MAT223 Notes

Linear Transformations

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December 9, 2021

1 Linear transformations

A function $T:\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if $\forall \mathbf{v},\ \mathbf{u} \in \mathbb{R}^n$ and $\forall r \in \mathbb{R}$ and has the following properties:

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

(ii) $T(r\mathbf{v}) = rT(\mathbf{v})$

Example: $f(x) = kx$ is a linear function $\mathbb{R}^1 \to \mathbb{R}^1$. The following properties hold:

$$f(x_1+x_2) = k(x_1+x_2) = f(x_1) + f(x_2)$$
$$f(rx_1) = k(rx_1) = rkx_1 = rf(x_1)$$

Second example: $f(x) = x^2$ is not a linear function $\mathbb{R}^1 \to \mathbb{R}^1$ a.k.a.

$$f(2x) \neq 2f(x)$$

RHS: $2f(x) \Rightarrow$ LHS $\neq$ RHS

Remark 1. To show that a function is linear, you can combine (i) and (ii) together.

$$(i)+(ii) = f(\mathbf{u}+r\mathbf{v}) = f(\mathbf{u}) + rf(\mathbf{v})\ \forall \mathbf{u},\ \mathbf{v} \in \mathbb{R}^n,\ \forall r \in \mathbb{R}$$

Capital letters may be used to denote a linear function. Don't confuse them and matrices!

1.1 What are functions?

A function *compacts* complex statements. For example:

Let $h(x) = 4x^2 + 6x$

Then we can write $4 + h(x)$ as $4 + 4x^2 + 6x$, and anytime we see $4x^2 + 6x$ appear in a statement, we can convert it back to $h(x)$. Note that $h(x)$ doesn't have to be represented in function notation; if we let $y = h(x)$, then we could write $4 + h(x)$ as $4 + y$. Knowing this is essential to prevent misunderstandings. That being said, if we define $g(x) = 1$ then whenever $g(x)$ appears anywhere in a statement we can replace it with 1 (isn't that just a constant?).

1.2 What are sets?

Sets are an unordered collection of anything. For example, $\{1,\ 2,\ 3,\ 4\}$ is a set. We can also have a set of real numbers, or a set of even numbers, which would be denoted as $\{x \in \mathbb{Z} \mid x\%2 = 0\}$. That's set builder notation, and here's how it works: {our set | if it meets what we write here}. Knowing this will help mitigate confusion further on.

1.3 Matrices as a form of a linear function

If these properties hold, then $f(\mathbf{x})$ is linear. A is a matrix. Remember that $A\mathbf{x}$ returns a vector.

$$\begin{aligned} f(\mathbf{x}) &= A\mathbf{x} = \mathbf{y} \Rightarrow f : \mathbb{R}^n \to \mathbb{R}^m \\ A(\mathbf{x}+\mathbf{y}) &= A\mathbf{x} + A\mathbf{y} \\ A(r\mathbf{x}) &= rA\mathbf{x},\ r \in \mathbb{R} \\ &\Rightarrow f(\mathbf{x}) \text{ is linear} \end{aligned}$$

1.4 Domain, Codomain, Image, and Kernel

A linear transformation's domain, codomain, image, and kernel are sets, and thus can be described in the set builder notation.

Definition: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then...

1. $\mathbb{R}^n$ is the domain of T. (The set of inputs)
2. $\mathbb{R}^m$ is the codomain of T. (The set of possible outputs)
3. Let $w \subseteq \mathbb{R}^n$, then the image of w under T is $T(w) = \{T(\mathbf{w}) | \mathbf{w} \in W\}$

a. $T(w)$ is a collection of all possible outputs corresponding to the inputs of w. $T(w)$ is the image.

4. The range of T is $T(\mathbb{R}^n) = \{T(\mathbf{v}) | \mathbf{v} \in \mathbb{R}^n\}$ (i.e., Range of T is the image of the domain.)

5. Let $w' \subseteq \mathbb{R}^m$. The inverse image of w' under T is $T^{-1}(w') = \{\mathbf{v} \in \mathbb{R}^n | T(\mathbf{v}) \in w'\}$ (is this the domain?)

6. Let $\mathbf{0}$ be the zero vector in $\mathbb{R}^m$.

$T^{-1}(\mathbf{0}) = \{\mathbf{v} \in \mathbb{R}^n | T(\mathbf{v}) = \mathbf{0} \in \mathbb{R}^m\}$ is called the kernel of T. **(The *kernel* of a linear transformation *T* is the set of inputs that *T* sends to the zero vector.)**

1.5 An example of a linear transformation

Example: Let $T: \mathbb{R}^3 \to \mathbb{R}^2$. $T([x_1,\ x_2,\ x_3]) = [x_1 - x_2,\ x_2 + x_3]$.

This is one way for us to define a function and is a definition of a function.

Here's a series of questions:

1. **DETERMINING WHETHER *T* IS LINEAR**

Is T a linear map/function/transformation? **Answer: Yes. Why?**

$T(\mathbf{u} + r\mathbf{v}) = T(\mathbf{u}) + rT(\mathbf{v})\ \forall \mathbf{u}, \mathbf{v},\ \ r \in \mathbb{R}$.

If we verify if LHS of the above equation is always equal to RHS, **then** T is a linear function. Our objective is to start with $T(\mathbf{u} + r\mathbf{v})$ and perform some operations to get us to

$$T(\mathbf{u}) + rT(\mathbf{v})$$

$$r \in \mathbb{R}$$

Rationale: $\mathbf{u}$ *and* $\mathbf{v}$ *are inputs to* T *and must be* $\mathbb{R}^3$ *.*

LHS:

$$\begin{aligned} T(\mathbf{u} + r\mathbf{v}) &= T([u_1,\ u_2,\ u_3] + r[v_1,\ v_2,\ v_3]) \\ &= T([u_1 + rv_1,\ u_2 + rv_2,\ u_3 + rv_3]) \end{aligned}$$

Running the function, we get this. We can do some algebraic manipulation to get what we have as close to the RHS,

$$\begin{aligned}
&= [u_1 + rv_1 - u_3 - rv_3,\ u_2 + rv_2 + u_3 + rv_3] \\
&= [u_1 - u_3 + r(v_1 - v_3),\ u_2 + u_3 + r(v_2 + v_3)] \\
&= [u_1 - u_3,\ u_2 + u_3] + [r(v_1 - v_3),\ r(v_2 + v_3)] \\
&= [u_1 - u_3,\ u_2 + u_3] + r[v_1 - v_3,\ v_2 + v_3]
\end{aligned}$$

Recall:

$$\begin{aligned}
T(x_1, x_2,\ x_3) &= (x_1 - x_3,\ x_2 + x_3) \\
&= T([u_1,\ u_2,\ u_3]) + rT([v_1,\ v_2,\ v_3])
\end{aligned}$$

$$T(\mathbf{u}) + rT(\mathbf{v}) = RHS$$

Therefore $LHS = RHS \Rightarrow T$ is a linear transformation.

2. **FINDING THE IMAGE OF A FUNCTION WITH AN INPUT DOMAIN**

Let $H = \left\{[x,\ x,\ x] \in \mathbb{R}^3\right\} \subseteq \mathbb{R}^3$. Find the **image** of H under T.

Recall: We let $T : \mathbb{R}^3 \to \mathbb{R}^2$. $T([x_1,\ x_2,\ x_3]) = [x_1 - x_2,\ x_2 + x_3]$.

Objective: Want to find the image of *H* under *T* in the most explicit way as possible.

$$T(H) = \left\{\mathbf{y} \in \mathbb{R}^2 \middle| \mathbf{y} = T(\mathbf{u}),\ \mathbf{u} \in H\right\}$$

The above notation says $\mathbf{y}$ is in $\mathbb{R}^2$ (because $T(H)$ is in $\mathbb{R}^2$), but only if $\mathbf{y}$ is an output of $T(\mathbf{u})$ where $\mathbf{u}$ is in H, a set of $\mathbb{R}^3$ vectors where all its entries are identical. We *could* stop at this step, but why not expand this further?

$$\begin{aligned}
&= \left\{\mathbf{y} \in \mathbb{R}^2 \middle| \mathbf{y} = T([u,\ u,\ u]),\ \mathbf{u} = [u,\ u,\ u] \in H\right\} \\
&= \{\mathbf{y} \in \mathbb{R}^2 | \mathbf{y} = [0,\ u + u] = [0,\ 2u],\ u \in \mathbb{R}\}
\end{aligned}$$

Conclude that $\left\{\mathbf{y} \in \mathbb{R}^2 \middle| \mathbf{y} = [0,\ 2u],\ u \in \mathbb{R}\right\}$

This is the image of H under T. This is an explicit representation of an image without $T(\mathbf{u})$ appearing anywhere within the set builder.

3. DEDUCING INVERSE IMAGES WITH AN IMAGE

Question: Let $u = \{[1,\ 2],\ [-1,\ 3]\} \subseteq \mathbb{R}^2$ (which is a subset of the codomain of T. Find the inverse image of u, under T.

We are given a **set** u with two elements – it's like two dots in a set, so what we actually have to do is compute a function's inverse with these inputs – we were given two inputs, which means double the work!

Recall: We let $T : \mathbb{R}^3 \to \mathbb{R}^2$. $T\left([x_1,\ x_2,\ x_3]\right) = [x_1 - x_2,\ x_2 + x_3]$.

The set builder below denotes our inverse image. We just have to compact that down, so we end up with an actual set.

$$\begin{aligned} &T^{-1}(u) = \left\{\mathbf{x} = [x_1,\ x_2,\ x_3] \in \mathbb{R}^3 \middle|\ T\left(\mathbf{x}\right) \in u\right\} \\ &\Rightarrow T\left(\mathbf{x}\right) = [1,\ 2]\ or\ [-1,\ 3] \end{aligned}$$

Case 1: $T\left(\mathbf{x}\right) = [1,\ 2]$. You'll need to find $[x_1,\ x_2,\ x_3]$ such that $T\left([x_1, x_2, x_3]\right) = [1,\ 2]$

$$\begin{aligned} [x_1 - x_3,\ x - 2 + x_3] = [1,\ 2] \Rightarrow & \begin{matrix} x_1 - x_3 = 1 \\ x_2 + x_3 = 2 \end{matrix} \\ \Rightarrow & \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right] \end{aligned}$$

So, you might be asking, where did the augmented matrix come from? That was because a system of linear equations appeared in your work, so you must solve it. It is a 3-unknown 2-equation system, so you will end up with a free variable.

3-variable, 2-equation systems typically have infinite solutions, and can be solved like any other linear system. If you're having trouble making sense out of this, you can always add a useless equation $0x_1 + 0x_2 + 0x_3 = 0$ giving you

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Which will have a free variable, resulting in infinitely many solutions.

This will give you: $\begin{cases} x_1 - x_3 = 1 \\ x_2 + x_3 = 2 \end{cases} \Rightarrow \begin{matrix} x_3 = s \\ x_1 = 1 + s \\ x_2 = 2 - s \end{matrix}$

We can thus conclude that:

$$\mathbf{x} = [x_1, x_2,\ x_3] = [1+s,\ 2-s,\ s],\ s \in \mathbb{R}$$

We deduced a part of the inverse image of u under T.

$$T\left(\left[\frac{1+s}{x_1}, \frac{2-s}{x_2}, \frac{s}{x_3}\right]\right) = \left[\frac{1+s}{x_1} - \frac{s}{x_3}, \frac{2-s}{x_2} + \frac{s}{x_3}\right] = [1,\ 2]$$

Case 2: $T(\mathbf{x}) = [x_1 - x_3,\ x_2 + x_3] = [-1,\ 3]$

$$\Rightarrow \begin{cases} x_1 - x_3 = -1 \\ x_2 + x_3 = 3 \end{cases} \Rightarrow \mathbf{x} = [-1+t,\ 3-t,\ t],\ t \in \mathbb{R}$$

That's case 2, complete. Now we can combine case 1 and 2 together and that is our inverse image, which is:

$$[1+s,\ 2-s,\ s]\ and\ [-1+t,\ 3-t,\ t]$$

Or we can represent it as a set: $\{[1+s,\ 2-s,\ s],\ [-1+t,\ 3-t,\ t]\}$

Because $\mathbf{u}$ is an "image", we can represent the inverse image like the following:

$$T\left(\{[1+s,\ 2-s,\ s],\ [-1+t,\ 3-t,\ t] \mid t,\ s \in \mathbb{R}\}\right) = u$$

$$T\left(\left\{\begin{bmatrix} 1+s \\ 2-s \\ s \end{bmatrix},\ \begin{bmatrix} -1+t \\ 3-t \\ t \end{bmatrix} \mid t,\ s \in \mathbb{R}\right\}\right) = u$$

4. **FINDING THE KERNEL OF A FUNCTION**

Find the kernel of *T*.

$$\ker(T) = \{\mathbf{x} \in \mathbb{R}^3 | T(\mathbf{x}) = 0\}$$

Figure out when input $\mathbf{x}$ can give you 0 as its output.

Recall: $T(\mathbf{x}) = T([x_1, x_2,\ x_3]) = [x_1 - x_3,\ x_2 + x_3] = [0,\ 0]$

This can be used to build a system of equations.

$$\left[\begin{array}{ccc|c}1 & 0 & -1 & 0\\ 0 & 1 & 1 & 0\end{array}\right] \Rightarrow [s,\ -s,\ s],\ s\in\mathbb{R}$$

$$T\left([s,\ -s,\ s]\right) = [s-s,\ -s+s] = [0,\ 0] \Rightarrow \mathbf{0}\in\mathbb{R}^2$$

$$\ker(T) = \left\{[s,\ -s,\ s]\in\mathbb{R}^3 \middle| s\in\mathbb{R}\right\}$$

1.6 Basis vectors

Proposition: Let $T:\mathbb{R}^n\to\mathbb{R}^m$ be a linear transformation. Then

$$\begin{aligned}&T\left(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \ldots + r_k\mathbf{v}_k\right)\\ &= rT\left(\mathbf{v}_1\right) + r_2T\left(\mathbf{v}_2\right) + \cdots + r_kT\left(\mathbf{v}_k\right)\end{aligned}$$

This is a linear combination between $\mathbf{v_1},\ \cdots,\ \mathbf{v}_k$

Remark: What if $\mathbf{v_1}\cdots\mathbf{v_n}$ is a set of **basis** vectors of $\mathbb{R}^n$, i.e., $span\left(\mathbf{v_1}\cdots\mathbf{v_n}\right)=\mathbb{R}^n$ and $\mathbf{v}_1\cdots\mathbf{v}_n$ are linearly independent

Then

$$\Leftrightarrow \begin{bmatrix}| & | & |\\ \mathbf{v}_1 & \cdots & \mathbf{v}_n\\ | & | & |\end{bmatrix} \sim \begin{bmatrix}1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 1\end{bmatrix}$$

Proposition: For any input $\alpha\in\mathbb{R}^n$. $\alpha = r_1\mathbf{v_1} + r\mathbf{v_2} + \cdots + r\mathbf{v_n}$ (given $\mathbf{v_1}\cdots\mathbf{v_n}$ are a set of basis vectors)

$$T\left(\alpha\right) = r_1T\left(\mathbf{v_1}\right) + r_2T\left(\mathbf{v_2}\right) + \cdots + r_nT\left(\mathbf{v}_n\right)$$

Tell me the image of the basis vectors without telling me the formula of T, and I would still be able to compute the value of the output for any given no input. If you have the image of basic for the T, then you can compute the output for any given input.

Property: The linear transformation must fix a zero. This means $T\left(\mathbf{0}\right)=\mathbf{0}$ must be in the codomain. This gives us a shortcut to check if a given function is linear or not. If a zero gives an output as a nonzero vector, then you can tell it is not a linear transformation.

Proof. Let $\mathbf{0}=\mathbf{v}-\mathbf{v}$. $\mathbf{v}\in\mathbb{R}^n$.

$$T(\mathbf{0}) = T(\mathbf{v} - \mathbf{v}) = T(\mathbf{v}) - T(\mathbf{v}) = \mathbf{0}$$

$\Rightarrow ker(T)$ always contains the zero vector.

2 Vector Spaces and Linear Independence

A homogeneous linear system can be represented as:

$$A\mathbf{x} = 0$$

Where A is an $m \times n$ matrix, where $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

$$T : \mathbb{R}^n \to \mathbb{R}^m : T(\mathbf{x}) = A\mathbf{x}$$

Remark

1. Every homogeneous system is consistent (≥ 1 soln)

If the system gives you no solutions – you know how that looks like. But, for a homogeneous equation, the right side of a partitioned matrix will always be zero for all components.

Moreover, the trivial solution always is a solution for $A\mathbf{x} = \mathbf{0}$. The reason why we call this the trivial solution is that $A(\mathbf{0}) = \mathbf{0},\ \mathbf{x} = 0$

2. If $A \sim H$ and H has a pivot in every column, then $A\mathbf{x} = 0$ only has the trivial solution ($\mathbf{x} = \mathbf{0}$).
3. If $m < n$ (number of rows < columns) and $A \sim H$ (*H* is in RREF) then it is impossible for H to have a pivot in every column, which means a free variable definitely exist. This means $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

2.1 Null Space, row space, col space

Definition 2.1. Let A be an $m \times n$ matrix

1. The null space of A is a set of vectors $\mathbf{x}$ such that $A\mathbf{x} = 0$ i.e., $\{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = 0\}$. It is the set of vectors that go to the zero vector when multiplying A from the right.

