable. Do not assume f is continuous.

Definition 19.3. A function which is increasing or decreasing would be called monotonic.

19.2.4 Debunking Question 7

Suppose $g(x)$ is integrable and bounded (by M) on $[a,\ b]$.

$$f(x) = \begin{cases} g(x) & \text{if } x \neq c \\ f(c) & \text{if } x = c \end{cases}, \text{ where } f(c) \neq g(c)$$

Proof. Let $\varepsilon > 0$. Choose $l = \frac{\varepsilon}{2(N-n)}$ (technically, we were supposed to define N and n earlier).

Since g is integrable, $\forall \varepsilon_2 > 0,\ \exists Q,$ such that $U_g(Q) - L_g(Q) < \varepsilon_2$ (there exists partition Q such that $U_g(Q) - L_g(Q)$.

Because g is integrable, we can remove parts of the partition in parts where we don't need it, such as the l-length region around c. This means the partition Q is only in the blue region.

Choose $\varepsilon_2 = \frac{\varepsilon}{2}$. This means the following:

- The error in the blue region must be at most $\frac{\varepsilon}{4}$.
- The error in the green region must be at most $\frac{\varepsilon}{4}$.
- The total error we will be making will be $\frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$.

$$\exists Q,\ U_g(Q) - L_g(Q) < \varepsilon_2$$

Choose $P = Q \cup \left\{c - \frac{l}{2},\ c + \frac{l}{2}\right\}$ (Q union the green rectangle). There is some way to pick rectangles in the partition of Q such that the total error in the blue region will be less than $\frac{\varepsilon}{2}$.

Focusing on the error in the green region (and we must see how much error at most there can be):

For $x \in \left[c - \frac{l}{2},\ c + \frac{l}{2}\right]$, what can be the error at most? It should be less than ε. Here, f can be at most $f(c)$, but g could be even larger, so

$$
\begin{aligned}
x &\le \max(f(x), M) \\
\min(f(x),\ -M) &\le x
\end{aligned}
$$

This is based on the bounds g, as the function g may not go higher than M or lower than $-M$. Also, here are our bounds for x if it is in $\left[c-\frac{l}{2},\ c+\frac{l}{2}\right]$.

$$
\min(f(c),\ -M) \le f(x) \le \max(f(c), M)
$$

These two values are simply two numbers, so let $n = \min(f(c),\ -M)$ and $N = \max(f(c),\ M)$.

So

$$
\begin{aligned}
&U_f(P) - L_f(P) \\
&= \overbrace{U_g(Q) - L_g(Q)}^{\text{blue region error}} \\
&+ \quad \overbrace{l \cdot N - l \cdot n}^{\text{green region error}} \\
&= \frac{\varepsilon}{4} + \qquad \overbrace{l(N-n)}^{\text{which we pick to be } \frac{\varepsilon}{4}} \\
&= \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2} < \varepsilon
\end{aligned}
$$

■

I don't get it. Our goal is to prove that f is integrable. This means we must prove that $\forall \varepsilon > 0,\ \exists P,\ U_f(P) - L_f(P) < \varepsilon$, knowing that we have g and the fact that it is integrable. Here's how we did it:

1. Name g's partition Q, which is actually from a to $c - \frac{l}{2}$, and then there is a region where we are not in partition Q, and then partition Q resumes at $c + \frac{l}{2}$ all the way to b. Yes, partitions can be split up.
2. Because we know g is integrable, we know the error it must not exceed: ε_2, which is universally quantified. We can let it be anything, so use that opportunity to assign it to something like $\frac{\varepsilon}{9999}$.

3. Choose the partition of f, which is the function we wish to prove integrable in the interval $[a,\ b]$. Pick it to be partition $Q\cup$ the gap in the middle, which is $\left\{c-\frac{l}{2},\ c+\frac{l}{2}\right\}$ – just one rectangle, and we have made gaps like this before because we want l to be the length of that rectangle.

4. This still concerns the partition of f we've just chosen. We know that $\forall x \in$ the gap $\left\{c-\frac{l}{2},\ c+\frac{l}{2}\right\}$, x can't:

   a. Go higher than $f(c)$ or M. We only worry about $f(c)$ when $x=c$, so M covers all the other cases: the maximum possible value of g.

   b. Go lower than $f(c)$ or $-M$. We only worry about $f(c)$ when $x=c$, so $-M$ covers all the other cases: the minimum possible value of g.

   c. We end up with this expression: $\min\left(f(c),\ -M\right) \leq f(x) \leq \max\left(f(c),\ M\right)$ We give aliases: $n=\min\left(f(c),\ -M\right)$ and $N=\max\left(f(c),\ M\right)$ although we totally didn't have to. The reason why we chose M and $-M$ instead of some $g(x)$ thing is because M and $-M$ are constants and will always be an overestimate/underestimate of where g always is.

5. We conclude that the error caused by that rectangle is $\leq (l\cdot N - l\cdot n)$. $l\cdot N$ is the upper height bound of that rectangle, and $l\cdot n$ is the lower height bound of the rectangle. A reminder that l is the length of the rectangle and remember that we overestimated the upper bound and underestimated the lower bound because instead of choosing $g(x)$, we just looked at the highest/lowest possible value of g.

6. We return to the partition P we've chosen because before after acquiring more information about the rectangle in the middle in the region $\left\{c-\frac{l}{2},\ c+\frac{l}{2}\right\}$. Because we overestimated the error bound, we can safely say that $U_f(P)-L_f(P) \leq U_g(Q)-L_g(Q)+(l\cdot N - l\cdot\ n)$ and can finally choose $U_g(Q)-L_g(Q) < \frac{\varepsilon}{9999999}$ and choose l such that $(l\cdot N - l\cdot n) < \frac{\varepsilon}{9999999}$ in the front of the proof and finally prove that $U_f(P)-L_f(P) < \varepsilon$.

19.2.5 Debunking Question 10: The integrable discontinuous function that looks like a triangle

$$f(x)=\begin{cases}\frac{1}{q} & x=\frac{p}{q} \text{ written as an irreducible fraction} \\ 0 & x \notin \mathbb{Q}\end{cases} \qquad x\in[0,\ 1]$$

Let $\varepsilon > 0$. There are finitely many x with $f(x) > \frac{\varepsilon}{4}$. There is a list of possible x for which $f(x) > \frac{\varepsilon}{4}$, and yes, for some epsilon values, that can be zero.

$$x:\{c_1,\ c_2,\ \ldots,\ c_n\}=\left\{\frac{1}{2},\frac{1}{3},\frac{2}{3},\ \ldots\right\}$$

Well, excluding duplicates.

Construct rectangles around each c_i length $l = \frac{\varepsilon}{4n}$ (we chose that later).

$$\begin{aligned}
&U_f(P) - L_f(P) \\
&= \sum_{\substack{\text{rectangles} \\ \text{including } c_i}} b_i h_i \\
&+ \sum_{\substack{\text{rectangles not} \\ \text{including } c_i}} b_i h_i
\end{aligned}$$

In the first (left) sum, there are n rectangles with the base being $n \cdot l$ with the height values all being less than $\frac{1}{2}$ (which implies they are all less than 1). There are n rectangles at most, where each base is chosen to have length l, and the height of each rectangle which includes one of these c_i points:

$$\begin{aligned}
&\sum_{\substack{\text{rectangles} \\ \text{including } c_i}} b_i h_i \\
&+ \sum_{\substack{\text{rectangles not} \\ \text{including } c_i}} b_i h_i \\
&< n_{\text{num rectangles}} \cdot l_{\text{length of each}} \cdot 1_{\text{max ht of each}} \\
&+ \sum_{\substack{\text{rectangles not} \\ \text{including } c_i}} b_i h_i
\end{aligned}$$

What about the sum on the right? These rectangles do not include any of the $c_{1\ldots i \ldots n}$, so the height of these rectangles is at most $\frac{\varepsilon}{4}$. So, and we will just be assuming the max height and that it does not cover a length over 1:

$$< n \cdot l + \sum_{\substack{\text{rectangles not} \\ \text{including } c_i}} b_i \left(\frac{\varepsilon}{4}\right)_{\text{max ht of each rect}}$$

$$< n \cdot l + (1)_{\text{dist from 0 to 1}} \frac{\varepsilon}{4} = \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$$
$$< \varepsilon$$

19.3 True area revisited

The true area is the minimum of all possible upper sum values and also the maximum of all possible lower sum values. However, the case where $S = (0,\ 1)$ (the supremum of all possible $L_f(P)$), the supremum will still exist.

Say our set is $S = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$. This set is infinite. To plot this set on the number line, it would be an infinite scattering of points close to 0, but never at 0. Even though it never touches 0, we can still define the infimum to exist, which is at 0. We decide that they exist; we don't prove it.

A number u is the supremum of a set if it is the last upper bound of S. It is the smallest number which can satisfy: $\forall s,\ s \leq u$.

What is the smallest number which can satisfy this? This is called the least upper bound, which for S, is at 1.

19.4 The epsilon characterization of supremum and infimum

For all $\varepsilon > 0$, there exists $s \in S$ such that $u - \varepsilon < s \leq u$. As u is the maximum of S, moving down even a tiny amount will bump into an element of S.

For all $\varepsilon > 0$, there exists $s \in S$ such that $l \leq s \leq l + \varepsilon$. As l is the minimum of S, moving up even a tiny amount will bump into an element of S.

Let $\int_a^b f(x)dx < \int_a^b g(x)dx$. Let P be a partition of $[a,\ b]$. Are the following statements true or false?

Q1: $L_f(P) < U_g(P)$.

Proof. $L_f(P) \leq \int_a^b f(x)dx < \int_a^b g(x)dx \leq U_g(P)$ ∎

One particular value of $L_f(P)$ would be less than or equal to the maximum value of all values of $L_f(P)$.

Q2: $L_f(P) < L_g(P)$.

Note the following:

We know that $L_g(P) \leq \int g$, but we can never say that $L_g(P) \geq \int f$. For example, the approximation of $L_g(P)$ can be made very bad. For example, if we pick one rectangle that is way below the maximum value of $L_g(P)$, it will be below $L_f(P)$. Q2 is false.

Q3: $L_f(P) < \int_a^b f(x)dx$

The problem is how we defined the supremum of lower sums: less than or equal to the true value. We can't say it is strictly less. Q3 is false. If $f(x)$ is flat, sup $L_f(P) = \int_a^b f(x)dx$.

Q4: $U_f(P) < U_g(P)$. Same case as Q2. $U_f(P)$ may be very terrible at approximating the area of f and end up higher than $U_g(P)$.

Q5: $\int_a^b f(x)dx < U_g(P)$. Because $\int f < \int g \leq U_g(P)$, this statement is true.

Q6: $U_f(P) < \int_a^b g(x)dx$ – this is false, as there are no upper bounds for $U_f(P)$.

19.4.1 Debunking question 61

TBA

19.4.2 Debunking question 63

Suppose f is **continuous**, $f \geq 0,\ \int_a^b f = 0$ (nonnegative everywhere).

Prove $\forall x \in [a,\ b],\ \ f(x) = 0$.

Proof. Suppose $f(c) > 0$. Since f is continuous, $\exists x \in \left(c - \frac{l}{2},\ c + \frac{l}{2}\right),\ f(x) > \frac{f(c)}{2}$. It at least has to decrease slowly towards 0.

$$L_f(P) = \sum b_i h_i \geq l \cdot \frac{f(c)}{2} > 0$$
$$L_f(P) \leq \sup\left\{L_f(P)\right\} = \int f = 0$$

This is a contradiction. ∎

Is this still true if we do not assume f to be continuous? Not.

20 The Fundamental Theorem of Calculus

Let $f(x)$ be continuous (which implies it is integrable). If you have constant a (which can be anything): Consider the function:

$$F(x) = \int_a^x f(t)dt$$

It is equal to the area under f between a and x. The input going into F is a number, and the output is an area. dt indicates that the integral is with respect to t.

Should that be the case, $F'(x) = f(x)$.

You can combine this with the chain rule:

$$F(x) = \int_a^{g(x)} f(t)dt,\ F'(x) = f(g(x)) \cdot g'(x)$$

Example:

$$F(x)=\int_5^x \frac{1}{1+\sin^2(t)}dt=\frac{1}{1+\sin^2(x)}$$

$$F(x)=\int_a^{x^2+1}\frac{2}{1+e^t}dt,\ F'(x)=\frac{2}{1+e^{x^2+1}}\cdot 2x$$

$$F(x)=\int_{e^x}^4\frac{5}{1+\cos(t)}dt,\ F(x)$$

$$=-\int_4^{e^x}\frac{5}{1+\cos(t)}dt,\ F'(x)$$

$$=-\frac{5}{1+\cos\left(e^x\right)}\cdot e^x$$

$$F(x)$$

$$=\int_{x^2-x}^{x^3+x^2}\frac{1}{1+\cos\left(t^3\right)}dt,\ F(x)$$

$$=\int_{x^2-x}^{a}\frac{1}{1+\cos\left(t^3\right)}dt$$

$$+\int_a^{x^3+x^2}\frac{1}{1+\cos\left(t^3\right)}dt$$

$$=-\int_a^{x^2-x}\frac{1}{1+\cos\left(t^3\right)}dt$$

$$+\int_a^{x^3+x^2}\frac{1}{1+\cos\left(t^3\right)}dt$$

$$=-\frac{1}{1+\cos\left(\left(x^2-x\right)^3\right)}(2x$$

$$-1)+\frac{1}{1+\cos\left(\left(x^3+x^2\right)^3\right)}\left(3x^2+2x\right)$$

Note that a does not need to be between anything.

20.1 Function definitions using integrals

$$\arccos(x)=x\sqrt{1-x^2}+2\int_x^1\sqrt{1+t^2}dt$$

How is cosine defined?

$$\arccos(x) = \theta = 2\left(\frac{x\sqrt{1-x^2}}{2} + \int_x^1 \sqrt{1-t^2}dt\right)$$

We have defined arccosine for $x \in [0,\ 1]$.

We can use this to obtain $\cos(\theta)$, then $\sin(\theta)$ (by shifting cosine 90 degrees to the right), then $\tan(\theta)$ through $\frac{\sin(\theta)}{\cos(\theta)}$.

21 The Fundamental Theorem of Calculus II

For f continuous (or just integrable), let $F'(x) = f(x)$ (don't connect this to FTC I; this isn't the F over there despite using the same notation).

$$\int_a^b f(x)dx = F(b) - F(a)$$

To prove this, we need FTC I, and it must be used to prove FTC II. Integral and derivative are not intended to be related to each other, the FTC connects them together.

Th compute an integral, we only need to find an anti-derivative: Find a function $F(x)$ such that $F'(x) = f(x)$.

$$\widehat{F}(x) = \int_a^x f(t)dt \Rightarrow \widehat{F}'(x) = f(x)$$

($\widehat{F}$ is used from FTC I)

$F(x)$ given in FTC II also has $F'(x) = f(x)$. While $\widehat{F}$ and F are different, their derivatives are the same.

Firstly, $f(x) - f(x) = 0$. Then by MVT, $\exists$ constant c such that

$$\widehat{F}(x) - F(x) = c$$

(From the fact that if $f'(x) = 0$ everywhere, then f is a constant function.)

We can rearrange what we've got above:

$$\widehat{F}(x) = F(x) + c,\ \forall x$$

21.1 Recall: MVT

If f is continuous on $[a,\ b]$ and differentiable on $(a,\ b)$, then there exists c between $(a,\ b)$ so that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

However, if $f(b) = f(a) \forall a,\ b$, then $f(x)$ is a constant function. This is obtained from the fact that $f'(x) = 0$ for all x in an interval.

$$\widehat{F}(b) = \int_a^b f(t)dt = F(b) + c$$

Notes:

$$\begin{aligned}\widehat{F}(a) &= \int_a^a f(t)dt = 0 = F(a) + c \\ \widehat{F}(b) &= \int_a^b f(t)dt = F(b) + c\end{aligned}$$

21.2 Definition of the indefinite integral

We define the indefinite-integral to be **one of the anti-derivatives.**

$$\int f(x) = F(x)$$

There are infinitely many anti-derivatives of $f(x)$; they are different up to an additive constant.

We will not use constant c here. Anyways:

$$\begin{aligned}&\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1 \\ &\int e^x dx = e^x \\ &\int \frac{1}{x}dx = \ln|x|\end{aligned}$$

The absolute value on the $|x|$ for ln will require an absolute value as it must take in negative values.

21.3 Defining natural logarithm (ln)

We have defined ln in this way: $\ln(x) : (0,\ \infty) \to (-\infty,\ \infty)$

$$\ln(x) = \int_1^x \frac{1}{t} dt$$

$$\ln(1) = \int_a^a \frac{1}{t} dt = 0$$

Now, what is $\ln^{-1}(x)$ (the inverse)?

$$\ln^{-1}(x) : (-\infty,\ \infty) \to (0,\ \infty)$$

Which happens to be $\exp(x)$.

