CSC320 Notes

Visual Computing

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Contents

1 Image Transformations

Transformations we can do to images:

- Scaling
- Warping (preserves straight lines)
  - The basic transformation that is used to scan documents; identify the corners (perhaps by hand) which should be enough to do the warp
  - A homography / linear transformation

What are the class of transformations used to perform the operation? First, we need to know more about affine transforms.

Summary

- Two homogeneous coordinates are the same if they're multiples of each other (except 0)

- A point at infinity is where the last element of a homogenous coordinate is 0; can be represented as an angle from 0 to 180 degrees
- We can represent a line by a vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that $\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$ or more familiarly $ax + by + c = 0$
- Given points in homogeneous coordinates p_1, p_2, the homogeneous coordinates of the line that passes through them is $p_1 \times p_2$
- Given two lines in homogeneous coordinates, their point of intersection is $l_1 \times l_2$
- Convert a homogeneous point coordinate to a regular coordinate by scaling it so that the last element is 1, then remove the last element
- Affine transformations preserve parallelism, and look like $\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & g \end{bmatrix}$

1.1 Notation Conventions

- Assume that an image is continuous. This means accessing a point on the image can be done with continuous $\mathbb{R}$ values.
- **Points are represented using column vectors:** $\begin{bmatrix} x \\ y \end{bmatrix}$, bottom 0 top image height
  - Row vectors are matrices that only contain a single row: $\begin{bmatrix} x & y \end{bmatrix}$
  - Transposing: $\begin{bmatrix} x \\ y \end{bmatrix}^T = \begin{bmatrix} x & y \end{bmatrix}$
- Homogenous coordinate representation of any point $p \in \mathbb{R}^2$: Euclidean coordinate to homogeneous coordinates
  - $\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

  - When we represent a point in homogeneous coordinates, we don't represent that point with that vector only. It's this vector and any scaled version of this vector, all represent the same 2D point. In other words, for any $\lambda \in \mathbb{R} \setminus \{0\}$, $\lambda \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ represents the same 2D point.
  - $p \cong \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \cong \begin{bmatrix} -2x \\ -2y \\ -2 \end{bmatrix} \cong \begin{bmatrix} 2x \\ 2y \\ 2 \end{bmatrix}$
  - Two vectors of homogeneous coordinates are called equal if they represent the same 2D point.
  - $\begin{bmatrix} x \\ y \\ w \end{bmatrix} \cong \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} \Leftrightarrow \exists \lambda \neq 0, \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \lambda \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix}$

- Homogeneous coordinates to Euclidean coordinates:
  - $\frac{1}{c} \begin{bmatrix} a \\ b \\ c \end{bmatrix} [0:2]$

1.2 Points at Infinity

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} \infty \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} \infty \\ \infty \end{bmatrix}$$

With homogeneous coordinates, we have a finite representation of a point that is infinitely far away. We can represent a point infinitely away only using $\mathbb{R}$. This leads to very stable geometric computations.

Points at infinity have their last coordinate equal to 0. Points at infinity are also called *ideal points* in textbooks.

What do points at infinity represent? The space described by these homogenous coordinates are called a projected plane: the Euclidean plane and a bit more.

You can encode them as a clock position with arrows on both hands (0-180 degrees)

1.3 Homogeneous 2D Line Coordinates

How do we represent a line?

We have a line l on the plane. Suppose there is a point p that lies on the line: $p \cong \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$.

What's the most general equation for a line?

$$ax + by + c = 0$$

($y = mx + b$ cannot encode vertical lines) In matrix form, the homogeneous coordinates of a line can be represented by:

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

So, a line can also be represented by this. If you multiply this equation by any non-zero scalar, the line remains the same. Meaning for all $\lambda \neq 0$, this represents the same line.

$$\lambda \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

The vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is the vector holding the line coordinate. It can be interpreted in any way.

1.3.1 Line Coordinates Conversion Examples

What are the homogeneous coordinates of the line $y = x$? They're written in $l^T p = 0$

$$\begin{aligned} y &= x \\ &\Leftrightarrow -x + y = 0 \\ &\Leftrightarrow -x + y + 0(1) = 0 \\ &\Leftrightarrow \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0 \end{aligned}$$

$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ are the homogeneous coordinates of this line.

1.3.2 Coordinates of the line passing through two points

What are the homogeneous coordinates of the line that connects two points?

The setup, given p_1, p_2:

$$l^T p = 0$$

The line passes through two points, so they must satisfy the line equation: l must satisfy $l^T p_1 = 0$, $l^T p_2 = 0$

The fact that $l^T p_1 = 0$, $l^T p_2 = 0$ implies that l must be perpendicular to p_1 and p_2, so it means that l must be the cross product between the two

So, the general expression is, if I know the homogeneous coordinates of p_1 and p_2, then I can get the homogeneous coordinates of the line that passes through both.

$$l = p_1 \times p_2$$

So, given two image points $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$?

1. Convert to homogeneous coordinates
2. Compute the cross product

This gives us an immediate expression for the equation of the line.

1.3.3 Calculating the cross product

As a matrix-vector product:

$$p_1 \times p_2 = \begin{bmatrix} 0 & -z_1 & y_1 \\ z_1 & 0 & -x_1 \\ -y_1 & x_1 & 0 \end{bmatrix} p_2$$

So, we have an analytical expression for computing the line coordinates from two points.

Alternatively, as a determinant:

$$\begin{aligned} p_1 \times p_2 &= \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \\ &= i \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} - j \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} + k \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \end{aligned}$$

i, j, k are short hands for vectors. $i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Feel free to use whatever you want for cross products.

1.4 Coordinates of the Intersection of Two Lines

We have two lines that we know, and we want to find the homogeneous coordinates of their intersection.

We know line $l_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$, $l_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$. Find $p = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. It must satisfy:

$$l_1^T p = 0,\ l_2^T p = 0$$

So, what is p? It's the cross product.

$$p = l_1 \times l_2$$

We have a very easy way to compute intersections.

1.4.1 In case they're parallel

Now, what happens when the two lines are **parallel?** You'll get a point at infinity, with a 0 at the third coordinate. Here's an example given $y = 1,\ y = 2$:

$$l_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, l_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

We have the homogeneous coordinates of two lines. Their intersection is going to be the cross product, $l_1 \times l_2$. What's the result?

$$l_1 \times l_2 = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

This represents negative infinity at the x-axis. Everything is completely finite from the perspective of homogeneous coordinates, but this is a point of infinity. In a different direction, you would get a different point at infinity. This is how we can check if two lines are parallel. If their intersection computed through the cross product gives us 0 at the last coordinate, they have to be parallel.

The order of the terms in the cross product do not matter due to how homogeneous coordinates work. A point at infinity could be seen as a clock with a handle that points in both directions

1.5 Affine Transformations

A matrix can transform all vectors in a space.

Scaling: Where x is the scale of x

$$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$$

Shearing:

$$\begin{bmatrix} 1 & \text{horizontal shear} \\ \text{vertical shear} & 1 \end{bmatrix}$$

Rotations: shear horizontally and vertically by the same amount. The cosine function prevents size changes from rotating.

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Translation: requires homogeneous coordinates

$$\begin{bmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Chaining multiplications **compose transformations**, the order being from right to left. You can encode many transformations in a single matrix. However, regardless of how you multiply, as long as you are multiplying transform matrices, it they will **always** be in this form:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$$

The last row will **always be** $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. It may look like $\begin{bmatrix} 0 & 0 & g \end{bmatrix}$ in some cases.