2. The row space of A, $row(A)$ is the span of row vectors of A.
3. The column space of A, $col(A)$ is the span of column vectors of A.

A linear system $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is in the column space of A.

2.1.1 Deducing Null Spaces

For example, suppose I have this matrix:

$$A = \begin{bmatrix} 2 & -6 & -2 & 4 \\ -1 & 3 & 3 & 2 \\ -1 & 3 & 7 & 10 \end{bmatrix}$$

$$\begin{aligned} row(A) &= span\left([2,\ -6,\ 2,\ 4],\ [-1,\ 3,\ 3,\ 2],\ [-1,\ 3,\ 7,\ 10]\right) \\ col(A) &= span\left(\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -6 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 10 \end{bmatrix}\right) \\ null(A) &= \left\{\mathbf{x} \in \mathbb{R}^4 \mid A\mathbf{x} = 0\right\} \end{aligned}$$

We need to solve $A\mathbf{x} = 0$.

$$\begin{bmatrix} 2 & -6 & -2 & 4 & 0 \\ -1 & 3 & 3 & 2 \mid & 0 \\ -1 & 3 & 7 & 10 & 0 \end{bmatrix} \sim \left[\begin{array}{cccc|c} 1 & -3 & 0 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

There are infinitely many solutions to this.

$$\begin{aligned} x_2 &= t \\ x_4 &= s \\ x_3 + 2x_4 &= 0 \Rightarrow x_3 = -2x_4 = -2s \\ x_1 - 3x_2 + 4x_4 &= 0 \Rightarrow x_1 = 3x_2 - 4x_4 = 3t - 4s \\ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= \begin{bmatrix} 3t - 4s \\ t \\ -2s \\ s \end{bmatrix} = \begin{bmatrix} 3t \\ t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -4s \\ 0 \\ -2s \\ s \end{bmatrix} \\ &= t \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \end{bmatrix} \end{aligned}$$

Therefore:

$$null(A) = span\left(\begin{bmatrix}3\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-4\\0\\-2\\1\end{bmatrix}\right)$$

Null space is the only space that requires computations.

2.1.2 Proofs regarding null spaces

Proposition: Let $A\mathbf{x} = \mathbf{b}$ be a linear system with a particular solution p. $(A\mathbf{p} = \mathbf{b})$

If $\mathbf{h}$ is a vector in the $null(A)$, then $\mathbf{p}+\mathbf{h}$ is also a solution for $A\mathbf{x} = \mathbf{b}$.

Proof. $A(\mathbf{p}+\mathbf{h}) = \mathbf{b}$

$$\begin{aligned}
LHS &= A\mathbf{p} + A\mathbf{h}\\
LHS &= A\mathbf{p} + 0\\
LHS &= \mathbf{b}
\end{aligned}$$

Remember that because h is a vector in the null space of A, it means $A\mathbf{h}$ is always equals to zero. $A\mathbf{p} = b$ is true because it is defined as true as part of the proof.

If $\mathbf{q}$ is any solution to $A\mathbf{x} = \mathbf{b}$, then $\mathbf{q} = \mathbf{p} + \mathbf{h}_1$ for some $\mathbf{h} \in null(A)$

$$A\mathbf{q} = A(\mathbf{p}+\mathbf{h}_1) = A\mathbf{p} + A\mathbf{h}_1 = \mathbf{b} + 0 \Rightarrow \mathbf{h}_1 \in null(A)$$

2.2 Linear independence

Definition: Linear independence (every single vector is useful; none of the other vectors can be made up through linear combinations of other vectors).

If $\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k \in \mathbb{R}^n$, $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \ldots + r_k\mathbf{v}_k = \mathbf{0}$ has exactly 1 solution (that is $\mathbf{r}$ being all zero), then we say that $\mathbf{v}_1,\ \ldots,\ \mathbf{v}_k$ are linearly independent.

(It is impossible that this system has no solutions. If this system has infinite solutions, then $\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k$ are linearly dependent.)

2.2.1 Linear independence properties

Remark: $r_1 = r_2 = r_3 = \cdots = r_k = 0$ is a solution for $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \ldots + r_k\mathbf{v}_k = 0$, and if that is the only solution then $\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k$ are linearly independent.

Suppose not. $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_k$ is another solution for $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \ldots + r_k\mathbf{v}_k = 0$. $\exists \alpha_i \neq 0$ (Why are we writing that? Because one solution has $r_1 \ldots r_k$ be all zero and we are trying to incorporate a non-all-zero solution for $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \ldots + r_k\mathbf{v}_k = 0$. Our work is here:

$$\begin{aligned}
\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_i\mathbf{v}_i + \ldots + \alpha_k\mathbf{v}_k &= 0 \\
&\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots \\
&+ \alpha_{i-1}\mathbf{v}_{i-1} + \ldots + \alpha_k\mathbf{v}_k \\
&= -\alpha_i\mathbf{v}_i \\
\frac{\alpha_1}{\alpha_i}\mathbf{v}_1 + \frac{\alpha_2}{\alpha_i}\mathbf{v}_2 + \cdots + \frac{\alpha_k}{\alpha_i}\mathbf{v}_k &= -\mathbf{v}_i
\end{aligned}$$

If $\mathbf{v}_i$ can be represented by the left equation, then $\mathbf{v}_i$ is redundant, which is *impossible* because we said that $\mathbf{v}_1,\ \ldots,\ \mathbf{v}_k$ are linearly independent. We thus arrive at a contradiction.

Definition: Two vectors in $\mathbb{R}^n$ are linearly independent if they are non-zero and non-parallel.

2.2.2 Testing for linear independence

Example: $\mathbf{v}_1 = [2,\ -1,\ -1],\ \mathbf{v}_2 = [-2,\ 3,\ 7],\ \mathbf{v}_3 = [2,\ 1,\ 5]$

Solution: If $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + r_3\mathbf{v}_3 = 0$ **has exactly one solution,** the vectors are linearly independent. You can represent this in a matrix $A\mathbf{r}$, where A's rows are formed by the column vectors $\mathbf{v}$.

If a column is missing a pivot, then it is impossible for this system to have a unique solution $\mathbf{r}$.

$$\begin{bmatrix} 2 & -2 & 2 \\ -1 & 3 & 1 \\ -1 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

While we did not reduce the above matrix to RREF, but this is good enough as we can see there is at least one column without the pivot.

2.2.3 Testing for linear dependence

Because $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + r_3\mathbf{v}_3 = 0$ has more than one solution, the vectors are linearly dependent.

Is this matrix invertible (if no, then we've got linear dependence)?

$$\begin{bmatrix} 2 & -1 & -1 \\ -2 & 3 & 7 \\ 2 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The rightmost column causes this matrix to have a free variable. If we let it equal to zero:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$A\mathbf{x} = 0$ has the solution $r \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ (we factored out the negatives)

$$[\mathbf{v}_3] = 2[\mathbf{v}_1] + [\mathbf{v}_2]$$

$\mathbf{v}_3$ is a redundant vector, and so could the others?

3 Subspaces and basis vectors

Hint: checking "in a subspace" WILL appear on an exam.

Definition: Let $W \subseteq \mathbb{R}^n$. W is called **a subspace** of $\mathbb{R}^n$ if these following properties hold:

1. $W \neq \varnothing$
2. If $\mathbf{u},\ \mathbf{v} \in W$, then $\mathbf{u} + \mathbf{v} \in W$ (Add two vectors in the "subspace" and your sum must still be within the subspace. This is the closure property regarding subspaces.)
3. If $\mathbf{u} \in W,\ r \in \mathbb{R}$, then $r \cdot \mathbf{u} \in W$ (Closure under scalar multiplication)

To check properties 2 and 3, you may need to pick variables arbitrarily and you must end up with a way to represent the sum or scalar product of the vectors in the way an individual vector in the set would be defined.

With all three holding W is a subspace.

3.1 Checking if a set is a subspace: Examples

Example: $W = \{\mathbf{v} \in \mathbb{R}^2 \mid \mathbf{v} = [x,\ 0],\ x \in \mathbb{R}\}$

A subspace is usually seen as a line or an infinite plane, and so on.

Is W a subspace?

1. $W \neq \varnothing; \varnothing \subset W$ (NOT the zero vector.)
2. Pick $\mathbf{u} = [x_1,\ 0] \in W$, $\mathbf{v} = [x_2,\ 0] \in W$, The sum? $\mathbf{u}+\mathbf{v} = [x_1+x_2,\ 0] \in W$ (it still lies on W and the x-axis.)
3. $r\mathbf{u} = r[x,\ 0] = [rx,\ 0] \in W$

Because these properties hold, W is a subspace in $\mathbb{R}^2$.

Second example: $H = \{\mathbf{v} \in \mathbb{R}^2 \mid \mathbf{v} = [0,\ y],\ y \in \mathbb{R}\}$

You can show that H is a subspace in $\mathbb{R}^2$.

Third example: $H \cup W$? No. The reason why this is not a subspace.

We can disprove this with vector addition:

$$[1,\ 0] + [0,\ 1] = [1,\ 1] \notin H \cup W$$

Therefore $H \cup W$ can't be a subspace.

Fourth example: (Try this out of lecture) $H \cap W$. That is just the zero vector. Scalar multiply a zero vector by any amount, and you will get the zero vector. Add the zero vector by another zero vector, and you get the zero vector. Therefore, the zero subspace is closed under addition and scalar multiplication.

Remark: I will check if $\mathbf{0}$ is contained in W **as a subspace must contain zero (but does not imply that a subspace is valid).**

Suppose W is a subspace. This implies that W is non-empty, implying $\exists \mathbf{u} \in W$. This also implies:

$$\begin{aligned}
&\Rightarrow -1\mathbf{u} \in W \Rightarrow \mathbf{u} + (-1\mathbf{u}) \in W \\
&\Rightarrow \mathbf{0} \in W
\end{aligned}$$

If $\overrightarrow{0}$ is not in the subspace, then I can immediately declare that W is not a subspace. Because all subspaces must be closed under addition and scalar multiplication you must be able to find zero when doing a combination of the two.

Example: $W = sp\{\mathbf{w}_1,\ \mathbf{w}_2,\ \ldots,\ \mathbf{w}_k\}$, where $w_1,\ w_2,\ \ldots,\ w_k \in \mathbb{R}^n \Rightarrow W$ is a subspace of $\mathbb{R}^n$.

Proof. Check if $\mathbf{0}$ belongs to W.

$$r_1\mathbf{w}_1 + r_2\mathbf{w}_2 + \ldots + r_k\mathbf{w}_k = \mathbf{0}$$

To show this is true, show that this system is homogeneous, and that there is always one solution: $r_1 = r_2 = \ldots = r_k = 0$.

Firstly, $\mathbf{0}$ is a vector in W, and a span means all possible linear combinations. Therefore, $\mathbf{0} \in W,\ \Rightarrow \mathbf{w} \neq 0$.

Step 2: Check if $\mathbf{u} + r\mathbf{v} \in W$ where $\mathbf{u} \cdot \mathbf{v} \in W,\ r \in \mathbb{R}$. If I can answer this, then $\mathbf{u},\ \mathbf{v}$ are closed under vector addition and scalar multiplication.

$$\mathbf{u} = \alpha_1\mathbf{w}_1 + \alpha_2\mathbf{w}_2 + \ldots + \alpha_k\mathbf{w}_k$$
$$\mathbf{v} = \beta_1\mathbf{w}_1 + \beta_2\mathbf{w}_2 + \ldots + \beta_k\mathbf{w}_k$$

Where $\alpha_1 \in \mathbb{R}\forall i,\ \beta_i \in \mathbb{R}\forall i$

$$\begin{aligned}
&\mathbf{u} + r\mathbf{v} \\
&= (\alpha\mathbf{w}_1 + \ldots + \alpha_k\mathbf{w}_k) \\
&+ (r\beta_1\mathbf{w}_1 + r\beta_2\mathbf{w}_2 + \ldots + r\beta_k\mathbf{w}_k) \\
\\
&= (\alpha_1 + r\beta_1)\,\mathbf{w}_1 + (\alpha_2 + r\beta_2)\,\mathbf{w}_2 \\
&+ \ldots + (\alpha_n + r\beta_n)\,\mathbf{w}_k
\end{aligned}$$

This is just another linear combination, showing that $\mathbf{u} + r\mathbf{v} \in sp\{\mathbf{w}_1 \ldots \mathbf{w}_k\} = W$ which means $\mathbf{u} + r\mathbf{v} \in W \Rightarrow W$ is closed under addition and scalar multiplication. Therefore, W is a subspace of $\mathbb{R}^n$.

■

3.2 Finding a basis for a set of vectors

In a nutshell:

If we need to find a basis for $W = sp(w_1,\ w_2,\ \ldots,\ w_k)$

1. Form the matrix A whose jth column vector is w_j
2. Row-reduce A to REF, which we will call H.
3. The set of all w_j such that the jth column of H contains a pivot is a basis for W.

Definition: Let W be a subspace of $\mathbb{R}^n$.

If $B = \{\mathbf{b}_1,\ \mathbf{b}_2,\ \ldots,\ \mathbf{b}_k\}$ is a subset of W, then B is a **basis** for W if every vector in W can be written **uniquely** as a linear combination of vectors in B. (All the vectors in W can be generated by B. If that holds, B has that property.)

Remark: Pick $\mathbf{w} \in W$.

$$\mathbf{w} = r_1\mathbf{b}_1 + r_2\mathbf{b}_2 + \ldots + r_k\mathbf{b}_k,\ r_i \in \mathbb{R}\forall i$$

And we need to prove this representation is unique, i.e., $\{r_1,\ \ldots,\ r_k\}$ is unique. This gives us another linear system, with unknowns $\mathbf{r}$. We need to check if this system has a unique solution; if it does, then those vectors $\mathbf{b}_1 \ldots \mathbf{b}_k$ are basis vectors.

$$\left[\begin{array}{cccc|c} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k & \mathbf{0} \end{array}\right]$$

This system having more than one solution indicates that $\{r_1 \ldots r_k\}$ is not unique.

Example: $W = sp\{\mathbf{w}_1,\ \mathbf{w}_2,\ \ldots,\ \mathbf{w}_5\}$

$$w_1 = \begin{bmatrix}1\\0\\1\\0\end{bmatrix},\ w_2 = \begin{bmatrix}4\\1\\4\\1\end{bmatrix},\ w_3 = \begin{bmatrix}3\\1\\3\\2\end{bmatrix},\ w_4 = \begin{bmatrix}-1\\0\\-1\\1\end{bmatrix}, w_5 = \begin{bmatrix}-6\\3\\-2\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0 & 0 & -6\\ 0 & 1 & 0 & -1 & 3\\ 0 & 0 & 1 & 1 & -2\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

Then only $\mathbf{w}_1,\ \mathbf{w}_2,\ \mathbf{w}_3$ are single-pivot row vectors; all of the other vectors are redundant. $\mathbf{w}_4 = -\mathbf{w}_2 + \mathbf{w}_3$ and $\mathbf{w}_5 = -6\mathbf{w}_1 + 3\mathbf{h}_2 - 2\mathbf{w}_3$

The basis vectors are $sp\{\mathbf{w}_1,\ \mathbf{w}_2,\ \mathbf{w}_3\}$.

This is equivalent to $sp\{\mathbf{w}_1,\ \mathbf{w}_2,\ \mathbf{w}_3,\ \mathbf{w}_4,\ \mathbf{w}_5\}$. The span is in $\mathbb{R}^3$; $dimension(sp\{\mathbf{w}_1,\ \mathbf{w}_2,\ \mathbf{w}_3\}) = 3$.

3.3 Basis Theorems

Theorem 3.1. *Let $A \in M_{n\times n}$. The following are equivalent (if you can show one of them all in this list will be true):*

1. The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b} \in \mathbb{R}^n$.
2. $A \sim I$
3. A is invertible.
4. The column vectors of A form a basis for $\mathbb{R}^n$.
5. The row vectors in A form a basis for $\mathbb{R}^n$.

3.3.1 Proofs for these Theorems

Proof. $1 \Rightarrow 2$ is the process of solving a linear system. Reducing A to RREF, if every single column has a pivot (n pivots), you'll get a unique solution. That matrix will look like I, thus forcing A to be row equivalent to the identity matrix.

This is trivially equivalent to 3. $[A \mid I] \sim \left[H \mid A^{-1}\right]$, where H is the identity matrix.

For no. 4: Column vectors form a basis $\Rightarrow$ they are linearly independent $\Rightarrow$ every column must contain a pivot $\Rightarrow$ you will have n pivots, meaning no free variables and a unique solution. If that is case, then $A \sim I \Leftrightarrow$ every single column has a pivot $\Leftrightarrow$ they can form a basis.

For no. 5: $A \sim I \Leftrightarrow A^T \sim I$

■

3.4 Rectangular cases

Theorem 3.2. *Let A be $m \times n$ matrix. $(m > n)$. The following are equivalent:*

1. Each **consistent** system $A\mathbf{x} = \mathbf{b}$ has a unique solution.
2. The RREF of A must contain the $n \times n$ identity matrix with $m - n$ rows of zeros (on the bottom). This means there are $m - n$ redundant equations.
3. The column vectors of A form a basis for the column space of A.

