# ADVANCED HIGH SCHOOL MATHEMATICS

## COMPLEX NUMBERS

## ARITHMETIC OPERATIONS WITH

## COMPLEX VARIABLES

You need to gain the ability to add, subtract, multiply and divide complex numbers.

The addition, subtraction, multiplication or division of two complex variables $z_1$ and $z_2$

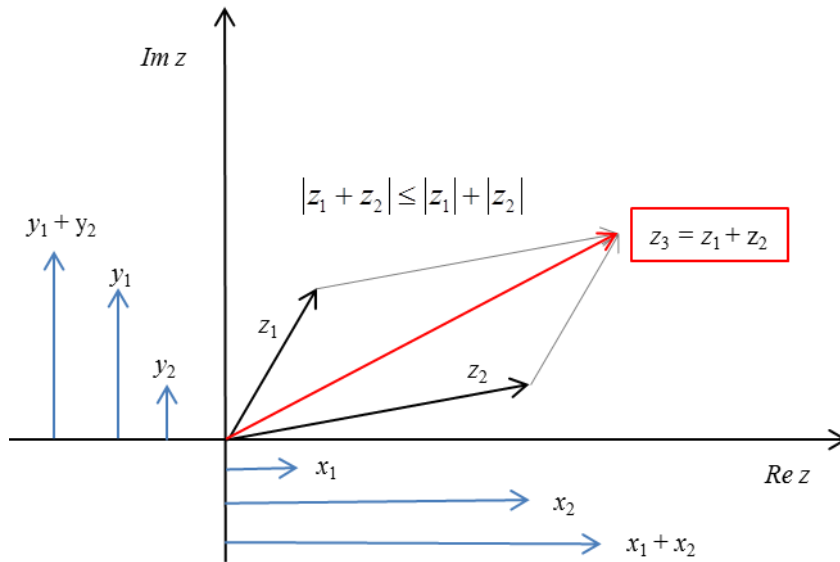$$z_1 = x_1 + i\ y_1 \qquad z_2 = x_2 + i\ y_2 \qquad x_1, y_1, x_2, y_2 \quad \text{real numbers}$$

produces other complex variables

### Addition

$$z_3 = z_1 + z_2 = x_1 + i\ y_1 + x_2 + i\ y_2$$
$$z_3 = (x_1 + x_2) + i\ (y_1 + y_2)$$

add real parts     add imaginary parts

The vector representing $z_3 = z_1 + z_2$ corresponds to the diagonal of a parallelogram with the vectors $z_1$ and $z_2$ as adjacent sides.

From the parallelogram, it is obvious that

$$|z_1 + z_2| \le |z_1| + |z_2|$$

## Subtraction

$$z_4 = z_1 - z_2 = x_1 + i \ y_1 - \left( x_2 + i \ y_2 \right)$$
$$z_4 = \left( x_1 - x_2 \right) + i \left( y_1 - y_2 \right)$$

subtract real parts        subtract imaginary parts

For subtraction using the Argand diagram simply add the vectors $z_1$ and $-z_2$

$$z_4 = z_1 - z_2 = z_1 + \left( -z_2 \right)$$

## Multiplication

$$z_5 = z_1 \ z_2 = \left( x_1 + i \ y_1 \right) \left( x_2 + i \ y_2 \right) = \left( x_1 \ x_2 - y_1 \ y_2 \right) + i \left( x_1 \ y_2 + x_2 \ y_1 \right)$$

## Division

$$z_6 = \frac{z_1}{z_2} = \frac{z_1 \ \bar{z}_2}{z_2 \ \bar{z}_2}$$

Multiply both the numerator and denominator by the complex conjugate of the denominator so that you are now dividing by the real number $z_2 \ \bar{z}_2$.