We can multiply this entire matrix by any scalar except 0 and the homogeneous transformation would remain the same.

Most general affine transform matrix:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & g \end{bmatrix}$$

1.5.1 Geometric Properties Preserved by Affine Transformations

PRESERVED

- Parallelism
  - Parallel line pairs stay parallel to each other. Remember, parallel lines intersect at infinity
  - The transformation maps point at infinity to points at infinity (it may not be the same point at infinity)
  - Why? Because check the bottom row $\begin{bmatrix} 0 & 0 & g \end{bmatrix}$, the first two are 0
  - What if they were not? Then, some non-infinity points might be mapped to infinity and vice versa. This implies that parallelism would not be preserved (possibly but not always a 3D rotation could do this).

NOT PRESERVED

- Angles
- Lengths

1.6 Projective Transforms

Any 2D transform of homogeneous coordinates that is **represented by an invertible** 3×3 **matrix**. Known as Homography. It will **not** preserve parallelism.

For example, the way scanner apps distort images is a projective transform.

This is a fundamental distinction between general linear transformations from affine transformations. Affine transformations are much more restrictive; scanner apps can't use affine transforms by themselves.

The scanning procedure depends on figuring out what is the homography the 3×3 matrix that allows us to take a raw image and convert it to a proper, scanned image.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ l & m & g \end{bmatrix}$$

Homographies preserve linearity. Any lines that lie before a homography, will remain on another line after a homography.

1.7 Forward Mapping Algorithm

Suppose that we have the homography transformation matrix H. Now, create an algorithm that creates an image given an old image.

There is a bunch of ways to do it. There's a very easy way that doesn't give great results, and there's a slight tweak that solves it.

Our input: `src_image`, H

Output: `dest_image`.

How does this work? Let's see how the array of pixels can be represented as points on a 2D plane. Every pixel is just a (row, column) coordinate and we need to convert it to an (x, y) point on the plane.

So, the forward mapping algorithm:

```
for c=1 to num_columns
  for r=1 to num_rows:
    x, y = pixel_xy(r, c) # get the source pixel's (x, y) coordinates
    p = homogeneous_coords(x, y)
    p_prime = H * p
    x_prime, y_prime = euclidean_coords(p_prime)
    r_prime, c_prime = pixel_rc(x_prime, y_prime) # floor, round, ceiling, I don't care for now
    dest_image(r_prime, c_prime) = src_image(r, c)
```

This algorithm already has problems.

If I stretch the image, we could have gaps that will never be filled. A lot of my pixels in the destination image will contain nothing.

If I shrink the image, I can have the opposite problem. Two pixels from the source might map overwrite an existing written pixel in the destination image. We're losing information; not a good thing either.

It's possible to have both happen in the same image.

There's a very simple fix for this: an algorithm that doesn't go forward; it goes backwards.

1.8 Backward Mapping Algorithm

We have a loop that goes over the destination pixels. We go over the (x, y) coordinates in the destination image and use the inverse homography H^{-1} to figure out what pixel to target in the source image.

Because we are looping over the destination pixels, we can look at every single pixel in the destination image, and we'll get a value for every one of them. The destination image will get filled. There will be no gaps, regardless of whether we're doing magnification, stretching, and so on.

This means that will every single pixel in the destination image be filled. **No**, there could be blank pixels. A pixel in the destination image might map to a pixel **outside** the source image.

```
for c_prime=1 to num_columns
  for r_prime=1 to num_rows:
     x_prime, y_prime = pixel_xy(r_prime, c_prime)
     p_prime = homogeneous(x_prime, y_prime)
     p = inverse(H)*p_prime
     x, y = euclid(p)
     r, c = pixel_rc(x, y)
     des_image(r_prime, c_prime) = source[r, c]
```

2 Image Projection

How do we relate 3D points in the world to 2D pixels in an image?

We'll look at

- The geometry of perspective projection and concepts of center of projection, focal length
- How to represent 3D rays and points in homogeneous 3D coordinates
- Proving that all perspective images of a plane can be stitched via Homographies
- Why is it that homography warping is enough
- How do we estimate the Homographies we need?
- Understanding what 3D information is lost by perspective projection
- Making 3D measurements on a planar surface by warping its photo to a canonical view

- So on

2.1 Camera Aperture

Why do cameras have an aperture? Why not just have the sensor and nothing in part of it? Isn't that good enough?

Can't orthographically ray-trace. You'll get an extremely blurry image. All pixels receive light from all possible points.

So, instead, let's use an aperture. This results in a 1-1 correspondence between world points and image points, ideally. Like what you've been told, you'll get an upside down horizontally reflected image. This way of getting images has been known for hundreds of years; this is called a camera obscura.

2.1.1 Adjusting the Aperture

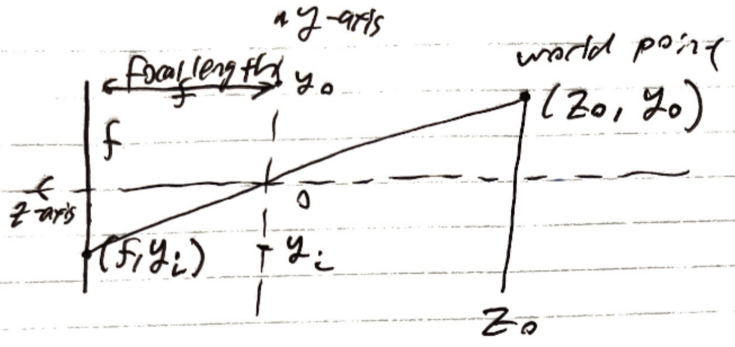
- $\uparrow$ Aperture size, $\uparrow$ Blurriness

2.2 Geometry of Perspective Projection

The focal length is the distance between the aperture and the image plane. Decreasing the focal length gives us a smaller image. It's a number that describes magnification.

- $\uparrow$ Focal length, $\uparrow$ image size, $\downarrow$ field of view

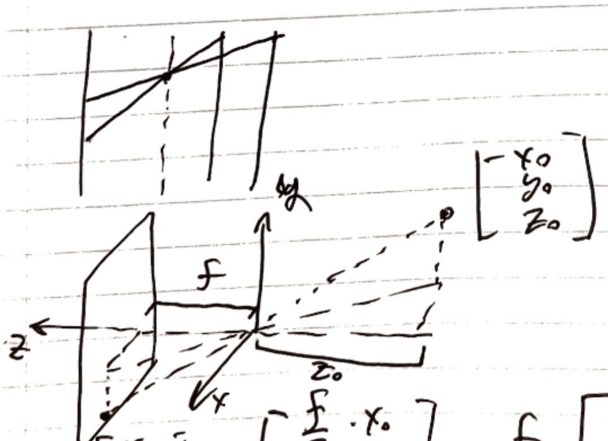
The lower the focal length, the image is being concentrated in a smaller set of pixels if the sensor's pixel count remains constant. Yet, it also gives us more field of view.



from similar triangles

$$\frac{y_i}{y_0} = \frac{f}{z_0} \Rightarrow y_i = \overset{\text{focal length}}{\frac{f}{\underset{\text{dist from origin}}{z_0}}} \cdot y_0$$