3.4.1 Proofs for these theorems

Proof. $1 \Rightarrow 2$

$A\mathbf{x} = \mathbf{b}$, and for a unique solution, free variables are not allowed to exist, which means every column must contain a pivot. All entries below the identity matrix must be all zero, implying $m-n$ rows of zeroes.

Proof. $2 \Rightarrow 3$

Every column has a pivot $\Rightarrow$ the column vectors are linearly independent $\Rightarrow$ can serve as basis.

Proof. $3 \Rightarrow 1$

Basis $\Rightarrow$ linearly independent. Building a linear system out of the basis guarantees no free variables $\Rightarrow$ unique solution.

Remark 2. The number of solutions will depend on whether free variables exist. If you have a rectangular matrix $M_{m<n}$ (wider than it is tall), then $A\mathbf{x} = \mathbf{b}$ is a linear system with fewer equations than unknowns. If $A\mathbf{x} = \mathbf{b}$ is consistent (ruling out no solution situations), then it has infinitely many solutions.

Theorem 3.3. *Let $B = \{\mathbf{b}_1,\ \ldots,\ \mathbf{b}_k\}$ be a basis for the subspace $W \subset \mathbb{R}^n$, if and only if [1] $B = \{\mathbf{b}_1,\ \ldots,\ \mathbf{b}_k\}$ is a linearly independent set and [2] it spans W.*

Proof. B is a basis for $W \Leftrightarrow$ Pick any $\mathbf{w} \in W$; $\mathbf{w}$ has a unique representation: $\mathbf{w} = r_1\mathbf{b}_1 + r_2\mathbf{b}_2 + \cdots + r_k\mathbf{b}_k$, and the solution for r must be unique. You could view this as a linear system $A\mathbf{x} = \mathbf{b}$, that has a unique solution where $\mathbf{b} = \mathbf{w}$, and A is a matrix with column vectors $\mathbf{b}_1 \ldots \mathbf{b}_k$, and $\mathbf{x} = [r_1,\ r_2,\ \ldots,\ r_k] \Leftrightarrow$ no free variable $\Leftrightarrow$ every column must contain a pivot $\Leftrightarrow$ none of any element in B are redundant $\Leftrightarrow \forall\mathbf{b}_i$ (column vectors), it cannot be represented by linear combination of other $\mathbf{b}_i \Leftrightarrow$ they are linearly independent.

∎

3.5 How to find a basis for any set of vectors

This: $W = sp\{\mathbf{w}_1 \cdots \mathbf{w}_k\}$ where $\mathbf{w}_i \in \mathbb{R}^n$ for $i = 1,\ 2,\ \ldots,\ k$

Step 1. Create a matrix A where the columns of it are formed by the $\imath$th vectors in W.

Step 2: $A \sim H$, where H is REF or RREF. (Idea: check if every column has a pivot.

Step 3: The basis for W consists only of all the columns of A, corresponding to the columns of H that contains the pivots.

Definition: Let W be a subspace of $\mathbb{R}^n$. The number of elements in a basis for W is called the dimension of W, denoted by $\dim(w)$. There are two facts supported by this definition:

[EXAM HINT] Fact 1: Let W be a subspace of $\mathbb{R}^n$. Let $\mathbf{w}_1,\ \mathbf{w}_2,\ \ldots,\ \mathbf{w}_k$ be vectors in W that span W and let $\mathbf{v}_1,\ \ldots,\ \mathbf{v}_m$ be linearly independent vectors in W. Then, $k \geq m$. (This fact is trivial – the dimension of $W \geq K$.)

$\mathbf{v}_1 \cdots \mathbf{v}_m$ being linearly independent could be a basis for either W or a subspace of W. In other words, $sp\{\mathbf{v}_1 \cdots \mathbf{v}_m\} = U \leq W \Rightarrow m \leq k$.

TL;DR: The quantity of linear independent vectors in W is always less or equal to the quantity of a list of vectors that have to at least span W.

Fact 2: Any two bases for a subspace $W \in \mathbb{R}^n$ contain the same number of vectors.

(The number of elements in a basis tells you the dimensions of the subspace it is in. If you have two bases with different numbers of vectors the dimensions formed by the span of it will be different.) For example, for $\mathbb{R}^2:\ sp([1,0],\ [0,1]) = \mathbb{R}^2$. Also, $sp([2,\ 0],\ [0,2]) = \mathbb{R}^2$, as $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{b}$ has a unique solution, meaning every $\mathbf{b}$ can uniquely be represented by $[2,\ 0],\ [0,\ 2]$. $sp([2,\ 0],\ [0,2],\ [a,b]),\ [a,b] = \frac{a}{2}[2,\ 0] + \frac{b}{2}[0,\ 2] \Rightarrow [a,\ b]$ is a redundant vector.

Proving Fact 1: Given:

1. $sp\{\mathbf{w}_1 \cdots \mathbf{w}_k\} = W \in \mathbb{R}^n$.
2. $\mathbf{v}_1 \cdots \mathbf{v}_m \in W$ are linearly independent

Show $k \geq m$.

Suppose $k < m$. By the idea of span, $\forall \mathbf{v}_i\ \ (i \in [1,\ 2,\ 3,\ \ldots,\ m]) \in W = sp\{\mathbf{w}_1 \cdots \mathbf{w}_k\}$. All of v_i are chosen from W. This means

$$
\begin{aligned}
\mathbf{v}_1 &= a_{11}\mathbf{w}_1 + a_{21}\mathbf{w}_2 + \cdots + a_{k1}\mathbf{w}_k \\
\mathbf{v}_2 &= a_{12}\mathbf{w}_1 + a_{22}\mathbf{w}_2 + \cdots + a_{k2}\mathbf{w}_k \\
&\vdots \\
\mathbf{v}_m &= a_{1m}\mathbf{w}_1 + a_{2m}\mathbf{w}_2 + \cdots + a_{km}\mathbf{w}_k
\end{aligned}
$$

This is true by definition of span (all in $\mathbf{v}$ are in the span of W).

Pick a vector $\mathbf{w} \in W$.

$$
\begin{aligned}
&\mathbf{w} \\
&= r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_m\mathbf{v}_m \\
&\in sp\{\mathbf{v}_1 \cdots \mathbf{v}_m\}
\end{aligned}
$$

By the definition of span. Now, we can rewrite $\mathbf{w}$ by replacing $\mathbf{v}_1$ with the first equation, and so on.

$$
\begin{aligned}
&\mathbf{w} \\
&= r_1\left(a_{11}\mathbf{w}_1 + a_{21}\mathbf{w}_2 + \cdots a_{k1}\mathbf{w}_k\right) \\
&+ r_2\left(a_{12}\mathbf{w}_1 + a_{22}\mathbf{w}_2 + \cdots + a_{k2}\mathbf{w}_k\right) \\
&+ \cdots + r_n(a_{1m}\mathbf{w}_1 + a_{2m}\mathbf{w}_2 \\
&+ \cdots + a_{km}\mathbf{w}_k) \\
\\
&= (r_1a_{11} + r_2a_{12} + \cdots + r_ma_{1m})\,\mathbf{w}_1 \\
&+ \cdots + (r_1a_{k1} + r_2a_{k2} + \cdots + r_ma_{km})\,\mathbf{w}_k
\end{aligned}
$$

If $\mathbf{w} = 0$, then $A\mathbf{r} = \mathbf{0}$, where A is a matrix formed by the column vectors of $\mathbf{v}_1 \cdots \mathbf{v}_m$. We know that $\mathbf{v}_1 \cdots \mathbf{v}_m$ is linearly independent, meaning the matrix has a unique and trivial solution. This means $\mathbf{r} = 0$. This means

$$
\begin{aligned}
r_1a_{11} + r_2a_{12} + \cdots + r_{m1}a_{1m} &= 0 \\
&\vdots \\
r_1a_{k1} + r_2a_{k2} + \cdots + r_ma_{km} &= 0
\end{aligned}
$$

This system only contains one solution, and that solution must be the trivial solution of $\mathbf{r} = 0$. However, you have k equations and m unknowns. Recall that we assumed $k < m$, meaning the number of equations is less than the number of unknowns, and you cannot have a unique solution that way – you would have infinitely many solutions.

~~You are supposed to have infinitely many $\mathbf{r}$ to satisfy~~ $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_m\mathbf{v}_m = \mathbf{0}$ ~~.~~ ~~Therefore~~ $\mathbf{v}_1 \cdots \mathbf{v}_m$ ~~are linearly independent.~~

[1] Remark. (Existence and determination of Base): Every subspace $W \subset \mathbb{R}^n$ has a basis and $\dim(w) \leq n$.

Pick $\mathbf{b} \in W$, and pick a collection of linearly independent vectors $\mathbf{w}_1,\ \mathbf{w}_2,\ \ldots,\ \mathbf{w}_k$, and use them to build a linear system. The objective is to make this system always have a solution.

$$\mathbf{w}_1 x_1 + \mathbf{w}_2 x_2 + \cdots + \mathbf{w}_k x_k = \mathbf{b}$$

Should have a unique solution $\Rightarrow$ pivot for every single column vector. If so, then they form a basis.

[2] Every linearly independent set of vectors in $\mathbb{R}^n$ can be enlarged if necessary to become a basis for $\mathbb{R}^n$.

$sp(\mathbf{v}_1 \cdots \mathbf{v}_k)$ gives you a k-dimensional space, if $k < n$. If you add some more linearly independent vectors (to all of the existing span) so that the number of vectors matches n, i.e.

$$sp(\mathbf{v}_1,\ \ldots,\ \mathbf{v}_k,\ \mathbf{v}_{k+1},\ \ldots,\ \mathbf{v}_n)$$

This forms a basis for $\mathbb{R}^n$.

Recall: Number of basis vectors of a subset gives you, its dimension.

[3] If W is a subspace of $\mathbb{R}^n, \dim(w) = k$. Then:

a) Every linearly independent k vectors in W is a basis for W

b) Every k vectors, the span of them gives you W, then those k vectors are basis for W.

3.6 Finding a basis from a matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 2 \\ 2 & -1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 & 3 \\ 2 & 1 & 3 & 3 & 3 \end{bmatrix}$$

What is the basis for the row space, and what is the basis for the column space?

The candidate bases for $col(A) = sp\{\mathbf{c}_1,\ \mathbf{c}_2,\ \mathbf{c}_3,\ \mathbf{c}_4,\ \mathbf{c}_5\}$, $row(A) = sp\{\mathbf{R}_1,\ \mathbf{R}_2,\ \mathbf{R}_3,\ \mathbf{R}_4\}$

To figure out the basis, we need to figure which $\mathbf{c}$ and which $\mathbf{r}$ is redundant. Row reducing this matrix:

$$A \sim H = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Columns 1, 2, and 4 have pivots. Columns 3 and 5 are redundant and can be made up by linear combinations of columns 1, 2, and 4. For example:

$$\begin{aligned}\mathbf{c}_5 &= \mathbf{c}_1 + \mathbf{c}_2 \\ \mathbf{c}_3 &= \mathbf{c}_1 + \mathbf{c}_2\end{aligned}$$

We can remove the redundant vectors from $col(A)$ to determine the basis for the column vectors of A:

$$col(A) = sp\{\mathbf{c}_1,\ \mathbf{c}_2,\ \mathbf{c}_4\} \Rightarrow \dim(col(A)) = 3$$

The row space of A: Remove $\mathbf{R}_4$; it is pivotless.

$$row(A) = sp\{\mathbf{R}_1,\ \mathbf{R}_2,\ \mathbf{R}_3\} \Rightarrow \dim(row(A)) = 3$$

3.6.1 The general procedure

How to find bases for spaces associated with a matrix?

Let $A \in M_{m\times n}$ matrix, and $A \sim H,\ H$ is RREF of A.

There are 3 types of spaces associated with A: A has

a) Row space

b) Column space

c) Null space (span of vectors for $\mathbf{x}$ that satisfies $A\mathbf{x} = 0$)

Procedure:

a) The non-zero rows of H form a basis for the row space.

b) The column has a pivot $\Rightarrow$ the column is a basis vector for the column space.

c) For a basis of null space $\mathscr{N}$, we need to solve this system: $H\mathbf{x} = 0$, to see how many basis vectors are in the solution space. The quantity of free variables in the system above will determine how many basis vectors in the solution space. If there are no free variables, then you only have one vector as the null space. One free variable generates a line; 2 generates a plane, 3 generates $\mathbb{R}^3$, and so on.

Recall: Number of basis vectors of a subset gives you, its dimension.

no. of basis for N $= \dim(\mathcal{N}) =$ no. of free variables $=$ no. of pivot-free columns

Remark 3.

$$\dim(col(A)) = \dim(row(A)) = \text{no. of pivot columns in H}$$

Example 3.1.

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 2 \\ 2 & -1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 & 3 \\ 2 & 1 & 3 & 3 & 3 \end{bmatrix} \sim H = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Compute $null(A)$. Firstly, column 3 and column 5 of H are free variables, so we have two free variables. Therefore, the dimension of the null space is 2.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solve this linear system:

$$\begin{aligned} 1 \cdot x_4 &= 0 \Rightarrow x_4 = 0 \\ x_3 &= t \\ x_5 &= s \\ x_1 &= -x_3 - x_5 = -t - s \\ x_2 &= -x_3 - x_5 = -t - s \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -t-s \\ -t-s \\ t \\ 0 \\ s \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, $Null(A) = sp \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

The rank of a matrix is how many basis vectors it can form. Also, it can be quantified by the number of pivots in the matrix when reduced to RREF.

3.6.2 Rank

Let $A \in M_{m\times n}$ matrix.

1. The dimension of $col(A) = \dim(row(A))$ is called the rank of A, denoted by $rank(A)$. The rank can also be seen as the pivot count for a matrix reduced to RREF.
2. The dimension of the nullspace of A, represented as $null(A)$ is called the nullity of A.

Theorem 3.4. *The rank equation of a matrix, where H is in REF or RREF:*

$$\begin{aligned} &A \sim H \\ &A,\ H \in M_{m\times n}\ (\mathbb{R}) \end{aligned}$$

Based on H, we have:

$$rank(A) + nullity(A) = n \text{ (width of matrix)}$$

Proof. If $A \sim H$

$$rank(A) = \text{no. of pivots in H} = \text{no. of columns in H with pivots}$$

$$nullity(A) = \text{no. of pivotless columns}$$

$$\begin{aligned} rank(A) + nullity(A) &= \text{col with pivots in H} + \text{col without pivots in H} \\ &= \text{no. of columns in A and H} \\ &= n \end{aligned}$$

If you have a matrix with a zero column on the right:

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 & 0 \\ 7 & 8 & 9 & 0 & 0 \\ 10 & 11 & 12 & 0 & a \end{array}\right] = \mathbf{0}$$

a is guaranteed to be mapped to zero. That is, the last element of $\mathbf{x}$.

$$x \cdot a = b \Rightarrow x = \frac{b}{a}; b \neq 0 \Rightarrow a \neq 0$$

■

4 Matrices and Linear Maps

Recall from section 1: A linear map T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a map that meets the following conditions:

$$T(\mathbf{x} + \alpha \mathbf{y}) = T(\mathbf{x}) + \alpha T(\mathbf{y})$$

As long as the map satisfies this then it is a linear map. This map can be modeled with a matrix of the appropriate size.

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, let $A \in M_{m \times n}\ (\mathbb{R})$.

$$A = \begin{bmatrix} | & | & | & \cdots & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) & \cdots & T(\mathbf{e}_n) \\ | & | & | & \cdots & | \end{bmatrix}$$

$T(\mathbf{e}_i) = \mathbf{v}_i \in \mathbb{R}^m; \mathbf{v}_i$ has m rows. $\mathbf{e}_1,\ \mathbf{e}_2,\ \ldots,\ \mathbf{e}_n$ are basis vectors that span $\mathbb{R}^n$. (i.e., $sp(\mathbf{e}_1,\ \mathbf{e}_2,\ \ldots,\ \mathbf{e}_n) = \mathbb{R}^n$).

Then $T(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$.

You can call A as the standard matrix representation of T.

Proof: Why do linear maps work?