Also: $\exp(a+b) = \exp(a) + \exp(b)$ (proof left as an exercise given $\ln(ab) = \ln(a) + \ln(b)$.) Also, note this:

$$2^{a+b} = 2^a 2^b$$

So $\exp(x) = b^x$, but what is b? So, define it. Make this the value where if you take the ln of it, you get 1. This means $\ln(e) = 1$.

Then $e = e^1 \Rightarrow \exp(x) = e^x$. This is how e is defined, and this is how e^x is defined.

We then derive log of any base using $y = \log_a(x) = \frac{\ln(x)}{\ln(a)}$.

Also:

$$\begin{aligned}
\ln(ab) = \ln(a) + \ln(b) &\Leftrightarrow \ln^{-1}(x+y) = \ln^{-1}(x) \cdot \ln^{-1}(y) \\
\ln^{-1}(x) = b^x &\text{ some base to a power}, \ln^{-1}(1) = b^1 = e \\
\text{meaning} \ln(e) = 1 &
\end{aligned}$$

21.3.1 Do I need to add dx?

Right now, dx is not as important, but later on, it may be required. If your function has multiple variables, you need to state what variable you are integrating using dy or dx.

21.4 Properties of integrals

Integrals are closed under addition.

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

Swapping integral bounds negate the integral.

$$\int_a^b f = -\int_b^a f$$

If $f \leq g$ on all $[a,\ b]$, then $\int_a^b f \leq \int_a^b g$.

Since $f(x)$ is integral, $f(x)$ must be **bounded:**

$$\exists m,\ M \text{ such that } m \leq f(x) \leq M\ \forall x \in [a,\ b]$$

(Bounded means $f(x)$ is not approaching infinity.)

$$m(b-a) \leq \int_a^b f \leq M(b-a)$$

21.4.1 Conditions that guarantee f is integrable

1. Bounded and monotone
2. **Bounded and continuous**
3. Bounded and discontinuous only at finitely many points (if infinitely, it needs to be minimized)

If the function is approaching infinity, we may not be able to integrate it for now, though we'll know how later. But we know for sure we can't integrate $\frac{1}{x}$ to infinity.

22 Antiderivatives

$$\int_a^b f(x)dx = F(b) - F(a)$$

Firstly: a simple case

$$\int_0^1 (2x-3)dx = x^2 - 3x \mid_0^1 = (1-3)-(0-0)$$

Cube roots:

$$\int_0^8 \sqrt[3]{x} = \int x^{\frac{1}{3}} = \frac{3}{4}x^{\frac{4}{3}} \mid_0^8 = \frac{3}{4}(8\cdot 2 - 0)$$

Cancelling derivatives out:

$$\int_1^5 \frac{d}{dx}\left(\sqrt{1+x^2}\right)dx = \sqrt{1+x^2} \mid_1^5$$

Piecewise:

$$\int_0^4 f(x) \text{ where } f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ -x & 1 < x \le 4 \end{cases}$$

Split them.

Denominator approaches infinity:

$$\int_{-2}^2 \frac{1}{x^2}\ DNE$$

Reciprocal function:

$$\int_1^3 \frac{1}{x} = \ln|x| \mid_1^3 = \ln 3 - \ln 1$$

Reciprocal but negative bounds:

$$\int_{-3}^{-1} \frac{1}{x} = \ln|x| \mid_{-3}^{-1} = \ln(1) - \ln(3)$$

Cases where it is not supposed to exist:

$$\int_{-3}^3 \frac{1}{x}\ DNE$$

22.1 Symmetric reciprocals

If the area you want to integrate passes the vertical asymptote of a reciprocal function, you can't integrate it.

22.2 Dealing with the constant of integration

All anti-derivatives are different up to a constant.

Finding $f(x)$ where $f'(x) = x^2$ and $f(2) = 1$

Well, it's $f(x) = \frac{x^3}{3} + C$ and find C.

23 Integration techniques

23.1 Substitution

Set a portion of $f(x)$ to be equal to $u = u(x)$

Compute $\frac{du}{dx} = u'(x)$

Rearrange for dx regarding the left side as a fraction $dx = \frac{du}{u'(x)}$

Replace dx in integral with the du expression above. Replace the portion in $f(x)$ you set to be u. Hopefully, there are no more quantities involving x. If there are, solve x in terms of u.

23.1.1 When to use substitution

Given $f(x)$, try to set the most troublesome piece of $f(x)$ to u.

EXAMPLE:

$$\int e^{-5x+1} dx$$

$$u = -5x+1$$

$$\frac{du}{dx} = -5$$

$$dx = \frac{du}{-5}$$

$$\frac{\int e^u du}{-5} = -\frac{1}{5}e^u$$

$$-\frac{1}{5}e^{-5x+1}$$

A HARDER EXAMPLE where we can't get rid of x entirely:

Set u such that it cancels out the remaining x -terms.

$$\begin{aligned}
\int x^2\sqrt{x^3+1}dx \\
u = x^3+1 \\
\frac{du}{dx} = 3x^2 \\
dx = \frac{du}{3x^2} \\
\frac{\int x^2\sqrt{u}du}{3x^2} = \frac{\int \sqrt{u}du}{3} = \frac{u^{\frac{3}{2}}}{\frac{3}{2}\cdot 3} = \frac{2}{9}u^{\frac{3}{2}} = \frac{2}{9}\left(x^3+1\right)^{\frac{3}{2}}
\end{aligned}$$

A complicated example

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \frac{\sin(x)}{\cos^3(x)}dx \\
u = \cos(x) \\
\frac{du}{dx} = -\sin(x) \\
dx = \frac{du}{-\sin(x)} \\
\int \left(\frac{\sin(x)}{u^3}\cdot\frac{du}{-\sin(x)}\right)du &= -\int\left(\frac{du}{u^3}\right) \\
&= -\left(\frac{1}{-2}\cdot\frac{1}{u^2}\right) = \frac{1}{2u^2} = \frac{1}{2\cos^2(x)}\ |_0^{\frac{\pi}{4}} \\
&= \frac{1}{2\cos^2\left(\frac{\pi}{4}\right)} - \frac{1}{2\cos^2(0)}
\end{aligned}$$

Okay so

$$\int_1^2 \left(\sqrt{x-1}(x+1)dx\right)$$

$$u = x-1$$

$$\frac{du}{dx} = 1$$

$$dx = du$$

$$\int \sqrt{u} \cdot (x+1)du$$

$$\int \sqrt{u} \cdot u + 2udu$$

$$\frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{4}{3}(x-1)^{\frac{3}{2}} \; |_1^2$$

Another one

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{x \ln x} dx$$

$$u = \ln(x)$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$dx = xdu$$

$$\int \left(\frac{1}{u}du\right)$$

$$= \ln|u| = \ln|\ln(x)|$$

Tip: always, just let $u = \ln(x)$ because of how differentiating $\ln(x)$ works

When considering fractions, the denominator is always more troublesome than the numerator.

$$\int \left(\frac{\sin(x) - \cos(x)}{\sin(x) + \cos(x)}\right)$$

This:

$$\int \left(\frac{\cos(\sqrt{x})}{\sqrt{x}}\right)$$

When you see $\sqrt{x}$ on the denominator, set $u = \sqrt{x}$ /

$$\begin{aligned}
u &= \sqrt{x} = x^{\frac{1}{2}} \\
\frac{du}{dx} &= \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \\
dx &= 2\sqrt{x}du \\
&= \int \text{cos}(u)2du \\
&= 2\sin u \\
&= 2\sin\left(\sqrt{x}\right)
\end{aligned}$$

Okay so

$$\begin{aligned}
&\int \left(\frac{x}{\sqrt{4-x^2}}dx\right) \\
u &= 4-x^2
\end{aligned}$$

Q10:

$$\begin{aligned}
&\int \left(\frac{2x-7}{x^2-7x+10}dx\right) \\
u &= x^2-7x+10 \\
\frac{du}{dx} &= 2x-7
\end{aligned}$$

Q12:

$$\int \left(\frac{x^2-2x}{x^3-3x^2-5}\right)$$

23.2 The involvement of arctangent in integrating

$$\int \left(\frac{x+2}{x^2+1}dx\right)$$
$$u = x^2+1$$
$$\frac{du}{dx} = 2x$$
$$dx = \frac{du}{2x}$$
$$\int \left(\frac{x+2}{u} \cdot \frac{du}{2x}\right)$$

Oops, that doesn't work. All you're doing is hoping that u happens to be the numerator, but what if it's not? $x+2$ does not cancel out with $2x$. What do you do? Force $u'(x)$ to be on the numerator – this is the only thing you can do. You've encountered this rational function that is too complicated, and you want to find its antiderivative.

In this case, the derivative of u is $2x$, but we have $x+2$ on the numerator. What we are going to do is put the 2 out of the integral, so we are going to do a bit of rewriting:

$$\int \left(\frac{x}{x^2+1}\right) dx + \int \left(\frac{2}{x^2+1}\right) dx$$
$$\int \left(\frac{x}{u} \cdot \frac{du}{2x}\right) = \frac{1}{2}\ln|u| = \frac{1}{2}\ln\left|x^2+1\right|$$

And also $\int \left(\frac{2}{x^2+1}\right) dx$. Now what? First, we know that this will be a lower degree of our previous term. This one can be solved in a different way. In this case, remember that $\int \left(\frac{1}{x^2+1}\right) = \arctan(x)$. So, this means:

$$\int \left(\frac{2}{x^2+1}\right) dx = 2\arctan(x)$$

Meaning:

$$\int \left(\frac{x+2}{x^2+1}dx\right) = \frac{1}{2}\ln\left|x^2+1\right| + 2\arctan(x)$$

(Anything between 1-13 will be fair game in a challenge.)

23.3 Integration by parts

Split a function to two pieces, u and dv. After choosing u, the rest of the function becomes dv.

- Compute the derivative and the anti-derivative of u and dv

Apply the formula:

$$\int u \cdot dv = u \cdot v - \int du \cdot v$$

The point of integration by parts is to move the derivative from one part to the other. If you choose u correctly, hopefully $\int du \cdot v$ would be easier to solve.

23.3.1 When am I supposed to use it?

When $f(x)$ is in the form:

- $x^n \sin(x)$
- $x^n \cos(x)$
- $x^n e^x$

Then set $u = x^n$, then du would have one power less than before. Continue the same process until there is no more powers of x left.

Example:

$$\int x \cdot \cos(x)$$

$$\begin{aligned} u &= x \\ dv &= \cos(x) \\ du &= 1 \cdot dx \\ v &= \int \cos(x) = \sin(x) \\ \int x \cdot \cos(x) &= u \cdot v - \int du \cdot v = x \cdot \sin(x) - \int 1 \cdot \sin(x) dx \end{aligned}$$

The same applies if you're dealing with an inverse trigonometric function.

Example:

$$\int \left(x^2 \cos(x)\right)$$

$$\begin{aligned}
u &= x^2,\ dv = \cos(x) \\
du &= 2x,\ v = \int dv = \int (\cos(x)) = \sin(x) \\
uv &- \int du \cdot v \\
&= x^2 \sin(x) - \int 2x \cdot \sin(x)
\end{aligned}$$

Go again:

$$\begin{aligned}
&\int 2x \cdot \sin(x) \\
u &= 2x,\ dv = \sin(x) \\
du &= 2,\ v = \int \sin(x) = -\cos(x) \\
&x^2 \sin(x) - (2x \cdot -\cos(x) - \int 2\,(-\cos(x)) \\
&= x^2 \sin(x) - (2x \cdot -\cos(x) + 2\sin(x) \\
&= x^2 \sin(x) + 2x\cos(x) - 2\sin(x)
\end{aligned}$$

23.3.2 Infinite loops

$$
\begin{aligned}
\int (e^x \cos(x) dx) \\
u = e^x,\ dv = \cos(x) \\
du = e^x,\ v = \int (\cos(x)) = \sin(x) \\
= uv - \int du \cdot v \\
e^x \sin(x) - \int (e^x \sin(x)) \\
u = e^x,\ dv = \sin(x) \\
du = e^x,\ v = \int (\sin(x) = -\cos(x) \\
= e^x \sin(x) - (e^x(-\cos(x)) - \int (e^x(-\cos(x))) \\
\int (e^x \cos(x) dx) = e^x \sin(x) + (e^x(\cos(x)) - \int (e^x(\cos(x))) \\
2\int (e^x \cos(x) dx) = e^x \sin(x) + (e^x(\cos(x))) \\
\int (e^x \cos(x) dx) = \frac{e^x \sin(x) + (e^x(\cos(x)))}{2}
\end{aligned}
$$

23.3.3 MASSIVE HINT

If you have an equation in the form $\int (e^{ax} \sin(bx))$ or $\int (e^{ax} \cos(bx))$ then it should be:

$$= Ae^{ax} \sin(bx) + Be^{ax} \cos(bx)$$

Solve $A,\ B$; take derivative.

Example:

$$
\begin{aligned}
\int (e^x \cos(x)) &= Ae^{ax} \sin(bx) + Be^{ax} \sin(bx) \\
&= Ae^x \sin(x) + Be^x \cos(x)
\end{aligned}
$$

Take the derivative:

$$
\begin{aligned}
e^x \cos(x) &= Ae^x \sin(x) + Ae^x \cos(x) + Be^x \cos(x) + Be^x(-\sin(x)) \\
1 &= A + B; 0 = A - B
\end{aligned}
$$

23.3.4 Another case to use integration by parts

When you have ln or any of the inverse trigonometric functions, set u to be this quantity. Then du should be way simpler.

Example:

$$
\begin{aligned}
&\int (\ln(x) \cdot 1) \\
&u = \ln(x),\ dv = 1 \\
&du = \frac{1}{x},\ v = \int 1 = x \\
&uv - \int du \cdot v \\
\ln(x) \cdot x - \int \left(\frac{1}{x} x\right) &= \ln(x) \cdot x - x
\end{aligned}
$$

Where you shouldn't integrate by parts directly

$$\int (\sin(\ln(x))$$

Don't implement integration by parts directly. You should probably use substitution, which is what you should use if you have a troublesome term. Also, u should be $\ln(x)$ because you can't really integrate ln.

$$
\begin{aligned}
&\int (\sin(\ln(x)) \\
&u = \ln(x) \to x = e^u \\
&\frac{du}{dx} = \frac{1}{x} \\
&dx = xdu \\
&= \int (\sin(u))\, xdu \\
&\int (\sin(u) e^u du) \\
&= Ae^u \sin u + Be^u \cos(u)
\end{aligned}
$$

(Integral by parts is what you would do by this point)

23.3.5 Integrating inverse trigonometric functions

$$
\begin{aligned}
&\int (\arcsin(x)\cdot 1)\\
u &= \arcsin(x),\ dv = 1\\
\frac{du}{dx} &= \frac{1}{\sqrt{1-x^2}},\ v = \int 1 = x\\
&= uv - \int du \cdot v\\
&= \arcsin(x)\cdot x - \int \left(\frac{1}{\sqrt{1-x^2}} x dx\right)\\
u &= 1-x^2\\
\frac{du}{dx} &= -2x\\
dx &= \frac{du}{-2x}\\
&= \arcsin(x)\cdot x - \int \frac{\frac{1}{\sqrt{u}} x du}{-2x} = \arcsin(x)\cdot x - \int \left(\frac{1}{\sqrt{u}}\cdot\left(-\frac{1}{2}\right)\cdot du\right)
\end{aligned}
$$

$$
-\int \left(\frac{1}{\sqrt{u}}\cdot\left(-\frac{1}{2}\right)\cdot du\right) = \frac{1}{2}\int u^{\frac{1}{2}} = \frac{1}{2}\cdot 2u^{\frac{1}{2}} = \left(1-x^2\right)^{\frac{1}{2}}
$$

$$
\arcsin(x)\cdot x - \left(1-x^2\right)^{\frac{1}{2}}
$$

I kind of want to do this again

$$
\begin{aligned}
&\int \left(\frac{x}{\sqrt{4-x^2}} dx\right)\\
u &= 4-x^2\\
\frac{du}{dx} &= -2x\\
dx &= \frac{du}{-2x}\\
&\int \left(\frac{x}{u}\frac{du}{-2x}\right)\\
&= \int \left(\frac{1}{u}\frac{du}{-2}\right)
\end{aligned}
$$

Question 8:

$$\int \left(\frac{\cos\sqrt{x}}{\sqrt{x}}\right) dx$$
$$u = \sqrt{x}$$
$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$$
$$dx = du \cdot 2\sqrt{x}$$
$$\int 2\left(\frac{\cos u}{u}\right) du \cdot u$$
$$\int (2cosu)\, du$$
$$\sin u = 2\sin\left(\sqrt{x}\right)$$

23.4 Integrating arctan

$$\int (x \cdot \arctan(x) dx)$$
$$u = \arctan(x),\ dv = x$$
$$du = \frac{1}{1+x^2},\ v = \frac{x^2}{2}$$
$$\arctan(x) \cdot \frac{x^2}{2} - \frac{1}{2}\int \left(\frac{x^2}{1+x^2}\right) dx$$
$$\int \left(\frac{x^2}{1+x^2}\right) dx = \int \left(\frac{x^2+1-1}{1+x^2}\right) dx$$
$$\int \left(\frac{x^2+1}{1+x^2} - \frac{1}{1+x^2}\right) dx = \int \left(1 - \frac{1}{1+x^2}\right) dx = x - \arctan(x)$$

Back:

$$\arctan(x) \cdot \frac{x^2}{2} - \frac{1}{2}(x - \arctan(x))$$

23.5 About long division

$$\frac{14}{3} = 4R2$$

This essentially means

$$\frac{14-2}{3} = 4$$

If I remove the remainder from the start before doing the division, it will properly divide. 4 is the quotient, and 2 is the remainder.

