Nonlinear Finite Volume Discretization Methods

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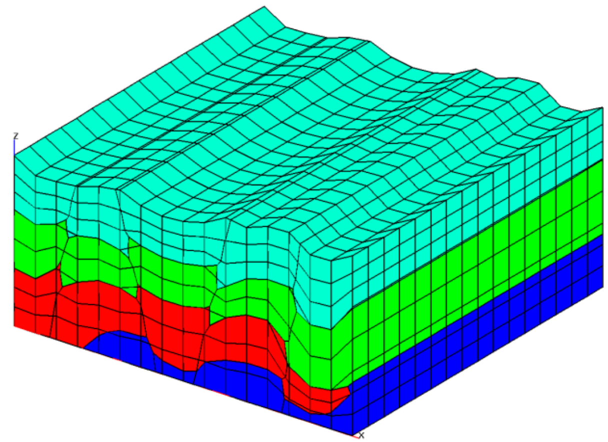
33rd SUPRI-B Annual Meeting

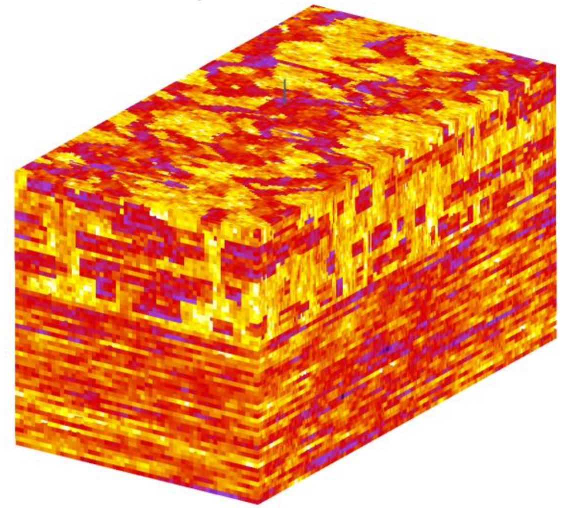
Anisotropic Diffusion Problem

$$\begin{cases} -\nabla \cdot \mathbb{K} \nabla p = g & \text{in} \quad \Omega \\ \mathbb{K} \dfrac{\partial p}{\partial \vec{n}} = g_N & \text{in} \quad \Gamma_N \\ p = g_D & \text{in} \quad \Gamma_D \end{cases}$$

Unstructured mesh



Heterogeneous tensor

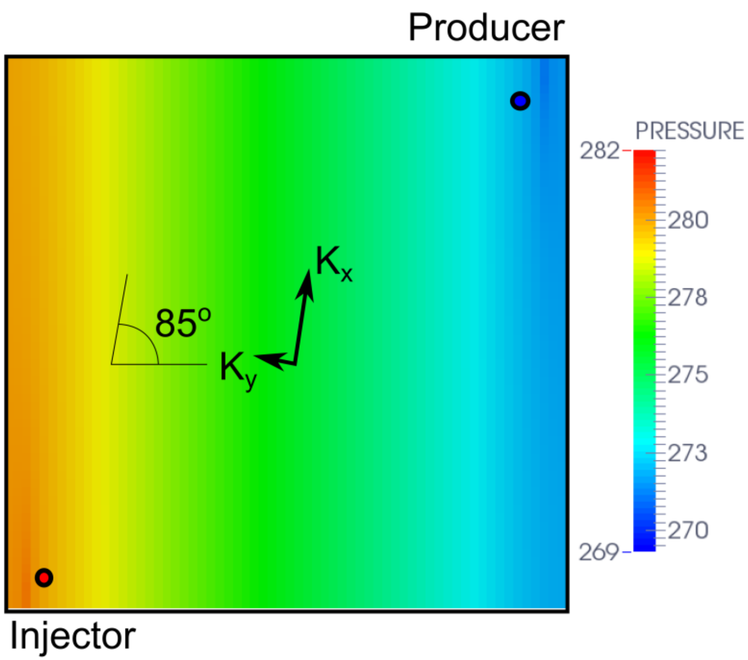


Domain Ω is decomposed into polyhedral mesh and $\mathbb{K}$ is symmetric positive definitive piecewise-constant tensor on every polyhedral.

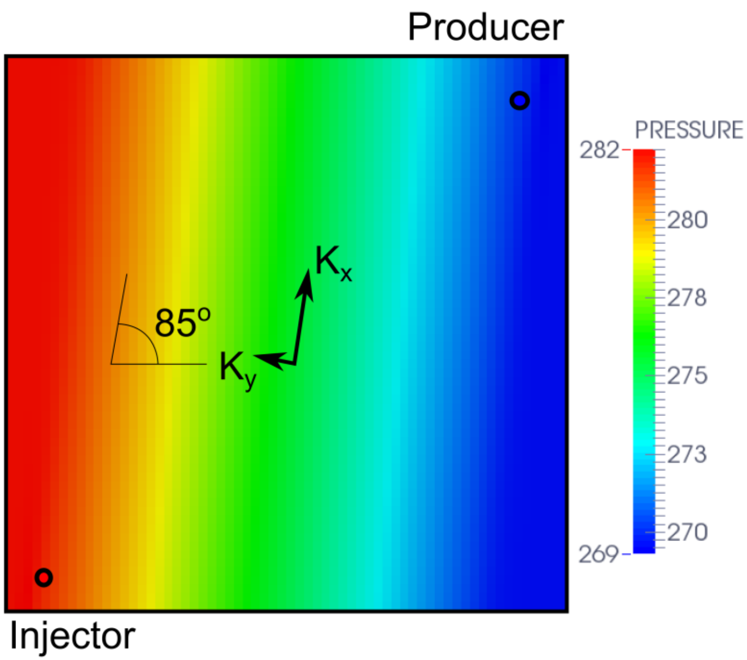
Discretization of the equation is a basis for any Black Oil simulator.

Loss of Approximation of Two-Point Flux Approximation

TPFA
O(1)-error, incorrect

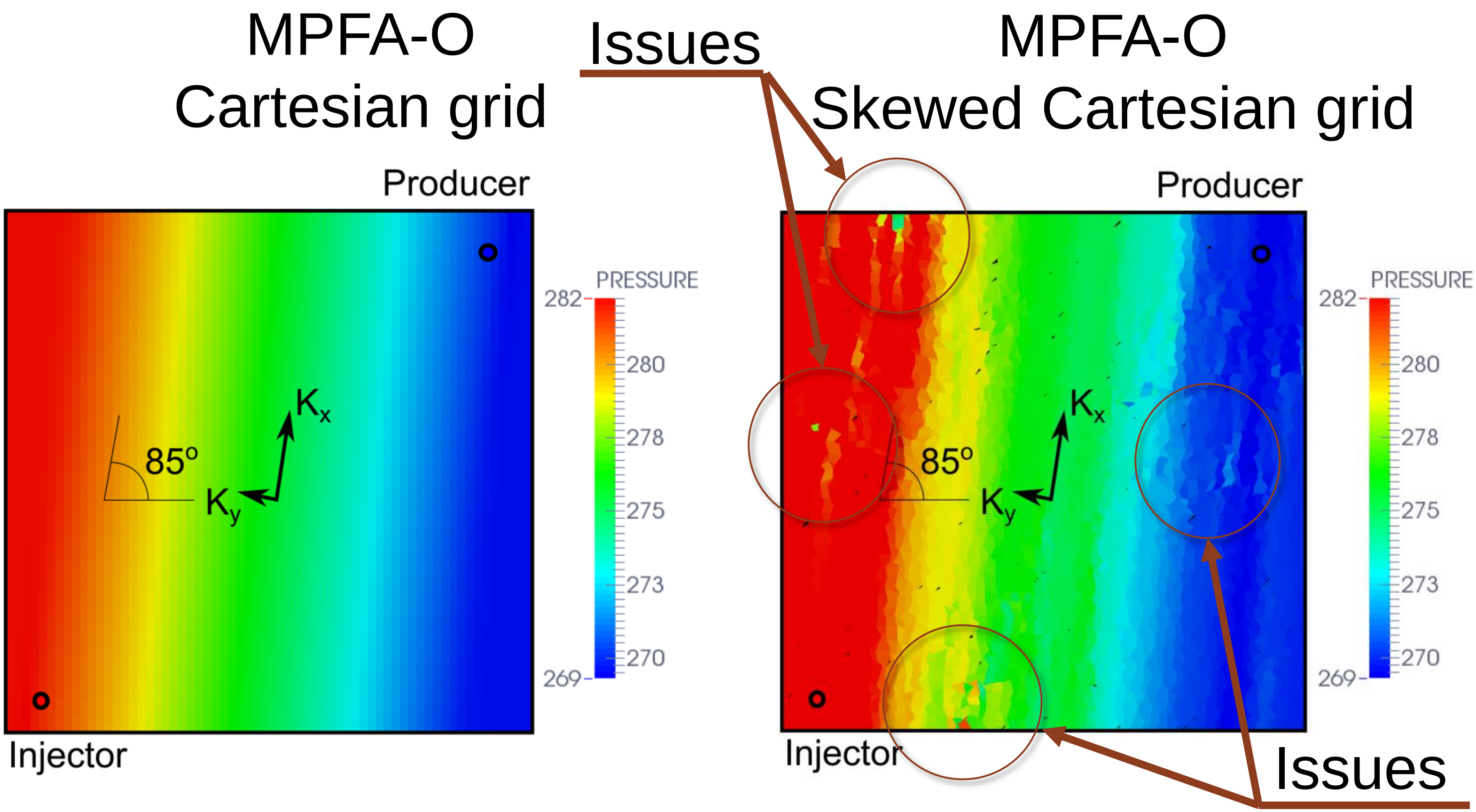


MPFA-O
O(h^2)-error, accurate



Single-phase problem with anisotropic permeability tensor.