For any possible input $\mathbf{x} \in \mathbb{R}^n,\ \exists! \mathbf{x} = x_1 e_1 + x_2 e_2 + \cdots + x_n \mathbf{e}_n$ ($\mathbf{x}$ can be represented *uniquely* in the fashion of a linear combination, by definition of basis.)

$$
\begin{aligned}
T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n) \\
&= x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) + \cdots + x_n T(\mathbf{e}_n) \\
&= \begin{bmatrix} | & | & | & \cdots & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) & \cdots & T(\mathbf{e}_n) \\ | & | & | & \cdots & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
&= x_1 \begin{bmatrix} | \\ T(\mathbf{e}_1) \\ | \end{bmatrix} + x_2 \begin{bmatrix} | \\ T(\mathbf{e}_2) \\ | \end{bmatrix} + \cdots + x_n \begin{bmatrix} | \\ T(\mathbf{e}_n) \\ | \end{bmatrix}
\end{aligned}
$$

Example: $T : \mathbb{R}^4 \to \mathbb{R}^3$ is a linear map.

$$
\begin{aligned}
&T : ([x_1,\ x_2,\ x_3,\ x_4]) \\
&= [x_2 - 3x_3 - x_4,\ 6x_1 + 5x_2,\ x_3 + 2x_1]
\end{aligned}
$$

How do I redefine T in terms of the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$, i.e.

$$
T([x_1,\ x_2,\ x_3,\ x_4]) = A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
$$

Solution: $domain(T) = \mathbb{R}^4$. $\mathbb{R}^4 = sp([1,\ 0,\ 0,\ 0],\ [0,\ 1,\ 0,\ 0],\ [0,\ 0,\ 1,\ 0],\ [0,\ 0,\ 0,\ 1])$

To construct A, compute the image of the basis: $T(\mathbf{e}_{1\ to\ 4})$:

$$
\begin{aligned}
T(\mathbf{e}_1) &= [0,\ 6,\ 2] \\
T(\mathbf{e}_2) &= [1,\ 5,\ 0] \\
T(\mathbf{e}_3) &= [-3,\ 0,\ 1] \\
T(\mathbf{e}_4) &= [-1,\ 0,\ 0]
\end{aligned}
$$

($\mathbf{e}_1 = [1,\ 0,\ 0,\ 0]$, and so on – we just ran those through the linear map.)

$$
\begin{aligned}
A &\\
&= \begin{bmatrix} | & | & | & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) & T(\mathbf{e}_4) \\ | & | & | & | \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} 0 & 1 & -3 & 1 \\ 6 & 5 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{bmatrix}}_{\text{this is A}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
\end{aligned}
$$

$$\Rightarrow T(\mathbf{x}) = A\mathbf{x}$$

Question: If we have a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$, how can the $A(T(\mathbf{x}) = A\mathbf{x})$ help you to find $\ker(T)$ and $range(T)$?

Answer: $\ker(T) = null(A)$, and $range(T) = col(A)$

Why?

$$\ker(T) = \{\mathbf{x} \in \mathbb{R}^n \mid T(\mathbf{x}) = 0\} = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = 0\} = null(A)$$

You can tell $null(A)$ by checking $(A \sim H_{RREF})$ the location of the pivots and the number of the pivots in H_{RREF}.

$$range(T) = \left\{T(\mathbf{x}) \in \mathbb{R}^M \mid \mathbf{x} \in \mathbb{R}^n\right\} = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\} = col(A)$$

(How many columns have a pivot?)

Example: The linear map $T : ([x_1,\ x_2,\ x_3,\ x_4]) = [x_2 - 3x_3 - x_4,\ 6x_1 + 5x_2,\ x_3 + 2x_1]$: Deduce the kernel and the range. You can do that by finding the basis.

Firstly, find matrix A (already deduced before). Reduce it to H.

$$\begin{bmatrix} 0 & 1 & -3 & 1 \\ 6 & 5 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -5/24 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & 5/12 \end{bmatrix} = H$$

$$T(\mathbf{x}) = A\mathbf{x}$$

$\mathbf{c}$ represents the column vectors of H.

$$\mathbf{c}_4 = -\frac{5}{24}\mathbf{c}_1 + \frac{1}{4}\mathbf{c}_2 + \frac{5}{12}\mathbf{c}_3$$

$$\mathbf{c}_4 + \frac{5}{24}\mathbf{c}_1 - \frac{1}{4}\mathbf{c}_2 - \frac{5}{12}\mathbf{c}_3 = 0$$

$$col(A) = span\,(c_1,\ c_2,\ c_3) = range(T)$$

$$\ker(T) = \{\mathbf{x} \mid T\,(\mathbf{x}) = 0\} = \{\mathbf{x} \mid A\mathbf{x} = 0\} \Leftrightarrow \{\mathbf{x} \mid H\mathbf{x} = 0\}$$

The $\Leftrightarrow$ statement is true as $null(A) = null(H)$ if $A \sim H$.

To figure out the kernel, I need to figure out what is happening to $H\mathbf{x} = 0$.

$$H\mathbf{x} = 0$$

$$\begin{bmatrix} 1 & 0 & 0 & -5/24 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & 5/12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - \frac{5}{24}x_4 = 0 \Rightarrow x_1 = \frac{5}{24}x_4 = \frac{5}{24}s$$
$$x_2 + \frac{1}{4}x - 4 = 0 \Rightarrow x_2 = -\frac{1}{4}x_4 = -\frac{1}{4}s$$
$$x_3 + \frac{5}{12}x_4 = 0 \Rightarrow x_3 = -\frac{5}{12}x_4 = -\frac{5}{12}s$$

x_4 is a free variable, $x_4 = s$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \left\{ s \begin{bmatrix} 5/24 \\ -1/4 \\ -5/12 \\ 1 \end{bmatrix} \mid s \in \mathbb{R} \right\} = sp \left\{ \begin{bmatrix} 5/24 \\ -1/4 \\ -5/12 \\ 1 \end{bmatrix} \right\} = \ker(T)$$

This is the basis for the kernel, or a basis for $null(A) = null(H)$. $nullity(A) = 1$

$$nullity(A) + rank(A) = 1 + 3 = 4 = \text{no. of columns in A}$$

Based on this observation:

Remark. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map.

$$T(\mathbf{x}) = A\mathbf{x}$$

We can define

1. $rank(T)$ is the dimension of the range of T, which is the same as $rank(A)$.
2. The nullity of T ($nullity(T)$) is the dimension of the kernel, or the same as $nullity(A)$.
3. If A is an invertible matrix, then T is invertible as well. (i.e., $\exists T^{-1}$). $T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x}$

Using these definitions, we can determine the fundamental theorem of the linear map:

Theorem 4.1. $rank(T) + nullity(T) = \dim(domain(T))$

This is the same as $rank(A) + nullity(A) =$ no. of columns in A

We have a new way to say matrix A is invertible:

$$\begin{aligned}
&(A \in M_{n\times n}) \\
rank(A) &= n \\
A\mathbf{x} &= \mathbf{b} \text{ has a unique sol'n} \Rightarrow \text{no free variable} \Rightarrow n \text{ pivots} \\
\Leftrightarrow nullity(A) &= 0 \\
\Leftrightarrow A\mathbf{0} &= \mathbf{0} \\
\Leftrightarrow rank(A) &= \text{no. of columns}
\end{aligned}$$

In this case, we call A full rank. The full rank is the highest possible value for the rank.

A is invertible if and only if A is full rank.

If a space only contains the zero vector, the dimension is zero.

Theorem 4.2. *Preservation of subspaces.*

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map.

If W is a subspace of $\mathbb{R}^n$, then $T(W)$ is a subspace in $\mathbb{R}^n$. (Linear transformations never knock inputs within a subspace out of the subspace.)

Proof. $W = \begin{cases} \neq \varnothing \\ \mathbf{x},\ \mathbf{y} \in W,\ \mathbf{x} + \alpha\mathbf{y} \in W \end{cases}$

How about $T(W)$? If $\alpha,\ \beta \in T(W)$, we want to check $\alpha + r\beta \in T(W)$, and $r \in \mathbb{R}$

$$\exists \mathbf{x},\ \mathbf{y} \in W,\ T(\mathbf{x}) = \alpha,\ T(\mathbf{y}) = \beta$$

This means you have two inputs in the domain of T such that (the above statement).

We can find the inputs corresponding to $\alpha + r\beta$

Consider $\mathbf{x} + r\mathbf{y} \in W$, as W is a subspace.

$$T(\mathbf{x} + r\mathbf{y}) = T(\mathbf{x}) + rT(\mathbf{y}) = \alpha + r\beta \in T(w)$$

This means $T(w)$ is closed under addition and scalar multiplication.

$T(W) \neq \varnothing$, as W is a subspace $\Rightarrow \mathbf{0} \in W$ (which again shows $W \neq \varnothing$)

$$T(\mathbf{0}) = \mathbf{0} \Rightarrow T(\mathbf{w}) \neq \varnothing$$

$\mathbf{0}$ is an input in W.

Homework (something you should be able to do for the final exam): Prove if W' is a subspace in $\mathbb{R}^m$, then the pre-image of W' under T is a subspace of the domain $\mathbb{R}^n$.

$f : V \to W$: if f is bijective, then $|V| = |W|$. (In terms of size).

This is how sizes of infinity are compared.

Definition 4.1. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. What is injective, surjective, and bijective? Injectivity is the same as one-to-one. Bijective means isomorphism (one-to-one correspondence from domain to codomain, for example, $y = mx + b$ – every single element in the domain has a counterpart in the codomain, and the other way around.)

5 Injective, onto, and bijective

Definition 5.1. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map.

T is **one to one** (injective) if:

$$T(\mathbf{v}) = T(\mathbf{u}) \Rightarrow \mathbf{u} = \mathbf{v}$$

Or if $\mathbf{u} \neq \mathbf{v}$, then $T(\mathbf{u}) \neq T(\mathbf{v})$.

For example: $\begin{cases} f(x) = x & \text{one to one} \\ f(x) = x^2 & \text{one to many} \end{cases}$

$$\ker(T) = \{\mathbf{0}\} \Rightarrow T \text{ is one to one.}$$

A function is **onto** if:

i) $T(\mathbb{R}^n) = \mathbb{R}^m$ (The image/range covers the entire codomain)

ii) $\forall v' \in \mathbb{R}^m,\ \exists v \in \mathbb{R}^n\ s.t.\ T(v) = v'$

iii) $Range(T) = \mathbb{R}^m$

iv) $T(\mathbf{x}) = A\mathbf{x}$, only if $col(A) = \mathbb{R}^m$ and $rank(A) = m$

These three statements are equivalent. (A function is onto if every element on the codomain can be traced back to the domain)

Example: $f(x) = x$ is onto, but $f(x) = e^x$ is **not** an onto function as outputs are strictly positive. If you try to reverse a negative value (which is in the codomain) it does not exist.

5.1 Isomorphism (Bijective):

If a linear map is one-to-one and onto, the linear map is bijective.

Remark 4. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Represent T as $T(\mathbf{x}) = A\mathbf{x}$. The input vector must be n-dimensional, so column count of A is n. Output vector must be m-dimensional, so row count of A is m.

To compute A, it is the image of the basis of $\mathbf{e}_1 \ldots e_n$.

$$\begin{bmatrix} | & | & | & \cdots & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) & \cdots & T(\mathbf{e}_n) \\ | & | & | & \cdots & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ \vdots \\ y_m \end{bmatrix}$$

1. A function is one to one $\Leftrightarrow \ker(T) = null(A) = \{\mathbf{0}\}$ (The kernel of T is strictly the zero vector.) **THE COLUMNS OF $\mathbf{A}$ ARE LINEARLY INDEPENDENT.**

2. A function is onto $\Leftrightarrow col(A) = \mathbb{R}^m$ or $Rank(A) = m$ (full rank.) (**COL SPACE MUST MATCH THE CODOMAIN or ALL ROWS HAVE PIVOTS or RANK MATCHES DIMENSION OF OUTPUT)**

3. Isomorphism: $A \sim I \Leftrightarrow A^{-1}$

Example: $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T\left(\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\right) = [x_1 - 2x_2,\ x_2 + x_3,\ -2x_3 - x_1]$

Check if this is one-to-one, onto, or isomorphic.

For example: $T(\mathbf{x} + r\mathbf{y}) = T(\mathbf{x}) + rT(\mathbf{y})$

$$\begin{aligned} T(\mathbf{x}) &= A\mathbf{x} \\ T(\mathbf{x} + r\mathbf{y}) &= A(\mathbf{x} + r\mathbf{y}) = A\mathbf{x} + rA\mathbf{y} \end{aligned}$$

Example: $H\mathbf{x} = 0$, $\mathbf{x}$ unique?

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_3 \\ x_2 + x_3 \\ 0 \end{bmatrix}$$

This implies that x_3 is a free variable, meaning anything in this vector in the map brings it to zero:

$$s \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix},\ s \in \mathbb{R}$$

This implies: $\mathbf{u}_1 \neq \mathbf{u}_2$. i.e., $\mathbf{u}_1 = [-2,\ -1,\ 1],\ \mathbf{u}_2 = [-4,\ 2,\ 2],\ T(\mathbf{u}_1) = 0 \textit{ and } T(\mathbf{u}_2) = 0$

Based on the kernel argument, you can find two different vectors, meaning at least two different vectors map to zero.

Onto: $\dim(codomain) = \dim\left(\mathbb{R}^3\right) = 3$

$$rank(A) = 2$$

(The rank is 2 because H only has two pivots.)

Isomorphic: NOT. A is not invertible.

CHECK FOR ISOMORPHISM

$$T : \mathbb{R}^N \to \mathbb{R}^n,\ A \in M_{n\times n}\left(\mathbb{R}^2\right)$$

(The corresponding matrix has to be square)

$$A\mathbf{x} = \mathbf{0} \to \text{unique } \mathbf{x} = 0$$

(There should be a pivot for every column)

$$A \sim I_{n\times n}$$

5.1.1 Example 1: One-to-one but not onto

$$T : \mathbb{R}^3 \to \mathbb{R}^4$$

$$\begin{aligned} &T\left([x_1,\ x_2,\ x_3]\right) \\ &= [x_1 - x_2 + x_3,\ x_2 - x_3,\ x_3 - x_1,\ x_1 + 2x_2 + x_3] \end{aligned}$$

Check if one to one / onto / isomorphic.

Firstly, this transformation is not isomorphic as the input and output dimension does not match. The size of A will be 4×3, as you expect the input to be a 3D vector meaning the matrix must contain three columns.

The matrix A: $\dim\left(col(A)\right) = 3$, which is the best case. The column space cannot cover $\mathbb{R}^4$.

$\dim\left(col(A)\right) = 3 \Rightarrow Range(T)$ is strictly a subset of R^4 and is NOT R^4. The linear transformation is not onto.

The map **can** be one-to-one: Just set up $A\mathbf{x} = \mathbf{0}$ and check the solution count. All you need to do is check if it has free variables.

Write down the image of the standard basis:

$$A = \begin{bmatrix} | & | & | \\ T\left(\mathbf{e}_1\right) & T\left(\mathbf{e}_2\right) & T\left(\mathbf{e}_3\right) \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

No free variable. We have a unique solution for $A\mathbf{x} = \mathbf{0}$, meaning T is one-to-one as $\ker(T) = null(A) = \{\mathbf{0}\}$.

5.1.2 Example 2: Onto but not one-to-one

$$T : \mathbb{R}^3 \to \mathbb{R}^2; T([x_1,\ x_2,\ x_3])$$
$$= [x_1 - 2x_2 + 3x_3,\ x_2 - 2x_3]$$

Matrix $A \in M_{2\times 3}$

The matrix being wider than it is tall $\to \exists$ free variable. The system $A\mathbf{x} = \mathbf{0}$ cannot give you a unique solution, implying infinitely as many solutions. This means

$$\ker(T) = Null(A) \neq \{\mathbf{0}\} \Rightarrow \text{T is not one-to-one}$$

Check for onto: $Rank(A) = 2 \Rightarrow col(A) = \mathbb{R}^2$

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}$$

We know every single **row** contains a pivot.

5.1.3 Example 3:

$$T : \mathbb{R}^3 \to \mathbb{R}^3,\ A \in M_{3\times 3}$$

$$T : ([x_1,\ x_2,\ x_3])$$
$$= [x_2 - 2x_2 + 3x_3,\ x_2 - 2x_3,\ x_1 + x_2 + x_3]$$

If the domain and the codomain have the same dimension computations have to be done to figure out if the transformation is one-to-one, onto, or isomorphic.

Solution: $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \end{bmatrix} \sim I$

This matrix is not RREF but can convince me that the matrix can be reduced to an identity matrix.

Therefore, the linear system is **isomorphic**, implying it is one-to-one and onto. (T is isomorphic).

$$
\begin{aligned}
A \sim I \Rightarrow Rank(A) = 3 = \dim\left(\mathbb{R}^3\right) \Rightarrow onto \\
&\text{No free variable} \Rightarrow A\mathbf{x} \\
&= \mathbf{0} \text{ has a unique solution} \Rightarrow null(A) \\
&= \{\mathbf{0}\} \Leftrightarrow \ker(T) = \{\mathbf{0}\} \\
&\Rightarrow T \text{ is one-to-one} \Rightarrow T \text{ is isomorphic.}
\end{aligned}
$$

(Longer explanation above)

One-to-one but not onto $\Rightarrow$ The matrix is taller than it is wide. $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$

- **Typically** has pivots in every column $\Leftrightarrow$ column space is typically linearly independent.
- **NO FREE VARIABLES**
- Will not have pivots in every row.
- Column space does not span the codomain. For example, in a 3×2 matrix, $\dim(col(A)) \leq 2$. It needs to be 3 to be onto.
- If you have a 2D input your image will only be a plane, or something that is worse than that, unless you try to compress a 3D space into 2D, in which you can't in this case.

Onto but not one-to-one $\Rightarrow$ The matrix is wider than it is tall. $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$

- In a 2×3 matrix, $rank(A) \leq 2$. If $rank(A) = 2$, which occurs most of the time, the matrix is onto. **(COLUMN PIVOT COUNT = HEIGHT OF MATRIX)**
- $rank(A) \leq 2 \Rightarrow rank(A) < 3$. The rank can never match the width of the matrix $\Leftrightarrow$ has pivotless columns $\Leftrightarrow$ matrix has free variable $\Leftrightarrow$ the span of the column space has redundant vectors.
- Compressing a 3D input to a 2D output, of course the existence of one of the entries in the 4D input will be redundant.

Neither onto nor one-to-one:

- Read the above two like a flow chart. There are conditions on the first point. If the condition fails, it is neither.

