MAT235 Notes

Multivariable Calculus

https://github.com/ICPRplshelp/

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1 Parametric equations

A function is something that takes in an input and returns an output. The set of acceptable inputs is the domain, and all the possible outputs based on the domain is the image. We can write a function as:

$$f\colon \mathbb{R} \to \mathbb{R}$$

To plot a single-variable, we need the y-axis, and we need the x-axis. We can use the vertical line test on a plot to test if something is a function or not.

Multi-dimensional maps include:

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto f(x, y) \in \mathbb{R} \end{aligned}$$

For some reason, you can slice through 3D shapes, and you can figure out the formula.

1.1 Mapping curves

Do we have a curve? Then, we can map it as:

$$x = f(t),\ y = g(t)$$

Where $f\colon \mathbb{R} \mapsto \mathbb{R}$, $g\colon \mathbb{R} \mapsto \mathbb{R}$. t is a parameter.

Example 1: sketch the curve defined by the following equations:

$$\begin{aligned} x &= t^2 - 3 \\ y &= t + 2 \end{aligned}$$

From $-3 \leq t \leq 3$.

How do we draw this curve? Draw a table.

t	x	y
-3	6	-1
-2	1	0
-1	-2	1
0	-3	2
1	-2	3
2	1	4
3	6	5

Feel free to extend or make this table contain more information whatsoever. Now, we want to draw the curve for these points, and connect the dots. How do we check the answer? Rewrite the formula you have, but in cartesian form.

$$\begin{aligned} y &= t+2 \Rightarrow t = y-2 \\ x &= t^2 - 3 = (y-2)^2 - 3 = y^2 - 4y + 4 - 3 \\ &= y^2 - 4y + 1 = x \end{aligned}$$

Use this with Desmos to check your answer. This is how you move from cartesian form to parametric. Maybe cartesian form is more convenient to imagine the curve for an equation.

Example 2 (harder): What curve is represented by the following parametric equations?

$$x = \cos(t),\ y = \sin(t),\ 0 \leq t \leq 2\pi$$

This makes a circle, as we know that $\cos^2(t) + \sin^2(t) = 1$:

$$x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1$$

But if we ever want to be precise:

t	$x = \cos(t)$	$y = \sin(t)$
0	1	0
$\frac{1}{2}\pi$	0	1
π	-1	0
$\frac{3}{2}\pi$	0	-1

Let's switch up the equation a bit: $0 \leq t \leq 2\pi$

$$x = \cos(2t),\ y = \sin\ \_2t$$

So:

$$x^2 + y^2 = \cos^2(2t) + \sin^2(2t) = 1$$

Do we get the same curve? The difference between this one and the previous one is that the circle is stacked twice.

What happens when we transform the circle by $(x_0,\ y_0)$ and make it have a radius of r? What is the parametric formula?

$$x = x_0 + r\cos(t)$$
$$y = y_0 + r\sin(t)$$

Let's bring up the formula for a circle on a cartesian plane, with center $(\alpha,\ \beta)$ and radius r:

$$(x-\alpha)^2+(y-\beta)^2=r^2$$

Do some substitution with the parametric equations, where we say $(\alpha,\ \beta)=(x_0,\ y_0)$:

$$\begin{aligned}(x_0+r\cos(t)-\alpha)^2+(y_0+r\sin(t)-\beta)^2&=r^2\\(r\cos(t))^2+(r\sin(t))^2&=r^2\\r^2\cos^2(t)+r^2\sin^2(t)&=r^2\\r^2\left(\cos^2(t)+\sin^2(t)\right)&=r^2\\r^2&=r^2\end{aligned}$$

Left side is the right side.

1.2 Calculus with parametric

To write the tangent line, you need the point, and the slope. The formula for a tangent line is $y=m(x-x_0)+y_0$, which is $\frac{dy}{dx}\mid_{x_0,\ y_0}$.

Suppose that we have two functions $f\colon\mathbb{R}\to\mathbb{R}$, $g\colon\mathbb{R}\to\mathbb{R}$, and that they are differentiable. Suppose that we have a parametric curve:

$$x=f(t),\ y=g(t)$$

$$\begin{gathered}\frac{dx}{dt}\neq 0\text{ and}\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}\Rightarrow\\\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\end{gathered}$$

The advantage of this is that we can easily compute $\frac{dy}{dt}$, because y is a function of t. We do not need to eliminate t. The same thing applies to $\frac{dx}{dt}$.

- A curve has a horizontal tangent when $\frac{dy}{dt}=0$ and $\frac{dx}{dt}\neq 0$.

- A curve has a vertical tangent line when $\frac{dx}{dt} = 0$, given that $\frac{dy}{dt} \neq 0$.

The second derivative is denoted by

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

(Grab the equation for the first derivative – that is how this was derived.)

Suppose we have:

$$x = t^2,\ y = t^3 - 3t$$

What is the tangent line for $t = 2$? Use the first derivative formula.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t}$$

Substitute $t = 2$:

$$\frac{dy}{dx} = \frac{3(2)^2 - 3}{2(2)} = \frac{3}{2}$$

1.3 Areas

You have a graph $y = f(x)$, and you want the area underneath the curve from a to b. That is, $A = \int_a^b ydx$ (and let us assign $g(t) = y$ and $f(t) = x$ for no reason).

So:

$$\begin{aligned} A &= \int_a^b ydx = \int g(t)d\left(f(t)\right) \\ &= \int_\alpha^\beta g(t)f'(t)dt \\ &\int_a^b y(t)x'(t)dt \end{aligned}$$

Where $a = f(\alpha),\ b = f(\beta)$. Good luck finding function inverses, or at least base the locations off t.

Example: Find the area of the region enclosed by the loop of the following curve (I swear the curve makes a loop):

$$x = 1 - t^2$$
$$y = t - t^3$$

$$\begin{aligned} x &= 0 \\ \Rightarrow 1 - t^2 &= 0 \\ \Rightarrow t &= \pm 1 \\ \Rightarrow y &= 0 \end{aligned}$$

At $t = 1$, then $(x, y) = (0,\ 0)$. For $t = -1$, $(x,\ y) = (0,\ 0)$. Unfortunately, the best way to find out about this is by finding two distinct t-values: $t_1,\ t_2$ such that $f(t_1) = f(t_2)$. Good thing is that this function happens to be symmetrical with respect to x.

$$A = \int_{-1}^{1} g(t) f'(t) dt$$

1.4 Arc length

Where S is the arc length, and α is the value of t where the arc starts, and β is t where the parametric point is where the arc ends.

$$S = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

How it is derived: Length of curve from a function, from $(a,\ f(a))$ to $(b,\ f(b))$:

$$S = \int_{a}^{b} \sqrt{1 + f'(x)} dx$$

To derive:

$$\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = \int \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = \int_\alpha^\beta \sqrt{1+\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\left(\frac{dx}{dt}\right)dt$$
$$= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt$$

Example. Find the length of the following parametric curves:

$$x = 1+3t^2$$
$$y = 4+2t^3$$
$$0 \le t \le 1$$

Solution:

$$S = \int_0^1 \sqrt{(6t)^2+(6t^2)^2}dt$$
$$= \int_0^1 \sqrt{36t^2+36t^4}dt$$

2 Polar coordinates

A coordinate in the polar coordinate system can be described as

$$P(r,\ \theta)$$

This means "the polar coordinates of P." Omit the P if you want.

2.1 Conventions

- Positive angles are measured counterclockwise

- Negative angles are measured in the clockwise direction
- If the point is placed at the origin, then $r = 0$ and $(0,\ \theta)$ represents the pole for any value of θ. In this case, what value θ takes will not matter.
- The points $(-r,\ \theta)$ and $(r,\ \theta)$ lie on the same line through the origin and at the same distance $|r|$ from the origin, but on the opposite sides of O (maybe a π angle difference). Moreover, $(r, \theta) = (-r,\ \theta \pm \pi)$

Polar to cartesian:

$$(r,\ \theta) = (r\cos(\theta),\ r\sin(\theta))$$

Cartesian to polar:

$$(x,\ y) = \left(\sqrt{x^2+y^2},\ \theta\right)$$

Where $\tan(\theta) = \frac{y}{x}$

When deciding the angle, it is better to visualize what quadrant $(x,\ y)$ is in, as the ratio $\frac{y}{x}$ itself may not be able to tell which quadrant is in unless you look at x and y as their own components.

2.2 Graphing polars

The graph of a polar equation $r = f(\theta)$ consists of all points that have at least once polar representations whose coordinates satisfy the equation.

Example: Polar equation $r = 2$? We need to find all points $(r,\ \theta)$ such that $r = 2$ – which happens to be a full 2-radius circle. Its equation is $x^2 + y^2 = r^2 = 4$.

Polar curve $\theta = \frac{\pi}{6}$. Solution:

$$\left\{(r, \theta) \mid \theta = \frac{\pi}{6}\right\}$$

Looks like $y = \arctan(\theta)$.

For a full equation, $r\cos(\theta)$: Make a table $\theta,\ r$ and connect the dots.

2.3 Calculus with polars – area of a slice of a circle

If you have a circle with the area $A=\pi r^2$, then the area of a section of the circle is $A=\frac{1}{2}r^2\theta$, where θ is the angle of the section of the circle you're taking.

If your circle isn't that much of a circle but something that is done through a polar curve, the area from angle a to b is:

$$A=\int_a^b \frac{1}{2}r^2 d\theta$$
$$A=\int_a^b \frac{1}{2}\left(f(\theta)\right)^2 d\theta$$
$$A\approx \lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{2}\left(f\left(\theta_j^*\right)\right)^2\Delta\theta$$

(You don't need to know the bottom one.)

Example: Find the area region enclosed by one loop (from one instance of $r=0$ to the next instance of $r=0$) of the curve $r=4\cos(3\theta)$.

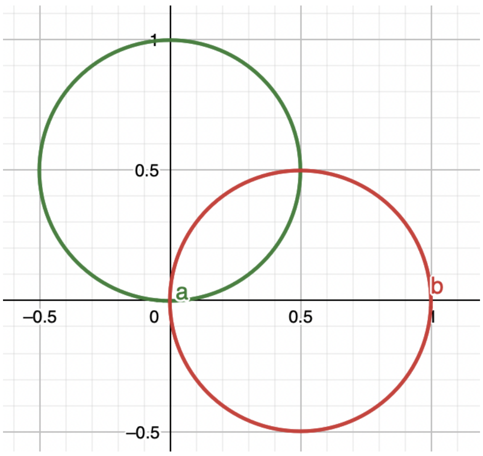
Here, $r=0\Rightarrow \cos 3\theta=0\Rightarrow 3\theta=\frac{\pi}{2}+n\pi,\ n\in\mathbb{Z}\Rightarrow\theta=\frac{\pi}{6}+\frac{n}{3}\pi$

We can choose θ to be $-\frac{\pi}{6},\frac{\pi}{6}$

This means:

$$\begin{aligned}
A &= \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (4\cos(3\theta))^2\, d\theta \\
&= 8\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2(3\theta) d\theta \\
&= 8\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1+\cos(6\theta)}{2} d\theta \\
&= 8\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2} + \frac{\cos(6\theta)}{2} d\theta \\
&= 8\left(\frac{1}{2}\theta + \frac{1}{12}\cdot\sin(6\theta)\right) \Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \\
&= 8\left(\frac{1}{2}\cdot\frac{\pi}{6} + \frac{1}{12}\sin(\pi) + \frac{1}{2}\cdot\frac{\pi}{6} - \frac{1}{12}\sin(-\pi)\right) \\
&= 8\left(\frac{1}{2}\cdot\frac{\pi}{6} + \frac{1}{12}\sin(\pi) + \frac{1}{2}\cdot\frac{\pi}{6} + \frac{1}{12}\sin(\pi)\right) \\
&= 8\left(\frac{\pi}{6} + \frac{1}{6}\sin(\theta)\right) \\
&= \frac{4}{3}\pi + \frac{1}{6}
\end{aligned}$$

2.3.1 Inside two circles



Find the area of the region that lies inside the curves:

$$A = \int_a^b \frac{1}{2} f(\theta)^2 d\theta$$

$r = \sin(\theta)$ and $r = \cos(\theta)$, $\theta = \frac{\pi}{4}$. This means the area is:

$$\frac{1}{2}\int_0^{\frac{\pi}{4}} \sin^2(\theta)d\theta + \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2(\theta)d\theta$$

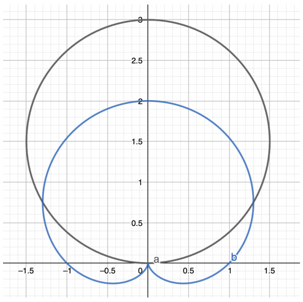
Visually, they look symmetrical, so we could say the answer is:

$$\begin{aligned}\int \sin^2(\theta)d\theta &= \left(\frac{1}{2} - \frac{\cos(2\theta)}{4}\right) |_0^{\frac{\pi}{4}} \\ &= \left(\frac{1}{2} - \frac{\cos\left(\frac{\pi}{2}\right)}{4}\right) - \left(\frac{1}{2} - \frac{\cos(0)}{4}\right) = \frac{1}{2} - \frac{1}{2} + \frac{1}{4} = \frac{1}{4}\end{aligned}$$

That seems to be the area – for now.

Another example:

Find the area of the region that lies **inside** the circle $r = 3\sin(\theta)$ ad **outside** the cardioid $r = 1 + \sin(\theta)$ (that's the shape)!!!



Area of the entire circle, from $\frac{\pi}{6}$ to $\frac{5\pi}{6}$:

$$\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 9\sin^2(\theta)d\theta$$

Area of the cardioid, from $\frac{\pi}{6}$ to $\frac{5\pi}{6}$:

$$\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin(\theta))^2 \, d\theta$$

Subtract the top from the bottom, and you have the answer. The final answer is π, and it's up to you to figure it out.

In general, when you want to find the area between two curves, here's the formula:

$$\forall a \leq \theta \leq b,\ g(\theta) \geq f(\theta) \Rightarrow \text{Area between} = \frac{1}{2}\int_a^b (g(\theta))^2 - (f(\theta))^2 \, d\theta$$

2.4 Tangents on a polar curve

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

To find the vertical/horizontal lines, we can use the same method that we use for parametric curves. This is when $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$ to find any vertical tangents; swap the $=$ and $\neq$ if you want vertical tangent lines.

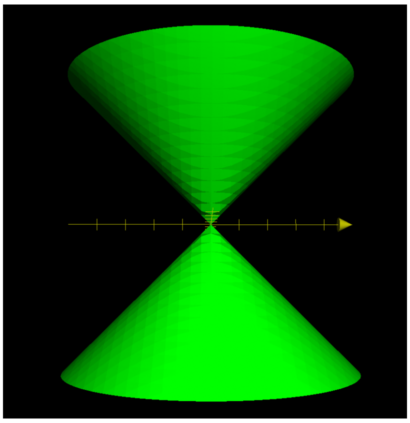
Alternatively, use the fact that

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned}$$

2.5 Arc length of a parametric curve

$$\begin{aligned} &L \\ &= \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta \\ &= \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta \end{aligned}$$

3 3D shapes and Conic sections



If I take a slice from it, what will I get? Here is the list of conic sections

- Circle
- Ellipse
- Hyperbola
- Parabola

3.1 Parabola

Definition (Parabola). A parabola is the set of points in a plane that are equidistant from a fixed-point F, called the focus, and a fixed line, called the direction.

The path an object travels when thrown creates a parabola. This type of curve has received a lot of attention.

The generic formula for a parabola with the vertex at the origin:

$$\begin{aligned} |y+p| = \left|\overline{pF}\right| &= \sqrt{x^2+(y-p)^2} \\ \Rightarrow (y+p)^2 &= x^2+(y-p)^2 \\ \Rightarrow y^2+2py+p^2 &= x^2+y^2-2py+p^2 \\ \Rightarrow x^2 &= 4py \end{aligned}$$

A parabola with a focus of $(0,\ p)$ is represented by this equation: $x^2 = 4py$, upwards or downwards. Swap x and y if the parabola rotates by $90°$.

Instead of using $4p$, we can instead use a. There are four kinds of parabolas:

1. $y = ax^2$, $a > 0$. U-shaped parabola.
2. $y = ax^2$, $a < 0$. ∩-shaped parabola.
3. $x = ay^2$, $a > 0$. C-shaped parabola.
4. $x = ay^2$, $a < 0$. ⊃-shaped parabola.

Here, the vertex is always the origin.

A parabola with a vertex at $(x_0,\ y_0)$:

$$(x - x_0)^2 = 4p\,(y - y_0)$$

Would be the formula of the parabola.

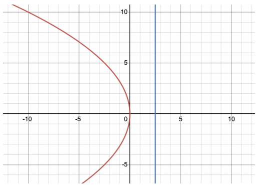
Example. Find the focus and directrix of this parabola. The focus is $(x_0,\ y_0 + p)$ and the directrix is $y = -p$:

$$y^2 + 10x = 0$$

The solution is as follows:

$$\begin{aligned} y^2 &= -10x \\ 4p &= -10 \\ p &= -\frac{5}{2} \\ x &= \frac{5}{2} \end{aligned}$$

The parabola is ⊃-shaped.



Example. Find the focus and directrix of this parabola: $x^2 - 6x + 3y = 18$

The solution is as follows:

$$\begin{aligned} x^2 - 6x + 3y + 18 &= 0 \\ (x-3)^2 - 9 + 3y + 18 &= 0 \\ (x-3)^2 + 3y &= -9 \\ (x-3)^2 &= -3y - 9 \\ (x-3)^2 &= -3(y+3) \end{aligned}$$

The vertex of this parabola is $(3,\ -3)$.

$$\begin{aligned} -3 &= 4p \\ p &= -\frac{3}{4} \end{aligned}$$

The directrix is at $y = \frac{3}{4}$. The focus is at $\left(3,\ -3 - \frac{3}{4}\right) = \left(3,\ -\frac{15}{4}\right)$.

3.2 Ellipse

An ellipse is the set of points in a plane that the sum of whose distances from two fixed points $f_1,\ f_2$ is a constant.

Instead of one focus, ellipses have two foci.

$$\begin{aligned}
|\overline{pf_1}| + |\overline{pf_2}| &= 2a > 0 \\
\Rightarrow \sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} &= 2a \\
\Rightarrow \sqrt{(x+c)^2+y^2} &= 2a - \sqrt{(x-c)^2+y^2} \\
\Rightarrow x^2+2cx+c^2+y^2 &= 4a^2 + (x-c)^2 + y^2 - 4a\sqrt{(n-c)^2+y^2} \\
&\Rightarrow \ldots \\
\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1
\end{aligned}$$

A horizontal ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a \geq b > 0$$

Has foci $(\pm c,\ 0)$ where $c^2 = a^2 - b^2$, and vertices $(\pm a,\ 0)$.

A vertical ellipse:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1,\ a \geq b > 0$$

Note that $a,\ b$ are swapped.

There is a simpler way to figure out an ellipse is vertical or horizontal: take $x,\ y = 0$ and figure out if $a,\ b$ has a higher value.

If $a = b$, then you have a circle. It has one focus.

The standard form of an equation of an ellipse with center $(x_0,\ y_0)$ is:

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$$

Example: sketch the graph of the following ellipses and locate the foci.

First:

$$9x^2 + 16y^2 = 144$$
$$\frac{9x^2}{144} + \frac{16y^2}{144} + \frac{144}{144}$$
$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

This is the standard form of an ellipse.

If we take $y = 0 \Rightarrow \frac{x^2}{16} = 1 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4,\ (4,\ 0),\ (-4, 0)$

If we take $x = 0 \Rightarrow \ldots \Rightarrow \frac{y^2}{9} = 1 \Rightarrow y = \pm 3,\ (0,\ 3),\ (0,\ -3)$

Therefore, our ellipse can be drawn by getting the points $(\pm 4,\ 0),\ (0,\ \pm 3)$ and tracing through the dots. We can find the focus f_1 and f_2 in some way.

Example 2: The standard form of this ellipse:

$$\begin{aligned}
3x^2 - 18x + 4y^2 + 16y &= -31 \\
3\left(x^2 - 6x\right) + 4\left(y^2 + 4y\right) &= -31 \\
3\left(x^2 - 6x + 9 - 9\right) + 4\left(y^2 + 4y + 4 - 4\right) &= -31 \\
3(x-3)^2 - 27 + 4(y+2)^2 - 16 &= -31 \\
3(x-3)^2 + 4(y+2)^2 &= -31 + 16 + 27 = 12 \\
\frac{(x-3)^2}{4} + \frac{(y+2)^2}{3} &= 1
\end{aligned}$$

Tip: take $x = 3$ and find all values of y in the ellipse such that $x = 0$, and so on.

3.3 To sketch an ellipse

You need four points. If you have the equation for an ellipse:

$$\frac{(x-2)^2}{16} + \frac{(y-1)^2}{9} = 1$$

Take $x = 2$ and find y, and take $y = 1$ and find x. The reason why we took x and y to be these values because it effectively eliminates one side of the sum.

To find the center of an ellipse, it is (x_0, y_0). Each component from each point is subtracted from x and y, respectively.

3.4 Hyperbola

A hyperbola is the set of all points in a plane the difference of whose distances from two fixed points f_1 and f_2 is any fixed constant.

$$|\overline{pf_1}| - |\overline{pf_2}| = \pm 2a$$

The standard formula for a hyperbola is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Has foci $(\pm c, 0)$, where $c^2 = a^2 + b^2$, vertices $(\pm a, 0)$ and asymptotes $y = \pm\left(\frac{b}{a}\right)x$.

You **only** need to know how to draw this hyperbola in a very simple way. Remember that what you need is the vertices and the asymptotes.

3.4.1 Vertical hyperbola

The hyperbola stated above is horizontal. For vertical, the equation is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$. Has foci $(0, \pm c)$ where $c^2 = a^2 + b^2$ and asymptotes $y = \pm\frac{a}{b}x$. **Note that the denominator of the positive part of the hyperbola equation (that that doesn't subtract) is always a.**

3.4.2 Hyperbola sketching

$$\begin{aligned} 9x^2 - 16y^2 &= 144 \\ \Rightarrow \frac{9x^2}{144} - \frac{16y^2}{144} &= 1 \\ \Rightarrow \frac{x^2}{16} - \frac{y^2}{9} &= 1 \end{aligned}$$

$$a^2 = 16,\ b^2 = 9$$
$$\Rightarrow a = \pm 4,\ b = \pm 3$$

The lines forming the hyperbola is $\pm\frac{b}{a}x$ and the vertices are at $(\pm a,\ 0)$.

4 3D coordinate system

Coordinate directions:

- x axis points to us
- y axis points to the right
- z axis points to the up.

The distance between two points is:

$$|\overline{p_1 p_2}|$$
$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

This is known as the norm: $||p_1 p_2||$.

4.1 Of a sphere

An equation of a sphere with center $(x_0,\ y_0,\ z_0)$ and radius r is:

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

Example: Find the center and the radius of the following sphere:

$$
\begin{aligned}
x^2+y^2+z^2+4x-6y+2z+6=0 \\
\Rightarrow x^2+4x+y^2-6y+z^2+2z=-6 \\
\Rightarrow \left(x^2+4x+4\right)-4+\left(y^2-6y+9\right)-9+\left(z^2+2z+1\right)-1=-6 \\
\Rightarrow (x+2)^2+(y-3)^2+(z+1)^2=8
\end{aligned}
$$

Another example: What region in $\mathbb{R}^3$ is represented by the following:

$$1 \leq x^2+y^2+z^2 \leq 9,\ z \geq 0$$

The results end up being:

$$\left\{(x,\ y,\ z) \in \mathbb{R}^3 : 1 \leq x^2+y^2+z^2 \leq 9 \wedge z \geq 0\right\}$$

A hollow circle sliced across the XY plane, and we only keep the thing in the positive z-axis.

4.1.1 Of an ellipsoid

Just an extra component.

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}+\frac{(z-z_0)^2}{c^2}=1$$

Where $a,\ b,\ c \in \mathbb{R} \setminus \{0\}$.

If $a=b=c$, it's a sphere.

4.2 Vector

A vector is often represented by an arrow or a directed line segment. Goes from A, an initial point, to B, a terminal point. Then, we have $\vec{AB}$.

Add two vectors using the triangle law or the parallelogram law. Vector addition is commutative.

4.3 Hollow, not hollow

Suppose we have these two equations in $\mathbb{R}^3$:

$$x^2 + y^2 = 1$$
$$x^2 + y^2 \leq 1$$
$$x^2 + y^2 < 1$$

The top one is a hollow cylinder. The middle one is a solid cylinder. The bottom one only allows you to consider the points inside the cylinder.

4.4 Vector bracket notation

- (a_1, a_2, a_3) is a point.
- $\langle a_1, a_2, a_3 \rangle$ is a vector, pointing from the origin.

4.5 Basis vectors

In $\mathbb{R}^2$, $\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$. In 3D space, the remaining one is $\vec{k} = \langle 0, 0, 1 \rangle$.

If I have components of two vectors, I can consider vector addition and scalar multiplication.

Any vector can be expressed by the sum of the basis vectors:

$$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

I can express the magnitude of a vector by:

$$||\vec{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

A unit vector is a vector whose length is 1, i.e., $||\vec{a}|| = 1$.

4.5.1 Unit vectors based on another vector

From this one: $\vec{a} = \langle 2,\ -1,\ -2 \rangle$ and $||\vec{a}|| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$

So, divide each component of $\vec{a}$ by 3. This means $\frac{1}{3}\vec{a} = \left\langle \frac{2}{3},\ -\frac{1}{3},\ -\frac{2}{3} \right\rangle$, which is the unit vector that points in the same direction of $\vec{a}$. In other words, it is produced by y

$$\frac{\vec{a}}{||\vec{a}||}$$

4.6 Properties of vectors

- Commutativity $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
- Associativity $\left(\vec{a} + \vec{b}\right) + \vec{c} = \vec{a} + \left(\vec{b} + \vec{c}\right)$
- Identity $\vec{a} + \vec{0} = \vec{a}$
- And so on.

4.7 Dot product

For dot product:

$$\langle a_1,\ a_2 \rangle \cdot \langle b_1,\ b_2 \rangle = a_1 b_1 + a_2 b_2$$

In general, in $\mathbb{R}^n$:

$$\vec{a} + \vec{b} = \sum_{i=1}^{n} a_i b_i$$

Dot products are also called inner products. They always return $\mathbb{R}$.

θ is the angle between the two vectors $\vec{a}$ and $\vec{b}$. Using magnitudes and angles,

$$\vec{a}\cdot\vec{b} = ||\vec{a}|| \left|\left|\vec{b}\right|\right| \cos(\theta)$$
$$\Rightarrow \theta = \arccos\left(\frac{\vec{a}\cdot\vec{b}}{||\vec{a}|| \left|\left|\vec{b}\right|\right|}\right)$$

Therefore, if you have two vectors such that they're perpendicular ($\theta = \frac{\pi}{2}, \ldots$), then the dot product between them is 0. In other words

$$\vec{a}\cdot\vec{b} = 0 \Rightarrow \theta = 0$$

4.7.1 Properties of the dot product

- Itself $\vec{a}\cdot\vec{a} = ||\vec{a}||^2$
- Commutative
- Associative
- Closed under scalar multiplication
- Null element $\vec{0}\cdot\vec{a} = 0$

Corollary. Two vectors $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\vec{a}\cdot\vec{b} = 0$.

Exercise: Find a unit vector that is orthogonal to both $\vec{i}+\vec{j}$ and $\vec{i}+\vec{k}$, in other words $\langle 1,\ 1,\ 0\rangle$ and $\langle 1,\ 0,\ 1\rangle$.

