Nonlinear Finite Volume Approximation Methods for Anisotropic Diffusion Equation

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Anisotropic Diffusion Problem

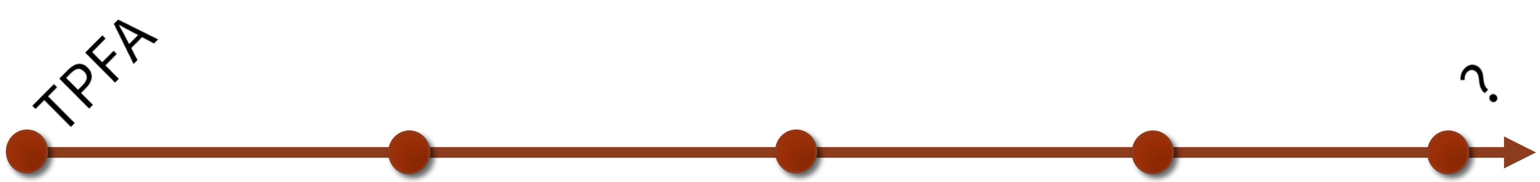
Consider a single-phase anisotropic diffusion problem:

$$\begin{cases} -\nabla \cdot \mathbb{K} \nabla p = g & \text{in} \quad \Omega \\ \mathbb{K} \dfrac{\partial p}{\partial \vec{n}} = g_N & \text{in} \quad \Gamma_N \\ p = g_D & \text{in} \quad \Gamma_D \end{cases}$$

Domain Ω is decomposed into polyhedral mesh and $\mathbb{K}$ is symmetric positive definitive piecewise-constant tensor on every polyhedral.

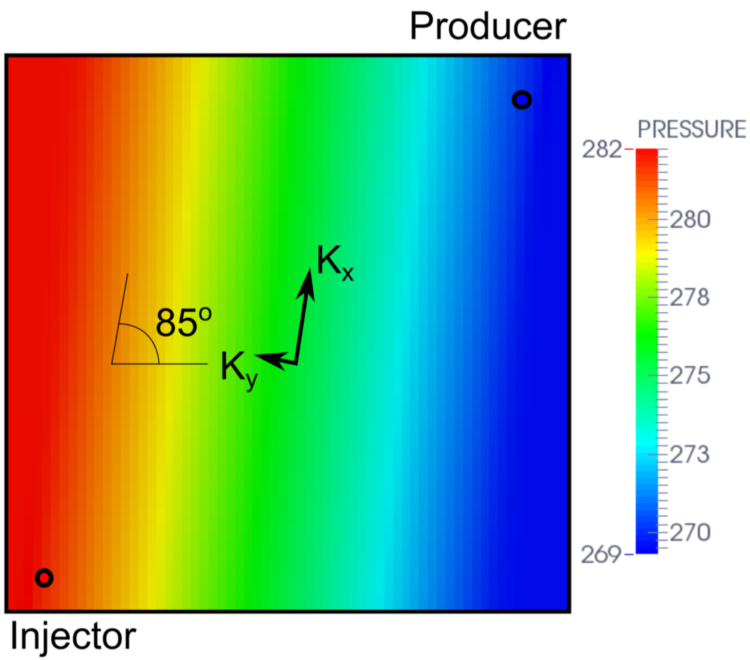
Overview of Cell-Centered Discretization Methods

	TPFA			
Approximation	**NO!**			

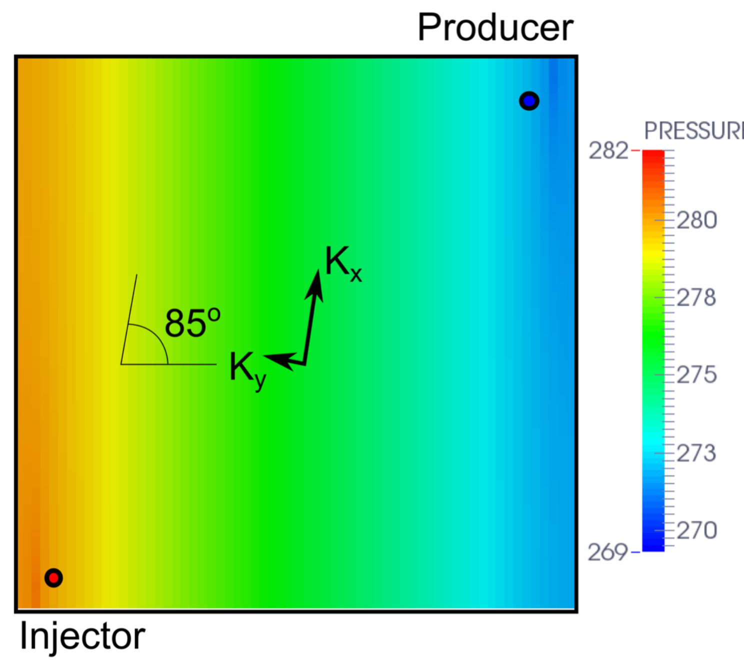


Lack of approximation of TPFA

Single-phase problem with full-permeability tensor.



MPFA-O $O(h^2)$-error



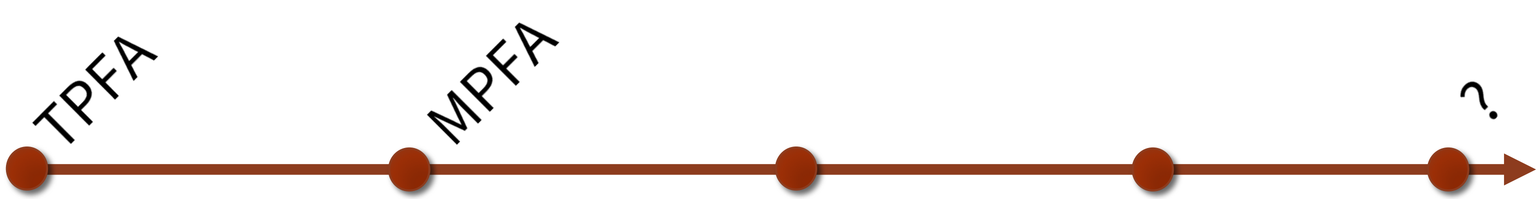
TPFA $O(1)$-error

History of development of MPFA methods

- 1996 Aavatsmark, Barkve, Boe, Mannseth
- 1998 Edwards, Rogers
- 2006 MPFA-O Aavatsmark, Eigstad, Klausen
- 2007 MPFA-L Aavatsmark, Eigstad, Mallison, Nordbotten
- 2009 MFD-O Lipnikov, Shashkov, Yotov
- 2010 MPFA-G Agelas, Pietro, Droniou
- 2010 MPFA-O(general) Agelas, Guichard, Masson
- And many more others…

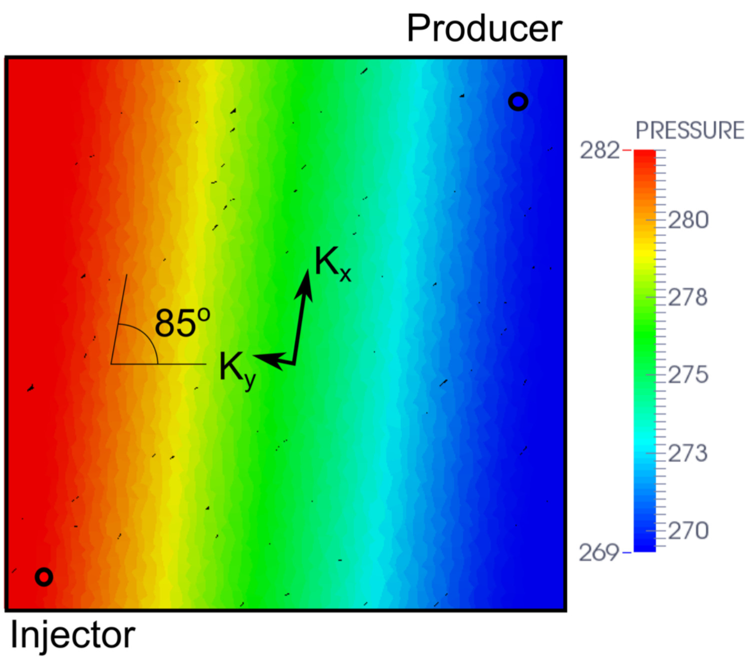
Overview of Cell-Centered Discretization Methods

	TPFA	MPFA		
Approximation	**NO!**	**YES**		
Robustness	YES	NO		

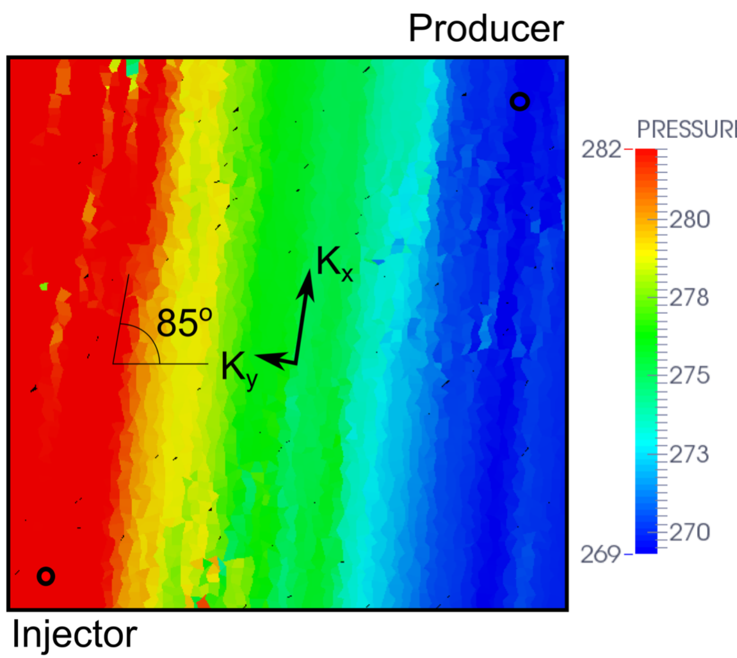


Robustness breakdown for MPFA

Same formulation but on distorted grid.



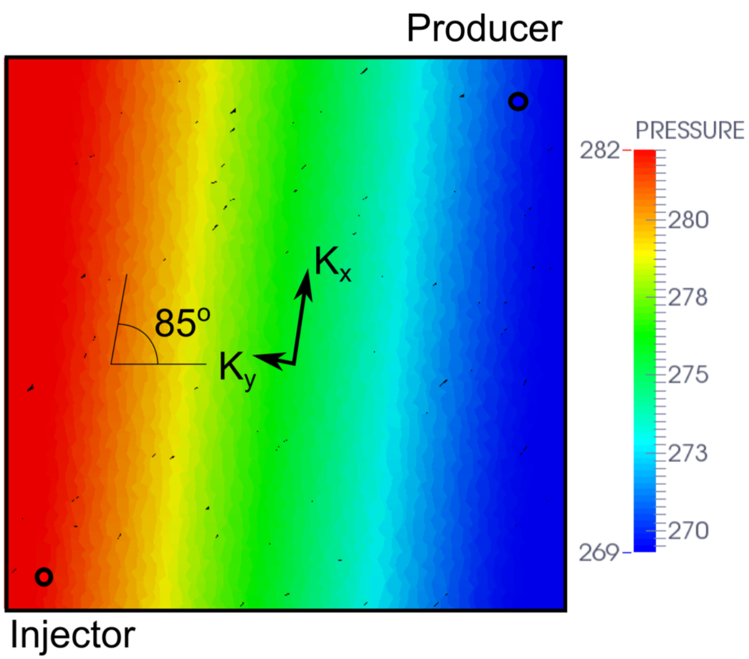
NTPFA-B



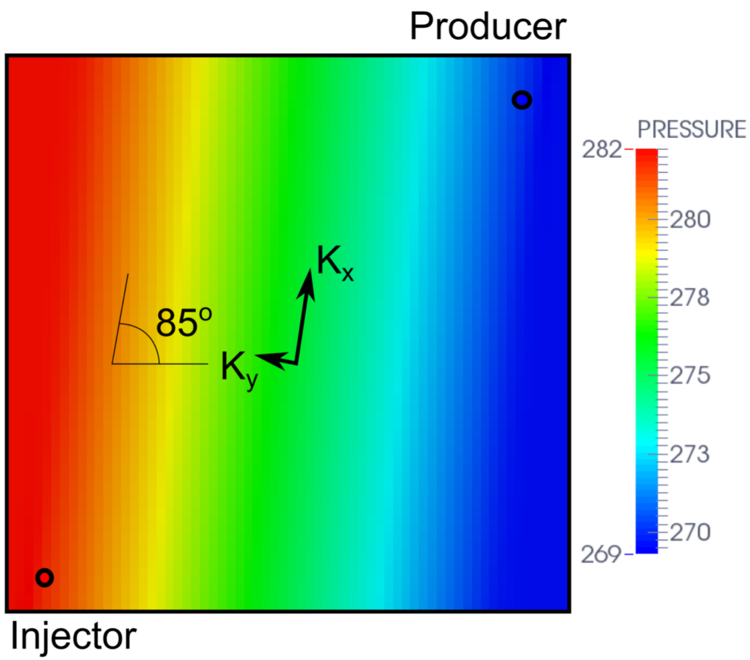
MPFA-O

Robustness breakdown for MPFA

Comparison to solution on Cartesian grid.