The same logic can be applied when doing polynomial long division.

23.5.1 Another example

$$\begin{aligned}
&\int \left(\arctan\left(\sqrt{x}\right) dx\right) \\
u &= \sqrt{x} \\
\frac{du}{dx} &= \frac{1}{2\sqrt{x}} \\
dx &= 2\sqrt{x} \cdot du \\
&\int \left(\arctan(u) 2\sqrt{x} \cdot du\right)
\end{aligned}$$

When doing integration by substitution, the derivative of u is in terms of u itself, so we can substitute again:

$$\int (\arctan(u) 2u \cdot du)$$

You can now proceed with integrating by parts using the previous example.

23.5.2 Another example

$$\int \left(x \sec^2(x) dx\right)$$

A bit stuck on substituting. What do I do? Whenever you encounter a difficult integral, you can only do integration by parts and hope to get somewhere.

$$\begin{aligned}
u &= x,\ dv = \sec^2(x) \\
du &= 1,\ v = \tan(x) \\
&x \cdot \tan(x) - \int (\tan(x))
\end{aligned}$$

What is the integral of $\tan(x)$? I forgot.

$$x \cdot \tan(x) - \int \left(\frac{\sin(x)}{\cos(x)}\right)$$

$$\begin{aligned}
u &= \cos(x) \\
\frac{du}{dx} &= -\sin(x) \\
dx &= \frac{du}{-\sin(x)} \\
\int \left(\frac{\sin(x)}{\cos(x)}\right) &= \int \left(\frac{-1}{u}\frac{du}{1}\right) = \int \left(-\frac{1}{u}du\right) = -\ln|u| = -\ln|\cos(x)|
\end{aligned}$$

We can complete this:

$$x \cdot \tan(x) + \ln|\cos(x)|$$

23.6 Tips when substituting

Say you are given

$$\int \left(\frac{x^3}{1+x^4}dx\right)$$

When taking u, make it:

$$u = 1 + x^4$$

This ensures the 1 is removed when taking the derivative.

When taking u-sub, you want to choose a value that would end up cancelling the numerator out, when dealing with algebraic variables. For example:

$$\int \left(\frac{x}{1+x^4}dx\right)$$

Then you want to take $u = x^2$ because $\frac{du}{dx} = 2x,\ dx = \frac{du}{2x}$, which ends up cancelling out the numerator.

24 Applications

24.1 Velocity

$$\sum\left(\frac{1}{n}\cdot\frac{i^2}{n^2}\right)$$
$$\sum\left(\frac{1}{n}\left(\frac{i}{n}\right)^2\right)$$
$$\frac{1}{n}\cdot\sum\left(\frac{i}{n}\right)^2$$
$$f(x) = x^2$$

25 Area

Given $y = f(x) > 0$ on $x \in [a,\ b]$, the area under the curve $y = f(x)$ and above the x-axis is given by

$$\int_a^b f(x)dx = \int_a^b ydx$$

And then you have area differences:

If $y = f(x) > g(x)$, then the area between the two curves is given by the larger area minus the smaller area.

$$\text{Area Difference} = \int_a^b f(x)dx - \int_a^b g(x)dx$$

This can be extended when the functions are as negative as well. The process is still the same if the area you are measuring crosses the x-axis.

In cases where intersection occurs, make the two functions equal to each other (similar to a system of equations). The solutions you find will be the intersections. Proceed to test points afterwards.

Example: total area between $y = x^3 - 2x + 2$ and $y = x^2 + 2$

Make them equal to each other:

$$\begin{aligned} x^3 - 2x + 2 &= x^2 + 2 \\ x^3 - x^2 - 2x &= 0 \\ x(x-2)(x+1) &= 0 \end{aligned}$$

$$x = 0,\ x = 2,\ x = -1$$

We end up with this:

The area is then:

$$\int_{-1}^{0} x^3 - 2x + 2dx - \int_{-1}^{0} x^2 + 2 +$$
$$\int_{0}^{2} x^2 + 2 + \int_{0}^{2} x^3 - 2x + 2$$

25.1 Integrating with respect to y

Sometimes, we are given a curve implicitly, such as $y^5 + y + 1 = x$. It may be difficult to solve for $y = f(x)$, and sometimes we cannot solve $y = f(x)$. We can still compute area by taking integral against y.

Given $x = f(y) > 0$ on y in $[c,\ d]$, the area between the curve $x = f(y)$ and the y-axis is given by

$$\text{area} = \int_{c}^{d} f(y)dy = \int_{c}^{d} xdy$$

If $x = f(y) > g(y)$ which means the curve $f(y)$ is to the right of $g(Y)$, then the area between the 2 curves is given by the larger area minus the smaller area.

$$\text{Area difference} = \int_{c}^{d} f(y)dy - \int_{c}^{d} g(y)dy$$

This can be extended to the case when the functions are negative as well.

Example:

Area between $y = x - 1$ and $y^2 = 2x + 6$

Then $x = y + 1,\ x = \frac{1}{2}y^2 - 3$

We need to find the intersection of the two curves by plugging one into the other one:

$$x = y + 1 = \frac{1}{2}y^2 - 3$$

Find all the solutions y.

$$\begin{aligned} y + 1 &= \frac{1}{2}y^2 - 3 \\ 0 &= \frac{1}{2}y^2 - y - 4 \\ &= (y-4)(y+2) \end{aligned}$$

$$y = 4, \; y = -2$$

So, the intercept occurs at these two y-points.

Then you would be integrating (go from low to high point):

$$\int_{-2}^{4} y + 1 dy - \int_{-2}^{4} \frac{1}{2}y^2 - 3dy$$

$x = y + 1$ is the larger one as it is more to the right and always like that.

25.2 Volume

Volume by revolution

There are 2 quantities: radius r and height h. Depending on the rotation axis, they will correspond to x and y.

25.2.1 The cheese wheel method

Rotation axis as y -axis: |

$$r = x, \; h = y$$

Given $x = f(y) \Rightarrow r = f(y)$

We can compute the volume bounded between $x = f(y)$ and the y-axis by adding up many horizontal circles on different values of y in (c, d). This is the case if you want to look sideways.

$$V = \int \left(\pi r^2 dy\right) = \int_c^d \pi \left(f(y)\right)^2 dy$$

y is the variable of interest.

Example:

Considering the region bounded by $x = y^3,\ x = 8,\ y = 0$, find the volume after revolving this region around the y-axis.

Firstly. $8 = y^3,\ \sqrt[3]{8} = 2$

So $\int_0^8 \pi(8)^2 dy - \int_0^8 \pi \left(y^3\right)^2 dy =$ the answer

25.2.2 The cylinder method

If we want to instead use $y = f(x)$:

Given $y = f(x) \Rightarrow h = f(x)$

We can compute the volume bounded between $y = f(x)$ and the x-axis by adding up many thin cylinders on different values of x in $(a,\ b)$.

$$V = \int (2\pi r \cdot h) dx = \int_a^b 2\pi x \cdot f(x) dx$$

Example:

$$y = x^2,\ y = x^{\frac{1}{3}}$$

$$\int_0^1 2\pi x \cdot x^{\frac{1}{3}} dx - \int_0^1 2\pi x \cdot x^2 dx$$

Rotation axis as the x -axis –

$$h = x,\ r = y$$

Given $y = f(x) \Rightarrow r = f(x)$

We can compute the volume bounded between $y = f(x)$ and the x-axis by adding up many vertical circles on different values in $(a,\ b)$.

$$V = \int \left(\pi r^2 dx\right) = \int_a^b \pi \left(f(x)\right)^2 dx$$

25.2.3 Volume surrounding the x-axis

$$h = x, \ r = y$$

Given $y = f(x) \Rightarrow r = f(x)$

Then

$$V = \int \left(\pi r^2 dx\right) = \int_a^b \pi \left(f(x)\right)^2 dx$$

$$\begin{aligned}
y = \sqrt{x} &= -x + 6 \\
\sqrt{x} &= -x + 6 \\
x &= x^2 - 12x + 36 \\
0 &= x^2 - 13x + 36 \\
0 &= (x-4)(x-9)
\end{aligned}$$

When we squared the step, it did not matter whether you started with $\pm\sqrt{x}$, so the act of squaring removes the information of whether or not you removed the $\pm$ from the square root of x. This means solutions caused by $-\sqrt{x}$ is called a superfluous solution. These are extra solutions that are not supposed to exist.

And then you have $y = 1$.

You end up having two parts.

Test which points are smaller where the split point is $x = 4$

$$x = 1, \ \sqrt{1} = 1, \ y = -1 + 6 = 5$$

So $\sqrt{x}$ is smaller at that point. Then, on the other side:

$x = 5$,

$$2 < \sqrt{5} < 3, \ y = -5 + 6 = 1$$

So $x + y = 6$ is smaller. We then need to check the points where they intersect with 1:

$1 = \sqrt{x}, \ x = 1$ and $x + 1 = 6 \Rightarrow x = 5$, so to integrate this:

$$\int_1^4 \sqrt{x} dx - \int_1^4 1 + \int_4^5 -x + 6 dx - \int_4^5 1 dx$$

Which is the area, but because we are dealing with volumes, we will have to switch up our calculations a bit:

$$\int_1^4 \pi\left(\sqrt{x}\right)^2 dx - \int_1^4 \pi 1^2$$
$$+ \int_4^5 \pi(-x+6)^2 dx - \int_4^5 \pi 1^2 dx$$

Solving this integral gives us the solution.

$$\int_1^4 \pi x dx - \int_1^4 \pi 1 + \int_4^5 \pi\left(x^2-12x+36\right)^2 dx - \int_4^5 \pi dx$$

$$\left[\pi \frac{1}{2}x^2\right]_1^4 - [\pi x]_1^4 + \left[\pi\left(\frac{1}{3}x^3-6x^2+36x\right)\right]_4^5 - [\pi x]_4^5$$

$$\left(\frac{\pi}{2}\left(4^2\right)-\frac{\pi}{2}\left(1^2\right)\right)$$
$$+\left(\pi\left(\frac{1}{3}\left(5^3\right)-6\left(5^2\right)+36(5)\right)-\pi\left(\frac{1}{3}\left(4^3\right)-6\left(4^2\right)+36(4)\right)\right)$$
$$-(5\pi-4\pi)$$

There we go. If we evaluate this, we get the answer.

$$7\pi+\frac{11\pi}{6}$$

25.2.4 Cylinder surrounding the x-axis

Given $x = f(y) \Rightarrow h = f(y)$

We can compute the volume bounded between $x = f(y)$ and the y axis by adding up many thin cylinders on different values of y in $(c,\ d)$.

$$V = \int(2\pi r \cdot h \cdot dy) = \int_c^d (2\pi y \cdot f(y) \cdot dy)$$

Example question:

$$y = x^2,\ y = 9$$

Revolve the region around the x-axis. The points of intersection are at $x = \pm 3$. Though I would do the cylindrical shell method:

$$x = \sqrt{y},\ x = -\sqrt{y}$$

Treat them as causing separate volumes then add them up.

In the setup of $x = f(y)$: $\int$ right $- \int$ left

$$\int_0^9 2\pi rhdy - \int_0^9 2\pi rhdy$$
$$= \int (2\pi y \cdot \sqrt{y}dy) - \int (2\pi y \cdot (-\sqrt{y})\, dy)$$

Solving this gives us the answer.

25.2.5 Cheese wheel or cylinder

The function should be in a form where it is easier to deal with. For example, you don't want to deal with $x = \pm\sqrt{y-4}$, which requires you to use the quadratic formula.

If you have $y^5 + y + 1 = x$, just solve with respect to y, because it is impossible to solve it by isolating y.

25.2.6 The harder questions

25.2.7 Volume of a circle

We have:

$$x^2 + y^2 = r^2$$

Then the upper curve can be modeled by:

$$y = \pm\sqrt{r^2 - x^2}$$

We can just only keep the positive side. Also:

$$V = \int \left(\pi r^2 dx\right)$$

Which is

$$
\begin{aligned}
&\int \left(\pi \sqrt{r^2 - x^2}^2\right) dx \\
&\int \left(\pi \left(r^2 - x^2\right)\right) \\
&\pi \int_{-r}^{r} r^2 - x^2 \\
&= \pi \left(r^2 x \mid_{-r}^{r} - \frac{x^3}{3} \mid_{-r}^{r}\right) \\
&= \pi \left(\left(r^2 r + r^2 r\right) - \left(\frac{r^3}{3} + \frac{r^3}{3}\right)\right) \\
&= \pi \left(2r^3 - \frac{2}{3} r^3\right) = \frac{4}{3} r^3 \pi
\end{aligned}
$$

25.2.8 Revolve around both cases

Bounded by $y = x^2 + 4,\ y = 6x - x^2 = x(6-x)$

Find points of intersection:

$$
\begin{aligned}
x^2 + 4 &= 6x - x^2 \\
2x^2 - 6x + 4 &= 0 \\
2\left(x^2 - 3x + 2\right) &= 2(x-1)(x-2)
\end{aligned}
$$

Intersection is $x = 1,\ x = 2$

Now we know: the area in-between is:

$$
\begin{aligned}
&\int_1^2 6x - x^2 - \int_1^2 x^2 + 4 \\
V &= \int \left(\pi r^2 dx\right) - \int \left(\pi r^2 dx\right) \\
&= \int_1^2 \pi \left(6x - x^2\right)^2 dx - \int_1^2 \pi \left(x^2 + 4\right)^2 dx
\end{aligned}
$$

Whatever is the solution to this is the volume.

Now: rotating around the y-axis. We have this formula:

$$
\begin{aligned}
&\int (2\pi rhdx) - \int (2\pi rhdx) \\
&= \int \left(2\pi x \cdot \left(6x - x^2\right) dx\right) - \int \left(2\pi x \cdot \left(x^2 + 4\right) dx\right)
\end{aligned}
$$

From [1, 2]. Whatever is the solution being the volume.

25.2.9 The rotated parabola cases

~~Consider the region bounded by~~ $x = 0,\ y = 0,\ \sqrt{x} + \sqrt{y} = 1$ ~~. This region is to be revolved around~~ $x = 3$ ~~, but there isn't a formula for that. Instead, we have to shift everything leftward by 3 units, then the axis of revolution will be on the~~ y ~~-axis.~~

~~This case:~~

$$\sqrt{x+3} + \sqrt{y} = 1$$

~~Solving it with:~~

$$
\begin{aligned}
\sqrt{y} &= 1 - \sqrt{x+3} \\
y &= \left(1 - \sqrt{x+3}\right)^2 \\
y &= \left(1 - 2\sqrt{x+3} + x + 3\right) = 4 - 2\sqrt{x+3} + x
\end{aligned}
$$

~~This is the area~~

$$\int_{-3}^{-2} 4 - 2\sqrt{x+3} + x \, dx$$

~~And this is the volume when revolved around the~~ y ~~-axis:~~

$$
\begin{aligned}
&\int_{-3}^{-2} 2\pi x \cdot \left(4 - 2\sqrt{x+3} + x\right) dx \\
&2\pi \int_{-3}^{2} 4x - 2x\sqrt{x+3} + x^2 dx \\
&2\pi \left[2x^2 - \int \left(2x\sqrt{x+3}\right) + \frac{1}{3}x^3\right]_{-3}^{-2}
\end{aligned}
$$

~~Because we have a harder integral~~ $\int\left(2x\sqrt{x+3}\right)$ ~~, we have to integrate it separately:~~

$$
\begin{aligned}
u &= x+3 \\
x &= u-3 \\
\frac{du}{dx} &= 1 \\
du &= dx \\
\int\left(2x\sqrt{u}\right) &= \int\left(2(u-3)\sqrt{u}\right) \\
&= \int\left(2u^{\frac{3}{2}}-6\sqrt{u}\right) = 2\cdot\frac{2}{5}u^{\frac{5}{2}}-6\cdot\frac{2}{3}u^{\frac{3}{2}} \\
&= \frac{4}{5}u^{\frac{5}{2}}-4u^{\frac{3}{2}}
\end{aligned}
$$

~~Returning back:~~

$$
2\pi\left[2x^2-\left(\frac{4}{5}u^{\frac{5}{2}}-4u^{\frac{3}{2}}\right)+\frac{1}{3}x^3\right]_{-3}^{-2}
$$

The method using the y-axis:

$$
\begin{aligned}
\sqrt{x+3}+\sqrt{y} &= 1 \\
\sqrt{x+3} &= 1-\sqrt{y} \\
x+3 &= (1-\sqrt{y})^2 \\
x &= (1-\sqrt{y})^2-3
\end{aligned}
$$

Then:

$$
\int_0^1 \pi(-3)^2dy-\int_0^1 \pi\left((1-\sqrt{y})^2-3\right)^2dy
$$

The vertical method revisited:

$$
\begin{aligned}
y &= \left(1-\sqrt{x+3}\right)^2 \\
V &= \int_{-3}^{-2} 2\pi r \left(1-\sqrt{x+3}\right)^2 dx \\
&= \int_{-3}^{2} 2\pi(-x)\cdot\left(1-\sqrt{x+3}\right)^2 dx
\end{aligned}
$$

x happens to be on the negative side, so we must negate it.