This means we must find a vector $\vec{a}$ such that:

$$\sqrt{a_1^2 + a_2^2 + a_3^3} = 1$$

So:

$$\begin{cases} \vec{a} \cdot \left(\vec{i} + \vec{j}\right) = 0 \\ \vec{a} \cdot \left(\vec{i} + \vec{k}\right) = 0 \end{cases} \Rightarrow a_1 = -a_2 = -a_3 \Rightarrow \langle 1,\ -1,\ -1 \rangle$$

The answer can be found there.

4.8 Cross product

For cross product, it returns a vector with the same dimensions of the two vectors that were used to compute the product. It is always perpendicular to both vectors. It is denoted as:

$$\vec{a} \times \vec{b}$$

It is:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

If we only have the magnitudes and direction:

$$\left|\vec{a} \times \vec{b}\right| = ||\vec{a}|| \left|\left|\vec{b}\right|\right| \sin(\theta)$$

Properties of the cross product

$$\begin{aligned} \vec{a} \times \vec{b} &= -\vec{b} \times \vec{a} \\ \vec{a} \times \left(\vec{b} + \vec{c}\right) &= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \\ \vec{a} \cdot \left(\vec{b} \times \vec{c}\right) &= \left(\vec{a} \times \vec{b}\right) \cdot \vec{c} \\ \vec{a} \times \left(\vec{b} \times \vec{c}\right) &= (\vec{a} \cdot \vec{c})\vec{b} - \left(\vec{a} \cdot \vec{b}\right)\vec{c} \end{aligned}$$

4.8.1 The right hand rules

Grab your right hand out flat with four of your fingers pointing in the direction of $\vec{a}$. Then, make sure your hand is angled in such a way that you can close your hand to reach $\vec{b}$. The direction of your thumb is the direction of the cross product.

If two vectors are parallel, the cross product between them is 0.

4.9 Determinant

Order 2:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Order 3, using cofactors:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}$$
$$- a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}$$
$$+ a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Example:

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix}$$
$$= (0\cdot 2 - 1\cdot 4) - 2(3\cdot 2 + 1\cdot 5) - 1(3\cdot 4)$$
$$= -38$$

Using determinant to find cross product: $\vec{a} = \langle 1,3,4\rangle$, $\vec{b} = \langle 2,7,-5\rangle$:

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{j} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = \vec{i}\begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix}$$
$$= \vec{i}((-5)\cdot 3 - 4\cdot 7) - \vec{j}((-5)\cdot 1 - 4\cdot 2) + \vec{k}(1\cdot 7 - 3\cdot 2)$$
$$= -43\vec{i} + 13\vec{j} + \vec{k}$$

4.10 Different solids

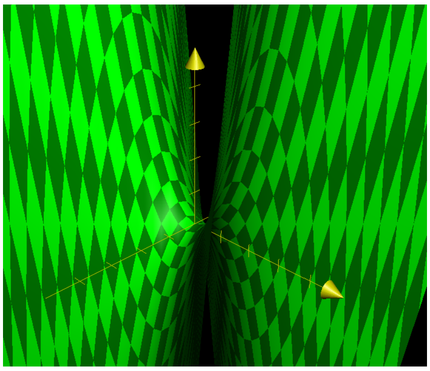
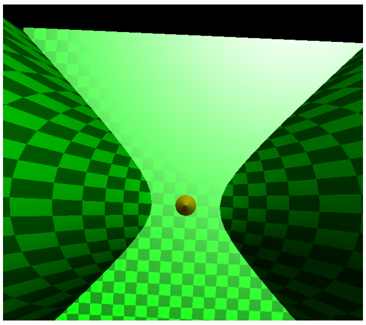


Figure 1: A picture containing green, light Description automatically generated

Intersect this shape with two planes: one on the XY-plane and one on the ZX plane. What do we get?

XY: Hyperbola



4.11 Intersecting two planes

How can we obtain equations for planes intersecting planes?

4.12 Shapes

- $x^2 - y^2 - z^2 = 1$ produces a hyperbola
- $x^2 + y^2 - z^2 = 1$ produces an hourglass
- $z = x^2 - y^2$ produces the pringle
- $z = x^2 + y^2$ produces a paraboloid
- $z^2 = x^2 + y^2$ produces a double cone

5 Lines and planes in 3D spaces

In 2D, to write down the formula of a line, write down a point and a slope. The slope is really the direction of the line. When we move to 3D space. In 2D,

- Two lines may intersect
- Or be parallel or on top of each other.

In 3D spaces, what are the intersection cases for lines in 3D space?

- Parallel or at the same place
- Cross
- Not parallel, no cross (askew)

For planes:

- Parallel or at the same place
- Intersects in a line at one place

This means the standard formula for lines and planes are important.

5.1 Equations of lines and planes in space

Parametric equations for a line through the point (x_0, y_0, z_0) and parallel to the direction vector $\langle a, b, c\rangle$ is:

$$\begin{bmatrix} x_0 + at \\ y_0 + bt \\ z_0 + ct \end{bmatrix}, t \in \mathbb{R}$$

For a plane, we need a point in the plane and one vector that is orthogonal to the plane (called the normal vector). A scalar equation of a plane through point (x_0, y_0, z_0) with normal vector $\vec{n} = \langle a, b, c\rangle$:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Example. Find parametric equations for the line that passes through the point $(5, 1, 3)$ and is parallel to the vector $\vec{i} + 4\vec{j} - 2\vec{k}$.

Well, $(x_0, y_0, z_0) = (5, 1, 3)$ and $\vec{a} = (1, 4, -2)$. So, the equation of the line i:

$$\begin{bmatrix} 5+t \\ 1+4t \\ 3-2t \end{bmatrix}$$

Therefore, $x = 5+t, y = 1+4t, z = 3-2t$

If we need to find some other points in this line, take t to be any value we want and plug it in for as many times you want given it is in $\mathbb{R}$.

- If we take $t = 1 \Rightarrow (6,\ 5,\ 1)$
- If we take $t = -1 \Rightarrow (4,\ -3,\ 5)$

That is the parametric formula for a line. Then, I can compute t by using any of the three formulas:

$$t = \frac{x-x_0}{a} \quad t = \frac{y-y_0}{b} \quad t = \frac{z-z_0}{c}$$

Thus, we have:

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

This is called the symmetric equations of a line.

If one of $a,\ b,\ c$ is 0, then we can still eliminate t. We just let $x = x_0$.

Example. Find parametric equations and symmetric equations of the line that passes through the points $p_0(2,\ 4,\ -3)$ and $p_1(3,\ -1,\ 1)$. At what point does this line intersect the xy plane?

We can find the direction vector by getting it from $p_1 - p_0$.

$$\vec{p}_1 - \vec{p}_0 = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$$

Then, grab one of p_0 and p_1, and the equation of the line is:

$$\begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$$

5.2 To show that two lines are skew

Not parallel and does not cross. To show that they aren't parallel, it is enough to show that their direction vectors are not parallel.

Example. Show that L_1 and L_2 do not cross. Then:

$$L_1 = \begin{bmatrix} 1+0t \\ -2+3t \\ 4-t \end{bmatrix}, L_2 = \begin{bmatrix} 0+2s \\ 3+s \\ -3+4s \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

To find whether the two vectors are parallel or not, take the cross product of them. If the cross product is 0, then they are parallel; otherwise, they're not.

To show that the two lines are askew, attempt to take a system of equations and conclude that there are no answers.

5.3 On planes

How can we get the standard formula of a plane?

Firstly, let $\vec{p}_1$ be any vector that is parallel on the plan (if we project it on the plane the length of it doesn't change). Then:

$$\vec{n} \cdot \vec{p}_1 = 0$$

As they are orthogonal.

THE STANDARD FORMULA OF A PLANE IS PRESENTED HERE:

If $n = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, then:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0$$

"NOT AS A VECTOR" FORM:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Where $x_0,\ y_0$, and z_0 form this point $(x_0,\ y_0,\ z_0)$ that lies on the plane – hint: take one of the points of the plane and sub them in $x_0,\ y_0$, and z_0.

Example. Find the plane passing through the following points: $(0,\ 1,\ 1),\ (1,\ 0,\ 1),\ (1,\ 1,\ 0)$.

What I need:

- One vector from one point to another
- A normal vector

Consider the two vectors that lie on the plane:

$$\vec{AB},\ \vec{AC}$$

Take the cross product, and you'll get the normal vector:

$$\vec{n} = \vec{AB} \times \vec{AC}$$

We have $\vec{AB} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\vec{AC} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Computing

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \vec{i}\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} + k\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix}$$
$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

This means the equation of the plane is:

$$(x-0)+(y-1)+(z-1)=0$$

Example. Find the angle between the following planes: $x+y+z=1$ and $x-2y+3z=1$. Hint: find the angle between the normal vectors.

The two planes can be represented as:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

And

$$\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Finding the angles between the normal vectors:

$$\cos(\theta) \left\| \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \right\| \left\| \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\| = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

$$\theta = \arccos\left(\frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \right\| \left\| \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\|} \right)$$

Whatever you get out of this is the angle between the two planes.

Example. Find the equation of the plane passing through $(2,\ 1,\ 0)$ and is parallel to the plane $x+4y-3z=1$.

The normal vector of the plane stated is:

$$\begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$$

The point is given, so the equation of the plane that is asked by the question is:

$$\begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} x-2 \\ y-1 \\ z \end{bmatrix} = 0$$
$$\Rightarrow (x-2)+4(y-1)-3z = 0$$

How to solve any problem involving finding the plane given one line, one point:

- Let $\vec{v}$ be any vector that points in the same direction as the line
- Let p be the point that we are given
- Let q be any point on the line
- Then the normal vector of the plane is $\vec{v} \times \vec{pq}$
- And the equation of the line is $(\vec{v} \times \vec{pq}) \cdot \begin{bmatrix} x-p_1 \\ y-p_2 \\ z-p_3 \end{bmatrix} = 0$

6 Vector-valued functions

A function takes an input and returns an output. It maps from a set to a set. In the context of single-variable calculus, functions we've seen are $f\colon \mathbb{R} \to \mathbb{R}$. This means if I input x to f, I won't get a vector.

A vector valued function is when I have a function that looks like $f\colon \mathbb{R} \to \mathbb{R}^n$. Here's an example:

$$r(t) = \begin{bmatrix} t^2 \\ \sin(t) \\ \cos(t) \end{bmatrix}$$

Then r: $\mathbb{R} \to \mathbb{R}^3$, because I input one $\mathbb{R}$ that is t and I get a vector that is $\begin{bmatrix} t^2 \\ \sin(t) \\ \cos(t) \end{bmatrix}$, which has 3 components.

6.1 Parametric equations

Remember parametric functions?

$$x^2 + y^2 = 1 \to \begin{array}{c} x = \cos(t) \\ y = \sin(t) \end{array}$$

We can represent it as:

$$\vec{r}(t) = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}$$

We can use the same method, but in 3D space.

6.2 Points of intersections of two 3D functions

Let $f,\ g,\ h$ be real-valued functions. For every t in the domain of $\vec{r}$, we have a unique vector $\vec{r}(t) = \begin{bmatrix} f(t) \\ g(t) \\ h(t) \end{bmatrix} \ldots ?$

Example. Describe the vector function given by $\vec{r}(t) = \begin{bmatrix} 1+2t \\ 1-5t \\ -2+6t \end{bmatrix}$

So, we have

$$\begin{aligned} x &= 1 + 2t \\ y &= 1 - 5t \quad \Rightarrow \text{parametric eq of a line} \\ z &= -2 + 6t \end{aligned}$$

Where the line's direction vector is $\begin{bmatrix} 2 \\ -5 \\ 6 \end{bmatrix}$ that passes through $(1,\ 1,\ -2)$.

Example. Sketch the curve whose vector equation is:

$$\begin{aligned} \vec{r}(t) &= 2\cos(t)\vec{i} + 2\sin(t)\vec{j} + t\vec{k} \\ &= \begin{bmatrix} 2\cos(t) \\ 2\sin(t) \\ t \end{bmatrix} \end{aligned}$$

If we project this onto the xy plane, we get $\begin{bmatrix} 2\cos(t) \\ 2\sin(t) \end{bmatrix}$, which looks like a radius-2 circle centered at the origin.

A point is $(x_0,\ y_0,\ z_0)$. A vector is $\langle x_0,\ y_0,\ z_0 \rangle$ or $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$.

We had this vector function: $\begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix}$

When projected on the xy plane, we have $\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$.

Example. Find a vector function that represents the curve of intersection of the cylinder $x^2 + z^2 = 1$ (going through the y-axis) and $z + y = 1$ (a plane).

As a parametric equation:

$$x = \cos(t)$$
$$z = \sin(t)$$

And $z + y = 1$ so $y = 1 - z = 1 - \sin(t)$

This means the vector function that represents the intersection of the curve, and the cylinder is

$$\begin{bmatrix} \cos(t) \\ 1 - \sin(t) \\ \sin(t) \end{bmatrix}, \ 0 \leq t \leq 2\pi$$

Example. The upper part of the sphere $x^2 + y^2 + z^2 = 1$ and the upper half of the cone $z^2 = x^2 + y^2$

We have

$$\begin{cases} z^2 = x^2 + y^2 \\ x^2 + y^2 + z^2 = 1 \end{cases}$$

$$\begin{aligned} \Rightarrow z^2 + z^2 &= 1 \\ 2z^2 &= 1 \\ z &= \frac{1}{\sqrt{2}} \end{aligned}$$

So:

$$
\begin{aligned}
x^2+y^2+\frac{1}{2}&=1\\
x^2+y^2&=\frac{1}{2}\\
x&=\frac{1}{\sqrt{2}}\cos(t)\\
y&=\frac{1}{\sqrt{2}}\sin(t)
\end{aligned}
$$

We thus have this parametric equation:

$$
\begin{bmatrix}\frac{1}{\sqrt{2}}\cos(t)\\ \frac{1}{\sqrt{2}}\sin(t)\\ \frac{1}{\sqrt{2}}\end{bmatrix}
$$

6.3 Limits of vector functions

If I have a function $f\colon \mathbb{R}\to\mathbb{R}$, then

$$
\lim_{x\to x_0} f(x)\in\mathbb{R}
$$

If the limit exists. If I swap $\mathbb{R}$ with $\mathbb{C}$, it should work the same. At least, if $\forall\varepsilon>0,\ \exists\delta>0,\ |x-a|<\delta\Rightarrow|f(x)-L|<\varepsilon$. I will never see that equation again.

If I have $f\colon \mathbb{R}\to\mathbb{R}^3$, then the limit is also in $\mathbb{R}^3$.

$$
\vec{r}\colon \mathbb{R}\to\mathbb{R}^3
$$

We can define $\lim_{t\to t_0}\vec{r}(t)\in\mathbb{R}^3$ if the limit exists.

Definition. Let $f,\ g,\ h\in\mathbb{R}\to\mathbb{R}$ be functions and let $\vec{r}(t)=\begin{bmatrix}f(t)\\ g(t)\\ r(t)\end{bmatrix}$

Then $\lim_{t\to a}\vec{r}(t) = \begin{bmatrix} \lim_{t\to a} f(t) \\ \lim_{t\to a} g(t) \\ \lim_{t\to a} h(t) \end{bmatrix}$, provided the limits of the component functions exist.

$\vec{r}$ is **continuous** at a if $\lim_{t\to a}\vec{r}(t) = \vec{r}(a)$.

Example. Find the $\lim_{t\to 0}\vec{r}(t)$, were

$$\begin{aligned}&\vec{r}(x)\\ &= \left(\frac{1}{\ln\left(x+\sqrt{x^2+1}\right)} - \frac{1}{\ln(x+1)}\right)\vec{i}\\ &+ \left((1+x)^{\frac{1}{x}}\right)\vec{j} + \left(\frac{\sin(5x)}{3x}\right)\vec{k}\end{aligned}$$

The answer is:

$$\left(-\frac{1}{2}\right)\vec{i} + e\vec{j} + \frac{5}{3}\vec{k}$$

You <u>may</u> have to use Hospital's rule for this.

Solution for the $\vec{k}$ component: TBA

6.4 Derivative

$$f\colon \mathbb{R} \to \mathbb{R}$$

Then $f^{'}(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

For a vector function, you have the same thing, but instead of real numbers, you must deal with vectors.

What is the appropriate notion of the derivative if $f\colon \mathbb{R} \to \mathbb{R}^3$? Or $f : \mathbb{R}^3 \to \mathbb{R}^3$?

If we have a vector-valued-function in 3D:

$$\frac{d\vec{r}}{dt} = \vec{r}^{'}(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

Theorem. The derivative of a vector function is the derivative of its components.

$$\vec{r}(t) = \begin{bmatrix} f(t) \\ g(t) \\ h(t) \end{bmatrix} \Rightarrow \vec{r}^{'}(t) = \begin{bmatrix} f^{'}(t) \\ g^{'}(t) \\ h^{'}(t) \end{bmatrix}$$

$$\vec{r}^{''}(t) = \begin{bmatrix} f^{''}(t) \\ g^{''}(t) \\ h^{''}(t) \end{bmatrix}$$

The **unit tangent vector,** denoted $\vec{T}$, is defined by:

$$\vec{T}(t) = \frac{\vec{r}^{'}(t)}{\left|\vec{r}^{'}(t)\right|}$$

6.5 The TNB frame

The normalized tangent vector, where the length is always 1.

The normal vector, $\vec{N}$, is defined by: The derivative of the normal vector, then normalized again for some reason

$$\vec{N}(t) = \frac{\vec{T}^{'}(t)}{\left|\vec{T}^{'}(t)\right|}$$

The binominal vector, defined by $\vec{B}$, is defined by the cross product of the previous two, $\vec{T}$ being first

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

Visually, suppose you have a curve. For each point of the curve, you want an orthogonal frame to the curve. Only $\vec{T}(t)$ will be asked on the exam, but it is best to know all three.

Example. Let $\vec{r}(t) = \begin{bmatrix} 1+t^3 \\ te^{-t} \\ \sin(2t) \end{bmatrix}$

Then $\vec{r}'(t) = \begin{bmatrix} 3t^2 \\ e^{-t} - te^{-t} \\ 2\cos(2t) \end{bmatrix}$, and $\vec{r}''(t) = \begin{bmatrix} 6t \\ -2e^{-t} + te^{-t} \\ -4\sin(2t) \end{bmatrix}$.

The normalized tangent vector is $\frac{\vec{r}'(t)}{\left|\vec{r}'(t)\right|} = \frac{\begin{bmatrix} 3t^2 \\ e^{-t} - te^{-t} \\ 2\cos(2t) \end{bmatrix}}{\sqrt{9t^4+(e^{-t}-te^{-t})^2+4\cos^2(2t)}}$

The normal is

$$\frac{\left(\frac{\begin{bmatrix} 6t \\ -2e^{-t}+te^{-t} \\ -4\sin(2t) \end{bmatrix}}{\sqrt{9t^4+(e^{-t}-te^{-t})^2+4\cos^2(2t)}}\right)}{\frac{\left\|\begin{bmatrix} 6t \\ -2e^{-t}+te^{-t} \\ -4\sin(2t) \end{bmatrix}\right\|}{\sqrt{9t^4+(e^{-t}-te^{-t})^2+4\cos^2(2t)}}}$$

$$= \frac{\begin{bmatrix} 6t \\ -2e^{-t}+te^{-t} \\ -4\sin(2t) \end{bmatrix}}{\left\|\begin{bmatrix} 6t \\ -2e^{-t}+te^{-t} \\ -4\sin(2t) \end{bmatrix}\right\|}$$

$$= \frac{\begin{bmatrix} 6t \\ -2e^{-t}+te^{-t} \\ -4\sin(2t) \end{bmatrix}}{\sqrt{(6t)^2+(-2e^{-t}+te^{-t})^2+(-4\sin(2t))^2}}$$

And

$$\vec{B}(t) = \frac{\begin{bmatrix} 3t^2 \\ e^{-t} - te^{-t} \\ 2\cos(2t) \end{bmatrix}}{\sqrt{9t^4 + (e^{-t} - te^{-t})^2 + 4\cos^2(2t)}} \times \frac{\begin{bmatrix} 6t \\ -2e^{-t} + te^{-t} \\ -4\sin(2t) \end{bmatrix}}{\sqrt{(6t)^2 + (-2e^{-t} + te^{-t})^2 + (-4\sin(2t))^2}}$$

You see, this isn't the cleanest thing to exist.

6.6 Parametric to cartesian

With $x = \sin(t),\ y = \cos(t)$, the cartesian form is $x^2 + y^2 = 1$.

With $x = \cos(t),\ y = \sin(t)$, the cartesian form is $x^2 + y^2 = 1$.

No, cartesian equations may not always be parameterized uniquely. Cartesian equations only have paths, whilst parametric equations have directions. It doesn't matter if we take $x = \sin(t)$ or $x = \cos(t)$ given y isn't the same as x.

In the future, because we want to look at green's theorem, we usually consider the direction of the parametric equation as **counterclockwise.** Hence, it is strongly recommended to consider that everything is supposed to go in the counterclockwise direction.

To find the direction of a curve, good luck tracing.

6.7 Derivative rules

Most of the rules from single-variable calculus apply.

1) Sum: $(\vec{u}(t)+\vec{v}(t))^{'} = \vec{u}^{'}(t)+\vec{v}^{'}(t)$
2) Closed under multiplication
3) Multiplication with a scalar and vector
4) Dot product: $(\vec{u}(t)\cdot\vec{v}(t))^{'} = \vec{u}^{'}(t)\cdot\vec{v}(t)+\vec{u}(t)\cdot\vec{v}^{'}(t)$
5) Cross product: $(\vec{u}(t)\cdot\times\vec{v}(t))^{'} = \vec{u}^{'}(t)\times\vec{v}(t)+\vec{u}(t)\times\vec{v}^{'}(t)$
6) Chain rule: $(\vec{u}\,(f(t)))^{'} = \vec{u}^{'}\,(f(t))\cdot f^{'}(t)$

Theorem. $\forall t\in\mathbb{R}$, If $|\vec{r}(t)| = c$ is a constant, then $\vec{r}^{'}(t)$ is orthogonal to $\vec{r}(t)$.

Proof. $\vec{r}(t)\cdot\vec{r}(t) = ||\vec{r}(t)||^2 = c^2$

Take the derivative on both sides:

$$\begin{aligned} &\vec{r}^{'}(t)\vec{r}(t)+\vec{r}(t)\vec{r}^{'}(t)\\ &= 2\vec{r}^{'}(t)\cdot\vec{r}(t) = 0\\ \Rightarrow &\vec{r}^{'}(t)\cdot\vec{r}(t) = 0\\ &\Rightarrow \vec{r}^{'}(t)\perp\vec{r}(t) \blacksquare \end{aligned}$$

Remember circular motion?

7 Functions of two variables

Motivation. Suppose that there's a sphere. Let T be the temperature of a point on the sphere depending on the longitude x and the latitude y. Then:

$$T = f(x,\, y)$$

We can think of T as functions of two variables.

In a mathematical example, suppose that you have a cylinder. If you want to compute the volume of this cylinder, if you know the radius of the base and the height of the edge, the volume is $\pi r^2 h$. If we substitute a different cylinder with different radius and height, the function is still the same: $\pi r^2 h$. We can consider volume as a function of h and r which is $\pi r^2 h$. This is a function of two variables. $f(r, h) = \pi r^2 h$.

Definition. A function of n variables is a rule that assigns to $x_1, x_2, \ldots x_n \in \mathbb{R}^n$ in a set $D \subseteq \mathbb{R}^j$, denoted by $f(x_1, \ldots, x_n)$. The set D is the domain of f and its image (range) is $\{f(x_1, \ldots, x_n) \mid (x_1, x_2, \ldots, x_n) \in D\}$. In this course, we'll only focus on 2-3 variables.

If we have a function of two variables, we often write $z = f(x, y)$ to make explicit the value taken by f of the general point (x, y). The domain is which point you can pick on the XY plane that returns something.

Example. For this function, evaluate $f(3, 2)$ and find and sketch the domain.

$$f(x, y) = \frac{\sqrt{x+y+1}}{x-1}$$

$$f(3, 2) = \frac{\sqrt{3+2+1}}{2-1} = \frac{\sqrt{5}}{1} = \sqrt{5}$$

The domain of this function is $\left\{(x, y) \in \mathbb{R}^2 \mid x+y+1 \geq 0 \land x \neq 1\right\}$

And

$$f(x, y) = \frac{\ln\left(x^2 - y\right)}{(x-1)\sqrt{1-x^2}}$$

The domain of this function is $\left\{(x, y) \in \mathbb{R}^2 \mid y < x^2 \land x \neq 0 \land -1 < x < 1\right\}$. The domain is an area in the XY plane that should be precisely shown.

7.1 Level curves

Definition. The level curves of a function f of two variables are the curves with equations $f(x, y) = c$, where c is a constant number in the image of f.

$$f^{-1}(c) = \{(x_1, x_2, \ldots) \mid f(x_1, x_2, \ldots) = c\}$$

A collection of level curves is called a contour map. These help us visualize functions of two or more variables.

The definition of a graph is $\{(x, f(x)) \mid x \in D_f\}$

For a 3D graph, it is:

$$\{(x, y, z) \mid z = f(x, y)\}$$

So, the result will be in a 3D curve.

Note that it is very difficult to visualize a function f of three variables by its graph.

Example. Sketch the graph of $f(x, y) = 6 - 3x - 2y$ and sketch the level curves for $k = -6, 0, 6$.

Solution:

$$\begin{aligned}
k = -6 &\Rightarrow f(x, y) = -6 \\
&\Rightarrow 6x - 3x - 2y = -6 \\
&\Rightarrow 3x + 2y = 12 \\
&\Rightarrow y = -\frac{3}{2}x + 6 \\
k = 0 &\Rightarrow f(x, y) = 0 \\
&\Rightarrow 6 - 3x - 2y = 0 \\
&\Rightarrow 3x + 2y = 6 \\
&\Rightarrow y = -\frac{3x}{2} + 3
\end{aligned}$$

$$
\begin{aligned}
k = 6 &\Rightarrow f(x, y) = 6 \\
&\Rightarrow 6 - 3x - 2y = 6 \\
&\Rightarrow -3x - 2y = 0 \\
&\Rightarrow 3x + 2y = 0 \\
&\Rightarrow y = -\frac{3x}{2}
\end{aligned}
$$

The question asks for me to sketch the level curves. These are just lines. This can be as simple as sketching the lines. A collection of these lines forms a contour map. The graph of this function is a plane, as it is $z = 6 - 3x - 2y$.

To sketch the 3D graph, we need three points. Choose two variables, find the last one, and that's a point on the graph.

You can approximate how a graph looks like using its contour maps if you have enough information.

7.1.1 Not a plane

Example. Let $f(x, y) = 4x^2 + y^2$.

Find the domain and range of f. There are no restrictions from the domain, but the image is $[0, \infty)$.

Sketch level curves. Plot $(0, 0)$, $1 = x^2 + \frac{y^2}{4}$, and $1 = \frac{x^2}{4} + \frac{y^2}{16}$.

Sketch graph. Looks like a stretched paraboloid, the longer side being on the y-axis.

Example. Sketch the contour map of $f(x, y) = ye^x$.

7.2 Limits in 1D space

Recall the epsilon delta:

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

That applies.

Theorem (Squeeze theorem). Also known as the sandwich theorem. Let $f,\ g,\ h : \mathbb{R} \to \mathbb{R}$ be functions and let $a \in \mathbb{R}$. If

$$f(x) \leq g(x) \leq h(x)$$

where x is near but not at a, and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$

Example. Show that $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

7.3 Limits in 2D and 3D space

Definition. Let $f : \mathbb{R}^2 \to \mathbb{R}$ and let $(a,\ b) \in D_f \subseteq \mathbb{R}^2$. We say that the limit of $f(x,\ y)$ as $(x,\ y)$ approaches $(a,\ b)$ is L, denoted $\lim_{(x,\ y)\to(a,\ b)} f(x,\ y) = L$:

$$\forall (x,\ y) \in D_f,\ 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x,\ y) - L| < \varepsilon$$

There are two ways to show the limit exists:

- Use the epsilon-delta method
- Use the squeeze theorem

To show that a limit doesn't exist at a point, show that the limit for two different paths don't match (like from the left to the right, but now you have 3 dimensions).