Monotonicity Breakdown of Multi-Point Flux Approximation



Single-phase problem with anisotropic permeability tensor.

History of Development of MPFA Methods

- 1996 Aavatsmark, Barkve, Boe, Mannseth
- 1998 Edwards, Rogers
- 2006 MPFA-O Aavatsmark, Eigstad, Klausen
- 2007 MPFA-L Aavatsmark, Eigstad, Mallison, Nordbotten
- 2009 MFD-O Lipnikov, Shashkov, Yotov
- 2010 MPFA-G Agelas, Pietro, Droniou
- 2010 MPFA-O(general) Agelas, Guichard, Masson
- And many more others…

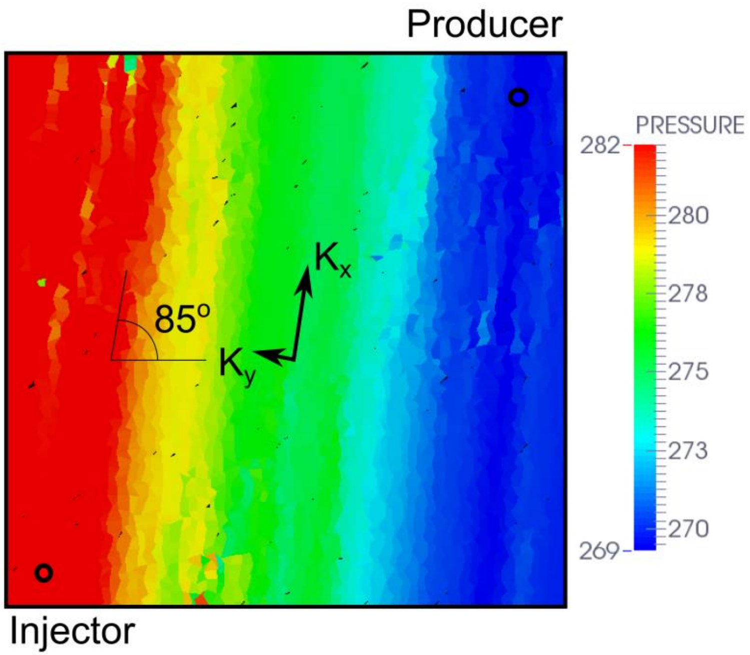
Transition to Nonlinearity

No linear small stencil scheme can be exact on linear solutions and respect the maximum principle.

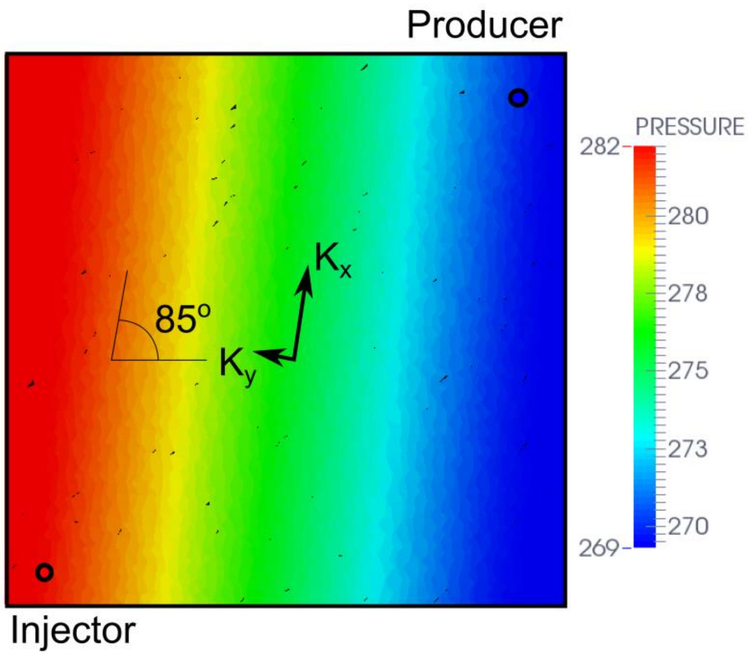
[Kershaw 1981, Nordbotton 2005]

Robustness of Nonlinear Two-Point Flux Approximation

MPFA-O
Skewed Cartesian grid



NTPFA
Skewed Cartesian grid



Single-phase problem with anisotropic permeability tensor.

History of Development of NTPFA Methods

- 2005 Le Potier
- 2007 Lipnikov, Shashkov, Svyatskiy, Vassilevski, Kapyrin
- 2009 Lipnikov, Svyatskiy, Vassilevski
- 2010 Danilov, Vassilevski
- 2012 Terekhov, Vassilevski
- 2013 Jimming, Zhimming
- 2013 Nikitin, Terekhov, Vassilevski
- 2014 Queiroz, Souza, Contreras, Lyra, Carvalho
- 2015 Zhimming, Jimming

Anisotropic Diffusion Problem

Focus on main equation.

$$
\begin{cases}
-\nabla \cdot \mathbb{K} \nabla p = g & \text{in} \quad \Omega \\
\mathbb{K} \dfrac{\partial p}{\partial \vec{n}} = g_N & \text{in} \quad \Gamma_N \\
p = g_D & \text{in} \quad \Gamma_D
\end{cases}
$$

Domain Ω is decomposed into polyhedral mesh and $\mathbb{K}$ is symmetric positive definitive piecewise-constant tensor on every polyhedral.

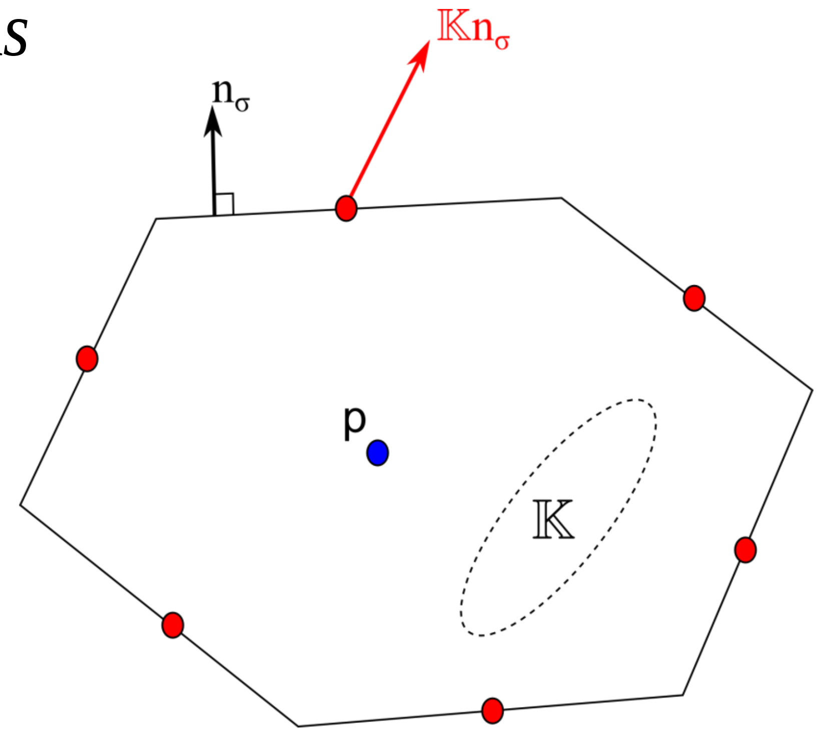
Finite Volume Discretization

$$\int_V -\nabla \cdot \mathbb{K}\nabla p \, \mathrm{dx} = \int_S -\mathbb{K}\nabla p \cdot \vec{n} \, \mathrm{ds}$$

Scalar product in red have an equivalent form:

$$\mathbb{K}\nabla p \cdot \vec{n} = \nabla p \cdot \mathbb{K}\vec{n}$$

We express the flux as gradient along co-normal $\mathbb{K}\vec{n}$.