25.2.10 The square-based pyramid example

$$V = \int \left(4\cdot r^2 dy\right)$$

The relationship between the radius of a circle and the height at where it was situated.

$$V = \int \text{area} dz = \int_0^H (2r)^2 dz$$

And that

$$
\begin{aligned}
z &= -\frac{H}{R}y + H \\
Z - H &= -\frac{H}{R}y \\
y &= -\frac{R}{H}(z-H)
\end{aligned}
$$

So, we get:

$$
\begin{aligned}
&\int_0^H \left(2 \cdot \frac{-R}{H}(z-H)\right)^2 dz \\
&= \frac{4R^2}{H^2}\int_0^H (z-H)^2 dz \\
u &= z-H \\
\frac{du}{dz} &= 1 \\
&= \frac{4R^2}{H^2} \cdot \frac{(Z-H)^3}{3} \,|_0^H \\
&= \frac{4R^2}{H^2} \cdot \left(0 - \frac{(-H)^3}{3}\right) = \frac{1}{3}4R^2H
\end{aligned}
$$

Where R is the radius of the square.

26 Trigonometric integrals

Note:

$$\int (\sin(x)) = -\cos(x),\ \int (\cos(x)) = \sin(x)$$

Integrating one power of sine or cosine:

Use substitution and set u to be the one with more power.

One of sine or cosine have an odd power:

Use the formula $\sin^2(x) + \cos^2(x) = 1$ to interchange $\cos^2(x)$ and $\sin^2(x)$ of the odd power so that one power remains.

Only $\sin^2(x)$ or $\cos^2(x)$

Use the identity again. Interchange the cosine and the sine.

Double angle identity:

$$\cos(2x) = \cos(x+x) = \cos(x)\cos(x) - \sin(x)\sin(x) = \cos^2(x) - \sin^2(x)$$

Interchanging:

$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$
$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$

Double angle:

$$\cos(2x) = \left(1-\sin^2(x)\right) - \sin^2(x)$$
$$\cos(2x) = \cos^2(x) - \left(1-\cos^2(x)\right)$$

Integrating even powers:

Use above formula to interchange all even powers into powers of $\cos(2x)$

Multiples of $\sec(x)$ and $\tan(x)$:

Note:

$$\frac{d}{dx}\tan(x) = \sec^2(x)$$
$$\frac{d}{dx}\sec(x) = \tan(x)\sec(x)$$

Starting with the formula and dividing by $\cos^2(x)$:

$$\sin^2(x) + \cos^2(x) = 1 \Rightarrow \tan^2(x) + 1 = \sec^2(x)$$

This allows $\tan^2(x)$ and $\sec^2(x)$ to be interchanged.

One power of $\tan(\mathbf{x})$ and at least two powers of $\sec(x)$

$$\int (\tan(x)\sec^{100}(x)dx$$
$$u = \sec(x),\ du = \tan(x)\sec(x)$$
$$\int \left(\tan(x)\sec^{100}(x)dx\right) = TBA$$

Even powers of $\sec(x)$:

Use the formula $\tan^2(x)+1=\sec^2(x)$ to interchange $\sec^2(x)$ with $\tan^2(x)$ so that two powers of $\sec(x)$ remain.

Odd powers of $\tan(x)$:

Replace so that only one power of $\tan(x)$ remain.

Integral of secant:

$$\int(\sec(x)) = \ln|\sec(x)+\tan(x)|$$

Multiples of $\csc(x)$and$\tan(x)$

Use $1+\cot^2(x)=\csc^2(x)$ and $csc'(x) = -\cot(x)\csc(x)$ and $\cot'(x) = -\csc^2(x)$

Use the relationships with cotangent and cosecant.

Different frequencies of $\sin(mx)$ and $\cos(nx)$

$$\begin{aligned}
\sin(A)\cos(B) &= \frac{1}{2}(\sin(A-B)+\sin(A+B)) \\
\sin(A)\sin(B) &= \frac{1}{2}(\cos(A-B)-\cos(A+B)) \\
\cos(A)\cos(B) &= \frac{1}{2}(\cos(A-B)+\cos(A+B))
\end{aligned}$$

The right side is easier to integrate.

26.1 Orthogonal relation for Fourier Sine and Cosine Series

Think of the dot product in linear algebra: if the two vectors are orthogonal (perpendicular), their dot products will be zero.

Generalization of the dot product: $\sum_{i=1}^{n} u_i v_i$

Guess what: There is a relationship between the sum and the integral. If you have a vector $\mathbf{u}$ in two dimensions, how much information do you need to define it? Two: its x component and its y-component. If it is n-dimensional, we need n components. Now, what if I ask a function: $f(x) : x \in [0,\ 1]$, how much pieces of information do I need to define the function? I need to know $f(x)$ for all $x \in [0,\ 1]$. That is infinitely many components. A function is like an infinite dimensional vector (if its domain isn't discrete).

Suppose we have $g(x) : x \in [0,\ 1]$. Then we can take the dot product of the two functions. Then:

$$f \cdot g = \sum f(i)g(i)$$

Where we are summing every possible i between 0 to 1. When you have to sum up infinitely as many variables, we're going to have to use the integral. Which is the inner product (dot product for functions):

$$\langle f,\ g\rangle = \int f(x)g(x)$$

Now, what about trigonometric functions? $\int (\sin(2x)\sin(3x)) = 0$

Sines and cosines of different frequencies are orthogonal to each other.

$$\begin{aligned}
\int_0^{\pi} \sin(mx)\sin(nx)dx &= \begin{cases} 0 & \text{if } m \neq n \\ \frac{\pi}{2} & \text{if } m = n \end{cases} \\
\int_0^{\pi} \cos(mx)\cos(nx)dx &= \begin{cases} 0 & \text{if } m \neq n \\ \frac{\pi}{2} & \text{if } m = n \end{cases} \\
\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx &= 0\ \forall n,\ m
\end{aligned}$$

26.1.1 Some examples

$$\begin{aligned}
&\int_0^{\pi} \sin(mx)\sin(nx),\ m \neq n \\
&= \frac{1}{2}\int (\cos(mx - nx) - \cos(mx + nx)dx \\
&= \frac{1}{2}\int \left(\cos\left((m-n)x\right) - \cos\left((m+n)x\right)dx\right) \\
&= \frac{1}{2} \cdot \left(\frac{\sin\left((m-n)x\right)}{m-n} - \frac{\sin\left((m+n)x\right)}{m+n}\ |_0^{\pi}\right) = 0
\end{aligned}$$

(Treat $(m-n)$ and $(m+n)$ as numbers. I swear it will work – as $\sin(k\pi) = 0\ \forall k \in \mathbb{Z}$)

Then the same question for $m = n$: If so, you're taking the dot product of $\sin(mx)$ and itself. So how does this work out?

$$\int_0^{\pi} \sin(nx)\sin(nx) = \int_0^{\pi} \sin^2(nx)$$

Grabbing out the even powers of $\sin(x)$ formula:

$$
\begin{aligned}
&\int_0^{\pi} \frac{1-\cos(2nx)}{2} dx \\
&= \frac{1}{2}x - \frac{\sin(2nx)}{2n} \mid_0^{\pi} \\
&= \frac{\pi}{2}
\end{aligned}
$$

(Reverse chain rule factor applies here. We aren't wasting time doing u-substitution.)

What does this mean? The dot product of a vector with itself is the length of the vector squared. So, the "length" of the function $\sin(nx) = \sqrt{\frac{\pi}{2}} = ||(\sin(nx))||$ hence $\sin(nx)\sin(nx) = \frac{\pi}{2}$.

27 Trigonometric substitution

The two integration techniques we've learned were substitution and by parts.

27.1 Type 1

Integrals looking like $\sqrt{a-x^2}$

- This looks like $\sqrt{1-\sin^2(\theta)} = \cos(\theta)$
- Try to turn the expression into $\sqrt{1-z^2}$ for some new z.
- Use substitution, set $z = \sin(\theta)$.

Particular example:

$$
\begin{aligned}
&\int \frac{x^2}{\sqrt{4-x^2}}dx \\
\sqrt{4-x^2} &= \sqrt{4-\frac{4x^2}{4}} = \sqrt{4\left(1-\frac{x^2}{4}\right)} = 2\sqrt{1-\left(\frac{x}{2}\right)^2} \\
\frac{x}{2} &= \sin\theta \Rightarrow x = 2\sin(\theta) \Rightarrow \arcsin\left(\frac{x}{2}\right) = \theta \\
\frac{dx}{d\theta} &= 2\cos(\theta) \Rightarrow dx = 2\cos(\theta)d\theta \\
&\int \frac{x^2}{\sqrt{4-x^2}}dx \\
&= \int \frac{x^2}{2\sqrt{1-\left(\frac{x}{2}\right)^2}}dx \\
&= \int \frac{x^2}{2\sqrt{1-\sin^2(\theta)}} \cdot 2\cos(\theta)d\theta \\
&= \int \frac{x^2}{2\cos\theta} \cdot 2\cos(\theta)d\theta = \int \frac{4\sin^2(\theta)}{2\cos\theta} \cdot 2\cos(\theta)d\theta \\
&= 4\int \sin^2(\theta)d\theta = 2\theta - \sin(2\theta) \\
&= 2\arcsin\left(\frac{x}{2}\right) - \sin(2\theta)
\end{aligned}
$$

To deal with $\sin(2\theta)$, use the double angle identity:

$$
\sin(2\theta) = 2\sin\theta\cos\theta = 2\cdot\frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2}
$$

About $\cos\theta$: use the triangle.

Meaning the final answer is:

$$
2\arcsin\left(\frac{x}{2}\right) - \left(x\cdot\frac{\sqrt{4-x^2}}{2}\right)
$$

27.2 Type 2

Integral involving $\sqrt{a+x^2}$

- This looks similar to $\sqrt{1+\tan^2(x)} = \sec(\theta)$
- Try to turn the expression into $\sqrt{1+z^2}$ for some new z.
- Using substitution, set $z = \tan(\theta)$.

Integral involving $\sqrt{x^2-a}$

- This looks similar to $\sqrt{\sec^2(\theta)-1} = \tan\theta$
- Try to turn the expression into $\sqrt{z^2-1}$ for some new z.
- Using substitution, set $z = \sec(\theta)$.

27.3 Type 3

Integral involving $\sqrt{ax^2+bx+c}$

Complete the square inside the square root. Then, depending on the situation, do a trigonometric substitution from above.

$$\int \frac{1}{(x+1)\sqrt{x^2+2x-3}} dx$$

Complete the square: add and subtract $\left(\frac{b}{2}\right)^2$: In general, in vertex form it will be $\left(x+\frac{b}{2}\right)^2$

$$x^2+2x-3 = x^2+3x-3+1-1 = (x+1)^2-4$$

Then:

$$\begin{aligned}
&\int \frac{1}{(x+1)\sqrt{(x+1)^2-4}} dx \\
&= \int \frac{1}{(x+1)\sqrt{4\left(\frac{(x+1)^2}{4}-1\right)}} dx \int \frac{1}{2(x+1)\sqrt{\left(\frac{x+1}{2}\right)^2-1}} dx
\end{aligned}$$

Now:

$$\begin{aligned}
\frac{x+1}{2} &= \sec(\theta) \Rightarrow x = 2\sec(\theta) - 1 \\
\frac{dx}{d\theta} &= 2\sec\theta\ \tan\theta \Rightarrow dx = 2\sec\theta\ \tan\theta\ d\theta \\
\int \frac{1}{(x+1)2\left(\sec^2\theta - 1\right)} \cdot 2\sec\theta\tan\theta d\theta &= \int \frac{1}{(x+1)2\tan(\theta)} \cdot 2\sec\theta\tan\theta d\theta \\
&= \int \frac{\sec\theta}{2\sec\theta} d\theta = \frac{\theta}{2} = \frac{1}{2} arcsec \frac{x+1}{2}
\end{aligned}$$

27.4 Some examples

1

$$\int \frac{1}{(x^2+2)^{\frac{3}{2}}} dx$$

Which can be written like

$$\begin{aligned}
&\int \frac{1}{\sqrt{x^2+2}^3} dx \\
&= \int \frac{1}{\sqrt{2}\sqrt{\frac{x^2}{2}+1}^3} dx \\
&= \int \frac{1}{\left(\sqrt{2}\sqrt{\left(\frac{x^2}{\sqrt{2}}\right)^2+1}\right)^3} dx
\end{aligned}$$

$$\begin{aligned}
\frac{x}{\sqrt{2}} &= \tan\theta,\ x = \sqrt{2}\tan\theta \\
\frac{dx}{d\theta} &= \sqrt{2}\sec^2\theta
\end{aligned}$$

So

$$
\begin{aligned}
\int \frac{1}{\left(\sqrt{2}\sqrt{\tan^2\theta+1}\right)^3}dx &= \int \frac{1}{\left(\sqrt{2}\sec\theta\right)^3}\sqrt{2}\sec^2\theta d\theta \\
&= \int \frac{1}{2\sec\theta}d\theta = \int \frac{\cos(\theta)}{2}d\theta = \frac{\sin\theta}{2} \\
l &= \sqrt{x^2+2} \\
\sin\theta &= \frac{x}{l} = \frac{x}{\sqrt{x^2+2}} \\
\frac{\sin\theta}{2} &= \frac{x}{2\sqrt{x^2+2}}
\end{aligned}
$$

3

$$
\begin{aligned}
&\int \frac{1}{e^x\sqrt{e^{2x}-4}}dx \\
&e^{2x} = (e^x)^2 = u^2,\ u = e^x \\
&\frac{du}{dx} = e^x, \frac{du}{e^x} = dx \\
&\int \frac{1}{u\sqrt{u^2-4}}\left(\frac{du}{u}\right) \\
&= \int \frac{1}{u\sqrt{4\left(\frac{u^2}{4}-1\right)}}\left(\frac{du}{u}\right) \\
&= \int \frac{1}{2u\sqrt{\left(\frac{u^2}{4}-1\right)}}\left(\frac{du}{u}\right)
\end{aligned}
$$

28 Partial fractions

We want to integrate any **rational function**:

$$
f(x) = \frac{p(x)}{q(x)}
$$

Where $p(x),\ q(x)$ are polynomials.

28.1 Long division

The degree is the highest power of the polynomial (remember big-O notation).

$$\frac{14}{3} = 4 + \frac{2}{3}$$

When dealing with polynomials $\frac{p(x)}{q(x)}$:

If $\deg p(x) < \deg q(x)$, then we skip long division.

If $\deg p(x) \geq \deg q(x)$, we perform **long division** on $\frac{p(x)}{q(x)}$.

For example:

$$\int \frac{2x^3 + x^2 - x - 4}{(x-1)^2} dx = \int (2x+5) dx + \int \frac{7x-9}{x^2+2x+1} dx$$

Meaning we get something that isn't a fraction and something where the top power is less than the bottom power.

28.2 Actually, doing partial fractions

We cannot prove why this works.

1. Completely factor the denominator into irreducible factors. The irreducible factors must have either degree 1 or degree 2.

$$q(x) = q_1(x) \dots q_n(x)$$

2. Assume each $q_i(x)$ are distinct. Write the original fraction into a sum of polynomials.

$$\frac{p(x)}{q(x)} = \frac{p_1(x)}{q_1(x)} + \cdots + \frac{p_n(x)}{q_n(x)}$$

   If $q_i(x)$ is degree 1, set $p_i(x)$ to be constant, like $p_i(x) = A$.

   If $q_i(x)$ is degree 2, we set $p_i(x)$ to be degree 1, such as $p_1(x) = Ax + B$.