7.4 Checking whether a limit exists

It exists

- Plug it in and it's not indeterminate
- $\varepsilon - \delta$
- Squeeze
- Polar cords

It doesn't

- Approaches (x, y) from different directions and obtain two different values for the limit
- Negation of $\varepsilon - \delta$

Limit properties:

- Limit of a constant, wherever it may approach is the constant
- You can sum and subtract limits
- You can multiply limits
- You can divide limits if it's not division by zero
- The continuity law of composition holds

7.4.1 Examples

$$\lim_{(x,y)\to(1,0)} \sin\left(\frac{1+x^2}{x^2+xy+1}\right) = \sin\left(\frac{1+1}{1+1}\right) = 1$$

$$\begin{aligned}\lim_{(x,y)\to(3,3)} \frac{\sqrt{x+1}-\sqrt{y-1}}{x-y} &= \lim_{(x,y)\to(3,3)} \frac{\sqrt{x+1}-\sqrt{y-1}}{x-y} \cdot \frac{\sqrt{x+1}+\sqrt{y+1}}{\sqrt{x+1}+\sqrt{y+1}} \\ &= \lim_{(x,y)\to(3,3)} \frac{(x+1)-(y+1)=x-y}{(x-y)\left(\sqrt{x+1}+\sqrt{y+1}\right)} = \frac{1}{\sqrt{x+1}+\sqrt{y+1}} = \frac{1}{\sqrt{4}+\sqrt{4}} =\end{aligned}$$

7.4.2 To show that a limit at a point doesn't exist

Show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$$

In two directions:

$$f(x,0) = \frac{x^2}{x^2} = 1,\ x \neq 0$$
$$f(0,\ y) = \frac{0}{y^2} = 0,\ y \neq 0$$

Therefore, the limit does not exist.

7.4.3 Using polar coordinates

$$x = r\cos(\theta) \quad y = r\sin(\theta)$$

This means:

$$x^2 + y^2 = r^2,$$
$$(x,\ y) \to (0,\ 0) \Rightarrow r \to 0^+$$

$$\lim_{r\to 0^+} \frac{x^2}{x^2+y^2} = \lim_{r\to 0^+} \frac{r^2\cos^2(\theta)}{r^2} = \cos^2(\theta)$$

The answer depends on θ, so the limit doesn't exist. It depends on θ if a different value of θ can change the limit.

7.4.4 It doesn't exist

Example. This:

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+3y^2}$$

$$\lim_{(x,x)\to(0,0)} \frac{2x^2}{x^2+3x^2} = \lim_{(x,x)\to(0,0)} \frac{2x^2}{4x^2} = \frac{1}{2}$$

$$\lim_{(x,3x=y)\to(0,0)} \frac{2x\cdot 3x}{x^2+3(3x)^2} = \lim_{(x,3x)\to(0,0)} \frac{6x^2}{28x^2} = \frac{3}{14}$$

Nope, not the same. Limit doesn't exist.

There's nothing stopping us from taking $y = mx,\ m\in\mathbb{R}$. Then:

$$\lim_{(x,mx)\to(0,0)} \frac{2mx^2}{x^2+3(mx)^2} = \lim_{(x,\,mx)\to(0,0)} \frac{2mx^2}{x^2+3m^2x^2}$$

$$= \lim_{(x,\,mx)\to(0,0)} \frac{2mx^2}{(1+3m^2)x^2} = \lim_{(x,\,mx)\to(0,0)} \frac{2m}{(1+3m^2)}$$

Because the answer of this limit depends on m, the limit doesn't exist.

Example.

$$\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2(x)}{x^4+y^4}$$

$$f(x,y) = f(x,x)$$

$$\lim_{(x,x)\to(0,0)} \frac{x^2\sin^2(x)}{x^4+x^4} = \lim_{(x,x)\to(0,0)} \frac{x^2\sin^2(x)}{2x^4}$$

$$\lim_{(x,x)\to(0,0)} \frac{\sin^2(x)}{2x^2} = \lim_{(x,x)\to(0,0)} \frac{x^2}{2x^2} = \frac{1}{2}$$

Take $y = 2x$

$$
\begin{aligned}
f(x,y) &= f(x,2x) \\
\lim_{(x,x)\to(0,0)} \frac{(2x)^2\sin^2(x)}{x^4+(2x)^4} &= \lim_{(x,x)\to(0,0)} \frac{4x^2\sin^2(x)}{17x^4} \\
\lim_{(x,x)\to(0,0)} \frac{4}{17x^2}\sin^2(x) &= \frac{4}{17}
\end{aligned}
$$

Nope, limit doesn't exist.

Example. Presented below:

$$\lim_{(x,y)\to(0,0)} \frac{2xy^2}{3x^2+y^4}$$

$$
\begin{aligned}
\lim_{(x,mx)\to(0,0)} \frac{2x(mx)^2}{3x^2+(mx)^4} &= \lim_{(x,mx)\to(0,0)} \frac{2x^3m^2}{3x^2+m^4x^4} \\
&= \lim_{(x,mx)\to(0,0)} \frac{2xm^2}{3+m^4x^2} = \frac{0}{3} = 0
\end{aligned}
$$

Well, look for another point:

$$x = y^2$$

$$\lim_{(y^2,y)\to(0,0)} \frac{2y^4}{3y^4+y^4} = \frac{2y^4}{4y^4} = \frac{1}{2}$$

Because we have two different limits when we take different paths, the limit doesn't exist.

Example. This requires a completely different method.

$$\lim_{(x,y)\to(1,1)} \frac{y-x}{1-y+\ln(x)}$$

You can expand this into three dimensions.

$y = x$:

$$f(x,\, y) = f(x,x) = \frac{0}{1 - x + \ln(x)} = 0,\; x \neq 1$$

Beware of pitfalls: there must exist a value where the limit to the point agrees. Try $x = e^{1-y}$:

$$\begin{aligned} f(x,\, y) = f\left(x,\, x^2\right) &= \frac{y - e^{1-y}}{1 - y + \ln\left(e^{1-y}\right)} \\ &= \frac{y - e^{1-y}}{1 - y + (1-y)}, \lim_{y\to 1} \frac{y - e^{1-y}}{2(1-y)} = \lim_{y\to 1} \frac{1 + e^{1-y}}{-2} = -1 \end{aligned}$$

Try any other expression, that when graphed, there's a point on the curve that holds $(1,\, 1)$. It's not always easy to find different paths, and many examples must be done.

7.5 Squeeze Theorem 2.0

Theorem. Let $a,\, b,\, L \in \mathbb{R}$ and let $f,\, g\ : \mathbb{R}^2 \to \mathbb{R}$ be functions. IF:

$$|f(x,\, y) - L| \leq g(x,\, y)$$

then $\forall (x,\, y) \in \mathbb{R}^2$ inside a disk centered at $(a,\, b)$, maybe except $(a,\, b)$:

$$\left(\lim_{(x,y)\to(a,b)} g(x,y) = 0\right) \Rightarrow \left(\lim_{(x,y)\to(a,b)} f(x,\, y) = L\right)$$

And like always

$$\lim_{x\to x_0} |f(x)| \text{ exists} \not\Rightarrow \lim_{x\to x_0} f(x)$$

So, proceed with caution.

Example. Compute this:

$$\lim_{(x,y)\to(0,0)} xy\sin\left(\frac{1}{x^2+y^2}\right)$$

Well, we know that $0 \leq \left|\sin\left(\frac{1}{x^2+y^2}\right)\right| \leq 1$, so:

$$\begin{aligned} 0 \leq \left|xy\sin\left(\frac{1}{x^2+y^2}\right)\right| &\leq xy \\ \lim_{(x,y)\to(0,0)} |xy| &= 0 \\ \Rightarrow \lim_{(x,y)\to(0,0)} xy\sin\left(\frac{1}{x^2+y^2}\right) &= 0 \end{aligned}$$

With another version of the squeeze theorem, I can say that this limit exists:

$$\begin{aligned} \left|xy\sin\left(\frac{1}{x^2+y^2}\right) - 0\right| \leq |xy|\left|\sin\left(\frac{1}{x^2+y^2}\right)\right| &= 0 \\ \Rightarrow \lim_{(x,y)\to(0,0)} xy\sin\left(\frac{1}{x^2+y^2}\right) &= 0 \end{aligned}$$

7.6 Continuity

No hole.

Definition. A function of two variables is continuous at $(a,\, b)$ if:

$$\lim_{(x,y)\to(a,b)} f(x,\, y) = f(a,\, b)$$

we say that f is continuous on D_f (domain of f) if f is continuous at every point $(a,\, b)$ in D.

Example. Where is $f(x,\, y) = \begin{cases} \frac{e^{-x^2-y^2}-1}{x^2+y^2} & (x,\, y) \neq (0,\, 0) \\ -1 & (x,\, y) = (0,\, 0) \end{cases}$

Firstly, find $\lim_{(x,\, y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2}$, which is:

$$\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2}$$

Use polar coordinates: $x = r\cos(\theta)$, $y = r\sin(\theta)$:

$$\lim_{r\to 0^+} \frac{e^{-r^2}-1}{r^2} = \frac{e^{-r^2}}{r^2} - \frac{1}{r^2}$$

$$\begin{aligned} & r^2\cos^2(\theta) + r^2\sin^2(\theta) \\ & = r^2\left(\cos^2(\theta)+\sin^2(\theta)\right) = r^2 \end{aligned}$$

Take Hospital's rule twice:

$$\ldots = -1$$

Then, the function is continuous at $(0,\ 0)$ and it is thus continuous everywhere.

Note that r goes to 0 from the right because $r = x^2 + y^2 \geq 0$.

$$\lim_{r\to 0^+} r\cos(\theta)\sin(\theta) = 0$$

Dependence on theta. Now, it doesn't

$$\lim_{r\to 0^+} \sin(\theta)\cos(\theta) \Rightarrow \text{The limit DNE}$$

If the limit when turned to something that r approaches doesn't depend on, r then the limit does not exist.

No Hospital's rule? Use Taylor series of e^{-r^2} and factor r out.

8 Partial derivative

The difference between derivatives and differential? Suppose we have

$$f\colon \mathbb{R} \to \mathbb{R}$$

The difference is that:

- To differentiate is an operator: $\frac{d}{dx}$
- The derivative is the function: f'.

$$f \to \frac{d}{dx} \to f'$$

For $f : \mathbb{R}^2 \to \mathbb{R}$, it is not clear which flavor of derivative we can have. Suppose I have this function:

$$f(x, y) = y + x$$

Take $f(1, y) = 1 + y$. This is a function based on y. If we take $y = 2$, then we would have $f(x, 2) = x + 2$. Therefore, this function $f(x, c)$ is a single-variable function based on x. If you take x or y (one of them in a certain value) then we will come up with a certain function based on x or y. Now, we can talk about $f'(1, y)$ and $f'(x, c)$ (where c is a constant).

What does that mean? We have $\frac{df}{dy}$. We can also talk about $f'(x, 2) = \frac{df(x, 2)}{dx}$.

For partial derivatives, we use this notation:

$$\frac{\partial f}{\partial x}$$

If I want a partial derivative with respect to x, I want y to be a certain number $(y = b)$, and I make a slice: now I see a parabola. Now I can consider the partial derivative.

If I want the partial derivative with respect to y, I slice something on the x-axis: $x = a$. Then I end up with another single variable function.

To make it clear:

$$\frac{\partial f}{\partial x}(a,\, b) = \frac{df(x,\, b)}{dx}\,|_{x=a}$$

This is the partial derivative, with respect to x, at $a,\ b$ is the differentiation of the function $f(x,\, b)$ computed at $x = a$.

Official definition for partials:

Let $f\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a function of two variables. Let $(a,\, b)\in D_f$.

The partial derivative of f with respect to x at $(a,\, b)$, denoted vy $f_x(a,b)$ or $\frac{\partial f}{\partial x}(a,\, b)$ is:

$$\frac{\partial f}{\partial x}(a,\, b) = \lim_{h\to 0}\frac{f(a+h,\, b)-f(a,\, b)}{h}$$

The partial derivative of f with respect to x at $(a,\, b)$ denoted by $f_x(a,\, b)$ denoted by $\frac{\partial f}{\partial x}\,(a,b)$ s:

$$f_y(a,b) = \lim_{h\to 0}\frac{f(a,\, b+h)-f(a,b)}{h}$$

8.1 Notation for partial derivatives

If $z = f(x,\, y)$, we write:

$$f_x(x,\, y) = \frac{\partial f}{\partial x}\,(x,\, y) = \frac{dz}{dx}$$
$$f_y(x,\, y) = \frac{\partial f}{\partial y}\,(x,\, y) = \frac{dz}{dy}$$

Example. Let $f(x,\, y) = \ln\left(2x^2+y^3+1\right)$. Find $f_x(2,\, 1)$ and $f_y(2,\, 1)$.

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,\,y) &= \frac{4x}{2x^2+y^3+1} \\
f_x(2,\,1) &= \frac{8}{10} = \frac{4}{5} \\
f_y(x,\,y) &= \frac{3y^2}{2x^2+y^3+1} \\
f_y(2,\,1) &= \frac{3}{8+1+1} = \frac{3}{10}
\end{aligned}$$

Example. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined as:

$$e^z = xyz$$

First one: use implicit differentiation. take the derivative of both sides, treating y as a constant and keeping in mind that z is a function of x: $\frac{\partial z}{\partial x}$

$$\begin{aligned}
\frac{\partial}{\partial x}(e^z) &= \frac{\partial}{\partial x}(xyz) \\
\frac{\partial z}{\partial x}e^z &= yz + xy\left(\frac{\partial z}{\partial x}\right) \\
\frac{\partial z}{\partial x}e^z - xy\left(\frac{\partial z}{\partial x}\right) &= yz \\
\frac{\partial z}{\partial x}(e^z - xy) &= yz \\
\frac{\partial z}{\partial x} &= \frac{yz}{e^z - xy}
\end{aligned}$$

Computing $\frac{\partial z}{\partial y}$:

$$
\begin{aligned}
\frac{\partial}{\partial y}(e^z) &= \frac{\partial}{\partial y}(xyz) \\
\frac{\partial z}{\partial y}e^z &= xz + xy\frac{\partial z}{\partial y} \\
\frac{\partial z}{\partial y}(e^z - xy) &= xz \\
\frac{\partial z}{\partial y} &= \frac{xz}{e^z - xy}
\end{aligned}
$$

8.2 With a curve

Suppose we have $z = f(x,\, y) = x^2 + 2y^2$.

Then:

$$
\begin{aligned}
f(1,\, y) &= 1 + 2y^2 = g(y) \\
g'(y) &= 4y
\end{aligned}
$$

This means

$$
\frac{df}{dy} = 4y
$$

This means:

$$
\frac{\partial f}{\partial y}(1,\, 1) = f_y(1,1) = 4y\,|_{y=1} = 4
$$

And:

$$
\begin{aligned}
g'(1) &= \lim_{h\to 0}\frac{g(1+h) - g(1)}{h} \\
\frac{\partial f}{\partial y}(1,\, 1) &= \lim_{h\to 0}\frac{f(1,\, 1+h) - f(1,\, 1)}{h}
\end{aligned}
$$

8.3 Four dimensional

Example. Find $f_x,\ f_y,\ f_z$ if $f(x,\ y,\ z) = e^{xy}\ln(z)$.

Then, treat uninvolved parameters as constants:

$$f_x = ye^{xy}\ln(z)$$
$$f_y = xe^{xy}\ln(z)$$
$$f_z = \frac{e^{xy}}{z}$$

9 Higher derivatives

Let $f\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a function of two variables. Then f_x and f_y are given functions of two variables. So, we consider their partial derivatives:

$$(f_x)_x,\ (f_x)_y,\ (f_y)_x,\ (f_y)_y$$

We call them higher derivatives. These are just different uses of notation.

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

Beware of notations: they involve nesting.

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\partial x} = \frac{\partial^2 z}{\partial y\partial x}$$

Note that the y is placed on the left.

Example. Find $f_{xy} - 2f_{yx}$ if $f(x,\ y) = e^{xy}\sin(y)$.

Solution: $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

$$
\begin{aligned}
&= \frac{\partial}{\partial y}\left(ye^{xy}\sin(y)\right) \\
&= \left(e^{xy} + xe^{xy}\right)\sin(y) + \left(ye^{xy}\cos(y)\right)
\end{aligned}
$$

And $f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

$$
\begin{aligned}
&= \frac{\partial}{\partial x}\left(xe^{xy}\sin(y) + e^{xy}\cos(y)\right) \\
&= e^{xy}\sin(y) + xe^{xy}\sin(y) + ye^{xy}\cos(y)
\end{aligned}
$$

Theorem (Clairaut's theorem). Suppose f is defined on a disk D that contains the point (a, b). If f_{xy} and f_{yx} are both continuous, then:

$$f_{xy}(a, b) = f_{yx}(a, b)$$

To find partial derivatives for functions with potential discontinuities, like piece-wise functions, you must use the definition of the derivative.

Example. Let

$$
f(x, y) = \begin{cases} \frac{x^3y - xy^3}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}
$$

find $f_x(x, y)$ and $f_y(x, y)$.

For $(x, y) \neq 0$:

$$
\begin{aligned}
f_x &= \frac{\left(3x^2y - y^3\right)\left(x^2+y^2\right) - \left(x^3y - xy^3\right)\left(2x + y^2\right)}{\left(x^2+y^2\right)^2} \\
f_y &= \frac{\left(x^3 - 3xy^2\right)\left(x^2+y^2\right) - \left(x^3y - xy^3\right)\left(x^2 + 2y\right)}{\left(x^2+y^2\right)^2}
\end{aligned}
$$

If $(x, y) = 0$:

$$
\begin{aligned}
f_x(0,0) &= \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} \\
&= \lim_{h\to 0}\frac{f(h,0)}{h} = \lim_{h\to 0}\frac{0}{h} = 0 \\
f_y(0,0) &= 0
\end{aligned}
$$

9.1 For potential discontinuities

Say we have:

$$
f(x,y) = \begin{cases} \frac{x^3y-xy^3}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}
$$

For $(x,y)\neq(0,0)$:

$$
f_x(x,y) = \frac{\left(3x^2y-y^3\right)\left(x^2+y^2\right)-\left(x^3y-xy^3\right)(2x)}{\left(x^2+y^2\right)}
$$

For $(0,0)$:

$$
f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0
$$

This means:

$$
\begin{aligned}
&f_x(x,y) \\
&= \begin{cases} \frac{\left(3x^2y-y^3\right)\left(x^2+y^2\right)-\left(x^3y-xy^3\right)(2x)}{\left(x^2+y^2\right)} & (x,y)\neq(0,0) \\ 0 & (x,y)=0 \end{cases}
\end{aligned}
$$

By the same method, you can compute f_y. Consider two cases, $(x,y)=0$ and $(x,y)\neq 0$.

$$f_{xy}(0,\,0) = \lim_{h\to 0} \frac{f_x(0,\,h) - f_x(0,\,0)}{h}$$
$$= \lim_{h\to 0} \frac{\frac{(-h^3)}{(h^2)}}{h} = \lim_{h\to 0} -\frac{h}{h} = -1$$

10 Partial Differential Equations

Laplace's equation:

For $u : D \subseteq \mathbb{R}^2 \mapsto \mathbb{R}$, the Laplace's equation is:

$$u_{xx} + u_{yy} = 0$$

Solution of these equations are called harmonic equations. If u can satisfy this equation, then it is a solution of Laplace's equation and u is harmonic.

1D wave equation:

For $u : D \subseteq \mathbb{R}^2 \mapsto \mathbb{R}$, the 1D wave equation is:

$$u_{yy} = a^2 u_{xx}$$

Where a is a constant. If your function u can satisfy this equation, then it is a 1D wave equation.

Example. Let $f(x,\,y) = e^{-x}\cos(y) - e^{-y}\cos(x)$. Show that f is a solution of Laplace's equation. In other words: (note that $\frac{\partial f}{\partial x} = f_x$)

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Well:

$$\begin{aligned} f_x &= -e^{-x}\cos(y) + e^{-y}\sin(x) \\ f_{xx} &= e^{x}\cos(y) + e^{-y}\cos(x) \\ f_y &= -e^{-x}\sin(y) + e^{-y}\cos(x) \\ f_{yy} &= -e^{-x}\cos(y) - e^{-y}\sin(x) \end{aligned}$$

If we add them, do they cancel?

$$\begin{aligned} u_{xx} + u_{yy} &= e^{x}\cos(y) + e^{-y}\cos(x) - e^{-x}\cos(y) - e^{-y}\sin(x) \\ &= 0 \end{aligned}$$

Now we know. f is a solution of Laplace's equation.

Example. Verify that $u(x,t) = \sin(x-at) + \ln(x+at)$ where a is a constant is a solution of the wave equation.

$$\begin{aligned} u_x &= \cos(x-at) + \frac{1}{x+at} \\ u_{xx} &= -\sin(x-at) - \frac{1}{(x+at)^2} \\ u_t &= -\cos(x-at)\cdot a + \frac{a}{x+at} \\ u_{tt} &= -\sin(x-at)\cdot a - \frac{a^2}{(x+at)^2} \end{aligned}$$

Putting them together:

$$-\sin(x-at) - \frac{1}{(x+at)^2} = \left(-\sin(x-at)\cdot a - \frac{a^2}{(x+at)^2}\right)k$$

I probably messed up. Maybe not. What do I do.

11 Tangent Planes

1. Let S be a surface whose equation is $z = f(x, y)$

2. Let first partial derivatives of f be continuous.
3. Let $p\left(x_0,\, y_0,\, z_0\right)$ be a point on S.
4. Let C_1 be the curve obtained by intersecting the vertical plane $y = y_0$. Let T_1 be the tangent line to the curve C_1 at the point p.
5. Let C_2 be the curve obtained by the curve obtained by intersecting the vertical plane $x = x_0$. Let T_2 be the tangent line ot the curve C_2 at the point p. What is the slope of T_2?

Equation of a tangent plane

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x,\, y)$ at the point $p\left(x_0,\, y_0,\, z_0\right)$ is:

$$\begin{aligned} &z - z_0 \\ &= \overset{\text{normal}}{f_x\left(x_0,\, y_0\right)}\left(x - x_0\right) \\ &+ \overset{\text{normal}}{f_y\left(x_0,\, y_0\right)}\left(y - y_0\right) \end{aligned}$$

The gradient of a function is:

$$\begin{gathered} f : \mathbb{R}^2 \to \mathbb{R} \\ \nabla f(x,\, y) = \left(f_x(x,\, y),\, f_y(x,\, y)\right) \end{gathered}$$

∇f is always orthogonal to the surface.

If $f : \mathbb{R}^3 \to \mathbb{R}$, then $\nabla f(x,\, y,\, z) = \begin{bmatrix} f_x(x,\, y,\, z) \\ f_y(x,\, y,\, z) \\ f_z(x,\, y,\, z) \end{bmatrix}$

Example. Find the equation of the tangent plane for $z = y^2 e^x$ at the point $(0,\, 3,\, 9)$.

Solution: I can say that

$$
\begin{aligned}
z = f(x, y) &= y^2 e^x \\
f_x(x, y) = y^2 e^x &\Rightarrow f_x(0, 3) = 9 \\
f_y(x, y) = 2ye^x &\Rightarrow f_y(0, 3) = 6
\end{aligned}
$$

We have

$$
\begin{aligned}
z - z_0 &= f_x(0, 3)\,(x - x_0) + f_y(0, 3)\,(y - y_0) \\
\Rightarrow z - 9 &= 9x + 6(y - 3) \\
9 &= 9x + 6y - z
\end{aligned}
$$

You may also wish to consider:

$$
\begin{aligned}
z &= y^2 e^x \\
z - y^2 e^x &= 0 \quad z = F(x, y, z) \\
\nabla F(0, 3, 9) &= \begin{bmatrix} f_x(0, 3, 9) \\ f_y(0, 3, 9) \\ f_z(0, 3, 9) \end{bmatrix} \\
F_x(x, y) &= -y^2 e^x \\
F_y(x, y) &= -2ye^x \\
F_z(x, y) &= 1 \\
\vec{N} &= \begin{bmatrix} -9 \\ -6 \\ 1 \end{bmatrix} \quad p_0 = (0, 3, 9)
\end{aligned}
$$

$$
\begin{bmatrix} -9 \\ -6 \\ 1 \end{bmatrix} \begin{bmatrix} x \\ y - 3 \\ z - 9 \end{bmatrix} = 0
$$

Example. Determine the equation of the tangent plane to the surface $z = x^3 - 3xy + y^3$ at point $(1,\ 2,\ 3)$. If $z = f(x,\ y)$, then

$$f_x = 3x^2 - 3y$$
$$f_y = 3x + 3y^2$$

$$z - z_0 = f_x(x_0,\ y_0)(x - x_0) + f_y(x_0,\ y_0)(y - y_0)$$
$$\ldots \Rightarrow 3x - 9y + z = -12$$

12 Linear Approximations

A linear function $f\colon \mathbb{R} \to \mathbb{R}$ can be represented as $f(x) = ax + b$, where a and b are constants. It looks like a map when graphed.

Perhaps you have a function $f\colon \mathbb{R} \to \mathbb{R}$ and the graph looks like some curve that isn't a line.

Say we have a point $(a,\ f(a))$. If the function is differentiable at a, then consider the tangent line at $(a,\ f(x))$, which we call $L(x)$. What is the formula for this tangent line? It is:

$$y - f(a) = f^{'}(a)(x - a)$$

If you consider some point b that is close to a, then:

- A point on the function is $(b,\ f(b))$
- Consider $L(b)$: $f(b)$ is close to $L(b)$. $L(b)$ is linear, so I can compute it easily.

If you consider a point c that is far from a, we can still approximate $f(c)$ with $L(c)$, but it may not be a good approximation. But if it is close, you can compute it.

The problem is, how can we extend what we said already if we have a function from $\mathbb{R}^2 \to \mathbb{R}$? What does linear mean in 3D space>

$f : \mathbb{R}^2 \to \mathbb{R}$ is a linear function if we can write f as the following:

$$f(x,\ y) = ax + by + c$$

Where $a,\ b,\ c$ are constants. This is the standard equation for a plane. If I have a surface that I don't know the formula for, but I want to consider the tangent plane at some point on the surface, if the visual gap is small then I have made a good approximation.

The linear function whose graph is the tangent plane is:

$$L(x,\ y) = f_x(a,\ b)(x-a) + f_y(a,\ b)(y-b) + f(a,\ b)$$

This is called the linearization of f at $(a,\ b)$.

The linear approximation of f at $(a,\ b)$ is:

$$f(x,y) \approx L(x,y) = f_x(a,\ b)(x-a) + f_y(a,\ b)(y-b) + f(a,\ b)$$

Example. Let $f(x,\ y) = xe^{xy}$. Find the linearization of f at $(1,\ 0)$. Approximate $f(1.1,\ -0.1)$

Solution: Firstly,

$$f_x(x,\ y) = e^{xy} + xe^{xy}y$$
$$f_y(x,\ y) = x^2e^{xy}$$

$$\begin{aligned} L(x,\ y) &= f_x(1,\ 0)(x-1) + f_y(1,\ 0)(y-0) + f(1,\ 0) \\ &= (1+0)(x-1) + (1)(y-0) + f(1,\ 0) \\ &= x-1+y+1 \\ &= x+y \end{aligned}$$

To approximate $f(1.1,\ -0.1)$, just plug it in the linear approximation: $1.1-0.1 = 1$. There we go.