Square matrices:

For square matrix $A \in M_{n\times n}$, A is one-to-one $\Leftrightarrow A$ is onto. **(A SQUARE MATRIX IS EITHER ISOMORPHIC, OR NEITHER ONE-TO-ONE OR ONTO.)**

RATIONALE: one-to-one requires pivots in all columns; onto requires pivots in every row. These notions become equivalent if the matrix is square.

Isomorphism:

$A \sim I \Leftrightarrow A$ is invertible $\Leftrightarrow A$ is one-to-one and A is onto.

6 Determinant

Represented by $\det(A)$ or $|A|$ (this isn't absolute value)

In lower dimensional cases ($\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^3$):

Definition 6.1. The determinant of a 1×1 matrix is just its entry.

$$A \in M_{1\times 1} \Rightarrow \det(A) = \alpha \in \mathbb{R}$$

The determinant of a 2×2 matrix is:

$$A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \Rightarrow \det(A) = \left| \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \right| = a_1b_2 - a_2b_1$$

The determinant of a 3×3 matrix is:

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$
\begin{aligned}
\det(A) &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \\
&+ a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}
\end{aligned}
$$

6.1 Cross product

If $\mathbf{a} = [a_1,\ a_2,\ a_3] \in \mathbb{R}^3,\ \mathbf{b} = [b_1,\ b_2,\ b_3] \in \mathbb{R}^3$

$$
\mathbf{a} \times \mathbf{b} = \det \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}
$$

In vector form:

$$
\mathbf{v} = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix},\ - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix},\ \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right)
$$

We also want to check if $\mathbf{v} \perp \mathbf{a},\ \mathbf{v} \perp \mathbf{b}$. Just use the dot product.

$$
\begin{aligned}
\mathbf{a} \cdot \mathbf{v} &= (a_1,\ a_2,\ a_3) \cdot (\mathbf{a} \times \mathbf{b}) \\
&= (a_1,\ a_2,\ a_3) \cdot \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix},\ - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix},\ \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right) \\
&= (a_1,\ a_2,\ a_3) \cdot (a_2 b_3 - a_3 b_2,\ -a_1 b_3 + a_3 b_1,\ a_1 b_2 - a_2 b_1) \\
&= a_1 a_2 b_3 - a_1 a_3 b_2 - a_2 a_1 b_3 + a_2 a_3 b_1 \\
&+ a_3 a_1 b_2 - a_3 a_2 b_1 \\
&= 0
\end{aligned}
$$

(All cancels out. Similar computations can be done for b.)

Lemma: $\mathbf{a} \times \mathbf{b} = \mathbf{v}$ such that $\mathbf{v} \perp \mathbf{a},\ \mathbf{v} \perp \mathbf{b}$

6.2 Geometric interpretation of determinants

Determinant of a 2×2 matrix

$$\begin{bmatrix} - & \mathbf{v}_1 & - \\ - & \mathbf{v}_2 & - \end{bmatrix}, \ \mathbf{v}_1, \ \mathbf{v}_2 \in \mathbb{R}^2$$

$$\det(A) = \text{The area defined by } \mathbf{v}_1, \ \mathbf{v}_2$$

If $\mathbf{v}_1 \parallel \mathbf{v}_2$, the area of the parallelogram formed by it is 0, so $\det(A) = 0$.

Proof.

$$\begin{aligned} \text{Area}^2 &= (||\mathbf{a}||\,||\mathbf{h}||)^2 \\ &= \left(||\mathbf{a}|| \cdot ||\mathbf{b}|| \cdot \sin^2\theta\right) \\ &= \left(||\mathbf{a}|| \cdot ||\mathbf{b}|| \cdot (1 - \cos^2\theta\right) \\ &= \cdots \end{aligned}$$

Dealing with three-dimensional vectors:

$$\mathbf{a}, \ \mathbf{b}, \ \mathbf{c} \in \mathbb{R}^3$$

If the three vectors are mutually independent, a parallelepiped can be formed.

The determinant of the three vectors tells us the volume of this thing.

The case where $\mathbf{a}, \ \mathbf{b}, \ \mathbf{c}$ are linearly dependent, the volume of the parallelepiped is zero.

If all vectors are parallel to each other, they define a 1-dimensional object.

Either case, $\det(A) = 0$, where $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

Example: All these operations have the same intent.

Find the volume of the parallelepiped determined by $\mathbf{a} = [1, \ 0, \ 3], \ \mathbf{b} = [0, \ 1, \ 2], \ \mathbf{c} = [3, \ 3, \ 1]$ and check if it isn't 0

$\Leftrightarrow$ Check of $\mathbf{a}, \ \mathbf{b}, \mathbf{c}$ are linearly independent

$\Leftrightarrow$ Check if $\{\mathbf{a}, \ \mathbf{b}, \ \mathbf{c}\}$ is a basis for $\mathbb{R}^3$

$\Leftrightarrow$ Check if $A\mathbf{x} = \mathbf{d}$ has a unique solution where $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

$\Leftrightarrow$ Check of $T = A\mathbf{x}$ is isomorphic, one-to-one, or onto

$\Leftrightarrow$ check if A is invertible.

Solution: Compute the determinant.

$$\text{Vol} = \det(A) = \begin{vmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 3 & 3 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} - 0 + 3 \cdot \begin{vmatrix} 0 & 1 \\ 3 & 3 \end{vmatrix} = |-14| = 14$$

The volume must be positive.

The position of the vectors must form something like this.

Therefore, these vectors are linearly independent, so they can span 3D space. The matrix $A \sim I$, has no free variable, the equation $A\mathbf{x} = \mathbf{d}$ has a unique solution, is full-rank, and the matrix is invertible.

THE VOLUME OF A PARALLELPIPED FROM THREE ADJACENT VECTORS IS THE DETERMINANT OF THE MATRIX FORMED BY THE ROW VECTORS

6.3 Generalization of determinant

The row vectors determine the edge of the polyhedron formed by the vectors. If the polyhedron has a nonzero volume, the row vectors are linearly independent.

Definition: The minor matrix A_{ij} for $A \in M_{n\times n}(\mathbb{R})$ is the $(n-1)\times(n-1)$ matrix. Obtain this matrix by removing the ith row and jth column of A (remove the column and row for the element we are targeting).

Example: The minor of b_3 is

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix} \to \begin{bmatrix} a_1 & a_2 & a_4 \\ c_1 & c_2 & c_4 \\ d_1 & d_2 & d_4 \end{bmatrix}$$

The cofactor of a_{ij} of A is:

$$a'_{ij} = (-1)^{1+j} \det(A_{ij}) \text{ where } A_{ij} \text{ is the minor for } a_{ij}$$

Based on these two, we are able to determine the determinant for an $n \times n$ matrix A.

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & \ddots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11} \cdot a'_{11} + a_{12} \cdot a'_{12} + \cdots + a_{1n} \cdot a'_{1n}$$

For example:

$$\det \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}$$

$$= a_1 a'_1 + a_2 a'_2 + a_3 a'_3 + a_4 a'_4$$

$$\begin{aligned} &= a_1 (-1)^{1+1} \det \left(\begin{bmatrix} b_1 & b_3 & b_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{bmatrix} \right) \\ &+ a_2 (-1)^{1+2} \det \left(\begin{bmatrix} b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{bmatrix} \right) \\ &+ a_3 (-1)^{1+3} \det \left(\begin{bmatrix} b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \\ d_1 & d_2 & d_4 \end{bmatrix} \right) \\ &+ a_4 (-1)^{1+4} \det \left(\begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{bmatrix} \right) \end{aligned}$$

The determinant of a 5×5 matrix:

$$\begin{aligned} \det(A) &= a_1 a'_1 + a_2 a'_2 + \cdots + a_5 a'_5 \\ &= a_1 (-1)^{1+1} \det(4 \times 4) + \cdots + a_5 (-1)^{1+5} \det(4 \times 4) \end{aligned}$$

To compute the determinant for a 5×5 matrix, you are required to compute the determinant of a 4×4 matrix 5 times.

You can work with any row you want as long as the rows stay the same.

Computing these are impossible by hand.

(You do not need to prove properties of determinants / use the properties to make computations easier)

$adj(A) = C^T$, where C is the minor matrix of A, a.k.a. $C = \begin{bmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{bmatrix}$ if $A \in M_{2\times 2}$

6.4 Tricks for computing determinant

- Compute $\det(A)$ along the row with the most zeroes
- Compute $\det\left(A^T\right)$ if a column has many zeroes (rotate)

6.4.1 Targeting different rows

Our target row is the row we are working with.

When calculating the determinant of a larger matrix, aim to compute it along the row with the most zeroes. For example:

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 5 \\ -2 & -6 & 7 \end{bmatrix}, \det(A) = 1 \cdot \begin{vmatrix} 1 & 5 \\ -6 & 7 \end{vmatrix} - 3 \begin{vmatrix} 0 & 5 \\ -2 & 7 \end{vmatrix} - 2 \begin{vmatrix} 0 & 1 \\ -2 & 6 \end{vmatrix}$$

You'll have to compute 2×2 determinants three times. However, when we target the second row:

$$\det(A) = 0(-1)^{2+1} \begin{vmatrix} 3 & -2 \\ -6 & 7 \end{vmatrix} + 1(-1)^{2+2} \begin{vmatrix} 1 & -2 \\ -2 & 7 \end{vmatrix} + 5(-1)^{2+3} \begin{vmatrix} 1 & 3 \\ 2 & -6 \end{vmatrix}$$

You will notice that one of the terms added is multiplied by zero, so we can get rid of that:

$$\det(A) = 1(-1)^{2+2} \begin{vmatrix} 1 & -2 \\ -2 & 7 \end{vmatrix} + 5(-1)^{2+3} \begin{vmatrix} 1 & 3 \\ 2 & -6 \end{vmatrix}$$

6.4.2 Transpose invariance

$$\det(A) = \det\left(A^T\right)$$

Working with the columns is the same as working with a row.

Example: $A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 4 & 1 & 5 \\ 0 & 2 & 2 & 6 \\ 0 & 1 & 4 & 7 \end{bmatrix}, \det(A) = \det\left(A^T\right) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 4 & 2 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 5 & 6 & 7 \end{vmatrix} = 1(-1)^{1+1} \begin{vmatrix} 4 & 2 & 1 \\ 1 & 2 & 4 \\ 5 & 6 & 7 \end{vmatrix}$

6.4.3 Triangular and diagonal matrices

If A is a triangular matrix (either upper or lower triangular), then $\det(A) =$ product of all its diagonal entries.

Example:

$$
\begin{aligned}
A = \begin{bmatrix} a_{11} & a_{12} & a_{12} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}&, \det(A) \\
&= a_{33} \cdot a'_{33} \\
&= a_{33}(-1)^{3+3} \begin{vmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{vmatrix} \\
&= a_{11} \cdot a_{22} \cdot a_{33} \\
&\begin{vmatrix} a_{11} & * & * & * \\ 0 & a_{22} & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & a_{nn} \end{vmatrix} \\
&= a_{nn}(-1)^{n+n} \begin{vmatrix} a_{11} & * & * \\ 0 & \ddots & * \\ 0 & 0 & a_{(n-1)(n-1)} \end{vmatrix} \\
&= \cdots = a_{nn} a_{(n-1)(n-1)} \cdots a_{11}
\end{aligned}
$$

If one of the diagonal entries are zero for an upper triangular matrix, then the determinant is zero.

The same applies to diagonal matrices. $\begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{vmatrix} = 2 \cdot 3 \cdot 4$

6.4.4 Row operations

1. **ROW SWAPS NEGATE:** Let $A \in M_{n \times n}$ matrix. Perform the row operation $R_i \leftrightarrow R_j$. Let A' be the $n \times n$ matrix after row interchange of one time. Then $\det(A) = -\det(A')$.

2. **LINEAR DEPENDENCE OF COLUMNS/ROWS MAKES IT ZERO:** If A has two equal or linearly dependent rows, then $\det(A) = 0$ (implies the column space of the matrix is linearly dependent and compresses everything to a lower dimension, which always makes the volume zero)

3. **ROW SCALING ALSO SCALES THE DETERMINANT:** $R_i \to r \cdot R_i$, $\det(A) = r \cdot \det(A)$ *Scaling one row of a matrix scales the determinant of the matrix by how much you scaled that row.*

4. **SCALING THE ENTIRE MATRIX APPLIES THE SQUARE-CUBE LAW:** $r \cdot A = A' \Rightarrow \det(A') = r^n \det(A)$ *(Based off the square-cube law – if a 2×2 matrix scales everything by two the determinant is 4 or the square of 2)*

5. **DETERMINANT WON'T CHANGE FOR ROW ADDITIONS:** $R_i \to R_i + rR_j$, $\det(A) = \det(A')$

Note: The determinant of a matrix's RREF is not necessarily the same as the original matrix, as row scaling could have been done.

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 5 \\ -2 & -6 & 7 \end{bmatrix} \sim A' = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 5 \\ -2 & -5 & 12 \end{bmatrix}$$

$$R_3 + R_2; \det(A) = \det\left(A'\right) = 3$$

$$R_3 - R_2, \det\left(A''\right) = 3$$

6. **SPLITTING MATRIX MULTIPLICATION:** Let $A,\ B \in M_{n\times n}$. $\det(AB) = \det(A) \cdot \det(B)$

7. **DETERMINANT OF $\mathbf{A^{-1}}$:** If A is invertible, then $\det\left(A^{-1}\right) = \frac{1}{\det(A)}$

Remark: A is invertible $\Rightarrow \exists A^{-1},\ A \cdot A^{-1} = A^{-1} \cdot A = I$

$$\det(I) = \det\left(A \cdot A^{-1}\right) = \det(A) \cdot \det\left(A^{-1}\right) = 1$$
$$1 = \det(A) \cdot \det\left(A^{-1}\right) \Rightarrow \frac{1}{\det(A)} = \det\left(A^{-1}\right)$$

7 Eigenvalues, Eigenvectors

Definition 7.1. Let $A \in M_{n\times n}(\mathbb{R})$ (square matrix). A scalar λ is an eigenvalue of A if there exists a non-zero vector $\mathbf{v}\backslash n\ \mathbb{R}^n$ such that $A\mathbf{v} = \lambda\mathbf{v}$. We will call $\mathbf{v}$ the eigenvector of A corresponding to the eigenvalue λ (eigenvectors must correspond to a specific eigenvalue).

7.1 Computation of eigenvalues

To find the eigenvalues of A, I am looking for λ such that $\lambda\mathbf{v} = A\mathbf{v}$, where $\mathbf{v}$ is a corresponding eigenvector, where $\mathbf{v} \neq 0$. We can call the pair $(\lambda,\ \mathbf{v})$.

$$
\begin{aligned}
\lambda\mathbf{v} &= A\mathbf{v} \\
\Leftrightarrow 0 &= A\mathbf{v} - \lambda\mathbf{v} \\
\Leftrightarrow 0 &= (A - I\lambda)\mathbf{v} \\
\Leftrightarrow \det(A - I\lambda) = 0
\end{aligned}
$$

$\mathbf{v} \neq 0$ is a solution for $(A - I\lambda)$. However, because $(A - I\lambda)\mathbf{v} = \mathbf{0}$, then $(A - I\lambda)$'s determinant is zero.

$\Rightarrow \lambda$ is an eigenvalue of A if it sastifies $\det(A - \lambda I) = 0$.

Recall: $A\mathbf{x} = \mathbf{0} \Rightarrow$ unique solution where $\mathbf{x} = \mathbf{0}$.

7.2 Computation of eigenspaces and eigenvectors

Definition 7.2. Let $A \in M_{n\times n}(\mathbb{R})$. The characteristic polynomial of A is $p(\lambda) = \det(A - \lambda I)$ (this expression itself, either side of the equality is the polynomial). The roots of this characteristic polynomial gives you the eigenvalues. The set $E_\lambda = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \lambda\mathbf{x}\}$ is the eigenspace of λ. This space contains the zero vector and all the eigenvectors of A corresponding to λ.

$$E_\lambda = null(A - \lambda I)$$

Why?

$$
\begin{aligned}
E_\lambda &= \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \lambda\mathbf{x}\} \\
\Leftrightarrow E_\lambda &= \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} - \lambda\mathbf{x} = 0\} \\
&= \{\mathbf{x} \in \mathbb{R}^n \mid (A - \lambda I)\mathbf{x} = 0\}
\end{aligned}
$$

7.2.1 Computational example

Suppose I have the square matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & 2 \end{bmatrix}$ and I want to compute the eigenvalues and the eigenspace.

1. Compute all eigenvalues for A.
2. For each eigenvalue λ of A, find its eigenspace E_λ.

FINDING EIGENVALUES

Find all λ for $0 = \det(A - I\lambda)$.

FINDING EIGENSPACES

For each eigenvalue, compute the nullspace of $A - I\lambda$.

The eigenvectors (nonzero) corresponding to that λ is spanned by the nullspace of $A - I\lambda$.