Multiplication or division of two complex numbers is accomplished most easily when they are in exponential form.

$$z_5 = z_1 \, z_2 = \left( R_1 \, e^{i\,\theta_1} \right)\left( R_2 \, e^{i\,\theta_2} \right) = R_1 \, R_2 \, e^{i\,(\theta_1 + \theta_2)} = R_1 \, R_2 \, \angle\left( \theta_1 + \theta_2 \right)$$
$$= R_1 \, R_2 \left[ \cos\left( \theta_1 + \theta_2 \right) + i \, \sin\left( \theta_1 + \theta_2 \right) \right]$$

$$z_6 = \frac{z_1}{z_2} = \left( \frac{R_1}{R_2} \right) e^{i\,(\theta_1 - \theta_2)} = \left( \frac{R_1}{R_2} \right) \angle\left( \theta_1 - \theta_2 \right)$$
$$= R_1 \, R_2 \left[ \cos\left( \theta_1 - \theta_2 \right) + i \, \sin\left( \theta_1 - \theta_2 \right) \right]$$

A complex number can be shown as a vector on an Argard diagram.

$$z_1 = x_1 + i \, y_1 = R_1 \, e^{i\,\theta_1} \qquad R_1 = \sqrt{x_1^{\,2} + y_1^{\,2}} \qquad \tan\theta_1 = \frac{y_1}{x_1}$$

We will consider the result of multiplying the complex number $z_1$ by the complex number $e^{i\theta}$

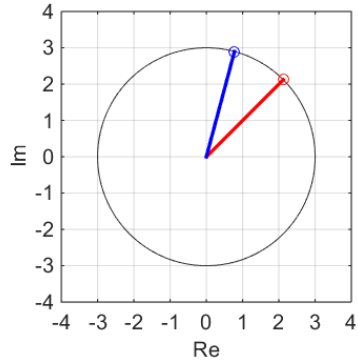$$z = e^{i\theta} \, R_1 \, e^{i\theta_1} = R_1 \, e^{i(\theta + \theta_1)}$$

This means that the vector for $z_1$ is rotated anticlockwise about the origin through an angle $\theta$ to produce the vector for the new vector $z$.
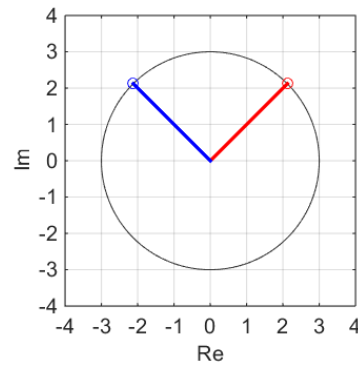


$$z_1 = x_1 + i \, y_1 = R_1 \, e^{i\theta_1}$$
$$z = x + i \, y = e^{i\theta} \, R_1 \, e^{i\theta_1} = R_1 \, e^{i(\theta + \theta_1)}$$
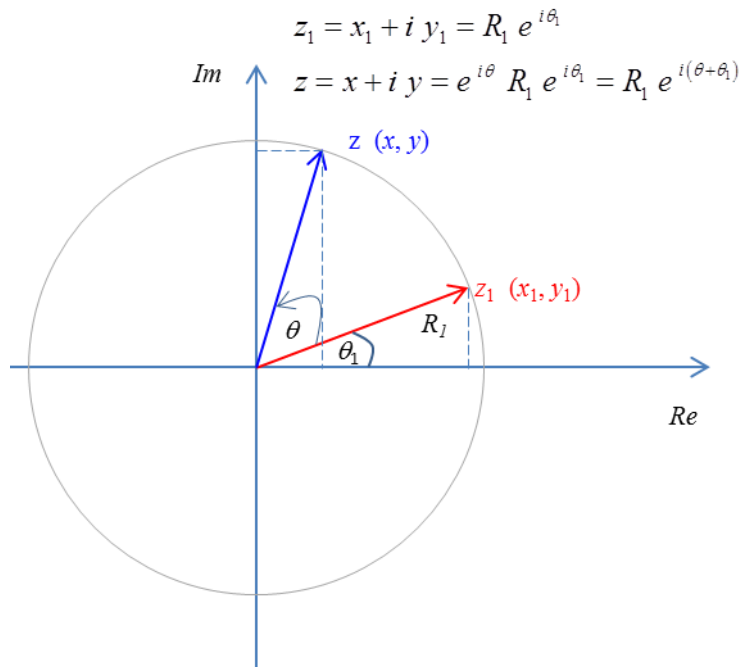
$$z = e^{i\theta} \quad z_1 = R_1 e^{i\theta_1} = R_1 e^{i(\theta+\theta_1)}$$

