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DSP Problem Set 5

1)

a) $\vec{K_{xy}} \in \mathbb{R}^2$

$K_z \in \mathbb{S}_{2\pi} = \frac{\mathbb{R}}{2\pi\mathbb{Z}}$

b) $G(K_{xy}, K_z) = \sum_{\vec{n}\in\mathbb{Z}} \left[\int_{\mathbb{R}^2} g(\vec{r}, n) e^{-j\vec{K_{xy}}\cdot\vec{r}} d\vec{r} \right] e^{-j\vec{K_z}\cdot\vec{n}}$

$g(\vec{r}, n) = \frac{1}{(2\pi)^3} \int_{\mathbb{S}_{2\pi}} \left[\int_{\mathbb{R}^2} G(K_{xy}, K_z) e^{j\vec{K_{xy}}\cdot\vec{r}} dK_{xy} \right] e^{j\vec{K_z}\cdot\vec{n}} dK_z$

c) $\langle u, v\rangle = \langle U, V\rangle$

inner product in spatial domain

inner product in frequency domain

d) $u * v = \sum_{m\in\mathbb{Z}} \left[\int_{\mathbb{R}^2} u(\vec{p}, m) v(\vec{r}-\vec{p}, n-m) d\vec{p} \right]$

$U * V = \frac{1}{(2\pi)^3} \int_{\mathbb{S}_{2\pi}} \left[\int_{\mathbb{R}^2} U(\vec{\Omega}_{xy}, \Omega_z) V(\vec{K}_{xy} - \vec{\Omega}_{xy}, K_z - \Omega_z) d\vec{\Omega}_{xy} \right] d\Omega_z$

f) $\sum_{n\in\mathbb{Z}} \int_{\mathbb{R}^2} |h(\vec{r}, n)| d\vec{r} < \infty$

e) $H(K_{xy}, K_z) = \sum_{n\in\mathbb{Z}} \left(\int_{\mathbb{R}^2} h(\vec{r}, n) e^{-j\vec{K_{xy}}\cdot\vec{r}} d\vec{r} \right) e^{-jK_z \cdot n}$

$= \sum_{n\in\mathbb{Z}} \left(\int_{\mathbb{R}^2} \left(\sum_{m\in\mathbb{Z}} \left(\int_{\mathbb{R}^2} u(\vec{p}, m) v(\underbrace{\vec{r}-\vec{p}}_{\vec{r}'}, n-m) d\vec{p} \right) \right) e^{-j\vec{K_{xy}}\cdot\vec{r}} d\vec{r} \right) \left(e^{-jK_z \cdot n} \right)$

$$= \sum_{n\in\mathbb{Z}} \left(\int_{\mathbb{R}^2} \left(\sum_{m\in\mathbb{Z}} \left(\int_{\mathbb{R}^2} u(\vec{\rho}, m)\, v(\vec{r}\,', \underbrace{n-m}_{n'})\, d\vec{\rho}\right)\right) e^{-j\vec{k}_{xy}\cdot\vec{r}\,'} e^{-j\vec{k}_{xy}\cdot\vec{\rho}}\, d\vec{r}\,' \cdot e^{-jk_z \cdot n}\right)$$

$$= \sum_{n'\in\mathbb{Z}} \left(\int_{\mathbb{R}^2} \left(\sum_{m\in\mathbb{Z}} \left(\int_{\mathbb{R}^2} u(\vec{\rho}, m)\, v(\vec{r}\,', n')\, d\vec{\rho}\right)\right) e^{-j\vec{k}_{xy}\cdot\vec{r}\,'} e^{-j\vec{k}_{xy}\cdot\vec{\rho}}\, d\vec{r}\,' \cdot \left(e^{-jk_z\cdot n'} e^{-jk_z\cdot m}\right)\right)$$

$$= \left(\sum_{n'\in\mathbb{Z}} \int_{\mathbb{R}^2} v(\vec{r}\,', n')\, e^{-j\vec{k}_{xy}\cdot\vec{r}\,'} e^{-jk_z\cdot n'}\, d\vec{r}\,'\right)\left(\sum_{m\in\mathbb{Z}} \int_{\mathbb{R}^2} u(\vec{\rho}, m)\, e^{-j\vec{k}_{xy}\cdot\vec{\rho}} e^{-jk_z\cdot m}\, d\vec{\rho}\right)$$

$$H(k_{xy}, k_z) = V(\vec{k}_{xy}, k_z)\, U(\vec{k}_{xy}, k_z)$$

2) a) $F(w_1, w_2) = \frac{1}{2}(-1 + \cos w_1 + \cos w_2 + \cos w_1 \cdot \cos w_2)$