NTPFA-B



MPFA-O

History of development of NTPFA methods

- 2005 Le Potier
- 2007 Lipnikov, Shashkov, Svyatskiy, Vassilevski, Kapyrin
- 2009 Lipnikov, Svyatskiy, Vassilevski
- 2010 NTPFA-C, Danilov, Vassilevski
- 2012 NTPFA-B, Terekhov, Vassilevski
- 2013 Jimming, Zhimming
- 2013 NTPFA-B , Nikitin, Terekhov, Vassilevski
- 2014 Queiroz, Souza, Contreras, Lyra, Carvalho
- 2015 Zhimming, Jimming

Anisotropic Diffusion Problem

Consider a single-phase anisotropic diffusion problem:

$$\begin{cases} -\nabla \cdot \mathbb{K}\nabla p = g & \text{in} \quad \Omega \\ \mathbb{K}\dfrac{\partial p}{\partial \vec{n}} = g_N & \text{in} \quad \Gamma_N \\ p = g_D & \text{in} \quad \Gamma_D \end{cases}$$

Domain Ω is decomposed into polyhedral mesh and $\mathbb{K}$ is symmetric positive definitive piecewise-constant tensor on every polyhedral.

Finite Volume Divergence Operator Discretization

Applying Green's formula we have:

$$\int_V -\nabla \cdot \mathbb{K}\nabla p \mathrm{dx} = \int_S -\mathbb{K}\nabla p \cdot \vec{n} \mathrm{d}s = \int_V g \mathrm{d}x$$

Scalar product in BLUE have an equivalent form:

$$\mathbb{K}\nabla p \cdot \vec{n} = \nabla p \cdot \mathbb{K}\vec{n}$$

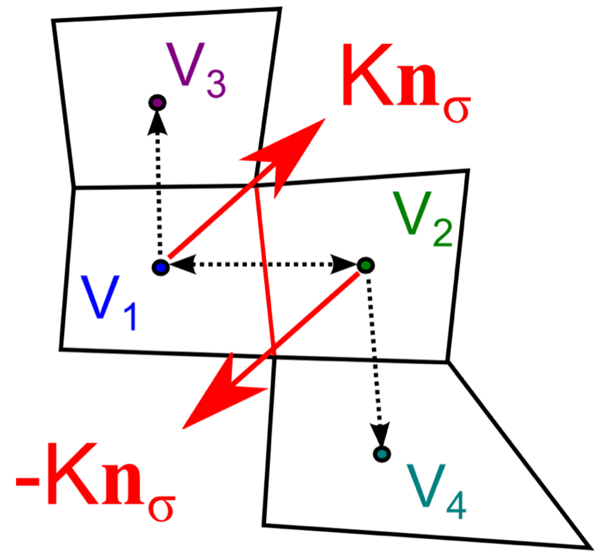
This form tells us that Total Flux can be expressed as gradient along co-normal $\mathbb{K}\vec{n}$.

NTPFA in Homogeneous Media

Suppose we can find a non-negative basis:

$$\nabla p \cdot \mathbb{K}\vec{n}_\sigma = \quad q_1 = \alpha(p_2 - p_1) + \beta(p_3 - p_1)$$

$$-\nabla p \cdot \mathbb{K}\vec{n}_\sigma = -q_2 = \gamma(p_1 - p_2) + \delta(p_4 - p_2)$$



To obtain total flux we will make convex combination of the two:

$$\begin{cases} q = \mu_1 q_1 + \mu_2 q_2 \\ \mu_1 + \mu_2 = 1 \end{cases}$$

Looking for the following form:

$$q = \mathbb{T}_1(p_3, p_4)p_2 - \mathbb{T}_2(p_3, p_4)p_1$$

Nonlinear Weighting: Nonlinear Two-Point Scheme

The weights depend on the solution:

$$\mu_1 = \frac{\delta p_4}{\beta p_3 + \delta p_4}, \quad \mu_2 = \frac{\beta p_3}{\beta p_3 + \delta p_4}$$

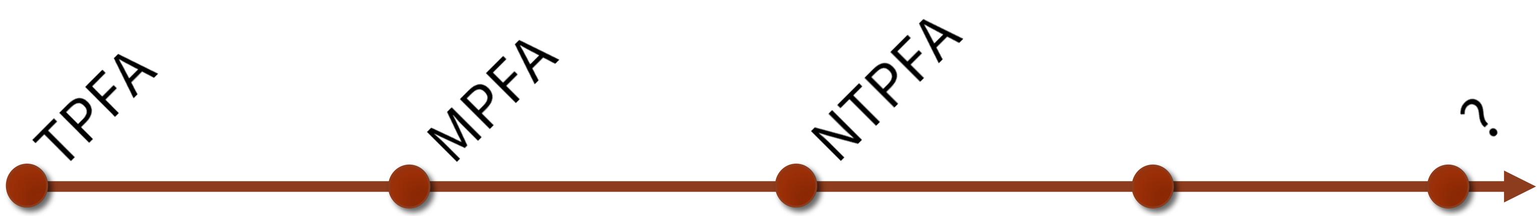
For Picard iterative method:

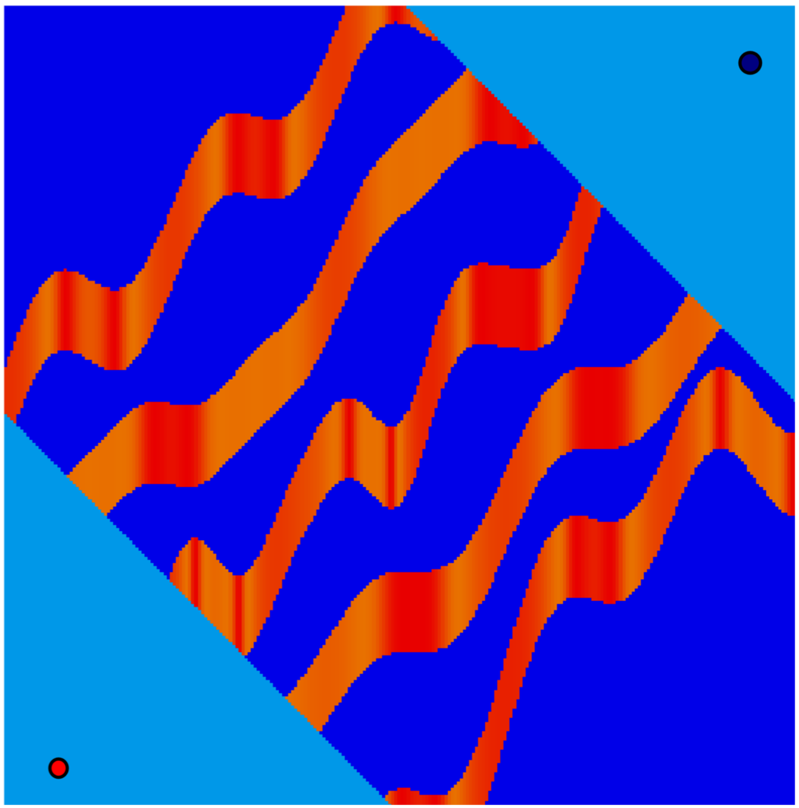
$$\begin{array}{cc} & \begin{array}{cc} p_1 & p_2 \end{array} \\ \begin{array}{c} V_1 \\ V_2 \end{array} & \begin{pmatrix} \mathbb{T}_1(p_3, p_4) & -\mathbb{T}_2(p_3, p_4) \\ -\mathbb{T}_1(p_3, p_4) & \mathbb{T}_2(p_3, p_4) \end{pmatrix} \end{array}$$

Column-sum of the M-matrix is non-negative, which gives us positive solution for positive right hand side. [R.S. Varga, Matrix iterative analysis, Prentice-Hall Inc., 1962]

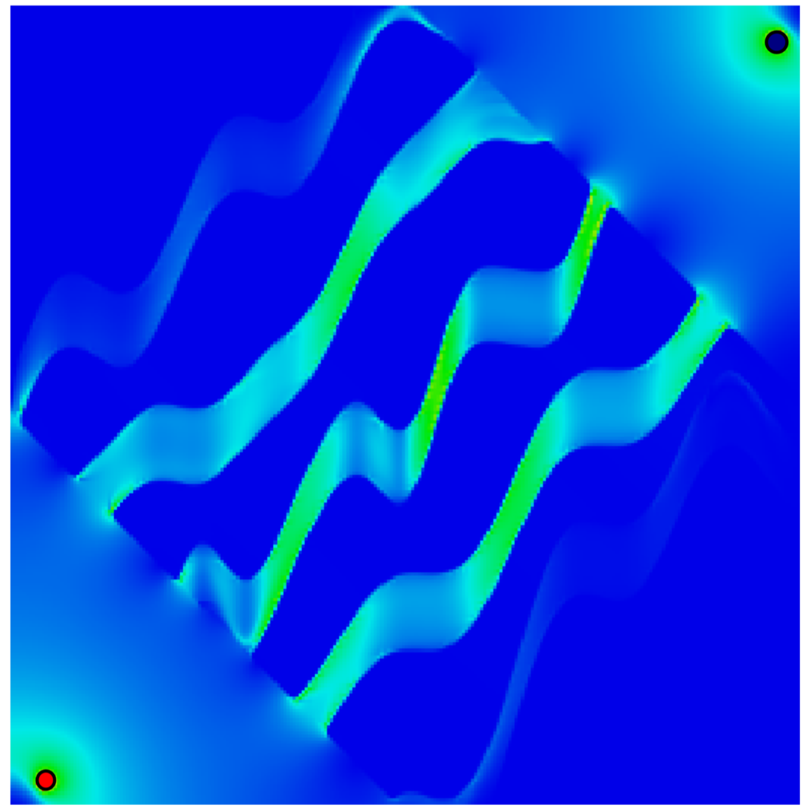
Overview of Cell-Centered Discretization Methods

	TPFA	MPFA	NTPFA	
Approximation	**NO!**	**YES**	**YES**	
Robustness	YES	NO	YES	
Locking-free	YES	NO	NO	





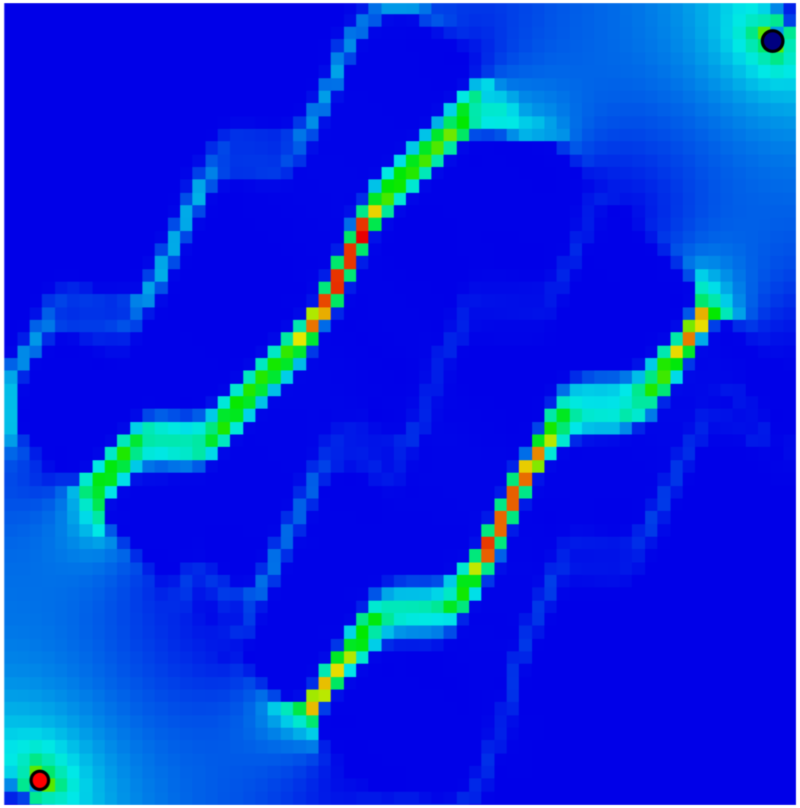
Magnitude of the
Permeability tensor



Magnetude of the reconstructed
Velocity field on fine grid

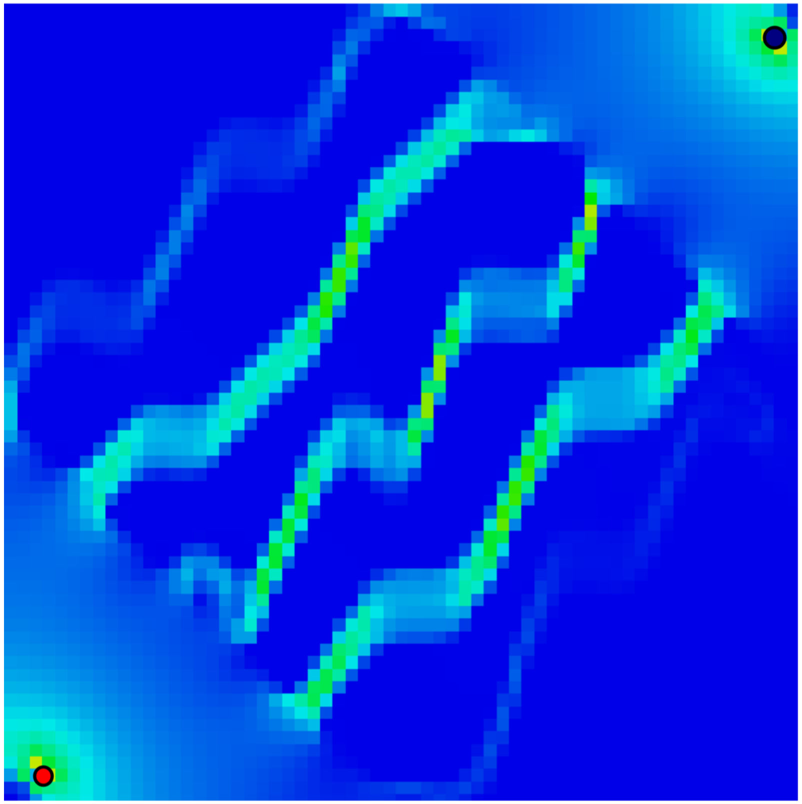
Synthetic problem with channels, velocity magnitude