Image of a point is a scaled version



$$\begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = \begin{bmatrix} \frac{f}{z_0} \cdot x_0 \\ \frac{f}{z_0} \cdot y_0 \\ f \end{bmatrix} = \frac{f}{z_0} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

$$\begin{bmatrix} x_i \\ y_i \\ z_i = f \end{bmatrix} = \frac{f}{z_0} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

Where:

- $\begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}$ is the location on the camera sensor
- $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ is the location of the actual image
- f is the focal length
- For the purposes of axis-alignment, $-z_0$ is the distance from the aperture to the image.
- $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is the pinhole / apeature

Observation: a lower magnitude of z_0 (but constant x_0, y_0) means the object is closer to the camera, so it appears larger on the sensor (points are more spread out)

If we scale $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ by a constant factor, how the image is projected **will look the same.**

This is how we can think of 2D homogeneous coordinates (think of superliminal). Interpretation of homogeneous equality: All 3D points having the same projection:

$$\begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \cong \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

2.3 Representing 3D Points in Homogeneous Coordinates

3D coordinates with scale invariance can only represent rays. So, how do we represent 3D points? Introducing homogeneous 3D coordinates, like always, defined up to a scale factor.

$$\underset{\text{euclidean}}{\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}} \rightarrow \underset{\text{homogeneous}}{\begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ w_0 \end{bmatrix}} \rightarrow \frac{1}{w_0} \underset{\text{euclidean}}{\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}}$$

And guess what? $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 0 \end{bmatrix}$ represents a point at infinity. Homogeneous coordinates allow us to represent points at infinity in 3D.

Let's look at how homogeneous coordinates help us simplify the expression for perspective projection.

Let's expand our *point in world space to sensor space* transformation using our new homogeneous coordinates system:

$$\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{f_0}{z_0} x_0 \\ \frac{f_0}{z_0} y_0 \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix}$$

(We can change the scaling of the input $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix}$ and we would still get the "same" sensor space coordinate.)

2.4 Alignment and Stitching

Let's talk about images of specific geometric objects (like lines and planes). A very important property of prospective project is that it preserves linearity. All the lines that were straight in the original source is straight in the output. It falls from the fact that our transformation has a matrix in between – it's a linear transformation. But it's also possible to reason geometrically why lines in the world map lines to the image.

In order to take two photographs of the same object in two different viewpoints and stitch them together (in order to do a homography), **two conditions must hold for the homography:**

- Lines map to lines
- Each line in one image is transformed to a **unique** line in the other (invertibility) – in other words, we can reverse the transformation (minus loss of quality).

2.4.1 Linearity of Perspective Projection

Why does perspective projection preserve linearity?

Projection goes through a center of projection. Every point of a line gets mapped to the sensor along the center of projection.

2.5 When can I Stitch Together?

- If the image can be aligned
- Viewpoint must remain the same (same lines **in real life**) must map to the same place in the sensor – **preserve the center of projection.** You may rotate your camera, but your camera **must be anchored at the center of projection (usually the pinhole).**
  - Beware, an iPhone panorama is **not** that because you are moving your phone very far. It's not a real photograph anymore. You wouldn't be able to create a camera with a wider field of view and capture the same image.
  - That blue fence in the slide is curved not because the requirements for stitching failed – it's some post processing just for visual intent. Maybe your sensor is not planar – that doesn't matter.

  - Your Minecraft screenshots by moving your character's camera angle can be stitched together (with some assumptions I'm making).
  - **What a 360 camera can do**

- Photos taken from pure camera rotation can be aligned and stitched

Place the camera (or at least the center of projection) in the same place and you can stitch.

Radial distortions break linearity. Images that have non-linear distortions are not stitch-able without first undistorting (correcting) them. This could be an artifact of the actual lens. The image plane (sensor) **is** a flat surface, but because you're trying to get an image with a very wide FOV, you must pack all that information on a flat surface, and it is the lens that creates these distortions.

Because these distortions do not depend on what the lens is taking a picture of, they can be undone. This is called radial distortion correction.

2.6 How do we compute Homographies?

Let's see.

The big picture – as long as you can find 4 points in one image, you can compute H – a 3×3 matrix that maps one image to the other.

A single correspondence means you have some point out there and you can map it somewhere else.

$$\underbrace{\begin{bmatrix} x_i' \\ y_i' \\ 1 \end{bmatrix}}_{\text{known}} \cong \underbrace{\begin{bmatrix} a & b & c \\ d & e & f \\ h & k & 1 \end{bmatrix}}_{\text{unknown}:H} \underbrace{\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}}_{\text{known}}$$

Make some conversions:

$$x_i'(hx_i + ky_i + 1) = ax_i + by_i + c$$
$$y_i'(hx_i + ky_i + 1) = dx_i + ey_i + f$$

If 1 point correspondence gives us two equations, a 4-point correspondence gives us **8 equations and 8 unknowns.** That gives us our homography. That is our source and destination.

2.7 What 3D Information is Lost in Perspective Projection?

2D photographs can't tell us too much. There is this relation between the 3D coordinates in the world and the projection in the image. It is a 3D → 2D map. This means:

- Depth is lost

We will not be able to look at a photograph and figure out the 3D coordinates projected. Parallelism is not preserved. This means a lot of illusions can be made.

However, if we have some extra information, we can make some inferences:

- Known dimensions or lengths or angles for some objects
- Surfaces known to be planar
- Lines known to be parallel

Homographies can correctly transform planes, but anything that isn't on the plane will transform weirdly.

2.8 Vanishing Points, Vanishing Lines, Parallelism

We can project points at infinity. This is how vanishing points can be drawn and can be computed using a homography.

Vanishing points in 3D are points at infinity, but in a photo they have a concrete location.

Each direction on a plane has its unique vanishing point.

A set of parallel lines in 3D (same or opposite direction) will **always have the same vanishing point.**

The horizon is the vanishing line of the ground plane.

Important to note:

- Parallel lines have a vanishing point, but converse is not true

Just because two lines appear to converge doesn't mean they converge at a point from infinity.

3 Image Filtering

Just a transformation of an image. But a different kind of transformation. We've looked at geometric transformations first.

3.1 A Taxonomy of Image Transforms

We start with image f. We transform it with T to get g.

$$g = T[f]$$

- What is being transformed?
  - Geometric: coordinates only
    - Linear transforms
    - Non-linear transforms
  - Intensity: intensities / colors only
    - Point-wise: for every pixel independently of all others, we map intensities: $g(x, y) = T\left[f(x, y)\right]$. For example, image darkening or brightening

    * Local: transformations where a value at a pixel will depend on the values of the pixels in a neighborhood of the input image. We'll be focusing on linear transforms.
  - Domain: Image representation is being changed (no more pixel coordinates)
- How is it being transformed?

3.2 Linear Filters

A linear filter, for the time being, is a black box that takes the image as an input and outputs another image. Our task is to model mathematically what the black box does.

A linear filter is a transformation that has the following properties:

- Linear scaling of the intensities of the input = that of the output
- Literally the same as what you've learned in linear algebra

A filter transforms one signal into another. Filters are used to describe image formation (lens, blur, etc.), as well as to implement operations on images (edge detection, denoising)

A transformation T is linear if and only if it sastifies

$$T\left[a_1 f_1(x) + a_2 f_2(x)\right] = a_1 T\left[f_1(x)\right] + a_2 T\left[f_2(x)\right]$$

3.3 The Superposition Integral

Any transformation that can be expressed as a linear filter MUST have this property:

- The result g must be writable in the following way: as a weighted sum of the input at t multiplied by a function that calculates the weighted coefficient of it.

$$g(x) = \int_{-\infty}^{\infty} h(x,\ t)\, f(t)dt$$

Where $h(x,\ t)$ is the filter. For h, it asks: how much does this particular part of the input affect the output of that pixel?

