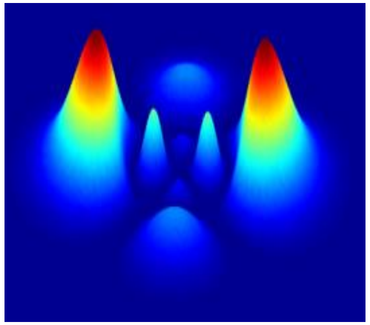

ADVANCED HIGH SCHOOL MATHEMATICS

VOLUMES

EXERCISES

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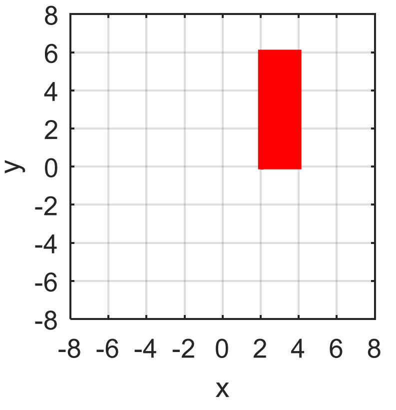
1

Let $\mathcal{R}$ be the region enclosed by the rectangle with dimensions 2x6 as shown in the diagram. The XY plane coordinates of the corners are (2,0), (4,0), (4,6) and (2,6).



The volume V of a cylinder is

(1) $V = \pi R^2 H$ where R is its radius and H is the height.

Find the volumes V of the solids produced from the following rotations using equation (1) and by the evaluation of a definite integral.

(A) The region $\mathcal{R}$ is rotated about the lines

$x = 3$, $x = 2$, $x = 4$ and $x = 0$ (Y-axis).

(B) The region $\mathcal{R}$ is rotated about the lines

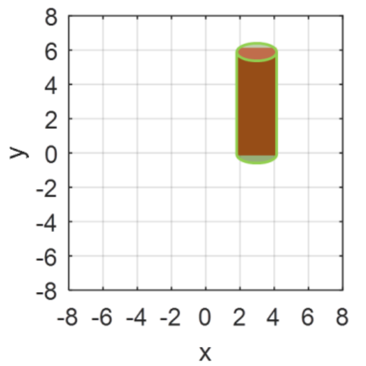
$y = 0$ (X-axis), $y = 6$ and $y = -2$.

Solution

(A)

$x = 3$

$R = 1 \quad H = 6 \qquad V = \pi R^2 H = 6\pi$



The solid of revolution is generated by rotating the line $x = 4$ in the interval $0 \le y \le 6$ about the axis $x = 3$. The volume V is

How to approach the problem:
Sketch the function and the solid.
Give the equations for the shape of the solid. Find the upper and lower limits for the bounded region.
Evaluation the definite integral to find the volume.

$$V = \int_{y_a}^{y_b} A(y)\,dy$$

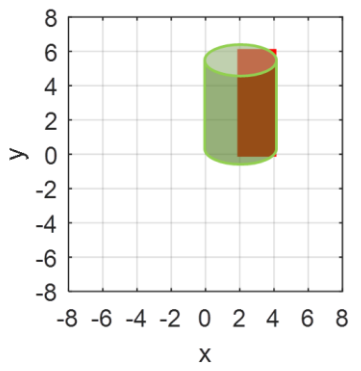
$$y_a = 0 \quad y_b = 6 \quad A(y) = \pi R^2 \quad R = 1 \quad \Rightarrow A(y) = \pi$$

$$V = \int_0^6 A(y)\,dy = \int_0^6 \pi\,dy = \pi\left[\,y\,\right]_0^6 = 6\pi$$

$x = 2$

$R = 2 \quad H = 6 \qquad V = \pi R^2 H = 24\pi$

The solid of revolution is generated by rotating the line $x = 4$ in the interval $0 \le y \le 6$ about the axis $x = 2$. The volume V is



$$V = \int_{y_a}^{y_b} A(y)\, dy$$

$$y_a = 0 \quad y_b = 6 \quad A(y) = \pi R^2 \quad R = 2 \quad \Rightarrow A(y) = 4\pi$$

$$V = \int_0^6 A(y)\, dy = \int_0^6 4\pi\, dy = 4\pi \left[y \right]_0^6 = 24\pi$$

$x = 4$

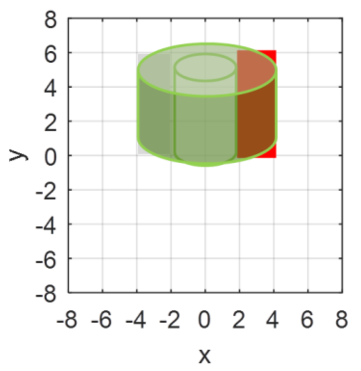
By symmetry, the volume of the cylinder is the same as the rotation about the line $x = 2 \quad \Rightarrow \; V = 24\pi$

$x = 0$

$$H = 6 \quad R_{out} = 4 \quad V_{out} = \pi R_{out}^{\;2} H = 96\pi$$

$$R_{in} = 2 \quad V_{in} = \pi R_{in}^{\;2} H = 24\pi$$

$$V = V_{out} - V_{in} = 72\pi$$



The volume of the solid of revolution by the method of cylindrical shells

$$V = \int_{x_a}^{x_b} (2\, \pi\, x)\, y\, dx = 2\, \pi \int_{x_a}^{x_b} x\, y\, dx$$

$$x_a = 2 \quad x_b = 4 \quad y = 6$$

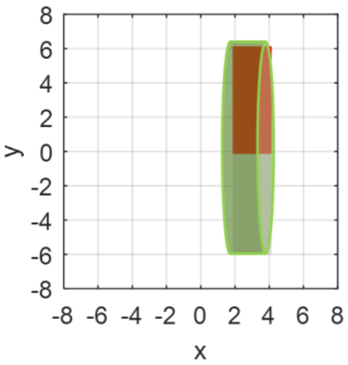
$$V = \; 12\pi \int_2^4 x \;\, dx = 12\pi \left(\tfrac{1}{2}\right) \left[x^2 \right]_2^4 = 6\pi \left(16 - 4\right) = 72\pi$$

(B)

$y = 0$

$R = 6 \quad H = 2 \qquad V = \pi R^2 \, H = 72\pi$

The solid of revolution is generated by rotating the line $y = 6$ in the interval $2 \le x \le 4$ about the axis $y = 0$. The volume V is



$$V = \int_{x_a}^{x_b} A(x)\, dx$$

$$a = 2 \quad b = 4 \quad A(x) = \pi R^2 \quad R = 6 \quad \Rightarrow A(y) = 36\pi$$

$$V = \int_2^4 A(x)\, dx = \int_2^4 36\pi\, dy = 36\pi \left[x \right]_2^4 = 72\pi$$