Let $z_1 = 3e^{i\left(\pi/4\right)} = 3\left[\cos\left(\pi/4\right) + i\sin\left(\pi/4\right)\right] = 2.1213(1+i)$

$$\theta = \pi/6 \quad z = 3\left(e^{i\left(\pi/6 + \pi/4\right)}\right) = 3e^{i\left(5\pi/12\right)}$$
$$= 0.7765 + i\,(2.8978)$$



$$\theta = \pi/2 \quad z = \left(e^{i\left(\pi/2\right)}\right)\left(3e^{i\left(\pi/4\right)}\right)$$
$$z = \left[\cos\left(\pi/2\right) + i\sin\left(\pi/2\right)\right]\left(3e^{i\left(\pi/4\right)}\right)$$
$$z = i\left(3e^{i\left(\pi/4\right)}\right) = -2.1213 + i\,(2.1213)$$
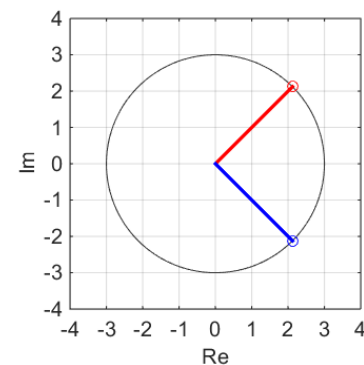
Multiplication by $i$ produces an anticlockwise rotation by $(\pi/2)$ rad.



$$\theta = \pi \quad z = \left(e^{i\pi}\right)\left(3e^{i\left(\pi/4\right)}\right)$$
$$z = \left[\cos\left(\pi\right) + i\sin\left(\pi\right)\right]\left(3e^{i\left(\pi/4\right)}\right)$$
$$z = -\left(3e^{i\left(\pi/4\right)}\right) = -\left[2.1213 + i\,(2.1213)\right] = -z_1$$
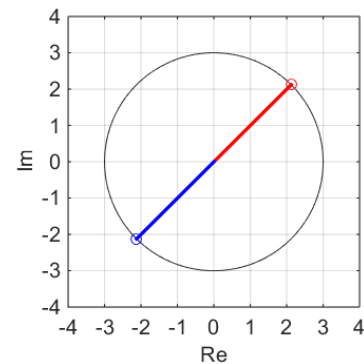


$$\theta = 3\pi/2 \quad z = \left(e^{i\left(3\pi/2\right)}\right)\left(3e^{i\left(\pi/4\right)}\right)$$
$$z = \left(e^{i\left(2\pi - \pi/2\right)}\right)\left(3e^{i\left(\pi/4\right)}\right)$$
$$z = \left(e^{i\left(-\pi/2\right)}\right)\left(3e^{i\left(\pi/4\right)}\right)$$
$$z = \left[\cos\left(-\pi/2\right) + i\sin\left(-\pi/2\right)\right]\left(3e^{i\left(\pi/4\right)}\right)$$
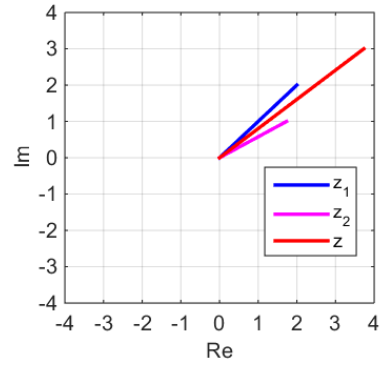$$z = -i\left(3e^{i\left(\pi/4\right)}\right) = 2.1213 - i\,(2.1213)$$



Clockwise rotation of $z_1$ through $(\pi/2)$ rad.