Every value of the input will contribute to the output and the way it contributes to it is the function h. T was the black box. Now, T is the function h. Now, as I have that function, I can write out the integral.

h is a function of two variables. It depends on the coordinate of the 1D image, and the coordinate of the image on the input. We only have one constraint: the transformations are linear. Next, what are linear shift-invariant filter?

3.4 Linear Shift Invariant Input

Shifting the image gives me the same output. It doesn't matter where the image is, it is always invariant to the shift.

A transformation is shift-invariant $\iff$ shifted inptus produce identical but shifted outputs: $f^{'}(x) = f(x - x_0)$. The output would be: $g^{'}(x) = g(x - x_0)$. So, the shift-invariance property formally is:

$$T\left[f(x - x_0)\right] = g(x - x_0)\ \ \forall x_0$$

3.5 The Box Function

The box function is 0 except for a small interval. The function is:

$$\text{box}_{\varepsilon}(\tau) = \begin{cases} 1 & |\tau| \leq \frac{\varepsilon}{2} \\ 0 & \text{else} \end{cases}$$

And the scaled box function, where the area under the curve is always 1

$$\frac{\text{box}_{\varepsilon}(\tau)}{\varepsilon}$$

3.6 The Impulse Function

The impulse function (AKA Dirac's delta function) is:

$$\delta(\tau) = \lim_{\varepsilon \to 0} \frac{\text{box}_{\varepsilon}(\tau)}{\varepsilon}$$

The property of this function is:

$$\delta(\tau) = 0 \ \forall \tau \neq 0$$
$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

3.7 Impulse Response of a Linear Filter

Let's send δ through the filter. We'll get the response of the filter to an impulse. Applying the integral to $\delta(\tau)$:

$$g_1(x) = \int_{-\infty}^{\infty} h(x, \tau)\delta(\tau)d\tau = h(x, 0)$$

The filter's response to the impulse tells us a lot about the filter itself. For an image, we could get the shape of the filter.

The delta function is shift-invariant. If we have $\delta(\tau - \tau_0)$, applying the superposition integral to it:

$$g_2(x) = \int_{-\infty}^{\infty} h(x, \tau)\delta(\tau - \tau_0)\, d\tau = h(x, \tau_0)$$

g_1 and g_2 are shifted versions of each other. They will be shifted in the same way.

$$g_2(x) = g_1(x - \tau_0)$$

In other words:

$$h\,(x,\ \tau_0) = h\,(x - \tau_0,\ 0)$$

So you could treat g_2 as a shifted version of the response.

So rather, the impulse response function is not a 2D function but rather a 1D function. This is why we can discard the second parameter of the impulse response.

A linear filter with impulse response h is shift-invariant if and only if for all shifts $\tau_0 \in \mathbb{R}$, $h\,(x,\ \tau_0) = h\,(x - \tau_0,\ 0)$. This comes from the shift-invariance property.

So:

$$g(x) = \int_{-\infty}^{\infty} h(x - \tau) f(\tau) d\tau$$

And this happens to be the convolution operation. Linearity gives us the superposition integral, and shift invariance gives us convolution.

$$g = f * h = (x) \Rightarrow \int_{-\infty}^{\infty} h(x - \tau) f(\tau) d\tau$$

The convolution operator takes in a function and outputs a function (I wrote this like a JS arrow function).

What you call a filter and what you call a signal can be completely interchanged. You can do variable substitution and create an equivalent expression.

The convolution operation is commutative, associative, and has distributivity over addition.

So, the convolution filter is what can be done to do an LSI filter.

3.8 Convolution in 2D

$$g(x,\ y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x - u,\ y - v) f(u, v) du dv$$

3.9 Convolving With An Impulse

Delta is 0 everywhere except where $\tau = x$. This is like the identity filter.

$$\begin{aligned} g(x) &= \int_{-\infty}^{\infty} \delta(x - \tau) f(\tau) d\tau \\ &= f(x) \end{aligned}$$

Shifted impulse? Suppose that δ was shifted to the right? Then:

$$\begin{aligned} g(x) &= \int_{-\infty}^{\infty} \delta\left(x - \tau - x_0\right) f(\tau) d\tau \\ &= f(x - x_0) \end{aligned}$$

The product of two shifted impulses? It's like overlaying, as shift invariant linear filters are shift invariant and sum invariant. You would get two copies of the image in different places.

An infinite sum of identically shifted impulses – the impulse train:

$$III_{\Delta}(x) = h(x) = \sum_{k=-\infty}^{\infty} \delta(x, -k\Delta)$$

If your images are Δ tall and wide, what you get is a tile that repeats the image, given k is greater than the side-length of the image.

3.10 Types of Filters

3.10.1 Box Filter

The box filter: $h(x) = \frac{1}{\varepsilon}\text{box}_{\varepsilon}(x)$. Forces integral to be under 1. What happens when I convolve a 1D function with the box filter?

$$g(x) = \int_{-\infty}^{\infty} f(u)\text{box}_{30}(x-u)du$$

This is the standard definition for convolution. Given that you already know that the function is non-zero in a small neighborhood of zero, we can change that to avoid redundant calculations.

$$\begin{aligned} g(x) &= \int_{x-15}^{x+15} f(u)\text{box}_{30}\left(\underset{\text{nonzero between } [x-15,\, x+15]}{x-u}\right)du \\ &= \frac{1}{30}\int_{x-15}^{x+15} f(u)du \end{aligned}$$

So, this box filter is really an averaging filter. We should expect the convolution to smooth out the signal. Convolving an image using the box filter blurs the image, as every result of the pixel is the result of averaging the neighborhood around that pixel.

3.10.2 The Pillbox Filter

Like the box filter, but it's the disk variant.

$$h(x,y) = \begin{cases} \frac{1}{\pi r^2} & \sqrt{x^2+y^2} \leq r \\ 0 & \text{otherwise} \end{cases}$$

$$\iint_{\mathbb{R}^2} h(x,y)dA = 1$$

Why? Most apertures are circular. If we want to model the blur that comes from an aperture, we can use this filter.

If I were to take my original image and pass it through the filter, vs. take the image, rotate it by 45 degrees, and pass it through the filter, I would get the same blur.

Visually, **you will not see a huge difference compared to the box filter.**

3.10.3 The Gaussian Filter

It averages pixels in a neighborhood, but it's a weighted average. Not all pixels will be weighted the same.

$$G_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

We will call σ the scale parameter. It controls the width of the gaussian. The higher the σ, the more spread out the gaussian will be.

In 2D, this is just the product of two 1D gaussians:

$$G_\sigma(x, y) = G_\sigma(x) \cdot G_\sigma(y)$$

It's a circular symmetric function. Convolving an image with a Gaussian will just give you a blurry version of the image, but for every pixel, if you move over 3σ pixels away, you might as well not count it as the Gaussian function is essentially zero.

Since I can control σ now if I increase σ it blurs the image even more.

So far, we've talked about smoothing. But there's more. We can compute image derivatives.

