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Supplementary material for: A computational tool for the analysis of 3D bending-active structures based on dynamic relaxation method

1. Introductory definitions

For each element connecting nodes a and b we define its direction being from a to b and denote v^k the vector of element k as

$$v^k = P_b - P_a \tag{1}$$
$$\|v^k\| = L^k \tag{2}$$

$$n^k = \frac{n_a + n_b}{\|n_a + n_b\|}, \tag{3}$$
$$t_1^k = \frac{P_b - P_a}{\|P_b - P_a\|}, \tag{4}$$
$$t_2^k = \frac{n^k \times t_1^k}{\|n^k \times t_1^k\|}, \tag{5}$$
$$t_3^k = \frac{t_1^k \times t_2^k}{\|t_1^k \times t_2^k\|} \tag{6}$$

For a vector quantity v we denote as v_x, v_y and v_z the x,y and z components respectively.
For computing the force of an element k we differentiate the energy terms

$$\begin{aligned} F_{I,i}^k = \frac{\partial U_I^k}{\partial u_{a,i}} &= \frac{\partial U_A^k}{\partial u_{a,i}} + \frac{\partial U_T^k}{\partial u_{a,i}} + \frac{\partial U_{B_2}^k}{\partial u_{a,i}} + \frac{\partial U_{B_3}^k}{\partial u_{a,i}} \\ &= F_{A,i}^k + F_{A,i}^k + F_{B_3,i}^k + F_{B_3,i}^k \end{aligned} \tag{7}$$

and from [SOA20] we get the following force expressions for the i^{th} dof in the 4 main ways of loading an element k.

$$F_{A,i}^k = \frac{EA}{\bar{L}_k} e^k \frac{\partial e_k}{\partial u_i} \tag{8}$$
$$F_{T,i}^k = \frac{GJ}{\bar{L}_k}\left(\theta_{1,b}^k - \theta_{1,a}^k\right)\frac{\partial\left(\theta_{1,b}^k - \theta_{1,a}^k\right)}{\partial u_i} \tag{9}$$
$$F_{B_2,i}^k = \frac{2EI_2^k}{\bar{L}_k}\left(\frac{2\partial\theta_{2,a}^k}{\partial u_i}\theta_{2,a}^k + \frac{\partial\theta_{2,b}^k}{\partial u_i}\theta_{2,a}^k + \frac{\partial\theta_{2,a}^k}{\partial u_i}\theta_{2,b}^k + \frac{2\partial\theta_{2,b}^k}{\partial u_i}\theta_{2,b}^k\right) \tag{10}$$
$$F_{B_3,i}^k = \frac{2EI_3^k}{\bar{L}_k}\left(\frac{2\partial\theta_{3,a}^k}{\partial u_i}\theta_{3,a}^k + \frac{\partial\theta_{3,b}^k}{\partial u_i}\theta_{3,a}^k + \frac{\partial\theta_{3,a}^k}{\partial u_i}\theta_{3,b}^k + \frac{2\partial\theta_{3,b}^k}{\partial u_i}\theta_{3,b}^k\right) \tag{11}$$

For eq.8-11 we will require some intermediate partial derivatives which we discuss next. From [SOA20] the partial derivative of the normal vector n_j of node j is defined as:

$$\frac{\partial n_j}{\partial u_i} = \begin{cases} \begin{matrix} \frac{\partial n_{j,x}}{\partial u_i} \\ \frac{\partial n_{j,y}}{\partial u_i} \\ -\frac{\frac{\partial n_{j,x}}{\partial u_i} n_{j,x} + \frac{\partial n_{j,y}}{\partial u_i} n_{j,y}}{\sqrt{1-\left(n_{j,x}^2+n_{j,y}^2\right)}} \end{matrix} & \text{for } n_{j,x}^2 + n_{j,y}^2 \leq 1 \\ \\ \begin{matrix} \frac{\partial n_{j,x}}{\partial u_i} \frac{1}{\sqrt{n_{j,x}^2+n_{j,y}^2}} - \frac{n_{j,x}\left(\frac{\partial n_{j,x}}{\partial u_i} n_{j,x} + \frac{\partial n_{j,y}}{\partial u_i} n_{j,y}\right)}{\left(n_{j,x}^2+n_{j,y}^2\right)^{3/2}} \\ \frac{\partial n_{j,y}}{\partial u_i} \frac{1}{\sqrt{n_{j,x}^2+n_{j,y}^2}} - \frac{n_{j,y}\left(\frac{\partial n_{j,x}}{\partial u_i} n_{j,x} + \frac{\partial n_{j,y}}{\partial u_i} n_{j,y}\right)}{\left(n_{j,x}^2+n_{j,y}^2\right)^{3/2}} \\ 0 \end{matrix} & \text{for } n_{j,x}^2 + n_{j,y}2 > 1 \end{cases}$$

If we differentiate with respect to $n_{j,x}$ we get:

$$\frac{\partial n_j}{\partial n_{j,x}} = \begin{cases} \begin{matrix} 1 \\ 0 \\ -\frac{n_{j,x}}{\sqrt{1-\left(n_{j,x}^2+n_{j,y}^2\right)}} \end{matrix} & \text{for } n_{j,x}^2 + n_{j,y}^2 \leq 1 \\ \\ \begin{matrix} \frac{1}{\sqrt{n_{j,x}^2+n_{j,y}^2}} - \frac{n_{j,x}^2}{\left(n_{j,x}^2+n_{j,y}^2\right)^{3/2}} \\ -\frac{n_{j,x} n_{j,y}}{\left(n_{j,x}^2+n_{j,y}^2\right)^{3/2}} \\ 0 \end{matrix} & \text{for } n_{j,x}^2 + n_{j,y}^2 > 1 \end{cases} \tag{12}$$

Similarly for $n_{j,y}$:

$$\frac{\partial n_j}{\partial n_{j,y}} = \begin{cases} \begin{matrix} 0 \\ 1 \\ -\frac{n_{j,y}}{\sqrt{1-\left(n_{j,x}^2+n_{j,y}^2\right)}} \end{matrix} & \text{for } n_{j,x}^2 + n_{j,y}^2 \leq 1 \\ \\ \begin{matrix} -\frac{n_{j,x} n_{j,y}}{\left(n_{j,x}^2+n_{j,y}^2\right)^{3/2}} \\ \frac{1}{\sqrt{n_{j,x}^2+n_{j,y}^2}} - \frac{n_{j,y}^2}{\left(n_{j,x}^2+n_{j,y}^2\right)^{3/2}} \\ 0 \end{matrix} & \text{for } n_{j,x}^2 + n_{j,y}^2 > 1 \end{cases} \tag{13}$$

and trivially $\frac{\partial n_j}{\partial X_{j,x}} = \frac{\partial n_j}{\partial X_{j,y}} = \frac{\partial n_j}{\partial X_{j,z}} = \frac{\partial n_j}{\partial n_{j,r}} = 0$ since the dof are considered independent.

From [SOA20] the partial derivative of t_1^k is computed as:

$$\frac{\partial t_1^k}{\partial u_i} = -\frac{1}{L^{k^2}} \frac{\partial L^k}{\partial u_i} \begin{bmatrix} P_{b,x} - P_{a,x} \\ P_{b,y} - P_{a,y} \\ P_{b,z} - P_{a,z} \end{bmatrix} + \frac{1}{L^k} \begin{bmatrix} \frac{\partial (X_{b,x} - X_{a,x})}{\partial u_i} \\ \frac{\partial (X_{b,y} - X_{a,y})}{\partial u_i} \\ \frac{\partial (X_{b,z} - X_{a,z})}{\partial u_i} \end{bmatrix}$$