   Example: $\frac{x^3+x^2-x+1}{x(x-1)\left(x^2+1\right)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}$

3. Take a common denominator on the right side. Solve for the constants in $p_i(x)$ based on $p(x)$ as a system of linear equations.

Example:

$$\begin{aligned}\frac{x^3+x^2-x+1}{x(x-1)\left(x^2+1\right)} &= \frac{A}{x}\cdot\frac{(x-1)\left(x^2+1\right)}{(x-1)\left(x^2+1\right)} \\ &+ \frac{B}{x-1}\cdot\frac{x\left(x^2+1\right)}{x\left(x^2+1\right)} \\ &+ \frac{Cx+D}{\left(x^2+1\right)}\cdot\frac{x(x-1)}{x(x-1)}\end{aligned}$$

$$x^3+x^2-x+1 = A(x-1)\left(x^2+1\right)+Bx\left(x^2+1\right)+(Cx+D)x(x-1)$$

Expand and factor out x^n (it gets messy in-between, but the steps make sense):

$$x^3+x^2-x+1 = (A+B+C)x^3+(-A-C+D)x^2+(A+B-D)x-A$$

We get

$$\begin{aligned}1 &= A+B+C \\ 1 &= -A-C+D \\ -1 &= A+B-D \\ 1 &= -A\end{aligned}$$

Solving this system, we see that $A=-1,\ B=1,\ C=1,\ D=1$

So, we attain:

$$\frac{x^3+x^2-x+1}{x(x-1)\left(x^2+1\right)} = -\frac{1}{x}+\frac{1}{x-1}+\frac{x+1}{x^2+1}$$

Integrating this:

$$\ln|x|+\ln|x-1|+\cdots$$

And

$$\int\frac{x+1}{x^2+1}dx = \int\frac{x}{x^2+1}+\frac{1}{x^2+1}dx$$

28.3 Indistinct quotients (multiplicity)

In the event that a factor has a multiplicity, that is to say $q(x)$ has a repeated irreducible factor, such as $(q_1(x))^n$:

Then to form a sum of functions for this factor, we must repeatedly form new fractions in the sum, with denominator $(q_1(x))^k$ with k increasing from 1 up to n.

The numerator is still determined by whether $q_1(x)$ is linear or quadratic. Meaning no $Mx+B$

Example:

$$\frac{p(x)}{q(x)} = \frac{3x^2+x+1}{x(2x-1)^3} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{(2x-1)^2} + \frac{D}{(2x-1)^3}$$

Remember that x^2 is factorable. It can be factored to $x \cdot x$.

Example:

$$\frac{p(x)}{q(x)} = \frac{3x^2+x+1}{x^2\left(x^2+1\right)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} + \frac{(Ex+F)}{\left(x^2+1\right)^2} + \frac{Gx+H}{\left(x^2+1\right)^3}$$

28.4 Cleanup

The partial function method writes $f(x)$ as a sum of fractions with degree 1 / 2 denominator, and each fraction can be integrated separately. We are left with solving integrals of the following forms:

$$\int \frac{A}{(ax+b)^k}$$
$$\int \frac{Ax+B}{\left(ax^2+bx+c\right)^k}$$

Integration:

28.4.1 Linear term on numerator

$$\int \frac{A}{(ax+b)^k}$$

$$
\begin{aligned}
\frac{du}{dx} = a,\ dx = &\int \frac{A}{u^k}\frac{du}{a} \\
&= \frac{A}{a}\int u^{-k} \\
&= \frac{A}{a}\cdot\frac{u^{-k+1}}{-k+1} \text{ or } \frac{A}{a}\ln|ax+b|
\end{aligned}
$$

28.5 Quadratic term on the denominator

If there is a linear term on the numerator, split the fraction using +. First, focus on the term with x in the numerator.

$$\int \frac{Ax}{(ax^2+bx+c)^k}dx$$

Set $u = ax^2+bx+c$ from the denominator. Then $du = (2ax+b)dx$ as a linear term. So we try to make the numerator to be $du = (2ax+b)dx$ exactly. (Note: $2ax = 2ax+b-b$

$$
\begin{aligned}
&= \frac{A}{2a}\int \frac{2ax}{(ax^2+bx+c)^k}dx \\
&= \frac{A}{2a}\left(\int \frac{2ax+b}{(ax^2+bx+c)^k}dx + \int -\frac{b}{(ax^2+bx+c)^k}dx\right)
\end{aligned}
$$

$$\int \frac{1}{u^k}du$$

The second integral is of the form

$$\int \frac{B}{(ax^2+bx+c)^k} = B\int \frac{1}{(ax^2+bx+c)^k}$$

28.6 Constant term over quadratic

$$\int \frac{1}{(ax^2+bx+c)^k}dx$$

This case is difficult. Focus on the case that $k = 1$:

$$\int \frac{1}{ax^2+bx+c}$$

By the partial fraction procedure, the quadratic must be irreducible. Complete the square:

$$\begin{aligned}
&1x^2+bx+c+\left(\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2\\
&x^2+bx+\left(\frac{b}{2}\right)^2+c-\left(\frac{b}{2}\right)^2\\
&\left(x+\frac{b}{2}\right)^2+c-\left(\frac{b}{2}\right)^2
\end{aligned}$$

You want to obtain this form:

$$\int \frac{1}{(gx+d)^2+1}$$

Take $u=cx+d$. This gives the integral of the form $\int \frac{1}{u^2+1}=\arctan(u)$ and substitute back.

28.6.1 Example:

$$\begin{aligned}
\int \frac{x+1}{x^3-2x^2+2x}dx\\
x^3-2x^2+2x=x\left(x^2-2x+2\right)
\end{aligned}$$

Then we have:

$$\frac{A}{x}+\frac{Bx+C}{x^2-2x+2}$$

$$\frac{x+1}{x\left(x^2-2x+2\right)}=\frac{A}{x}\cdot\frac{\left(x^2-2x+2\right)}{x^2-2x+2}+\frac{Bx+C}{x^2-2x+2}\cdot\frac{x}{x}$$

Since now the denominators are equal, so are the numerators. Then:

$$
\begin{aligned}
&A\left(x^2-2x+2\right)+\left(Bx^2+Cx\right)\\
0 &= (A+B)x^2\\
1 &= \ (-2A+C)x\\
1 &= \ 2A
\end{aligned}
$$

Solve this system.

$$
\begin{aligned}
A &= \frac{1}{2}\\
A+B &= \frac{0}{x^2}=0,\ B=-\frac{1}{2}\\
1 &= -2\cdot\frac{1}{2}+C,\ C=2\\
&\int\left(\frac{A}{x}+\frac{Bx+C}{x^2-2x+2}\right)\\
&= \frac{1}{2}\int\frac{1}{x}dx+\int\frac{-\frac{1}{2}x+2}{x^2-2x+2}dx\\
u &= x^2-2x+2\\
du &= (2x-2)dx\\
\int\frac{-\frac{1}{2}x+2}{x^2-2x+2}dx &= \frac{1}{-4}\left(\int\frac{2x-8}{x^2-2x+2}dx\right)\\
&= -\frac{1}{4}\left(\int\frac{2x-2}{x^2-2x+2}dx+\int-\frac{6}{x^2-2x+2}\right)
\end{aligned}
$$

$$
\int\frac{2x-2}{x^2-2x+2}dx=\int\frac{du}{u}=\ln|u|
$$

$$
-\int\frac{6}{x^2-2x+2}=-6\int\frac{1}{x^2-2x+2}
$$

Complete the square:

$$
\begin{aligned}
-6\int\frac{1}{x^2-2x+2+\left(\frac{2}{2}\right)^2-\left(\frac{2}{2}\right)^2} &= -6\int\frac{1}{(x^2-2x-1)+1}\\
&= -6\int\frac{1}{(x-1)^2+1}\\
&u=x-1,\ du=1,\ -6\int\frac{du}{u^2+1}=-6\arctan(u)=-6\arctan(x-1)
\end{aligned}
$$

Combine the two sides:

$$-\frac{1}{4}\ln\left|x^2-2x+2\right|+\frac{3}{2}\arctan(x+1)$$

28.7 Quartic on the denominator

$$\int \frac{1}{\left(x^2+1\right)^2}dx$$

Another integration technique is to be clever.

$$\begin{aligned}
&\int \frac{1}{\left(x^2+1\right)^2}\\
&=\int \frac{1+x^2-x^2}{\left(x^2+1\right)^2}\\
&=\int \frac{1+x^2}{\left(x^2+1\right)^2}-\int \frac{x^2}{\left(x^2+1\right)^2}\\
&=\int \left(\frac{1}{1+x^2}\right)-\int \frac{x^2}{\left(x^2+1\right)^2}=\arctan(x)-\int \frac{x^2}{\left(x^2+1\right)^2}
\end{aligned}$$

$$\begin{aligned}
\int \frac{x^2}{\left(x^2+1\right)^2}&=\int \frac{x\cdot x}{\left(x^2+1\right)^2}\\
u=x,\ dv&=\frac{x}{\left(x^2+1\right)^2}\\
du=1,\ v&=\int \frac{x}{\left(x^2+1\right)^2}\\
z=x^2+1,\ dz&=2xdx\\
v=\frac{\int \frac{1}{z^2}dz}{2}&=\frac{1}{2}\int z^{-2}=\frac{\frac{1}{2}z^{-1}}{-1}=-\frac{1}{2\left(x^2+1\right)}
\end{aligned}$$

So:

$$
\begin{aligned}
\int \frac{x^2}{(x^2+1)^2} &= -\frac{x}{2(x^2+1)} + \int \frac{1}{2(x^2+1)} \\
&= -\frac{x}{2(x^2+1)} + \frac{1}{2}\int \frac{1}{(x^2+1)} \\
&= -\frac{x}{2(x^2+1)} + \frac{1}{2}\arctan(x)
\end{aligned}
$$

29 Sequences – the basics

A note about the epsilon characterization of the inf or the sup, it can only really be used if we know for a fact that we are dealing with something that is confirmed to be the inf or sup. Don't try to prove that something is the sup / inf using the epsilon characterization.

A sequence $S = \{a_n\}_{n=1}^{\infty}$ is like a regular function $f(n) = a_n$, except you can only ask its limit as $n \to \infty$. Sequences are discrete, so you can't take the limit anywhere but ∞.

Properties:

Definition 29.1 (Bounded above)**.** A sequence S is bounded above if every number in the sequence is less than some big number M. For example:

$$a_n = \frac{1}{n}$$

Is bounded above.

If a sequence S is bounded above, there exists a quantity that is very similar to the maximum of S, It is called the **supremum** of S, denoted $\sup S$. A supremum is a value if you take an ε-sized step to the left, you will bump into something inside S.

If a sequence S is bounded below, there exists a quantity similar to the minimum of S, called the infimum of S, denoted $\inf S$. An infimum is a value in which if you add ε, then you're in S.

For example:

$$S_1 = \{x : |x-1| < 2\}$$

Then $\sup S_1 = 3, \inf S_1 = -1$. This is like a 2-radius circle centered at 1.

$$S_2 = \{0.9,\ 0.99,\ 0.999,\ \ldots\}$$

Then $\sup S_2 = 1, \inf S_2 = 0.9$.

$$S_3 = \left\{(-1)^n\left(1-\frac{1}{n}\right)\right\}$$

Then $\sup S_3 = 1$ and $\inf S_3 = -1$.

29.1 Limits for sequences to infinity

Same as regular functions. The definition is:

$$\forall \varepsilon > 0,\ \exists N,\ n > N \Rightarrow |a_n - L| < \varepsilon$$

If the limit exists, then S gets close to L, so it is not going to infinity, so it is bounded above and below. However, being bounded above and below is not enough for the limit to exist – which occurs if the sequence is oscillating.

29.2 Convergent, divergent

If the limit of $S = \{a_n\}$ DNE, we say it diverges. It could:

- Approach ∞
- Oscillate
- Increasing means $\forall n,\ a_{n+1} > a_n$ (successive terms are larger)
- Non-decreasing means $\forall n,\ a_{n+1} \geq a_n$ (nothing goes down as n goes up)
- If S is increasing, then it's non-decreasing.

29.3 Bounded above and increasing implies convergence

If S is bounded above and it's **non-decreasing,** then it converges to its **supremum.** It will be stuck at the supremum. It will be bounded by the least upper bound.

Proof. Suppose S is bounded above, meaning $\exists \sup S = L$ as the least upper bound. Show that $\lim_{n\to\infty} a_n = L$. This means to show:

$$\forall \varepsilon > 0,\ \exists N,\ n > N \Rightarrow |a_n - L| < \varepsilon$$
$$L - a_n < \varepsilon \text{ because we are proving the supremum}$$

Let $\varepsilon > 0$. We know:

$$\exists a_N \in S \text{ such that}$$
$$L - \varepsilon < a_N \leq L$$

Because it is between a_n, which is greater than $L - \varepsilon$.

Show that $\forall n > N,\ \ L - a_n < \varepsilon$

We know N is fixed. If $N = 100$, we need to show all n above 100. To prove this question, use induction.

Base: $n = N + 1$. We're not trying to show for every natural number n is true – we're trying to show every natural number over N is true.

- Show $L - a_{N+1} < \varepsilon$. Our assumption is we know $L - \varepsilon < a_N \leq L$ (within epsilon distance). But a_{N+1} should be the right (or the same position) as a_N. Manipulate the inequalities a bit:
- $a_{N+1} \geq a_N$
- $-a_{N+1} \leq -a_N$ (Multiply both sides by -1)
- $L - a_{N+1} \leq L - a_N$

Remember $L - \varepsilon < a_N \leq L$ (which we assumed true)? Rearrange this:

$$L - \varepsilon < a_N \leq L$$
$$-\varepsilon < a_N - L \leq 0$$
$$\varepsilon > L - a_N$$

So:

- $L - a_{N+1} \leq L - a_N < \varepsilon$

We showed what we wanted to show.

Inductive step:

- Suppose $L - a_n < \varepsilon$
- Show $L - a_{n+1} < \varepsilon$

We know

- $a_{n+1} \geq a_n$ because the sequence is increasing
- $-a_{n+1} \leq -a_n$
- $L - a_{n+1} \leq L - a_n$
- $L - a_{n+1} \leq L - a_n < \varepsilon$ by the inductive hypothesis.

Since $L = \sup S$, $a_n \leq L$, so $|a_n - L| = L - a_n$

■

What is a_N? The thing you bump into when taking a small step to the left from the supremum. Then you will bump into something from S. Then you can assume it is greater than $L - \varepsilon$ because the step is very small.

Flip this definition for decreasing equivalents. If S is bounded below and it's **non-increasing,** then it converges to its **infimum.**

Being **non-increasing** or **non-decreasing** is called monotone.

The limit of a sequence has nothing to do with how it behaves at the beginning. Feel free to ignore the first few terms (or more, precisely finitely as many).

29.4 Finding limits to infinity

Given in a regular formula, compute the limit to infinity like normal. For example:

$$\frac{n + (-1)^n}{n} = \frac{n}{n} + \frac{(-1)^n}{n} = 1 + 0 = 1$$

And

$$1.01^n = \left(e^{\ln(1.01)}\right)^n = e^{n \cdot \ln(1.01)} = e^{\infty} = \infty$$

And

$$\frac{4n}{\sqrt{4n^2+2}} = \frac{\frac{4n}{n}}{\sqrt{\frac{4n^2}{n^2}+\frac{2}{n^2}}} = \frac{4}{\sqrt{4}} = \frac{4}{2} = 2$$

29.5 Recursively defined sequences

Consider the recursively defined sequence. Determine whether it converges. If it converges, find its limit.

$$\begin{aligned} a_1 &= 1, \\ a_{n+1} &= \frac{1}{2}a_n + 1 \end{aligned}$$

This means:

$$\begin{aligned} a_2 &= \frac{1}{2} + 1 = \frac{3}{2} \\ a_3 &= \frac{3}{4} + 1 = \frac{7}{4} \\ a_4 &= \frac{7}{8} + 1 = \frac{15}{8} \end{aligned}$$

How can we show that it converges? We need to show that this function is:

- Bounded above
- Increasing / non-decreasing
- And find $\sup S = L$ (the supremum)

So, what do we do? Take the limit of both sides in the recursive part of the function.

$$\begin{aligned} \lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \frac{1}{2}a_n + 1 \\ L &= \frac{1}{2}L + 1 \\ L &= 2 \end{aligned}$$

Here, we kind of assumed that $\infty + 1 = \infty$, but assuming that isn't sufficient. We need to show that it is bounded above, so we're going to use induction. We technically aren't supposed to do that first, but we're going to do it anyways, because we need to so we can prove it.

Show bounded above: show $\forall n,\ a_n \leq 2$.