3.11 Computing Derivatives by Filtering

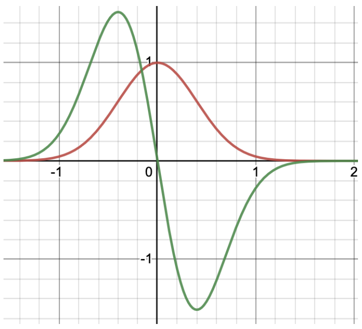
Because of the linearity of convolution, I can bring the derivative inside the integral.

$$\begin{aligned}\frac{d}{dx}(f * h)(x) &= \frac{d}{dx}\int_{-\infty}^{\infty} h(x-\tau)f(\tau)d\tau \\ &= \int_{-\infty}^{\infty} \frac{d}{dx}h(x-\tau)f(\tau)d\tau \\ &= \int_{-\infty}^{\infty} \left(\frac{d}{dx}h(x-\tau)\right) f(\tau)d\tau \\ &= \left(\frac{d}{dx}h\right) * f\end{aligned}$$

Let's do this for a Gaussian. Here, we've taken our function, smoothed it with our Gaussian, and then we take the derivative of the result. We could do the exact same thing by convolving f with the derivative of the gaussian.

$$\frac{d}{dx}(f * G_\sigma) = f * \left(\frac{d}{dx}G_\sigma\right)$$

How does the derivative of the Gaussian look like?



Now, how do we represent negative pixels? We're just going to color code it now and assume that pixels are greyscale. Our heatmap would be

- Red > 0
- White = 0
- Blue < 0

If I convolve the filter with the image, I will not expect the resultant image to be positive, as the convolution would also contain negative components. What would we expect the result to be?

$$f * \frac{\partial}{\partial x} G_3(x, y)$$

What would I expect the output to look like? Well, if an image goes from dark → bright from left to right, the derivative will be positive. However, at a sharp edge:

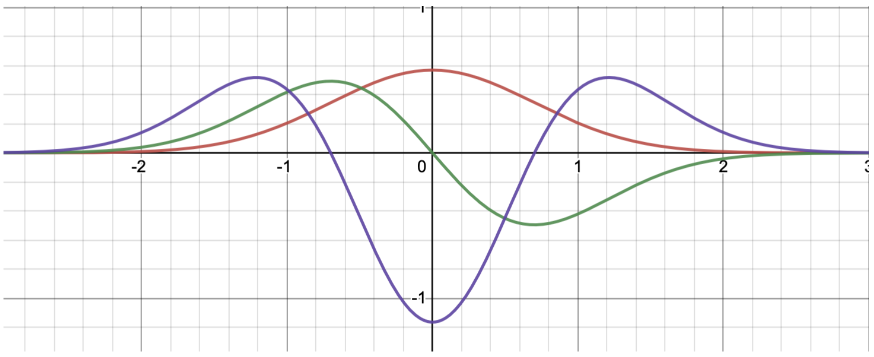
- An abrupt change from a bright left to a dark right would have a negative derivative.

Our resultant image would be zero nearly everywhere expect for an abrupt change in luminance somewhere in the image.

3.11.1 The Gaussian Second Derivative Function

$$\frac{d^n}{dx^n}(f * G_\sigma) = f * \left(\frac{d^n}{dx^n} G_\sigma\right)$$

The second derivative of the Gaussian function is:



The second derivative function is purple in the figure above. This function is symmetric.

The point is, we can compute second derivatives just as easily as computing the first derivative, just as easily as smoothing the image. Everything applies to either direction.

3.12 The DoG Filter

- Take our image
- Convolve it with the Gaussian σ
- Convolve it with a slightly larger σ
- Subtract the two images

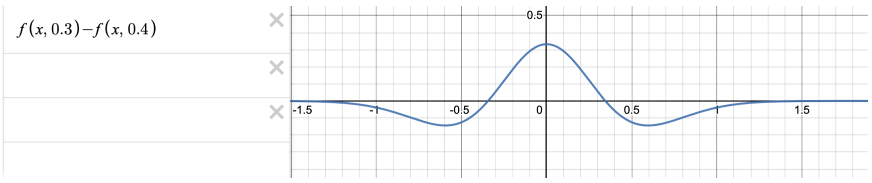
What would I get?

- Areas that are already smooth would just get 0
- Areas where smoothing by a slightly larger Gaussian would smooth more, well, I get another edge detector.

$$G_3 - G_4$$

Why is this called the DoG filter? Because it's called the difference of gaussians.

$$f * G_{\sigma_1} - f * G_{\sigma_2} = f * \left(G_{\sigma_1} - G_{\sigma_2}\right)$$



(This would be rotationally symmetric in 3D)

3.13 Sharpening

$$f + \left(f * \left(G_{\sigma_1} - G_{\sigma_2}\right)\right)$$

Gives you something that is slightly sharper. You could put a coefficient:

$$f + \underbrace{s}_{\text{adjustable sharpening parameter}} \left(f * \left(G_{\sigma_1} - G_{\sigma_2}\right)\right)$$

This would control the sharpness of the image. It **will** reduce the contrast of the images – if we push the edges to be stronger, the camera will have to squint (by squashing intensities above 255 to 255).

Now, how would I compact this statement?

$$
\begin{aligned}
&f * \delta + \left(f * \left(G_{\sigma_1} - G_{\sigma_2}\right)\right) s \\
&= f * \left(\delta + \left(G_{\sigma_1} - G_{\sigma_2}\right) s\right)
\end{aligned}
$$

So, now we have an expression for the filter that sharpens the image.

Problem? Sharpening images causes a halo effect. Sharpening an image too much will cause it to no longer look natural.

3.14 Bounded Domain and Out of Bounds Filtering

Images are typically defined over a bounded domain. We need some convention to define the concept of convolution on a boundary.

There are two ways we can handle integrals when the image is not defined over the entire range.

3.14.1 0-Padding

Assume $f(x, y) = 0$ at the image border.

One way: for pixels out of bounds, assume that the image value is 0. What this really means, is that I've defined my image to be a function that is zero everywhere except in the actual image. If I were to convolve the image with an edge enhancement filter, I would get a very large brightness change in the borders, which will cause the edge-enhancement filters to respond.

3.14.2 Tiling / Wrap Around

The other approach would be to wrap around the pixel values:

Assume that $f(x, y) = f(x\%W, y\%H)$. Another way to think about this, is the image is not the image that you're seeing, but an infinitely tiled version of the image. If you're trying to convolve an image using this convention, you are convolving your filter with an image that extends infinitely in both dimensions and has this tiling corresponding to the tiled version of the image. It is a periodic function where the same image repeats – but this is the version of the image I'm applying my filter to.

If I have some change of brightness between edges, if I apply a filter, I will get strong responses around that area.

This particular way to think about an image is very common. We'll come back to this when we talk about Fourier transforms – which assumes that are images are tiled like this.

3.15 Edge-Degrading Behavior of Smoothing LSI Filters

How do we improve the quality of a noisy photo? What can I do to get a less noisy image back?

I can say, if I were to apply a blurring filter that will take all pixels in the neighborhood and average them together (discrete or gaussian), it will make an image that looks smoother, but you start losing details from the edges.

At any given position, when we're trying to compute the value of the image at this position, it takes the weighted sum of all pixels around it. If I'm at an edge, the same weighted sum will apply.

In other words, LSI filters will always act the same way no matter where I am in the image. Can we do better than that? Totally, but we can't maintain the shift invariance property. By definition, shift invariance is agnostic to the image content.

3.16 The Bilateral Filter

It can denoise and preserve sharpness.