**Example**

Consider the two vectors $z_1 = 2 + 2i$ $\qquad z_2 = \sqrt{3} + i$

Find $\quad z = z_1 + z_2$

$$z = z_1 + z_2 = \left(2 + \sqrt{3}\right) + \left(2 + 1\right)i = 3.7321 + 3.0000\,i$$

Find the magnitudes and arguments of $z_1$, $z_2$ and $z$

$$\text{Know} \qquad z = x + i\,y \quad |z| = R = \sqrt{x^2 + y^2} \qquad Arg(z) = \theta = a\tan\left(\frac{y}{x}\right)$$

$$|z_1| = R_1 = \sqrt{2^2 + 2^2} = 2.8284 \qquad Arg(z) = \theta_1 = a\tan\left(\frac{2}{2}\right) = 0.7854 \text{ rad} = 45^\circ$$
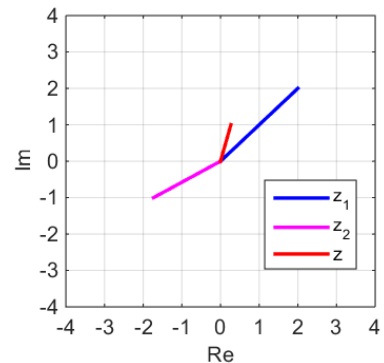
$$|z_2| = R_2 = \sqrt{\sqrt{3} + 1^2} = 2.0000 \qquad Arg(z) = \theta_2 = a\tan\left(\frac{1}{\sqrt{3}}\right) = 0.5236 \text{ rad} = 30^\circ$$

$$|z| = R = \sqrt{3.7321^2 + 3^2} = 4.7883 \qquad Arg(z) = \theta = a\tan\left(\frac{3}{3.7321}\right) = 0.6771 \text{ rad} = 38.7940^\circ$$

Find $z = z_1 - z_2$

$$z_1 = 2x + 2i \qquad -z_2 = -\left(\sqrt{3}\,x + i\right)$$

$$z = z_1 - z_2 = z_1 + \left(-z_2\right) = \left(2 - \sqrt{3}\right) + \left(2 - 1\right)i$$

$$= 0.28284 + 1.0000\,i$$

- $|z| = |z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ hence the

    magnitude of the complex number $|z_1 - z_2|$ is equal to the distance between the two points $z_1(x_1, y_1)$ and $z_2(x_2, y_2)$ on the Argand diagram
    $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ .

- $\theta = Arg\left(z_1 - z_2\right) = \text{atan}\left(\dfrac{y_2 - y_1}{x_2 - x_1}\right)$ but $\left(\dfrac{y_2 - y_1}{x_2 - x_1}\right)$ is the slope of the line joining

    the two points $z_1(x_1, y_1)$ and $z_2(x_2, y_2)$ on the Argand diagram and so,
    $\theta = Arg\left(z_1 - z_2\right)$ is equal to the angle this line makes with the horizontal direction (line parallel to the real axis) as measured in an anticlockwise sense with respect to the horizontal.

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**Find**    $z = z_1\,z_2$

$$z = x + i\,y \quad z_1 = x_1 + i\,y_1 \quad z_2 = x_2 + i\,y_2$$

Know    $z = z_1\,z_2 = \left( x_1\,x_2 - y_1\,y_2 \right) + i\left( x_1 y_2 + x_2 y_1 \right)$

$$|z| = \sqrt{x^2 + y^2} \qquad Arg(z) = \theta = a\tan\left(\frac{y}{x}\right)$$

$$z = \left( (2)\left(\sqrt{3}\right) - (2)(1) \right) + i\left( (2)(1) + \left(\sqrt{3}\right)(2) \right) = 1.4641 + 5.4641\,i$$
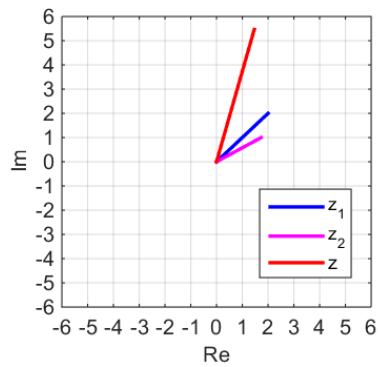
$$|z| = R = \sqrt{1.4641^2 + 5.4641^2} = 5.6569$$

$$Arg(z) = \theta = a\tan\left(\frac{5.4641}{1.4641}\right) = 1.3090\ \text{rad} = 75^\circ$$