Comparisons: $f(1.1,\ -0.1) = (1.1)e^{(1.1)y} = 0.985$. It's close to 1.

If $f : \mathbb{R}^2 \to \mathbb{R}$ does not have continuous first partal derivatives, the linearization for the function f fails.

12.1 The Nabla Operator

If you have an $\mathbb{R}^n$ cartesian system $(x_1, x_2, \ldots, x_n)$, we define nabla (∇) as:

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix}$$

This means, in $\mathbb{R}^2$:

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$$

In $\mathbb{R}^3$:

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

If we have $f : \mathbb{R}^2 \to \mathbb{R}$, we can define

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

If we have $f : \mathbb{R}^3 \to \mathbb{R}$, we can define:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

We call this the gradient of f.

12.2 Upcoming

If we have $f : \mathbb{R}^n \to \mathbb{R}$, we may have:

$$div(f) = \nabla \cdot f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$

We also have $curl$

For Laplace's equation, this satisfies it: $\nabla^2 f = 0$.

13 Multivariable derivatives

In f: $\mathbb{R} \to \mathbb{R}$, $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$,

$$\Leftrightarrow \lim_{h \to 0} \frac{f(x+h) - f(x) - f'(x)h}{h} = 0$$

If the limit exists. Now, we can say that f' is a 1-by-1 matrix (a number). We can also say that it is some sort of operator:

$$A(h) := f'(x) \cdot h$$

$A(h)$ is a linear operator (a linear transformation). A **linear** transformation is, if you have a mapping $T : \mathbb{R}^n \to \mathbb{R}^m$, $T(\vec{x} + \lambda \vec{y}) = T(\vec{x}) + \lambda T(\vec{y})$. This means linear transformations always fix the origin.

If we want to extend this notion for functions $f:\ \mathbb{R}^2 \to \mathbb{R}$, what should be done? How should I define the notion of derivative?

$$\lim_{h \to (0,\,0)} \frac{\left| f\left(\vec{x}+\vec{h}\right) - f\left(\vec{x}\right) - A\left(\vec{h}\right) \right|}{\left|\vec{h}\right|}$$

Then, A is a 1×2 matrix (column vector) and is a transformation from $\mathbb{R}^2 \to \mathbb{R}$. **If** I can find A:

Then:

$$f^{'}(x) = A$$

If $f_x,\ f_y$ are continuous, then

$$A = \begin{bmatrix} f_x & f_y \end{bmatrix} = \nabla f$$

If we have $\mathbb{R}^3$, the same formula applies, and if $f_x,\ f_y,\ f_z$ are continuous, then $A = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix}$

In a much more general setting (which we won't focus on), if you have a function $\mathbb{R}^m \to \mathbb{R}^n$, then A is an $n \times m$ matrix and is composed of a matrix I would like to avoid seeing in this course.

To show that something is differentiable at a point, I just need to show that $\langle f_x,\ f_y \rangle$ is continuous.

Definition (Differentiability at a point). Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function and let $x \in E$. If there exists a 1×2 matrix A such that

$$\lim_{h \to 0} \frac{|f(x+h) - f(x) - A(h)|}{|h|} = 0 \quad h = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \in \mathbb{R}^2$$

$$\lim_{h\to 0}\frac{|f(x+h)-f(x)-A(h)|}{|h|}=0 \quad h=\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}\in\mathbb{R}^2$$

then we say that f is differentiable at x and we can write $f^{'}(x)=A$, which is a 1×2 matrix.

Theorem (I can tell that it is differentiable there). Let $f:\mathbb{R}^2\to\mathbb{R}$ be a function. If f_x and f_y exist near the point $(a,\, b)$, and f_x, f_y are continuous at $(a,\, b)$, then f is differentiable at $(a,\, b)$ and $f^{'}(a,\, b)=\langle f_x(a,\, b),\, f_y(a,\, b)\rangle$ (column vector).

Let me say this again:

- If $f_x,\ f_y$ exist near $(a,\, b)$
- And $f_x,\ f_y$ are continuous at $(a,\, b)$
- Then f is differentiable at $(a,\, b)$ and $f^{'}(a,\, b)=\begin{bmatrix} f_x(a,\, b) & f_y(a{,}b)\end{bmatrix}$.

The **converse** of this theorem is **NOT TRUE.** Simple counterexample:

$$f(x,\, y)=\begin{cases}(x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & (x,\, y)\neq(0,\, 0)\\ 0 & (x,\, y)=(0,\, 0)\end{cases}$$

We can see that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ is not continuous.

Compared with $\nabla f(a,\, b)$, it means that f is in the class of C_1.

Definition (Smoothness). Let $n,\, k\in\mathbb{Z}_{>0}$. Let $E\subseteq\mathbb{R}^n$ be an open set and let $f:\mathbb{R}^n\to\mathbb{R}$ be a function. We say that f is of the class C^k on E if all kth partial derivatives if f is continuous on E. We cay that f is of the class C^∞ on E if all partial derivatives of all orders of f are continuous on E.

Example. Let $f(x,\, y)=1+x\ln(xy-5)$.

1. Is f differentiable at the point $(2,\, 3)$?
2. What is the linear approximation of f at $(2,\, 3)$?

3. Find an appropriate approximate for $f(2.1, 3.1)$

Solution: Checking differentiability:

$$f_x = \ln(xy-5) + x \cdot \frac{y}{xy-5}$$
$$f_y = x \cdot \frac{x}{xy-5}$$

Continuous at the point $(2, 3)$? It is – substitute!! Then f is differentiable at $(2, 3)$ and it is $\langle f_x(2, 3), f_y(2, 3)\rangle = \langle 6, 4\rangle$

14 Differential

Motivation: Let $y = f(x)$ be a function.

For $f: \mathbb{R} \to \mathbb{R}$, $f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. Then $dy = f'(x)dx$. If you change x by a bit, then can you quantify the value of y, which may have also changed?

1. Change of f as x changes form a to $a + \Delta x$: $\Delta f = f(a + \Delta x) - f(a)$.

If you have a function from $\mathbb{R}^2 \to \mathbb{R}$, the definition may have changed by a bit:

$$\begin{aligned} z &= f(x, y) \\ \Rightarrow dz &= \nabla f(x, y) \cdot \langle dx, dy\rangle \\ &= f_x dx + f_y dy \end{aligned}$$

Example. Let $z = f(x, y) = x^2 - 3xy - y^2$. Find the total differential of f. If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

Then: $f_x = 2x + 3y,\ f_y = 3x - 2y,\ df = (2x + 3y)dx + (3x - 2y)dy.$

Then:

$$\begin{aligned}(x,\, y) = (2, 3),\, dx = 0.5,\, dy = -0.4 \\ dz = f_x(2,\, 3) \cdot 0.5 + f_y(2, 3) \cdot (-0.4) \\ = \frac{13}{2} + 13 \cdot (-0.4)\end{aligned}$$

In other words:

$$\Delta z = f(2.05,\, 2.96) - f(2,\, 3) = TBA$$

15 Chain Rule

If $g,\, f\colon \mathbb{R} \to \mathbb{R}$

Then, $f\left(g(t)\right)' = f'\left(g(t)\right) g'(t)$

$$\begin{aligned}y = f(x) = f\left(g(t)\right),\, x = g(t) \\ \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}\end{aligned}$$

Now suppose you have a function $\mathbb{R} \xrightarrow{g} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$ and $f(x,\, y) = f\left(x(t),\, y(t)\right)$. $g(t) = (x(t),\, y(t))$

$$t \mapsto \left(x(t),\, y(t)\right),\, (x,\, y) \mapsto f(x,\, y)$$

Then:

$$f \circ g(t) = f\left(x(t),\, y(t)\right)$$

Now, I want to compute $(f \circ g)'(t)$. Then:

$$f^{'}(g(t))\,g^{'}(t) = f^{'}(g(t))\,g^{'}(t) = \begin{bmatrix} f_x(g(t)) & f_y(g(t)) \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \end{bmatrix}$$

$$= f_x g(t) \cdot \frac{dx}{dt} + f_y g(t) \cdot \frac{dy}{dt}$$

If I have $z = f(x, y)$, then:

z is a function of x and y; $x = x(t)$ and $y = y(t)$. x, y are the intermediate variables and t is the independent variable (t does not depend on anything).

If I want to define $\frac{dz}{dt}$:

- $z \to \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$
- Each leading to $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

Then:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Because $z = f(x, y)$, then $\frac{\partial z}{\partial x} = f_x$. If you change t, both x and y will be changed. Because z is the result of a function of x and y, z will be changed. This means you'll have to consider the changes in both x and y should t change by just a bit.

Example. Let $z = x^2y^2 + 3x^2y^4$ and let $x = \sin(2t)$ and $y = \cos(2t)$. Find $\frac{dz}{dt}$.

We note that z is a function of x, y and x, y are functions of t. All you need to know to compute the partial derivatives is to draw this diagram:

Tree display:

- z
  - $\frac{\partial z}{dx} \to x$
    - $\frac{dx}{dt} \to t$
  - $\frac{\partial z}{dy} \to y$
    - $\frac{dy}{dt} \to t$

So:

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}\\ &= \left(2xy^2+6xy^4\right)(2\cos(2t))+\left(2x^2y+4x^2y^3\right)(-2\sin(2t))\end{aligned}$$

$$\begin{aligned}&\left(2\sin(2t)\cos^2(2t)+6\sin(2t)\cos^4(2t)\right)(2\cos(2t))\\ &+\left(2\sin^2(2t)\cos(t)+4\sin^2(2t)\cos^3(2t)\right)(-2\sin(2t))\end{aligned}$$

15.1 Chain Rule Case 2

Suppose that $z = f(x, y)$ is differentiable, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s, t. Then:

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}\\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}\end{aligned}$$

The tree diagram goes like the following:

Tree display:

- z
  - $\frac{\partial z}{\partial x} \to x$
    - $\frac{\partial x}{\partial s} \to s$
    - $\frac{\partial x}{\partial t} \to t$
  - $\frac{\partial z}{\partial y} \to y$
    - $\frac{\partial y}{\partial s} \to s$
    - $\frac{\partial y}{\partial t} \to t$

So, I want to compute $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$ and $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$.

If not writing dot products hurt my hand too much, $\frac{\partial z}{\partial s} = \begin{bmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial s} \end{bmatrix}$.

Example. Let $z = x^2 + 2y^2$ and let $x = \frac{r}{s}$ and $y = r^2 + \ln(s)$. Find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial s}$. Note that

$$\begin{aligned} \frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r} \\ &= \left(2\frac{r}{s}\right)\left(\frac{1}{s}\right) + \left(4(r^2 + \ln(s)\right)\left(2r^2\right) \end{aligned}$$

And

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} \\ &= (2x)\left(-\frac{r}{s^2}\right) + (4y)\left(\frac{1}{s}\right) \end{aligned}$$

Example. Suppose that $z = f(x, y)$ has continuous second-order partial derivatives and $x = r^2 + s^2$ and $y = 2rs$. Find $\frac{dz}{dr}$ and $\frac{d^2z}{dr^2}$. It is:

$$\begin{aligned} \frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x} \cdot 2r + \frac{\partial z}{\partial y} \cdot 2s \\ \frac{\partial^2 z}{\partial r^2} &= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x} \cdot 2r + \frac{\partial z}{\partial y} \cdot 2s\right) = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x} \cdot 2r\right) + \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y} \cdot 2s\right) \\ &= 2 \cdot \frac{\partial z}{\partial x} + 2r \cdot \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + 0 + 2s \cdot \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right) \\ &= 2 \cdot \frac{\partial z}{\partial x} + 2r \cdot \left(2r \cdot \frac{\partial^2 z}{\partial r^2} + 2s \cdot \frac{\partial^2 z}{\partial y \partial x}\right) + \\ &2s \cdot \left(\frac{\partial^2 z}{\partial x \partial y}\left(\frac{\partial x}{\partial r}\right) \cdot \frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y^2} \cdot \frac{\partial y}{\partial r}\right) \\ &= 2 \cdot \frac{\partial z}{\partial x} + 4r^2 \cdot \frac{\partial^2 z}{\partial x^2} + 8rs \cdot \frac{\partial^2 z}{\partial x \partial y} + 4s^2 \cdot \partial^2 z/\partial y^2 \end{aligned}$$

We know that

$$\begin{aligned}\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) &= \frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial x}{\partial r}+\frac{\partial^2 z}{\partial y\partial x}\cdot\frac{\partial y}{\partial r}\\ &= 2r\cdot\frac{\partial^2 z}{\partial r^2}+2s\cdot\frac{\partial^2 z}{\partial y\partial x}\end{aligned}$$

And

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial x\partial y}\left(\frac{\partial x}{\partial r}\right)\cdot\frac{\partial x}{\partial r}+\frac{\partial^2 z}{\partial y^2}\cdot\frac{\partial y}{\partial r}$$

Based on this tree diagram:

Tree display:

- $\frac{\partial z}{\partial x}$
  - $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\to x$
    - $\frac{\partial x}{\partial r}\to r$
    - $\frac{\partial x}{\partial s}\to s$
  - $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$
    - $\frac{\partial y}{\partial r}\to r$
    - $\frac{\partial y}{\partial s}\to s$

Example. The radius of a right circular clone is increasing at a rate of 1.8 while height decreases at rate of 2.5. What are is the volume of the cone changing when radius is 120 and height is 140?

$$\begin{aligned}v(r,\,h) &= \frac{1}{3}(r\cdot h)\\ \frac{dv}{dt} &= \frac{\partial v}{\partial r}\cdot\frac{dr}{rt}+\frac{\partial v}{\partial h}\cdot\frac{dh}{dt}\\ \frac{dv}{dt} &= \frac{1}{3}h\cdot 1.8-\frac{1}{3}r\cdot 2.5\end{aligned}$$

Plug in $h = 140$ and $r = 120$ to find the answer. (Note that if I get a negative answer, the question is illegal)

Suppose we have

$$\begin{aligned} &f(x, y) \\ &= x^2 e^{\sin(4x)} + \left(x^2 + y^2\right) e^{\cos(5x)} \\ &+ \sin(10x) e^{(\sin(x) + \cos(y))^2} \end{aligned}$$

Find

$$(f_{xy} + f_{yx})^2 - 4 f_{xy}$$

f is continuous everywhere, so Clairaut's theorem works. So, the answer is 0.

16 Implicit Function Theorem

Theorem. Suppose that $F(x, y) = 0$ is differentiable and the equation $F(x, y) = 0$ defines y as a differentiable function of x. Then, at any point where $F_y \neq 0$:

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Proof. $0 = F(x, y) \overset{\text{differentiate both sides}}{\Rightarrow} 0 = F_x\left(\frac{dx}{dx}\right) + F_y\left(\frac{dy}{dx}\right)$

$$\begin{aligned} &\Rightarrow 0 = F_x + F_y\left(\frac{dy}{dx}\right) \\ &\overset{F_y \neq 0}{\Rightarrow} \frac{dy}{dx} = -\frac{F_x}{F_y} \ \blacksquare \end{aligned}$$

For functions of three variables:

1. Suppose that $z = f(x, y)$.

2. z can implicitly be given as $F(x, y, z) = 0$. This means $F(x, y, f(x, y)) = 0$. Note that F is a different function than f.
3. Assume that F and f are differentiable.
4. Using the chain rule for $F(x, y, 0)$, we get:

$$
\begin{aligned}
F(x, y, z) = 0 &\overset{\text{diff w/ x}}{\Rightarrow} 0 = F_x\left(\frac{dx}{dx}\right) + F_z\left(\frac{\partial z}{\partial x}\right) \\
&\overset{\text{if } F_z \neq 0}{\Rightarrow} \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}
\end{aligned}
$$

Example. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(0, 0, 0)$ if $x^3 + z^2 + ye^{xz} + z\cos(y) = 0$.

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= -\frac{F_x}{F_z} = -\frac{3x^2 + zye^{xz}}{2z + xye^{xz} + \cos(y)} \\
\frac{\partial z}{\partial y} &= -\frac{F_y}{F_z} = \frac{e^{xz}}{2z + xye^{xz} + \cos(y)}
\end{aligned}
$$

To calculate it without the implicit function theorem:

$$
\begin{aligned}
0 &= \frac{\partial}{\partial x}\left(x^3 + z^2 + ye^{xz} + z\cos(y)\right) \\
&= 3x^2 + 0 + \frac{\partial}{\partial x}\left(ye^{xz}\right) + 0 \\
\frac{\partial}{\partial x}\left(e^{xz}\right) &= ze^{zx} + x\frac{\partial z}{\partial x}e^{xz} \\
&\Rightarrow \\
&= 3x^2 + yze^{zx} + \frac{\partial z}{\partial x}\left(2z + yxe^{xz} + \cos(y)\right) = 0 \\
\Rightarrow \frac{\partial z}{\partial x} &= -\frac{yze^{xz} + 3x^2}{2z + yxe^{xz} + \cos(y)} = -\frac{F_x}{F_z}
\end{aligned}
$$

16.1 Implicit function theorem with a single variable function

Suppose we have a function $f(x) = x^2$. Then, $f'(x) = 2x = \frac{dy}{dx}$. We can see that x is the independent variable, and y depends on x. Yet, this implies $\underbrace{y - x^2}_{F(x,\, y)} = 0$. Now, consider $F(x,\, y) = y - x^2 = 0$. In this context for capital F, x and y are both independent variables. Applying the **implicit function theorem,** we get:

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-2x}{1} = 2x$$

17 The Gradient Vector

Definition. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. The gradient vector at a point $(x_0,\, y_0)$ is the vector:

$$\nabla f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0,\, y_0)\,\widehat{i} + \frac{\partial f}{\partial y}(x_0,\, y_0)\,\widehat{j}$$

$$\nabla f(x_0,\, y_0,\, z_0) = \frac{\partial f}{\partial x}(x_0,\, y_0,\, z_0)\,\widehat{i} + \frac{\partial f}{\partial y}(x_0,\, y_0,\, z_0)\,\widehat{j} + \frac{\partial f}{\partial z}(x_0,\, y_0, z_0)\,\widehat{j}$$

$$\nabla f(x,\, y,\, z,\, w) = \begin{bmatrix} f_x \\ f_y \\ f_z \\ f_w \end{bmatrix}$$

17.1 Algebraic rules for gradients

It's linear, and most properties from derivatives cover over. Instead of prime, we use ∇.

1. $\nabla(f \pm g) = \nabla f \pm \nabla g$

2. $\nabla(\lambda f) = \lambda \nabla f$
3. Product rule is $\nabla(fg) = \nabla f \cdot g + f \cdot \nabla g$
4. Quotient rule also applies

17.2 The direction of the Gradient, and Why It's Normal

Theorem. Let $k \in \mathbb{R}$ be a constant. Let S be a surface with an equation $f(x, y, z) = k$, and that f is a differentiable function. Let $p(x_0, y_0, z_0)$ be a point on S. Let C be any curve that lies on S and passes p (the path of C must crawl on S). Let $\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$ be a continuous vector function describing C. Then:

$$\underbrace{\nabla F(x_0, y_0, z_0)}_{\text{normal}} \cdot \vec{r}'(t) = 0$$

Recall that $\vec{r}'(t)$ is always tangent to the surface S that it is on, meaning it is always perpendicular to ∇F. Also, this theorem holds regardless of k, as ∇F is not impacted by k.

∇F is orthogonal to the tangent plane. We can consider the gradient as the normal vector of the plane, at the point.

Proof. $\vec{r}'(t) = \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix}$. Do a bit of substituting:

$$f(x, y, z) = f(x(t), y(t), z(t)) = k$$

$$\begin{aligned}
0 &= \frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}+\frac{\partial f}{\partial z}\cdot\frac{dz}{dt} \\
0 &= \left\langle \frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right\rangle\cdot\left\langle\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right\rangle \\
&= \nabla f\left(x_0,\, y_0,\, z_0\right)\cdot \mathbf{r}^{'}(t) \blacksquare
\end{aligned}$$

Remark. Let $\nabla F\left(x_0,\, y_0,\, z_0\right) \neq 0$. Then, ∇F at the point $p\left(x_0,\, y_0,\, z_0\right)$ is orthogonal to the tangent plane to the level surface $F(x,\, y,\, z) = k$ at $p\left(x_0,\, y_0,\, z_0\right)$.

$$\begin{aligned}
&\frac{\partial F}{\partial x}\left(x_0,\, y_0,\, z_0\right)\left(x-x_0\right)+\frac{\partial F}{\partial y}\left(x_0,\, y_0, z_0\right)\left(y-y_0\right)+\frac{\partial F}{\partial z}\left(x_0,\, y_0,\, z_0\right)\left(z-z_0\right) \\
&= 0
\end{aligned}$$

Definition (Level surface). $\{(x,\, y,\, z)\ : f(x,\, y,\, z) = c\}$, for $c \in \mathbb{R}$.

Compare it with the case that $z = f(x,\, y)$. Rearrange things. What do we have?

$$\begin{aligned}
F(x,\, y,\, z) &= z - f(x,\, y) = 0 \\
\Rightarrow \nabla F &= \left(-f_x,\, -f_y,\, 1\right)
\end{aligned}$$

Note: $\frac{\partial}{\partial x}(z - f(x,\, y)) = \frac{\partial}{\partial x}(-f(x,\, y)) = f_x(x,\, y)$. That should clarify things.

Definition. The normal line to S at P is the line passing through P and orthogonal to the tangent plane at P.

Remark. The direction of the normal line is given by the gradient $\nabla F\left(x_0,\, y_0,\, z_0\right)$.

$$\begin{aligned}
&\frac{x-x_0}{F_x\left(x_0,\, y_0,\, z_0\right)} \\
&= \frac{y-y_0}{F_y\left(x_0,\, y_0,\, z_0\right)} \\
&= \frac{z-z_0}{F_z\left(x_0,\, y_0,\, z_0\right)}
\end{aligned}$$

where $F_{x\ldots y\ldots z}\left(x_0,\, y_0,\, z_0\right) \neq 0$.

This comes from this form of the equation of a line:

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

Where $\begin{aligned} a &= F_x(x_0, y_0, z_0), b \\ &= F_y(x_0, y_0, z_0), c \\ &= F_z(x_0, y_0, z_0) \end{aligned}$

Parametric is:

$$\vec{r}(t)=\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}+\begin{bmatrix} F_x(x_0, y_0, z_0) \\ F_y(x_0, y_0, z_0) \\ F_z(x_0, y_0, z_0) \end{bmatrix} t$$

17.3 The VERY Long Example

Let $f(x, y) = x^2 + y^2$ and $F(x, y, z) = x^2 + y^2 - z$.

1. What is the difference between f and F?
   a. f is $\mathbb{R}^2 \to \mathbb{R}$ and F is $\mathbb{R}^3 \to \mathbb{R}$.
2. Find $\nabla f(x, y)$ and $\nabla F(x, y, z)$. Difference?
   a. $\nabla f(x, y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$ and $\nabla F(x, y, z) = \begin{bmatrix} 2x \\ 2y \\ -1 \end{bmatrix}$.
   b. Note that $\nabla F(x, y, z) = \begin{bmatrix} \frac{d\mathbf{F}}{dx}(x, y, \mathbf{z}) \\ \frac{d\mathbf{F}}{dy}(x, y, \mathbf{z}) \\ \frac{\mathbf{dF}}{\mathbf{dz}}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \end{bmatrix}$ (the boldface is what is separate from $\nabla f(x, y)$).
3. Find $\nabla f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $\nabla F\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1\right)$.

a. $\nabla f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$ and $\nabla F\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1\right) = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ -\mathbf{1} \end{bmatrix}$

4. Find the level **curve** $f(x, y) = 1$ and the graph of f.

   a. The **level curve** is $\left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\right\}$. Hence, it's a 1-radius circle centered at the origin. If we imagine $z = f(x, y)$, the level curve is that function projected at $z = f(x, y) = 1$.

   b. The **graph** of f: form a contour map or attempt to sketch the 3D surface.

      i. $z = 1 \Rightarrow x^2 + y^2 = 1$

      ii. $z = 2 \Rightarrow x^2 + y^2 = 2$ (Radius $\sqrt{2}$)

      iii. The surface forms a paraboloid.

5. Find the level surface of $F(x, y, z) = 0$ and plot the graph.

   a. What is the level surface? **The same as the level curve of** $f(x, y)$.

   b. $F(x, y, z) = 0 \Leftrightarrow z = x^2 + y^2$. The level surface is $\left\{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\right\}$, where I could replace $z = x^2 + y^2$ with $F(x, y, z) = 0$ and the level surface will still be the same.

6. Is there any relationship between the graph of f and the level surface $F(x, y, z) = 0$?

   a. The **level surface** of F and the **graph** of f look the same.

7. Is $\nabla f(x, y)$ orthogonal to any level **curve** $f(x, y) = c$ where c is constant?

   a. $\nabla f(x, y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$. Plot this as a **vector field**, and they all point away from any level curve $f(x, y) = c$ (if it forms a circle, and if it does it would be centered at the origin).

b. Or I could use the theorem $\nabla f(x,y)\cdot\vec{r}'(t)=0$. I could use the chain rule. If I have a parametric equation $\begin{matrix} x=x(t) \\ y=y(t) \end{matrix}$, then $f(x(t),y(t))=1$ implies $\frac{\partial f}{\partial t}=\frac{\partial f}{\partial x}\cdot x'(t)+\frac{\partial f}{\partial y}y'(t)=\begin{bmatrix}\frac{\partial f}{\partial x}\\ \frac{\partial f}{\partial y}\end{bmatrix}\cdot\begin{bmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\end{bmatrix}=\nabla f(x,y)\cdot\vec{r}'(t)\Rightarrow\nabla f(x,y)$ is orthogonal to any level curve.

8. Is $\nabla F(x,y,z)$ orthogonal to the level **surface** of $F(x,y,z)=1$?

   a. The level surface is $\left\{(x,y,z)\in\mathbb{R}^3 : F(x,y,z)=1\right\}$. Which is, $x^2+y^2-z=1$. I mean, **this surface is the same as the graph of** $f(x,y)$, so $\nabla F(x,y,z)$ anywhere should be orthogonal to the surface.

9. Find the tangent plane to the graph of f at the point $\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},1\right)$.

   a. $z-1=\sqrt{2}\left(x-\frac{\sqrt{2}}{2}\right)+\sqrt{2}\left(y-\frac{\sqrt{2}}{2}\right)$

   b. By the way, when we set $G(x,y,z)=x^2+y^2-z=0$, then $\nabla G\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},1\right)=\begin{bmatrix}\sqrt{2}\\ \sqrt{2}\\ -1\end{bmatrix}=\vec{N}$. Which is a normal vector to the tangent plane at $\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},1\right)$.

   So, the equation of the tangent plane is $\begin{bmatrix}\sqrt{2}\\ \sqrt{2}\\ -1\end{bmatrix}\cdot\begin{bmatrix}x-\frac{\sqrt{2}}{2}\\ y-\frac{\sqrt{2}}{2}\\ 1\end{bmatrix}=0$.

10. Find the points (a,b,c) in which the tangent plane to the graph of f is orthogonal to the xz-plane (which is $y=0$).

    a. Two planes are orthogonal if and only if their normal vectors are perpendicular.

b. A normal vector for the xz plane might be $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

c. The normal vector for the tangent plane at any point is $\nabla G(x, y, z) = \begin{bmatrix} 2x \\ 2y \\ -1 \end{bmatrix}$.

Now, all we need to do is figure out what values of x, y, z make $\begin{bmatrix} 2x \\ 2y \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$. Which is $2y = 0 \Rightarrow y = 0$. This means the answer is $\{(x, y, z) \in \mathbb{R}^3 : y = 0\}$.