Determining eigenvalues Solution: $p(\lambda) = 0 \to$ roots of $p(\lambda)$ gives you all eigenvalues of A.

$$0 = p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & -1 \\ 1 & 3 & -2-\lambda \end{vmatrix}$$

$$\begin{aligned} &= (2-\lambda)(-1)^{1+1} \begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} \\ &= (2-\lambda)\left((2-\lambda)(-2-\lambda)+3\right) \\ &= (2-\lambda)\left(\lambda^2-1\right) = (2-\lambda)(\lambda-1)(\lambda+1) \end{aligned}$$

Based on this factorization $p(\lambda) = 0 \Rightarrow \lambda = 2,\ \lambda = 1,\ \lambda = -1$

Simplifying this gives you a polynomial to solve, where it is equated to zero.

The eigenvalues of A gives 2, 1, and -1. We have three different eigenspaces:

$$E_{\lambda=2},\ E_{\lambda=1},\ E_{\lambda=-1}$$

Determining eigenspaces For $\lambda = 2$, compute the nullspace of $A - \lambda I = A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & 3 & -4 \end{bmatrix}$. Nullspace: $(A - \lambda I)\mathbf{x} = \mathbf{0}$

Reduce the matrix. We will obtain $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. (We have $x_2 - x_3 = 0$ and $x_1 - x_3 = 0$)

This reduced matrix has precisely one free variable, so $x_3 = s,\ x_2 = s,\ x_1 = s$, so the general solution is $\mathbf{x} = \begin{bmatrix} s \\ s \\ s \end{bmatrix},\ s \in \mathbb{R}$. Therefore, the set of vectors in $E_{\lambda=2} = \left\{ \mathbf{x} \in s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ s \in \mathbb{R} \right\}$ (A set of vectors spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and has a subspace structure, as it is generated by span meaning closed under addition and scalar multiplication.)

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow \text{eigenvector}$$

An eigenvector corresponding to $\lambda = 2$ is where $A\mathbf{v} = \lambda\mathbf{v}$.

$$A\mathbf{v} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \Rightarrow A\mathbf{v} = \lambda\mathbf{v}$$
$$\lambda\mathbf{v} = 2 \cdot \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

For $E_{\lambda=1}$: $null(A - 1I) = null\left(\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 3 & 3 \end{bmatrix}\right)$

The nullspace is $\mathbf{x}$ such that $\beta\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore $x_3 = s,\ x_2 = s,\ x_1 = 0$ and $\mathbf{x} = sp\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

This means $E_{\lambda=1} = \left\{ \mathbf{x} = sp \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

This means $A\mathbf{v} = \lambda$ given $\mathbf{v}$ is in the span of $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

For $\lambda = -1$

$$null(A+I) = null \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & -1 \\ 1 & 3 & -1 \end{bmatrix} \sim null \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

To compute the solution, reduce the matrix.

$$\mathbf{x} = sp \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

This is one concrete example for computation of eigenvalues and corresponding eigenspaces. The eigenvectors are nonzero vectors in the corresponding eigenspaces.

7.3 Properties of eigenvalues

1. $A^k\mathbf{v} = \lambda^k\mathbf{v}$ (Proof: $A^2\mathbf{v} = AA\mathbf{v} = A\lambda\mathbf{v} = \lambda A\mathbf{v} = \lambda\lambda\mathbf{v} = \lambda^2\mathbf{v}$, then proceed with induction)

2. Zero (0) cannot be an eigenvalue for an invertible matrix.

Proof. A is invertible $\Leftrightarrow \det(A) \neq 0 \Leftrightarrow \det(A - 0 \cdot I) \neq 0 \Leftrightarrow (A - \lambda I) \neq 0 \Leftrightarrow \lambda = 0$ is not a solution for $p(\lambda) = 0 \Leftrightarrow \lambda = 0$ is **not** an eigenvalue of A.

3. A is invertible $\Rightarrow \frac{1}{\lambda}$ is an eigenvalue of A^{-1} corresponding to the eigenvector of $\mathbf{v}$

$A\mathbf{v} = \lambda\mathbf{v} \Rightarrow A^{-1}\mathbf{v} = \frac{1}{\lambda}\mathbf{v}$ (if A is invertible)

Proof.

Firstly, $\lambda \neq 0$, so $\frac{1}{\lambda}$ makes sense.

$$
\begin{aligned}
A\mathbf{v} &= \lambda\mathbf{v} \\
A^{-1}A\mathbf{v} &= A^{-1}\lambda\mathbf{v} \\
\mathbf{v} &= A^{-1}\lambda\mathbf{v} \\
\mathbf{v} &= \lambda A^{-1}\mathbf{v} \\
\frac{\mathbf{v}}{\lambda} &= A^{-1}\mathbf{v} \\
\frac{1}{\lambda}\mathbf{v} &= A^{-1}\mathbf{v}
\end{aligned}
$$

■

8 Diagonalization

Let $A,\ B \in M_{n\times n}$ (STRICTLY SQUARE MATRICES). $A,\ B$ are similar if:

1. The determinants match ($\det(A) = \det(B)$)
2. Both matrices are either invertible or not invertible (A is invertible iff B is invertible)
3. The rank and nullity of them match ($rank(A) = rank(B)$; $nullity(A) = nullity(B)$)
4. $\det(A-\lambda I) = \det(B-\lambda I)$
5. Same solutions for their characteristic polynomial meaning $p(\lambda_A) = p(\lambda_D)$
6. Same eigenvalues

Proofs:

1. –
2. Recall A is invertible if and only if $\det(A) \neq 0$. $\Rightarrow$ If $A,\ B$ give you the same determinant, A is invertible if and only if B is invertible.
3. $rank(A) + nullity(A) = n$, and $rank(D) + nullity(D) = n$ (n is the column/row count of the matrix, and they have to at least be the same size) We need to show that the rank agrees with each other (which implies the nullities agree with each other).

   a. Show $rank(A) = rank(D)$.

Lemma: $rank(QM) = rank(M)$ if Q is invertible (because we only care about the number of pivots in the matrix – reduce it to RREF and based off it we can count how many pivots there are for the given matrix.)

You can review Q as a linear transformation, and if Q isn't invertible it will squish everything into a lower dimension, removing its bijectivity. That's why Q needs to be a bijective (isomorphic) linear map $\Rightarrow$ **Q PRESERVES ALL INFORMATION**. (Note that for matrix multiplication, the order of application of linear maps are from right to left)

If this is true, then $rank(B) = rank\left(P^{-1}AP\right)$. By the lemma, $rank\left(P^{-1}AP\right) = rank(AP) = rank(A)$ (as P is invertible).

$$\begin{aligned} &= \det\left(P^{-1}AP - P^{-1}\cdot\lambda I\cdot P\right) = \det\left(P^{-1}(A-\lambda I)P\right) \\ &= \frac{1}{\det(P)}\cdot\det(A-\lambda I)\cdot\det(P) = det(A-\lambda I) \end{aligned}$$

4. –
5. –

8.1 Diagonal matrices

The determinant of a diagonal matrix is the product of its diagonal entries. Computing the inverse matrix for the diagonal matrix is also very easy (is the reciprocal of all entries in the diagonal matrix).

1. Recall: A matrix A is diagonalizable if $\exists$ an invertible $n\times n$ matrix P such that $D = P^{-1}AP$ where D is a diagonal matrix.
2. If A and D are similar, then $\exists P$ such that $D = P^{-1}AP$ or $PDP^{-1} = A$
   a. *Proof:* $\det(D) = \det\left(P^{-1}AP\right) = \det\left(P^{-1}\right)\det(A)\det(P) = \frac{\det(P)}{\det(P)}\det(A) = \det(A)$
3. We should know how to check if A is real-diagonalizable. $p(\lambda) = 0$

8.2 Check if a matrix is diagonalizable

Information about diagonalization: $A\in M_{n\times n}$ (strictly square)

1. A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors.

$P(\lambda) = \det(A - \lambda I) = 0 \Rightarrow \lambda_1 \cdots \lambda_k$ eigenvalues for A

For $E(\lambda_1)$, basis vector $\mathbf{v}_i$ is a good representation for the eigenvector corresponding to λ_1, $\lambda_1 \mathbf{v}_i = A\mathbf{v}_i$. ($\mathbf{v}_i$ is a basic vector, which changes on which eigenvalue you are checking) However, eigenspaces do not necessarily contain one eigenvector, so you'll have to figure it out yourself.

The total number of eigenvector spaces must be equal to n. Once you've deduced n eigenvectors, you must verify that they are linearly independent by putting them as column vectors in a matrix, then row reducing it.

2. If $A \in M_{n\times n}$ and A has n **distinct** eigenvalues, $\Rightarrow$ then A is diagonalizable. Write down the characteristic equation. **(This is a strictly an if then statement and not an iff)**

$$p(\lambda) = 0, \deg(p(\lambda)) = \text{no. of distinct roots} \Rightarrow A \text{ is diagonalizable}$$

All distinct eigenvalues imply all the basis of all eigenvectors $\mathbf{v_i}$ are linearly independent.

8.3 The procedure of diagonalization

1. Find $\det(A - \lambda I) = 0 \Rightarrow$ roots $\lambda_1 \ldots \lambda_n$ (check if A is diagonalizable; all roots must be distinct)

2. Find $\mathbf{v}_1,\ \mathbf{v}_2 \ldots \mathbf{v}_n$ for $\mathbf{v}_1 \ldots \mathbf{v}_n$ such that $A\mathbf{v}_i = \lambda \mathbf{v}_i \forall i$

3. Use them to construct matrix $P = \begin{bmatrix} | & | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \cdots & \mathbf{v}_n \\ | & | & | & | & | \end{bmatrix}$ and $D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$

4. $P^{-1} \cdot A \cdot P = D$

Why is this true? *Proof.* $P^{-1}AP = D \Leftrightarrow AP = PD$ (pre-multiply P on both sides)

Verify if $AP = PD$. Compute LHS and RHS.

$$
\begin{aligned}
LHS &= AP \\
&= A\begin{bmatrix} | & | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \cdots & \mathbf{v}_n \\ | & | & | & | & | \end{bmatrix} \\
&= \begin{bmatrix} | & | & | & | & | \\ A\mathbf{v}_1 & A\mathbf{v}_2 & A\mathbf{v}_3 & \cdots & A\mathbf{v}_n \\ | & | & | & | & | \end{bmatrix}
\end{aligned}
$$

$$
A\mathbf{v}_i = \lambda_i \mathbf{v}_i
$$

$$
\begin{aligned}
&= \begin{bmatrix} | & | & | & | & | \\ \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \lambda_3\mathbf{v}_3 & \cdots & \lambda_n\mathbf{v}_n \\ | & | & | & | & | \end{bmatrix} \\
&= \begin{bmatrix} | & | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \cdots & \mathbf{v}_n \\ | & | & | & | & | \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix} \\
&= PD = RHS
\end{aligned}
$$

Example: $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -3 & 5 & 4 \end{bmatrix}$. Check if A is diagonalizable. If yes, find $D,\ P$ such that $P^{-1}AP = D$.

1. Find all eigenvalues.

$$
\det\left(\begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 3-\lambda & 0 \\ -3 & 5 & 4-\lambda \end{bmatrix}\right) = (\lambda-2)(\lambda-3)(\lambda-4) = 0
$$

The eigenvalues: $\lambda_1 = 2,\ \lambda_2 = 3,\ \lambda_3 = 4$

For $\lambda = 2$, $A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 5 & 2 \end{bmatrix}$; $null(A - \lambda I) = sp\left(\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}\right)$. Your basis is $\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \mathbf{v}_1$;
$[A]\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ where $\lambda = 2$.

For $\lambda = 3$, $A - 3I = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ -3 & 5 & 1 \end{bmatrix}$, $null(A - 3I) = sp\left(\begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}\right)$. Your basis is $\begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix} = \mathbf{v}_2$.

For $\lambda = 4$, $A - 4I = \begin{bmatrix} -2 & 0 & 0 \\ -1 & -1 & 0 \\ -3 & 5 & 0 \end{bmatrix}$, $null(A - 4I) = sp\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$. Your basis is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{v}_3$.

$\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$ should be linearly independent, as the eigenvalues are distinct.

$$P = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 4 & -5 & 1 \end{bmatrix}$$

Fact: Let $\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k$ be eigenvectors corresponding to distinct eigenvalues $\lambda_1 \ldots \lambda_k$. Then, $\mathbf{v}_1 \ldots \mathbf{v}_k$ are linear independent $\Rightarrow P = \begin{bmatrix} | & | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \cdots & \mathbf{v}_k \\ | & | & | & | & | \end{bmatrix} \Rightarrow P$ is invertible. Then

$$P^{-1}AP = D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Where A is similar to D.

$$\det(A) = \det(D) = 2 \cdot 3 \cdot 4 = 24$$

Definition 8.1. Let A be an $n \times n$ matrix with characteristic polynomial

$$p(\lambda) = (\lambda - \lambda_1)^{\alpha_1} (\lambda - \lambda_2)^{\alpha_2} \cdots (\lambda - \lambda_i)^{\alpha i}$$

$p(\lambda) = 0 \Rightarrow \lambda_1,\ \lambda_2,\ \cdots,\ \lambda_i$ with order $\alpha_1 \cdots \alpha_i$.
Algebraic multiplicity of λ_i is α_i
Geometric multiplicity of λ_i is $\dim\left(E_{\lambda_i}\right) = \beta_i$

Fact: For every λ_i : If $alg(\lambda_i) = geo(\lambda_i) \Rightarrow A$ is diagonalizable **(Matching algebraic and geometric multiplicity FOR ALL eigenvalues will guarantee you are diagonalizable matrix)**

8.4 The diagonalization process

Let $A \in M_{n\times n}$.

1. Find the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$
2. Factor. $p(\lambda) = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_i)^{n_i}$

$$
\begin{aligned}
&alg(\lambda_i) = n_i \\
&geo(\lambda_i) = \dim\left(E_{\lambda_i}\right) = \dim(null(A - \lambda_i I)) = nullity(A - \lambda_i I)
\end{aligned}
$$

To figure $null(A - \lambda_i I)$, $(A - \lambda_i I)\mathbf{x} = \mathbf{0}$ and figure out how many free variables, which tells us the dimension of the nullspace. This tells us the dimension of the eigenspace.

Once you know the dimension for the eigenspace, you will know the geometric multiplicity.

3. Check if $alg(\lambda_i) = geo(\lambda_i)$ for every λ_i. If that is the case the matrix is diagonalizable.
4. If A is diagonalizable, then you can diagonalize A by the following:

$$
A \sim D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}
$$

An example:

If A is an $n \times n$ matrix and A is diagonalizable. Then there is a shortcut to compute A^k.

Solution: $P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$

$$
\begin{aligned}
A &= PDP^{-1} \\
A^k &= \left(PDP^{-1}\right)\left(PDP^{-1}\right)\cdots\left(PDP^{-1}\right) \text{ k times} = P\cdot D^k\cdot P^{-1}
\end{aligned}
$$

Note that because matrix multiplication is associative you have a bunch of $P^{-1}P$ cases.

Then $D^k = \begin{bmatrix} \lambda_1^k & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n^k \end{bmatrix}$

When you mention an eigenvector, it must correspond to an eigenvalue

The rank-nullity theorem will appear in the final exam.

Determine whether matrix $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ is real diagonalizable.

$$
P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}
$$

$$
\begin{aligned}
p(A) &= \det(A-\lambda I) \\
(a-\lambda)(b-\lambda)-cd &= 0 \\
\Delta &< 0 \Rightarrow \text{Can't be real diagonalized}
\end{aligned}
$$

Solution:

$$
p(\lambda) = \det(A-\lambda I) = \begin{vmatrix} -\lambda & 0 & -2 \\ 1 & -2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda)\left(\lambda^2-3\lambda+2\right) = (2-\lambda)(\lambda-2)(\lambda-1)
$$

$$
\lambda_1 = 2,\ \lambda_2 = 2,\ \lambda_3 = 1
$$

You have a repeated eigenvalue. This means the eigenvalue 2 repeats twice, meaning the algebraic multiplicity for $\lambda = 2$ to be two. This means we expect the geometric multiplicity to be two as well.

Meaning $\dim\left(E_{\lambda=2}\right) = 2$. If this were not the case, $alg(A) \neq geo(A) \Rightarrow A$ wouldn't be diagonalizable.

Checking eigenspaces: For $\lambda = 2, null(A - \lambda I)$ (the nullspace is the solution space of $(A - \lambda I)\mathbf{x} = \mathbf{0}$)

$$A - \lambda I = \begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow nullity(A - \lambda I) = 2$$

Based on this I can tell there are two free variables, meaning $\dim(\mathbf{x}) = 2 \Rightarrow \dim(E_{\lambda=2}) = 2$.

For $\lambda = 1$

$$A - \lambda I = \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Two columns have a pivot $\Rightarrow$ 1 free variable $\Rightarrow \dim(\mathbf{x}) = 1 \Rightarrow \dim(E_{\lambda=1}) = 1 \Rightarrow geo(E_{\lambda=1}) = 1$

The eigenvalue only appears one time during the factorization of the characteristic polynomial, so the algebraic multiplicity of this eigenvalue is 1. Therefore, $alg(\lambda = 1) = geo(\lambda = 1)$.

$\Rightarrow A$ is diagonalizable; $\Rightarrow A$ is similar to $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\det(D) = 2 \cdot 2 \cdot 1 = 4$

What can we get out of this?