This tells us how better we can get when we remove shift-invariance. It is a transformation of the intensity of the image, it is local, but it is a non-linear operation.

Now, the bilateral filter does not behave the same way in all parts of the image. It behaves almost like a gaussian filter when the image is smooth. However, when we are in a detailed area, the filter behaves different. Depending on where I am in the image, the weighting changes. But how is the weighting computed?

It's done by revising the expression for the output image. We write that as:

$$g(x,y) = \iint \text{Weight}(u,\ v,\ x,\ y) \cdot f(u,\ v)dudv$$

We will be assigning a different weight depending on where the pixels are in the image. All that matters is figuring out what the weight is.

This is a 4D function. It depends on the destination pixel, and which pixel in the source image we are looking at. This weight function will depend on two factors – a product of two factors:

- Like the original gaussian filter
- Something else

Our weight expression can be decomposed into:

$$\text{Weight}(u,\ v,x,y) = w_{\text{spatial}}(x,y,u,v) \cdot w_{\text{intensity}}(x,y,u,v)$$

With the following decomposition being:

- w_{spatial}: weight goes down the further the pixel is, like the gaussian filter
  - $= G_{\sigma_s}(||(x,y) - (u,\ v)||)$
  - If we only have this, then our filter would be LSI. But we don't.
- $w_{\text{intensity}}$: gives more weights to pixels with similar intensities.

  - This is a very special term. Two pixels that are roughly similar in brightness, then it will contribute. If its brightness is very different, it won't contribute. Note that the higher the input passed into G_{σ_r}, the closer it will be to zero.
  - $= G_{\sigma_r}(|f(x, y) - f(u, v)|)$

So, our final expression is:

$$G(x, y) = \iint_{\mathbb{R}^2} G_{\sigma_s}(||(x, y) - (u, v)||) \cdot G_{\sigma_r}(|f(x, y) - f(u, v)|) \cdot f(u, v) du dv$$

So, it's based on intensity and proximity.

Closer intensity and proximity $\Rightarrow$ higher of the value of the filter at that point

This is a very expensive operation. For every pair of pixels, I have a weight. If I try to do this in a naïve way, with an $n \times n$ image, computing the weight function has an n^4 time complexity. This is not a super useful filter unless there is a way to compute it efficiently. It took a couple of years after this filter was introduced for people to come up with efficient solutions, and at least it exists.

What this allows us to do, is that once we've processed the image, our filter is edge preserves. It does not blur edges. Very important filter has some nice properties, but we lose the clean mathematical properties of convolution.

4 2D Digital Images

How can we represent images that are captured with a sensor? These are discrete, but it's important to understand how these discrete images we capture relate to the underlying continuous functions. When we want to display an image, we may need to think about it in a continuous fashion first. Suppose I have an image that is 10000 × 10000 pixels and display it with my phone, with a smaller screen pixel count. The expressions we've talked about 2 lectures ago introduce artifacts and issues.

4.1 Aliasing

If you downscale an image, you are changing the image that is being displayed while maintaining the fixed sampling (pixel grid). What happens is that you're spacing out a high-frequency sinusoidal wave which could then be represented by a completely different wave with just some matching points. That is the underlying problem of aliasing. This is caused by inherently sampling a signal.

4.2 In a Camera

Behind all cameras, there exists a sensor. If you zoom into the sensor, it looks like an array of individual sensing elements. Each of these, it corresponds to a pixel. Notice in front of the pixel, there's a color filter. The sensor records the intensity of the light after it goes through a red, blue, or green filter. It will return an array of scalar values – it will not send you color images. All you have is the brightness of the light that fell onto the pixel, passed through the red filter, and was passed through the sensor.

We'll use twice as many green filters as red and blue as the human eye is more sensitive to the color green.

4.2.1 The Micro-Lens Array

There is a lens in the front and redirects the light to the part of the sensor that can turn down input light down into a discrete value. What's really important, is that what gets recorded is **the integral of all the light that fell onto the surface.** There is some continuous representation of light that falls onto the sensor, but we just get the integral of it with, through the pixel's footprint. What the sensor gives you is the intensity, and they have to be turned into RGB values. There is processing involved that takes the scalar values passed in by the camera into RGB values. All cameras do this. No camera would record just the red, green, and blue.

The point here, is that all we end up with is a discrete array of digital numbers. You can think these numbers of being computed from a continuous function. Whatever happened in a pixel footprint was integrated (or averaged), and what we end up getting is one value.

There is a connection between the continuous function and the discrete representation. The connection is that there is integration involved.

4.3 Digital Images Expressed as Convolution and Sampling

We start with a continuous function of brightness that is defined on the plane of the sensor. It is subdivided into a grid of footprints of the original pixels. For each of the footprints, we get the values of it. What's the value? The average of the rightness.

What did we capture? One measurement per pixel. Mathematically, they are represented as a set of δ functions. The height of the function is the value of the pixel. This is a more accurate way of picturing what the sensor captured. A pixel's footprint you see in a pixelated image is **not constant,** conceptually.

So, what goes on?

- I take my original image
- I convolve it with the box filter

Inside a single pixel, we are computing one value: the **average** of the intensity of the image within that footprint. What's the expression?

$$\begin{aligned} \text{pixel}(r,\ c) &= \text{pixel center} \\ \Delta x,\ \Delta y &= \text{width/height of a pixel} \end{aligned}$$

$$f_{rc} = \int_{c\Delta x - \frac{\Delta x}{2}}^{c\Delta x + \frac{\Delta x}{2}} \int_{r\Delta y - \frac{\Delta y}{2}}^{r\Delta y + \frac{\Delta y}{2}} f(x,y)dxdy$$

So, this whole integration can be written as the convolution of the original image with a box filter whose dimension is the same as the footprint of a pixel. But the convolved image is not captured, as we don't have the continuous convolved image. All we have is the values at the pixel center. It is not f_{rc} because the sensor only measures the averages at the pixel centers.

What we get is:

$$\widetilde{f}(x,\ y) = \left(f(x,y) * \left(\text{box}_{\Delta x}(x) \cdot \text{box}_{\Delta y}(y) \cdot \frac{1}{\Delta x \Delta y}\right)\right) \cdot \left(III_{\Delta x}(x) \cdot III_{\Delta y}(y)\right)$$

Note:

- $f(x,y) * \left(\text{box}_{\Delta x}(x) \cdot \text{box}_{\Delta y}(y) \cdot \frac{1}{\Delta x \Delta y}\right)$ is what the camera "sees"
- $\widetilde{f}(x,\ y)$ is what the camera captures to us. This is the sampled image, the collection of images that is being acquired. $\widetilde{f}$ is expressed as a function over a continuous domain but it is non-zero only at a discrete set of points.

It may look like a complicated expression, but it's not too bad. **But why do we care?**

In this convention, $\delta(t) = \begin{cases} 1 & t = 0 \\ 0 & \text{otherwise} \end{cases}$

4.4 The Image Resampling Problem

What we ultimately need to do is resampling. Somehow, I want to display an image on a screen with a completely different resolution. How do I up-sample (supersample) an image, on a screen with more pixels than I started with?

Sub-sampling is the opposite – how do I display a higher-res image on a screen with fewer pixels?

This is something that has to happen when I display an image on **any** device. When I zoom in or zoom out, you end up having to take that image and display it on some *finite* neighborhood of pixels. We need to be able to perform this image resampling problem. Want to print an image? This depends on the resolution of your printer. Want to rotate the image? Suddenly, your pixels, which was originally nice, is now rotated and it's no longer on a regular grid anymore. In assignment 1, your homography did not put your pixels right on the center.