11. Instead of orthogonal, where we use parallel, then use proportions – one is a scalar multiple of the other, a.k.a. $\vec{N}_1 = \lambda \vec{N}_2$.

18 Directional Derivative

A derivative in a certain direction.

Definition. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function and let $\alpha = (\alpha_1, \alpha_2)$, $x = (x_1, x_2) \in \mathbb{R}^2$. Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ be a unit vector. Then, the directional derivative of f at $x = (x_1, x_2)$ in the direction of $\vec{u}$ is denoted by $D_{\vec{u}} f(x)$, is:

$$\begin{aligned} D_{\vec{u}} f(x) &= \lim_{t \to 0} \frac{f((x_1, x_2) + t(u_1, u_2)) - f(x_1, x_2)}{t} \\ &= \lim_{t \to 0} \frac{f(x_1 + tu_1, x_2 + tu_2) - f(x_1, x_2)}{t} \end{aligned}$$

Assume $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then:

$$\lim_{t\to 0} \frac{f(x_1+t, x_2) - f(x_1, x_2)}{t}$$

Assume $u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then:

$$\lim_{t\to 0} \frac{f(x_1, x_2+t) - f(x_1, x_2)}{t}$$

So, partial derivatives are a special case of the directional derivative. $\vec{u}$ tells us the direction.

Definition. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and let $a = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Let $\vec{u}$ be a unit vector in $\mathbb{R}^n$. The directional derivatives of f at $\vec{x}$ in the direction of $u\ldots$ (TBA)

18.1 Interpretation of directional derivatives

$\vec{u}$ is always a unit vector. If I slice a surface by a plane (a contains subspace of $\mathbb{R}^3$), the plane **should** include $\vec{u}$, the tangent line to the curve is called the directional derivative in direction $\vec{u}$.

Example. Find the derivative of $f(x, y) = xy$ at $p_0(1, 2)$ in the direction of the vector of the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

To solve this, you should always normalize the direction vector:

$$\frac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\sqrt{2}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

And now, we can compute the directional derivative:

$$\begin{aligned}
&\lim_{t\to 0} \frac{f\left(1+t\frac{1}{\sqrt{2}},\, 2+t\frac{1}{\sqrt{2}}\right) - f(1,2)}{t} \\
&= \lim_{t\to 0} \frac{\left(\left(1+t\frac{1}{\sqrt{2}}\right)\left(2+t\frac{1}{\sqrt{2}}\right)\right) - 1\cdot 2}{t} \\
&= \frac{\left(2+3\frac{t}{\sqrt{2}}+t^2\cdot\frac{1}{2}\right) - 2}{t} \\
&= \frac{t\left(\frac{3}{\sqrt{2}}+t\frac{1}{2}\right)}{t} = \frac{3}{\sqrt{2}}
\end{aligned}$$

Fortunately, we have a **theorem.**

Theorem. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a **differentiable** function. Let $\vec{u} \in \mathbb{R}^n$ be a unit vector and let $p \in \mathbb{R}^n$. Then:

$$D_{\vec{u}} f(p) = \nabla f(p) \cdot \vec{u}$$

This theorem does not work if f isn't differentiable.

Example. Find the derivative of $f(x,\, y) = xy$ at $p_0(1,\, 2)$ in the direction of the vector of the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, using the above theorem.

Solution: $f(x,\, y) = xy$ is differentiable everywhere as $\frac{\partial f(x,\, y)}{\partial y}$ and $\frac{\partial f(x,\, y)}{\partial x}$ exist and are continuous everywhere. So, where $\vec{u} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$:

$$\begin{aligned}
D_{\vec{u}} f(1,\, 2) &= \begin{bmatrix} y \\ x \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{y}{\sqrt{2}} + \frac{x}{\sqrt{2}} = \frac{\sqrt{2}}{2}(x+y) \\
&= \frac{\sqrt{2}}{2}(1+2) = \frac{\sqrt{2}}{2} \cdot 3 = \frac{3}{\sqrt{2}}
\end{aligned}$$

But why may we only use this theorem if the function is differentiable? There's a counterexample.

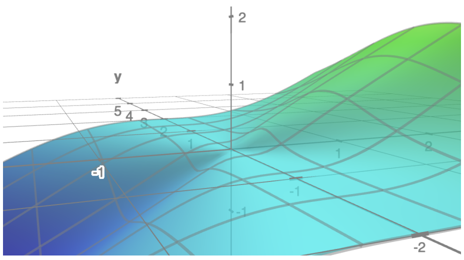
Proposition. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. If all directional derivatives of f exist at (a, b) then f is differentiable at (a, b). This statement is false.

Here's a counterexample: Say we have $f(x) = |x|$. It is **not** differentiable at 0, because there could be two tangent lines there. In $\mathbb{R}^3$, we would say there could be two or more possible tangent planes.

Example. Show that all the directional derivatives at $(0, 0)$ exist. Show that f is not differentiable at $(0, 0)$.

$$f(x, y) = \begin{cases} \frac{x^3}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Graph it out. Look at the origin. I can't look for a distinct tangent plane I could give it.



Behaves weirdly near the origin.

$$D_{\vec{u}} f(0, 0) = u_1^3$$

Now, suppose that f is differentiable at $(0, 0)$, we can apply the theorem (somehow, we chose a random directional derivative):

$$D_{\vec{u}} f(0, 0) = \nabla f(0, 0) \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$2 = 1$$

Contradiction. With the assumption that f is differentiable at $(0,\ 0)$.

Example. $f(x,\ y) = xe^y + \cos(xy)$. Find the directional derivative of f at the point $(2,\ 0)$ in the direction of $v = 3i - 4j$.

This function is continuous as it is composed of continuous functions. $\frac{\vec{v}}{|\vec{v}|} = \frac{\langle 3,\ -4\rangle}{\sqrt{3^2+4^2}} = \frac{\langle 3,\ -4\rangle}{\sqrt{25}} = \left\langle \frac{3}{5},\ -\frac{4}{5}\right\rangle$.

18.2 Finding the Direction of Maximum Change

Theorem. If we have a function that is **differentiable**, the maximum value of the directional derivative $D_{\vec{u}}f(x)$ is $|\nabla f(x)|$ and it occurs when $\vec{u}$ has the same direction as the gradient vector $\nabla f(x)$. (Remember 3b1b)?

The minimum value of the directional derivative is $-|\nabla f(x)|$ and occurs when $\vec{u}$ has the opposite direction as ∇f

The value of the direction derivative is zero if u is orthogonal to $\nabla f(x) \neq 0$.

Proof. Since f is differentiable, it follos that

$$D_{\vec{u}}f(x) = \nabla f(x) \cdot u = |\nabla f(x)|\,|u|\cos(\theta) = |\nabla f(x)|\cos(\theta)$$

Where θ is the angle between ∇f and u. Maximize it by making θ zero. Minimize it by making θ a right angle, π. ■

Example. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function given by $f(x\ ,y) = \frac{x^2}{2} + \frac{y^2}{2}$. Find the directions in which f increases the most rapidly at $(1,\ 1)$.

Well, $\nabla f(x,\ y) = \begin{bmatrix} x \\ y \end{bmatrix}$. At $(1,\ 1)$, then in direction $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. That's the direction where it increases the most rapidly. And in the opposite direction $-\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ gives us the minimum value.

19 Minimum and Maximum Values

Motivation. $f\colon \mathbb{R} \to \mathbb{R}$

Definition. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and let D_f be the domain of f (a set):

1. f has an absolute (or global) maximum value on D_f at the point $c \in D_f$ if $f(x) < f(c)$, well, if it exists.
2. f has an absolute (or minimum) maximum value on D_f at the point $c \in D_f$ if $f(x) < f(c)$, well, if it exists.

Notation. Let $r \in (0, \infty)$ and let $a \in \mathbb{R}$. A neighborhood of a of radius r, denoted by $N_r(a)$, is:

$$N_r(a) = \{x \in \mathbb{R} : |x-a| < r\} = \{x \in \mathbb{R} : x \in (a-r, a+r)\}$$

(Some interval centered around a, on the number line, with radius r).

19.1 Local Min Max

Local minimum on D_f at $c \in D_f$ if $f(x) \leq f(c)$, for all $x \in N_r(c) \subseteq D_f$ for some $r \in (0, \infty)$.

19.2 Critical Point

- c is a critical point if $f'(c) = 0$ or $f'(c)$ is undefined
- c is a stationary point if $f'(c) = 0$

Definition. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and let D_f be the domain of f. Let $c \in D_f$. c is an inflection point if the graph of f has a tangent line and concavity changes. I mean critical point for f'.

- All local max points, local min points, all inflexion points $\subseteq$ stationary points (0 derivative) $\subseteq$ all critical points

- Any continuous function on a compact set can attain a minimum and maximum value.

19.3 Neighborhoods in $\mathbb{R}^2$

Open disk:

$$N_r(a, b) := \left\{(x, y) \in \mathbb{R}^2 : (x-a)^2 + (y-b)^2 < r^2\right\}$$

Closed disk:

$$\overline{N_r(a, b)} := \left\{(x, y) \in \mathbb{R}^2 : (x-a)^2 + (y-b)^2 \leq r^2\right\}$$

Theorem (the first derivative theorem for local extreme values). Let $c \in D_f$ be a local max or local min. If f' is defined at c, then $f'(c) = 0$.

Theorem (EVT). If $f \colon \mathbb{R} \to \mathbb{R}$ is **continuous** in a compact subset of $\mathbb{R}$ like $[a, b]$ (any closed **and** bounded interval), then f attains both an absolute max and absolute min.

19.4 Max and Mins in Two Dimensions

"Tip of a hill in all directions" vs. "pringle"

The "pringle" gives us saddle points: where it may be a min/max if we cut through either axis.

Definition (critical points for multivariable functions). A critical point at any $\mathbb{R}^2 \to \mathbb{R}$ function is all (x, y) when $\nabla f(x, y) = \vec{0}$. This implies that $f_x(x, y) = 0$ and $f_y(x, y) = 0$. Also, if one partial doesn't exist so does the gradient (flat tangent plane or tangent plane DNE?).

The definition for $\mathbb{R}^2 \to \mathbb{R}$:

f has a local max at (a, b) if $f(x, y) \geq f(a, b)$ for all $(x, y) \in N_r(a, b) \subseteq D_f$ for some $r \in (0, \infty)$ (can be epsilon-sized a.k.a. infinitely small)

f has a local min at (a, b) if $f(x, y) \leq f(a, b)$ for all $(x, y) \in N_r(a, b) \subseteq D_f$ for some $r \in (0, \infty)$

19.5 CRITICAL POINTS IN R2 – Testing for local extremes

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function and let $(a, b) \in D_f$ be a local max or min point. If the first order partial derivatives of f exist, then $\nabla f(a, b) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Note the following equivalences:

$$\nabla f(a, b) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} f_x(a, b) \\ f_y(a, b) \end{bmatrix} \Leftrightarrow f_x(a, b) = 0,\ f_y(a, b) = 0$$

19.6 Hessian Matrix, Finding Critical Points, and Min/Max Values

We have some critical points. What do they mean?

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function. The hessian matrix of f at (a, b), denoted $Hf(a, b)$, is:

$$Hf(a, b) = \begin{bmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{bmatrix}$$

$Hf(a, b)$ happens to be exactly what the second derivative of f would be, which is formed by taking the derivative of ∇f.

By the way, an $\mathbb{R}^n \rightarrow \mathbb{R}^m$ function, the matrix would be $m \times n$.

Second partial derivatives are packed here and are arranged into a matrix.

If f is C^2 at (a, b), then:

$$\det(Hf(a,\, b)) = f_{xx}(a,\, b)f_{yy}(a,\, b) - (f_{xy}(a,\, b))^2$$

Theorem (Second derivative test for local extreme values). Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function and let $(a,\, b) \in D_f$. Suppose first and second partial derivatives of f are continuous on $\overline{N_r(a,\, b)}$ (consider the boundary of the disk as well). Suppose that $f_x(a,\, b) = f_y(a,\, b) = 0$ (so $(a,\, b)$ is a critical point). **Then:**

- $\det(Hf(a,\, b)) > 0$
  - $f_{xx}(a,\, b) > 0 \Rightarrow (a,\, b)$ is local min $\cup$
  - $f_{xx}(a,\, b) < 0 \Rightarrow (a,\, b)$ is local max $\cap$
  - Neither otherwise
- $\det(Hf(a,\, b)) < 0$
  - $f(a,\, b)$ is a saddle point
- $\det(Hf(a,\, b)) = 0$
  - The test is inconclusive.

Example. Obtain all the critical points at $xy - x^2 - y^2 - 2x - 2y + 4$.

This requires a system of equations. $\nabla f(x,\, y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ at when? $f_x(x,\, y) = y - 2x - 2 \quad f_y(x,\, y) = x - 2y - 2$

$$\begin{cases} y - 2x - 2 = 0 \\ x - 2y - 2 = 0 \end{cases} \Rightarrow \begin{cases} y = 2x + 2 \\ x = 2y + 2 \end{cases} \Rightarrow x = -2 \Rightarrow y = -2$$

$\Rightarrow (-2,\, -2)$ is a critical point. But what type? Compute the Hessian matrix. By the way, $f_{xy}(x,\, y) = 1$. So:

$$Hf(x,\, y) = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$
$$\det\left(Hf(x,\, y)\right) = 4 - 1 = 3$$

The result from the hessian matrix is positive. $f_{xx}(-2,\, -2) = -2$, which is less than 0, so $(-2,\, -2)$ is a local maximum.

Example. Find and classify the critical points for this: $f(x,\, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$

Firstly: $\nabla f(x,\, y) = \begin{bmatrix} -6x + 6y \\ 6y - 6y^2 + 6x \end{bmatrix}$

And the hessian matrix: $\begin{bmatrix} -6 & 6 \\ 6 & 6 - 12y \end{bmatrix}$

Match $\begin{bmatrix} -6x + 6y \\ 6y - 6y^2 + 6x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$:

$$\begin{cases} -6x + 6y = 0 \\ 6y - 6y^2 + 6x = 0 \end{cases}$$
$$\Rightarrow 12y - 6y^2 = 0$$
$$\Rightarrow 6\left(2y - y^2\right) = 0$$
$$\Rightarrow 2y - y^2 = 0$$
$$2y\left(1 - \frac{1}{2}y\right) = 0$$
$$\Rightarrow y = 0 \text{ or } 1 - \frac{1}{2}y = 0$$
$$\Rightarrow -\frac{1}{2}y = -1 \Rightarrow y = 2$$

If $y = 0,\ -6x = 0 \Rightarrow x = 0$. If $y = 2$, $-6x + 12 = 0 \Rightarrow -6x = -12 \Rightarrow x = 2$

So, our critical points are $(0,\, 0)$ and $(2,\, 2)$. I want to classify these points. Firstly, take the determinant of the hessian matrix:

$$\begin{aligned}\begin{vmatrix} -6 & 6 \\ 6 & 6-12y \end{vmatrix} &= -6(6-12y)-36 \\ &= -36+72y-36 \\ &= -72+72y = 72(y-1)\end{aligned}$$

When $(x, y) = (0, 0)$, the result of the determinant of the hessian matrix is $72(0-1) = -72$, so it is negative and $(0, 0,)$ is a **saddle point** of f**.**

When $(x, y) = (2, 2)$, the result of the determinant of the hessian matrix is $72(2-1) = 72$. As $f_{xx}(2, 2)$ is negative, $(2, 2)$ is a **local maximum** of f**.**

19.7 To find the absolute min and max

For $\mathbb{R}^2 \to \mathbb{R}$ in a certain region (closed and bounded set D):

1. Find the values of f at the **critical points** of f in D
2. Find the extreme values of f at the **boundary** of D
3. The largest of the values from step 1 and step 2 is the absolute maximum value
4. The smallest of the values from step 1 and step 2 is the absolute minimum value

Example. Find the absolute max and min values of $f(x, y) = 2+2x+4y-x^2-y^2$ on the triangular region

$$D = \{(x, y)\ : x = 0,\ y = 0,\ y = 9-x\}$$

<u>Solution:</u> Get the area. Find all critical points in the area. Then, inspect everything in the boundary.

Firstly, compute the critical points of f.

$$\nabla f(x, y) = \begin{bmatrix} 2-2x \\ 4-2y \end{bmatrix}$$

$$\nabla f(x, y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 2-2x=0 \\ 4-2y=0 \end{matrix} \Rightarrow \begin{matrix} x=1 \\ y=2 \end{matrix}$$

We have one critical point: $(1, 2)$. $f(1, 2) = 7$.

This is an interior point in this region.

Then, consider the boundaries. There's a theorem saying any continuous function on any compact region can attain maximum and minimum values. The function given is continuous, for sure, and we have a compact (closed) region.

The first boundary is $x = 0$. Because of this, I'll have to compute $f(0, y) = 4y - y^2$. $f^{'}(0, y) = 4 - 2y = 0 \Rightarrow y = 2$

Therefore, the next potentially max or min point: $(0, 2)$.

AND the boundary: $(0, 0)$, $(0, 2)$, $(0, 9)$.

Go over lines $y = 0$ and $y = 9 - x$ but make sure you're inside the set. After that, you'll get an entire graph:

$$f(0, 0) = 2$$
$$f(0, 9) = -43$$

And the lowest one is the min; the highest one is the max.

Find the absolute min and max values of

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$

On $D = \{(x, y) \in \mathbb{R}^2 : x = 0,\ y = 0,\ y = 9 - x\}$ (the boundaries)

You're going to slice the graph of f three times. We've already found the critical points: $(1, 2)$. $f(1, 2) = 7$

Case 1. $x = 0$. Then, $f(x, y) = f(0, y) = 2 + 4y - y^2$. Max/min points in $[0, 9]$? $f'(0, y) = 0 \Rightarrow 4 - 2y = 0 \Rightarrow y = 2$. When $y = 2,\ x = 0$, $f(0, 2) = 6$. $f(0, 0) = 2,\ f(0, 9) = -43$.

Case 2. $y = 0$, inside $[0, 9]$. Then, $f(x, 0) = 2 + 2x - x^2$. $f'(x, 0) = 2 - 2x$. When $f'(x, 0) = 0 \Rightarrow 2 - 2x = 0 \Rightarrow x = 1$. Then, $f(1, 0) = 3$, $f(9, 0) = 2 + 18 - 81 = 20 - 81 = -61$. The lowest is at $(9, 0)$ with a value of -61.

Case 3. $y = 9 - x$. $\begin{aligned} f(x, y) &= f(x, 9 - x) \\ &= 2 + 2x + 4(9 - x) - x^2 - (9 - x)^2 \\ &= 2 + 2x - 36 - 4x - x^2 - 81 + 18x - x^2 \\ &= -43 + 16x - 2x^2 \end{aligned}$. Now, find the min/-max on values such that $x \in [0, 9]$. $f'(x, 9 - x) = 16 - 4x = 0 \Rightarrow x = 4$. When $x = 4$, $y = 9 - 4 = 5$, so $f(4, 5) = 2 + 8 + 20 - 16 - 25 = -11$.

The absolute minimum is -61 located at $(9, 0)$ and the absolute maximum is 7, at $(1, 2)$.

19.8 The Nabla Operator for Over Two Dimensions

$$\nabla g(x, y, z) = \begin{bmatrix} f_x & f_y & f_z \\ h_x & h_y & h_z \end{bmatrix}$$

$$\text{where } g(x, y, z) = (f(x, y, z), h(x, y, z))$$

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$\nabla f(x, y) = \begin{bmatrix} f_x \\ f_y \end{bmatrix}$$

...

20 Lagrange Multipliers for Constrained Optimization

With a single constraint vs. with multiple constraints

Motivation. Sometimes, we must find the extreme. Values of a function whose domain is constrained to some **subset of the plane like a disk or along a curve.**

Find the max. of f subject to it being on a curve.

Theorem. Let $S \subseteq \mathbb{R}^n$ and let $f,\ g : S \to \mathbb{R}$ be functions of class C^1. Let $x \in \mathbb{R}^n$ be a local minimum point or local maximum point of f to the constraint $g(x) = 0$. If $\nabla g(x) \neq \vec{0}$, then $\exists \lambda \in \mathbb{R}$ such that the following system of equation is satisfied by x and λ:

$$\begin{cases} \nabla f(x) = \lambda \nabla g(x) \\ g(x) = 0 \end{cases}$$

Theorem. Let $S \subseteq \mathbb{R}^n$ and let $f,\ g,\ h : S \to \mathbb{R}$ be C^1. Let $x \in \mathbb{R}^n$ be a local minimum point or local maximum point of f constraint to $g(x) = 0$ and $h(x) = 0$. Suppose that ∇g is not parallel to ∇h. Then, $\exists \lambda,\ \mu \in \mathbb{R}$ such that

$$\begin{cases} \nabla f(x) = \lambda g(x) + \mu \nabla h(x) \\ g(x) = 0 \\ h(x) = 0 \end{cases}$$

Example. Find the extreme values of $f(x,\ y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

Set $g(x,\ y) = x^2 + y^2 + 1$. We'll apply the Lagrange theorem to solve this.

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \begin{bmatrix} 2x \\ 4y \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix} \\ x^2 + y^2 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} 2x = \lambda 2x \\ 4y = \lambda 2y \\ x^2 + y^2 = 1 \end{cases}$$

Solve this system. Number the equations on the matrix from 1 to 3 from top to bottom.

Suppose that $2x = 2\lambda x$. There are two possibilities:

- $x = 0$
  - Then, $0 = 0$ and any λ works… but that wouldn't work.
- $x \neq 0$
  - Then, $\lambda = 1$ by canceling stuff out.
  - Then by the second equation, we get $4y = 2y \Rightarrow y = 0$
  - Therefore, I have one candidate: $y = 0,\ \lambda = 1,\ x = TBA$
  - By the bottom-most equation, we have $x^2 + y^2 = 1 \Rightarrow x^2 = 1 - y^2 \Rightarrow x = \pm 1$
  - Okay, we have two candidates:
    * $A(1, 0) \quad B(-1, 0)$
- $y \neq 0$
  - $\Rightarrow 4y = \lambda 2y \Rightarrow \lambda = 2$
    * $\Rightarrow x = \pm 1$. It satisfies the first equation.

        * So, we have $C(0,\ 1) \quad D(0,\ -1)$

- $y = 0$
    - Then, $x = 0$, but this results in a contradiction.

Try plugging in points:

- $f(A) = 1$
- $f(B) = 1$
- $f(C) = 2$
- $f(D) = 2$

The maximum value of f is 2 and the function f attains the maximum values at points C and D. The minimum value is 1 and the function f attains the minimum value at A and B.

The solution above is a bit of a mess and you probably shouldn't use this to learn about LaGrange multipliers for the first time.

20.1 Taking a Ratio

$$\begin{cases} 2x = \lambda 2x \\ 4y = \lambda 2y \\ x^2 + y^2 = 1 \end{cases}$$

If $x \neq 0$, then there is a case where $\lambda = 1$

If $y \neq 0$, then there is a case where $\lambda = 2$

20.2 Another Example

Lagrange theorem can only find potential max/mins on the boundary. In the interior, you'll have to use $\nabla f(x,\ y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Example. Find the points on the hyperbolic cylinder

$$x^2 - z^2 - 1 = 0$$

that are the closet to $(0,\ 0,\ 0)$. Note that $d = \sqrt{x^2 + y^2 + z^2}$.
So, the function is $\sqrt{x^2 + y^2}$ and the constraint is $x^2 - z^2 - 1 = 0$.

$$f(x,\ y) = \sqrt{x^2 + y^2 + z^2}$$
$$g(x,\ y) = x^2 - z^2 - 1 = 0$$

Apply the LaGrange theorem. I can amend $f(x,\ y)$ to $f(x,\ y) = x^2 + y^2 + z^2$ because regardless, the maximum I get will at least be the maximum.

$$\begin{cases} \nabla f(x,\ y,\ z) = \lambda \nabla g(x,\ y,\ z) \\ g(x,\ y,\ z) = 0 \end{cases}$$

What I have is

$$\begin{cases} \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 0 \\ -2z \end{bmatrix} \\ x^2 - z^2 - 1 = 0 \end{cases}$$

So, we have:

$$\begin{cases} 2x = 2\lambda x \\ 2y = 0 \\ 2 = -2\lambda z \\ x^2 - z^2 - 1 = 0 \end{cases}$$

Immediately, we can obtain $y = 0$. Now, $2x = 2\lambda x$. We have two possibilities.

- $x = 0$
  - Then $z^2 = -1$. Because we're only working with $\mathbb{R}^3 \to \mathbb{R}$, this is a contradiction.
- $x \neq 0$
  - $\Rightarrow \lambda = 1$
  - $2z = -2z \Rightarrow 4z = 0 \Rightarrow z = 0$
  - If $z = 0$, then, we have $x^2 - 1 = 0 \Rightarrow x = \pm 1$
  - So, our potential points are $(1,\ 0,\ 0)$ and $(-1,\ 0,\ 0)$
- For $2z = -2\lambda z$:
  - $z = 0$
    * TBA
  - $z \neq 0$
    * $\lambda = -1$
    * $x = 0$
    * Contradiction; does not satisfy bottom equation

Then: $f(A) = 1,\ f(B) = 1$. As $\sqrt{1} = 1$, the maximum value is 1.

20.3 When is the Gradient Is Orthogonal to the Curve

If we have $r(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, and we want $f\left(x(t),\ y(t)\right) = 1$, differentiate both sides:

$$0 = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$

We can write this as two vectors:

$$\begin{aligned}
&\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \end{bmatrix} \\
&= \nabla f(x, y) \cdot \begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \end{bmatrix} \\
&= \nabla f(x, y) \cdot \vec{r}(t) = 0
\end{aligned}$$

So, $\vec{r}^{'}(t)$ is always tangent to $\vec{r}$, and the gradient is always orthogonal to $r^{'}$.

Now, if we have two vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x^{'} \\ y^{'} \end{bmatrix}$, what can we say about the relationship between them?

$$\exists \lambda \in \mathbb{R} : \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x^{'} \\ y^{'} \end{bmatrix}$$

Suppose we have a function $f(x, y) = 1$ and suppose you have another function $g(x, y) = 1$. g and f at some point is tangent (they share the exact same tangent line, including the slope and a point of it).

If we disregard g, ∇f at the point where f and g share the same tangent line x_0, y_0 will always be perpendicular to the tangent line (at x_0, y_0).

If we disregard f, ∇g at the point where f and g share the same tangent line will always be perpendicular to the tangent line.

Hence:

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

In LaGrange multipliers, we have a function $z = f(x, y)$ and we have a level curve of a function $f(x, y) = 1, 2, 3, \ldots$ and so on, and we also have a constraint, $g(x, y)$. We want to maximize (know about the maximum value of the graph of f) subject to $g(x, y)$.

As we see, if we move from one level to the other, the value of f moves from 1 to 2.

Look for the level curve where a level curve of $g(x, y) = c$ has a point that shares the same tangent line to somewhere in the level curve of $f(x, y)$.

And I end up with this system:

$$\begin{aligned} \nabla f(x_0, y_0) &= \lambda g(x_0, y_0) \\ g(x, y) &= 1 \end{aligned}$$

For g, consider a single level curve. For f, consider all level curves and look for one that works.

20.4 Gradient Descent

(Will not be tested)

Machine learning concepts are typically optimization problems. Input is something, and output is something else. For each example in the dataset, we compare the output of the neural network to the ideal output. Take the squared sum difference, we get the cost. Then, we get the overall cost function. How do we find the right input value that makes the cost unction as small as possible.

Here comes gradient descent. It feels like making Newton iterations. You want to make the function decrease in each iteration. It feels like dropping a ball over a surface and hoping that it gets to the lowest value.