$y = 6$

By symmetry, the volume of the cylinder is the same as the rotation about the line $y = 0 \quad \Rightarrow \quad V = 72\pi$

$y = -2$

$$H = 2 \quad R_{out} = 8 \quad V_{out} = \pi R_{out}^{\ 2} H = 128\pi$$

$$R_{in} = 2 \quad V_{in} = \pi R_{in}^{\ 2} H = 8\pi$$

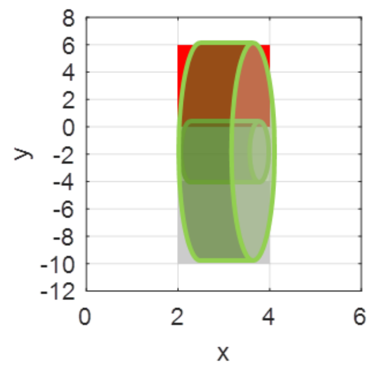
$$V = V_{out} - V_{in} = 120\pi$$

The volume of the solid of revolution by the method of cylindrical shells is

$$V = \int_{y_a}^{y_b} (2\,\pi\, y)\, x\, dy = 2\,\pi \int_a^b x\, y\, dy$$

$$y_a = -2 \quad y_b = 8 \quad x = 2$$

$$V = 4\pi \int_{-2}^{8} y \ dy = 4\pi\left(\tfrac{1}{2}\right)\left[y^2\right]_{-2}^{8} = 2\pi\left(64 - 4\right) = 120\pi$$



2

Find the volume V of an ellipsoid formed by the rotation of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$ about the X-axis.

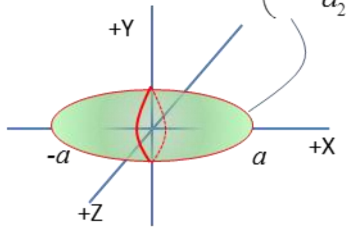
Solution

Volume of solid of revolution about the X axis is

$$V = \pi \int_a^b y^2 \, dx$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$y = b \left(1 - \frac{x^2}{a_2} \right)^{1/2}$$



The limits of integration are x_a = - a and x_b = a.

The function $y = f(x) \geq 0$ in the interval [- a a] is

$$y = b \left(1 - \frac{x^2}{a_2} \right)^{1/2} \qquad y^2 = b^2 \left(1 - \frac{x^2}{a^2} \right)$$

The volume of the ellipsoid is

$$V = \pi \int_{-a}^{a} b^2 \left(1 - \frac{x^2}{a^2} \right) dx = \frac{2\pi b^2}{a^2} \int_0^a \left(a^2 - x^2 \right) dx$$

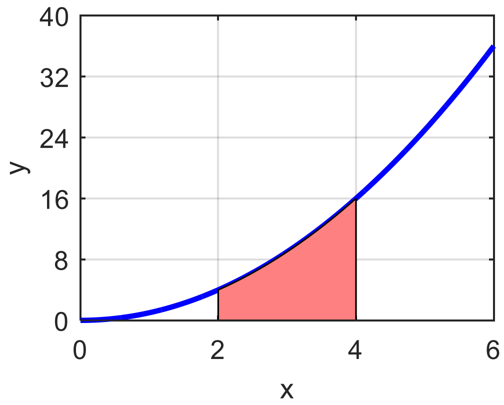
$$V = \frac{2\pi b^2}{a^2} \left[a^2 x - \tfrac{1}{3} x^3 \right]_0^a$$

$$V = \frac{4\pi a b^2}{3}$$

For a sphere of radius a $\qquad a = b \qquad V_{sphere} = \dfrac{4\pi a^3}{3}$ ***QED***

3

Find the volumes of the solids generated by revolving the given region $\mathcal{R}$ under the curve $y = x^2$ and above the X-axis in the interval $x = 2$ to $x = 4$ about the specified axes.



(A)
The region $\mathcal{R}$ is rotated about the lines $y = 0$ (X-axis) and $y = -10$.

(B)
The region $\mathcal{R}$ is rotated about the lines $x = 0$ (Y-axis), $x = 2$ and $x = 6$.

Solution

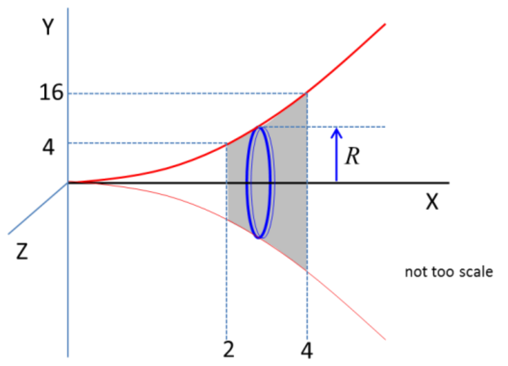
(A)

$y = 0$

The solid of revolution is generated by rotating the curve $y = x^2$ the interval $2 \le x \le 4$ about the X-axis ($y = 0$). The volume V is

How to approach the problem:

Sketch the function and the solid.
Give the equations for the shape of the solid.
Find the upper and lower limits for the bounded region.
Evaluation the definite integral to find the volume.



$$V = \int_a^b A(x)\,dx$$

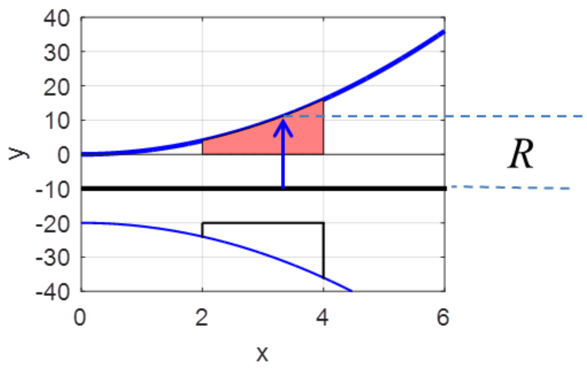
$$a = 2 \quad b = 4 \quad A(x) = \pi R^2 \quad R = y = x^2 \quad \Rightarrow A(x) = \pi\, x^4$$

$$V = \int_2^4 A(x)\,dx = \pi \int_2^4 x^4 dx = \pi \left[\tfrac{1}{5} x^5\right]_2^4 = \frac{\pi}{5}\left(4^5 - 2^5\right)$$

$$V = \frac{992}{5}\,\pi = 198.4\,\pi$$

$y = -10$

The solid of revolution is generated by rotating the curve $y = x^2$ the interval $2 \le x \le 4$ about the line $y = -10$. The volume V is



$$V = \int_a^b A(x)\,dx$$

$$a = 2 \quad b = 4 \quad A(x) = \pi R^2 \quad R = y + 10 = x^2 + 10$$
$$\Rightarrow A(x) = \pi\left(x^4 + 20x^2 + 100\right)$$

$$V = \int_2^4 A(x)\,dx = \pi\int_2^4 \left(x^4 + 20x^2 + 100\right)dx$$
$$V = \pi\left[\tfrac{1}{5}x^5 + \tfrac{20}{3}10x^3 + 100x\right]_2^4$$
$$V = \pi\left(\tfrac{1}{5}\left(4^5 - 2^5\right) + \left(\tfrac{20}{3}\right)\left(4^3 - 2^3\right) + 200\right)$$

$$V = \frac{11576}{15}\pi = 771.73\pi$$

(B)