Proof. Base: $n = 1,\ a_1 = 1 \leq 2$

Suppose $a_n \leq 2$. Show $a_{n+1} \leq 2$

$$a_{n+1} = \frac{1}{2}a_n + 1 \leq \frac{1}{2}2 + 1 = 1 + 1 = 2$$

■

29.5.1 How to show its increasing

For $a_1 = 1,\ a_{n+1} = \frac{1}{2}a_n + 1$

- Show $\forall n,\ a_{n+1} > a_n$. Use induction:
- Base: $n = 1$. In this case, $a_2 = \frac{3}{2},\ a_1 = 1$. Then $a_2 > a_1$
- Suppose $a_{n+1} > a_n$. Then show $a_{n+2} > a_{n+1}$.
- Because we know $a_{n+1} > a_n$,
- $a_{n+2} = \frac{1}{2}a_{n+1} + 1 > \frac{1}{2}a_n + 1 = a_{n+1}$

29.6 The big theorem

Constant < Logs < Polynomials < Exponentials < Factorials < Power of itself

We get the ascending growth rates:

$$1 \ll \ln(n) \ll \sqrt{n} \ll n \ll n^2 \ll \cdots \ll 2^n \ll e^n \ll n! \ll n^b$$

Where $a_n \ll b_n$,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$$

30 Improper integrals

The integral is the area of many rectangles, which is base times height.

Type 1: Base is infinite

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$

Type 2: Height is infinite

The function is approaching infinity. Assume $f(x)$ approaches infinity at $x = k$ and $a < k$. Then:

$$\int_a^k f(x) = \lim_{b\to k^-}\int_a^b f(x)dx$$

Example: The harmonic series:

$$\int_0^1 \frac{1}{x}dx = \ln|x| \,|_0^1 = \ln 1 - \ln 0 = 0 - (-\infty) = \infty$$

Alternatively:

$$\lim_{b\to 0^+} \ln|x| \,|_b^1 = 0 - (-\infty) = \infty$$

Some more problems regarding improper integrals:

$$\int_{-1}^\infty \frac{1}{x^2+1} = \arctan(x) \,|_{-1}^\infty \to \lim_{b\to\infty} \arctan(x) \,|_{-1}^b$$

$$= \arctan(\infty) - \arctan(-1) = \frac{\pi}{2} - \left(-\frac{\pi}{4}\right) = \frac{3\pi}{4}$$

Example 2:

$$\int_2^\infty \frac{1}{x+2}dx = \ln|x+2| \,|_2^\infty = \infty$$

Example 3:

$$\int_0^\infty \cos(x)dx = \sin(x) \,|_0^\infty \to \lim_{b\to\infty} \sin(x) \,|_0^b = \text{DNE}$$

Example 4:

$$\begin{aligned}\int_0^1 \frac{1}{x^2}dx &= \int_0^1 x^{-2} \\ &= \frac{x^{-1}}{-1}\Big|_0^1 = -\frac{1}{x}\Big|_0^1 \\ &= -\frac{1}{1} - \left(-\frac{1}{0}\right) = +\infty\end{aligned}$$

Of course, we should have used $\lim_{b\to 0^+} \frac{1}{x} |_b^1 = \infty$

Example 5:

$$\int_0^1 \frac{1}{\sqrt{x}}dx = \int x^{-\frac{1}{2}} = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x}\,|_0^1 = 2-0 = 2$$

It turns out that somehow when the function is $\frac{1}{\sqrt{x}}$. The area is finite. You can't see that the area is finite, but it is finite. Of course, we should have used $\lim_{b\to 0^+} 2\sqrt{x}\,|_b^1$, but that's just a technical detail.

30.1 Multiple infinite points

Split the integral into several pieces, so that in each piece, only 1 of the 2 bounds is approaching an infinite point. For example:

$$\int_0^\infty \frac{1}{(x-1)^3}$$

Just integrate from 0 to 1, 1 to some finite location C, and C to infinity.

This changes the direction of limits if you choose to evaluate some. In this case:

$$
\begin{aligned}
&\lim_{b\to 1^-}\int_0^b \frac{1}{(x-1)^3}dx \\
&+\lim_{b\to 1^+}\int_b^c \frac{1}{(x-1)^3}dx \\
&+\lim_{b\to\infty}\int_c^b \frac{1}{(x-3)^3}dx \\
&= -\frac{2}{2(x-1)^2}\mid_0^1 + \frac{-2}{2(x-1)^2}\mid_1^c + \left(-\frac{2}{2(x-1)^2}\mid_c^\infty\right) \\
&\dots \\
&= -\infty + \infty + \frac{1}{2(c-1)^3} = \pm\infty
\end{aligned}
$$

When one piece of the integral results in infinity, the entire integral diverges.

The problem is you cannot cancel infinities. You need to split the integral up if there are multiple infinite points.

31 P-series

Let $p > 0$. For what values of p would the integral converge or diverge?

$$\int_1^\infty \frac{1}{x^p}dx$$

Case 1: $p = 1$. Then $\int \frac{1}{x} = \ln|x| \mid_1^\infty = \infty$

$$
\begin{aligned}
&= \int x^{-p} = \frac{x^{-p+1}}{-p+1} \\
&\lim_{b\to\infty} b^{-p+1}
\end{aligned}
$$

$$\lim_{x\to\infty} x^n = \infty,$$

What if $n < 0$? $n = -k$. Then:

$$\lim_{x\to\infty} x^{-k} = \frac{1}{x^k} = 0$$

So, we get (b is like the upper bound of the integral, and avoids us from typing in ∞):

$$\lim_{b\to\infty} b^{-p+1} = \begin{cases} \infty & p+1>0 \text{ diverge} \\ 0 & -p+1<0 \text{ converge} \end{cases}$$

So, when $-p+1<0$ is true, then we get a convergent interval. So:

$$\int_1^\infty \frac{1}{x^p}$$

Will converge if $p>1$ and will diverge if $p\leq 1$.

32 Integral comparison test

Integrals can't always be computed, but we can ask whether an integral is convergent.

Comparison test for improper integrals:

If $f,\ g$ is continuous, and $0\leq f(x)\leq g(x)$ for $x\geq a$, then:

$$0\leq \int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx$$

In particular:

- To check for convergence, look for another integral that is larger and base it on that.
- To check for divergence, look for another integral that is smaller

Example question:

$$\int_1^\infty \frac{\sin x + 2\cos(x)+10}{x^2} \leq \int_1^\infty \frac{1+2+10}{x^2} < \infty$$

32.1 Limit comparison test for improper integrals

If $f,\ g$ are continuous and both test positive for $x\geq a$, IF

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \neq 0 \neq \infty$$

Then:

$$f(x) \approx g(x) \text{ (The rate of growth is similar)}$$

$$\int_a^\infty f(x)dx \text{ converges if and only if } \int_a^\infty g(x)dx \text{ converges}$$

For example:

$$\int \frac{1}{x^2} < \infty,\ \int \frac{1}{2x^2} < \infty,\ \int \frac{1}{x^2+1} < \infty$$

Checking limits:

$$\lim\left(\frac{f}{g}\right) = \lim\left(\frac{1}{x^2}\left(x^2+1\right)\right) = 1$$

Hence why we don't want f to grow faster than g and g to grow faster than f. Similar to them having the same Θ.

Example:

$$\int_0^\infty \frac{(x-7)}{x^2+x+5}dx \approx \int_1^\infty \frac{x}{x^2} = \int \frac{1}{x} = \infty$$

Does $x-7$ and x^2+x+5 grow at the same rate?

$$\frac{(x-7)}{x^2+x+5}$$

Then we do some:

$$\frac{(x-7)}{x^2+x+5} \cdot \frac{x^2}{x}$$

33 Series

Consider the sequence $s_1 = a_1,\ s_2 = a_1 + a_2,\ s_3 = a_1 + a_2 + a_3,\ \cdots$

The sequence $\{s_n\}$ are the partial sums. We say that the series converges if the sequence $\{s_n\}$ converges.

Usually, infinite series are very difficult to calculate, but for geometric series:

$$\sum_{x=0}^{\infty} x^n = \frac{1}{1-x},\ |x|<1$$

Only when series converge, we can perform regular algebra with them like non-infinite series.

Proof.

$$\sum_{x=0}^{\infty} x^n = \lim_{N\to\infty} \sum_{n=0}^{N} x^n$$

First step: $\sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x}$.

Then as $N \to \infty$, $x^{N+1} = 0$, as $|x| < 1$. Then we get $\frac{1}{1-x}$.

$$\sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x} = 1 + x + x^2 + \ldots + x^N$$

$$(1-x)\left(1+x+x^2+\cdots+x^N\right) = 1 - x^{N+1}$$

If we expand the left side:

$$1+x+x^2+\cdots+x^N - x - x^2 - \cdots - x^N - x^{N+1} = 1 - x^{N+1}$$

Now that we have the formula, we can then figure out the sum of infinite series.

■

33.1 When series converge

We can ignore finitely as many terms in the front of the sequence.

33.1.1 The zero test for divergence

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges. This test is almost never useful due to oscillations or divergence for other reasons.

33.1.2 The integral test

For a_n positive and decreasing eventually:

$$\sum_{n=1}^{\infty} a_n \approx \int_1^{\infty} a_n$$

The series converges if and only if the integral converges. Almost never useful due to how hard it is to calculate integrals.

Harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty$$
$$\int_1^{\infty} \frac{1}{x} dx = \ln|x| \;|_1^{\infty} = \ln\infty$$

33.1.3 P-series

$$\sum^{\infty} \frac{1}{n^p} \approx \int^{\infty} \frac{1}{x^p} dx$$

Converges for $p > 1$, diverges for $p \leq 1$

Given this result, what can we say about:

$$\int^{\infty} \frac{1}{n^2 - 1} \approx \sum^{\infty} \frac{1}{n^2}$$

This will converge as $p > 1$. When a_n is a rational function, we only need to take the leading powers on the top and bottom to create a new series $\sum^{\infty} b_n$.

33.1.4 Limit comparison test

For $a_n,\ b_n$, positive eventually:

$$\lim_{n\to\infty} \frac{a_n}{b_n} \neq 0 \neq \infty \Rightarrow \sum^{\infty} a_n \approx \sum^{\infty} b_n$$

Example: $a_n = n^2,\ b_n = n^2 - 1$

$$\frac{a_n}{b_n} = \frac{1}{n^2 - 1} \cdot \frac{n^2}{1} = 1$$

33.1.5 Comparison tests

Suppose $a_n \leq b_n$ and positive eventually.

If $\sum^\infty b_n$ converges then $\sum^\infty a_n$ converges.

If $\sum^\infty b_n$ diverges then $\sum^\infty a_n$ diverges...?

For example:

$$\sum^\infty \frac{\arctan(n)}{n^2} \leq \sum^\infty \frac{\pi}{2} \cdot \frac{1}{n^2}$$

This converges.

Using $\ln(n) < n^p$ for any $p > 0$ (pick p to be small):

$$\sum^\infty \frac{\ln(n)}{n^2} < \sum^\infty \frac{\sqrt{n}}{n^2} = \sum^\infty \frac{1}{n^{\frac{3}{2}}}$$

Which converges.

33.1.6 Ratio test

For a_n positive eventually, compute:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$$

If $L < 1$, then $\sum^\infty a_n$ converges.

If $L > 1$, then $\sum^\infty a_n$ diverges.

If $L = 1$, it is inclusive.

Example:

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= L < 1 \Rightarrow \\ &\lim_{n\to\infty} \left(\frac{1}{2}\right)^{n+1} \cdot \frac{2^n}{1} = \frac{1}{2} \\ a_{n+1} &\approx \frac{1}{2} a_n \end{aligned}$$

$L = 1$ example:

$$\frac{a_{n+1}}{a_n} = \frac{1}{n+1} \cdot \frac{n}{1} = 1$$

Not a huge difference between a_{n+1} and a_n. So, this test does not give us any conclusions, as we don't know how fast this is decreasing. This means this test does not work for polynomials, but it does for exponentials and factorials. For polynomials, use the p-series.

$L > 1$ example: If $\frac{a_{n+1}}{a_n} > 1$, it's getting larger and larger. The series is divergent.

33.2 Exponential vs. Polynomial series

Exponential series can be calculated easily. Polynomials, on the other hand, seem to be very hard.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

33.3 Convergence of series involving positive or negative terms

$$\sum a_n,\ a_n \sim \frac{1}{n} = \frac{1}{1} - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \frac{1}{10} - \frac{1}{11} \pm \cdots$$

In general, this is a difficult problem. Thus, it is much easier to just get rid of all the negative signs. Now, with this series, the negative terms are somehow cancelling some of the positive terms. This means it is possible for this series to not be infinite. We've said that:

$$\sum a_n \leq \sum |a_n|$$

There may be cancellation on the left side, for our above series. But there is no cancellation on the right.

Test 5: Absolute Convergence Test

$$\text{if} \sum^{\infty} |a_n| \text{ converges} \Rightarrow \text{then} \sum^{\infty} a_n \text{ converges}$$

When $\sum^{\infty} |a_n|$ converges, we say $\sum^{\infty} a_n$ is **absolutely convergent.** Also, if $a_n > 0\ \forall n$, convergent and absolutely convergent mean the same thing.

There is a distinction when $\exists a_n,\ a_n < 0$. In the event where the original series involving positive and negative terms converge, but if we absolute value a_n, then we say that it is conditionally convergent. In formal logic:

$$\sum^{\infty} a_n \text{ converges and } \sum^{\infty} |a_n| \text{ diverges}$$
$$\Rightarrow \sum^{\infty} a_n \text{ is conditionally convergent}$$

An example of an erratic sequence:

$$\sum^{\infty} \left| \frac{\sin(n)}{n^2} \right| \leq \sum^{\infty} \frac{1}{n^2} \text{ which converges}$$

The sine function is erratic, as its period is 2π, which happens to not be a rational number or even an integer. Determining a pattern for $\sin(n)$ being positive or negative with increasing ns can be very hard to determine by hand, so the distributions of + and $-$s are very erratic.

Well:

$$\sin(n) \leq 1 \Rightarrow$$
$$\sum^{\infty} \left| \frac{\sin(n)}{n^2} \right| \leq \sum^{\infty} \frac{1}{n^2} \Rightarrow \sum^{\infty} \frac{\sin(n)}{n^2} \text{ converges absolutely}$$

By the theorem we stated in the beginning of this subsection.

33.3.1 The alternating series test

A special case is when the positive and negative terms alternate in turn:

$$a_1 - a_2 + a_3 - a_4 \ldots$$

This can be written with the notation $(-1)^n$ or $(-1)^{n+1}$.

For example:

$$\sum (-1)^n \times \frac{1}{n} \sim \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

We know:

$$\sum^{\infty}\left|\frac{(-1)^n}{n}\right| = \sum^{\infty}\frac{1}{n} = \infty$$

(When you have infinitely as many things, you cannot change the order in which you add terms).

And also, we have this theorem, which requires us to only compute the limit of a_n:

$$\lim_{(n\to\infty)} a_n = 0 \text{ if and only if } \sum^{\infty}(-1)^n a_n \text{ converges}$$

In the event, when you have positive and negative terms alternating, we don't need to worry how fast the terms head towards zero. Essentially what happens, is that even if 0 is being approached slowly, with adjacent positive and negative terms, we still get convergence.

$$\begin{aligned}&\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\dots\\&=\frac{1}{2}+\frac{1}{12}+\dots\end{aligned}$$

Because $\lim_{n\to\infty}\frac{1}{n} = 0$ we know that $\sum^{\infty}\frac{(-1)^n}{n}$ converges, using the theorem from above.

The test also holds with $\cos(n\pi) = (-1)^n = \sin\left(\left(n+\frac{1}{2}\right)\pi\right) = (-1)^{n+1}$

A note about looping convergence $a_1 - a_2 + a_3 - a_4$.

Add a_1 to 0. Then subtract a_2. But since you said the sequence is decreasing:

$$a_1 > a_1 - a_2 > 0$$

And then you add back a_3. Then you take back most of what you subtracted back in a_2. So:

$$a_1 > a_1 - a_2 + a_3 > a_1 - a_2 > 0$$

The cycle repeats.

And that's another reason to why $\sum^{\infty}\frac{1}{n}$ converges.

34 Summary of Tests for Series

Terms lower under $+$ can be ignored, but terms under multiplication can't be ignored. You can say that the numerator / denominator is order something.