NTPFA-B



An old formulation

NTPFA-A



A new formulation

History of mesh locking problem

- <1980 Zienkiewicz, Pawsy, Clough, MacNeal…
- 1986 Koh, Kikuchi
- 1987 Belytschko, Bachrach
- 1992 Babuska, Suri
- 2001 Havu, Pitkaranta
- 2003 Havu
- 2004 Slimane, Renard
- 2006 Manzini, Putti
- 2013 Ambroziak

Why the scheme locks?

The primary reason is a contribution of the transverse gradient to the flux. This transverse gradient may act reducing the total flux.

In our case the nonlinear weighting procedure seems to worsen the situation of locking by artificially locking the total flux.

To gain more control over the issue we will split the total flux into normal and transverse part in the same fashion as in work of Manzini and Putti 2006.

New Formulation

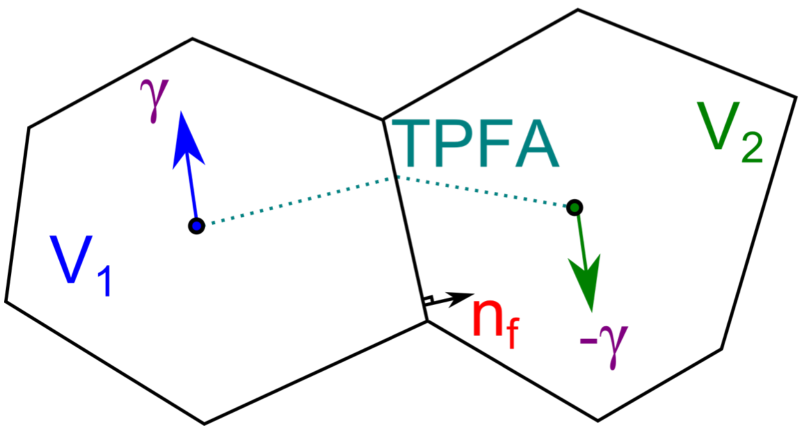
Could we measure the amount of inconsistency in TPFA?

TPFA Approximation is:

$$\mathrm{TPFA} = \frac{\lambda_1 \lambda_2}{\lambda_1 d_2 + \lambda_2 d_1}(p_2 - p_1)$$

Missing transverse part of the flux:

$$\vec{\gamma} = \frac{\lambda_1 \lambda_2 (\mathbf{y}_1 - \mathbf{y}_2) + \lambda_2 d_1 \vec{\gamma}_1 + \lambda_1 d_2 \vec{\gamma}_2}{\lambda_1 d_2 + \lambda_2 d_1}$$



$$\lambda = \vec{n}_\sigma \cdot \mathbb{K} \vec{n}_\sigma$$

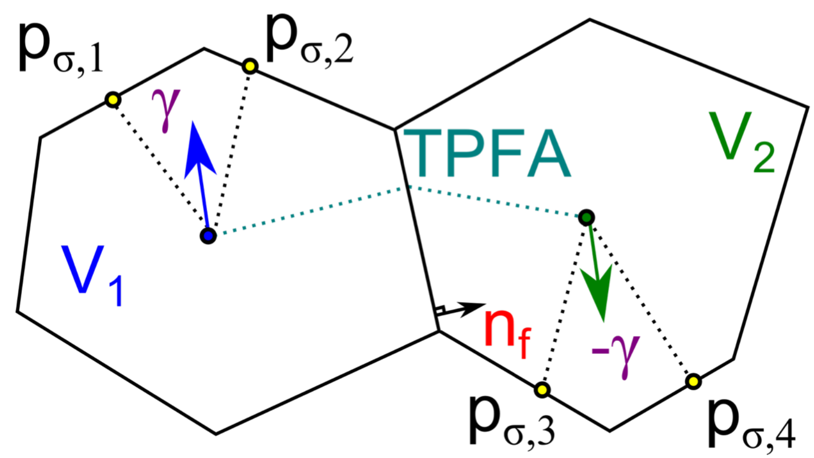
Full derivation is in backup slides.

Transverse Flux Approximation

It is hard to approximate without breaking robustness!

Suppose we can write:

$$\gamma_1 = \alpha_1(p_{\sigma,1} - p_1) + \alpha_2(p_{\sigma,2} - p_1),$$
$$-\gamma_2 = \beta_1(p_{\sigma,3} - p_2) + \beta_2(p_{\sigma,4} - p_2).$$

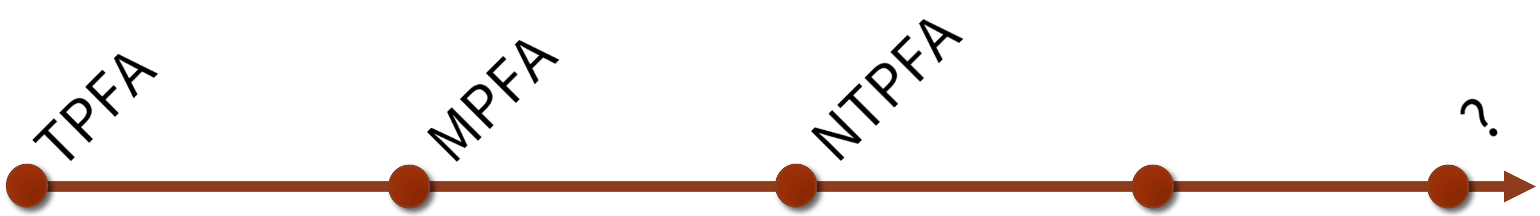


Then we will approximate transverse part of the flux by choosing the weights:

$$\begin{cases} \gamma = \mu_1\gamma_1 + \mu_2\gamma_2 \\ \mu_1 + \mu_2 = 1 \end{cases}$$

Overview of Cell-Centered Discretization Methods

	TPFA	**MPFA**	**NTPFA**	
Approximation	**NO!**	**YES**	**YES**	
Robustness	YES	NO	YES	
Locking-free	YES	NO	IMPROVED	
Discrete Maximum Principle	YES	NO	NO	



Discrete maximum principle: problem setup

Solution is expected to be bounded in [0,1].

TPFA: [0,1]

MPFA-O: [-0.076,1.076]

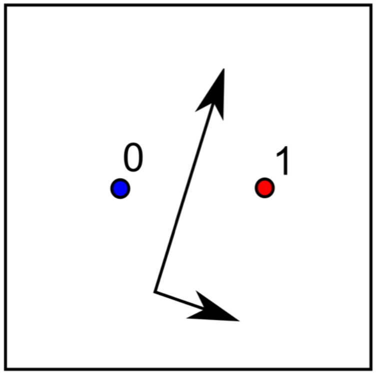
MPFA-L: [-0.67,1.67]

MPFA-G: [-1.04,1.97]

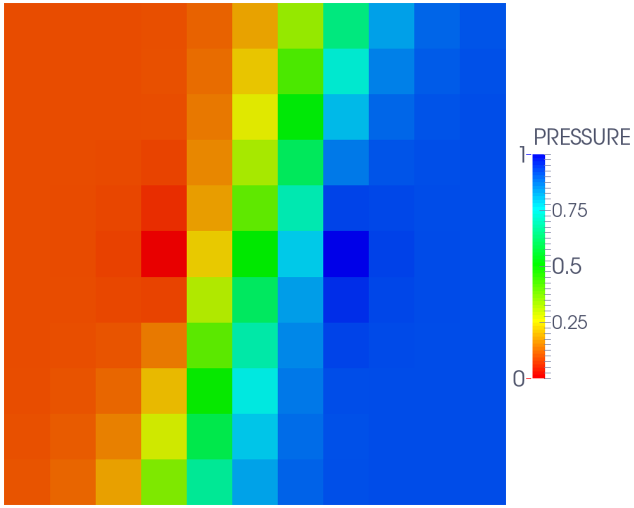
NTPFA-A: [0,2.42]

NTPFA-B: [0,2.45]

NMPFA-A: [0,1]



Problem setup



Expected solution

Discrete Maximum Principle: Problem Setup

A local minima or maxima should be attained only at the boundary or at the source.

Single-phase problem with homogeneous permeability:

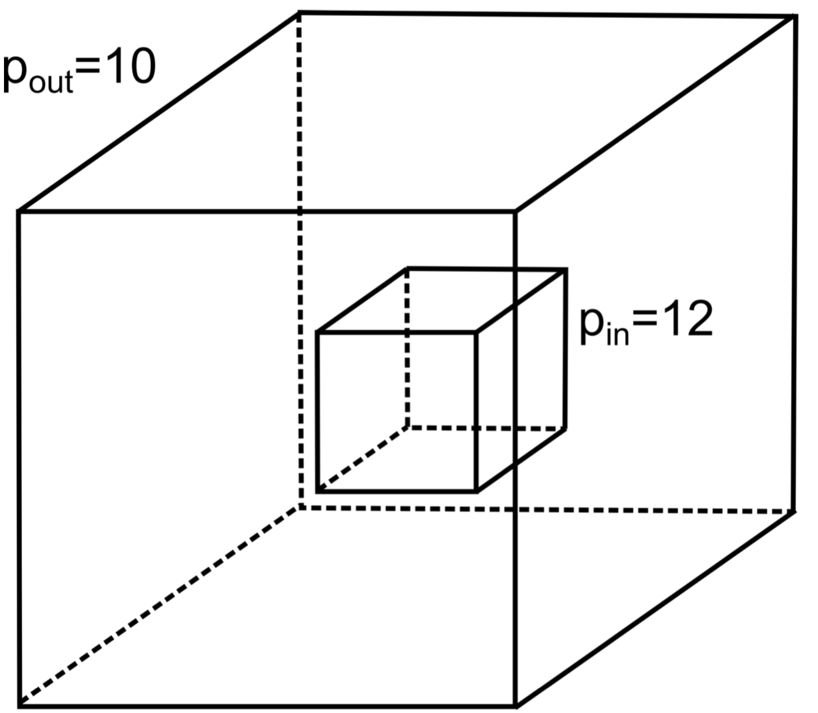
$$\mathbb{K} = R \cdot \begin{pmatrix} 300 & & \\ & 15 & \\ & & 1 \end{pmatrix} \cdot R^T$$