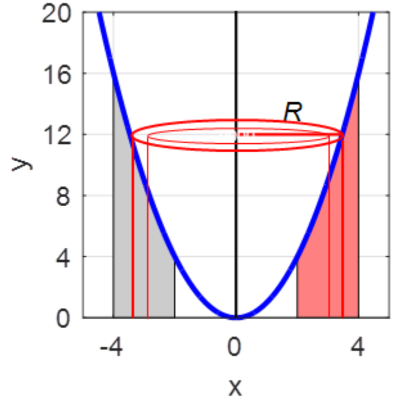
$x = 0$ (Y-axis)

$$V = \int_a^b (2\pi x)\, y\, dx$$ **method: cylindrical shells**

$$a = 2 \quad b = 4 \quad y = x^2$$

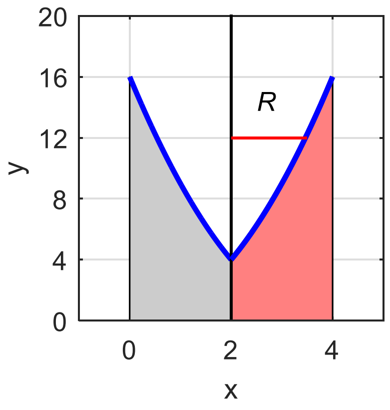
$$V = 2\pi \int_2^4 x^3\, dx = \frac{\pi}{2}\left[x^4\right]_2^4 = \frac{\pi}{2}\left(4^4 - 2^4\right)$$

$$V = 120\pi$$



$x = 2$

The solid of revolution is generated by rotating the curve $y = x^2$ the interval $2 \le x \le 4$ about the line $x = 2$. The volume V is



$$V = \int_a^b A(y)\,dy \quad \textbf{method: volumes based on cross-sections}$$

$$a = 4 \quad b = 16 \quad A(y) = \pi R^2 \quad y = x^2 \quad x = y^{1/2}$$

$$R = x - 2 = y^{1/2} - 2 \quad R^2 = y - 4y^{1/2} + 4$$

$$A(y) = \pi\left(y - 4y^{1/2} + 4\right)$$

$$V = \pi\int_4^{16}\left(y - 4y^{1/2} + 4\right)dy = \pi\left[\tfrac{1}{2}y^2 - \tfrac{8}{3}y^{3/2} + 4y\right]_4^{16}$$

$$V = \pi\left(\tfrac{1}{2}\left(16^2 - 4^2\right) - \tfrac{8}{3}\left(16^{3/2} - 4^{3/2}\right) + 4(16-4)\right)$$

$$V = \frac{56}{3}\pi = 18.667\,\pi$$

Alternatively

$$V = \int_a^b 2\pi\left(x - 2\right)y\,dx \quad \textbf{method: cylindrical shells}$$

$$a = 2 \quad b = 4 \quad y = x^2$$

$$V = 2\pi\int_2^4\left(x^3 - 2x^2\right)dx = 2\pi\left[\tfrac{1}{4}x^4 - \tfrac{2}{3}x^3\right]_2^4 = 2\pi\left(\tfrac{1}{4}\left(4^4 - 2^4\right) - \tfrac{2}{3}\left(4^3 - 2^3\right)\right)$$

$$V = \frac{136}{3}\pi = 45.33\pi$$

4

Find the volumes of the solids of revolution for the function $y = x/2$ and bounded by the X-axis and the vertical lines x_a = 2 and x_b = 4 for the following axes of rotation

(A) X-axis y_R = 0

(B) Y-axis x_R = 0

(C) y_R = - 2

(D) y_R = +2

Solution

(A) **rotation around X-axis**

Volume of solid of revolution about the X-axis is

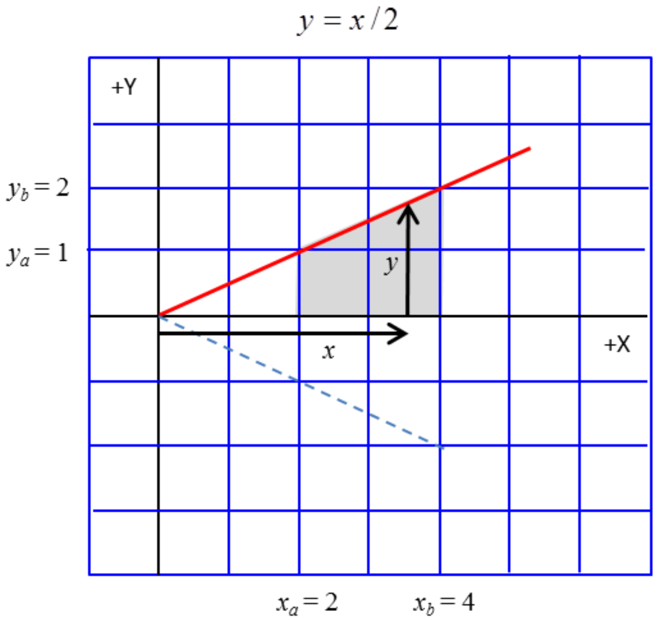
$$V = \pi \int_{x_a}^{x_b} y^2 \, dx \qquad \textbf{Disk Method}$$

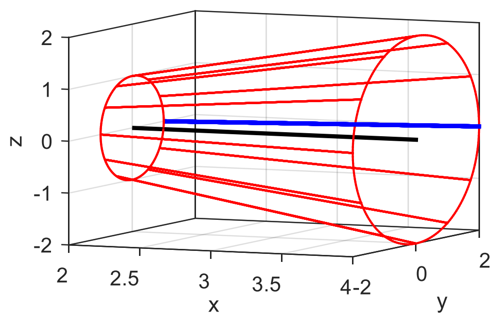
The limits of integration are $x_a = 2$ and $x_b = 4$