Nonlinear Two-Point Scheme in Homogeneous Media

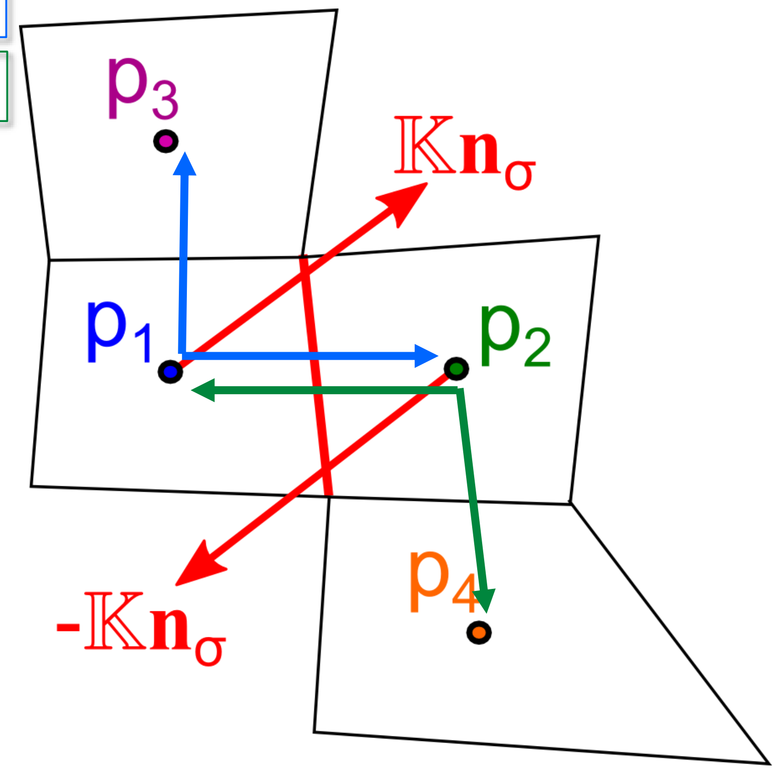
$$\nabla p \cdot \mathbb{K}\vec{n}_\sigma = \quad q_1 \boxed{= \alpha(p_2 - p_1) + \beta(p_3 - p_1)}$$

$$-\nabla p \cdot \mathbb{K}\vec{n}_\sigma = -q_2 \boxed{= \gamma(p_1 - p_2) + \delta(p_4 - p_2)}$$

With $\alpha, \beta, \gamma, \delta \geq 0$. The flux is a convex combination of the two:

$$q = \mu_1 q_1 + \mu_2 q_2$$

$$\mu_1 + \mu_2 = 1$$



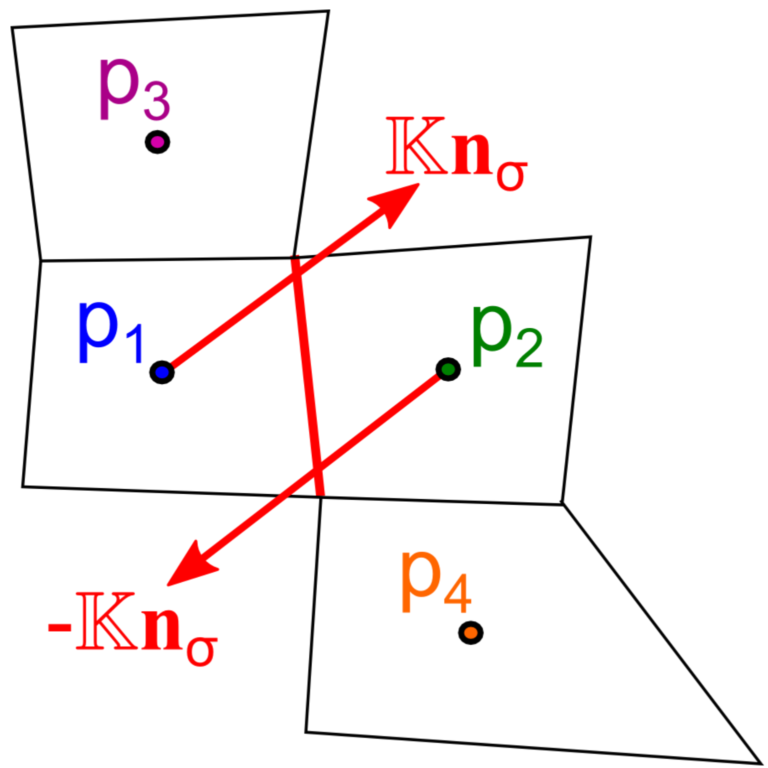
We are looking for the expression:

$$q = \mathbb{T}_1(p_3, p_4)\,p_2 - \mathbb{T}_2(p_3, p_4)\,p_1$$

$$= \big(\mu_1(\alpha+\beta) + \mu_2\gamma\big)\,p_2 - \big(\mu_1\alpha + \mu_2(\gamma+\delta)\big)\,p_1 + (\mu_1\beta\,p_3 - \mu_2\delta\,p_4)$$

Construction of Nonlinear Two-Point Scheme

$$\begin{cases} \mu_1 \beta p_3 - \mu_2 \delta p_4 = 0 \\ \mu_1 + \mu_2 = 1 \end{cases}$$



Results in:

$$\mu_1 = \frac{\delta p_4}{\beta p_3 + \delta p_4}, \quad \mu_2 = \frac{\beta p_3}{\beta p_3 + \delta p_4}$$

For Picard iterative method, solution to the problem is positive for positive right hand side. [Varga 1962]

$$\begin{array}{cc} & \begin{array}{cc} \quad p_1 & \qquad\qquad p_2 \end{array} \\ \begin{array}{c} V_1 \\ V_2 \end{array} & \begin{pmatrix} \mathbb{T}_1(p_3, p_4) & -\mathbb{T}_2(p_3, p_4) \\ -\mathbb{T}_1(p_3, p_4) & \mathbb{T}_2(p_3, p_4) \end{pmatrix} \end{array}$$

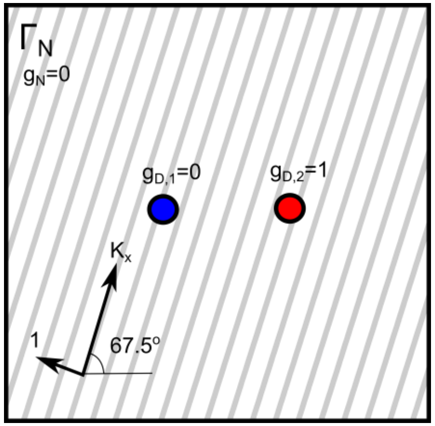
Discrete Maximum Principle: Two Wells Problem

Solution is expected to be bounded in [0,1].

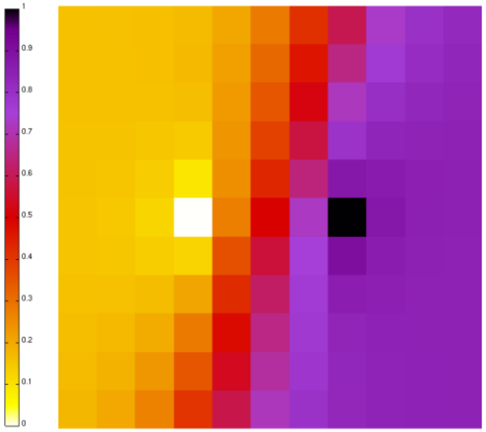
Scheme	P_{min}	P_{max}
TPFA	0	1
MPFA-O	-0.076	1.076
MPFA-L	-0.676	1.676
NTPFA	0	1.923

Overshot!

Problem setup



Good solution



Construction of Nonlinear Multi-Point Scheme

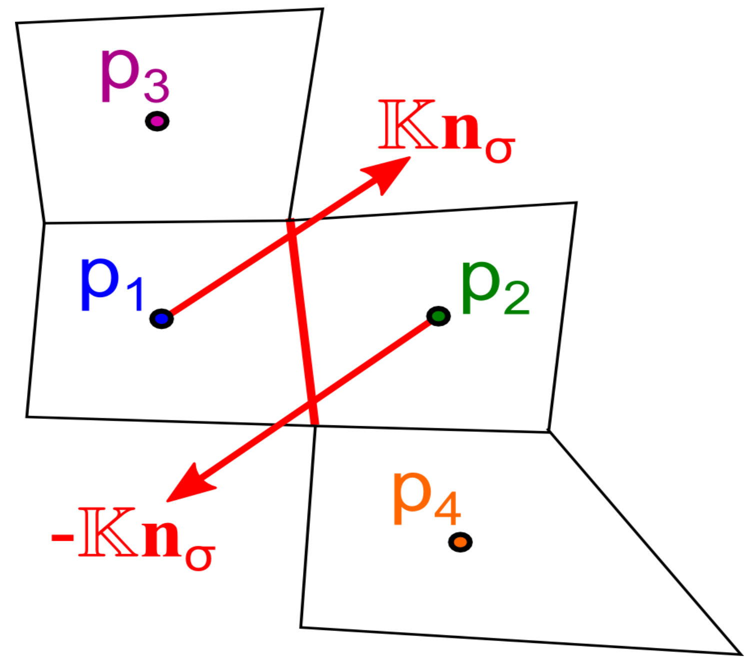
Different convex combination [Bertolazzi 2005]:

$$\mu_1 = \frac{|q_2|}{|q_1| + |q_2|}, \quad \mu_2 = \frac{|q_1|}{|q_1| + |q_2|}.$$

When q_1 and q_2 have the same sign:

$$q = \mu_1 q_1 + \mu_2 q_2 = 2\mu_1 q_1 = 2\mu_2 q_2$$

For Picard iterative method, solution to the problem satisfy discrete maximum principle. [Ciarlet 1969]



$$\begin{array}{ccccc} & p_1 & p_2 & p_3 & p_4 \\ \begin{array}{c} V_1 \\ V_2 \end{array} & \multicolumn{4}{c}{\begin{pmatrix} 2\mu_1(\alpha+\beta) & -2\mu_1\alpha & -2\mu_1\beta & \\ -2\mu_2\gamma & 2\mu_2(\gamma+\delta) & & -2\mu_2\delta \end{pmatrix}} \end{array}$$

History of Development of NMPFA Methods

- 2005 Bertolazzi, Manzini
- 2009 Le Potier
- 2011 Droniou, Le Potier
- 2012 Lipnikov, Svyatskiy, Vassilevski, NMPFA-B
- 2012 Zhiqiang, Guangwei
- 2013 Droniou (review) (presentation)

Discrete Maximum Principle: Single Well Problem

Solution is expected to be bounded in [10,12].