A rotation matrix:

$$R = R_{Oyz}(60^\circ) \cdot R_{Oxz}(45^\circ) \cdot R_{Oxy}(30^\circ)$$

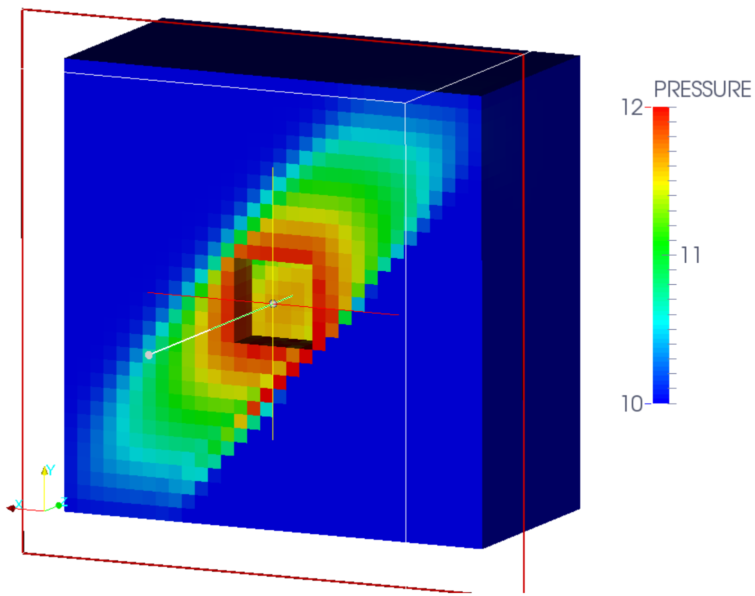
Example of rotation matrix:

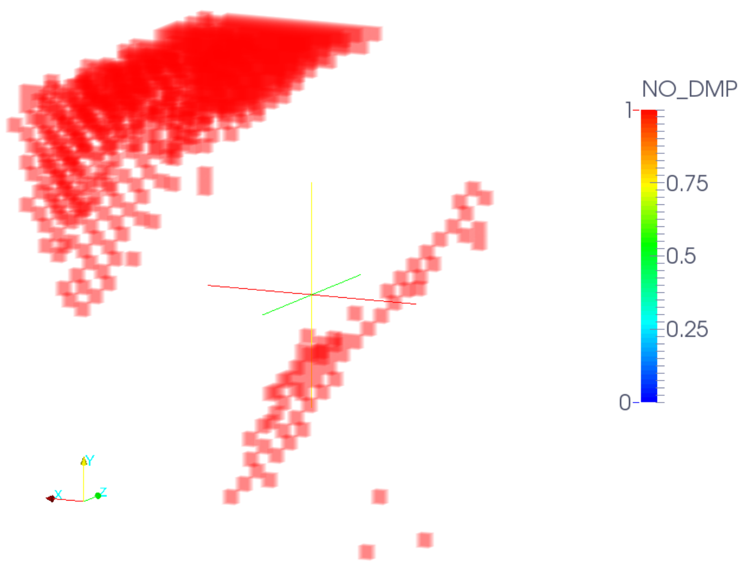
$$R_{Oxy}(\phi) = \begin{pmatrix} \cos(\phi) & \sin(\phi) & \\ -\sin(\phi) & \cos(\phi) & \\ & & 1 \end{pmatrix}$$



Discrete Maximum Principle: MPFA-O

Values span: [1.41:16.68]
Maximal violation: 3.82 Violation points: 800





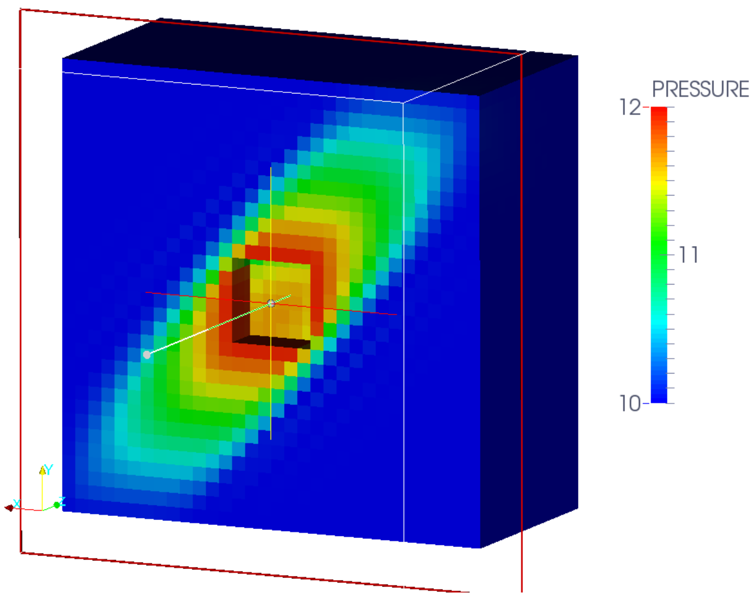
Pressure in slice

Violation zones

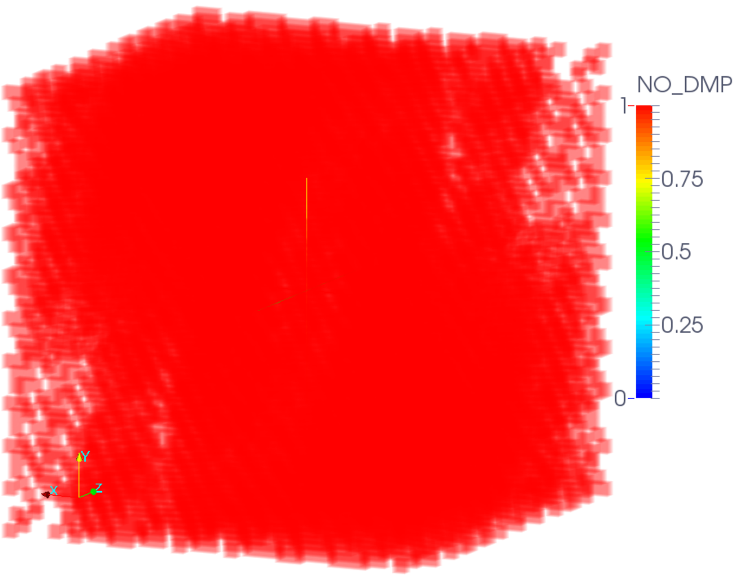
Discrete Maximum Principle: MPFA-L

Values span: [9.96:11.91]

Maximal violation: 0.11 Violation points: 12289



Pressure in slice

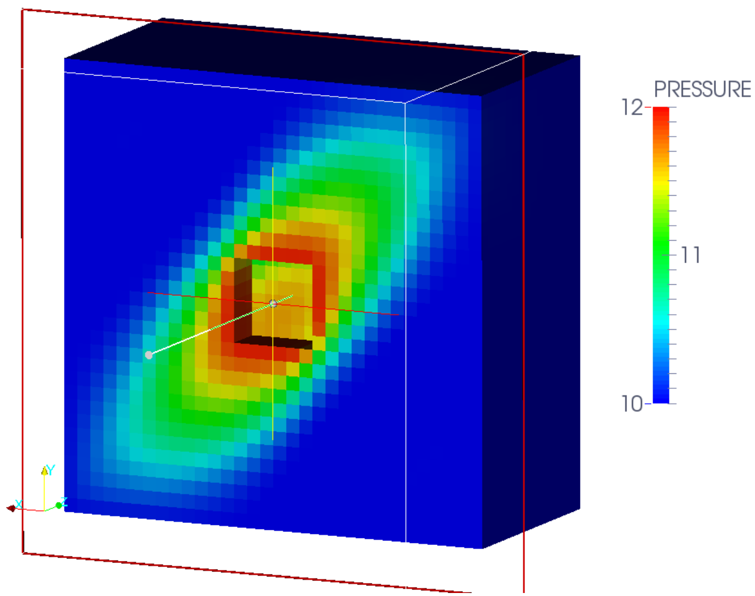


Violation zones

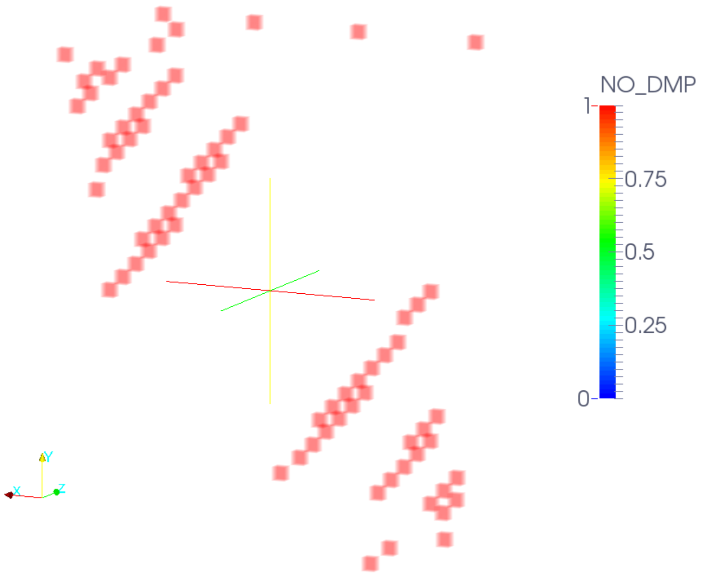
Discrete Maximum Principle: MPFA-G

Values span: [9.94:11.92]

Maximal violation: 0.004 Violation points: 75



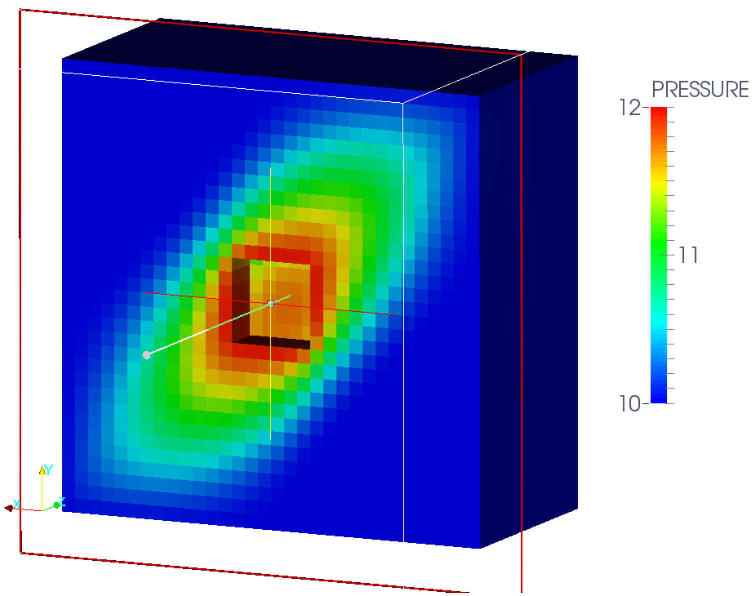
Pressure in slice

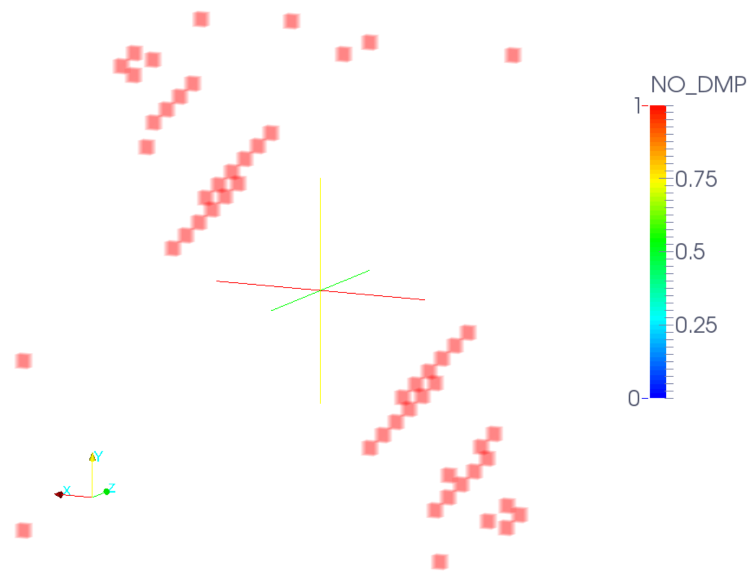


Violation zones

Discrete Maximum Principle: NTPFA

Values span: [9.95:11.94]
Maximal violation: 0.0025 Violation points: 53



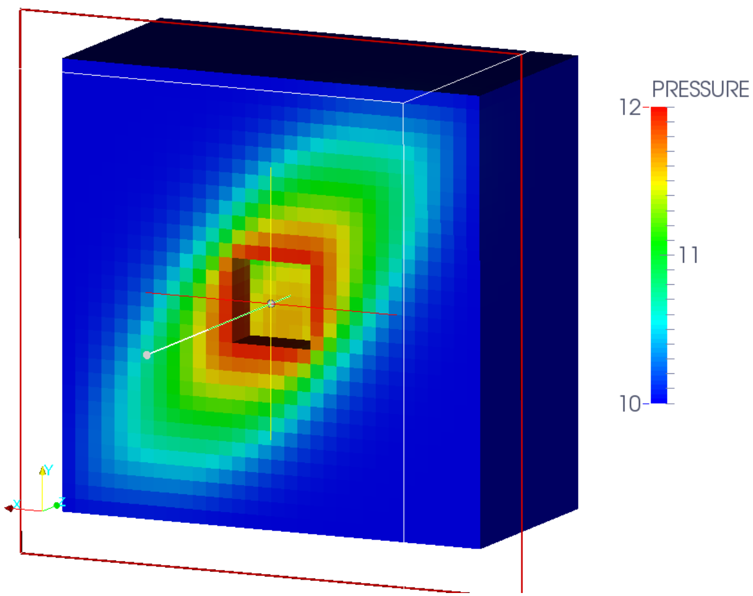


Pressure in slice | Violation zones

Converged in 2 Newton iterations

Discrete Maximum Principle: NMPFA

Values span: [10:11.92]
No DMP violation.