The function $y = f(x) \geq 0$ in the interval [2 4] is

$$y = x/2 \qquad y^2 = x^2/4$$

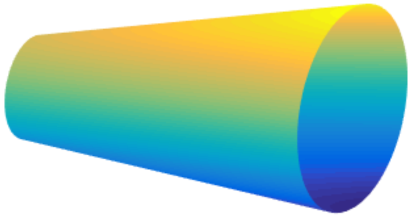
The volume of the cone is





$$V = \frac{\pi}{4}\int_2^4 x^2\,dx = \frac{\pi}{4}\left[\tfrac{1}{3}x^3\right]_2^4 = \frac{\pi}{12}\left[64-8\right]$$

$$V = \frac{14\pi}{3}$$



(B) **rotation around Y-axis**

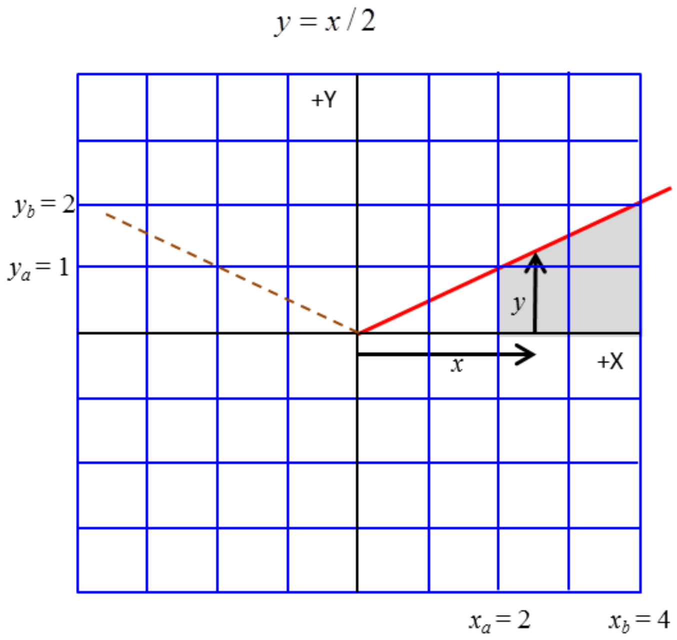
Volume of solid of revolution about the Y-axis is

$$V = 2\pi \int_{x_a}^{x_b} y\, x\, dx$$ **Cylindrical shell method**

The limits of integration are $x_a = 2$ and $x_b = 4$

The function $y = f(x) \geq 0$ in the interval [2 4] is

$$y = x/2$$



The volume of the cone is

$$V = 2\pi \int_2^4 \tfrac{1}{2} x^2\, dx = \pi \left[\tfrac{1}{3} x^3 \right]_2^4 = \frac{\pi}{3}[64 - 8]$$

$$V = \frac{56\pi}{3}$$

(C) **rotation about the horizontal line y_R = -2**

There are a number of ways in which this type of problem can be solved. The method we will use starts with

$$V = \int_{x_a}^{x_b} A(x)\,dx$$

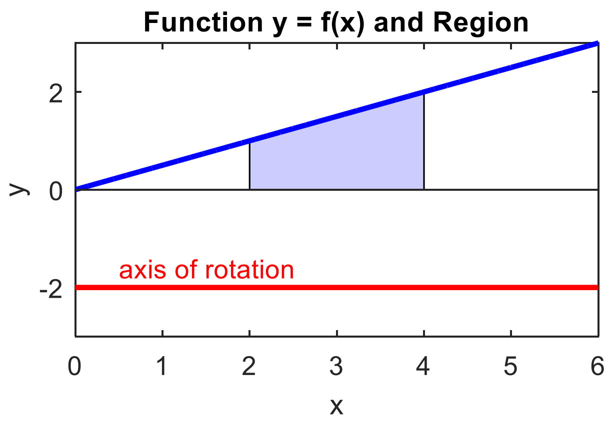
The solid is generated by a rotation through 360°, therefore the cross-sections of the solid of revolution will be circles, hence

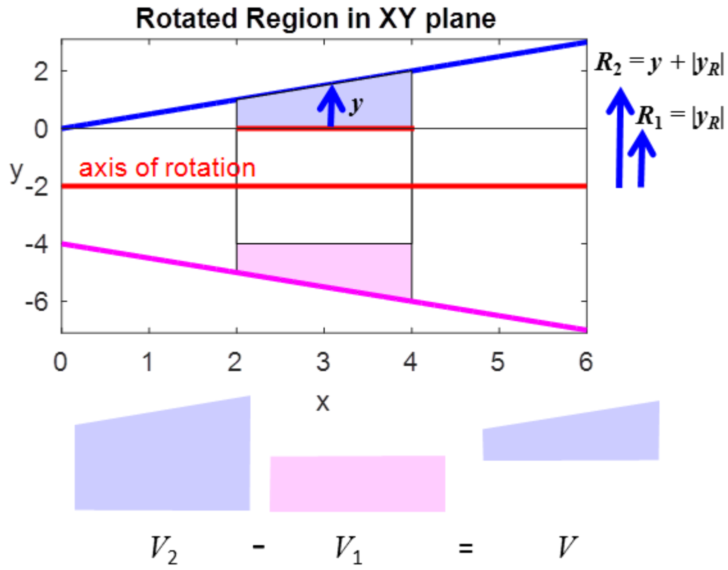
$$A(x) = \pi\, R(x)^2$$

and the volume V is

$$V = \pi \int_{x_a}^{x_b} R(x)^2\,dx$$

You need to carefully draw a set of sketches to clearly identify the region, the axis of rotation and determine the radius $R(x)$.





The function is $y = 0.5x \quad 2 \le x \le 4$ and the axis of rotation is $y_R = -2$. The rotation of the function around the axis of rotation generates a volume V_2.

$$R_2(x) = y + |y_R| = 0.5x + 2 \quad R_2(x)^2 = x^2/4 + 2x + 4 \quad x_a = 2 \quad x_b = 4$$

$$V_2 = \pi \int_2^4 \left(x^2/4 + 2x + 4\right) dx$$

$$V_2 = \pi \left[\tfrac{1}{12} x^3 + x^2 + 4x \right]_2^4$$

$$V_2 = \left(\tfrac{1}{12}\left(4^3 - 2^3\right) + \left(4^2 - 2^2\right) + 4\left(4 - 2\right) \right) \pi$$

$$V_2 = \frac{74}{3} \pi$$

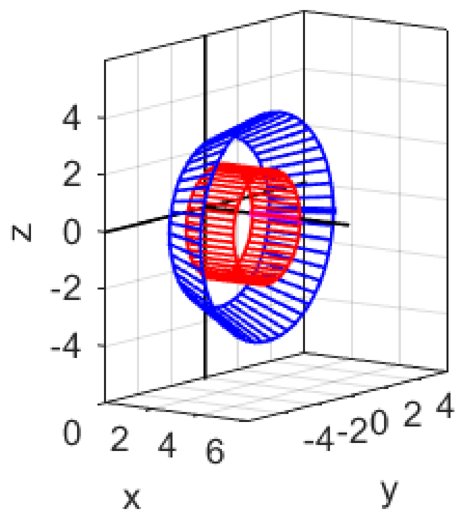
We need to subtract the volume V_1 generated by revolving the interval $2 \le x \le 4$ along the X-axis (y = 0) around the axis of rotation

