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ECE310 Digital Signal Processing
Problem Set V: Multidimensional Signals & Systems
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1. Consider a medical CT (computerized tomography) scan which results in a set of two-dimensional lateral image "slices" taken along a longitudinal axis. With reference to (x, y, z) coordinates, x, y are the lateral coordinates, and z the longitudinal coordinate. For simplicity assume the spacing between the slices has unit length.

   We will model the resulting 3-D image as a signal $g(\vec{r}, n)$ with coordinate system $\vec{r} \in \mathbb{R}^2$ (the continuous lateral coordinates), and $n \in \mathbb{Z}$ (the discrete longitudinal coordinate). In what follows, do not separate $\vec{r}$ into x, y coordinates, or join it with n to form a 3-D vector: keep $\vec{r}$,n separate. Let the corresponding frequency coordinates be denoted $\left(\vec{k}_{xy}, k_z\right)$, where $\vec{k}_{xy} = (k_x, k_y)$ is kept separate from k_z. Do not explicitly write x, y, k_x, k_y in any of the following,

   (a) Specify the domains of $\vec{k}_{xy}$ and k_z, respectively.

   (b) Write the Fourier and inverse Fourier transforms.

   (c) State Parseval's theorem.

   (d) Write the formulas for convolution in the spatial and wavenumber domains.

   (e) Prove that convolution in the spatial domain leads to multiplication in the wavenumber domain.

   (f) Let $h(\vec{r}, n)$ denote the impulse response of an LTI system applied to such a 3-D object. Write the condition for stability as an explicit mathematical expression (i.e., do not simply write $h \in L^1$).

2. **McClellan Transform Design Method for Multidimensional FIR Filters**

   The problem presents a technique for designing multidimensional FIR filters based on 1-D "prototypes". The approach described here can be used for 2-D, 3-D (or higher!) filters, but we will stay with a 2-D example. The idea is to start with a 1-D FIR filter "prototype", $H(z)$, and make a substitution $z \longleftarrow F(z_1, z_2)$ to obtain a 2-D filter:

   $$G(z_1, z_2) = H(F(z_1, z_2))$$

   The McClellan transform proposes the form of $F(\cdot)$ that we chose. Usually, the target filter G should be zero-phase with real coefficients, so we chose H to be zero-phase with real coefficients. The function F is also zero-phase with real coefficients, and it is usually designed to map "DC" to "DC" and high frequencies (say the points $(\omega_1, \omega_2) = (\pm\pi, \pm\pi)$) to $\omega = \pm\pi$. Thus, for example, if H is lowpass, so is G, if H is bandpass, so is G. In addition, F is often chosen to achieve certain desirable symmetries. For example, if F is nearly isotropic, so is G.

If we start with a zero-phase FIR filter with real coefficients, it can be expressed (in the frequency domain) as a polynomial[1] in $\cos\omega$. Specifically, if $h[\cdot]$ has support $\{-N \le n \le N\}$, $h[n] = h[-n]$, then we can write:

$$\begin{aligned} H(\omega) &= h_0 + 2h_1 \cos\omega + \cdots + 2h_N \cos N\omega \\ &= h_0 + 2\sum_{m=1}^{N} h_m T_m(\cos\omega) \end{aligned}$$

where $T_m(\cdot)$ is the Chebyshev polynomial of degree m. Note that $H(\omega) = H(-\omega)$ is real. With the mapping $\cos\omega \longrightarrow (z + z^{-1})/2$ we get:

$$H(z) = h_0 + 2\sum_{m=1}^{N} h_m T_m\left(\frac{z+z^{-1}}{2}\right) \tag{1}$$

Next, we make a substitution:

$$\cos\omega = F(\omega_1, \omega_2)$$

where $F(\omega_1, \omega_2)$ is a low degree polynomial in $\cos\omega_1, \cos\omega_2, \sin\omega_1, \sin\omega_2$. This results in $F(z_1, z_2)$ being a 2-D FIR filter (i.e., involving relatively low order positive and negative powers of z_1, z_2). For example, $\cos\omega_1 \longrightarrow \frac{z_1+z_1^{-1}}{2}$, and $\sin\omega_1 \longrightarrow \frac{z_1-z_1^{-1}}{2j}$, and similarly for $\cos\omega_2, \sin\omega_2$. Note that symmetries in F will ensure its final form will involve only real coefficients, however. This gives us the following formulas for the 2-D filter G in the frequency and transform domains:

$$\begin{aligned} G(\omega_1, \omega_2) &= h_0 + 2\sum_{m=1}^{N} h_m T_m(F(\omega_1, \omega_2)) \\ G(z_1, z_2) &= h_0 + 2\sum_{m=1}^{N} h_m T_m(F(z_1, z_2)) \end{aligned}$$