Nothing to Display

Pressure in slice

Converged in 11 QN+AA iterations

Nonlinear Weighting: Nonlinear Multi-Point Scheme

The weights that make row-sum of M-matrix non-negative [Enrico Bertolazzi, Gianmarco Manzini: A second-order maximum principle preserving finite volume method for steady convection-diffusion problems, 2005]:

$$\mu_1 = \frac{|\gamma_2|}{|\gamma_1| + |\gamma_2|}, \quad \mu_2 = \frac{|\gamma_1|}{|\gamma_1| + |\gamma_2|}.$$

For Picard iterative method:

$$\begin{array}{cc} & \begin{array}{cccccc} p_1 & p_2 & p_{\sigma,1} & p_{\sigma,2} & p_{\sigma,3} & p_{\sigma,4} \end{array} \\ \begin{array}{c} V_1 \\ V_2 \end{array} & \begin{pmatrix} \mu_1(\alpha_1 + \alpha_2) & & -\mu_1\alpha_1 & -\mu_1\alpha_2 & & \\ & \mu_2(\beta_1 + \beta_2) & & & -\mu_2\beta_1 & -\mu_2\beta_2 \end{pmatrix} \end{array}$$

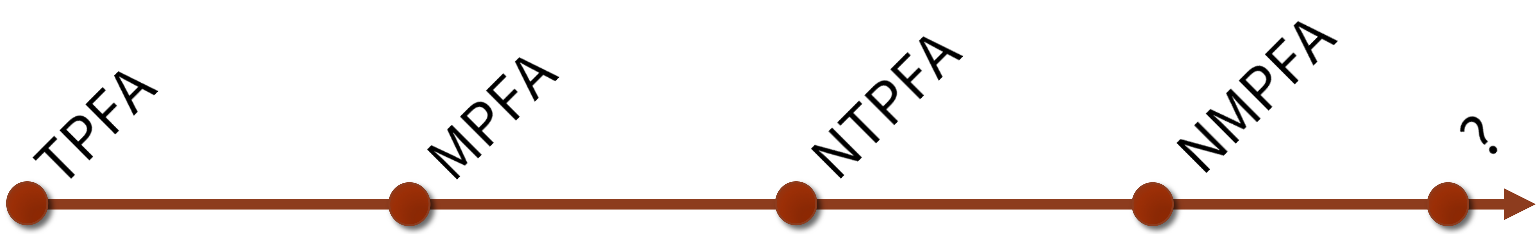
Solution to the problem satisfy discrete maximum and minimum principle.
[Philippe G. Ciarlet Discrete Maximum Principle for Finite-Difference Operators, 1969]

History of development of NMPFA methods

- 2005 Bertolazzi, Manzini
- 2009 Le Potier
- 2011 Droniou, Le Potier
- 2012 Lipnikov, Svyatskiy, Vassilevski, NMPFA-B
- 2012 Zhiqiang, Guangwei
- Today

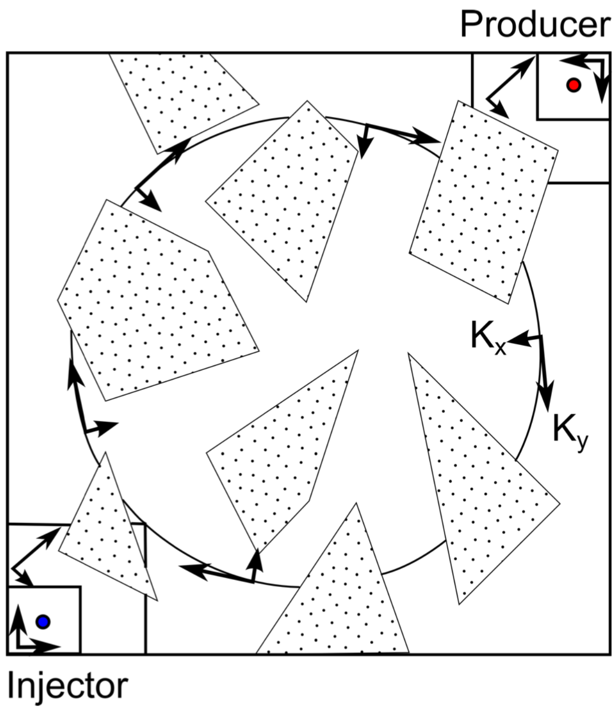
Overview of Cell-Centered Discretization Methods

	TPFA	MPFA	NTPFA	NMPFA
Approximation	**NO!**	**YES**	**YES**	**YES**
Robustness	YES	NO	YES	YES
Coercivity	YES	NO	IMPROVED	IMPROVED
Discrete Maximum Principle	YES	NO	NO	YES
Efficiency	YES	YES	YES	NO

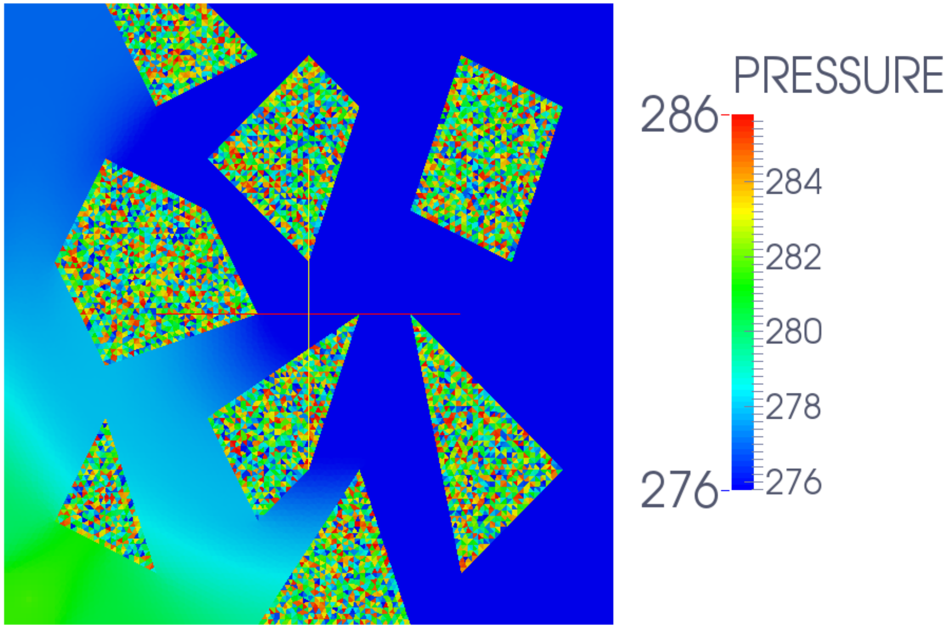


NMPFA Efficiency

Synthetic single-phase anisotropic diffusion problem:

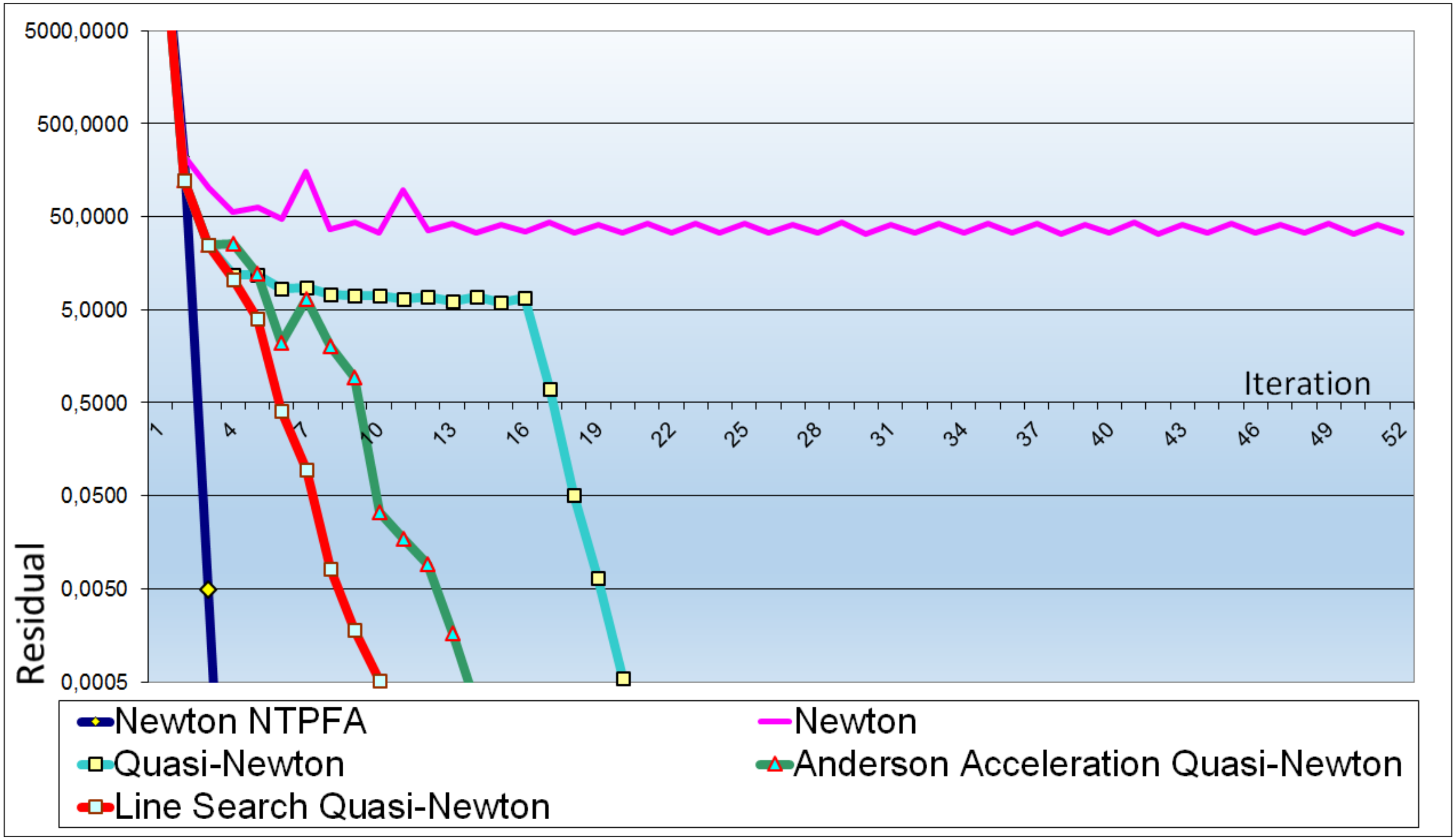


Setup



Solution

NMPFA Efficiency



NTPFA: 3 Newton iterations.

NMPFA: 11 Quasi-Newton iterations with Line Search.

Overview of Cell-Centered Discretization Methods

	TPFA	MPFA	NTPFA	NMPFA
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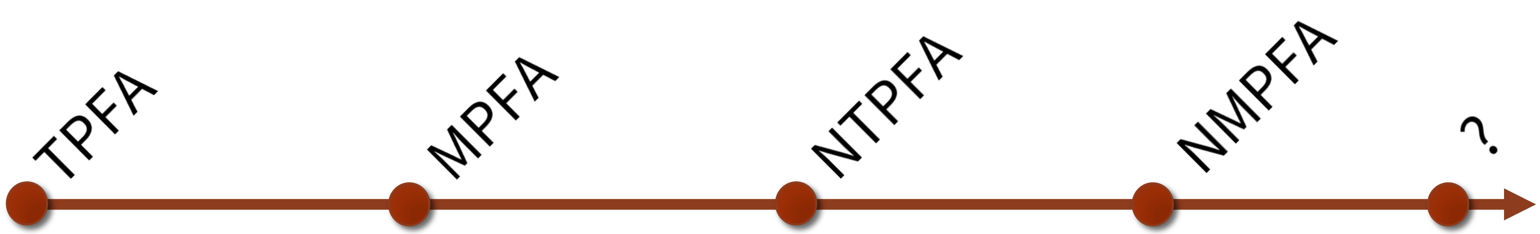
Robustness	YES	NO	YES	YES
Coercivity	YES	NO	IMPROVED	IMPROVED
Discrete Maximum Principle	YES	NO	NO	YES
Efficiency	YES	YES	YES	IMPROVED



Next slides: Discretization Toolkit

Summery

- A new formulation was proposed that seemingly resolves numerical locking and improves efficiency of nonlinear methods. NTPFA and NMPFA schemes were proposed based on new formulation.