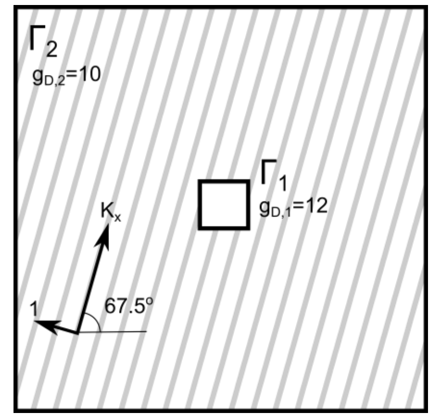
Scheme	P_{min}	P_{max}
TPFA	10	12
MPFA-O	9.7	12.22
MPFA-L	9.94	11.97
NTPFA	9.95	11.97
NTPFA*	10	12.17
NMPFA	10	11.96

*NTPFA respects lower bound.

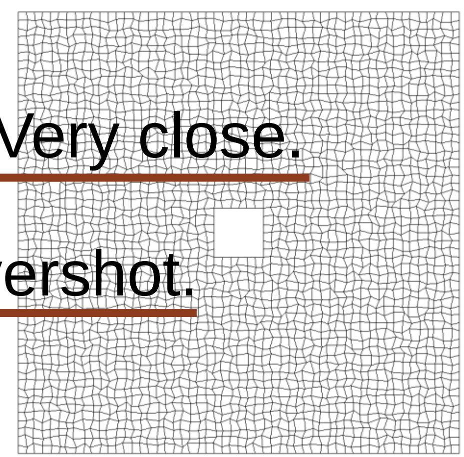
Very close.

Overshot.

Problem setup

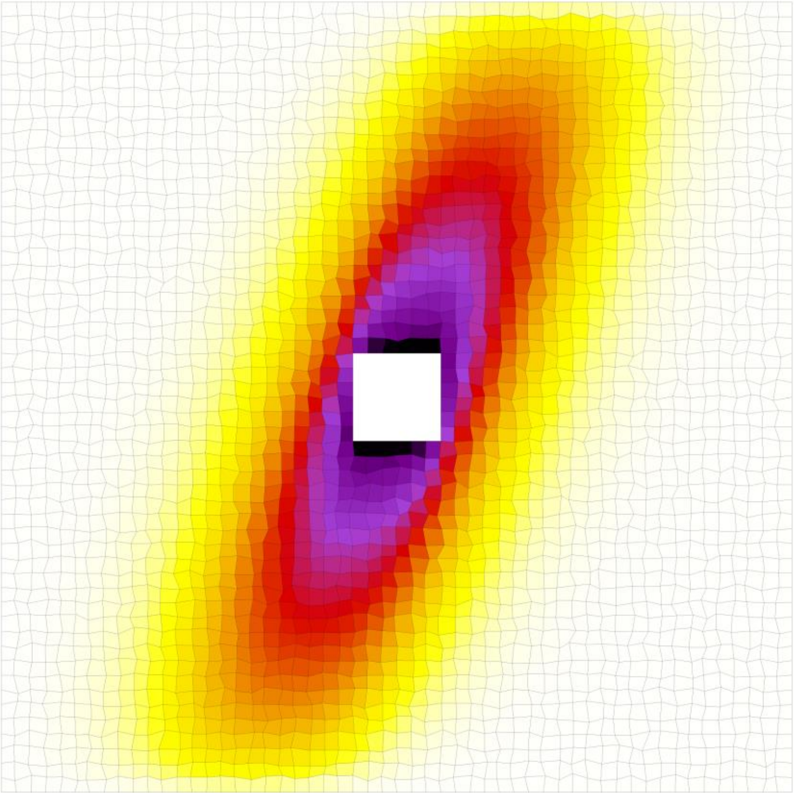


Mesh

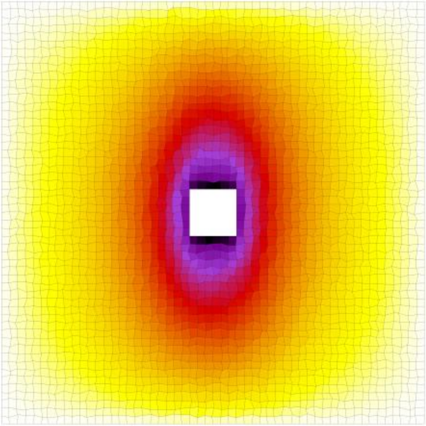


Discrete Maximum Principle: Single Well Problem

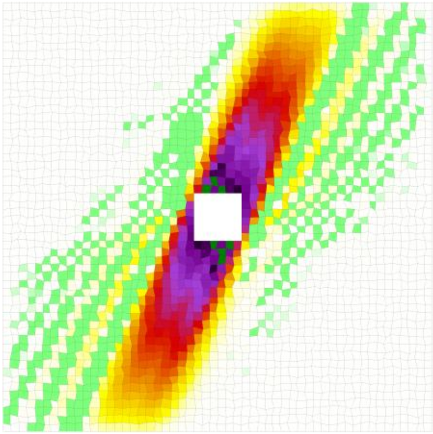
NMPFA, monotone, accurate



TPFA, inaccurate



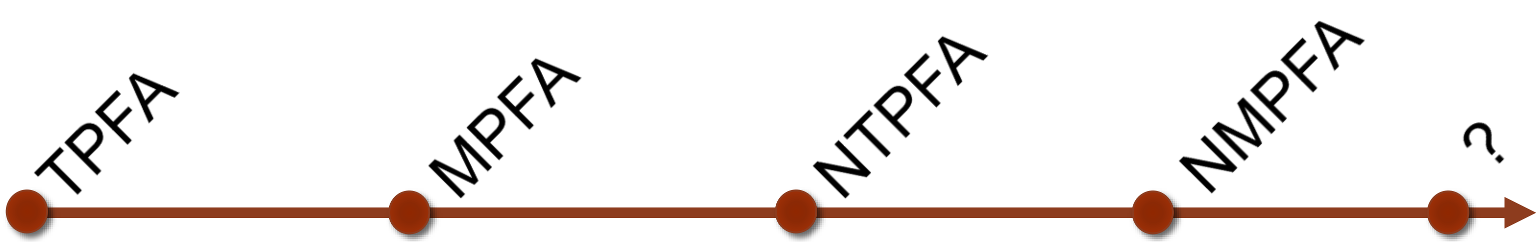
MPFA-O, non-monotone



Overview of Cell-Centered Discretization Methods

	TPFA	MPFA	NTPFA	NMPFA
Consistent	**NO!**	**YES**	**YES**	**YES**
Robustness	YES	NO	YES	YES
Discrete Maximum Principle	YES	NO	NO	YES

Convergence = Stability + Consistency

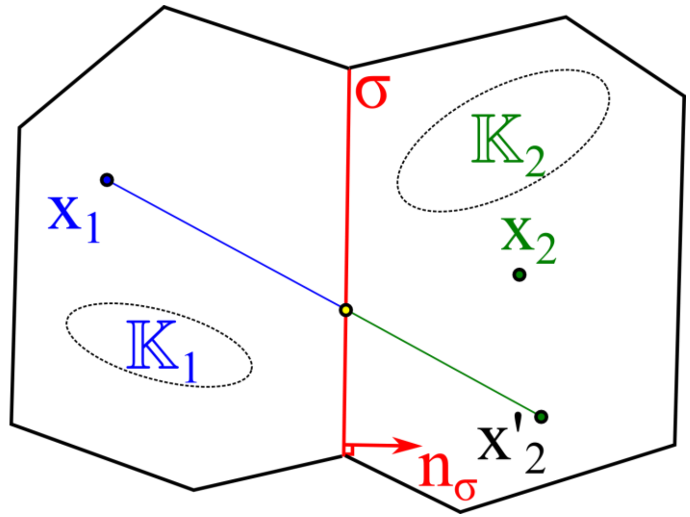


We’ve Just Opened a Pandora’s Box

A number of puzzling issues:

- Heterogeneous media: how to construct non-negative flux accurately?

Heterogeneous Media: Interpolation

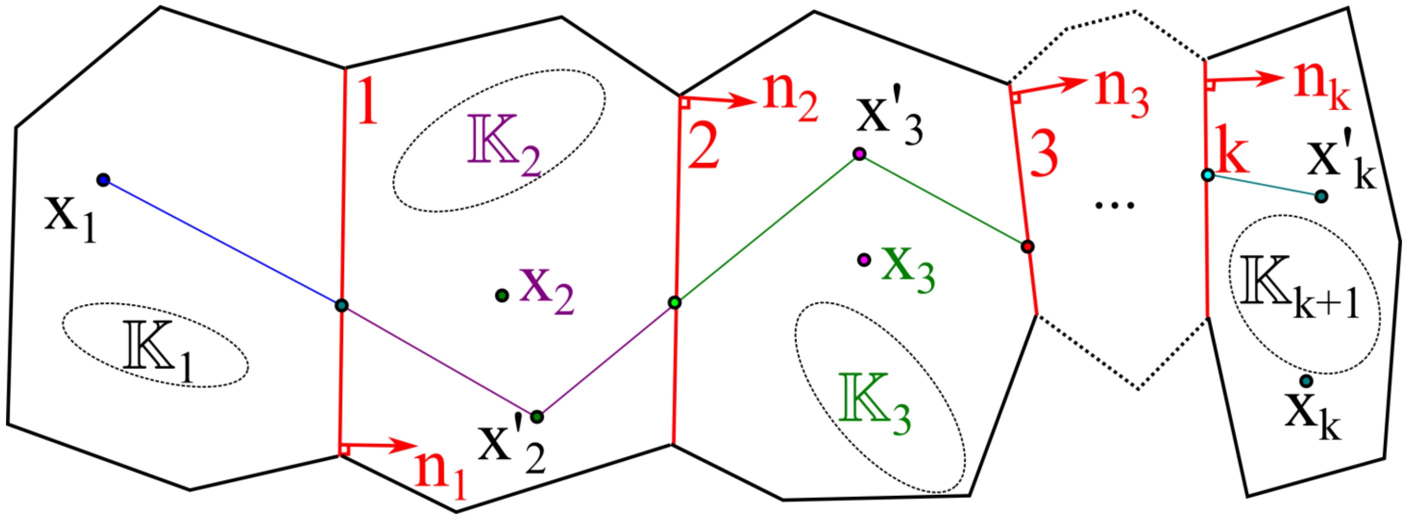


Second-order accurate interpolation over the discontinuity of permeability at one interface:

$$p(\mathbf{x}_2) = p_1 + \nabla_1 p \cdot \left(\left[\mathbb{I} + \frac{1}{\lambda_2^\sigma} [\mathbb{K}_1 - \mathbb{K}_2] \vec{n}_\sigma (\vec{n}_\sigma)^T \right] (\mathbf{x}_2 - \mathbf{x}_1) - \frac{d_1}{\lambda_2^\sigma} [\mathbb{K}_1 - \mathbb{K}_2] \vec{n}_\sigma \right)$$

Construct the scheme as in homogeneous case.