Know    $z = R\,e^{i\theta} = \left( R_1\,e^{i\theta_1} \right)\left( R_2\,e^{i\theta_2} \right) = R_1\,R_2\,e^{i(\theta_1 + \theta_2)}$

$$z = (2.8284)(2))\,e^{\,i(0.7854 + 0.5236)} = 5.6569\,e^{\,i(1.3090)}$$

$$|z| = R = 5.6569 \qquad Arg(z) = \theta = 1.3090\ \text{rad} = 75^\circ \qquad \left( 75^\circ = 45^\circ + 30^\circ \right)$$



6

**Find**   $z = \dfrac{z_1}{z_2}$

$$z = x + i\,y \quad z_1 = x_1 + i\,y_1 \quad z_2 = x_2 + i\,y_2$$

Know
$$z = \frac{z_1}{z_2}\frac{\overline{z_2}}{\overline{z_2}} = \frac{z_1\,\overline{z_2}}{\left|\overline{z_2}\right|^2} = \frac{\left(x_1\,x_2 + y_1\,y_2\right) + i\left(-x_1 y_2 + x_2 y_1\right)}{x_2^{\,2} + y_2^{\,2}}$$

$$\left|z\right| = \sqrt{x^2 + y^2} \qquad Arg(z) = \theta = a\tan\left(\frac{y}{x}\right)$$

$$x_1 = 2 \quad y_1 = 2 \quad x_2 = \sqrt{3} \quad y^2 = 1 \quad \left|z_2\right| = 2$$

$$z = \frac{\left(2\sqrt{3} + 2\right) + i\left(-2 + 2\sqrt{3}\right)}{4} = 1.3660 + 0.3660$$

$$\left|z\right| = R = \sqrt{1.3660^2 + 0.3660^2} = 1.4142$$

$$Arg(z) = \theta = a\tan\left(\frac{0.3660}{1.3660}\right) = 0.2618 \text{ rad} = 15^\circ$$

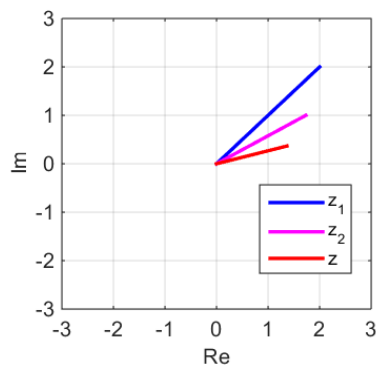Know   $z = R\,e^{i\theta} = \left(R_1\,e^{i\theta_1}\right)\left(R_2\,e^{i\theta_2}\right) = R_1\,R_2\,e^{i(\theta_1 + \theta_2)}$

$$R_1 = 2.8284 \qquad \theta_1 = 0.7854 \text{ rad} = 45^\circ$$

$$R_2 = \sqrt{\sqrt{3} + 1^2} = 2.0000 \qquad \theta_2 = 0.5236 \text{ rad} = 30^\circ$$

$$z = \frac{z_1}{z_2} = \left(\frac{R_1}{R_2}\right)e^{i(\theta_1 - \theta_2)} = \left(\frac{2.8284}{2}\right)e^{i(0.7854 - 0.5236)} = 1.4142\,e^{i(0.2618)}$$

$$\left|z\right| = R = 1.4142 \qquad Arg(z) = \theta = 0.2618 \text{ rad} = 15^\circ \qquad \left(15^\circ = 45^\circ - 30^\circ\right)$$