20.5 Easiest Optimization Problem

The temperature at point (x, y, z) on the unit sphere is given by $f(x, y, z) = 2xy + z^2 - z$ in the constraint $g(x) = x^2 + y^2 + z^2 = 1$.

$$\begin{aligned} &\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \\ &x^2 + y^2 + z^2 = 1 \end{aligned}$$

$$\begin{bmatrix} 2y \\ 2x \\ 2z-1 \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$$
$$\Rightarrow \begin{cases} 2y = \lambda 2x \\ 2x = \lambda 2y \\ 2z-1 = \lambda 2z \end{cases}$$

- $2y = \lambda \cdot \lambda 2y$
- $\Rightarrow 2y = \lambda^2 2y$
- $\Rightarrow y = \lambda^2 y$
- Case 0. $z = 0$.
  - $2z - 1 = 0$
  - $-1 = 0$
  - Not permissible.
- Case 1. $y = 0$
  - Then, $2x = 0 \Rightarrow x = 0$.
  - $z^2 = 1 \Rightarrow z = \pm 1$, so we get $(0,\ 0,\ 1)$ and $(0,\ 0,\ -1)$
  - But we need to check for contradiction on the other equations.
    * $2 - 1 = \lambda 2 \Rightarrow 1 = \lambda 2$. There exists a λ, so no contradiction for $z = 1$.
    * $-2 - 1 = -2\lambda \Rightarrow -3 = -2\lambda$. There exists a λ, so no contradiction for $z = -1$.
- Case 2. $y \neq 0$.
  - Then, $1 = \lambda^2 \Rightarrow \lambda = \pm 1$
  - Sub-case 1: $\lambda = 1$
    * $2y = 2x \Rightarrow x = y$

    * $2x^2 + z^2 = 1$
      $2z - 1 = 2z$
    * No solutions for z exist.
  - Sub-case 2: $\lambda = -1$
    * $2y = -2x \Rightarrow x = -y$. Then, $x \neq 0$.
    * $2z - 1 = -2z$
    * $4z = 1 \Rightarrow z = \frac{1}{4}$
    * Then, $2x^2 + \frac{1}{16} = 1$
    * $\Rightarrow x^2 = \frac{15}{32}$
    * $x = \pm\sqrt{\frac{15}{32}}$
    * Therefore, we get the points
    * $\left(\sqrt{\frac{15}{32}}, -\sqrt{\frac{15}{32}}, \frac{1}{4}\right)$ and $\left(-\sqrt{\frac{15}{32}}, \sqrt{\frac{15}{32}}, \frac{1}{4}\right)$

Plug and chug to figure out what is the max and min.

Question: If I don't end up coincidentally find a case where $x = 0$, do I have to do make up a case? Or do I have to automatically do it all the time?

20.6 Closest To The Origin

> Find the points closest to the origin on the curve of the intersection of the plane $2y + 2z = 5$ and the cone $z^2 = 4x^2 + 4y^2$.

We have these equations

$$\begin{aligned} \text{set } f(x, y, z) &= x^2 + y^2 + z^2 \\ g(x, y, z) &= 2y + 2z - 5 = 0 \\ h(x, y, z) &= 4x^2 + 4y^2 - z^2 = 0 \end{aligned}$$

Using LaGrange multipliers, set this up:

$$\begin{cases} \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} 8x \\ 8y \\ -2z \end{bmatrix} \\ 2y+2z-5=0 \\ 4x^2+4y^2-z^2=0 \end{cases}$$

So, we get:

$$\begin{cases} x = 4\mu x \\ y = \lambda + \mu 4y \\ z = \lambda - \mu z \\ 2y+2z-5=0 \\ 4x^2+4y^2-z^2=0 \end{cases}$$

We need to solve this system:

- Case 1: $x=0$
  - $\Rightarrow z^2 = 4y^2 \Rightarrow z = \pm 2y$
  - Case 1A: $z = 2y$
    * If $z=2y$, then $2y+4y=5 \Rightarrow y=\frac{5}{6}$
    * And $\frac{5}{3}+2z-5=0 \Rightarrow 2z = 5-\frac{5}{3} \Rightarrow z = \frac{5-\frac{5}{3}}{2} \Rightarrow z = \frac{5}{3}$
    * Hence, we have a point: $A\left(0, \frac{5}{6}, \frac{5}{3}\right)$
  - Case 1B: $z=-2y$ and $-z=2y$
    * Then, $-2y=5 \Rightarrow y = -\frac{5}{2}$, $z=5$
    * We get $B\left(0,\ -\frac{5}{2}, 5\right)$

- Case 2: $x \neq 0$
  - $\mu = \frac{1}{4}$
  - $y = \lambda + y \Rightarrow \lambda = 0$
  - $\Rightarrow z = \frac{1}{4}z \Rightarrow z = 0$
  - $2y = 5 \Rightarrow y = \frac{5}{2}$
  - $4x^2 + 4y^2 = 0 \Rightarrow 4y^2 = -4x^2 \Rightarrow y^2 = -x^2 \Rightarrow$ the only working values that can satisfy this equation is when $x = 0$ and $y = 0$, which is a contradiction as we initially said $x \neq 0$ and $y = \frac{5}{2}$.

I'm not going to do any calculations, but I'm pretty sure $A\left(0, \frac{5}{6}, \frac{5}{3}\right)$ is the minimum and $B\left(0, -\frac{5}{2}, 5\right)$ is the maximum.

$$f\left(0, \frac{5}{6}, \frac{5}{3}\right) = \frac{25}{36} + \frac{25}{9}$$
$$f\left(0, -\frac{5}{2}, 5\right) = \frac{25}{4} + 25$$

WARNING – if we are asked for the distance, keep in mind that $f(x, y, z) = x^2 + y^2 + z^2$. Hence, f actually gives us the distance squared, so we need to square root anything we get from f to call it distance.

20.7 Cuts

> The plane $x + y + z = 1$ cuts the cylinder $x^2 + y^2 = 1$ in an ellipse. Find the points on the ellipse that lie closest and farthest from the origin.

We have the distance from the origin squared:

$$f(x, y, z) = x^2 + y^2 + z^2$$

And we have the constraints:

$$g(x, y, z) = x^2 + y^2 = 1$$
$$h(x, y, z) = x + y + z = 1$$

Leaving us with this system:

$$\begin{cases} \nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z) \\ x^2 + y^2 = 1 \\ x + y + z = 1 \end{cases}$$

We may expand it and get:

$$\begin{cases} 2x = \lambda 2x + \mu & (1) \\ 2y = \lambda 2y + \mu & (2) \\ 2z = \mu & (3) \\ x^2 + y^2 = 1 & (4) \\ x + y + z = 1 & (5) \end{cases}$$

It is often in our best interests to solve for λ and μ first.

From (3), we can make our substitutions into (1) and (2):

$$\begin{cases} 2x = \lambda 2x + 2z & (1) \\ 2y = \lambda 2y + 2z & (2) \\ 2z = \mu & (3) \\ x^2 + y^2 = 1 & (4) \\ x + y + z = 1 & (5) \end{cases}$$

I then subtract (1) and (2):

$$2x-2y=\lambda(2x-2y)$$

Case 1: $x \neq y$. Then:

- $\frac{2x-2y}{2x-2y}=\lambda \Rightarrow \lambda=1$
- We then get the relation $2x=2x+2z$ and thus $0=2z$. Hence, $z=0$.
- Hence, $\begin{cases} x^2+y^2=1 \\ x+y=1 \Rightarrow y=1-x \end{cases}$ and we get the solutions $A(0,\ 1,\ 0)$ and $B(1,\ 0,\ 0)$.

Case 2: $x=y$. Then:

- $2x^2=1 \Rightarrow x^2=\frac{1}{2} \Rightarrow x=\pm\frac{1}{\sqrt{2}}$ and $y=x$. We also know that $z=1-x-y$.
  - Sub-case 1: $x,\ y=\frac{1}{\sqrt{2}}$. Then, $z=1-\frac{2}{\sqrt{2}}$, and we get the point $C\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}},\ 1-\sqrt{2}\right)$
  - Sub-case 2: $x,\ y=-\frac{1}{\sqrt{2}}$. Then, $z=1+\frac{2}{\sqrt{2}}$, and we get the point $D\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}},\ 1+\sqrt{2}\right)$.

Then, I plug in each point into $f(x,\ y,\ z)=x^2+y^2+z^2$:

- $f(0,\ 1,\ 0)=f(1,\ 0,\ 0)=1$. The distance is 1.
- $$\begin{aligned} &f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}},\ 1-\sqrt{2}\right) \\ &=\left(\frac{1}{2}+\frac{1}{2}+\left(1-2\sqrt{2}+2\right)\right) \end{aligned}$$
  $\approx\ 4-2.8=1.2$
- $$\begin{aligned} &f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}},\ 1+\sqrt{2}\right) \\ &=\left(\frac{1}{2}+\frac{1}{2}+1+2\sqrt{2}+2\right) \\ &=\left(5+2\sqrt{2}\right) \end{aligned}$$

Hence, the longest distance is $\sqrt{5+2\sqrt{2}}$ and the shortest distance is 1.

20.8 Extreme values

> Find the extreme values of $f(x, y) = \frac{x+y}{1+x^2+y^2}$ subject to the constraint $x^2 + y^2 = R^2$ where $R \in \mathbb{R}_{\geq 0}$.

$$g(x) = x^2 + y^2 = R^2$$

Hence:

$$\begin{cases} \nabla f(x, y) = \lambda \nabla g(x) \\ x^2 + y^2 = R^2 \end{cases}$$

As an alternative way, we can consider $x = R\cos(\theta)$ and $y = R\sin(\theta)$. Then, find the extreme values of $f(R\cos(\theta), R\sin(\theta))$. Then, we have:

$$f(R\cos(\theta), R\sin(\theta)) = \frac{R}{1+R^2} \cdot (\cos(\theta) + \sin(\theta))$$

Take $\frac{df}{d\theta}$ = some single variable function. We should end up with $\sin(\theta) = \cos(\theta)$ which implies that $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$. Plug in those values in f, with $x = R\cos(\theta)$ and $y = R\sin(\theta)$.

21 Integrals

In single variable calculus, the area under the graph of f between a and b is $\int_a^b f(x)dx$. This is defined using Riemann sums.

We want to approximate them by:

1. Subdividing the interval $[a, b]$ into sub-intervals $[x_{i-1}, x_i]$, $a = x_0 < x_1 < x_2 < \ldots < x_n = b$.
   a. Assume that all the sub-intervals are equally spaced. Then, Δx, the length between each interval, is $\Delta x = \frac{b-a}{n}$.

2. Choose a point inside each sub-interval x_i^*. In theory, it doesn't matter, but when using computers, it does.
   a. Leftmost: $x_i^* = x_{i-1}$
   b. Right: $x_i^* = x_i$
   c. Middle: $x_i^* = \frac{x_{i-1}+x_i}{2}$
3. Evaluate $f(x_i^*)$
   a. The area of the tall rectangular strip in the sub-interval is $A = f(x_i^*) \cdot \Delta x$
4. The area under the graph between a, b is $A = \sum_{i=1}^{n} f(x_i^*) \Delta x$
5. Take a limit: $\lim_{n\to\infty} f(x_i^*) \Delta x \to \int_a^b f(x)dx$

For functions you can do this, they are integrable.

21.1 Integrating in Multivariable Calculus

Integrating $z = f(x, y)$. Instead of an interval, I have a rectangle. The rectangle region is $R = [a, b] \times [c, d] = \{(x, y) : a \leq x \leq b \text{ and } c \leq y \leq d\}$. We want to find the volume under the curve.

If we want to calculate the volume under the surface inside the rectangle R:

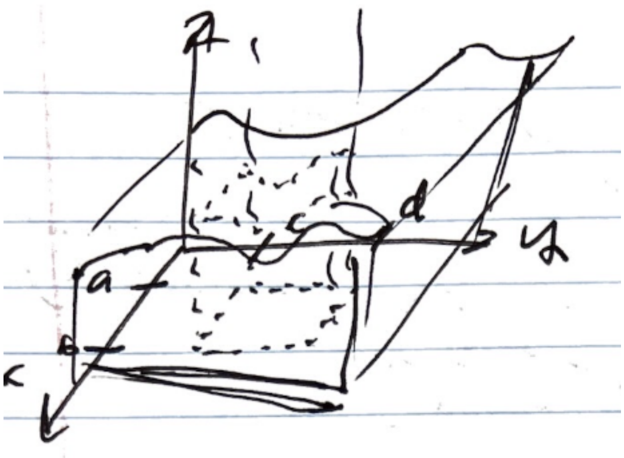
$$\iint_R f(x, y)dA$$

The procedure goes as follows:

1. Subdivide R into sub-rectangles
   a. $a = x_0 < x_1 < \cdots < x_n = b$
   b. $c = y_0 < y_1 < \cdots < y_m = d$
   c. Assume the subdivisions are equally spaced: $\Delta x = \frac{b-a}{h}$ and $\Delta y = \frac{d-c}{m}$
   d. The area of each sub-rectangle $\Delta A = \Delta x \cdot \Delta y$.

2. Choose a point in each sub-rectangle.
   a. $\left(x_{ij}^*, y_{ij}^*\right)$
3. Evaluate $f\left(x_{ij}^*, y_{ij}^*\right)$, which can be seen as a rectangular prism. The volume of that piece is $f\left(x_{ij}^*, y_{ij}^*\right) \cdot \Delta A \cdot$
4. Volume is approximately$\sum_{j=1}^{m}\sum_{i=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$
   a. If you can get a number out of this (the limit converges), then f is an integrable function.

This is just the definition; we will never calculate integrals this way. Almost all functions we will encounter here are integrable.



21.2 To Integrate, Exactly

Iterated integrals

$$f(x, y) \quad R = [a, b] \times [c, d]$$

Fix an x-location y. Along the slice, I'll get a single variable function.

$$A(x) = \int_c^d f(x,\, y)dy$$
$$\int_a^b A(x)dx = \int_a^b \int_c^d f(x,\, y)dydx$$

21.3 Fubini's Theorem

If f is integrable on the rectangle $R = \underbrace{[a,\, b]}_{x\text{-axis}} \times \underbrace{[c,\, d]}_{y\text{- axis}}$, then:

$$\begin{aligned}&\iint_R f(x,\, y)dA\\ &= \int_a^b \int_c^d f(x,\, y)dydx\\ &= \int_c^d \int_a^b f(x,y)dxdy\end{aligned}$$

Takeaway: iterated integral gives us the answer and **the order of integration does not matter.** However, order does impact the difficulty of the integral.

Integrals are a lot of work. You must integrate multiple times to get a single question, and speed matters.

21.4 Examples

EXAMPLE 1

$$\iint_R \left(x - 3y^2\right) dA,\; R = \{(x,\, y)\; : x \in [0,\, 2],\, y \in [1,\, 2]\}$$

By Fubini's theorem:

$$
\begin{aligned}
&\int_0^2 \int_1^2 \left(x-3y^2\right) dy dx \\
&= \int_0^2 \left(xy-y^3\right)_{y=1}^{y=2} dx \\
&= \int_0^2 \left((2x-8)-(x-1)\right) dx \\
&= \int_0^2 (x-7) dx \\
&= \left(\frac{1}{2}x^2-7x\right)_{x=0}^{x=2} \\
&= \left(\frac{1}{2}\cdot 4-7\cdot 2\right) \\
&= -12
\end{aligned}
$$

EXAMPLE 2

$$\iint_R y\sin(xy)dA \quad R=[1,\,2]\times[0,\,\pi]$$

Integrating with y first is a bad idea. Why?

$$\int_1^2 \int_0^\pi y\cdot\sin(xy)dydx$$

You can see that you'll need to do integration by parts. Not doing this.

Integrating with x first?

$$
\begin{aligned}
&\int_0^\pi \int_1^2 y\cdot\sin(xy)dxdy \\
&= \int_0^\pi y\int_1^2 \sin(xy)dxdy \\
&= -\int_0^\pi y\cdot\left(\frac{1}{y}\cos(xy)\right)_{x=1}^{x=2} dy \\
&= -\int_0^\pi \left(\cos(2y)-\cos(y)\right) dy
\end{aligned}
$$

Your main goal is to choose the dimensions such that you'll get less work.

21.5 Multiplying Instead Of Double Integrating

You may be able to split a double integral into a product of two single integrals... but why would you do that?

PROPERTY

$$\begin{aligned}\iint_R g(x)h(y)dA &= \int_a^b \int_c^d \underbrace{g(x)}_{\text{constant WRT } y} h(y)dydx \\ &= \int_a^b g(x) \underbrace{\left(\int_c^d h(y)dy\right)}_{\text{constant WRT } x} dx \\ &= \left(\int_c^d h(y)dy\right)\left(\int_a^b g(x)dx\right)\end{aligned}$$

21.6 A Double Integral That Doesn't Need To Be Double

If I'm asked to calculate $\iint_R \sqrt{1-x^2}dA$ where $R = [-1, 1] \times [2, 2]$:

- The function $\sqrt{1-x^2}$ does not depend on y
- So, it is as if we're stretching something on the y-axis, getting us half of a cylinder
- The integral is the volume of the half cylinder

$$\underbrace{4}_{\text{height of cylinder}} \cdot \underbrace{\frac{\pi r^2}{2}}_{\text{half cylinder}}$$

21.7 Visualization, Deducing What To Integrate

It's great to draw pictures. They're just hard to do.

Calculate the volume of the solid bounded by $x^2+2y^2+z=16$, the elliptic paraboloid, the planes $x=2$ and $y=2$, and all three coordinate planes. So, we get:

$$\iint_R 16-x^2-2y^2 dA \quad R=[0,\ 2]\times[0,2]$$

So, the paraboloid gives us what to integrate, and everything else gives us the bounds.

Okay maybe you don't need to draw for this one, but for some harder questions, you may need to draw.

Finishing up the calculation:

$$\begin{aligned}
&\int_0^2\int_0^2 \left(16-x^2-2y^2\right)dxdy\\
&=\int_0^2\left(16x-\frac{1}{3}x^3-2y^2x\right)_{x=0}^{x=2}dy\\
&=\int_0^2\left(32-\frac{8}{3}-4y^2\right)dy\\
&=\left(\left(32-\frac{8}{3}\right)y-\frac{4}{3}y^3\right)_{y=0}^{y=2}\\
&=48
\end{aligned}$$

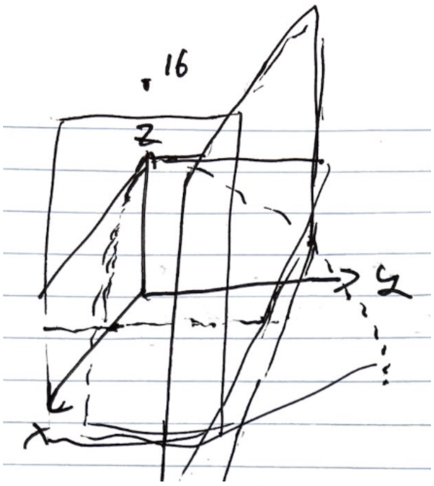


Figure 2: Function and Domain Visualization

21.8 The Midpoint Rule

With 4 sub-rectangles of equal size, I divide the rectangle R (the region I want to integrate) into 4 equal sides: ⊞, and so on:

1. Divide R into 4 sub rectangles
2. Find the midpoint at each of the rectangles
3. Evaluate the midpoints

And we get:

$$\begin{aligned}&\iint_R f(x, y)dy\\&=\sum_{j=1}^{2}\sum_{i=1}^{2} f\left(\bar{x}_{ij}, \bar{y}_{ij}\right)\Delta A\end{aligned}$$

Where $\bar{x}_{ij}$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_{ij}$ is the midpoint of $\left[y_{j-1}, y_j\right]$.

With an actual example:

$$
\begin{aligned}
f(x, y) &= 16 - x^2 - 2y \\
\Delta A &= 1 \times 1 \text{ based on the prev question} \\
\bar{x}_{11} &= \frac{1}{2} \quad \bar{y}_{11} = \frac{1}{2} \Rightarrow \left(\frac{1}{2}, \frac{1}{2}\right) \\
\bar{x}_{21} &= \frac{3}{2}, \bar{y}_{21} = \frac{1}{2} \Rightarrow \left(\frac{3}{2}, \frac{1}{2}\right) \\
&\cdots \Rightarrow \left(\frac{1}{2}, \frac{3}{2}\right) \\
&\cdots \Rightarrow \left(\frac{3}{2}, \frac{3}{2}\right)
\end{aligned}
$$

ΔA is the area of each of the sub-rectangles. So, the integral is APPROXIMATELY:

$$
\begin{aligned}
\iint_R f(x, y) dA &\approx f\left(\frac{1}{2}, \frac{1}{2}\right) \cdot \underbrace{1}_{=\Delta A} + f\left(\frac{3}{2}, \frac{1}{2}\right) + f\left(\frac{1}{2}, \frac{3}{2}\right) + f\left(\frac{3}{2}, \frac{3}{2}\right) \\
&= 49
\end{aligned}
$$

This is the midpoint rule. It's painful. But if your functions aren't nice, when integrating on a computer, you'll have to do some variation.

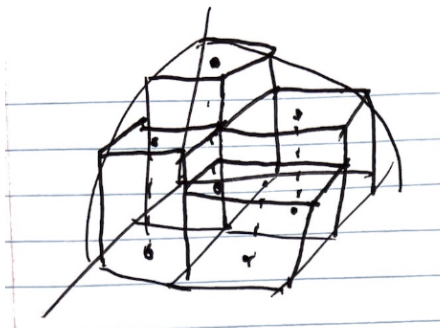


Figure 3: Midpoint visualization

21.9 Average Value

The average value of $f(x, y)$ in region R is:

$$\frac{1}{\text{Area}(R)} \iint_R f(x, y) dA$$

22 Irregular Regions

What happens if the region we're integrating over is **NOT** a rectangle? This still holds:

$$\iint_D f(x, y) dA$$

Is the volume under $f(x, y)$ inside domain D. How do we define this, for any D, if D is not a rectangle?

In theory, we can define this easily:

- Enclose the domain with a rectangular box
- Define a new function $F(x, y) = \begin{cases} f(x, y) & \text{if } f(x, y) \in D \\ 0 & \text{if } f(x, y) \notin D \end{cases}$
  - This new function is not continuous anymore.
  - But it is still integrable.
- And finally:

$$\iint_D f(x, y) dA = \iint_R F(x, y) dA$$

22.1 Types of Domains

Domains can be type I, type II, both, and none. It just matters if I can represent it like...

- Type I region: $D = \{(x, y) \ : a \leq x \leq b, \ g_1(x) \leq y \leq g_2(x)\}$

  - A paper with sharp/clean edges vertically but the horizontal edges look like a mess
  - g_1, g_2 are continuous
- Type II region: $D = \{(x, y) : a \leq y \leq b, h_1(x) \leq x \leq h_2(x)\}$
  - A paper with sharp/clean edges horizontally but the vertical edges look like a mess
  - h_1, h_2 are continuous

If we focus on type Is: Can I draw a vertical line from the top curve to the bottom curve with the whole line in the domain? (i.e., g_1, g_2 won't cross each other)

To integrate things in these domains, we just adjust the bounds of integration.

22.1.1 Integrating Type 1

You MUST integrate with respect to y first.

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx$$

The y-values you are considering depends on x, so you'll have to integrate y first.

Scan from down to up before scanning from up to down.

To say how we can define this integral where D is a general domain:

$$\begin{aligned} F(x, y) = \begin{cases} F(x, y) & (x, y) \in D \\ 0 & \text{otherwise} \end{cases} &= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx \\ &= \iint_R F(x, y)dA \end{aligned}$$

22.1.2 Integrating Type 2

You MUST integrate with respect to x first.

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

That is, scan from left to right first before scanning down to up.

22.1.3 Trying it out

FIRST EXAMPLE

$$\iint_D x + 2y dA$$

Where D is the region bounded by parabolas $y = 2x^2,\ y = 1 + x^2$. Draw it out to figure which curve goes on the top/bottom. Use the intersection of parabolas to find the bounds for x. This forms a type I domain. So

$$D = \left\{(x, y)\ :-1 \leq x \leq 1,\ 2x^2 \leq y \leq 1 + x^2\right\}$$

Then, the integral. I'm integrating with respect to y first as I'm dealing with a type I (looking down to up first):

$$
\begin{aligned}
&\int_{-1}^{1}\int_{2x^2}^{1+x^2}(x+2y)dydx \\
&=\int_{-1}^{1}\left(xy+y^2\right)_{y=2x^2}^{y=1+x^2}dx \\
&=\int_{-1}^{1}\left(\left(x\left(1+x^2\right)+\left(1+x^2\right)^2\right)-\left(x2x^2+\left(2x^2\right)^2\right)\right)dx \\
&=\int_{-1}^{1}\left(\left(x+x^3+1+2x^2+x^4\right)-\left(2x^3+4x^4\right)\right)dx \\
&=\int_{-1}^{1}x-x^3+1+2x^2-3x^4dx \\
&=\left(\frac{1}{2}x^2-\frac{1}{4}x^4+x+\frac{2}{3}x^3-\frac{3}{5}x^5\right)_{-1}^{1} \\
&=\cdots=\frac{32}{15}
\end{aligned}
$$

Figuring out **what to integrate** is the difficult part.

It is possible to integrate this as a type-2 domain, you'll have to transform this into **multiple** type-2 domains and solve multiple double integrals. This takes extremely long. In this case, it will result in 3 type-2 domains.

SECOND EXAMPLE

Find the volume of a solid under paraboloid $z=x^2+y^2$ above region D in the xy plane that is bounded by curves

$$y=2x,\ y=x^2$$

Firstly, the point of intersection:

$$
\begin{aligned}
2x&=x^2 \\
0&=x^2-2x=x(x-2) \\
x&=0,\ x=2
\end{aligned}
$$

This region can be both described as a type I and type II region.

$$
\begin{aligned}
&\int_0^2 \int_{x^2}^{2x} \left(x^2+y^2\right) dydx \\
&= \int_0^2 \left(x^2y+\frac{1}{3}y^3\right)_{x^2}^{2x} dx \\
&= \int_0^2 \left(x^2 2x+\frac{1}{3}8x^3-x^2x^2-\frac{1}{3}x^6\right) dx \\
&= \int_0^2 \left(2x^3+\frac{8}{3}x^3-x^4-\frac{1}{3}x^6\right) dx \\
&= \int_0^2 \left(2x^3-x^4+\frac{8x^3-x^6}{3}\right) dx \\
&= \left(\frac{1}{2}x^4-\frac{1}{5}x^5+\left(\frac{8}{4}x^4-\frac{1}{7}x^7\right)\right)_0^2 \\
&= \left(\frac{1}{2}x^4-\frac{1}{5}x^5+\frac{2x^4}{3}-\frac{1}{21}x^7\right)_0^2 \\
&= \frac{1}{2}\left(2^4\right)-\frac{1}{5}\left(2^5\right)+2\left(2^4\right)-\frac{1}{7}\left(2^7\right) \\
&= 8-\frac{32}{5}+\frac{32}{3}-\frac{128}{21}=\frac{216}{35}
\end{aligned}
$$

As a type-2 region, invert the functions.

$$
\begin{aligned}
&y=2x \leftrightarrow x=\frac{y}{2} \\
&y=x^2 \leftrightarrow x=\pm\sqrt{y} \rightarrow x=\sqrt{y} \quad x\text{-region} \\
&D=\left\{(x,\,y)\;:0\le y\le 4, \frac{y}{2}\le x\le \sqrt{y}\right\} \\
&\int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} \left(x^2+y^2\right) dxdy
\end{aligned}
$$

You will get the same answer if you solve it.

EX: Find the volume of the tetrahedron bounded by the planes:

$$x+2y+z=0,\; x=2y,\; x=0,\; z=0$$

Remember what you learned from your stats course. And you'll need to find the line that intersects $x+2y+z=0$ and $z=0$.