Combining all the tests, the steps for determining convergence of a series $\sum^{\infty} a_n$:

1. Take absolute value of a_n: $|a_n|$
2. If there are exponents or factorials, apply the ratio test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$ or the root test: $\lim_{n\to\infty}\sqrt[n]{|a_n|}$
    a. $L < 1$ concludes $\sum^{\infty} a_n$ absolutely converges.
    b. $L > 1$ concludes $\sum^{\infty} a_n$ diverges.
    c. $L = 1$ and we are unclear.
3. Upper bounds:
    a. If $\ln(n)$ in the numerator, compare $1 < \ln(n) < n^p$ with p small.
    b. $\ln(n)$ in the denominator, compare $1 > \frac{1}{\ln(n)} > \frac{1}{n^p}$ with p small.
    c. For sine and cosine in the numerator, conclude $|\sin(n)| \leq 1,\ |cos(n)| \leq 1.$
    d. For arctangent in the numerator, compare $\arctan(n) < \frac{\pi}{2}$
4. For series in rational function form, take b_n to be leading powers of top and bottom. Attain $b_n = \frac{1}{n^p}$.
    a. $p > 1 \Rightarrow \sum^{\infty} a_n$ absolutely converges.
    b. $p \leq 1 \cdot \sum^{\infty} |a_n|$ diverges, but $\sum^{\infty} a_n$ may still converge.
5. If the original series can be written in the form $\sum^{\infty} (-1)^n a_n$, compute $\lim_{n\to\infty} a_n = L$.
    a. $L = 0 \Rightarrow \sum^{\infty} (-1)^n a_n$ conditionally converges.
    b. $L \neq 0 \Rightarrow \sum^{\infty} (-1)^n a_n$ diverges.

$$\sum^{\infty} a_n \text{ converges and } \sum^{\infty} |a_n| \text{ diverges}$$

$$\Rightarrow \sum^{\infty} a_n \text{ is conditionally convergent}$$

Otherwise, it is absolutely convergent.

34.1 Linearithmic on the denominator

For $a_n = \frac{1}{n \cdot \ln(n)}$, there is one power of n in the bottom so use the integral test: $\int \frac{1}{x \ln(x)} dx$

Use small angle approximation on (note that $n^p \to \infty \Rightarrow \frac{1}{n^p} \to 0$):

$$\sin\left(\frac{1}{n^p}\right) \approx \frac{1}{n^p} \approx \tan\left(\frac{1}{n^p}\right) \approx \arcsin\left(\frac{1}{n^p}\right) \approx \arctan\left(\frac{1}{n^p}\right)$$

34.2 The very obvious

If $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum^{\infty} a_n$ diverges.

34.3 Some examples

Special case 1: From something you know, try to get to the original function.

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{2+\cos(n)} \\
-1 &\leq \cos(n) \leq 1 \\
1 &\leq 2+\cos(n) \leq 3 \\
\frac{1}{3} &\leq \frac{1}{2+\cos(n)} \leq 1 \\
\lim_{n\to\infty} \frac{1}{2+\cos(n)} &\neq 0
\end{aligned}$$

Divergent.

Special case 2:

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{1}{n \ln(n)} &\approx \int_2^{\infty} \frac{1}{x \ln(x)} dx \\
u = \ln(x),\ du &= \frac{1}{x} dx \\
\int \frac{1}{u} du &= \ln|u| = \ln|\ln(x)| \,\Big|_a^{\infty} \\
&= \lim_{b\to\infty} \ln|\ln b| = \infty
\end{aligned}$$

Divergent.

(3)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$
$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum \frac{1}{n}$$

Because $p = 1 \leq 1$, this one diverges.

Consider $\lim_{n\to\infty} \frac{1}{n} = 0$. Then we can tell the series of (3) is conditionally convergent. Conditionally converges $\Rightarrow$ converges.

(4)

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Converges due to the p-series

(5)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2-1},\ \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2-1} \right| \approx \sum_{n=1}^{\infty} \left| \frac{1}{n^2} \right| = \sum \frac{1}{n^2}$$

This absolutely converges. Meaning it converges.

(6)

$$\sum_{n=1}^{\infty} \frac{n^2+4n}{\sqrt{n^5+4n^4+2n}}$$
$$\sum_{n=1}^{\infty} \left| \frac{n^2+4n}{\sqrt{n^5+4n^4+2n}} \right|$$
$$\approx \sum_{n=1}^{\infty} \left| \frac{n^2}{\sqrt{n^5}} \right|$$
$$= \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}},\ p = \frac{1}{2} \leq 1$$

Diverges due to the p-series.

(7)

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}\sin(n)}{n^2}$$
$$\sum_{n=1}^{\infty} \left|\frac{\sqrt{n}\sin(n)}{n^2}\right| \leq \sum_{n=1}^{\infty} \left|\frac{\sqrt{n}}{n^2}\right| = \sum_{n=1}^{\infty} \left|\frac{1}{n^{\frac{3}{2}}}\right|$$

Converges due to the p-series. This converges absolutely.

(8)

$$\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$$
$$\sum_{n=1}^{\infty} \left|(-1)^n \sin\left(\frac{1}{n}\right)\right| = \sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right) \approx \sum_{n=1}^{\infty} \frac{1}{n}$$

Divergent due to the p-series. But does it conditionally converge?

$$\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0 \text{ converges}$$

So, it conditionally converges.

(9)

$$\sum \left|(-1)^n \sin\left(\frac{1}{n^2}\right)\right| = \sum \sin\left(\frac{1}{n^2}\right) \approx \sum \frac{1}{n^2}$$

Converges absolutely.

34.3.1 An aside on small angle approximation

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right) \approx \sum_{n=1}^{\infty} \frac{1}{n^2}$$
$$\lim_{n\to\infty} \sin\left(\frac{1}{n^2}\right) \cdot n^2 \approx \frac{1}{n^2} \cdot n^2$$

(10)

$$\sum_{n=1}^{\infty} \frac{(n+1)2^n}{n!}$$

Time for the ratio test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(n+2)2^{n+1}}{(n+1)!}\cdot\frac{n!}{(n+1)2^n}\right| = \left(\frac{n!}{(n+1)!}\cdot\frac{n+2}{n+1}\cdot\frac{2^{n+1}}{2^n}\right)\\ &= \left(\frac{1}{n+1}\cdot\frac{n+2}{n+1}\cdot\frac{2^{n+1}}{2^n}\right) = 0\cdot 1\cdot 2 = 0\end{aligned}$$

Meaning this converges absolutely.

The point of the ratio test is to take the next term and divide it by the previous term. Then, you can say that the series decreases fast enough.

(11)

$$\sum_{n=1}^{\infty} \frac{n!}{2^{3n}}$$

Try

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(n+1)!}{2^{3n+3}}\cdot\frac{2^{3n}}{n!}\right|\\ &= \left|\frac{(n+1)}{2^3}\right|\end{aligned}$$

Diverges

(12)

$$\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$$

$$\begin{aligned}&\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\\ &= \frac{(n+1)!}{(2(n+1))!}\cdot\frac{(2n)!}{n!} = \frac{(n+1)n!}{(2n+2)(2n+1)(2n)!}\cdot\frac{(2n)!}{n!} = \frac{n+1}{(2n+2)(2n+1)} \approx \frac{n}{n^2} \approx \frac{1}{n}\end{aligned}$$

Diverges

(13)

$$\sum_{n=1}^{\infty} \frac{n!(2n)!}{(3n)!}$$

The ratio test:

$$\begin{aligned}
&\frac{(n+1)!(2n+2)!}{(3n+3)!} \cdot \frac{(3n)!}{n!(2n)!} \\
&= \frac{(3n)!}{(3n+3)!} \cdot \frac{(n+1)!}{n!} \cdot \frac{(2n+2)!}{(2n)!} \\
&= \frac{(3n)!}{(3n+3)(3n+2)(3n+1)(3n)!} \cdot \frac{(n+1)n!}{n!} \cdot \frac{(2n+2)(2n+1)(2n)!}{(2n)!} \\
&= \frac{1}{(3n+3)(3n+2)(3n+1)} \cdot \frac{(n+1)}{1} \cdot \frac{(2n+2)(2n+1)}{1}
\end{aligned}$$

There are 3 powers of the numerator and 3 powers on the denominator.

Expand:

$$\frac{\left(2n^2+4n+2\right)(2n+1)}{(9n^2+15n+6)(3n+1)} = \frac{4n^3+\cdots}{27n^3+\cdots} = \frac{4}{27} < 1$$

It's convergent, absolutely. You can calculate this quickly by only treating the leading coefficients as existing:

$$\frac{(1\cdot 2\cdot 2)n^3}{(3\cdot 3\cdot 3)n^3}$$

(14)

$$\sum_{n=1}^{\infty} \frac{1}{n^n}$$

Let's try this:

$$\lim_{n\to\infty} \frac{1}{(n+1)^{n+1}} \cdot n^n$$

This becomes problematic. If you have n^n as your series, if you try to use the ratio test, the answer may not be obvious. This means we have to factor out $n+1$ in the exponent:

$$\frac{1}{(n+1)^{n+1}} \cdot \frac{1}{n+1} = \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1} = \frac{1}{e} \cdot \frac{1}{n+1} = 0 < 1$$

Okay, that isn't the best decision. However, since n^n sits on the top of the growth hierarchy, we can do comparisons:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} > \sum_{n=1}^{\infty} \frac{1}{n^n}$$

Because $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so does $\sum_{n=1}^{\infty} \frac{1}{n^n}$. However, if we want to do the ratio test:

Choose strategically, but make sure to choose something high in the hierarchy that retains the information that n^n grows very fast. For example:

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n^n} : \sum_{n=1}^{\infty} |a_n| \\
&= \sum_{n=1}^{\infty} \frac{2^n}{n^n} \sim \sum_{n=1}^{\infty} \frac{2^n}{3^n},\ x \\
&= \frac{2}{3},\ |x| < 1 \Rightarrow \text{converges}
\end{aligned}$$

Try this again:

$$\sum_{n=1}^{\infty} \frac{1}{n^n}$$

$$\lim_{n\to\infty} \left(\frac{n^n}{(n+1)^{n+1}}\right)$$

34.3.2 Evaluating n to the n

From $\frac{1}{n^n}$: Do the ratio test. We get $\left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1}$. Also: $\frac{1}{(n+1)^{n+1}} \cdot n^n = \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1}$

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{1}{e}$$
$$\left(\frac{n}{n+1}\right) = e^{\ln\left(\frac{n}{n+1}\right)}$$
$$\left(\frac{n}{n+1}\right)^n = \left(e^{\ln\left(\frac{n}{n+1}\right)}\right)^n = e^{n\cdot\ln\left(\frac{n}{n+1}\right)}$$

So what's $\lim_{n\to\infty} n\cdot\ln\left(\frac{n}{n+1}\right)$? Perhaps use L'Hopital's rule. We have the form of something like $\infty\cdot 0$.

$$\frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}}$$

Take the derivative of both sides:

$$\frac{\frac{1}{\frac{n}{n+1}}\cdot\frac{(n+1)-n}{(n+1)^2}}{-\frac{1}{n^2}} = -\frac{n+1}{n}\cdot\frac{1}{(n+1)^2}\cdot n^2 = -\frac{1}{n}\cdot\frac{1}{n+1}\cdot n^2 = -\frac{n}{n+1} = -1$$

The answer to this exponent limit is -1 so the answer is $e^{-1} = \frac{1}{e}$

34.3.3 The compound interest limit

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$
$$\left(\frac{n+1}{n}\right)^n = \left(\frac{n}{(n+1)^{-1}}\right)^n = \left(\left(\frac{n}{n+1}\right)^n\right)^{-1} = e$$

Now, why is this e? $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

It is not defined directly. We have to define ln:

$$\ln(x) = \int_1^x \frac{1}{t}dt$$

But that was not how e was originally discovered. It was done through this formula.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

100% interest? With $1, one year later, you have $1+$1=$2.

Now, you want a different plan. What if you withdraw mid-year? Well, then I withdraw $1.50. Keep it in the bank. $\$1.50$. Half a year later – I was supposed to receive 100% interest per year. So you get $1.5 + $0.75. The total you now have is $2.25. While previously you had $2 when the arrangement you had with the bank has always been the same.

Say compounding happens $\frac{1}{4}$ years. Then precisely at the $\frac{1}{4}$ th point of the year, my money gets multiplied by 1:

$$\begin{aligned}
&\$1\\
&\$1+\$0.25\\
\$1.25+\frac{1}{4}\$1.25=\$1.25+\$0.3125= &
\end{aligned}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Must exist. So call it to be e.

Why is this significant? These numbers are computationally infeasible at high values.

Say we have $r\%$ interest. For example, $r=0.05$.

Then $\$1+\$0.05=1+\frac{r}{1}$

So, for continuous compounding, we would have:

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n$$

If you want an initial deposit:

$$P_1\approx\lim_{n\to\infty}P_0\left(1+\frac{r}{n}\right)^n=P_0e^r$$

35 Taylor series

To approximate the function $f(x)$ with a polynomial.

We choose the point a as the center of expansion. We require the polynomial to have the same derivatives as the function at a. This infinite polynomial is called the **Taylor (power) series of $f(x)$.**

$$f(x) \approx P(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Note that $f'(a),\ f''(a),\ f^{(n)}(a)$ are constants. If $a = 0$, take the derivative and plug in 0.

$$\begin{aligned}
P(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \\
P'(x) &= f'(a) + f''(a)(x-a) + \frac{f'''(a)}{2!}(x-a)^2 + \cdots \\
P''(x) &= f''(a)(1) + \frac{f'''(a)(x-a)}{2!} + \cdots \\
P'(a) &= f'(a) \\
P''(a) &= f''(a)
\end{aligned}$$

We defined the derivative of the Taylor polynomial at a point to be exactly the derivative of the original function... at a. The same applies for the second derivative, and so on. The $n!$ is here to cancel out the powers that come out from $(x-a)^n$.

For $a = 0$, the most important Taylor Series to know are:

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
\sin(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \\
\cos(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots,\ |x| < 1
\end{aligned}$$

Usually, the approximation only holds in a symmetric interval centered at a. This is called the interval of convergence. Half the length of the interval is called the radius of convergence.

It turns out, that the approximation for $e^x, \sin(x)$ and $\cos(x)$ works for every x. Oftentimes, these functions are just defined like these.

Since the formula holds for any x inside the interval of convergence, we can substitute x with any other quantity.

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{\left(x^2\right)^n}{n!} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!}$$
$$\frac{1}{3-x} = \frac{1}{3} \cdot \frac{1}{1-\frac{x}{3}} = \frac{1}{3}\sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n = \frac{1}{3}\left(1 + \frac{x}{3} + \left(\frac{x}{3}\right)^2 + \left(\frac{x}{3}\right)^3 + \cdots\right)$$

35.1 Derivatives and Integrals

$$\int \left(\sum a_n\right) dx = \sum \int a_n dx$$
$$\frac{d}{dx}\sum a_n = \sum \frac{d}{dx} a_n$$

Example:

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} \left(-x^2\right)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Take the integral of both sides:

$$\int \frac{1}{1+x^2} dx = \arctan(x) = C + \sum_{n=0}^{\infty} \int (-1)^n x^{2n},\ |x| < 1$$
$$\arctan(x) = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

This is the case where C is needed. To solve for C, plug in $x = 0,\ x^{2n+1} = 0$, so $\arctan(0) = 0 = C + \sum 0 = 0$

Similarly:

$$
\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\
\int \frac{1}{1-x} dx &= C + \sum_{n=0}^{\infty} \int x^n dx,\ |x| < 1 \\
-\ln(1-x) &= C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1},\ |x| < 1 \\
-\ln(1-x) &= C + \sum_{k=1}^{\infty} \frac{x^k}{k} \text{ to get rid of } n+1,\ k = n+1s
\end{aligned}
$$

$$
x = 0,\ -\ln(1) = C + \sum_{n=0}^{\infty} \frac{0^{n+1}}{0+1} = 0,\ C = 0
$$

$$
\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3}
$$

35.2 Radius of convergence

Involves n and x as a variable. Has an interval of convergence and radius of convergence. For example, find the radius of convergence for this in the power series:

$$
\sum_{n=1}^{\infty} n(x-2)^n
$$

Ratio test:

$$
\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)(x-2)^{n+1}}{n(x-2)^n} \right| = 1 \times |x-2| \\
|x-2| < 1 \Rightarrow 1 < x < 3
\end{aligned}
$$

Another example:

$$
\begin{aligned}
\sum_{n=2}^{\infty} \frac{x^{3n}}{2^n} \\
\left| \frac{x^{3n+3}}{2^{n+1}} \cdot \frac{2^n}{x^{3n}} \right| = \left| \frac{1}{2} x^3 \right| < 1 \Rightarrow \left| x^3 \right| < 2 \Rightarrow |x| < \sqrt[3]{2}
\end{aligned}
$$

Then 2 is the radius of convergence, and the interval of convergence is $-\sqrt[3]{2} < x < \sqrt[3]{2}$
Isolate x when it is multiplied by a constant? Do it.

$$\begin{aligned}
&\sum_{n=3}^{\infty} (-1)^n (x-3)^{2n} \\
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(-1)^{(n+1)}(x-3)^{2n+2}}{(-1)^n (x-3)^{2n}}\right| \\
\left|(x-3)^2\right| &< 1 \\
|x-3| &< \sqrt{1} = 1 \\
2 &< x < 4
\end{aligned}$$

35.3 Taylor series of some functions:

$$\cos\left(x^4\right) = 1 - \frac{x^8}{2!} + \frac{x^{16}}{4!} + \cdots,\ x \in \mathbb{R}$$