## Modulus and Complex conjugate relationships

- $|z_1\,z_2| = |z_1||z_2|$ $\qquad \arg(z_1\,z_2) = \arg(z_1) + \arg(z_2)$

  Proof

  $$z_1 = R_1\,e^{i\,\theta_1} \qquad z_2 = R_2\,e^{i\,\theta_2} \qquad z_1\,z_2 = \left(R_1\,e^{i\,\theta_1}\right)\left(R_2\,e^{i\,\theta_2}\right) = R_1\,R_2\,e^{i\,(\theta_1+\theta_2)}$$

  $$|z_1| = R_1 \quad |z_2| = R_2 \quad |z_1\,z_2| = R_1\,R_2 = |z_1||z_2|$$

  $$\arg(z_1) = \theta_1 \quad \arg(z_2) = \theta_2 \quad \arg(z_1\,z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$$

- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ $\qquad \arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

  Proof

  $$z_1 = R_1\,e^{i\,\theta_1} \qquad z_2 = R_2\,e^{i\,\theta_2} \qquad \frac{z_1}{z_2} = \frac{R_1\,e^{i\,\theta_1}}{R_2\,e^{i\,\theta_2}} = \frac{R_1}{R_2}\,e^{i\,(\theta_1-\theta_2)}$$

  $$|z_1| = R_1 \quad |z_2| = R_2 \quad \left|\frac{z_1}{z_2}\right| = \frac{R_1}{R_2} = \frac{|z_1|}{|z_2|}$$

  $$\arg(z_1) = \theta_1 \quad \arg(z_2) = \theta_2 \quad \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$$

- $|z^n| = |z|^n$ $\qquad \arg(z^n) = n\arg(z)$

  Proof

  $$z = Re^{i\,\theta} \qquad z^n = R^n e^{i\,n\theta}$$

  $$|z|^n = R^n = |z|^n \qquad \arg(z^n) = n\,\theta = n\arg(z)$$

- $z_1^{*} + z_2^{*} = \left(z_1 + z_2\right)^{*}$ $\qquad \left(z_1\,z_2\right)^{*} = z_1^{*}\,z_2^{*}$ $\qquad$ * used for complex conjugate

  Proof

  $$z_1 = x_1 + i\,y_1 \qquad z_2 = x_2 + i\,y_2 \qquad z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2)$$

  $$z_1^{*} = x_1 - i\,y_1 \qquad z_2^{*} = x_2 - i\,y_2 \qquad (z_1 + z_2)^{*} = (x_1 + x_2) - i\,(y_1 + y_2)$$

  $$z_1^{*} + z_2^{*} = (z_1 + z_2)^{*}$$

  $$z_1 = R_1\,e^{i\,\theta_1} \quad z_1^{*} = R_1\,e^{-i\,\theta_1} \qquad z_2 = R_2\,e^{i\,\theta_2} \quad z_2^{*} = R_2\,e^{-i\,\theta_1 2}$$

  $$(z_1\,z_2)^{*} = R_1 R_2\,e^{-i\,(\theta_1+\theta_2)} = \left(R_1\,e^{-i\,\theta_1}\right)\left(R_2\,e^{-i\,\theta_1 2}\right) = z_1^{*}\,z_2^{*}$$

## QUADRATIC FUNCTION

A quadratic function has the general form

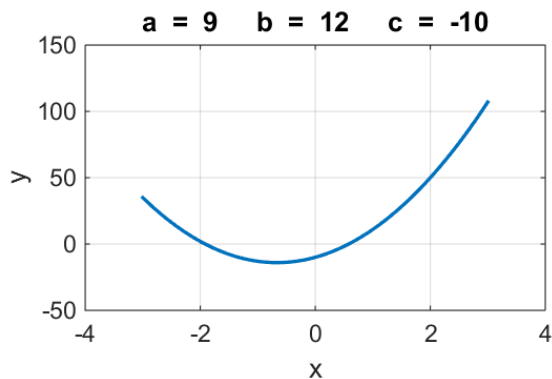$$a\,x^2 + b\,x + c = 0 \qquad a \neq 0$$

and its graph is a **parabola**.