So, how do you sample?

4.4.1 How to sample

- Start with a discrete representation
- Interpolate it: $\widetilde{f}(x,y) \longrightarrow \widetilde{f}_{\text{interpolated}}(x,\ y) \longrightarrow \widetilde{f}_{\text{resampled}}(x,\ y)$
  - Resampled uses another grid. $\widetilde{f}_{\text{interpolated}}$ is continuous.
  - Before you resample, you may need to re-filter the image to prevent aliasing artifacts. It's a very important step.

We're going to use filtering to do this. All the tools we've learned so far are going to be very helpful.

4.5 Function Interpolation

Given a discrete set of samples $f_k = f(x_k)$ of a function $f(x)$ at $x_1,\ x_2,\ \ldots,\ x_k$, construct a new function $g(x)$ that satisfies $g(x_k) = f(x_k)\ \forall i \in 1 \ldots k$, and we can evaluated for any x

There are two forms of sampling:

- Uniform sampling: the samples we have are spaced Δ apart (this is regular sample). We start with δ functions where the height (**multiplier** of δ) is the value of f at that location.
- Non-uniform interpolation: where the spaces between the samples are not the same. We're not covering this.

This interpolation should be shifting invariant.

Whether I shift first then interpolate, it would make no difference than if I did them the other way around.

4.6 The Expression of Function Interpolation

Given a uniformly sampled function f with period Δ, an infinite number of it so the samples extend to infinity.

We want to compute the interpolated function g. The expression is going to be as follows:

$$g(x) = \sum_{k=-\infty}^{\infty} f_k \cdot h(x - k\Delta)$$

Where f_k is f sampled at x_k, read the definition of it above in the previous section.

Where h is the interpolation filter (kernel). Treat it as a weighted combination of the samples.

Say $h(x)$ is my interpolation filter. How would our interpolation filter look like? It'll probably look like a bell curve, or a bell curve with humps below its first descent to 0. Normally, the higher the absolute value of what is passed into h, the closer to zero h returns (with some exceptions). However, regardless, $h(x)$ **must be zero at the sample locations except for the center one** so if we're sampling at an existing defined pixel, it won't mess up the image.

4.7 Derivation of the Interpolation Equation

Let $\widetilde{f}(x) = \sum_{k=-\infty}^{\infty} f_k \delta(x - k\Delta)$

We want a shift-invariant transformation $\widetilde{f}(x) \to g(x)$ such that $g(x) = f * h$. If we plug in this expression for f, we end up with:

$$\begin{aligned}
&\left(\sum_{k=-\infty}^{k} f_k\delta(x-\Delta k)\right) * h \\
&= \sum_{k=-\infty}^{\infty} f_k \cdot (\delta(x-k\Delta) * h) \\
&= \sum_{k=-\infty}^{\infty} f_k \cdot h(x-k\Delta)
\end{aligned}$$

g is written in this way as shift invariance requires the transformation for f to g to be a convolution, so it has to be written this way. Linear shift invariance implies convolution which implies the existence of this particular expression.

4.7.1 Conditions of the interpolation filter

What can this h be? It cannot be arbitrary. The very important properties:

$$h(0) = 1$$
$$h(k\Delta) = 0 \; \forall k \neq 0$$

This ensures that when I evaluate g at some discrete sample (original pixel), only that original pixel contributes. It applies everywhere that has a pixel defined at it.

- Ideally, we want h to be a smooth function (continuous derivatives) everywhere. It is not a must.
- And we want local support: $h(x) = 0$ outside a small neighborhood of 0. Computationally, this makes a difference and ensures that $g(x, y)$ depends only on a few samples.

4.8 Types of Interpolation Filters

Okay. Let's do this.

4.8.1 Nearest Neighbors

Use the nearest sample. For any x, look at the closest sample and get the value of it. Well, our function becomes piecewise constant.

$$h(x) = \text{box}_\Delta(x) = \begin{cases} 1 & |x| \leq \frac{\Delta}{2} \\ 0 & \text{else} \end{cases}$$

It's the box filter. We're taking shifted copies of h.

Our image will just look like a mosaic. The visualization of a discrete image is simply its nearest neighbor interpolation.

4.8.2 Linear Interpolation

We compute the in-between values of two samples. Between two samples, the values blend. **Connect neighboring samples with a straight line.**

We can write an expression for this for any given point:

$$g(x) = \left(1 - \frac{x}{\Delta}\right) f_0 + \frac{x}{\Delta} f_1 \text{ for } 0 \leq x < \Delta$$

If we wanted to treat interpolation as a filtering operation, what filter gives us this linear interpolation result when we convolve with the original function? It all comes down to this expression:

$$g(x) = \sum_{k=-\infty}^{\infty} f_k h(x - k\Delta)$$

Where:

$$h(x) = \begin{cases} 1 - \frac{|x|}{\Delta} & |x| < \Delta \\ 0 & |x| \geq \Delta \end{cases}$$

Notice how a maximum of two sampled points can contribute to $g(x)$.

The actual filter looks like a triangle: $h(x) = \text{box}_\Delta(x) * \text{box}_\Delta(x) \cdot \frac{1}{\Delta}$. In 2D, this is going to be a pyramid:

$$g(x,y) = \widetilde{f}(x,y) * ((\text{box}_\Delta(x) * \text{box}_\Delta(x)) \cdot (\text{box}_\Delta(y) * \text{box}_\Delta(y)))$$

This is the most common form of 2D interpolation. It is very fast, results are okay.

4.8.3 Cubic Interpolation

How do I find a function that is a cubic polynomial and has the properties we need (1 at 0, 0 at all interval multiples of delta)?

$$h(x) = \begin{cases} \frac{3}{2}|s|^3 - \frac{5}{2}|s|^2 + 1 & 0 \leq |s| < 1 \\ -\frac{1}{2}|s|^3 + \frac{5}{2}|s|^2 - 4|s| + 2 & 1 \leq |s| < 2 \\ 0 & |s| > 2 \end{cases}$$

With $s = \frac{x}{\Delta}$

This looks *like* the DoG filter but analytically they're not the same. Why is there a dip? Please don't ask. It can cause "ringing" due to the negative dips.

Why isn't there a quadratic? We don't have much control over a quadratic function. You can move the vertex, but you can't move anything else. You can use a piecewise function but expect discontinuities at a boundary.

4.9 Could you compare?

Bilinear? Bicubic? Bicubic is smoother. The larger region of support, you are getting smoother results.

5 Image Morphing

- I am going to become Switzerland
- That one effect in kaput
- Except deep learning tends to outperform all of this, so most of this is useless anyways.

We have two different images, and we would like to perform a warp between the two. Morphing involves two steps:

1. Pre-warp the two images
    a. Into a common space, so at the half-way point, they look structurally the same. The cheek size, the face composition is roughly in the same place
    b. This prevents ghosting
2. Cross-dissolve their colors
    a. Simple cross-dissolve

We note

- An image pre-wrap is a re-positioning of all pixels in an image to avoid the double-image effects much as possible.
- The field morphing algorithm is a pre-warping algorithm that offers intuitive warp control.