22.2 Solving the Unsolvable

$$\int_0^1 \int_x^1 \sin\left(y^2\right) dydx$$

This is solvable: swap this to a type-2 region (you need to draw the region out):

$$\int_0^1 \int_0^y \sin\left(y^2\right) dxdy$$

This can be solved using substitution

23 Properties of Double Integrals

LINEARITY

$$\iint_D cf(x, y) \pm g(x, y)dA = c\iint_D f(x, y)dA \pm \iint_D g(x, y)dA$$

COMPARING VOLUMES

If $\forall (x, y) \in D,\ f(x, y) \geq g(x, y)$, then $\Rightarrow \iint_D f(x, y)dA \geq \iint_D g(x, y)dA$

THE UNION OF REGIONS THAT DON'T OVERLAP

If $D = D_1 \cup D_2$ where D_1 and D_2 only intersect at boundaries (almost disjoint, disregarding boundaries), then:

$$\iint_D f(x, y)dA = \iint_{D_1} f(x, y)dA + \iint_{D_2} f(x, y)dA$$

We did this before. If you don't have type-1 and type-2 regions, you want to split the regions into type-1 and type-2 regions.

INTEGRATING THE CONSTANT TO GET THE AREA

For a D-region in the xy-plane, the area of D is:

$$\text{Area}(D) = \iint_D 1dA$$

BOUNDING INTEGRALS WITH AREAS

Let m and $M \in \mathbb{R}$. If $\forall (x, y) \in D,\ m \le f(x, y) \le M$, then

$$m \cdot \text{Area}(D) \le \iint_D f(x, y)dA \le M \cdot \text{Area}(D)$$

24 Double Integrals over Polar Regions

I want to integrate $\iint_D f(x, y)dA$, but D is the disk

$$\{(x, y)\ : x^2 + y^2 \le 1\}$$

If I were to draw this out, it would be a circle.

I could do this as a type 1 or type 2 integral, using $\sqrt{1 - x^2}$. That's too difficult.

If I swap to polar coordinates, with the polar coordinate transformation:

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \\ x^2 + y^2 &= 1 \\ 0 &\le r^2 \le 1 \quad 0 \le \theta \le 2\pi \end{aligned}$$

Now, the bounds become easy (essentially, they become constants). And you get:

$$\int_0^{2\pi \text{ (arc)}} \int_0^{1 \text{ (radius)}} \underbrace{r}_{\text{THIS GETS MULTIPLIED}} \cdot f\left(r\cos(\theta),\, r\sin(\theta)\right)\, dr d\theta$$

The reason why we get the r factor is something we need to go back to Riemann sums to figure out. It has to do with the idea as if we go further from the origin, the scale widens.

24.1 Disk Restrictions

D restrictions (the disk):

- $a \le r \le b \to \ r \ge 0$ (Radius must be nonnegative)
- $\alpha \le \theta \le \beta \to \ \beta - \alpha \le 2\pi$ (Arc length cannot be more than one entire period)

For these types of questions, always integrate with respect with r first; it gets very messy otherwise

24.2 Performing an Example

EXAMPLE

$$\iint 3x^2 + 4y^2 dA$$

R is in the upper half plane (above the x-axis), bounded by circles inner $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. This is much better expressed using polar coordinates, where the set is:

$$D = \left\{(x,\, y)\ : x^2 + y^2 \ge 1 \text{ and } x^2 + y^2 \le 4\right\}$$

The bounds for r:

$$1 \leq r^2 \leq 4 \Rightarrow r \in [1, 2]$$
$$0 \leq \theta \leq \pi \quad \text{if you draw it}$$

Let's write out the transformed integral by replacing x with $r\cos(\theta)$ and y with $r\sin(\theta)$:

$$\int_0^\pi \int_1^2 \left(3r\cos(\theta) + 4r^2\sin^2(\theta)\right) r dr d\theta$$

And you can now solve this integral:

$$\begin{aligned}
&\int_0^\pi \int_1^2 3r^2\cos(\theta) + 4r^3\sin^2(\theta) dr d\theta \\
&= \int_0^\pi \left[r^3\cos(\theta) + r^4\sin^2(\theta)\right]_1^2 d\theta \\
&= \int_0^\pi \left(8\cos(\theta) + 16\sin^2(\theta) - \cos(\theta) - \sin^2(\theta)\right) d\theta \\
&= \int_0^\pi 7\cos(\theta) + 15\sin^2(\theta) d\theta \\
&= \int_0^\pi 7\cos(\theta) + \frac{15(1-\cos(2\theta))}{2} d\theta \\
&= \left[7\sin(\theta) + \frac{15}{2}\left(\theta - \frac{\sin(2\theta)}{2}\right)\right]_0^\pi \\
&= \frac{15}{2}\pi
\end{aligned}$$

The integration is "routine" – integrate with respect to your variables. Your questions will end up a lot with sin and cos.

EXAMPLE 2

Find volume of solid bounded by surfaces $z = 0$, $z = 1 - x^2 - y^2$

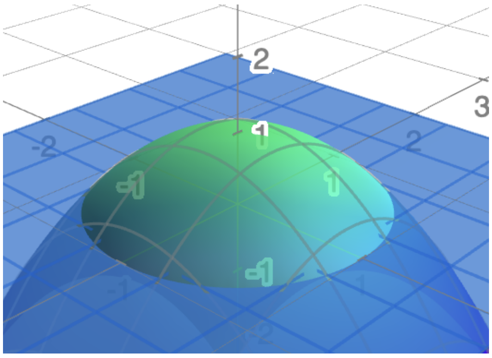


Figure 4: The surface in question (from Math3D)

We can determine the boundary of intersection by finding the intersection of $z = 0$ and $z = 1 - x^2 - y^2$:

$$\begin{aligned} 0 &= 1 - x^2 - y^2 \\ -1 &= -x^2 - y^2 \\ 1 &= x^2 + y^2 \end{aligned}$$

So that is the boundary of the disk. So $r \in [0,\ 1]$, $\theta \in [0,\ 2\pi]$ as we're going around the full circle.

$$\begin{aligned} V &= \iint_D \left(1 - x^2 - y^2\right) dA \\ &= \int_0^{2\pi} \int_0^1 \left(1 - r^2\cos^2(\theta) - r^2\sin^2(\theta)\right) r dr d\theta \end{aligned}$$

If we get to this step, the rest follows. Then integrate.

$$\begin{aligned} &= \int_0^{2\pi} \int_0^1 \left(r - r^3\right) dr d\theta \\ &= \int_0^{2\pi} \left(\frac{1}{2}r^2 - \frac{1}{4}r^4\right)\Bigg|_0^1 d\theta \\ &= \int_0^{2\pi} \left(\frac{1}{4}\right) d\theta = \frac{\pi}{2} \end{aligned}$$

EXAMPLE 3

Calculate the volume of a solid under the paraboloid $z = x^2 + y^2$ above the xy-plane, inside cylinder $x^2 + y^2 = 2x$

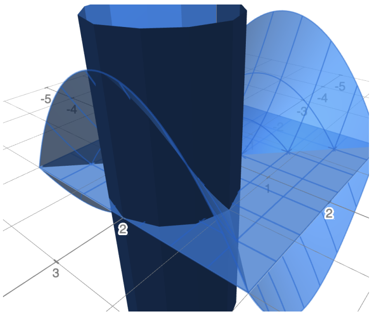


Figure 5: Chart Description automatically generated

The polar curve's radius:

$$x^2 + y^2 = 2x \to r^2 = 2r\cos(\theta)$$
$$r = 2\cos(\theta)$$

We may need to draw this to figure out what bounds θ may take.

If we project on the xy-plane. With some visual intuition

Setting up integration:

$$\int_0^{\pi} \int_0^{2\cos(\theta)} r^2 r \, dr d\theta = \frac{3\pi}{2}$$

24.3 General Polar Regions

This is a general polar region: θ still works the same, but this time r can vary instead of remaining constant. The radius can fluctuate as the angle changes.

$$D = \{(r, \theta) : \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$$

If h_1 and h_2 cross, you may need to break up the integral. The integral bounds will change:

$$\begin{aligned}&\iint_D f(x, y)dA\\&= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f\left(r\cos(\theta),\, r\sin(\theta)\right) r\, drd\theta\end{aligned}$$

24.4 Area of Region

Surely I've learned this before, and I hope I haven't forgotten about it. The area of the region is obtained when you integrate the constant function 1:

$$\begin{aligned}\iint_D 1dA &= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} r\, drd\theta\\&= \int_\alpha^\beta \frac{1}{2}\left(h_1(\theta)^2 - h_1(\theta)^2\right) d\theta \quad \text{it follows}\\&= \int_\alpha^\beta \frac{1}{2}\left(h_1(\theta)^2\right) d\theta - \int_\alpha^\beta \frac{1}{2}\left(h_1(\theta)^2\right) d\theta\end{aligned}$$

You change the bounds of the outer integral such that $\forall x \in [\alpha, \beta]$, $h_1(\theta) \leq h_2(\theta)$. An instance that you would do this is if you wanted to integrate the 4-leaf clover.

25 Applications of Double Integrals

25.1 Density and Mass

The density function is $\rho(x, y)$. Density is $\frac{M}{V}$.

$$\text{Total Mass} = m = \iint_{\text{lamina } D} \rho(x, y)dA$$

You are adding over all the area.

CHARGED DENSITY

$$Q = \iint_D \underbrace{\sigma(x,\, y)}_{\text{charge density}} \; dA$$

PROBABILITY

Define a probability density $f(x,\, y)$. If I want the total probability hitting a certain region: $P\,(\text{my dart lands inside D}) = \iint_D f(x,y)dA$

MOMENTS OF CENTER OF MASS

Moments about the x-axis, moment about the y-axis.

$$M_x = \iint_D y\rho(x,\, y)dA$$
$$M_y = \iint x\rho(x,\, y)dA$$

To take this and calculate the center of mass:

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\rho(x,\, y)dA$$
$$\bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\rho(x,\, y)dA$$

25.1.1 Example 1

Find the mass and the center of mass of a triangular lamina with vertices $(0,\, 0),\, (1,\, 0),\, (0,\, 2)$ with density $\rho(x,\, y) = 1 + 3x + y$

$$
\begin{aligned}
m = \iint_D \rho(x,\, y) dA &= \int_0^1 \int_0^{2-2x} (1+3x+y) dy dx \\
&= \int_0^1 \left(y + 3xy + \frac{1}{2}y^2 \right)_{y=0}^{y=2-2x} dx \\
&= \int_0^1 \left(2 - 2x + 3x(2-2x) + \frac{1}{2}(2-2x)^2 \right) dx \\
&= \int_0^1 \left(4 - 4x^2\right) dx \\
&= \left(4x - \frac{4}{3}x^3 \right) = 4 - \frac{4}{3} \\
&= \frac{8}{3}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{x} &= \frac{1}{m} \iint_D x(1+3x+y) dA \\
&= \frac{1}{m} \int_0^1 \int_0^{2-2x} x(1+3x+y) dy dx \\
&= \frac{1}{m} \int_0^1 x \int_0^{2-2x} 1 + 3x + y dy dx \\
&= \frac{1}{m} \int_0^1 x \left(2 - 2x + 3x(2-2x) + \frac{1}{2}(2-2x)^2 \right) dx \\
&= \frac{1}{m} \int_0^1 x \left(4 - 4x^2\right) dx = \frac{1}{m} \int_0^1 4x - 4x^3 dx \\
&= \frac{1}{m} \left(2x^2 - x^4\right)_0^1 = \frac{1}{m}(2-1) = \frac{1}{m} = \frac{3}{8}
\end{aligned}
$$

What the $\overline{x}$ means: if I balanced it among $x = \frac{3}{8}$, it won't fall

$$
\overline{y} = \frac{1}{m} \int_D y(1+3x+y) dA
$$

No shortcuts. Do that exactly. The answer comes out to be $y = \frac{11}{16}$.

25.1.2 Example 2

Density on a semicircular lamina in the region $y \geq 0$ is proportional to the distance to the center. The radius of the circle is a. And $\rho(x, y) = k\sqrt{(x-0)^2 + (y-0)^2} = k\sqrt{x^2 + y^2}$. Its mass is, using polar coordinates. Note that $\sqrt{x^2 + y^2} = r$ as $x^2 + y^2 = r^2$, and this works at any time.

$$\begin{aligned} 0 \leq r \leq a \\ 0 \leq \theta \leq \pi \end{aligned}$$

$$\begin{aligned} &\int_0^a \int_0^\pi kr \cdot r \, dr \, d\theta \\ &= k \int_0^\pi \int_0^a r^2 dr d\theta \\ &= k \int_0^\pi \left(\frac{1}{3} r^3 \right)_0^a d\theta \\ &= \frac{k}{3} a^3 \int_0^\pi d\theta = \frac{k\pi a^3}{3} \end{aligned}$$

Note that $\bar{x} = 0$ as the function is symmetric

For $\bar{y}$:

$$\begin{aligned} \bar{y} &= \frac{1}{m} \iint_D y\rho(x, y) dA \\ &= \cdots \end{aligned}$$

I'm just extremely error prone when solving integrals.

(Term test 3 content cuts off here)

26 Surface Area

Surface area over a rectangle: Instead of trying to calculate a volume under a surface, we try to calculate its surface area for the function $f(x, y)$ over the rectangle R.

We break up the region into multiple parts, find an approximation for the surface area for each of the small regions, and if we add all of them up, we get the area of the whole thing. If we take a limit, the formula will pop out.

1. Split R into small sub-rectangles R_{ij}
2. Approximate surface area of $f(x, y)$ in R_{ij}
3. Add them up
4. Take a limit

Use a tangent plane, make a parallelogram, and the area will be close to the curved region that we have.

$$\vec{a} = \begin{bmatrix} \Delta x \\ 0 \\ f_x(x_{ij}, y_{ij})\,\Delta x \end{bmatrix}$$
$$\vec{b} = \begin{bmatrix} 0 \\ \Delta y \\ f_y(x_{ij}, y_{ij})\,\Delta y \end{bmatrix}$$

To calculate the area of the parallelogram, use the cross product: $\left|\vec{a} \times \vec{b}\right|$. This turns into:

$$\begin{aligned}
\vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \Delta x & 0 & f_x \Delta x \\ 0 & \Delta y & f_y \Delta y \end{vmatrix} \\
&= \begin{bmatrix} -f_x \Delta x \Delta y \\ -f_y \Delta x \Delta y \\ \Delta x \Delta y \end{bmatrix} \\
\left|\vec{a} \times \vec{b}\right| &= \sqrt{(f_x \Delta x \Delta y)^2 + (f_y \Delta x \Delta y)^2 + (\Delta x \Delta y)^2} \\
&= \Delta x \Delta y \sqrt{f_x^2 + f_y^2 + 1}
\end{aligned}$$

Then, the total area is approximately

$$\sum_{j=1}^{m} \sum_{i=1}^{n} \Delta x \Delta y \sqrt{f_x^2 + f_y^2 + 1}$$

The limits as $M, N \to \infty$:

$$SA = \iint_R \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2} dA$$

This means if the rectangle is $R = [a, b] \times [c, d]$, then the surface area of the graph of $f(x, y)$ over the rectangle R is:

$$\iint_R \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2} dA$$

Which I could reinterpret it as, for $f(\vec{x}) : \mathbb{R}^n \to \mathbb{R}$

$$\iint_R \sqrt{1 + \sum_{i=1}^{n} f_i(\vec{x})^2} dA$$

Another way to write this is:

$$\iint_R \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} dA$$

Note how this is very similar to the arc length formula:

$$\text{Arc length} = \int_a^b \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2}$$

For a parametric curve $\vec{r}(t)\colon \mathbb{R} \to \mathbb{R}^n$ and $\alpha \leq t \leq \beta$, then

$$\text{Arc length} = \int_\alpha^\beta \sqrt{\sum_{i=1}^{n}\left(\frac{\partial r_i}{\partial t}\right)^2} dt$$

26.0.1 Examples

Calculate the surface area of the surface given by $z = x^2 + 2y$ that lies above the triangular region T with verticies $(0,\ 0)$, $(1,\ 0)$, and $(1,\ 1)$.

$$\int_0^1 \int_0^x \sqrt{1+(2x)^2+(2)^2} dydx$$
$$\int_0^1 \int_0^x \sqrt{4x^2+5} dydx$$

As a reminder of how to solve this integral:

$$\int_0^1 y \cdot \sqrt{5+4x^2} dx \mid_{y=0}^{x}$$
$$= \int_0^1 x\sqrt{5+4x^2} dx$$

Use substitution: $u = x^2$: $du = 2xdx$, $dx = \frac{du}{2x}$

$$\frac{1}{2}\int \sqrt{5+4u}du$$
$$=\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{2}{3}[5+4u]^{\frac{3}{2}}\ |_0^1$$
$$=\frac{1}{12}\left(27-5\sqrt{5}\right)$$

This requires a bit of creativity with the reverse chain rule.

EXAMPLE 2

SA of paraboloid $z=x^2+y^2$ under the plane $z=9$.

Our domain is $\left\{(x,\ y)\ : x^2+y^2\leq 9\right\}$.

Wait, we can solve that without flipping anything? We can integrate from that.

$$\iint_D \sqrt{1+(f_x)^2+(f_y)^2}dA$$
$$=\iint_D \sqrt{1+4x^2+4y^2}dA$$

Swap to polar coordinates:

$$=\int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\cdot r\cdot drd\theta$$

Getting to this step is the hardest part. Use substitution to solve this. The answer is $\frac{\pi}{6}\cdot 37^{\frac{3}{2}}-1$.

27 Triple Integrals

I can't visualize 4D.

Suppose we want to integrate $f(x,\ y,\ z)$ over domain $E\in\mathbb{R}^3$ (not using D again). E is a 3D region, and it is hard to visualize, so use a heat map to visualize it. When you add up all the "heat" over the entire domain, you get to know how much energy is in that blob.

$$\iiint_E f(x, y, z)dV$$

The theory goes like this:

Define for boxes $B = \{(x, y, z) \ : a \leq x \leq b,\ c \leq y \leq d,\ g \leq z \leq h\}$. Split up the boxes, add them up, take a limit. You'll eventually come out to this integral:

$$\iiint_B f(x, y, z)dV = \int_g^h \int_c^d \int_a^b f(x, y, z)dxdydz$$

All orderings are good, and there are six of them. With Fubini's theorem, you can integrate in ANY ORDER.

If $f(x, y, z) = 1$, then $\iiint_E 1dV =$ volume of the region.

EXAMPLE

$$\iiint_B xyz^2 dV,\ B = \{(x, y, z) \ : x \in [0, 1],\ y \in [-1, 2],\ z \in [0, 3]\}$$

The integral becomes

$$\begin{aligned}
&\int_0^3 \int_{-1}^2 \int_0^1 xyz^2 dxdydz \\
&= \int_0^3 z^2 \int_{-1}^2 y \int_0^1 xdxdydz \\
&= \int_0^3 z^2 \int_{-1}^2 y \cdot \frac{1}{2} dydz \\
&= \int_0^3 z^2 \cdot \frac{3}{4} dz \\
&= \frac{1}{4} z^3 \mid_0^3 \\
&= \frac{27}{4}
\end{aligned}$$

27.1 Triple Integrals for General Regions

They all involve, and if D was a cookie-cutter that looks like a cylinder. If it forms a cylinder, I should be able to shoot a bullet through the top of it, and the bullet may only enter and exit the region at most once.

- Type I: $\{(x, y, z) : (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$
  - The cylinder goes from bottom to top
  - $\iint_D \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dzdxdy$
- Type II: $\{(x, y, z) : (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$
  - The cylinder faces towards us
  - $\iint_D \int_{u_1(y, z)}^{u_2(y, z)} f(x, y, z) dxdydz$
- Type III: $\{(x, y, z) : (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\}$
  - The cylinder goes from the left to the right
  - $\iint_D \int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z) dydxdz$

EXAMPLE

$$\iiint_E z dV$$

And E is a tetrahedron bounded by planes

$$x = 0,\ y = 0,\ z = 0,\ x + y + z = 1$$

This is a type-1 region, so:

$$\begin{aligned}
&\underbrace{0 \leq z \leq 1 - x - y}_{\text{isolate } z} \\
&0 \leq y \leq 1 - x \\
&0 \leq x \leq 1
\end{aligned}$$

So, our integral is:

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} z dz dy dx$$

Solving this:

$$\iint_D \frac{1}{2}(1-x-y)^2 dx dy = \frac{1}{24}$$

And solve the double integral.

EXAMPLE 2

$$\iiint_E \sqrt{x^2+z^2} dV$$

The E region bounded by $y = x^2 + z^2$ and plane $y = 4$

You will have to draw this out. This is a paraboloid, and this is a type-III region. This means I should integrate with respect to the y-variable first.

$$x^2 + z^2 \leq y \leq 4$$

Along the x-z plane:

$$x^2 + z^2 = 4$$

So $D: x^2 + z^2 = 4$. The integral we must do is:

$$\iint_D \int_{x^2+z^2}^4 \sqrt{x^2+z^2} dy dx dz$$
$$= \iint_D 4\sqrt{x^2+z^2} - \left(x^2+z^2\right)^{\frac{3}{2}} dx dz = \frac{128\pi}{15}$$

Use polars to continue. Convert one variable to $r\cos(\theta)$ and the other to $r\sin(\theta)$ – it doesn't matter which one.

27.2 Applications of Triple Integrals

Center of mass: the same as double integrals. The direct center of mass is $(\bar{x}, \bar{y}, \bar{z})$, with:

$$\bar{x} = \frac{\iiint_E x\rho(x, y, z)dV}{\iiint_E \rho(x, y, z)dV}$$

28 Cylindrical Coordinates

Sometimes, regions are better expressed in other coordinates.

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \\ z &= z \end{aligned}$$

If your cylinder is $y^2 + z^2 = 1$, then it is a cylinder centered around the x-axis. Choose your variables correctly:

$$\begin{aligned} x &= x \\ y &= r\cos(\theta) \\ z &= r\sin(\theta) \end{aligned}$$

Choose which variables to change to polar coordinates.

If a region $E = \{(x, y, z) : (x, y) \in D, z \in [u_1(x, y), u_2(x, y)]\}$ where D is a polar region:

$$\int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos(\theta), r\sin(\theta))}^{u_2(r\cos(\theta), r\sin(\theta))} f(r\cos(\theta), r\sin(\theta), z)\, dzr \cdot drd\theta$$

29 Spherical Coordinates

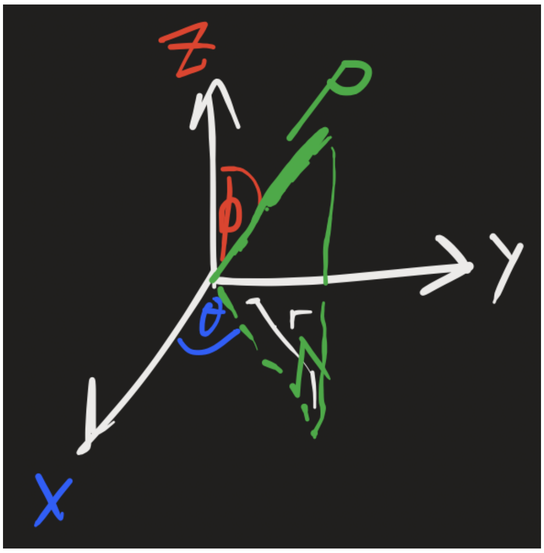


Figure 6: Spherical Coordinates picture

- ρ: "radius" – distance from $(0, 0, 0)$.
  - $\rho > 0$
- θ: angle with x-axis
  - $\theta \in [0, 2\pi]$
- ϕ: angle with z-axis
  - $\phi \in [0, \pi]$. Note that if $\phi = \frac{\pi}{2}$, then my point lies in the xy-plane.

We get these identities:

$$x = \rho \sin(\phi) \cos(\theta)$$
$$y = \rho \sin(\phi) \sin(\theta)$$
$$z = \rho \cos(\phi)$$

For $x^2 + y^2 + z^2$, this gives us the norm squared so this equation makes sense:

$$x^2+y^2+z^2=\rho^2$$

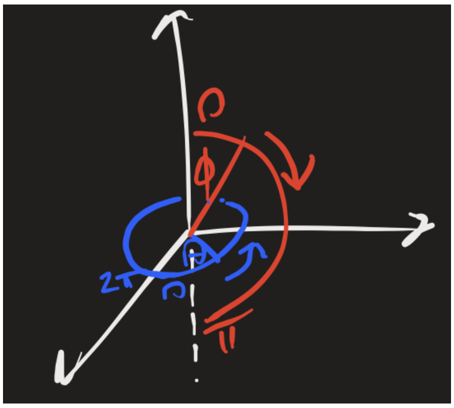


Figure 7: Direction of angles

29.1 Spherical to Cartesian

EX: Find the cartesian coordinates if $\left(\underset{\rho\uparrow}{2}, \underset{\theta\uparrow}{\frac{\pi}{4}}, \underset{\phi\uparrow}{\frac{\pi}{3}}\right)$ is the spherical coordinates.

$$x=\rho\sin(\phi)\cos(\theta)=2\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)=\sqrt{\frac{3}{2}}$$
$$y=\rho\sin(\phi)\sin(\theta)=2\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)=\sqrt{\frac{3}{2}}$$
$$z=\rho\cos(\phi)=2\cos\left(\frac{\pi}{3}\right)=1$$

29.2 Rectangular to Spherical

EX: $(0,\,2\sqrt{3},\,-2)$ to spherical:

$$\rho^2=0^2+\left(2\sqrt{3}\right)^2+(-2)^2-16\Rightarrow\rho=4$$

Find ϕ:

$$
\begin{aligned}
z &= \rho\cos(\phi) \\
-2 &= 4\cos(\phi) \\
-\frac{1}{2} &= \cos(\phi) \Rightarrow \phi = \frac{2\pi}{3}
\end{aligned}
$$

Find θ:

$$
\begin{aligned}
x &= \rho\sin(\phi)\cos(\theta) \\
0 &= 4\cdot 4\sin\left(\frac{2\pi}{3}\right)\cdot\cos(\theta) \\
\theta &= \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{one of them}
\end{aligned}
$$

Use the y to disambiguate θ (it would've been smarter to do this first to prevent ambiguity):

$$
\begin{aligned}
-2 &= 4\sin\left(\frac{2\pi}{3}\right)\sin(\theta) \\
\frac{2\sqrt{3}}{4\cdot\frac{\sqrt{3}}{2}} &= \sin(\theta) \\
\sin(\theta) = 1 &\Rightarrow \theta = \frac{\pi}{2}
\end{aligned}
$$

So our spherical coordinates are:

$$
\left(4, \frac{\pi}{2}, \frac{2\pi}{3}\right)
$$

29.3 Integrating Spherical Coordinates

Use change of variables to spherical coordinates.

$$\iiint_E f(x,y,z)dV$$
$$= \int_0^{2\pi\leftarrow\theta} \int_0^{\pi\leftarrow\phi} \int_0^{\rho} f\begin{pmatrix}\rho\sin(\phi)\cos(\theta)\\ \rho\sin(\phi)\sin(\theta)\\ \rho\cos(\phi)\end{pmatrix} \rho^2\sin(\phi)d\rho d\phi d\theta$$

$\rho^2\sin(\phi)$ is called the determinant of the Jacobian when performing change of variables. LOOK AT IT CAREFULLY

29.4 For more general spherical regions

$$E = \{(\rho,\theta,\phi) : \theta\in[\alpha,\beta], c\in[c,d], \rho\in[g_1(\theta,\phi), g_2(\theta,\phi)]\}$$

You may have to change the bounds of integration depending on what question you get. For instance, if you are asked to integrate on the region $\{(x,y,z) : y\geq 0, x^2+y^2+z^2\leq 9\}$, your bounds of integration for θ is from 0 to π.