If there are real values of $x$ for which the quadratic function $a\,x^2 + b\,x + c = 0$ then the curve will intersect the X axis at the values of $x$ given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a}$$

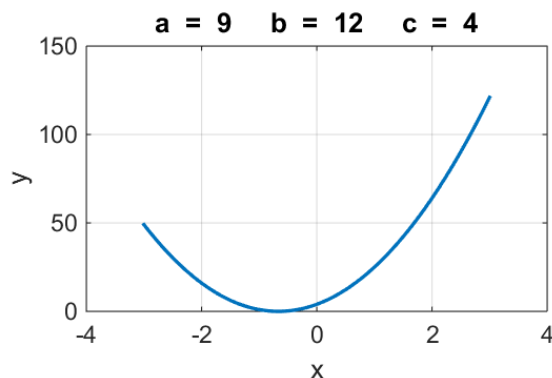Two real roots when the parabola crosses the X axis twice

$$x_1 = 0.5806$$

$$x_2 = -1.9139$$

a = 9   b = 12   c = -10

When the parabola touches the X axis at one point, there is only one real root
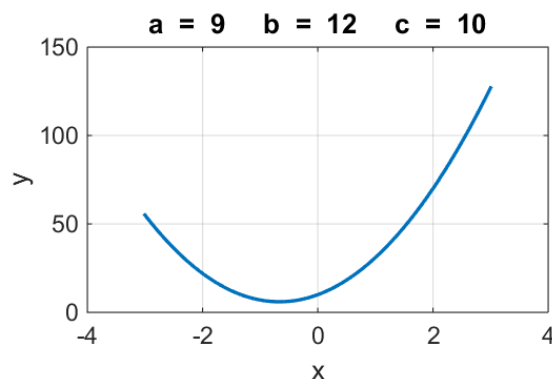
$$x_1 = x_2 = -0.6667$$

a = 9   b = 12   c = 4

The are no real roots when the parabola does not cut the X axis. The **two** roots are now **imaginary**

$$x_1 = -0.6667 + i\,(0.8165)$$

$$x_2 = -1.9139 - i\,(0.8165)$$

a = 9   b = 12   c = 10

9

0ABC is a square on an Argand diagram. The point A is given by the ordered pair (has the ordered $\left(\sqrt{3}, 1\right)$. Find the complex numbers for the points B and C.

$$z_A = x_A + i\ y_A = \sqrt{3} + i$$

$$\left|z_A\right| = R_A = \sqrt{x_A^{\ 2} + y_A^{\ 2}} = \sqrt{\sqrt{3}^2 + 1^2} = 2$$

$$Arg(z_A) = \theta_A = a\tan\left(\frac{1}{\sqrt{3}}\right) = 0.5236 \text{ rad} = 30^\circ$$

The angle between 0A and OC must be $90^\circ$. Multiplication by $i$ produces an anticlockwise rotation of $90^\circ$ ($\pi/2$ rad). Therefore, we can determine the coordinates of the point C from

$$z_C = i\ z_A = -1 + \sqrt{3}\ i$$

Hence the order pair for C is $\left(-1, \sqrt{3}\right)$ or $\left(-1, 1.7321\right)$.

$$\left|z_C\right| = R_C = \sqrt{x_C^{\ 2} + y_C^{\ 2}} = \sqrt{1^2 + \sqrt{3}^2} = 2$$

$$Arg(z_C) = \theta_C = \pi - a\tan\left(\left|\frac{-\sqrt{3}}{1}\right|\right) = 2.0944 \text{ rad} = 120^\circ$$

0B must correspond to the diagonal of the square whose side length is $|z| = R_A = 2$.