Heterogeneous Media: Interpolation



Second-order accurate interpolation over the discontinuity of permeability at multiple interfaces:

$$p(\mathbf{y}) = p_1 + \nabla_1 p \cdot \left(\prod_{i=1}^{k} \left[\mathbb{I} + \frac{1}{\lambda_{i+1}^{\sigma_i}} [\mathbb{K}_i - \mathbb{K}_{i+1}] \vec{n}_{\sigma_i} (\vec{n}_{\sigma_i})^{\mathrm{T}} \right] (\mathbf{y} - \mathbf{x}_1) - \sum_{j=1}^{k} \left(\prod_{i=1}^{j-1} \left[\mathbb{I} + \frac{1}{\lambda_{i+1}^{\sigma_i}} [\mathbb{K}_i - \mathbb{K}_{i+1}] \vec{n}_{\sigma_i} (\vec{n}_{\sigma_i})^{\mathrm{T}} \right] \right) \frac{d_1^{\sigma_j}}{\lambda_{j+1}^{\sigma_j}} [\mathbb{K}_j - \mathbb{K}_{j+1}] \vec{n}_{\sigma_j} \right)$$

History of Interpolation in Heterogeneous Media

- 2007 Lipnikov, Svyatsky, Vassilevski, inverse distance
- 2008 Yue, Yuan, Sheng, least-squares
- 2008 Chen, Wan, Yang, Muffin, divergence-free
- 2009 Lipnikov, Svyatskiy, Vassilevski, interpolation-free
- 2009 Danilov, Vassilevski, nonlinear two-point
- 2009 Agelas, Di Pietro, Masson, harmonic averaging points
- 2013 Vidovic, Dotlic, Dimkic, Pusic, Pokorni, homogenization function

Current approach: combination of the two.

We've Just Opened a Pandora's Box

A number of puzzling issues:

- Heterogeneous media: how to construct non-negative flux accurately?
- Mesh locking: convergence of nonlinear schemes deteriorates with strong anisotropy.

Locking Issue

- A numerical scheme for the approximation of parameter-dependent problem is said to exhibit locking, if the accuracy deteriorates as the parameter tends to a limiting value. [Babuska 1992]

Permeability tensor, anisotropy ratio.

  - Use flux splitting [Manzini 2006]

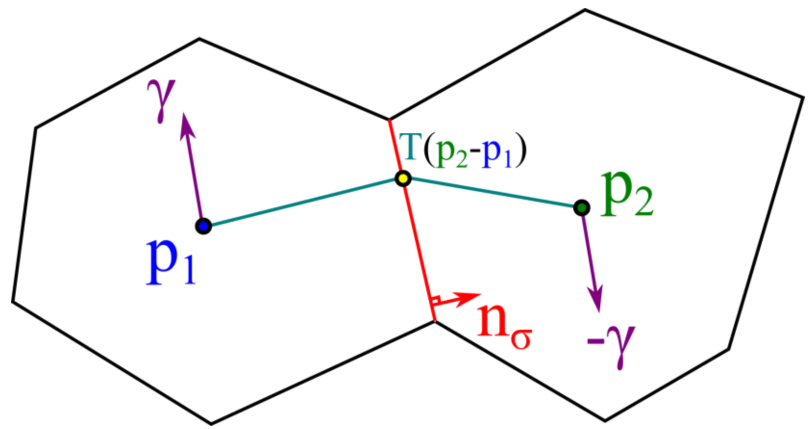
Locking Issue: Flux Splitting

Two-Point Flux Approximation is:

$$\mathrm{T}(p_2 - p_1) = \frac{\lambda_1 \lambda_2}{\lambda_1 d_2 + \lambda_2 d_1}(p_2 - p_1)$$

Missing transverse part of the flux:

$$\vec{\gamma} = \frac{\lambda_1 \lambda_2 (\mathbf{y}_1 - \mathbf{y}_2) + \lambda_2 d_1 \vec{\gamma}_1 + \lambda_1 d_2 \vec{\gamma}_2}{\lambda_1 d_2 + \lambda_2 d_1}$$



Now:

$$q_1 = \nabla p \cdot \vec{\gamma}$$

$$-q_2 = -\nabla p \cdot \vec{\gamma}$$

$$q = \mathrm{T}(p_2 - p_1) + \mu_1 q_1 + \mu_2 q_2$$

Harmonic part

Transversal part (nonlinear)

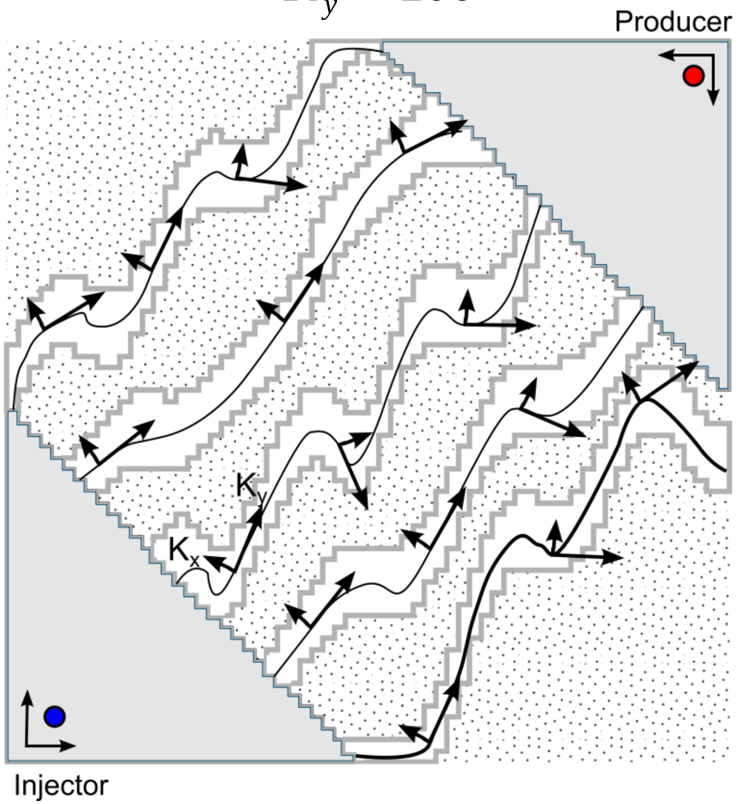
$$\lambda_i = \vec{n}_\sigma \cdot \mathbb{K}_i \vec{n}_\sigma$$

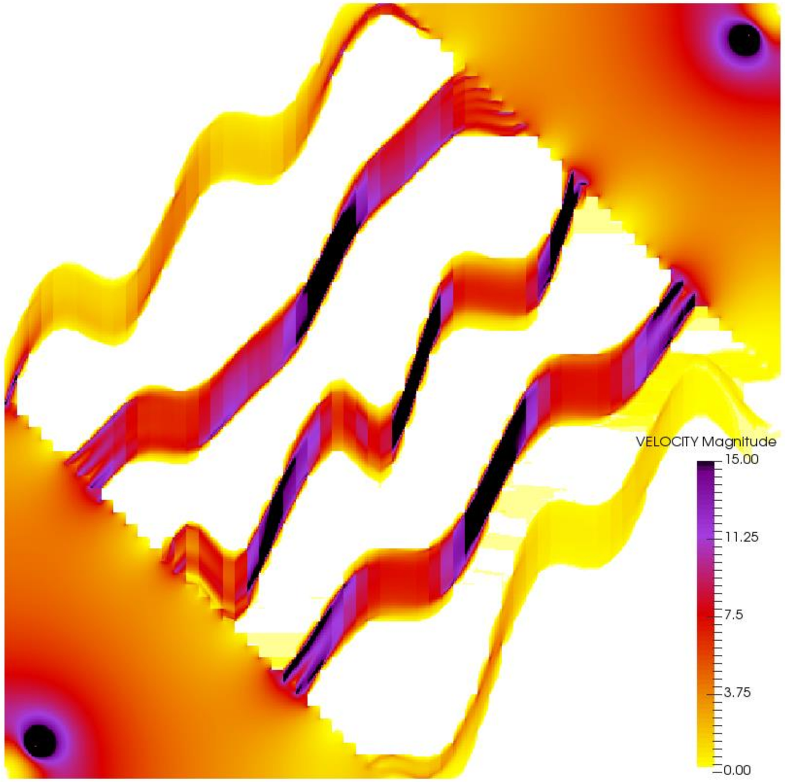
$$\vec{\gamma}_i = \left(\mathbb{K}_i - \lambda_i \mathbb{I}\right)\vec{n}_\sigma$$

Full derivation is in backup slides.