$$R_1(x) = |y_R| = 2 \quad R_1(x)^2 = 4 \quad x_a = 2 \quad x_b = 4$$

$$V_1 = \pi \int_2^4 4\,dx$$

$$V_1 = 4\pi \left[x \right]_2^4$$

$$V_1 = 8\pi$$



The volume V of the solid of revolution of the region is thus

$$V = V_2 - V_1 = \left(\frac{74}{3} - 8 \right)\pi$$

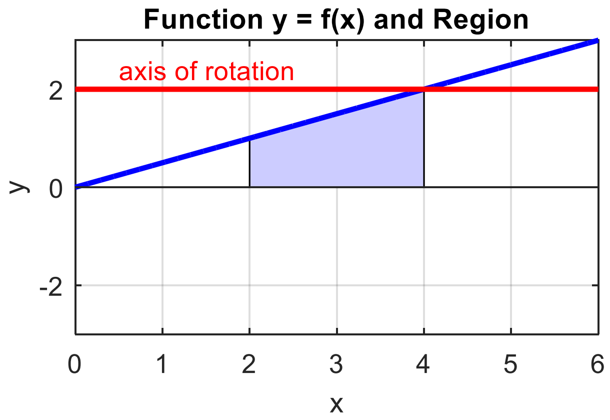
$$V = \frac{50}{3}\pi$$

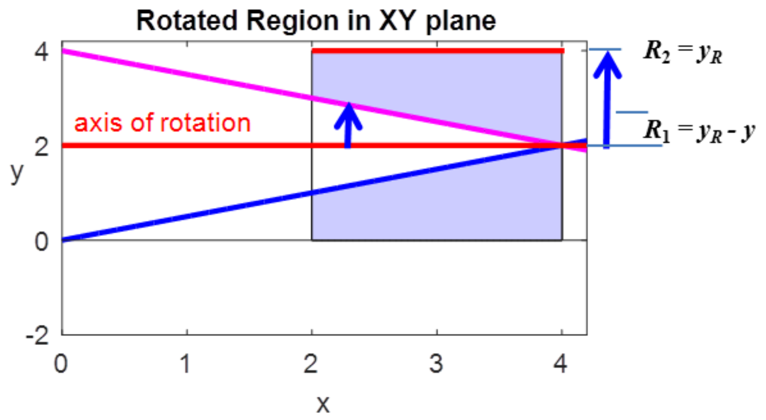
(D) rotation about the horizontal line y_R = +2

The solid is generated by a rotation through 360°, therefore the cross-sections of the solid of revolution will be circles, hence the volume of the rotated region can be found by evaluating the definite integral

$$V = \pi \int_{x_a}^{x_b} R(x)^2\,dx$$

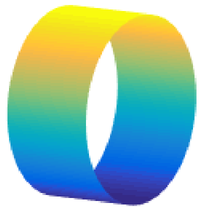
You need to carefully draw a set of sketches to clearly identify the region, the axis of rotation and determine the radius $R(x)$.





The function is $y = 0.5x \quad 2 \le x \le 4$ and the axis of rotation is $y_R = +2$.

The volume V_2 generated by revolving the interval $2 \le x \le 4$ along the X-axis (y = 0) around the



axis of rotation is

$$R_2(x) = |y_R| = 2 \quad R_2(x)^2 = 4 \quad x_a = 2 \quad x_b = 4$$

$$V_2 = \pi \int_2^4 4\,dx$$

$$V_2 = 4\pi [\,x]_2^4$$

$$V_2 = 8\pi$$

We need to subtract the volume V_1 generated by the rotation of the function around the axis of

rotation

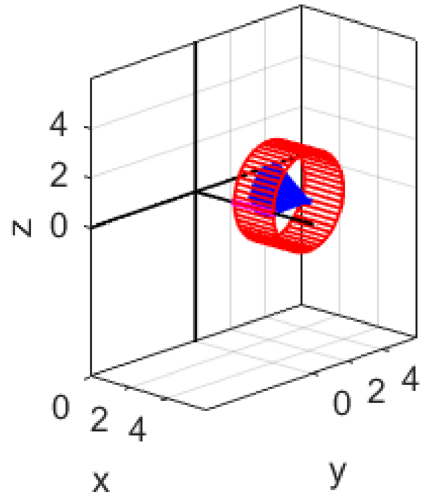
$$R_1(x) = y_R - y = 2 - 0.5x \quad R(x)^2 = x^2/4 - 2x + 4 \quad x_a = 2 \quad x_b = 4$$

$$V_1 = \pi \int_2^4 \left(x^2/4 - 2x + 4\right) dx$$

$$V_1 = \pi \left[\tfrac{1}{12} x^3 - x^2 + 4x\right]_2^4$$

$$V_1 = \left(\tfrac{1}{12}\left(4^3 - 2^3\right) - \left(4^2 - 2^2\right) + 4(4-2)\right)\pi$$

$$V_1 = \frac{2}{3}\pi$$



The volume V of the solid of revolution of the region is thus

$$V = V_2 - V_1 = \left(8 - \frac{2}{3}\right)\pi$$

$$V = \frac{22}{3}\pi$$

5

Find the volume V of the solid of revolution generated by the rotation about the X-axis of the region bounded by the curves

$f_2(x) = 42 - 5x$ and $f_1(x) = 2x^2 - 5x + 10$

solution

How to approach the problem:
Sketch the function and the solid.
Give the equations for the shape of the solid.
Find the upper and lower limits for the bounded region.
Evaluation the definite integral to find the volume.

We need to find the points of intersection of the two functions.

$$f_1(x) = f_2(x)$$
$$2x^2 - 5x + 10 = 42 - 5x$$
$$x^2 = 16$$
$$x = \pm 4$$

The limits of integration are x_a = - 4 and x_b = 4.