Hence, $\qquad \left|z_B\right| = R_B = \sqrt{2^2 + 2^2} = 2.8284$

The angle between the side 0A and the diagonal OB is $45^\circ$ ($\pi/4$ rad). Therefore,

$$\theta_B = \theta_A + 45^\circ = 30^\circ + 45^\circ = 1.3090 \text{ rad}$$
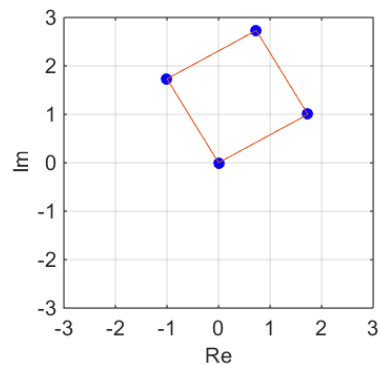
Hence the complex number $z_B$ is

$$z_B = 2.8284\, e^{\,i\,(1.3090)}$$

$$x_B = R_B \cos\theta = (2.8284)\cos(1.3090) = 0.7320$$

$$y_B = R_B \sin\theta = (2.8284)\sin(1.3090) = 2.7320$$

Hence the order pair for B is $(0.7320, 2.7320)$.

The square is now rotated about O through an angle of $60^\circ$ ($\pi/3$ rad) in a clockwise direction. Find the ordered pairs for the vertices of the square $A_1$, $B_1$ and $C_1$.

*know*      multiplication of a complex number $z$ by the complex number $e^{-i\left(\frac{\pi}{3}\right)}$ corresponds to a rotation of the vector representing $z$ on an Argand diagram through an angle ($\pi/3$) rad in a clockwise direction.

$$z_1 = e^{-i\left(\frac{\pi}{3}\right)} z = e^{-i\left(\frac{\pi}{3}\right)} R e^{i\theta} = R e^{i\left(\theta - \frac{\pi}{3}\right)}$$

The new vertices of the square $A_1$, $B_1$ and $C_1$ are determined by the rotation of the vectors representing $z_A$, $z_B$ and $z_C$ by ($\pi/3$) rad

$$z_{A1} = R_A\, e^{i\left(\theta_A - \frac{\pi}{3}\right)} = 2\, e^{i\left(0.5236 - \frac{\pi}{3}\right)} = 2\, e^{i(-0.5236)}$$

$$z_{B1} = R_B\, e^{i\left(\theta_B - \frac{\pi}{3}\right)} = 2.8284\, e^{i\left(1.3090 - \frac{\pi}{3}\right)} = 2.8284\, e^{i(0.2618)}$$

$$z_{C1} = R_C\, e^{i\left(\theta_C - \frac{\pi}{3}\right)} = 2\, e^{i\left(2.0944 - \frac{\pi}{3}\right)} = 2\, e^{i(1.0472)}$$

$$\theta_{A1} = -0.5236 \text{ rad} = -30^\circ \quad \theta_{B1} = 0.2618 \text{ rad} = 15^\circ \quad \theta_{C1} = 1.0472 \text{ rad} = 60^\circ$$

*know*      The rectangular form of the complex numbers is found from
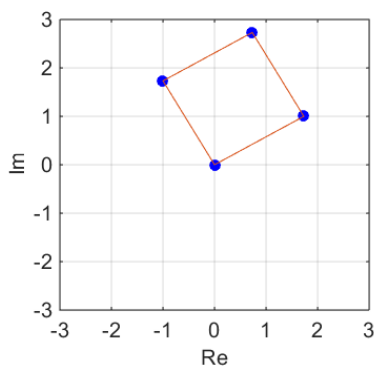
$$x = R\cos\theta \quad y = R\sin\theta$$

$$x_{A1} = 2\cos(-0.5236) = 1.7320 \quad y_{A1} = 2\sin(-0.5236) = -1.0000$$
$$x_{B1} = 2.8284\cos(0.2618) = 2.7320 \quad y_{B1} = 2.8284\sin(0.2618) = 0.7320$$
$$x_{C1} = 2\cos(1.0472) = 1.0000 \quad y_{C1} = 2\sin(1.0472) = 1.7321$$

The order pairs are $A_1$ (1.7320, -1.0000), $B_1$ (2.7320, 0.7320), $C_1$ (1.0000, 1.7321).



rotation by

$60^\circ$ clockwise →