Locking Issue: Channels

$$\frac{K_x}{K_y} = \frac{1}{100}$$

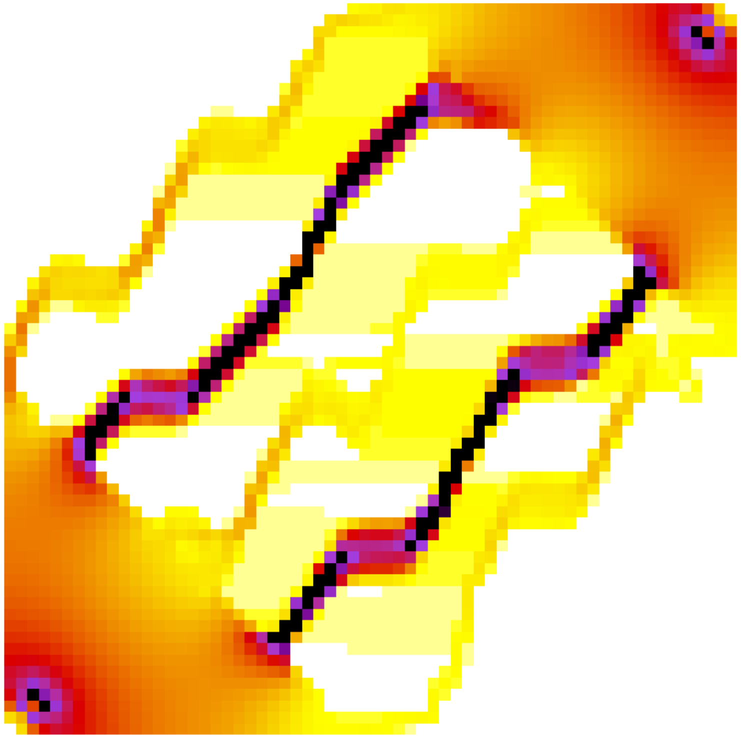




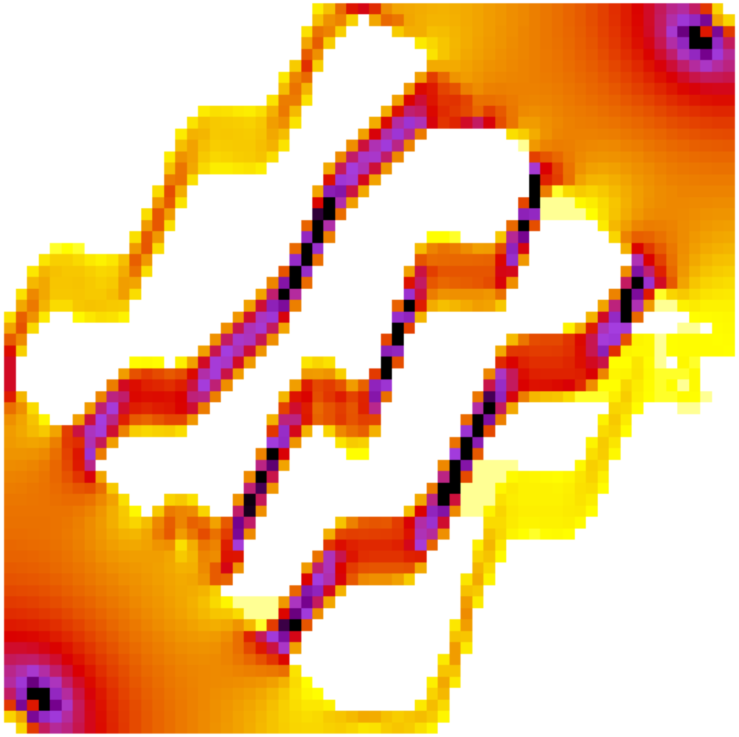
Permeability tensor pattern

Velocity magnitude

Locking Issue: Channels



NMPFA-B



NMPFA-A

Locking Issue: Channels

Scheme	Rate on well
MPFA-O, fine mesh	158,652
MPFA-B	146,5
NTPFA-B	136,02
NMPFA-B	128,98
MPFA-A	156,9
NTPFA-A	156,55
NMPFA-A	155,78

Nonlinearity induces locking

Locking Issue

- A numerical scheme for the approximation of parameter-dependent problem is said to exhibit locking, if the accuracy deteriorates as the parameter tends to a limiting value. [Babuska 1992]
  - Use flux splitting [Manzini 2006]
- Schemes that satisfy M-matrix property fall back to first order accuracy at extrema even when they realize higher order accuracy elsewhere. [Jameson 1994] (context: convection problem)
  - Nonlinear convex combination introduces another parameter.
  - Relax monotonicity constraint. (future research)
  - Avoid approximation across extrema. (future research)

History of Mesh Locking Problem

- <1980 Zienkiewicz, Pawsy, Clough, MacNeal…
- 1992 Babuska, Suri
- 2001 Havu, Pitkaranta
- 2003 Havu
- 2004 Slimane, Renard
- 2006 Manzini, Putti
- 2011 Arnautu, Mosneagu
- 2013 Ambroziak (review)

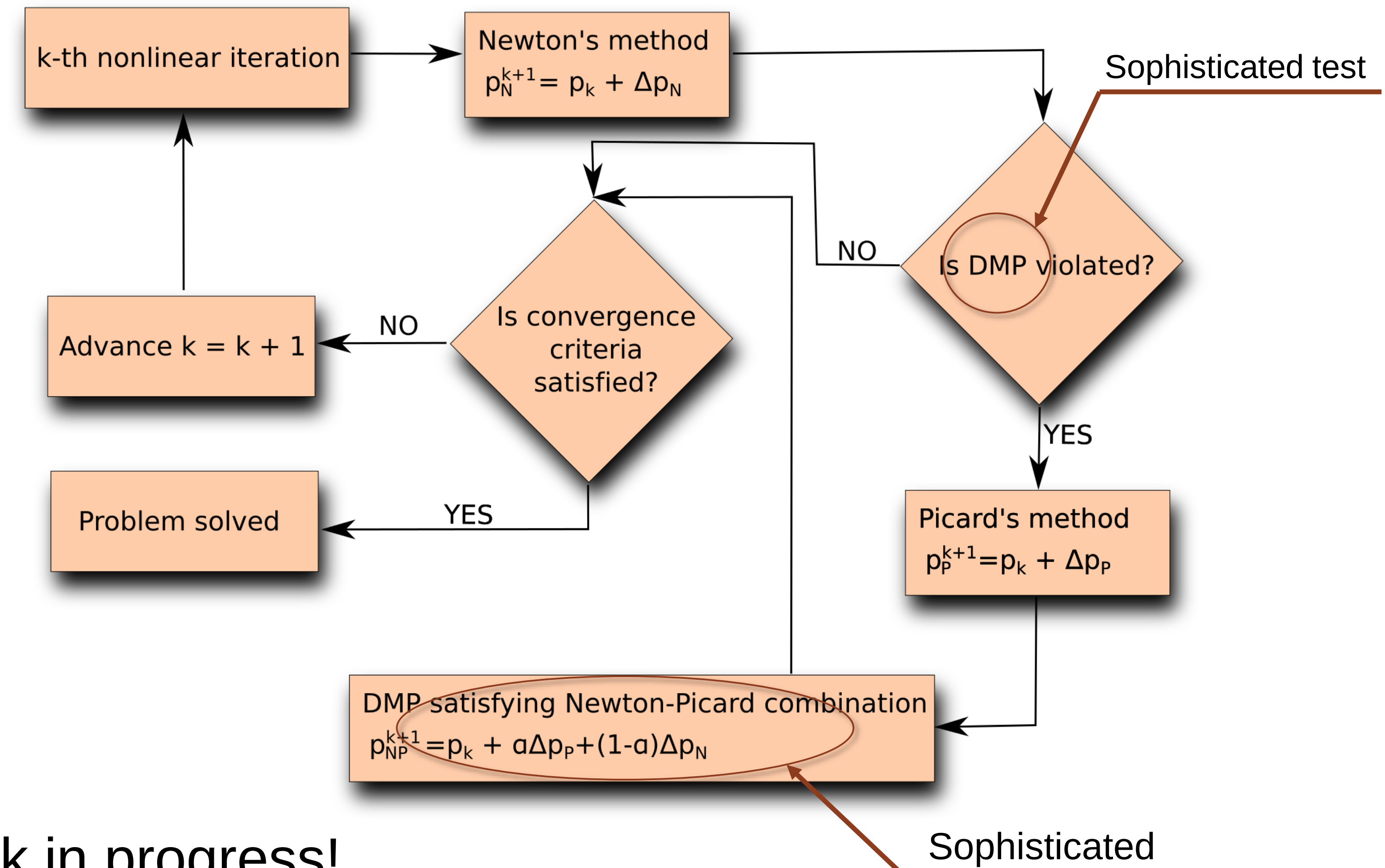
+ harmonic averaging = current approach

We've Just Opened a Pandora's Box

A number of puzzling issues:

- Heterogeneous media: how to construct non-negative flux accurately?
- Mesh locking: convergence of nonlinear schemes deteriorates with strong anisotropy.
- Nonlinear solver: how to solve the problem efficiently?