Volume of solid of revolution about the X-axis is

$$V = V_2 - V_1$$
$$V_2 = \pi \int_{x_a}^{x_b} f_2(x)^2 \, dx$$

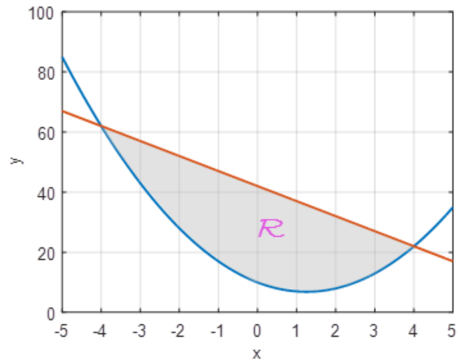
$$f_2(x) = 42 - 5x \qquad \left(f_2(x) \right)^2 = 25x^2 - 420x + 1764$$

$$V_2 = \pi \int_{-4}^{4} \left(25x^2 - 420x + 1764 \right) dx$$

$$V_2 = \pi \left[\tfrac{25}{3} x^3 - 210 x^2 + 1764 x \right]_{-4}^{4}$$

$$V_2 = \pi \left[\tfrac{25}{3} \left(4^3 + 4^3 \right) - 210 \left(4^2 - 4^2 \right) x^2 + 1764 \left(4 + 4 \right) \right]$$

$$V_2 = 15179 \pi$$



$f_2(x) = 42 - 5x$

$f_1(x) = 2x^2 - 5x + 10$

$$f_1(x) = 2x^2 - 5x + 10$$

$$\left(f_1(x) \right)^2 = 4x^4 - 20x^3 + 65x^2 - 100x + 100$$

$$V_1 = \pi \int_{-4}^{4} \left(4x^4 - 20x^3 + 65x^2 - 100x + 100 \right) dx$$

$$V_1 = \pi \left[\tfrac{4}{5} x^5 - 5 x^4 + \tfrac{65}{3} x^3 - 50 x^2 + 100x \right]_{-4}^{4}$$

$$V_1 = \pi \left[\tfrac{4}{5} \left(4^5 + 4^5 \right) + \tfrac{65}{3} \left(4^3 + 4^3 \right) + 100 \left(4 + 4 \right) \right]$$

$$V_1 = 5212 \pi$$

$$V = V_2 - V_1 = (15179 - 5212) \pi$$

$$V = 9967 \pi$$

QED

6

Find the volumes of the solids of revolution for the region bounded by the function $y = 2\sqrt{x}$, the X-axis and the vertical lines x_a = 0 and x_b = 4 for the following axes of rotation

(A) X-axis y_R = 0

(B) Y-axis x_R = 0

(C) y_R = - 2

Solution

(A) **rotation around X-axis**

Volume of solid of revolution about the X-axis is

$$V = \pi \int_{x_a}^{x_b} y^2 \, dx \qquad \textbf{Disk Method}$$

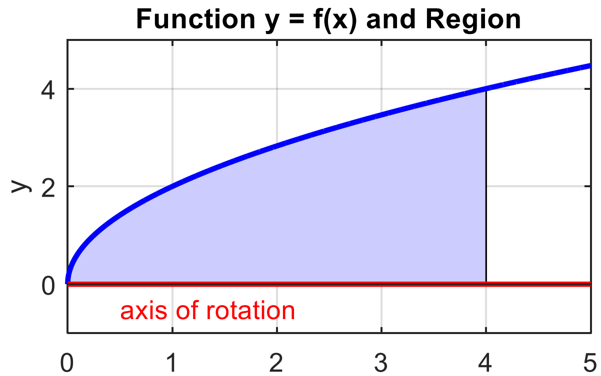
The limits of integration are $x_a = 0$ and $x_b = 4$

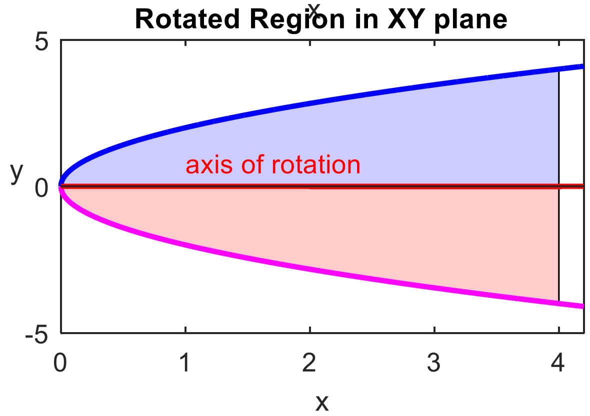
The function $y = f(x) \geq 0$ in the interval [0 4] is

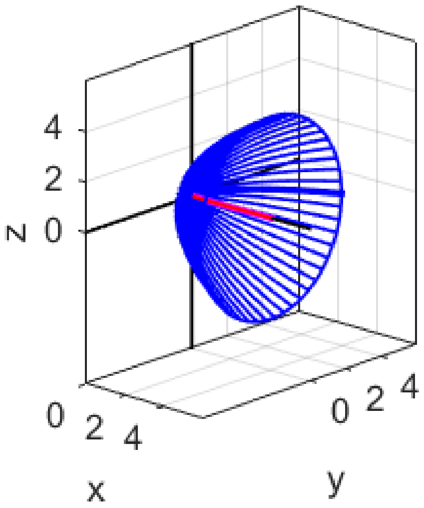
$$y = 2\sqrt{x} \qquad y^2 = 4x$$

The volume of the cone is

$$V = 4\pi \int_0^4 x \, dx = 4\pi \left[\tfrac{1}{2} x^2 \right]_0^4 = 32\pi$$







We can also find the volume of solid of revolution about the X-axis using the cylindrical shell method

$$V = 2\pi \int_{y_a}^{y_b} y\, x\, dy$$ **Cylindrical Shell Method**

The limits of integration are $y_a = 0$ and $y_b = 4$ and the function is

$$y = 2\sqrt{x} \quad x = \frac{y^2}{4} \quad x\, y = \frac{y^3}{4}$$

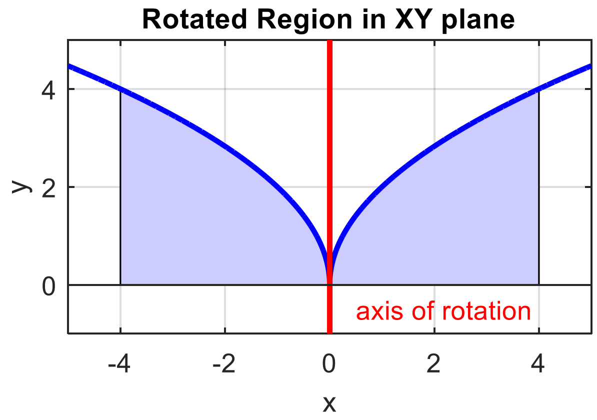
The volume V of the solid of revolution is

$$V = 2\pi \int_0^4 \frac{y^3}{4} dy = 2\pi \left[\tfrac{1}{4} y^3 \right]_0^4$$

$$V = 32\pi$$

(B) rotation around Y-axis

Volume of solid of revolution about the Y-axis is $V = 2\pi \int_{x_a}^{x_b} y\,x\,dx$