Future work

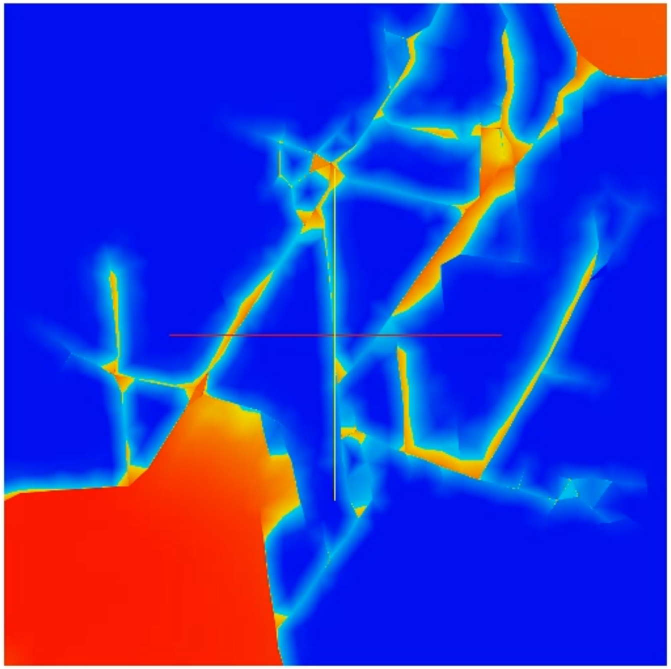
- Attempt to make the scheme unconditionally coercive:
  - Search for a *nonlinear* variational technique.
  - Look into mimetic methods and discrete calculus methods for ideas.
- Extend schemes for solution-dependent permeability tensor.

Thank you for Attention!

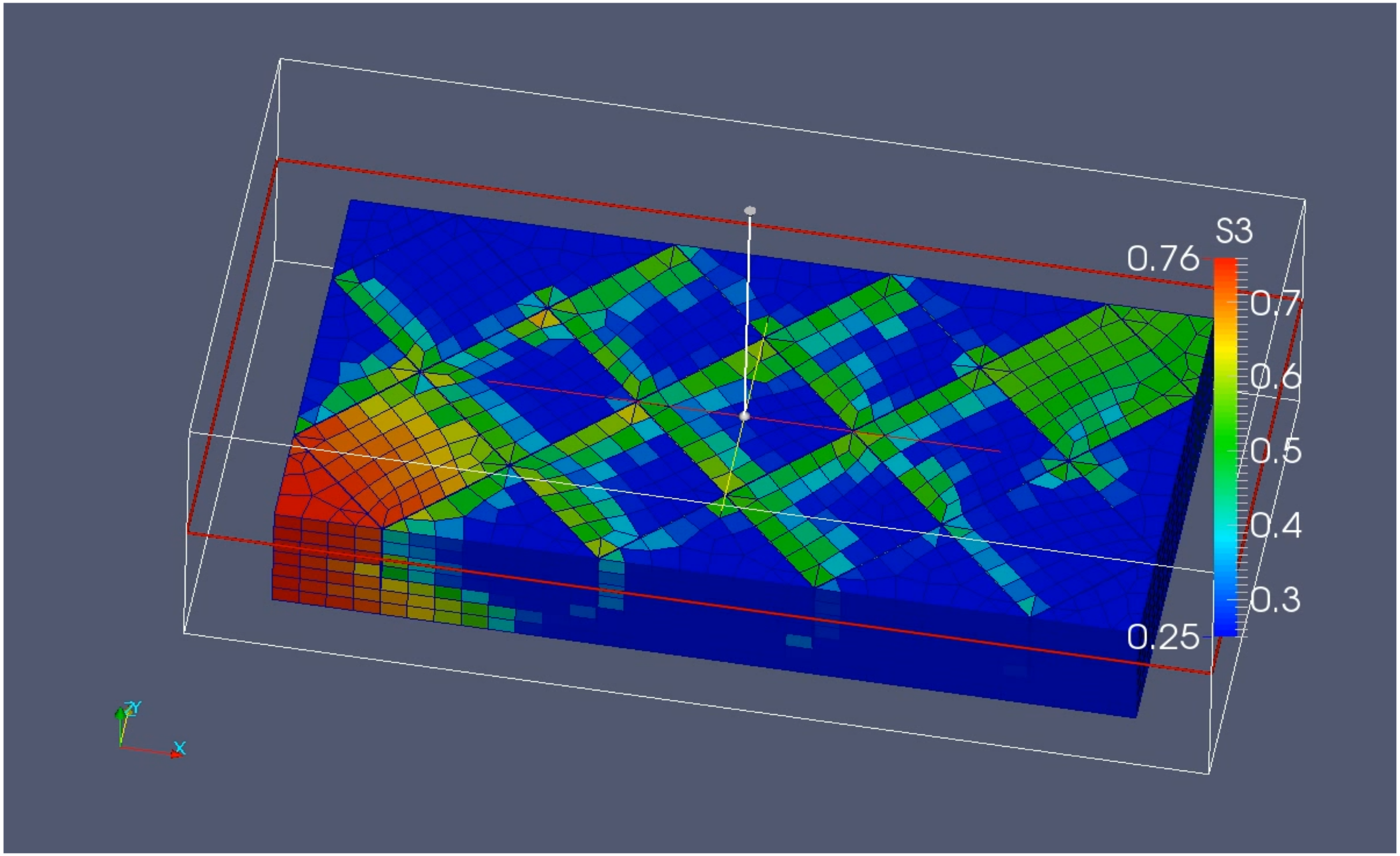
ACKNOWLEDGEMENTS:

- DENIS VOSKOV
- CHEVRON

Flow on sample mesh with fractures

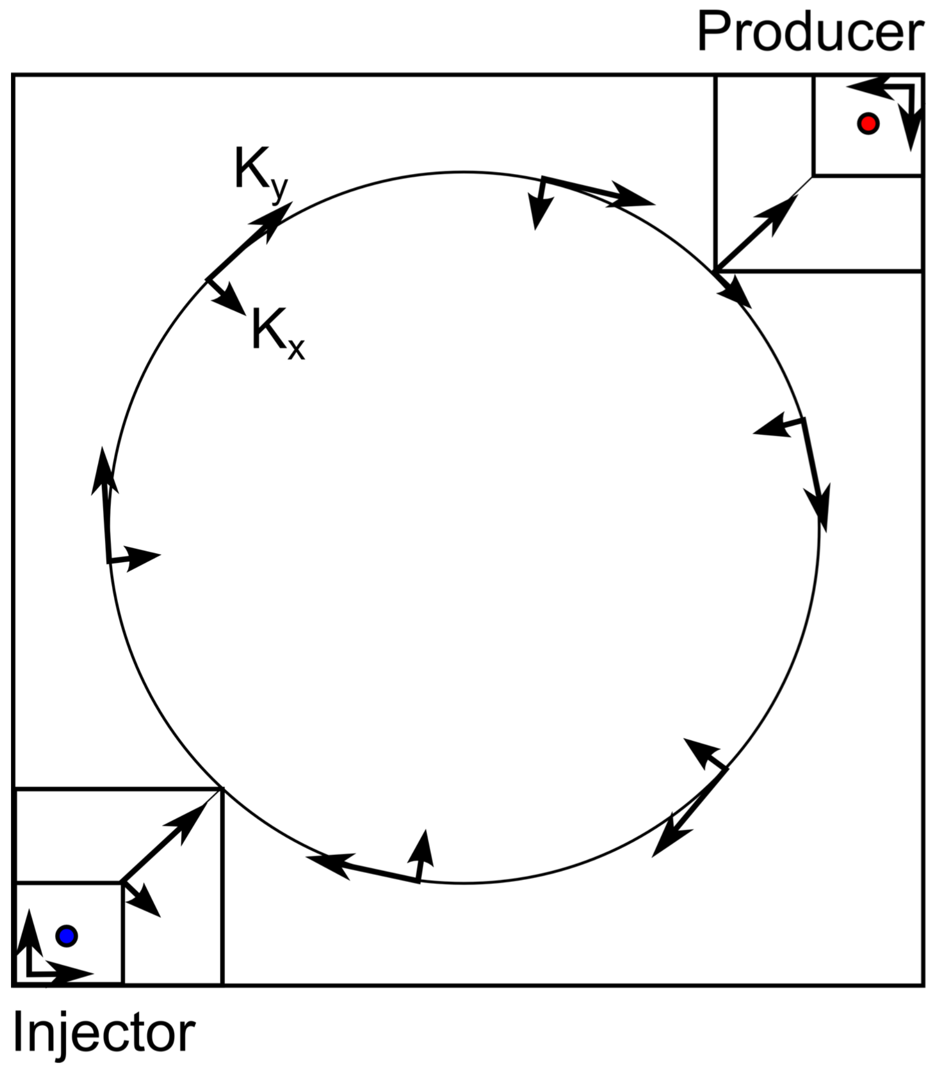


Flow on sample mesh with fractures



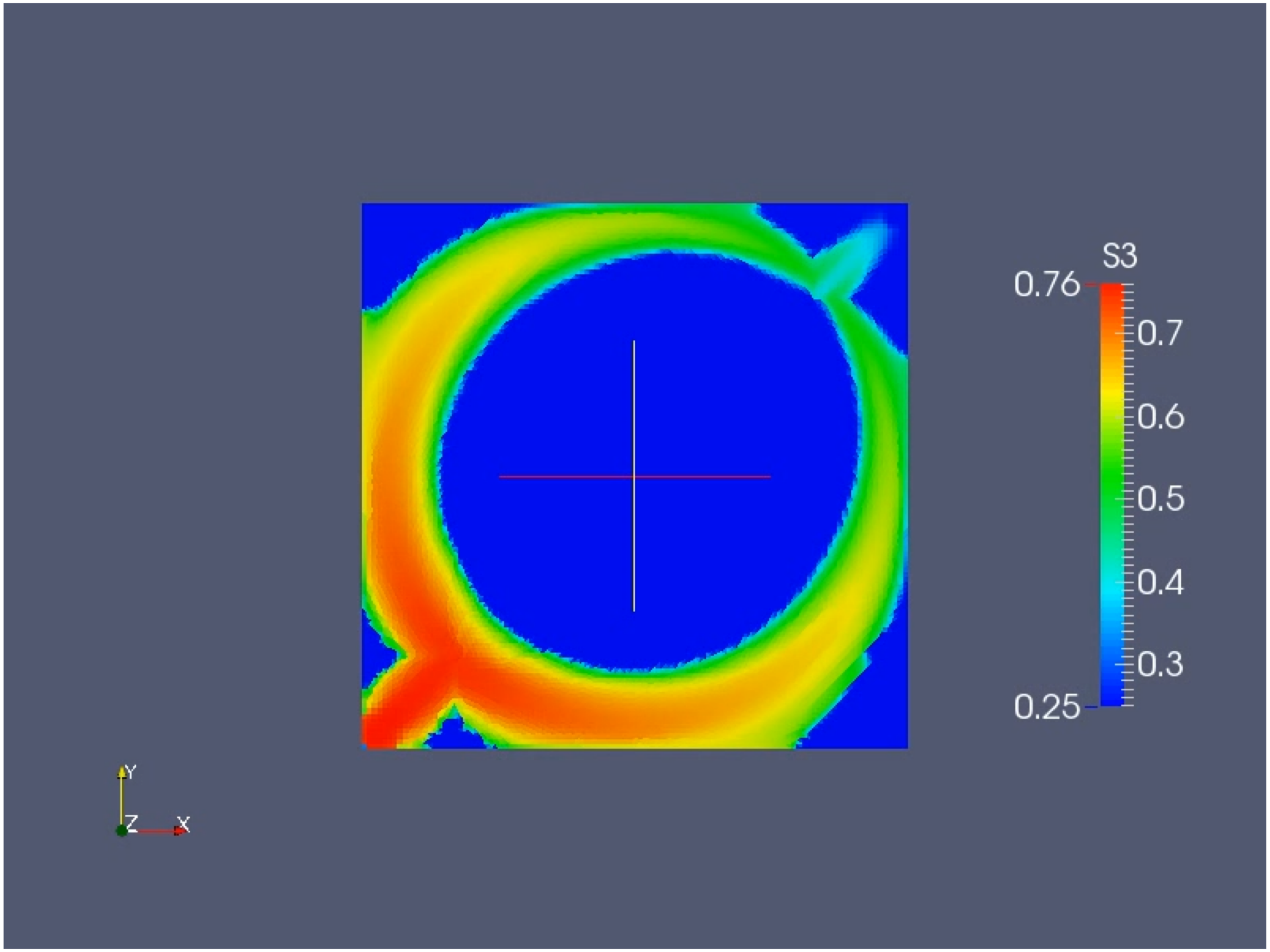
Mesh by Timur Garipov

Synthetic test problem

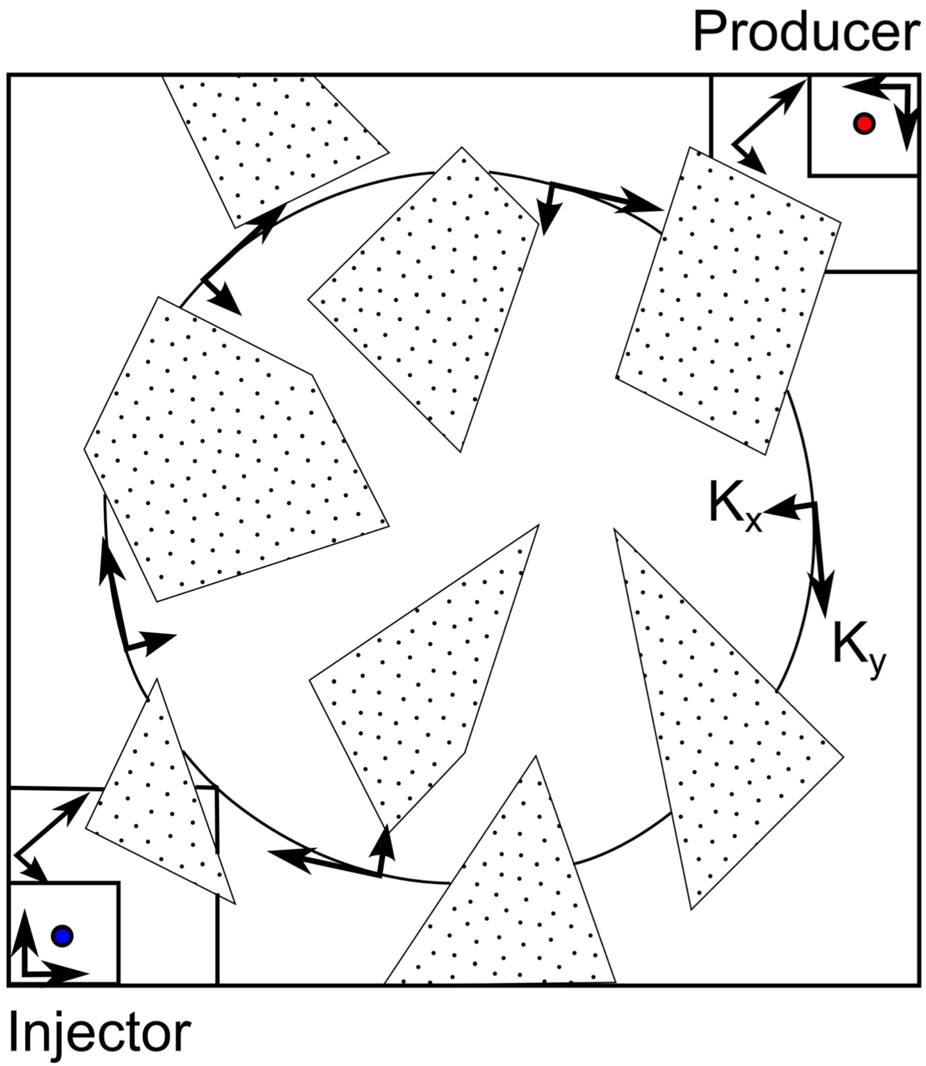


$$\frac{K_x}{K_y} = \frac{1}{100}$$

Simple synthetic black-oil test problem (NTPFA)

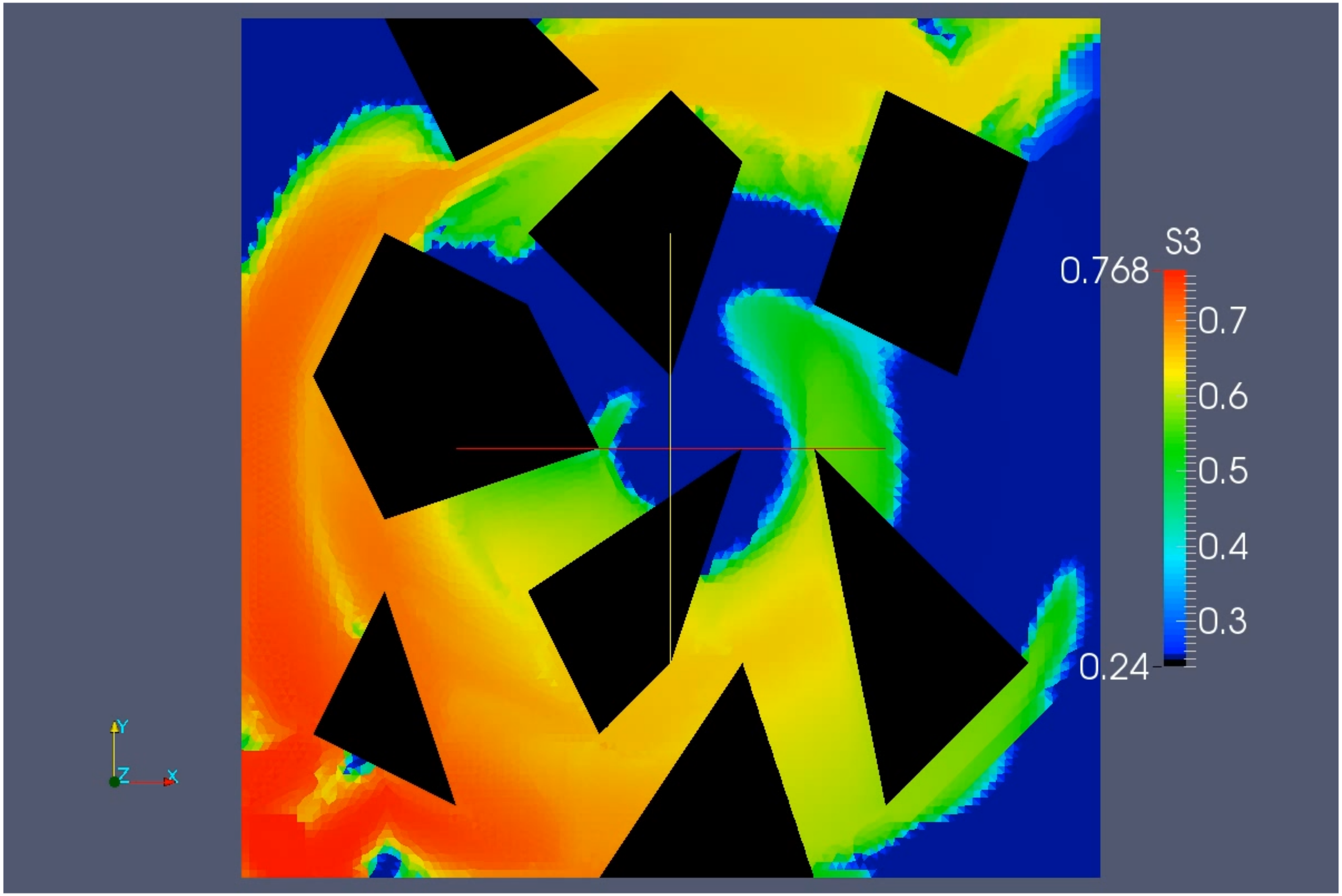


Synthetic test problem

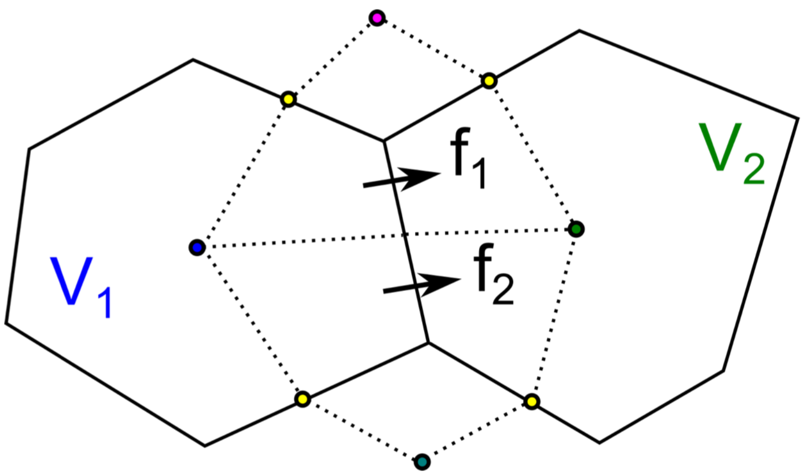


$$\frac{K_x}{K_y} = \frac{1}{100}$$

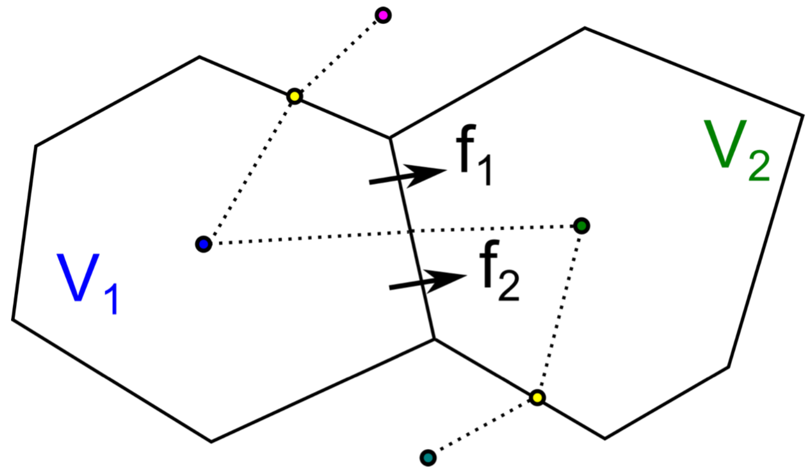
Harder synthetic black-oil test problem (NTPFA)