Nonlinear Solver: Newton-Picard Method

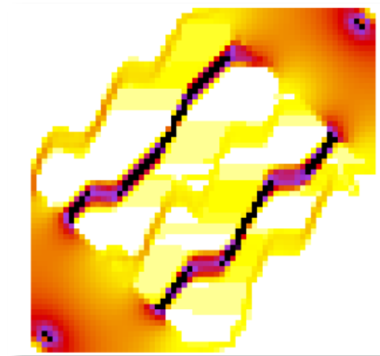


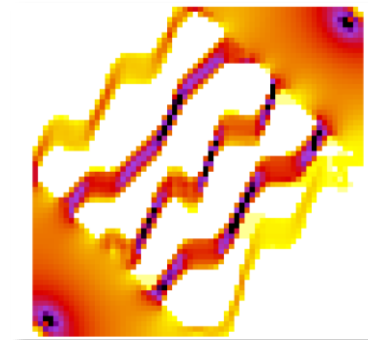
Work in progress!

Nonlinear Solver Iterations: Channels

Appears in literature, AA = Anderson Acceleration

Scheme	Newton-Picard	AA-Picard
MPFA-O, fine mesh	-	-
	-	-
MPFA-B	-	-
NTPFA-B	2	60
NMPFA-B	193 (111)	∞
MPFA-A	-	-
NTPFA-A	2	33
NMPFA-A	11 (4)	159



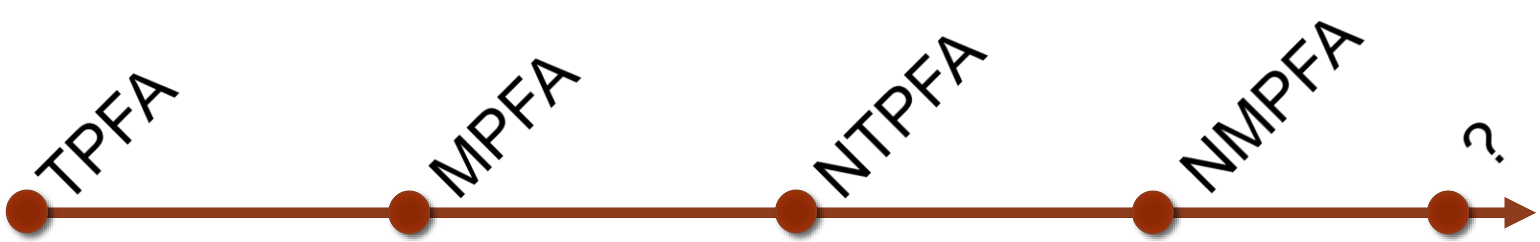


Scheme formulation is important

Picard iterations

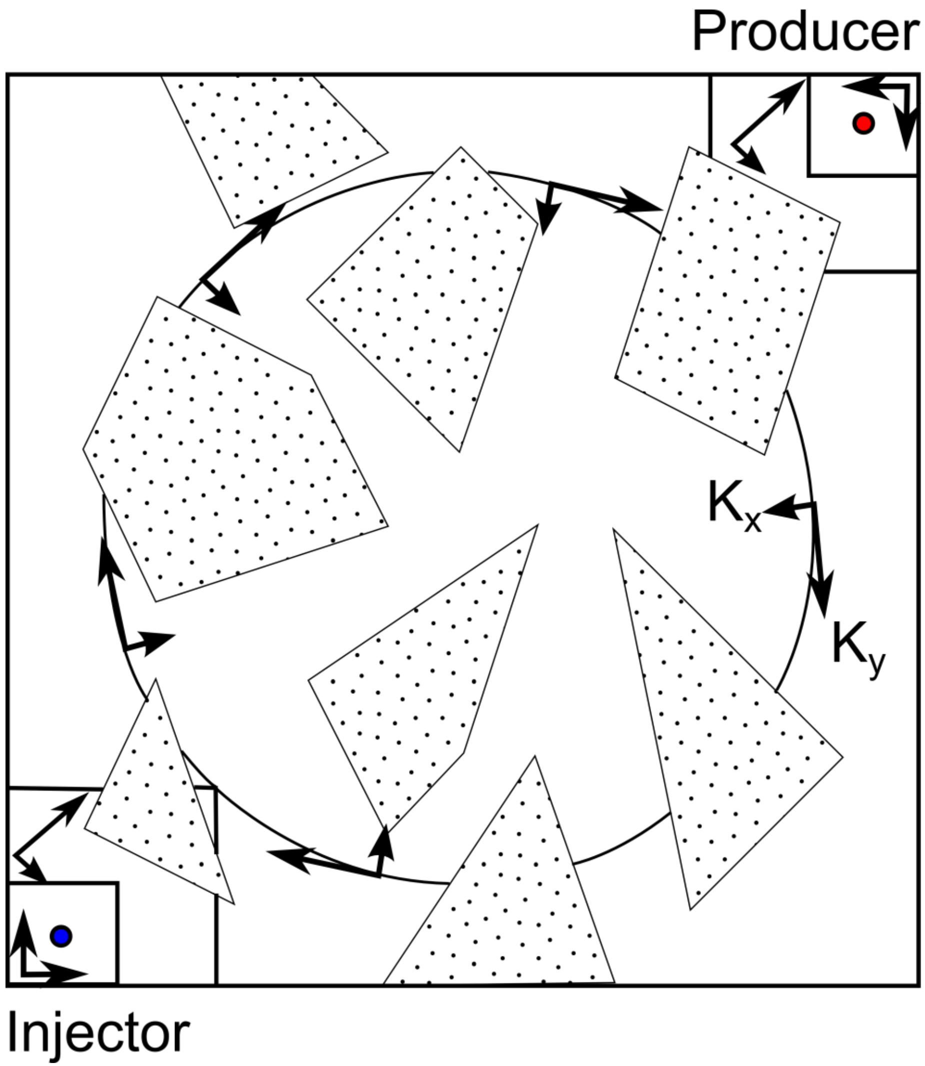
Overview of Cell-Centered Discretization Methods

	TPFA	MPFA	NTPFA	NMPFA
Approximation	**NO!**	**YES**	**YES**	**YES**
Robustness	YES	NO	YES	YES
Discrete Maximum Principle	YES	NO	NO	YES
Locking-free	YES	NO	IMPROVED	IMPROVED
Efficiency	YES	YES	YES	IMPROVED



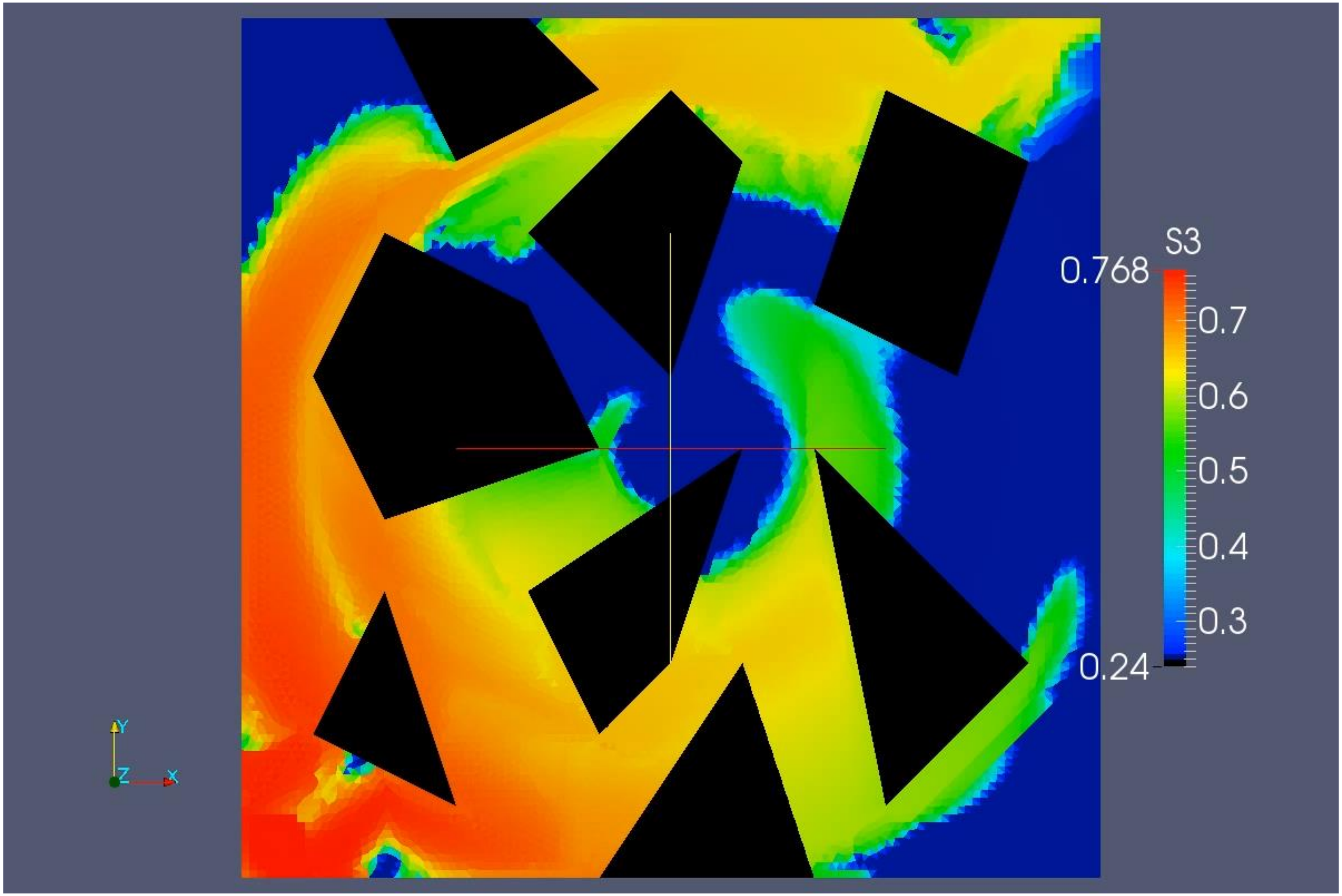
Available in AD-GPRS with DiscretizationToolkit