The function $F(\omega_1, \omega_2)$ is real and must satisfy:

$$-1 \le F(\omega_1, \omega_2) \le 1 \quad \text{for} \quad (\omega_1, \omega_2) \in [-\pi, \pi]^2$$

Then, under the mapping $\cos\omega = F(\omega_1, \omega_2)$, for each $(\omega_1, \omega_2) \in [-\pi, \pi]^2$, there is a unique $\omega \in [0, \pi]$ to which it corresponds. Let us denote that as $\omega_{(\omega_1,\omega_2)}$. Then the value of the frequency response $G(\omega_1, \omega_2)$ at (ω_1, ω_2) is the value of the frequency response H at $\pm\omega_{(\omega_1,\omega_2)}$.

So here is some help with MATLAB. If x is a symbolic variable, *T=chebyshevT(1:N,x)* will create a vector of symbolic Chebyshev polynomials $T_m(x)$ for $1 \le m \le N$. Also, I have emailed you a MATLAB function *sym2z2d* that will map a symbolic 2D FIR transfer function $H(z_1, z_2)$ into the impulse response matrix $h[n_1, n_2]$; the script *testsym2z2d* gives an example of its use.

Let us take a specific example. Take:

$$F(\omega_1, \omega_2) = \frac{1}{2}(-1 + \cos\omega_1 + \cos\omega_2 + \cos\omega_1 \cdot \cos\omega_2)$$

[1] Since $\cos(m\omega) = T_m(\cos\omega)$ where T_m is the Chebyshev polynomial of degree m, the given formula is indeed a polynomial in $\cos\omega$.

(a) Check the following:

1. F has 8-fold symmetry, in that $F(\pm\omega_1, \pm\omega_2) = F(\pm\omega_2, \pm\omega_1)$ (i.e., replacing either ω_1 or ω_2 by its negative, or swapping them, does not change F). This is what leads to "approximately" isotropic behavior.
2. Either $\omega_1 = \pi$ or $\omega_2 = \pi$ corresponds to $\omega = \pi$.
3. $F(\omega_1, 0) = \cos\omega_1$, which means $G(\omega_1, 0) = H(\omega_1)$, i.e., the "slice" of G along the ω_1 axis is the same as the 1-D filter H, and similarly $G(0, \omega_2) = H(\omega_2)$.

(b) Compute $F(\omega_1, \omega_2)$ on a 128x128 grid over $[-\pi, \pi]^2$, and graph a contour plot of it. Check that its values are bounded between -1 and 1.

(c) Find $F(z_1, z_2)$ as a 2-D polynomial.

(d) If the 1-D filter has support $\{-N \leq n \leq N\}$, what is the support of the 2-D filter designed using this method?

(e) Write a MATLAB function that will do the following: given a zero-phase real 1-D FIR filter $h[n]$ with support $\{-N \leq n \leq N\}$, it will use THIS SPECIFIC $F(\omega_1, \omega_2)$ (don't try to write a general purpose McClellan transform function!) and:

1. Compute and graph the magnitude response of $H(\omega)$ over $[-\pi, \pi]$ (128 points equally spaced is fine). Use a linear (not decibel) scale.
2. Compute and graph the magnitude response of $G(\omega_1, \omega_2)$ over $[-\pi, \pi]^2$. (a 128×128 grid). Also on a linear scale.
3. Use the symbolic toolbox to find $G(z_1, z_2)$, and compute $g[n_1, n_2]$ (represented as a matrix). These two quantities should be returned by the function.

(f) Apply your function to the following. Let h be a Hamming window of length 7, but centered so as to have support $\{-3 \leq n \leq 3\}$. This will serve as a lowpass prototype filter.

(g) Now apply your function to the following bandpass filter: find the impulse response $h_0[n]$ of an ideal bandpass filter with $H_0(\omega) = 1$ if $\frac{\pi}{3} \leq \frac{2\pi}{3}$, and 0 otherwise (it will have the property $h_0[n] = h_0[-n]$); I expect you to use the exact formula (you can look it up); truncate it with your Hamming window function to obtain the length 7 prototype filter $h_{BP}[n]$.