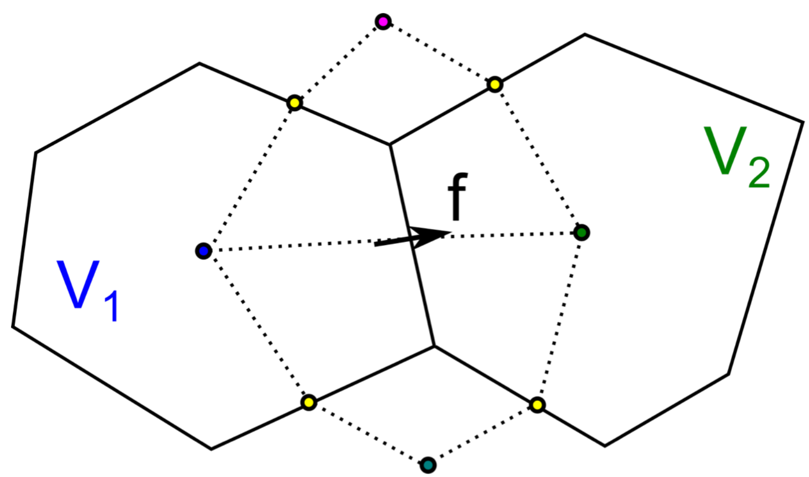
Schemes in the comparison: Linear



MPFA-O

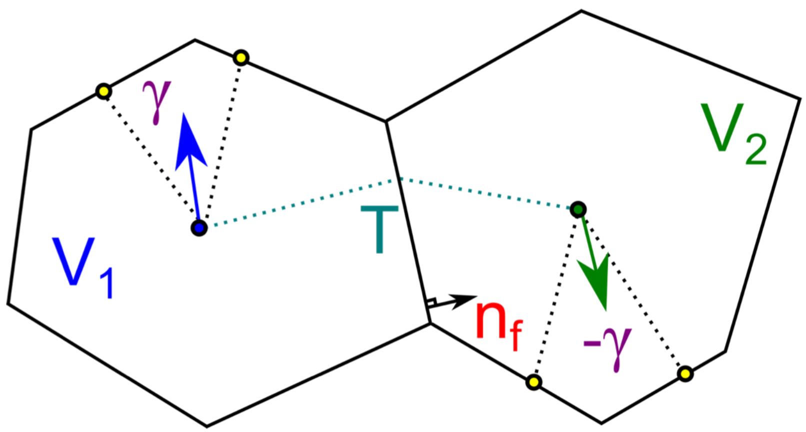


MPFA-L

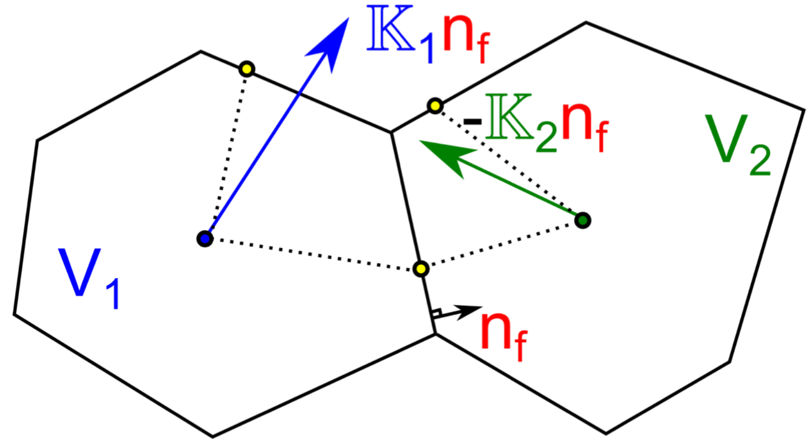


MPFA-G

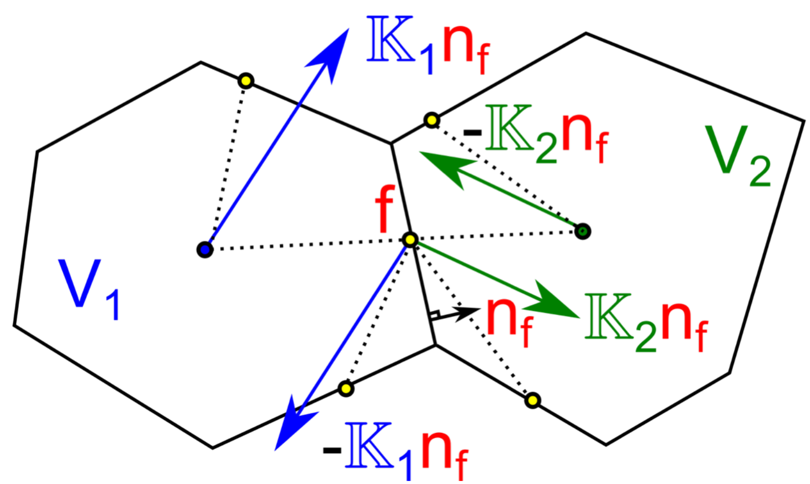
Schemes in the comparison: Nonlinear



NTPFA-A, NMPFA-A



NTPFA-B, NMPFA-B



NTPFA-C

Harder black-oil test problem

Scheme	Nonlinear iterations	Linear iterations
TPFA(inconsistent)	180	4941
MPFA-O	157	2790
MPFA-G	188+20	9475+7021
MPFA-L	180+20	12764+4053
NTPFA-A	159	4743
NTPFA-B	158	5170
NTPFA-C	170	3449
NMPFA-A*	219	5936
NMPFA-B*	246	7903

First 5 days with milder anisotropy.
With GMRES_CPR0 solver.
*Quasi-Newton iterative method was used.

Harder black-oil test problem

Scheme	Total Time**
TPFA(inconsistent)	95
MPFA-O	134
MPFA-G	355
MPFA-L	306
NTPFA-A	123
NTPFA-B	133
NTPFA-C	126
NMPFA-A*	165
NMPFA-B*	205

Total time in seconds.
*Quasi-Newton iterative method was used.
**With improvement on Automatic Differentiation.

Co-Normal Representation

For each interface and each material we will decompose normal into:

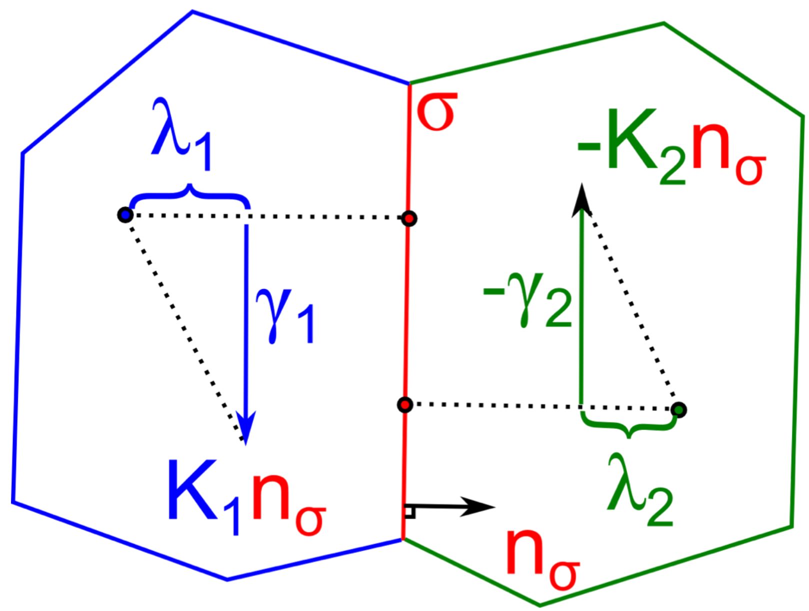
$$\mathbb{K}\vec{n}_\sigma = \lambda\vec{n}_\sigma + \vec{\gamma}$$

Lambda is a projection of co-normal onto normal:

$$\lambda = \vec{n}_\sigma \cdot \mathbb{K}\vec{n}_\sigma$$

Gamma is made coplanar to interface:

$$\vec{\gamma} = \left(\mathbb{K} - \lambda\mathbb{I}\right)\vec{n}_\sigma$$



Total Flux Continuity

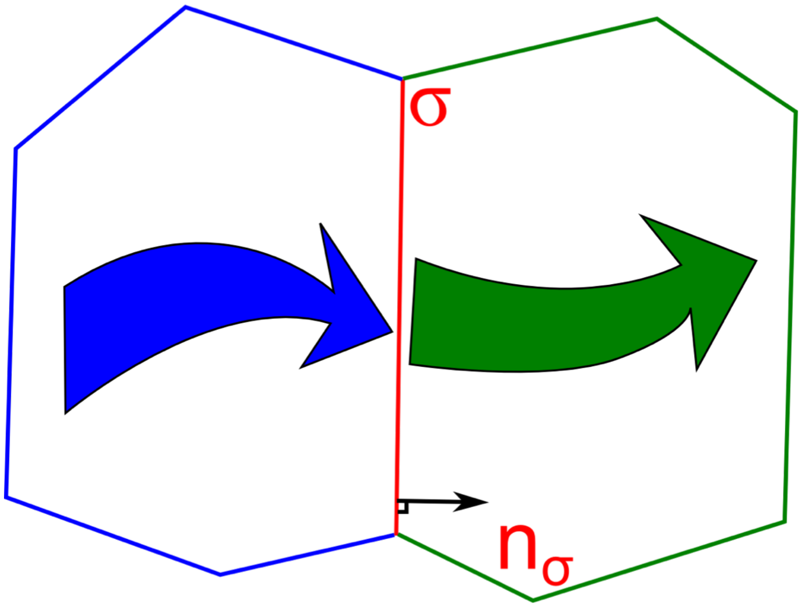
We enforce total flux continuity on each interface:

$$\nabla p \cdot \mathbb{K}_1 \vec{n}_\sigma = \nabla p \cdot \mathbb{K}_2 \vec{n}_\sigma$$

For a point y on interface with coplanar gradient $\vec{g}_\sigma$ we can write an approximation:

$$\lambda_1 \frac{p(\mathbf{y}) + (\mathbf{y}_1 - \mathbf{y}) \cdot \vec{g}_\sigma - p_1}{d_1} + \vec{\gamma}_1 \cdot \vec{g}_\sigma$$

$$= \lambda_2 \frac{p_2 - p(\mathbf{y}) - (\mathbf{y}_2 - \mathbf{y}) \cdot \vec{g}_\sigma}{d_2} + \vec{\gamma}_2 \cdot \vec{g}_\sigma$$



Harmonic Averaging Point

By regrouping terms in flux continuity equation we get:

$$p(\mathbf{y}) = \frac{\lambda_1 d_2 p_1 + \lambda_2 d_1 p_2}{\lambda_1 d_2 + \lambda_2 d_1} + \frac{d_1 d_2}{\lambda_1 d_2 + \lambda_2 d_1}(\vec{\gamma}_2 - \vec{\gamma}_1) \cdot \vec{g}_\sigma + \left(\mathbf{y} - \frac{\lambda_1 d_2 \mathbf{y}_1 + \lambda_2 d_1 \mathbf{y}_2}{\lambda_1 d_2 + \lambda_2 d_1} \right) \cdot \vec{g}_\sigma$$

From this we find a harmonic averaging point:

$$\mathbf{y}_\sigma = \frac{\lambda_1 d_2 \mathbf{y}_1 + \lambda_2 d_1 \mathbf{y}_2 + d_1 d_2 (\vec{\gamma}_1 - \vec{\gamma}_2)}{\lambda_1 d_2 + \lambda_2 d_1}$$

$$p(\mathbf{y}_\sigma) = \frac{\lambda_1 d_2 p_1 + \lambda_2 d_1 p_2}{\lambda_1 d_2 + \lambda_2 d_1}$$

[Leo Agelas, Robert Eymard, Raphaele Herbin: A nine-point finite volume scheme for the simulation of diffusion in heterogeneous media, 2009]

Transverse Flux Derivation

By construction for $y = y_\sigma$ in normal part of total flux we obtain:

$$\lambda_1 \frac{p(\mathbf{y}_\sigma) - p_1}{d_1} = \lambda_2 \frac{p_2 - p(\mathbf{y}_\sigma)}{d_2} = \frac{\lambda_1 \lambda_2}{\lambda_1 d_2 + \lambda_2 d_1}(p_2 - p_1)$$

Which is exactly two-point flux approximation. What is missing?

$$\mathbb{K}_1 \vec{n}_\sigma - \frac{\lambda_1}{d_1}(\mathbf{y}_\sigma - \mathbf{x}_1) = \mathbb{K}_2 \vec{n}_\sigma - \frac{\lambda_2}{d_2}(\mathbf{x}_2 - \mathbf{y}_\sigma) = \frac{\lambda_1 \lambda_2 (\mathbf{y}_1 - \mathbf{y}_2) + \lambda_2 d_1 \vec{\gamma}_1 + \lambda_1 d_2 \vec{\gamma}_2}{\lambda_1 d_2 + \lambda_2 d_1}$$

This gives us the direction for transverse part of the total flux that is missing in two-point flux approximation.

$$\vec{\gamma} = \frac{\lambda_1 \lambda_2 (\mathbf{y}_1 - \mathbf{y}_2) + \lambda_2 d_1 \vec{\gamma}_1 + \lambda_1 d_2 \vec{\gamma}_2}{\lambda_1 d_2 + \lambda_2 d_1}$$

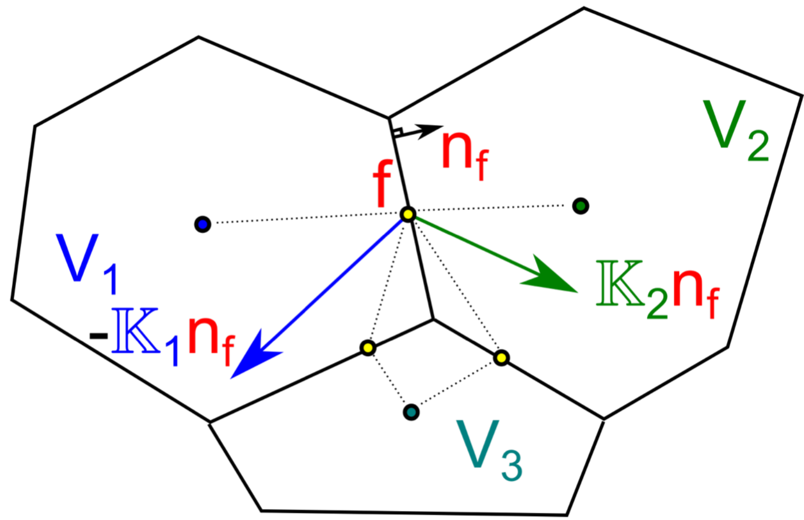
Robustness Improvement

Relaying only on harmonic averaging points is not very robust.

We can use flux continuity equation with respect to some point on interface to produce non-negative interpolation:

$$-\nabla p \cdot \mathbb{K}_1 \vec{n}_\sigma = \alpha(p_1 - p_f) + \beta(p_3 - p_f),$$

$$\nabla p \cdot \mathbb{K}_2 \vec{n}_\sigma = \gamma(p_2 - p_f) + \delta(p_3 - p_f).$$



It then can be expressed as:

$$p_f = \frac{\alpha p_1 + \gamma p_2 + (\beta + \delta) p_3}{\alpha + \beta + \gamma + \delta}.$$

Heterogeneous Inverse Distance Weighting

When it is not possible to obtain an all-positive second-order interpolation we use inverse distance weighting with the following choice of weights:

$$w_i = \frac{(\mathrm{x}_i - \mathrm{x}_e)\cdot \mathbf{K}_i \dfrac{\mathrm{x}_i - \mathrm{x}_e}{|\mathrm{x}_i - \mathrm{x}_e|}}{(\mathrm{x}_i - \mathrm{x}_e)^2}$$

It then can be expressed as:

$$p_e = \frac{\sum_i^n w_i p_i}{\sum_i^n w_i}.$$

One can notice that for a face in a locally K-orthogonal grid the method reproduces harmonic averaging.