IF THE RAIDUS ENDS UP BEING A FUNCTION OF ϕ and θ, then $0\leq\rho\leq sf(\phi,\theta)$, where sf is your function. This is usually the case when your sphere isn't on the origin. You can find ρ by solving a system of equation with the surfaces that bound the region. The way spherical coordinates are set up is that if you square all of x, y, z and add them up, you get ρ^2.

For example:

$$\begin{aligned} x^2+y^2+z^2 &= 2z\\ \rho^2 &= 2\rho\cos(\phi)\\ \rho &= 2\cos(\phi)\end{aligned}$$

Now you have these bounds, you can integrate right away.

30 Change of Variables

The single-variable calculus analogue of it is called integration using substitution. Typically, when you write up the substitution rule, we write $x = g(u), dx = g^{'}(u)$ instead of $u = g^{-1}(x), du = \left(g^{-1}\right)^{'}(x)dx$. So:

$$\int_a^b f(x)dx = \int_d^c f(g(u))\, g^{'}(u)du$$

Why do we use the substitution rule?

- Because we can't integrate the function otherwise
- Use substitution to change the function into something you can integrate
- Used to help with finding the antiderivative

However, change of variables, abbreviated COV, is used to change the domain (if it is awkward to work with). We've already seen examples of this:

- Polar coordinates: $\iint_R f(x, y)dA = \iint_S f(r\cos(\theta),\ r\sin(\theta))\, r\, drd\theta$ with the variables $\begin{matrix} x = r\cos(\theta) \\ y = r\sin(\theta) \end{matrix}$
- The r you see above is the analogue of the derivative ($g^{'}(u)$) you see in the substitution rule
- The region changes $R \to S$
- This gets you from working with something regarding a circle to something regarding a box or rectangle.

30.1 Change of Variables in the General Case

$T : \mathbb{R}^2 \to \mathbb{R}^2$ is the transformation from the $u - v$ coordinate plane to the $x - y$ coordinate plane. This means:

$$x = g(u, v)$$
$$y = h(u, v)$$

We do need some conditions on T for T to work with change of variables:

- T is C^1
- T is one-to-one
- The image of S: $T(S)$ is $T(S) = \{(x, y) : (u, v) \in S \text{ where } T(u, v) = (x, y)\}$

For instance, we have a transformation $T(u, v) = (x, y)$. x is given as $u^2 - v^2$ and $y = 2uv$. Find the image of $S = \{(u, v) : 0 \leq u \leq 1, 0 \leq v \leq 1\}$

You could simply model the edges of S and transform them. You'll get a parametric equation out of it.

$$\iint_R f(x, y)dxdy = \iint_S f(u, v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv$$

We're going to abbreviate $\left| \frac{\partial(x, y)}{\partial(u, v)} \right|$ as the Jacobian: $|J|$. Note that

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right|$$

This is the Jacobian of the transformation $(x, y) = T(u, v)$. The Jacobian matrix is the matrix, but the Jacobian may be referred to as the determinant of the matrix. The Jacobian matrix of T is just $\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$.

I want the **absolute value of the determinant of the Jacobian matrix as I don't want to negate things when I mirror things.**

30.2 The Change of Variables Theorem

Suppose T is a C^1 transformation $(x, y) = T(u, v)$ whose Jacobian is non-zero (transformation is non-degenerate) and T maps the region S in the uv plane to R in the xy plane.

Suppose $f(x, y)$ is continuous on R. Then

$$\iint_R f(x, y)dxdy = \iint_S f(u, v)\left|\frac{\partial(x, y)}{\partial(u, v)}\right|dudv$$

$$\left|\frac{\downarrow \partial(x, y)}{\rightarrow \partial(u, v)}\right| = \left|\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\right|$$

For example, polar coordinates

$$x = r\cos(\theta)$$

$$y = r\cos(\theta)$$

$$\begin{aligned} |J| &= \left|\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}\right| = \left|\begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix}\right| = \left|r\cos^2(\theta) + r\sin^2(\theta)\right| \\ &= \left|r\left(\cos^2(\theta) + \sin^2(\theta)\right)\right| = |r| = r \end{aligned}$$

As $r \geq 0$

30.3 An Example With Change of Variables

$\iint_R ydA$ where R is the region above the x-axis bounded by parabolas

$$y^2 = 4 - x$$
$$y^2 = 4 + x$$

Using the transformation $x = u^2 - v^2$, $y = 2uv$ ($T(u,\ v) = \begin{bmatrix} u^2 - v^2 \\ 2uv \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$).

Note: the bounds of the integral just happen to coincidentally be $[0,\ 1] \times [0,\ 1]$ in this question. Think of the Jacobian as a gradient of an $\mathbb{R}^2 \to \mathbb{R}^2$ transformation – though determinants are transpose invariant.

$$\begin{aligned}
&\Rightarrow \int_0^1 \int_0^1 2uv \left\| \begin{matrix} 2u & -2v \\ 2v & 2u \end{matrix} \right\| dudv \\
&= \int_0^1 \int_0^1 2uv \left(4\left(u^2 + v^2\right)\right) dudv \\
&= \dots \\
&= 2
\end{aligned}$$

30.4 Two Families of Curves

Pattern matching and substituting

Another type of problem is if we are given a different bound:

$$\iint_R y^2 dxdy$$

Where R is bounded by curves $xy = 1$, $xy = 2$, $xy^2 = 1$, $xy^2 = 2$

The region I end up drawing is sandwiched between $y = \frac{1}{x}$ (bottom) and $y = \frac{2}{x}$ (top), and $y = \frac{1}{\sqrt{x}}$ and $y = \frac{\sqrt{2}}{\sqrt{x}}$. Not a great idea to stick with xy coordinates. However, you can see what I want as my transformation:

$$u = xy$$
$$v = xy^2$$

Then $1 \leq u \leq 2,\ 1 \leq v \leq 2$. The problem is that u and v is a function of xy, so I have to invert it.

$$y = \frac{v}{u}$$
$$x = \frac{u^2}{v}$$

And there is my transformation, and I have my bounds.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{2u}{v} & -\frac{u^2}{v^2} \\ -\frac{v}{u^2} & \frac{1}{u} \end{vmatrix} = 2\left(\frac{u}{v}\right)\left(\frac{1}{u}\right) - \left(-\frac{u^2}{v^2}\right)\left(-\frac{v}{u^2}\right) = \frac{1}{v}$$

If you see the same difficult expression be used multiple times (precisely twice), then you could reassign them by using change of variables.

30.5 Change of Variables For 3D

You'll need:

$$x(u,\ v,\ w)$$
$$y(u,\ v,\ w)$$
$$z(u,\ v,\ w)$$

The 3D integral is

$$\begin{aligned}&\iiint_E f(x, y, z)dxdydz \\ &= \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x, y, z)}{\partial(u,v,w)}\right| dudvdw\end{aligned}$$

Where the Jacobian is:

$$\left|\frac{\partial(x, y, z)}{\partial(u,v,w)}\right| = \left\|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{matrix}\right\|$$

For spherical coordinates, you had:

$$\begin{aligned}x &= \rho\sin(\phi)\cos(\theta) \\ y &= \rho\sin(\phi)\sin(\theta) \\ z &= \rho\cos(\phi)\end{aligned}$$

Taking the Jacobian: $\left|\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)}\right| = \rho^2\sin(\phi)$, so in spherical coordinates, you get:

$$\iiint_S f(\cdots)\rho^2\sin(\phi)d\rho d\phi d\theta$$

30.6 Inverting The Determinant

$$\left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \frac{1}{\left|\frac{\partial(u, v)}{\partial(x, y)}\right|}$$

Might not be a great time saver.

30.7 Transformations Showing In The Function

Integrate this:

$$\iint_R e^{\frac{x+y}{x-y}} dA$$

Where R is the trapezoidal region with vertices $(1,\,0),\,(2,\,0),\,(0,\,-1),\,(0,\,-2)$

If I were to transform:

$$u = x + y$$
$$v = x - y$$

Then our integral transforms into:

$$\iint_S e^{\frac{u}{v}} |J| du dv$$

Because we're working with a trapezoidal region, plug in the points to get the trapezoid in the U-V space. If the region isn't that nice, you may have to do something more complicated.

$$T^{-1}((1,\,0)) = (1,\,1)$$
$$T^{-1}((2,\,0)) = (2,\,2)$$
$$T^{-1}((0,\,-2)) = (-2,\,2)$$
$$T^{-1}((0,\,-1)) = (-1,\,1)$$

We get a type-2 region. Sketch it out to determine your bounds. To figure out the stretching factor:

$$x = \frac{u+v}{2} \quad y = \frac{u-v}{2}$$

The Jacobian is:

$$|J| = \left\| \begin{matrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{matrix} \right\| = \left| -\frac{1}{2} \right| = \frac{1}{2}$$

Our integral is:

$$\begin{aligned}
&\int_1^2 \int_{-v}^{v} \frac{e^{\frac{u}{v}}}{2} du dv \\
&= \frac{1}{2} \int_1^2 \left[v e^{\frac{u}{v}} \right]_{-v}^{v} dv \\
&= \frac{1}{2} \int_1^2 v \left(e - \frac{1}{e} \right) dv \\
&= \frac{1}{2} \left(e - \frac{1}{e} \right) \int_1^2 v dv \\
&= \frac{\frac{1}{2} \left(e - \frac{1}{e} \right) 1}{2} \left[v^2 \right]_1^2 \\
&= \frac{3}{4} \left(e - \frac{1}{e} \right)
\end{aligned}$$

30.8 Integrating Over Ellipses

To find:

$$\iint_R \sin\left(9x^2 + 4y^2\right) dA$$

Where R is the region bounded by the ellipse $9x^2 + 4y^2 = 1$ in the first quadrant ($\frac{1}{3}$ radius horizontally, $\frac{1}{2}$ radius vertically)

1. Transform so the ellipse turns into a circle
2. Then to a square

We want to put the integral into the form of a circle: $u^2 + v^2 = 1$. What's the transformation?

$$u^2 = 9x^2 \Rightarrow u = 3x$$
$$v^2 = 4y^2 \Rightarrow v = 2y$$

That will get me my transformation to my U-V space. We get this region:

$$\iint_{u^2+v^2\leq 1} \sin\left(u^2+v^2\right)|J|dudv$$
$$|J| = \frac{1}{6}$$

We'll have to do another polar coordinate transformation again.

You could try to incorporate all of this in one go:

$$u = 3r\cos(\theta) \quad \text{as } u = 3x \text{ and } x = r\cos(\theta)$$
$$v = 2r\sin(\theta) \quad \text{as } v = 2y \text{ and } y = r\sin(\theta)$$

But I wouldn't suggest it.

31 Vector Calculus

Vector functions: $F : \mathbb{R}^2 \to \mathbb{R}^2$. $\vec{F}(x, y) = \begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix}$. It's really about the input being a vector as well, so we may use $\vec{F}(\vec{x})$ instead, where $\vec{F}(\vec{x}) = \begin{bmatrix} P(\vec{x}) \\ Q(\vec{x}) \end{bmatrix}$. We may also use the $\widehat{i}, \widehat{j}$ notation: $\vec{F}(\vec{x}) = P(\vec{x})\widehat{i} + Q(\vec{x})\widehat{j}$.

31.1 Applications To Gravitational Force

Newton's law of gravitation states that:

$$\vec{F}(\vec{x}) = -\frac{GMm}{||\vec{x}||^3}\vec{x}$$

A vector field is conservative if $\exists f$ such that

$$\vec{F}(,\ y) = \nabla f(x,\ y)$$

For instance:

$$\begin{aligned}
\vec{F}(x,\ y,\ z) &= -\frac{GMm}{(x^2+y^2+z^2)^{\frac{3}{2}}} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\
&= \nabla f(x,\ y,\ z) = \nabla\left(\frac{GMm}{\sqrt{x^2+y^2+z^2}}\right) \\
&= \begin{bmatrix} -\frac{mMG}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}x \\ -\frac{mMG}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}y \\ -\frac{mMG}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}z \end{bmatrix} = -\frac{GMm}{(x^2+y^2+z^2)^{\frac{3}{2}}} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\end{aligned}$$

So we do have a potential for gravitational force. The gravitational force respects the conservation of energy.

32 Line Integrals (Path Integrals)

There are two types of line integrals:

32.1 Type 1

Integrating scalar functions over curves (back to the start). The arc length is an integral like this.

Suppose we have some scalar function $f : \mathbb{R}^2 \to \mathbb{R}$ and we have a path r: $\mathbb{R} \to \mathbb{R}^2$. Then, I want to integrate

$$\int_S f(x, y)ds$$

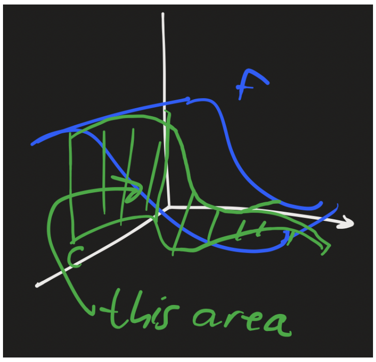


Figure 8: This integral finds the area of the shaded green. Note that green represents the path r

Recall the arc length formula: If C is a parameterized curve $\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, then the length of C is, where $t \in [a, b]$:

$$\int_C ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

The ds notation is just $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$

If I were to just integrate a scalar function, I would get the area of that sheet. The integral

$$\int_C f(x, y)ds = \int_a^b f(x(t), y(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

If you have a curve, subdivide it, and estimate the length of each subdivided line (linearize the problem), and do the whole Riemann sum construction, you can approximate the length of a curve using the length of each subdivided lines. Add more points and make your approximation better, you will end up converging into the integral.

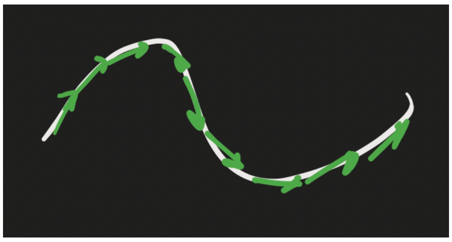


Figure 9: Subdividing a curve into many line segments

$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ parameterizes C. So:

$$\int_C f(x, y)ds = \int_a^b f(\vec{r}(t)) \underbrace{\left\|\vec{r}^{\,\prime}(t)\right\| dt}_{ds}$$

It **does not matter** if I flip the order of parameterization (make the parameterization traverse in the opposite direction). This only matters for this type of integrating.

32.1.1 Example

Evaluate $\int_C (2 + x^2 y)\, ds$ where C is the upper half of the unit circle from $(1, 0) \to (-1, 0)$.

We can parameterize this curve:

$$\vec{r}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}, t \in [0, \pi]$$

$$\vec{r}^{\,\prime}(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$$

So I can transform the integral (**substitute** x and y):

$$\int_0^\pi \left(2+\cos^2(t)\sin(t)\right)\cdot\left|\left|\vec{r}'(t)\right|\right|dt$$
$$=\int_0^\pi \left(2+x^2y\right)\cdot\sqrt{(-\sin(t))^2+(\cos(t))^2}dt$$

If you have multiple curves and you want the area below all of them, just add them all.

When finding the center of mass:

$$\bar{x}=\frac{\int_a^b x\rho(x,\,y)ds}{\int_a^b \rho(x,\,y)ds}$$

And sub $x=r[0]$, $y=r[1]$ (vector indexing)

32.2 Type 2

Line integral of a scalar function with respect to dx and dy

$$\int_C f(x,\,y)dx \quad \int_C f(x,\,y)dy$$

For example, if our curve is still parameterized by $\vec{r}(t)=\begin{bmatrix}x(t)\\y(t)\end{bmatrix}$, then do these substitutions whenever you see it:

$$\begin{aligned} x&=x(t) & y&=y(t)\\ dx&=x'(t)dt & dy&=y'(t)dt \end{aligned}$$

This means:

$$\int_C f(x,y)dx = \int_a^b f(x(t),y(t))x'(t)dt$$
$$\int_C f(x,y)dy = \int_a^b f(x(t),y(t))y'(t)dt$$

Projection of the shape onto the x-direction (and if it gets projected twice because I'm squishing a zigzagged piece of paper, count all of them)

Applies to type 2 only.
Orientation (direction) matters!! Only impacts if your answer is negative or not.
If $-C$ denotes the same curve but opposite direction, then the integral $\int_C f(x,y)dx = -\int_{-C} f(x,y)dx$
(On the other hand, $\int_C f(x,y)ds = \int_{-C} f(x,y)ds$ for type 1s)

A formula for line segments (percentage of progression):

$$\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1 \quad 0 \le t \le 1$$

Where $\vec{r}_0$ is the starting point and $\vec{r}_1$ is the ending point.
For curves with starting points and ending points:

- Parameterize the curve
- Focus on one dimension (x or y): find t such that $x(t) = x_0$ for the lower bound and find t such that $x(t) = x_1$ for the upper bound. You can swap x with y if you need to.

Coordinate transformations are done for regular integrals, while parameterization is done for line integrals.

32.3 Line Integrals Over a Vector Field

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}(t) dt$$

This can be seen as the work done moving something along a vector field (it can be negative if you're experiencing a tailwind). The direction of C does matter, only to a negative sign, so if your path is reversed, multiply the entire integral by -1. If you want to integrate over a circle clockwise, that's what you're going to have to do.

HINT:

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Works only when $\vec{F}$ is conservative, then $\nabla f = \vec{F}$

32.4 2D Parameterization

An $\mathbb{R}^2 \to \mathbb{R}^3$ transformation. Plotting an area in the domain gives us a surface in $\mathbb{R}^3$.

If we look at 1D parameterization, the way we set up the parameterization from $y = g(x)$ from a to b and we say that $\alpha(t) = \begin{bmatrix} t \\ g(t) \end{bmatrix}$ by saying that $t = x$. As if we lifed up a line segment. The arc length is $\int_a^b 1 \cdot \left| a'(t) \right| dt$

Figure 10: Bring up a line segment

Using this analogy, in three dimensions:

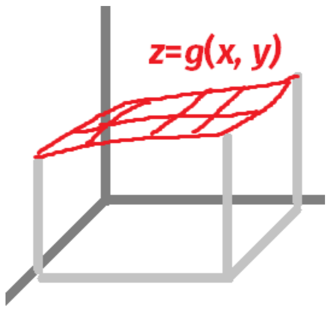


Figure 11: Lifting a surface in 3 dimensions

Where $\alpha(u, v) = \begin{bmatrix} u \\ v \\ g(u, v) \end{bmatrix}$ if your surface can be described using $z = g(x, y)$. The surface area of this lifted surface is if the domain is A:

$$SA = \int_S 1 = \int_A 1 \cdot a \left| \frac{\partial \alpha}{\partial u} \times \frac{\partial \alpha}{\partial v} \right| dA$$

Usually, $\left| \frac{\partial \alpha}{\partial u} \times \frac{\partial \alpha}{\partial v} \right| = \sqrt{\left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2 + 1}$

This turns into a double integral, and you will be integrating with $dudv$ or $dvdu$, whichever is easier.

33 Conservative Vector Fields

If C is piecewise smooth, and $\vec{F}$ is conservative, then $f(\vec{r}(b)) - f(\vec{r}(a))$ where C is given by the part $\vec{r}(t),\ a \le t \le b$, and $D_f = \vec{F}$.

33.1 Independence of Path

If $\vec{F}$ is **conservative**, and we have C_1, C_2, being paths with the same initial and terminal points, then

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

If I have a potential function, I can use this theorem. If I know that a function is conservative, then I can use this. If the function is NOT conservative, I cannot use this.

This has a nice consequence:

If C is a closed path (terminal and initial points are the same – a LOOP), and $\vec{F}$ is conservative, then there is a nice formula for the line integral:

It's zero

$$\int_C \vec{F} \cdot d\vec{r} = 0$$

Because the work done after all is 0

Why is this true? I could break up the paths: $A \to B \to A$ could be split into $A \to B$ and $B \to A$. Regardless of how the paths are set up, $\begin{aligned} &\int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} \\ &= \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_1} \vec{F} \cdot d\vec{r} \\ &= 0 = \int_C \vec{F} \cdot d\vec{r} \end{aligned}$

This means that C_1 and $-C_2$ have the same initial and termina paths.

33.2 Conservative Inspection

How do I know that a vector field is conservative? How do I know if/when $\vec{F}$ is conservative? There are two theorems that relate a vector field to the partials of its component. Give me any $\vec{F}(x, y) = \begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix}$. If I want to check that it is conservative, I want to make use of their partials.

Theorem 33.1. *If $\vec{F}$ is conservative, and P, Q have continuous partials, then*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

The converse has additional conditions to hold. See the theorem below.

Proof. We find f so that $\nabla f = \vec{F}$.

This means that $\frac{\partial f}{\partial x} = P$ and $\frac{\partial f}{\partial y} = Q$.

Then $\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial Q}{\partial x}$, using Clairaut's theorem. $\blacksquare$

Theorem 33.2. *If $\vec{F}$*

- *is some vector field*
- *P, Q have continuous partials (I usually don't need to check). Note that $P = \vec{F}[0]$ and $Q = \vec{F}[1]$*
- *$\vec{F}$ is defined on an **open, simply connected domain, and** $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ (the last one is the one you'll usually check)*

THEN $\vec{F}$ is conservative.

$$\begin{bmatrix} & \blacksquare \\ \blacksquare & \end{bmatrix}$$

What does open, simply connected mean? It's a technical requirement. Really, it means:

Definition 33.1 (Open, simply connected)**.** The domain D is in one piece (not split) and does not have any holes.

Examples:

- $\mathbb{R}^2$

- A blob
- NOT two blobs
- NOT a donut

The takeaway is to check if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Note that $P = \vec{F}[0]$ and $Q = \vec{F}[1]$. I could rewrite this as

$$\frac{\partial \vec{F}[0]}{\partial y} = \frac{\partial \vec{F}[1]}{\partial x}$$

33.3 Examples

Find the potential functions:

How do I find f so that $\nabla f = \vec{F}$?

$$\vec{F}(x, y) = \begin{bmatrix} 3 + 2xy \\ x^2 - 3y^2 \end{bmatrix} = \begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix}$$

33.3.1 First example

Is $\vec{F}$ conservative? If so, find the potential function.

Check if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

$$\frac{\partial P}{\partial y} = 2x$$
$$\frac{\partial Q}{\partial x} = 2x$$

So $\vec{F}$ is conservative. At least, if we assume the other conditions.

The potential function requires f such that $\nabla f = \vec{F}$. This means

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 3+2xy \\ x^2-3y^2 \end{bmatrix}$$

What we want to do is integrate either of the equations with respect to the corresponding variable. Integrating the first function, P, with respect to x:

$$f(x,\, y) = \int \frac{\partial f}{\partial x} dx = \int (3+2xy)dx = 3x + x^2y + g(y)$$

$g(y)$ is some function that depends only on y or a constant. If I want to conclude something about g, then I should take the partial of f with respect to y

$$\begin{aligned} x^2 - 3y &= \frac{\partial f}{\partial y} = x^2 + g^{'}(y) \\ \Rightarrow g^{'}(y) &= -3y^2 \\ \Rightarrow g(y) &= -y^3 \end{aligned}$$

This means that the result is

$$f(x,\, y) = 3x + x^2y - y^3$$

33.3.2 Second Example

Calculate $\int_{C_1} \vec{F} \cdot d\vec{r}$ where C_1 is $\vec{r}(t) = \begin{bmatrix} e^t \sin(t) \\ e^t \cos(t) \end{bmatrix}$, $0 \le t \le \pi$

I don't need to compute these directly. I know that F is conservative, and I have a potential function. By FTC for line integrals, if I want $\int_{C_1} \vec{F} \cdot d\vec{r}$, it becomes

$$f(\vec{r}(\pi)) - f(\vec{r}(0))$$

It doesn't matter what happens between 0 to π as I only care about the endpoints.

$$\vec{r}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \vec{r}(\pi) = \begin{bmatrix} 0 \\ -e^{\pi} \end{bmatrix}$$

This means

$$f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = -1$$
$$f\left(\begin{bmatrix} 0 \\ -e^{\pi} \end{bmatrix}\right) = e^{3\pi}$$
$$\int_{C_1} \vec{F} \cdot d\vec{r} = e^{3\pi} + 1$$

33.3.3 Third Example

Calculate $\int_{C_2} \vec{F} \cdot d\vec{r}$ where C_2 is the ellipse given by $\frac{x^2}{4} + \frac{y^2}{25} = 1$ oriented counterclockwise.

The answer is 0 because the starting and ending points are the same and $\vec{F}$ is a conservative vector field.

34 Green's Theorem

Note that $\oint$ means integral over a closed curve.

FTC says that we only need information on the boundary.

34.1 Green's Theorem

SO I DON'T NEED TO DO MULTIPLE LINE INTEGRALS – A MASSIVE TIME SAVER

Green's theorem states that

If $\vec{F} = \begin{bmatrix} P \\ Q \end{bmatrix}$, then

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_C P dx + Q dy = \int_C \vec{F} \cdot d\vec{r}$$
$$= \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

Where D is the **region** bounded by the curve C, and curve C must be:

- Positively oriented
  - As you walk on the curve, the region is on the left. This applies to all boundaries. This means if you have a donut, the outer boundary must be counterclockwise and the inner boundary must be clockwise.
  - Changing orientation will multiply your answer by a factor of -1
- Curve has to be simple and closed (no loops or self-intersections, meaning no ∞-shaped curves)

$C = \partial D$ means C is the boundary of D.

34.1.1 Example

$$\int_C (x^4 dx + xy dy)$$

Where C is the curve given by the line segments from $(0, 0) \to (1, 0) \to (0, 1) \to (0, 0)$, which looks like a triangle

Figure 12: The region

So:

$$P(x,\, y) = x^4$$
$$Q(x,\, y) = xy$$

We get

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y - 0$$

Green's theorem gives us

$$\begin{aligned}&\int_C \left(x^4 dx + xydy\right)\\ &= \iint_D ydA = \int_0^1 \int_0^{1-x} ydydx\end{aligned}$$

34.1.2 Another Example

$$\oint_C \left(3y - e^{\sin(x)}\right) dx + \left(7x + \sqrt{y^4 + 1}\right) dy$$

Note that the above is in the form of $\oint_C Pdx + Qdy$

Where C is the circle $x^2 + y^2 = 9$ in the counterclockwise direction. Then:

$$\begin{aligned}&= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D (7-3) dA\\ &= 4 \iint_D dA = 4 \cdot \pi \cdot 9 = 36\pi\end{aligned}$$

34.2 My Regions Have Holes In It

In other words, not simply connected. Then make sure that both curves are going in the correct direction.

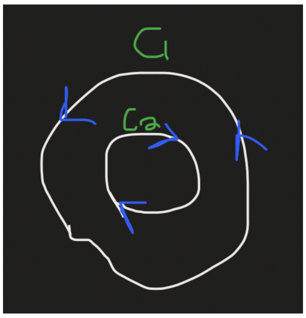


Figure 13: Calculating the line integral of this. D is the region bounded within

Then

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$$
$$= \oint_C Pdx + Qdy$$
$$= \oint_{C_1} (Pdx + Qdy) + \oint_{C_2} (Pdx + Qdy)$$

34.2.1 More Examples?

$$\oint_C (y^2 dx + 3xydy)$$

Where C is the boundary of D and D is the region

Inside circle $x^2 + y^2 = 4$

Outside circle $x^2 + y^2 = 1$

And $y \geq 0$

Instead of doing many line integrals, I could use green's theorem

$$\iint_D 3y - 2ydA$$
$$= \iint_D ydA$$

So, if you are given a line integral on a closed region, maybe turn it into a line integral.