1. $\det(A) = \det(D) = 2 \cdot 2 \cdot 1 = 4$ *As A is similar to D, their determinants match.*
2. Is A invertible? Yes. A is invertible as D is invertible *(and also because we just computed the determinant of A already and it is nonzero)*
3. $T(\mathbf{x}) = A\mathbf{x} : \mathbb{R}^3 \to \mathbb{R}^3$ (And this is bijective because A is invertible - $A \sim I; A\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{0}$.)
4. $nullity(A) = 0$; A is full rank as $rank(A) = 3$.

Rank-nullity theorem: $nullity(A) + rank(A) = n$; $0 + 3 = 3$

5. $A\mathbf{x} = \mathbf{b}$ will always have a unique solution.

8.5 Every symmetric matrix is diagonalizable $\mathbf{A^T} = \mathbf{A}$

Example: $A = \begin{bmatrix} 3 & -3 & -3 \\ -3 & 3 & -3 \\ -3 & -3 & 3 \end{bmatrix}$; $A^T = A \Rightarrow A$ is symmetric $\Rightarrow A$ is diagonalizable $\Rightarrow A$ is similar to $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

(Excuse: from lecture, every symmetric matrix is diagonalizable.)

Computing the eigenvalues for A:

$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -3 & -3 \\ -3 & 3-\lambda & -3 \\ -3 & -3 & 3-\lambda \end{vmatrix} = (6-\lambda)(\lambda+3)(\lambda-6) = 0 \Rightarrow$ Eigenvalues: $\lambda_1 = 6,\ \lambda_2 = -3,\ \lambda_3 = 6$

In this case, you have one eigenvalue repeating twice, and one eigenvalue appearing only once. Normally, you would have to verify if $alg(\lambda = 6) = geo(\lambda = 6) = 2$ and the one for 3, but because the matrix is diagonalizable, you don't need to.

$$\dim(E_{\lambda=6}) = 2$$

$$A - 6I = \begin{bmatrix} -3 & -3 & -3 \\ -3 & -3 & -3 \\ -3 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; nullity(A - 6I) = 2$$

$$x_3 = s; x_2 = t; \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t-s \\ t \\ s \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\dim(E_{\lambda=1}) = 1$$

$$E_{\lambda=1} = sp\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

Note: To make P nicely, exactly 3 linearly independent eigenvectors must appear in your solutions. In this case, we have exactly 3, so we can use them to make up P:

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

Composed from the column vectors of $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$. The order of what vectors from the eigenvalues you put does not matter, but **it will affect** the column ordering of D. For example, if $\mathbf{v}_1$ is an eigenvector of $\lambda = 1$, 1 has to be put on the top left of the matrix.

$$P^{-1}AP = D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

8.6 Justifying order

$$\begin{aligned} & A \cdot \begin{bmatrix} | & | & | \\ \mathbf{v}_3 & \mathbf{v}_2 & \mathbf{v}_1 \\ | & | & | \end{bmatrix} \\ = & \begin{bmatrix} | & | & | \\ \mathbf{v}_3 & \mathbf{v}_2 & \mathbf{v}_1 \\ | & | & | \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \\ = & \begin{bmatrix} | & | & | \\ A\mathbf{v}_3 & A\mathbf{v}_2 & A\mathbf{v}_1 \\ | & | & | \end{bmatrix} \\ = & \begin{bmatrix} | & | & | \\ d_1\mathbf{v}_3 & d_2\mathbf{v}_2 & d_3\mathbf{v}_1 \\ | & | & | \end{bmatrix} \end{aligned}$$

Order of $\mathbf{v}_i$ will determine the order of eigenvalues you put in D.

8.7 Property of matrix exponentiation with diagonalization:

Theorem.

$$\begin{aligned} & A = PDP^{-1} \\ \Rightarrow & A^k = \left(PDP^{-1}\right)\left(PDP^{-1}\right)\left(PDP^{-1}\right)\cdots\left(PDP^{-1}\right)\ k \text{ times} \\ \Rightarrow & A^k = PD\left(P^{-1}P\right)D\left(P^{-1}P\right)D\left(P^{-1}P\right)\cdots\left(P^{-1}P\right)DP^{-1} \\ \Rightarrow & A^k = PD^kP^{-1} \end{aligned}$$

9 Learning objectives

Definition for some important concepts (particularly those after the midterm – worth 10% of the exam): The strongest way to do these questions is to memorize them term-by-term, but if you understand it, you should make sure you do.

EXAMPLE: ***Define eigenvalues.*** λ is a **real nonzero** ($\mathbb{R}$) and A is a real square matrix, and $\mathbf{v}$ is a **nonzero** vector such that such that number such that $A\mathbf{v} = \lambda\mathbf{v}$.

Define eigenvalues. Let $A \in M_{n\times n}(\mathbb{R})$. A scalar $\lambda \in \mathbb{R}$ is an eigenvalue for A if $\exists$ a non-zero vector $\mathbf{v} \in \mathbb{R}^n$ such that $A\mathbf{v} = \lambda\mathbf{v}$. *Optional: Moreover:* $\mathbf{v}$ *is an eigenvector of* A *corresponding to eigenvalue* λ *.*

a. A must be a square matrix only containing real values ($n \times n$)

b. $\lambda \in \mathbb{R}$ (Any real number)

c. $\exists\mathbf{v}$ must be non-zero, and $\mathbf{v} \in \mathbb{R}^n$ (must have the same dimensions as the matrix A)

d. Such that $A\mathbf{v} = \lambda$

e. Every symbol, operation, and unknown needs to be declared. The reader cannot be assumed to know any of the properties: assume they know nothing about the symbols and clearly define every single term before using them in your statement (all quantifiers – e.g. vector: can it be any vector, or can't be a specific type of vector? n-dimensional or any-dimensional? Nonzero?)

9.1 List of important definitions (they, or at least some will all be tested on)

You must be able to clearly define these terms. Just having the ideas of them are not enough. **Typically, the top result on Wikipedia is the definition.**

Either you must understand their intrinsic definition and write the definition in your own words to write a polished definition or memorize the definitions.

- Invertible (This term is subject to square matrices having real values)
  - Eigenvalue
  - Eigenvector
  - Algebraic / Geometric multiplicity
  - Characteristic polynomial
  - Eigenspace
  - Similar matrices (WHAT is the idea of $A \sim B$? Explain this mathematically.)

- The rank-nullity theorem: $rank(A) + nullity(A) = n$. You must define every term in this equation such as rank, nullity, acceptable properties of A, and that n is limited to naturals (from zero or one?)
- Nullspace: $A\mathbf{x} = \mathbf{0}$ for matrix $A \in M_{n\times n},$ all $\mathbf{x} \in \mathbb{R}^n$ such that this holds
- How to define basis (BASIS VECTORS CAN'T BE ZERO and the set should be linearly independent) (and the idea of linear independence)
  - Linear combinations
- Linear maps
  - Range (image)
  - Domain
  - Codomain
  - Kernel
  - The linear map can be represented as a matrix $T(\mathbf{x}) = A\mathbf{x}$
- Order of basis

9.2 Concepts

- REF (strictly speaking)
- RREF (strictly speaking)
  - When a matrix is presented in front of you, you should tell if it is REF or RREF.
  - Matrix with some parameters, which if you substitute values such that it will be REF / RREF. Check the online midterm.
- The idea of the basis (how to use it, not just the definition)

You should also know:

- Linear transformations: $T : \mathbb{R}^n \to \mathbb{R}^m : T(\mathbf{x}) = A\mathbf{x}$. The range, kernel, codomain, and domain of T is highly related to A. Indicate if the map is 1-1, onto, or bijective.
  - Is T 1-1, onto, bijective, and what is its range?

- The idea of determinant. Invertibility and determinant can relate to the linear map.
- Graphing linear maps in 2D case.
- Eigenvalue and eigenvectors, together with diagonalization.
- Implications of similar matrix (same det, same rank eq, same eigenvalues) (IMPORTANT) no need to prove but know the facts and how to use that.
  - Example: $A \sim B \Rightarrow$? – given A, and knowing that A is similar to B, what is $\det(B)$?

9.2.1 Matrix representation of linear maps

All linear transformations can be represented as a matrix. $T(\mathbf{x}) = A(\mathbf{x}),\ A \in M_{n\times n}$. I can then do analysis on that matrix if I want to do anything with the linear map.

$$\begin{aligned}&= [5x + yv_1,\ 5x + yv_2],\ \mathbf{v} = [v_1,\ v_2]\\ &= \begin{bmatrix} 5 & v_1 \\ 5 & v_2 \end{bmatrix} = [T]_{\{\mathbf{e}_1,\ \mathbf{e}_2\}}\end{aligned}$$

Some questions you might get: For what values of $v_1,\ v_2 \ldots$

1. T is invertible?
2. $rank(T) = 1$
3. $nullity(T) = 0,\ 1,\ 2,$ or 3 (And not possible is an option)
4. $rank(T) = 2 \wedge nullity(T) = 1$: is it possible? (No)
5. T is injective/surjective or bijective

This is a way to test you on the rank-nullity theorem, which states that $nullity(T) + rank(T) = n = 2$ (same as matrix width). If you know two of the nullity, rank, or dimensions of matrix, you can find the third one.

Solutions to the 4 questions:

Q1: T is invertible $\Leftrightarrow T^{-1}$ exists $\Leftrightarrow \exists A^{-1} \Leftrightarrow A \sim I \Leftrightarrow$ Every row/column has a pivot. Check that by reducing the matrix. $\begin{bmatrix} 5 & v_1 \\ 5 & v_2 \end{bmatrix} \sim \begin{bmatrix} 5 & v_1 \\ 0 & v_2 - v_1 \end{bmatrix}$. Based on this matrix, T will be invertible if $v_1,\ v_2 \in \mathbb{R}$ such that $v_1 \neq v_2 \Rightarrow T$ is an invertible map.

9.2.2 Graphing linear maps

Remark. Linear maps must fix zero $\Rightarrow T(\mathbf{0}) = \mathbf{0}$ regardless of the linear map. If a map is linear, $T : (\mathbf{x} + \alpha\mathbf{y}) = T(\mathbf{x}) + \alpha T(\mathbf{y})$. To fix zero: $T(\mathbf{0} + \mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0}) \Rightarrow T(\mathbf{0}) = 2T(\mathbf{0}) \Rightarrow T(\mathbf{0}) = \mathbf{0}$

We should be able to tell properties based on graphs.

9.3 Linear transformation stuff

Not a linear transformation: 0 doesn't map to 0

Reflection

$$T([x,\ y]) = (x,\ -y)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$T_e = \hat{}$$

Compute det

Determinant of rotation matrix is always 1

If your linear map is bijective, it is supposed to preserve all information. If it is not, and you start with 4 dots on the linear transformation, and one dot goes away, the linear map is no longer bijective.

Function composition: linear maps are applied from the right to the left. They have to go in order.

For not full rank linear transformations you end up sending some things to the kernel, and if you perform enough of that linear transformation, you will eventually get 0 if you keep applying the linear transformation. Sent to the kerel is information lost.

9.4 The idea of subspace

A given pace is a subspace if:

- The given subset is non-empty.
- The subspace must contain $\mathbf{0}$. Because subspace is closed under addition and scalar multiplication. So let $\mathbf{v} \in S$, $-\mathbf{v} \in S$, as $-\mathbf{v} = \mathbf{v} \cdot (-1)$, and because subspaces are closed under scalar multiplication, that has to be the case.

- Closure under scalar addition
- Meaning
- Pick $\mathbf{v},\ \mathbf{u} \in S$. WTS: If $\mathbf{u}+\mathbf{v} \in S$ then it is a subspace.
- Pick $r \in \mathbb{R}$. $r\mathbf{v} \in S$.
- $\mathbf{u}+r\mathbf{v} \in S$ – check that.

Examples

$$S = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} > 0\}$$

Not a subspace – does not contain the zero vector.

Pick $[x_1,\ 0]$ and $[x_2,\ 0]$ from S. Now:

$[x,\ 0]+\alpha\,[x_2,\ 0] = [x_1+\alpha x_2,\ 0]$ is in the space. So yes, it is closed under vector addition and scalar multiplication.

10 Change of basis

How to check if a given set of vectors can be set as a basis for a space V

- None of the given vectors can be zero, if so remove them from the basis set
- They must be linearly independent
- The spawn of those given vectors must span V (note that V must be a subset, as they can only span vector spaces provided it is a subset of some larger vector space).

Remark. To do this, pick any $\mathbf{v} \in V$. And $\begin{bmatrix} | & | & | & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \\ | & | & | & | \end{bmatrix} \begin{bmatrix} | \\ \mathbf{x} \\ | \end{bmatrix} = \mathbf{v}$, $\mathbf{b}_1 \ldots \mathbf{b}_k$ must be linearly independent.

Remark. The basis for a vector space of subspace is NOT unique. i.e., $\mathbb{R}^2 = sp\{[1,\ 0],\ [0,\ 1]\} = sp\{[2,\ 0,\ [0,\ 2]\}$

$\mathbb{R}^2$ is a two-dimensional space, but it is also a subspace for $\mathbb{R}^3$. This means $\mathbb{R}^2 \subseteq \mathbb{R}^3$ (a plane in 3-dimensional space)

NEW: $[T]_E = A\mathbf{x},\ A = \begin{bmatrix} | & | & | & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) & T(\mathbf{e}_4) \\ | & | & | & | \end{bmatrix}$. E is a set of basis vectors.

$$[T]_\beta = A'\mathbf{x},\ A \neq A$$

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^3$. $T(x,\ y,\ z) \to (x+y,\ 2x-y,\ 3x+5y)$. *Remark:* Can you rewrite this linear map differently other than matrix representation? Finding the matrix representation in standard basis and alternative basis.

Find A by using $\mathbf{e}_1 = [1,\ 0],\ \mathbf{e}_2 = [0,\ 1]$. $\alpha = \{\mathbf{e}_1,\ \mathbf{e}_2\}$ (Spans $\mathbb{R}^2$). Then $[T]_\alpha = \begin{bmatrix} | & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) \\ | & | \end{bmatrix}$. $T(\mathbf{e}_1) = T([1,\ 0]) = (1,\ 2,\ 3)$ and $T(\mathbf{e}_2) = T([0,\ 1]) = (1,\ -1,\ 5)$. Meaning our matrix using the standard basis α is $\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 3 & 5 \end{bmatrix}$. Therefore $T(\mathbf{x}) = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} | \\ \mathbf{x} \\ | \end{bmatrix}$. You could try a different basis β to produce another identical matrix representation?

Find A by using $\{\mathbf{b}_1 = [1,\ 1],\ \mathbf{b}_2 = [0,\ -1]\}$. Check: $\{\mathbf{b}_1,\ \mathbf{b}_2\}$ is a basis for $\mathbb{R}^2$. Firstly, $\mathbf{b}_1,\ \mathbf{b}_2 \neq 0$. To check if they are linearly independent, put them in a matrix and reduce it. $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow$ they are linearly independent. Once they are linearly independent, it tells you that every vector in $\mathbb{R}^2$ can be represented as a unique linear combination of the basis vectors $\mathbf{b}_1,\ \mathbf{b}_2$.

Pick any $\mathbf{v} \in \mathbb{R}^2,\ \mathbf{v} = [a_1,\ a_2]$. $\begin{bmatrix} | & | \\ \mathbf{b}_1 & \mathbf{b}_2 \\ | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} | \\ \mathbf{b}_1 \\ | \end{bmatrix} x_1 + \begin{bmatrix} | \\ \mathbf{b}_1 \\ | \end{bmatrix} x_2 = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $x_1,\ x_2$ has a unique solution $\forall \mathbf{v} = [a_1,\ a_2]$ (once that vector is fixed). This shows that $sp\{\mathbf{b}_1,\ \mathbf{b}_2\} = \mathbb{R}^2$. They are nonstandard but are alternative basis. In this case, how can I find the linear transformation with respect to this basis?

Let $\beta = \{\mathbf{b}_1,\ \mathbf{b}_2\}$. The idea is just to compute $[T]_\beta = \begin{bmatrix} | & | \\ T(\beta_1) & T(\beta_2) \\ | & | \end{bmatrix}$, which ends up being $\begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 8 & 5 \end{bmatrix}$. This is a representation of T with respect to β. This means if you want to work with β, you should write down $[T(\mathbf{x})]_\beta = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 8 & 5 \end{bmatrix} \mathbf{x} = \mathbf{y}$.