And the process goes like this when you see it:

- Pre-warp the first image
- Cross dissolve
- Then undo the pre-warp on the second image

5.1 Cross-Dissolving two images

A weighted combination of two images, pixel by pixel

$$morph(t) = (1 - t)warp_{1(t)} + t\, warp_2(t)$$

5.2 Warping Images by Backward Mapping

The coordinate map: two functions $R(r, c)$, $C(r, c)$ that map each pixel (r, c) in a destination image to its (potential fractional) location in the source image. Coordinate maps are essential for specifying the image warp. We are backward mapping to prevent holes in the picture.

Warp: $(r, c) \longrightarrow$ source $(R(r,c), C(r,c))$

5.3 The Backward Mapping Algorithm With Footprints

Given a map R, C

```
• For $r$ in rmin to rmax
    – For $c$ in cmin to cmax
        * $r' = R(r,c)$
        * $c' = C(r,c)$
        * Sample source image at $(r', c')$
        * Copy that sample to the destination pixel
```

Some areas in the warped image are going to be smaller than other areas, and some areas are going to have more information. What we often do, is use super-sampling. So, let's do this again:

```
• For $r$ in rmin to rmax
    – For $c$ in cmin to cmax
```

  * $r' = R(r, c)$
  * $c' = C(r, c)$
  * Define $K \times K$ grid of samples within footprint of destination pixel $(r,\ c)$
  * Use grid to super-sample the source image by interpolation
  * Copy the average of these samples to the destination pixel $(r,\ c)$

5.4 The Beier-Neely field-warping Algorithm

We want to map the two coordinate systems: the original grid to the warped image. It is an algorithm for specifying the coordinate maps of a warp interactively.

Its input is a set of corresponding line segments drawn on the source images. Yes, there is some work you'll have to do here. It's not all automatic.

For every t, the endpoints of corresponding segments are linearly blended to define their position in $\text{warp}_i(t)$

$$\vec{P}(t) = (1-t)\vec{P} + t\vec{P}'$$
$$\vec{Q}(t) = (1-t)\vec{Q} + t\vec{Q}'$$

For $warp_1(t)$ and $warp_2(t)$, the segments should be identical.

The field warping algorithm computes $R_i(r,\ c)$ and $C_i(r,\ c)$ for $i = 1, 2$ from the line segment positions at t

For a single segment:

1. Destination pixel $(r,\ c)$ assigned coordinates $(u,\ v)$ relative to the segment (that is, $\vec{P}(t)$ to $\vec{Q}(t)$ and we make an orthogonal line).
2. These $(u,\ v)$ coordinates and line segments are converted to $\left(r',\ c'\right)$ in the source image

The destination $X = (r, c)$ expressed in a local coordinate system defined by $\vec{P}(t)\vec{Q}(t)$

$$
\begin{aligned}
X \\
&= \underbrace{\vec{P}(t)}_{\text{origin}} + \underbrace{u\left(\vec{Q}(T) - \vec{P}(t)\right)}_{\text{The } u \text{ axis is along this segment}} \\
&+ \underbrace{v\frac{\text{Perpendicular}\,(Q(t) - P(t))}{||Q(t) - P(t)||}}_{v\text{–axis defined by this unit length vector}}
\end{aligned}
$$

Which, then we can show that $u,\ v$ can be given by these simple formulas, where $\vec{X} = (r,\ c)$ in our destination image:

$$
\begin{aligned}
u &= \frac{\left(\vec{X} - \vec{P}(t)\right)\left(\vec{Q}(t) - \vec{P}(t)\right)}{\left|\left|\vec{Q}(t) - \vec{P}(t)\right|\right|^2} \\
v &= \frac{\left(\vec{X} - \vec{P}(t)\right)\left(\text{Perp}\left(\vec{Q}(t) - \vec{P}(t)\right)\right)}{\left|\left|\vec{Q}(t) - \vec{P}(t)\right|\right|}
\end{aligned}
$$

Change of basis detected

5.5 Dealing with Multiple Lines

Apply the single-segment algorithm to each segment separately to obtain N coordinates $\left(r_1',\ c_1'\right)\cdots\left(r_N',\ c_N'\right)$

Compute r', c' as:

$$
\left(r', c'\right) = \sum_{n=1}^{N} w_n \cdot \left(r_n',\ c_n'\right)
$$

Where $w_n = \frac{\widetilde{w}_n}{\sum_{n=1}^{N}\widetilde{w}_n}$ with $\widetilde{w}_n = \left(\frac{\left(\text{length of segment}^{??}\right)}{a + \text{dist of } (r,c) \text{from segment}}\right)^{??}$

a: controls influence of segment for pixels near it. Just view the assignment page and view the slides. I'm not repeating the code again. You can just translate pseudocode into actual code.

6 Fourier Transforms

Involve representing images in a completely different way. Down with functions of x, y, we're using functions of spatial frequencies.

The idea is that any function can be expressed as an (infinite) weighted sum of sinusoids. We have to look at sinusoids of many different frequencies. We'll see the function as a collection of weights where we have one weight per frequency. Depending on how many frequencies you choose to use, you get a representation that gets increasingly more accurate. Get out the high frequencies, and the function isn't exactly what you see.

When you Fourier-transform an image, you'll get two images: magnitude and phase. The center is a constant, and moving away gives you images of sinusoids with increasing frequencies.

If we remove high frequency components, you'll start blurring the image.

6.1 Normalizing a Sinusoid

Why does $\sin(2\pi x)$ have a period of 1? The period of $\sin(x)$ is 2π, and by instead passing $\sin(2\pi x)$ into the argument, we are compressing it by a factor of $2\pi x$ which reduces the period down to 1.

$\sin\left(\frac{2\pi x}{c}\right)$ stretches the sinusoid to have a period of c.

And then:

$$\sin\left(\omega\frac{2\pi x}{c}\right)$$

Here, we get ω phases through a distance of c, and the phase will always reset when we hit c (given ω is an integer).

6.2 The 2D Fourier Domain

We'll think of an image as a function of frequencies. We have two spatial frequencies ωx and ωy.

- Frequency $(\omega_x, \omega_y) = (0,\ 0)$ represents a constant-intensity image (the "DC component")
- The image corresponding to $(5,\ 0)$, this is an image that is a sinusoid in the x-direction and constant in the y-direction. It has 5 periods from the beginning from the start to the end of the image (again, assume we're dealing with cosine waves).

Oh wait, we can't just use the same frequencies over and over. Images may have differences in their phases. When an image being composed as a weighted sum of sinusoidal or cosine image, we also have to consider their phase.

The general expression for the intensity of pixel $(x,\ y)$, where x goes from 0 to c:

$$I(x,y) = \cos\left(\text{periods} \times \frac{x}{c}2\pi + \text{phase}\right)$$

What we need to be able to really express an image as a weighted sum of these different cosines, it turns out we need complex-valued weights.

$$I(x,y) = \cos\left(s\frac{2\pi}{c}x + \phi\right) + i\sin\left(s\frac{2\pi}{c}x + \phi\right)$$

The real component is the cosine, the imaginary component is the sine.

A real-valued cosine image would be described as the sum of two terms, for $(5,\ 0)$: sum up $(-5,\ 0)$ and $(5,\ 0)$. So, the general expression for images where we have a certain frequency represented by $(w_x,\ 0)$ and $(-w_x,\ 0)$:

$$I(x,y) = \cos\left(\frac{w_x 2\pi}{c} + \phi\right)$$

Note that the real-valued representation of an image is symmetric over the cartesian plane, as you need that to create a real-valued image.