Synthetic test problem: AD-GPRS Black-Oil



$$\frac{K_x}{K_y} = \frac{1}{100}$$

Synthetic test problem: AD-GPRS Black-Oil



Future Work

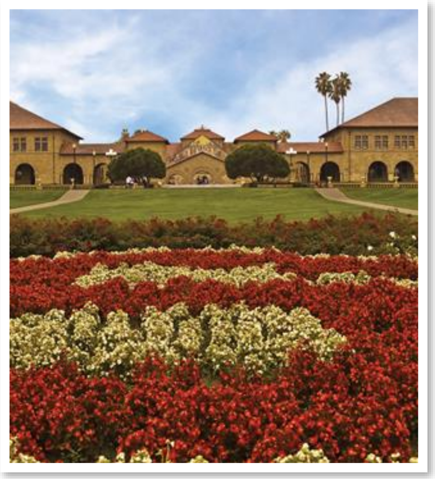
- Modifications to NMPFA formulation, to improve on locking and linear solver robustness:
  - Nonlinear parameterization of approximation.
  - Different choice of nonlinear convex combination.

- Nonlinear finite volume techniques for other physics:
  - Advection in heterogeneous media.
  - Higher-order mobility upstream weighting.
  - Heterogeneous linear elasticity.

Thank you for Attention!

ACKNOWLEDGEMENTS:

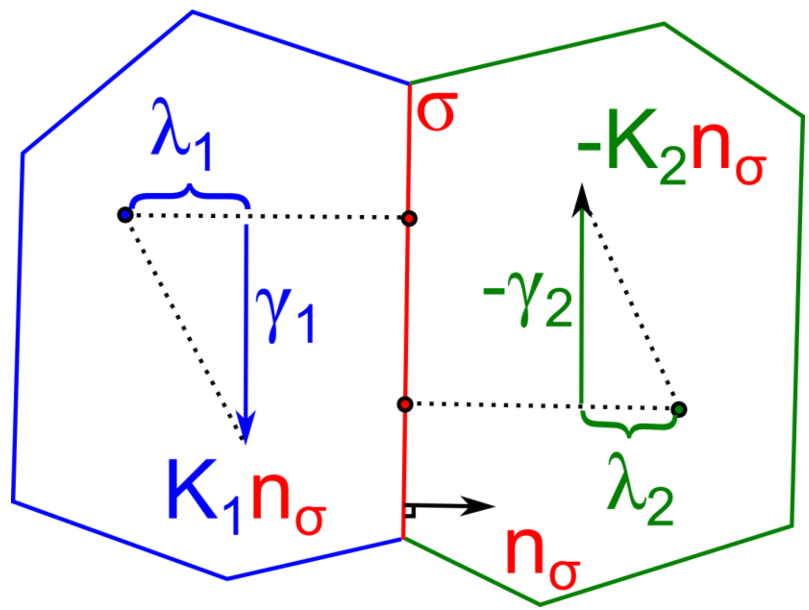
- OLEG VOLKOV
- CHEVRON



Co-Normal Representation

For each interface and each material we will decompose normal into:

$$\mathbb{K}\vec{n}_\sigma = \lambda\vec{n}_\sigma + \vec{\gamma}$$



Lambda is a projection of co-normal onto normal:

$$\lambda = \vec{n}_\sigma \cdot \mathbb{K}\vec{n}_\sigma$$

Gamma is made coplanar to interface:

$$\vec{\gamma} = \left(\mathbb{K} - \lambda\mathbb{I}\right)\vec{n}_\sigma$$

Flux Continuity

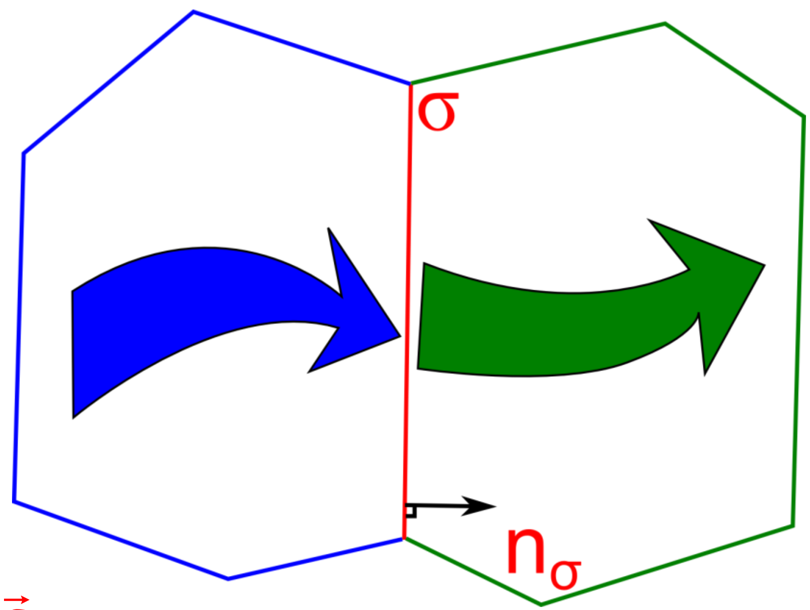
We enforce flux continuity on each interface:

$$\nabla p \cdot \mathbb{K}_1 \vec{n}_\sigma = \nabla p \cdot \mathbb{K}_2 \vec{n}_\sigma$$

We assume continuity of tangential derivative $\vec{g}_\sigma$. Then for a point y on interface:

$$\lambda_1 \frac{p(\mathbf{y}) + (\mathbf{y}_1 - \mathbf{y}) \cdot \vec{g}_\sigma - p_1}{d_1} + \vec{\gamma}_1 \cdot \vec{g}_\sigma$$

$$= \lambda_2 \frac{p_2 - p(\mathbf{y}) - (\mathbf{y}_2 - \mathbf{y}) \cdot \vec{g}_\sigma}{d_2} + \vec{\gamma}_2 \cdot \vec{g}_\sigma$$



Harmonic Averaging Point

By regrouping terms in flux continuity equation we get:

$$p(\mathbf{y}) = \frac{\lambda_1 d_2 p_1 + \lambda_2 d_1 p_2}{\lambda_1 d_2 + \lambda_2 d_1} + \frac{d_1 d_2}{\lambda_1 d_2 + \lambda_2 d_1}(\vec{\gamma}_2 - \vec{\gamma}_1) \cdot \vec{g}_\sigma + \left(\mathbf{y} - \frac{\lambda_1 d_2 \mathbf{y}_1 + \lambda_2 d_1 \mathbf{y}_2}{\lambda_1 d_2 + \lambda_2 d_1} \right) \cdot \vec{g}_\sigma$$

From this we find a harmonic averaging point [Agelas 2009]
:

$$\mathbf{y}_\sigma = \frac{\lambda_1 d_2 \mathbf{y}_1 + \lambda_2 d_1 \mathbf{y}_2 + d_1 d_2 (\vec{\gamma}_1 - \vec{\gamma}_2)}{\lambda_1 d_2 + \lambda_2 d_1}$$

$$p(\mathbf{y}_\sigma) = \frac{\lambda_1 d_2 p_1 + \lambda_2 d_1 p_2}{\lambda_1 d_2 + \lambda_2 d_1}$$

Transverse Flux Derivation

By construction for $y = y_\sigma$ in normal part of total flux we obtain:

$$\lambda_1 \frac{p(\mathbf{y}_\sigma) - p_1}{d_1} = \lambda_2 \frac{p_2 - p(\mathbf{y}_\sigma)}{d_2} = \frac{\lambda_1 \lambda_2}{\lambda_1 d_2 + \lambda_2 d_1}(p_2 - p_1)$$

Which is exactly two-point flux approximation. What is missing?

$$\mathbb{K}_1 \vec{n}_\sigma - \frac{\lambda_1}{d_1}(\mathbf{y}_\sigma - \mathbf{x}_1) = \mathbb{K}_2 \vec{n}_\sigma - \frac{\lambda_2}{d_2}(\mathbf{x}_2 - \mathbf{y}_\sigma) = \frac{\lambda_1 \lambda_2 (\mathbf{y}_1 - \mathbf{y}_2) + \lambda_2 d_1 \vec{\gamma}_1 + \lambda_1 d_2 \vec{\gamma}_2}{\lambda_1 d_2 + \lambda_2 d_1}$$

This gives us the direction for transverse part of the total flux that is missing in two-point flux approximation.

$$\vec{\gamma} = \frac{\lambda_1 \lambda_2 (\mathbf{y}_1 - \mathbf{y}_2) + \lambda_2 d_1 \vec{\gamma}_1 + \lambda_1 d_2 \vec{\gamma}_2}{\lambda_1 d_2 + \lambda_2 d_1}$$