Are the two representations compatible with each other / do they tell us the same thing? Check:

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = [4,\ 2,\ 16]$$

$$[T]_\beta = [b_1,\ b_2]$$

$$\mathbf{x} = x_1 \mathbf{b}_1 + x_2 \mathbf{b}_2$$

$$[T(\mathbf{x})]_\beta = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 8 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 16 \end{bmatrix}$$

Okay, so

$$\gamma = [b_1 = [2,\ 0],\ b_2 = [0,\ 2]]$$

$$[\mathbf{x}]_\gamma = [1,\ 1]$$

$$\left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$$

$$[T]_\gamma = \begin{bmatrix} 2 & 2 \\ 4 & -2 \\ 6 & 10 \end{bmatrix}$$

$$[T(\mathbf{x})] = \begin{bmatrix} 2 & 2 \\ 4 & -2 \\ 6 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 16 \end{bmatrix}$$

11 The definition tables

11.1 Definitions

Here are the table of definitions:

DEF	EXPLAIN
Invertible matrix (property)	Let $A \in M_{n\times n}$. If $\exists B \in M_{n\times n}$ such that $AB = I$, where I is the $n \times n$ identity matrix, then A is invertible.
Eigenvalue	Let $A \in M_{n\times n}(\mathbb{R})$. A scalar $\lambda \in \mathbb{R}$ is an eigenvalue for A if $\exists$ (there exists) a non-zero vector $\mathbf{v} \in \mathbb{R}^n$ such that $A\mathbf{v} = \lambda\mathbf{v}$.
Eigenvectors	Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$, and that $\mathbf{v} \neq \mathbf{0}$. $\mathbf{v}$ is an eigenvector of matrix $A \in M_{n\times n}$ if there exists ($\exists$) $\lambda \in \mathbb{R}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.
Algebraic multiplicity	For an $n \times n$ matrix A, for its characteristic polynomial $p(\lambda) = \det(A - \lambda I) = (\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_i)^{n_i}$, the algebraic multiplicity for λ_i is n_i.
Geometric multiplicity	For an $n \times n$ matrix A, for its characteristic polynomial $p(\lambda) = \det(A - \lambda I) = (\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_i)^{n_i}$, the geometric multiplicity for λ_i is equal to: $nullity(A - \lambda_i I)$
Characteristic polynomial	Let $A \in M_{n\times n}$. The characteristic polynomial is $\lvert A - \lambda I \rvert,\ and\ so\ on \ldots$
Eigenspace	The space generated by the eigenvectors corresponding to the same eigenvalue – the space of all vectors that can be written as linear combination of these eigenvectors. (IMPROVE)
Similar matrices	Two $n \times n$ matrices A and D are similar if there exists an invertible matrix P such that $P^{-1}AP = D$ (the placement of P and P^{-1} do not matter). Similar matrices have the same determinant and eigenvalues.

DEF	EXPLAIN
The rank nullity theorem $rank(A) + nullity(A) = n$	The rank is the number of linearly independent columns of any real matrix $A \in M_{m\times n}$. The nullity is the dimension of then nullspace of A. The value n is the width of the matrix, which is a nonnegative integer. Both rank and nullity are nonnegative integers.
Nullspace	$A\mathbf{x} = \mathbf{0}$ for matrix $A \in M_{n\times n}$, nullspace is all $\mathbf{x} \in \mathbb{R}^n$ such that this holds
Basis vectors	A set B of vectors in $\mathbb{R}^n$ a vector space V if: 1. All of them are linearly independent 2. None of them are zero vectors 3. Every element in V can be written as a unique linear combination of basis vectors B
Linear combinations	Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$. $\mathbf{a}$ is a linear combination of them if $\exists r_1, r_2, \ldots, r_k$ such that $\mathbf{a} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \ldots + r_k\mathbf{v}_k$.
Linear maps	A linear map T is a mapping of vector spaces $V \in \mathbb{R}^n \to W \in \mathbb{R}^m$ where the following properties hold: $T(\mathbf{u} + r\mathbf{v}) = T(\mathbf{u}) + rT(\mathbf{v}) \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^n, r \in \mathbb{R}$.
Order of basis	The number of basis vectors in a set of basis vectors.

11.2 Things to know, again

- Eigenvalues can be zero
  - If so, matrix isn't invertible
- Non-invertible matrices can be diagonalized (diagonalization does not imply invertibility) – compute characteristic polynomial, find roots, and compute its algebraic multiplicity and geometric multiplicity, and check if they agree with each

other. If so, $\exists P,\ P^{-1},\ P^{-1}AP = D$ and P is constructed by the corresponding eigenvectors. Eigenvectors may not be zero as it will make P not invertible.

- Alg multiplicity means: if you construct your characteristic polynomial $P(\lambda)$, you can do your factorization:

$$\lambda_1 = a_1,\ \lambda_2 = a_2.\ \cdots,\ \lambda_n = a_n$$

- Order of $\lambda_1 = n_1 \Rightarrow alg(\lambda_1) = n_1$, and so on.
  - The order of the nth root is n_n
- For each $\lambda_i \to E_{\lambda_i} =$ nullspace of $A - \lambda_i I = \{\mathbf{x} \mid \mathbf{x}(A - \lambda_i I) = 0\}$

11.3 Subspace proofs

Prove that something is a subspace. If it is a subspace, you can get the basis that span it. The number of basis is the geometric multiplicity. $\dim\left(E_{\lambda_i}\right) = geo(\lambda_i)$.

This exam really emphasizes eigenvalues and eigenvectors.

Suppose I have a square matrix and I want to do diagonalization. Now, is it diagonalizable:

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -3 & 5 & 4 \end{bmatrix}$$

Find its eigenvalues and eigenvectors. Step 1. Construct the characteristic polynomial.

$$\begin{aligned} &\det(A - \lambda I) \\ (2-\lambda)(\lambda-3)(\lambda-4) &= 0 \\ \lambda &= 2 \\ \lambda &= 3 \\ \lambda &= 4 \end{aligned}$$

Order of all roots are 1, so algebraic multiplicity for all eigenvalues are 1.

For $\lambda_i : null(A-2I)$ Which is spanned by $sp \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ which is $\mathbf{v}_1 \to \lambda_1$. We can verify if this is correct if $A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$. As the dimension of this nullspace is 1 (span of the basis gives you the eigenspace), then you can tell the geometric multiplicity of $\lambda_1 = 2$ is 1. This means λ_1's algebraic and geometric multiplicity match. For the rest of the two eigenvalues, repeat the same computation twice.

For $\lambda_2 = 3 \Rightarrow [0,\ 1,\ 5]$ and for $\lambda_3 = 4 \Rightarrow [0,\ 0,\ 1]$

Meaning A is diagonalizable. IF so, ho w can you diagonalize A?

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 4 & 5 & 1 \end{bmatrix}$$

Remark. If diagonalization exists $P^{-1}AP = D$ then you can be clear that A is similar to D. If they are similar, they share the following properties.

- Determinants match
- Both are invertible, or both are not (invertibility is the same) but that is simply a consequence of no. 1
- Ranks match
- Nullity match
- $\det(A-\lambda I) = \det(A-\lambda I)$

12 Subspace proofs

To prove that a given subset is non-empty, **check if the subset is non-empty by checking if $\mathbf{0}$ is in the subset.**

- Why must subspaces contain the zero vector:
- If S is a subspace, pick $\mathbf{x} \in S \Rightarrow -\mathbf{x} \in S$. Then $\mathbf{x} + (-\mathbf{x}) \in S \Rightarrow \mathbf{0} \in S$
- Set builder notation: $\{\mathbf{x} \in \mathbb{R}^n \mid ||\mathbf{x}|| > 0\}$ or $\{\mathbf{x} \in \mathbb{R}^n \mid ||\mathbf{x}+\mathbf{y}|| > 0\},\ \mathbf{y} \in \mathbb{R}^n$

- This subset, the zero vector does not belong to this subset $\Rightarrow \{\mathbf{x} \in \mathbb{R}^n \mid ||\mathbf{x}|| > 0\}$ can never be a subspace.

- We also want to show that S is closed under + and scalar $\times$. Pick $\mathbf{x},\ \mathbf{y} \in S,\ r \in \mathbb{R}$. Show that $\mathbf{x} + r\mathbf{y} \in S$
- If you can show that this is true for any $\mathbf{x}, \mathbf{y}, r$, then S is closed under vector addition and scalar multiplication. That is the three steps. There is an advanced way to prove a subspace. Create a linear map that maps from a set to another vector space:
  - $T : S \to \mathbb{R}^n,\ n \in \mathbb{Z}$.
  - If you can build a bijective map between them, then $\Rightarrow S \in \mathbb{R}^n \Rightarrow S$ is a subspace. The dimension of a vector space of a subspace is equal to the number of basis vectors that span the subspace.
  - If subspace $V = \{\mathbf{0}\}, \dim(V) = 0$.
  - Basis vectors **cannot** be zero ($\mathbf{0}$) vectors. This is the reason why eigenvectors cannot be zero. For the idea of diagonalization, you make eigenvectors act like basis. However, **eigenvalues** can be zero.

Example:

Suppose I have a linear map $T :\ \mathbb{R}^n \to \mathbb{R}^m$.

First, show $\ker(T)$ or the nullspace of T is a subspace of $\mathbb{R}^n$.

Similarly, show the range of T is a subspace of $\mathbb{R}^n$.

Solution:

Let S be the nullspace.

When talking about kernel or nullspace, similar the set of the vectors: $\{\mathbf{x} \in \mathbb{R}^n \mid T(\mathbf{x}) = 0\}$. This is the definition of the null space (and this is set-builder notation).

Kernel proof: Firstly, it is a fact that for all linear maps: $T(\mathbf{0}) = \mathbf{0}$. This means $\mathbf{0}$ is in the kernel.

Secondly, we want to show that S is closed $+,\ \times$. Pick $\mathbf{x}, \mathbf{y} \in S,\ r \in \mathbb{R}$. Want to check: $\mathbf{w} = \mathbf{x} + r\mathbf{y} \in S$. If you can show that this is true, it means S is closed under addition and scalar multiplication. Let's try this:

$$T(\mathbf{w}) = \mathbf{0} \Rightarrow T(\mathbf{w}) = T(\mathbf{x} + r\mathbf{y}) = T(\mathbf{x}) + rT(\mathbf{y})$$

It is a given that $\mathbf{x}$ is an element in the nullspace, meaning the image of $\mathbf{x}$ under T is 0 (in other words $T(\mathbf{x}) = 0$). Also, $\mathbf{y}$ is in the kernel/nullspace **as defined by the set**

builder notation, so $T(\mathbf{y}) = 0$. This means $r \cdot \mathbf{0} = \mathbf{0}$. Therefore, the given subset is a subspace, and we can thus confirm that S is a subspace.

Proving that Range of $\mathbf{T}$ is a subspace:

Pick $\mathbf{u},\ \mathbf{v} \in Range(T)$

What to show: $\mathbf{u} + r\mathbf{v} \in Range(T)$ for any $r \in \mathbb{R}$. Then:

To prove a subspace, show closure and non-empty. Firstly, $T(\mathbf{0}) = \mathbf{0}$ (for zero element in the range, you can always find the inverse). (0 maps 0 to 0)

$\exists \mathbf{x},\ \mathbf{y} \in domain(T)$ such that $T(\mathbf{x}) = \mathbf{u},\ T(\mathbf{y}) = \mathbf{v}$. Then we want to show:

$$\mathbf{u} + r\mathbf{v} \in Range(T) \Rightarrow \exists \phi \in domain(T) \text{ such that } T(\phi) = \mathbf{u} + r\mathbf{v}$$

If we can find such ϕ, then $\mathbf{u} + r\mathbf{v}$ is an element in the range.

Claim $\phi = \mathbf{x} + r\mathbf{y}$. This is a good input where it can generate an output in the form $\mathbf{u} + r\mathbf{v}$, which can show that $\mathbf{u} + r\mathbf{v}$ is an element in the range.

$$T(\phi) = T(\mathbf{x} + r\mathbf{y}) = T(\mathbf{x}) + rT(\mathbf{y}) = \mathbf{u} + r\mathbf{y}$$

We can thus confirm $\mathbf{u} + r\mathbf{v}$ is an output that corresponds to $T(\phi)$. Therefore, $\mathbf{u} + r\mathbf{v} \in Range(T)$, and therefore $Range(T)$ is closed under $+,\ \times$ and we can thus confirm the range of T is a subspace.

12.1 Another subspace questions

Let V be a vector space of $k \times k$ matrices (all of them, meaning the zero matrix is one element of it).

$$V = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & \ddots & \ddots & a_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ a_{k1} & \cdots & \cdots & a_{kk} \end{bmatrix} \mid a_{ij} \in \mathbb{R} \forall i,\ j \right\}$$

Definition. $W = \{R \cdot A \cdot S \mid A \in V\}$ and $S,\ R$ are two fixed matrices in V.

Show that W is a subspace of V. *Proof.*

Firstly, show that $\mathbf{0} \in W$. But $\mathbf{0} = 0$ matrix $\in V$, and if $A = [0]$, then $R \cdot A \cdot S = R \cdot 0 \cdot S = [0] \in W$. Therefore, W is non-empty.

Closure under addition and scalar multiplication. This means I have to pick two elements: $X,\ Y \in W$. This means $\exists A,\ B \in V$ such that $RAS = X$ and $RBS = Y$, by definition of W. Pick $r \in \mathbb{R}$, and WTS:

$$X + rY \in W$$

If you show that this is true, then W has closure under addition and scalar multiplication.

$$X + rY = RAS + r \cdot RBS = R(AS + rBS) = R(A + rB)S$$

It is a given that A and B are two matrices chosen from vector space V, which has closure under addition and scalar multiplication, so $A + rB \in V$. Therefore $R(A + rB)S = X + rY \in W$.

$\Rightarrow W$ is closed under vector addition and scalar multiplication $\Rightarrow W$ is a subspace.

12.2 About isomorphism

For a bijective map:

The sets $V = W$ (From $V \subseteq W,\ W \subseteq V$)

13 Final exam preparation list

13.1 Q1

Definitions (they must be formally written down – which means every variable you mention must be defined.)

- Inverse
- Diagonalization
- Eigenvalues
  - $\lambda \in \mathbb{R}$ is an eigenvalue if $\forall v,\ v \neq 0 \Rightarrow A\mathbf{v} = \lambda \mathbf{v}$
- Eigenvectors
- Linear map
- Nullity

- Rank equation
- Linear combination
- Linearly independent
  - $\mathbf{v}_1 \ldots \mathbf{v}_n \in \mathbb{R}^m$ is linearly independent if $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \ldots + r_n\mathbf{v}_n = 0$ has only one solution: $r_1 = r_2 = \ldots = r_n = 0$. We can also instead construct this matrix: $\begin{bmatrix} | & | & \cdots & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & \cdots & | \end{bmatrix}$ and say the nullspace of this matrix only contains the 0 vector ($\mathbf{0}$).
- Linearly dependent
  - Above, but say the nullity of this matrix is greater than 0. (1 or more)

13.2 Q2

Recall matrix but letters are in some of the entries. Revisit this question.

Make sure to fully understand this question. And don't mess it up this time. You could also be asked something more than REF or RREF. For example, asking if a matrix is similar, asking the determinant, rank, eigenvalues, characteristic polynomial.

13.3 Right hand / left hand rule

What do you do to check the orientation of the basis? We just use $(\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3$. It follows the RHR if cross product is **positive** and the LHR if the cross product is negative.

Cross product is a special version of the determinant:

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ - & \mathbf{v}_1 & - \\ - & \mathbf{v}_2 & - \end{vmatrix}$$

13.4 The idea of basis and its span

For example:

$B = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\}$ and I want to write down $[\mathbf{e}_1]_B$. Then the solution is to find r_1 and r_2 such that:

$$\begin{bmatrix}1\\0\end{bmatrix} = r_1\begin{bmatrix}2\\1\end{bmatrix} + r_2\begin{bmatrix}0\\3\end{bmatrix}$$

This creates a linear system. Solve it. In other words, solve this augmented matrix:

$$\left[\begin{array}{cc|c}2 & 0 & 1\\1 & 3 & 0\end{array}\right] \sim \left[\begin{array}{cc|c}2 & 0 & 1\\0 & 3 & -\frac{1}{2}\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 0 & \frac{1}{2}\\0 & 1 & -\frac{1}{6}\end{array}\right]$$

The answer is $\begin{bmatrix}r_1\\r_2\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\-\frac{1}{6}\end{bmatrix}$

13.5 The idea of a linear map

Is a given map linear or nonlinear?

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) \to \begin{bmatrix}x+1\\y\end{bmatrix}$$

No, it does not fix 0.

$$T\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = \begin{bmatrix}0+1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

Proof for that zero must be fixed:

$$\begin{aligned}T(\mathbf{u}) + T(-\mathbf{u}) &= T(\mathbf{u}-\mathbf{u})\\&= T(\mathbf{0}) \text{ and } T(\mathbf{0}) + T(\mathbf{0})\\&= T(\mathbf{0}) = \mathbf{0} = 2T(\mathbf{0}) = T(\mathbf{0})\\\Rightarrow T(\mathbf{0}) &= \mathbf{0}\end{aligned}$$

13.6 Rank, nullspace, and rank equation of the linear map, and that linear maps can always be represented by matrices

$$T(\mathbf{x}) = A\mathbf{x}$$

And using this, we can represent the eigenvalues and eigenvectors for this linear map.

1-1, onto, and bijective, and draw a graph to represent these concepts.

13.7 Function composition of linear maps

After the composition, what happens to the range? What happens to the kernel or the nullspace? If you know the range and the kernel of T? What will happen to the result mapping?

$$T \circ G$$

Remember to study by cases.

13.8 Graph-type problems of linear maps

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13.9 How to represent a linear map by nonstandard basis

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13.10 Idea of eigenvalues of eigenvectors

How to check if a given matrix is diagonalizable or not and create a matrix that is NOT diagonalizable. Diagonalization gives us the similarity between two square matrices. If they are similar, what are the consequences? (They share the same rank eigenvalues ,determinant, invertibility)

13.11 Convexity

It is just a special way to do a linear combination. Linear combinations are just $r_! \mathbf{v}_1 + \cdots + r_n \mathbf{v}_n$

A convex combination means $r_1 + r_2 + \cdots + r_n = 1$

Just impose one more condition for these scalars. This way generates a convex combination. Together with this condition this is a convex linear combination.