Note that $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$$\sin(2x) = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!}$$

$$e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \cdots = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$\begin{aligned}
e^x - e^{-x} &= 2x + \frac{2x^3}{3!} + \cdots \\
\frac{e^x - e^{-x}}{2} &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sinh(x) \\
e^x + e^{-x} &= 2 + 2\left(\frac{x^2}{2!}\right) + 2\left(\frac{x^4}{4!}\right) + \cdots \\
\frac{e^x + e^{-x}}{2} &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} = \cosh(x)
\end{aligned}$$

35.3.1 Euler's formula

$$
\begin{aligned}
e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots \\
&= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \cdots
\end{aligned}
$$

Note that

$$
i \cdot \sin(x) = ix - \frac{ix^3}{3!} + \frac{ix^5}{5!} - \cdots
$$

So:

$$
\cos(x) + i \cdot \sin(x) = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots
$$

Note:

$$
\begin{aligned}
\frac{\left(e^{ix} + e^{-ix}\right)}{2} &= \cos(x) \\
\frac{e^{ix} - e^{-ix}}{2i} &= \sin(x)
\end{aligned}
$$

35.4 Taylor series expansion

Find the Taylor series centered at $a = 0$:

$$
\frac{1}{x^2 - 3x + 2}
$$

We'll have to do partial fractions. If the quadratic is irreducible, then you can complete the square (don't try otherwise). $1 + x^2 = 0$ is the quintessential irreducible quadratic, where its roots are $\pm i$.

$$
\begin{aligned}
\frac{1}{x^2-3x+2} &= \frac{1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} \\
&= \frac{A}{x-1} \cdot \frac{x-2}{x-2} + \frac{B}{x-2} \cdot \frac{x-1}{x-1} \\
0x+1 &= (A+B)x + (-2A-B) \\
A+B &= 0,\ -2A-B = 1 \\
A &= -B \\
&\vdots \\
B &= 1,\ A = -1 \\
&-\frac{1}{x-1} + \frac{1}{x-2} \\
&= \frac{1}{1-x} - \frac{1}{2-x} = \frac{1}{1-x} - \frac{1}{2} \cdot \frac{1}{1-\frac{x}{2}} \\
&= \sum_{n=0}^{\infty} x^n - \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n \text{ given } |x| < 1 \text{ and } \left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 1
\end{aligned}
$$

The series will be the following above as long as the above conditions are met. If x is multiplied by anything in $\frac{1}{1-x}$, then x is swapped with what x is multiplied by.

$$\frac{\sin(x)}{x}$$

We know:

$$
\begin{aligned}
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \\
\frac{\sin(x)}{x} &= 1 - \frac{x^2}{2!} + \frac{x^4}{5!} - \cdots
\end{aligned}
$$

Now, for $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n},\ |x| < 1$

So what is $\ln(1+x)$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

So what about:

$$f(x) = \int_0^x e^{-t^2} dt$$

So $f'(x) = e^{-x^2}$. If we look for the Taylor series for e^{-x^2} and integrate it, we will get back f.

$$\begin{aligned}
e^{-x^2} &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
\int_0^x & e^{t^2} dt \\
&= \int_0^x \sum_{n=0}^{\infty} \frac{t^n}{n!} dt = f(x) \\
&= C + \sum_{n=0}^{\infty} \int_0^x \frac{(-1)^n t^{2n}}{n!} dt \\
&= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(n!)(2n+1)}
\end{aligned}$$

To find C, plug in $x = 0$.

$$\begin{aligned}
f(0) = 0 = C + \sum_{n=0}^{\infty} \frac{(-1)^n 0^{2n+1}}{(n!)(2n+1)} \\
= C + \sum_{n=0}^{\infty} \frac{0}{(n!)(2n+1)} = C \Rightarrow C = 0
\end{aligned}$$

As the integral from 0 to 0 is 0, always

Meaning

$$\int_0^x e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(n!)(2n+1)}$$

35.5 Convergence or divergence on the boundary of the interval

The series may converge or diverge on the boundary of the interval of convergence. How would we determine if the series converges or diverges on the boundary?

Test for convergence individually on the boundaries, from the series itself. Do not use ratio tests.

The moment when the ratio test fails, other steps will have to be taken. Working with a sum involving factorial, p-series will not work.

To check whether it converges or diverges is very complicated. For example:

$$\sum_{n=3}^{\infty} (-1)^n (x-3)^{2n}$$

Ratio test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-3)^{2n+2}}{(-1)^n(x-3)^{2n}}\right| \\ &= (x-3)^2 < 1 \Rightarrow |x-3| < 1 \Rightarrow 2 < x < 4\end{aligned}$$

What will always happen, is for x values in the boundary of the interval, the series may converge or diverge. That is, when $x = 2$ or $x = 4$. To check for this, we would need to do something else. Considering $x = 4$, the ratio will exactly be 1, which may not be possible to run the ratio test with. First check $x = 4$. Our series is of the form:

$$\sum_{n=3}^{\infty} (-1)^n (4-3)^{2n} = \sum_{n=3}^{\infty} (-1)^n (1)^{2n} \text{ diverges}$$

If we keep going to infinity, our total is not going closer to a particular number. The more we add, the closer and closer we get to 2. As we continue to add 1 and subtract 1, we are not getting closer to any number. We are bounding between 0 and 1. This is why it counts as divergent.

We may plug in $x = 2$:

$$\sum_{n=3}^{\infty} (-1)^n (-1)^{2n} = \sum_{n=3}^{\infty} (-1)^n \text{ diverges}$$

This means that $(2,\ 4)$ is the interval of convergence.

For this course, boundaries should be inspected, but it may not be done for future courses.

Another one:

$$\sum_{n=1}^{\infty} n(x-2)^n$$

$$\lim_{n\to\infty}\left|\frac{(n+1)(x-2)^{n+1}}{n(x-2)^n}\right| = |x-2| < 1$$
$$\Rightarrow |x-2| < 1$$

Do some tests:

$$x = 1$$

$$\sum_{n=1}^{\infty} n(-1)^n, \lim_{n\to\infty} n \neq 0 \Rightarrow \text{diverges}$$

$$x = 3 \Rightarrow \sum_{n=1}^{\infty} n \Rightarrow \text{diverges}$$

Let's change the question up:

$$\sum_{n=1}^{\infty} \frac{1}{n}(x-2)^n$$

What would change? Not much.

$$\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}n}{(n+1)(x-2)^n}\right| = |x-2| < 1$$
$$1 < x < 3$$

Check $x = 3$

$$\sum_{n=1}^{\infty} \frac{1}{n}(1)^n \ \Rightarrow \text{diverges}$$

Check $x = 1$

$$\sum_{n=1}^{\infty}\left|\frac{1}{n}(-1)^n\right| = \sum_{n=1}^{\infty}\frac{1}{n} \Rightarrow \text{div.}$$

$$\sum_{n=1}^{\infty}\frac{1}{n}(-1)^n \Rightarrow \text{cond. conv.}$$

The interval of convergence is $[1,\ 3)$

The same question, but:

$$\sum_{n=1}^{\infty}\frac{1}{n^2}(x-2)^n$$

$$\lim_{n\to\infty}\left|\frac{1}{(n+1)^2}(x-2)^{n+1}\cdot\frac{n^2}{(x-2)^n}\right|$$

$$\lim_{n\to\infty}|x-2| = |x-2| < 1$$

$$1 < x < 3$$

Testing boundaries:

$$x = 3$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}(3-2)^n \Rightarrow \text{conv}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^n$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \text{conv}$$

If a series is absolutely convergent, it is convergent. The interval of convergence is $[1,\ 3]$.

Anything out of the interval of convergence, the sum doesn't exist, so there's no approximation going on.

35.6 Interval of convergence for the geometric series

If we try to plot $\frac{1}{1-x}$, we have a function that looks like

We can only approximate the function up to $|x| < 1$. The function approaching infinity gives us the radius of convergence. In the real plane and the complex plane, anything inside the circle formed by the radius of convergence will converge.

36 Taylor series at a given value

To find the Taylor series of the functions centered at the given value of a, you'll need $x-a$.

$$\begin{aligned}
e^x &= e^{(x-a)+a} = e^{x-a} \cdot e^a \\
\Rightarrow e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{(x-a)^n}{n!} \cdot e^a \\
&= \sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n \\
\frac{1}{1-x} &= \frac{1}{1-((x-3)+3)} = \frac{1}{-2} \cdot \frac{1}{1-\frac{x-3}{-2}} \\
&= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x-3}{-2}\right)^n
\end{aligned}$$

Also: $\left|\frac{x-3}{-2}\right| < 1 \Rightarrow |x-3| < 2$

Note that x will work for every interval as exponentials' intervals...

36.1 To deal with a constant

$$f(x) = f\left((x-a)+a\right)$$

Example with $\sin(x)$ at $a = \frac{\pi}{4}$. The sum of angles formula must be used:

$$
\begin{aligned}
\sin(x) &= \sin\left(\left(x-\frac{\pi}{4}\right)+\frac{\pi}{4}\right) \\
&= \sin\left(x-\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)+\cos\left(x-\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(x-\frac{\pi}{4}\right)^{2n+1}\cdot\frac{1}{\sqrt{2}} \\
&+ \sum_{n=0}^{\infty}\frac{-1}{(2n)!}\left(x-\frac{\pi}{4}\right)^{2n}\cdot\frac{1}{\sqrt{2}}
\end{aligned}
$$

36.2 Exact value of some series

What does $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converge to? $\frac{\pi^2}{6}$. We'll never get this result.

The geometric series formula is:

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

Which happens to be the Taylor series. If you encounter a series in the form

$$\sum_{n=1}^{\infty}nx^n$$

Work backwards.

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

Take the derivative of both sides:

$$-\frac{1}{(1-x)^2}\cdot(-1)=\sum_{n=1}^{\infty}nx^{n-1}$$

Keep in mind: $\frac{d}{dx}\sum_{n=0}^{\infty}x^n=\frac{d}{dx}\left(1+x+x^2+x^3\right)=1+2x+3x^2+4x^3+\cdots=\sum_{n=1}^{\infty}nx^{n-1}$ meaning we can get rid of the starting term. That is when the former starting term ends up as 0 after the derivative is taken.

Multiply x on both sides:

$$\frac{x}{(1-x)^2} = x\sum_{n=1}^{\infty} nx^{n-1}$$
$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n$$

There we go.

Another one:

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$xe^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!!}$$

Another one:

$$\sum_{n=0}^{\infty} \frac{x^n}{(n+2)n!}$$
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$xe^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

Take the derivative of both sides with respect to x:

$$e^x + xe^x = \sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!}$$

Another example to get $n+$something on the denominator:

$$\sum_{n=0}^{\infty} \frac{x^n}{(n+2)n!}$$

$$
\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
xe^x &= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}
\end{aligned}
$$

$$
xe^x - e^x = C + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)n!}
$$

When you integrate both sides, add the constant of integration. Most of the time, it is 0, but best to check it every time by setting x to 0.

$x = 0$:

$$
\begin{aligned}
0e^0 - e^0 = C + \sum_{n=0}^{\infty} \frac{1}{n+2} \cdot \frac{0^{n+2}}{n!} &= C + 1 + 0 \\
-1 = C
\end{aligned}
$$

Be a bit careful when you think sums evaluate to zero – there may be causes where something might be raised to the power of zero. You may need to expand sums first.

Some things:

$$
\begin{aligned}
xe^x - e^x + 1 &= \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)n!} \\
\frac{xe^x - e^x + 1}{x^2} &= \sum_{n=0}^{\infty} \frac{x^n}{(n+2)n!}
\end{aligned}
$$

Hardest example:

$$
\begin{aligned}
\sin(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \\
x\sin(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+2} \\
\sin(x) + x\cos(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2n+2) x^{2n+1}
\end{aligned}
$$

36.3 Finding limits using Taylor series

$$\begin{aligned}
&\lim_{x\to 0}\frac{6\sin(x)-6x+x^3}{x^5}\\
&=\lim_{x\to 0}\frac{6\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}\cdots\right)-6x+x^3}{x^5}\\
&=\lim_{x\to 0}\frac{6\left(\frac{x^5}{5!}-\frac{x^7}{7!}\cdots\right)}{x^5}\\
&=\lim_{x\to 0}\frac{6}{5!}-\frac{6x^2}{7!}+\frac{6x^4}{9!}-\cdots\\
&=\frac{6}{5!}=\frac{1}{20}
\end{aligned}$$

Because $x=0$, all terms that are multiplied by x cancel out.

Another:

$$\begin{aligned}
&\lim_{x\to 0}\frac{e^{x^2}-\cos(2x)-3x^2}{x^2\sin\left(x^2\right)}\\
&=\lim_{x\to 0}\frac{\left(1+x^2+\frac{x^4}{2}+\frac{x^6}{3!}+\cdots\right)-\left(1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\cdots\right)-3x^2}{x^2\left(x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\cdots\right)}
\end{aligned}$$

37 Telescoping series

A **telescoping series** is when adjacent positive and negative terms cancel so the Nth partial sum: $\sum_{n=1}^{N} a_n$ would only consist of the first and the last term, and we can compute the series by taking the limit $N\to\infty$.

$$\sum_{n=1}^{\infty} a_n=\lim_{N\to\infty}\sum_{n=1}^{N} a_n$$

Example:

$$\sum_{n=1}^{\infty} \arctan(n) - \arctan(n+1) = \lim_{N\to\infty} \sum_{n=1}^{N} \arctan(n) - \arctan(n+1)$$

$$\begin{aligned}&= \arctan(0) - \arctan(1) + \arctan(1) - \arctan(2) + \arctan(2)\\ &- \arctan(3) + \cdots + \arctan(N) - \arctan(N+1)\end{aligned}$$

Because we're adding one positive and one negative, the adjacent terms cancel out. We're left out with the first and last term:

$$\begin{aligned}&= \arctan(0) - \arctan(N+1)\\ &= -\frac{\pi}{2}\end{aligned}$$

38 Power series

$$g(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

is called a power series.

For any power series $g(x)$, there exists a unique radius of convergence $R \geq 0$.

1. The power series $g(x)$ converges absolutely for $|x-a| < R$ (in the circle).
2. The power series $g(x)$ diverges for $|x-a| > R$ (outside the circle).
3. You must check whether the series diverges or converges on the boundaries.

If $R > 0$, then the power series $g(x)$, as a function of x, is analytic in $|x-a| < R$ and $g'(x)$ is given by term-by-term differentiation:

$$g'(x) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$$

Within the same radius of convergence R.

38.1 Solving for the radius of convergence

$$R = \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}$$

38.2 Taylor polynomial

Why is the series equal to the function? If we only take finitely many terms of the Taylor Series, we get a polynomial called the Nth order polynomial. The Taylor polynomial will not be equal to $f(x)$, but it will be an approximation. The difference between the Taylor Polynomial and $f(x)$ which is called the remainder $R_N(x)$.

$$\begin{aligned}
P_N(x) &= \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n \\
f(x) &= \sum_{n=n}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + R_N(x) \\
R_N(x) &= \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}
\end{aligned}$$

Why does adding up a lot of terms give you a precise quantity? Because adding more terms give you a more precise approximation. Because $n!$ grows extremely quickly, as long as your $N+1$th derivative doesn't get too big, it will grow slower than $(N+1)!$ – since $(N+1)!$ is extremely large, all other values: $f^{(N+1)}(c)\cdot(x-a)^{N+1}$ will be small.

38.3 First order Taylor polynomial

The first-order Taylor polynomial is the tangent line at a.

$$P_1(x) = f(a) + f'(a)(x-a)$$

Why is this first-order Taylor polynomial giving the best approximation? Why couldn't we give the line a different slope?

The definition of the derivative, $f'(a) = \lim_{x\to a}\frac{f(a)-f(a)}{x-a}$ gives:

$$\begin{aligned}
\lim_{x\to a}\frac{f(x)-P_1(x)}{x-a} &= \lim_{x\to a}\frac{f(x)-f(a)-f'(a)(x-a)}{x-a} \\
&= f'(a)-f'(a) = 0
\end{aligned}$$

This is why the tangent line is good at making approximations.

Let's say I want to use the second-order Taylor approximation. Why is the second order Taylor approximation more special than any other quadratic? We can see that the error we're making is tending towards 0?

$$\lim_{x \to a} \frac{f(x) - P_2(x)}{(x-a)^2} = 0$$

The tangent line is a good approximation of $y = f(x)$ near $(a,\ f(a))$:

$$y - f(a) = f'(a)(x-a)$$

As N increases, the speed for which $(x-a)^N$ is going to zero increases. In general, the Nth order Taylor Polynomial would have its error go to zero **faster than N powers** $(x-a)$:

$$\lim_{x \to a} \frac{f(x) - P_N(x)}{(x-a)^N} = 0$$