Cylindrical Shell Method

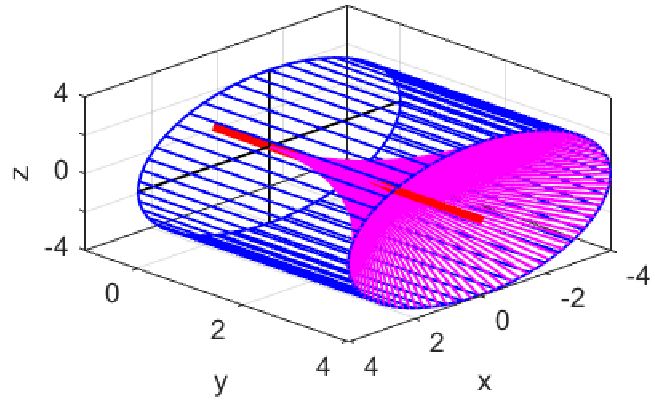
The limits of integration are $x_a = 0$ and $x_b = 4$. The function $y = f(x) \geq 0$ in the interval [0 4] is

$$y = 2\sqrt{x} \qquad x\,y = 2\,x^{3/2}$$

The volume of the solid of revolution is

$$V = 2\pi \int_0^4 2x^{3/2}\,dx = 4\pi\left[\tfrac{2}{5}x^{5/2}\right]_0^4 = \frac{8\pi}{5}[32] = \frac{256\pi}{5}$$

We can also find the volume of revolution using the **Disk Method** but it is a little more difficult. V_2 is the volume of revolution for the rotation of the line $x_b = 4$ in the interval $0 \leq y \leq 4$



$$V_2 = \pi \int_{y_a}^{y_b} x^2\,dy = \pi \int_{y_a}^{y_b} x^2\,dy = \pi \int_0^4 4^2\,dy = 16\pi\,[y]_0^4 = 64\pi$$

V_1 is the volume of revolution of the region bounded by the function and the Y-axis

$$V_2 = \pi \int_{y_a}^{y_b} x^2\,dy = \pi \int_{y_a}^{y_b} \left(y^4/16\right) dy = \frac{1}{16}\pi\left[\tfrac{1}{5}y^5\right]_0^4 = \frac{64}{5}\pi$$

The volume of revolution of the region is

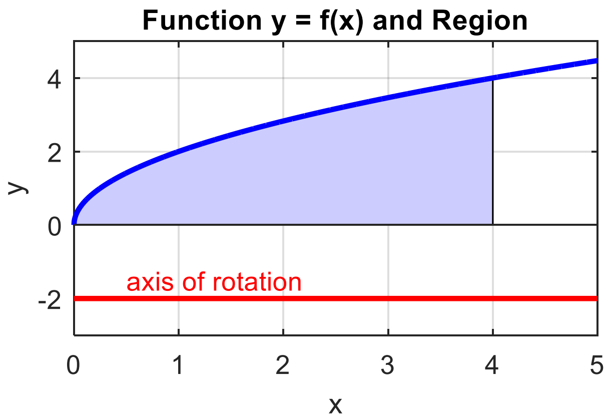
$$V = V_2 - V_1 = \left(64 - \frac{64}{5}\right) = \frac{256\pi}{5}$$

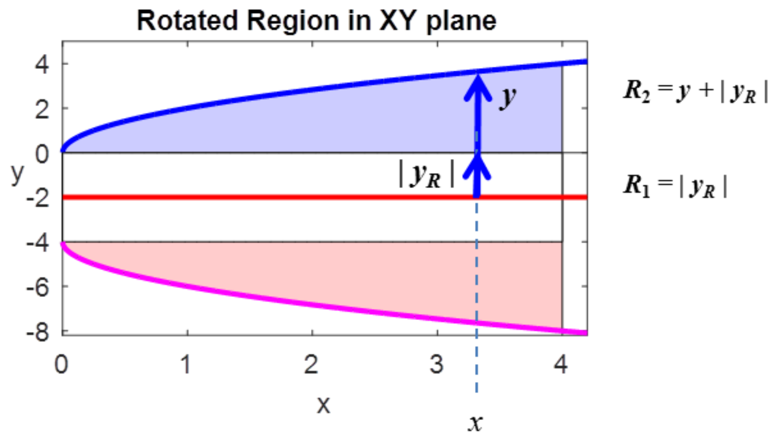
(C) rotation about the horizontal line y_R = -2

The solid is generated by a rotation through 360°, therefore the cross-sections of the solid of revolution will be circles, hence, the volume of the rotated region can be found by evaluating the definite integral

$$V = \pi \int_{x_a}^{x_b} R(x)^2 \, dx$$

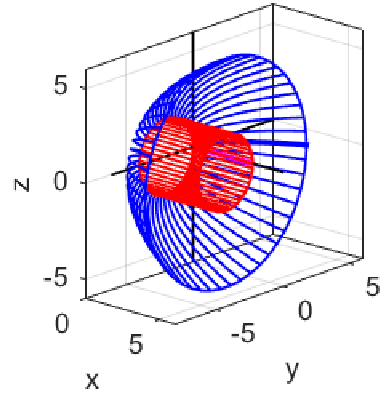
You need to carefully draw a set of sketches to clearly identify the region, the axis of rotation and determine the radius $R(x)$.





The function is $y = 2\sqrt{x}$ $0 \le x \le 4$ and the axis of rotation is $y_R = -2$.

The volume V_2 generated by revolving the function $y = 2\sqrt{x}$ in interval $0 \le x \le 4$ around the around the axis of rotation y_R = +2 is



$$R_2(x) = y + |y_R| = 2 \quad R_2(x)^2 = 4x + 8x^{1/2} + 4 \quad x_a = 2 \quad x_b = 4$$

$$V_2 = \pi \int_2^4 \left(4x + 8x^{1/2} + 4\right) dx$$

$$V_2 = \pi \left[2x^2 + \frac{16}{3} x^{3/2} + 4x \right]_0^4$$

$$V_2 = \frac{272}{3}\pi$$

We need to subtract the volume V_1 generated by the rotation of the X-axis in the interval $0 \le x \le 4$ around the axis of rotation

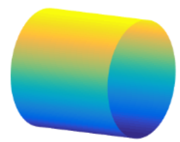
$$R_1(x) = 2 \quad R(x)^2 = 4 \quad x_a = 2 \quad x_b = 4$$