(i) $F(w_2, w_1) = \frac{1}{2}(-1 + \cos w_2 + \cos w_1 + \cos w_2 \cdot \cos w_1) = F(w_1, w_2)$ ✓

$F(-w_1, w_2) = \frac{1}{2}(-1 + \underbrace{\cos(-w_1)}_{\cos(-x) = \cos(x)} + \cos(w_2) + \cos(-w_1) \cdot \cos w_2)$

$= \frac{1}{2}(-1 + \cos(w_1) + \cos(w_2) + \cos(w_1) \cdot \cos(w_2)) = F(w_1, w_2)$ ✓

$F(w_1, -w_2) = \frac{1}{2}(-1 + \cos(w_1) + \cos(-w_2) + \cos(w_1) \cdot \cos(-w_2))$

$= \frac{1}{2}(-1 + \cos(w_1) + \cos(w_2) + \cos(w_1) \cdot \cos(w_2)) = F(w_1, w_2)$ ✓

$F(-w_1, -w_2) = \frac{1}{2}(-1 + \cos(-w_1) + \cos(-w_2) + \cos(-w_1) \cdot \cos(-w_2))$

$= \frac{1}{2}(-1 + \cos(w_1) + \cos(w_2) + \cos(w_1) \cdot \cos(w_2)) = F(w_1, w_2)$ ✓

~~F(-w~~

Replacing either w_1 or w_2 by its negative, or swapping them, the result is still $F(w_1, w_2) \Rightarrow$ 8-fold symmetry

$F(-w_1, -w_2)$, $F(-w_2, -w_1)$, $F(-w_2, w_1)$, and $F(w_2, -w_1)$ are the same as the above, because $\cos(-w) = \cos(w)$ simplifies all of them the same way.

(ii) $w_1 = \pi$

$\cos w = F(w_1, w_2)$

$\cos w = \frac{1}{2}(-1 + \cos(\pi) + \cos(w_2) + \cos(\pi)\cdot\cos(w_2))$

$= \frac{1}{2}(-1 - 1 + \cos(w_2) - \cos(w_2))$

$= \frac{1}{2}(-2)$

$\cos w = -1$

$w = \pi$

$w_2 = \pi$

$\cos w = F(w_1, w_2)$

$\cos w = \frac{1}{2}(-1 + \cos(w_1) + \cos(\pi) + \cos(w_1)\cdot\cos(\pi))$

$\cos w = \frac{1}{2}(-1 + \cos(w_1) - 1 - \cos(w_1))$

$= \frac{1}{2}(-1 - 1)$

$\cos w = -1$

$w = \pi$

(iii) $F(w_1, 0) = \frac{1}{2}(-1 + \cos(w_1) + \cos(0) + \cos(w_1)\cdot\cos(0))$

$= \frac{1}{2}(-1 + \cos(w_1) + 1 + \cos(w_1))$

$= \frac{1}{2}(2\cos(w_1)$

$F(w_1, 0) = \cos(w_1)$

$G(w_1, 0) = H(F(w_1, 0)) = H(\cos(w_1)) = H(w_1)$ ✓

$H(w_1) = h_0 + 2h_1 \cos w_1 + \ldots$

$G(0, w_2) = H(F(0, w_2)) = H(\cos w_2) = H(w_2)$ ✓

$F(0, w_2) = \frac{1}{2}(-1 + \cos(0) + \cos(w_2) + \cos(0)\cdot\cos(w_2))$

$= \frac{1}{2}(-1 + 1 + \cos(w_2) + \cos(w_2)) = \frac{1}{2}(2\cos w_2) = \cos w_2$

c) ~~[illegible]~~

$$P(w_1,w_2) = \frac{1}{2}\left(-1 + \cos(w_1) + \cos(w_2) + \cos(w_1)\cdot\cos(w_2)\right)$$

$$F(z_1,z_2) = \frac{1}{2}\left(-1 + \frac{z_1+z_1^{-1}}{2} + \frac{z_2+z_2^{-1}}{2} + \left(\frac{z_1+z_1^{-1}}{2}\right)\left(\frac{z_2+z_2^{-1}}{2}\right)\right)$$

$$= \frac{1}{2}\left(-1 + \frac{z_1+z_1^{-1}+z_2+z_2^{-1}}{2} + \frac{(z_1+z_1^{-1})(z_2+z_2^{-1})}{4}\right)$$

$$= \frac{-4 + 2z_1 + 2z_1^{-1} + 2z_2 + 2z_2^{-1} + z_1z_2 + z_1z_2^{-1} + z_1^{-1}z_2 + z_1^{-1}z_2^{-1}}{8}$$

d) $-N \le n \le N$,

$-N \le m \le N$