$$V_1 = 4\pi \int_0^4 dx$$

$$V_1 = 4\pi [x]_0^4$$

$$V_1 = 16\pi = \frac{48}{3}\pi$$

The volume V of the solid of revolution of the region is thus



$$V = V_2 - V_1 = \left(272 - 48\right)\frac{\pi}{3}$$

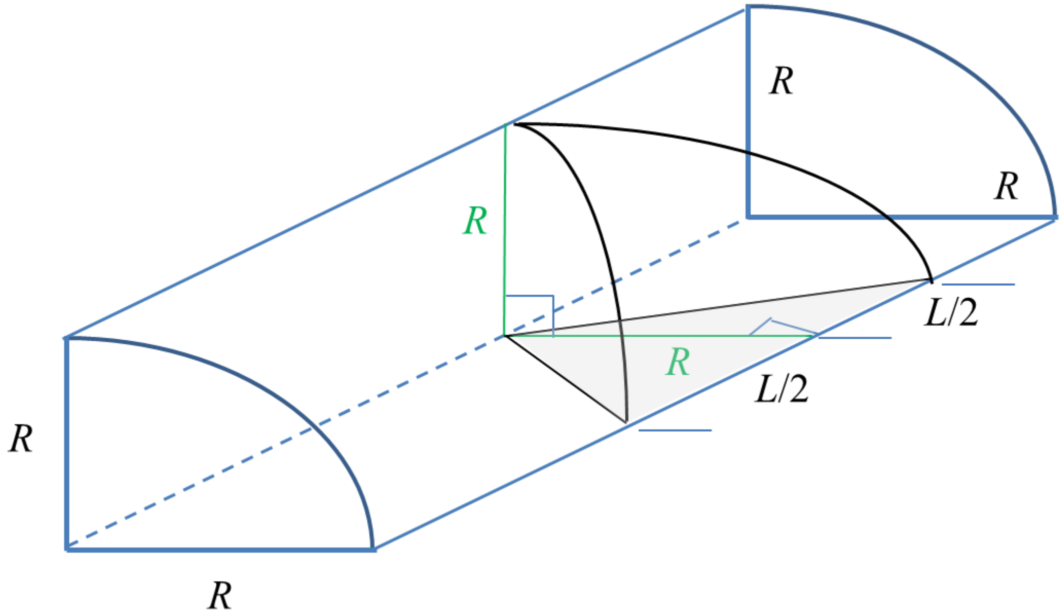
$$V = \frac{224\pi}{3}$$

7

A solid is cut from a quarter cylinder of radius R. The solid's base is an isosceles triangle. The width of the isosceles triangle is L and its height is R.

Show that the volume of the solid is

$$V = \frac{1}{3} L\, R^3$$



Solution

The volume of the solid can be calculated from the formula

$$V = \int_{x_a}^{x_b} A(x)\, dx$$

where the cross-sections of the solid are rectangles in the YZ plane. The area of the rectangle at x is

$$A(x) = (2\,y)\,z$$

The base of the solid is an isosceles triangle of height R and width L. From similar triangles at x

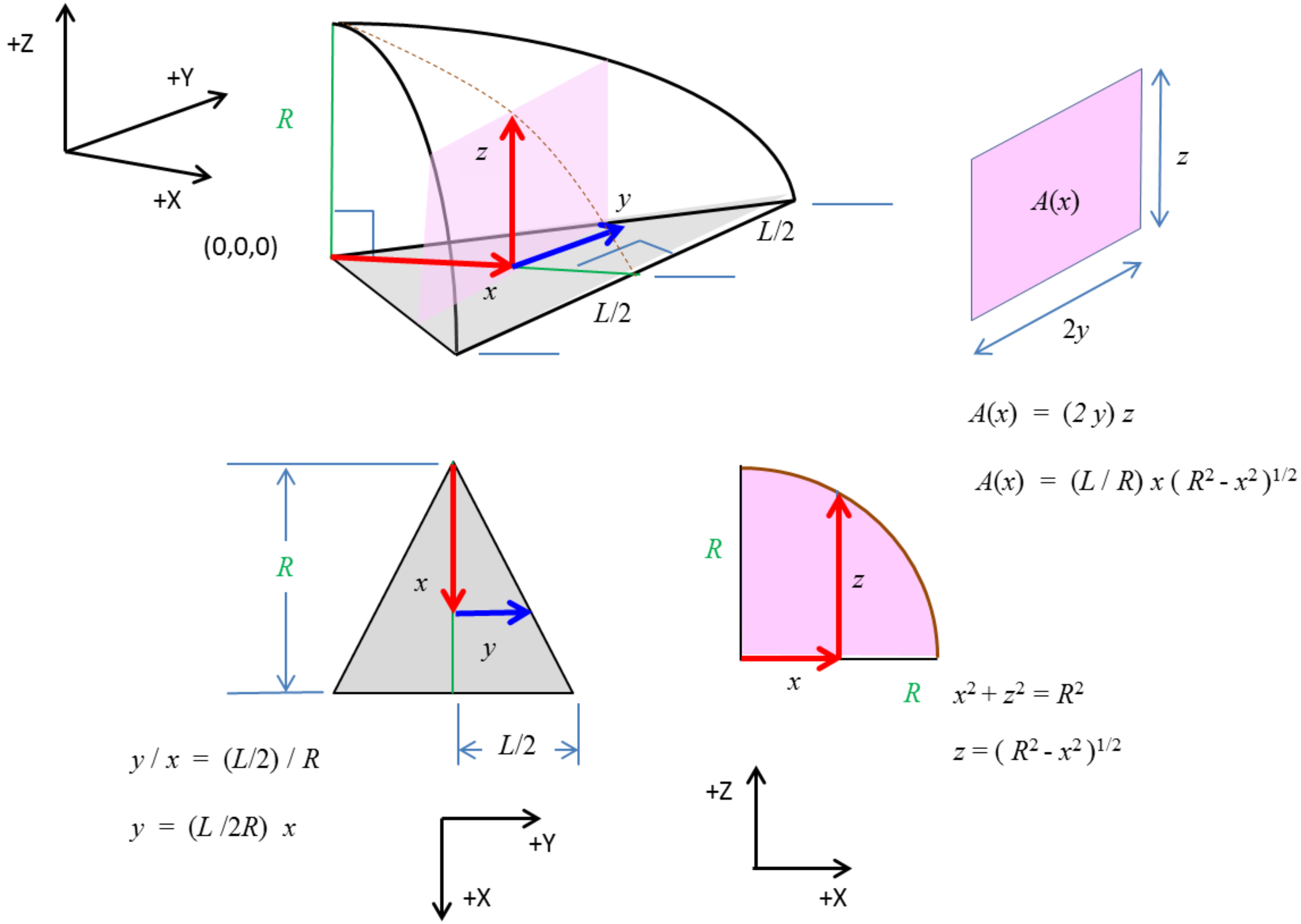
$$y = \left(\frac{L}{2}\right) x$$

In the XZ plane, the height of the rectangle at x is

$$z = \left(R^2 - x^2\right)^{1/2}$$

Area of the rectangle at x is

$$A(x) = \left(\frac{L}{R}\right) x \left(R^2 - x^2\right)^{1/2}$$



The limits of the integration are $x_a = 0$ and $x_b = R$.

The volume of the solid is

$$V = \int_{x_a}^{x_b} A(x)\, dx$$

$$V = \int_0^R \left(\frac{L}{R}\right) x \left(R^2 - x^2\right)^{1/2} dx$$

$$V = \left(\frac{L}{R}\right) \left[\left(\frac{2}{3}\right)\left(\frac{-1}{2}\right)\left(R^2 - x^2\right)^{3/2} \right]_0^R$$

$$V = \frac{1}{3} L\, R^2$$

QED