From eq.1 and eq.16 we can rewrite as:

$$\frac{\partial t_1^k}{\partial u_i} = -\frac{1}{L^{k^3}} \left(v^k \cdot \frac{\partial \left(X_b - X_a\right)}{\partial u_i} \right) \cdot v^k + \frac{1}{L^k} \cdot \frac{\partial \left(X_b - X_a\right)}{\partial u_i} \tag{14}$$

the derivatives of the element frame vectors are

$$\frac{\partial t_2^k}{\partial u_i} = -\frac{n^k \times t_1^k}{\left\|n^k \times t_1^k\right\|^2} \frac{\partial \left\|n^k \times t_1^k\right\|}{\partial u_i} + \frac{1}{\left\|n^k \times t_1^k\right\|} \frac{\partial \left(n^k \times t_1^k\right)}{\partial u_i},$$

$$\frac{\partial t_3^k}{\partial u_i} = -\frac{t_1^k \times t_2^k}{\left\|t_1^k \times t_2^k\right\|^2} \frac{\partial \left\|t_1^k \times t_2^k\right\|}{\partial u_i} + \frac{1}{\left\|t_1^k \times t_2^k\right\|} \frac{\partial \left(t_1^k \times t_2^k\right)}{\partial u_i},$$

$$\frac{\partial n^k}{\partial u_i} = -\frac{n_a + n_b}{\|n_a + n_b\|^2} \frac{\partial \|n_a + n_b\|}{\partial u_i} + \frac{1}{\|n_a + n_b\|} \frac{\partial (n_a + n_b)}{\partial u_i}.$$

2. Force computation

In this section using the expressions of the previous section we formulate the per dof force expressions which are needed in each DRM step.

2.1. Axial force

From [SOA20] we have

$$\frac{\partial e^k}{\partial u_i} = \frac{\partial \left(L^k - \bar{L}^k\right)}{\partial u_i} = \frac{\partial L^k}{\partial u_i}$$

which is obtained as

$$\frac{\partial L^k}{\partial u_i} = \frac{1}{L^k} \left[P_{b,x} - P_{a,x}, \quad P_{b,y} - P_{a,y}, \quad P_{b,z} - P_{a,z}\right] \begin{bmatrix} \frac{\partial (X_{b,x} - X_{a,x})}{\partial u_i} \\ \frac{\partial (X_{b,y} - X_{a,y})}{\partial u_i} \\ \frac{\partial (X_{b,z} - X_{a,z})}{\partial u_i} \end{bmatrix} = \frac{1}{L^k} v^k \cdot \frac{\partial (X_b - X_a)}{\partial u_i}$$

From eq.1 and eq.4:

$$v^k = t_1^k \cdot L^k \tag{15}$$

Using equation 15 we can rewrite as:

$$\frac{\partial L^k}{\partial u_i} = t_1^k \cdot \frac{\partial (X_b - X_a)}{\partial u_i} \tag{16}$$

By sustituting eq. 16 in eq. 8 we get

$$F_{A,i}^k = \frac{EA}{\bar{L}_k} e^k \frac{1}{L^k} v^k \cdot \frac{\partial (X_b - X_a)}{\partial u_i} = \kappa_A e^k t_1^k \cdot \frac{\partial (X_b - X_a)}{\partial u_i}$$

The only derivatives which are non-zero are the ones with respect to the translational degrees of freedom of the endpoints a and b of element k. As such for each element we compute in total 6 axial-force components as:

$$F_{A,X_{a,x}}^k = \kappa_A e^k t_1^k \cdot \frac{\partial (X_b - X_a)}{\partial X_{a,x}} = -\kappa_A e^k t_{1,x}^k$$

$$F_{A,X_{b,x}}^k = \kappa_A e^k t_1^k \cdot \frac{\partial (X_b - X_a)}{\partial X_{b,x}} = \kappa_A e^k t_{1,x}^k$$

$$F_{A,y_{a,y}}^k = \kappa_A e^k t_1^k \cdot \frac{\partial (X_b - X_a)}{\partial X_{a,y}} = -\kappa_A e^k t_{1,y}^k$$

$$F_{A,y_{b,y}}^k = \kappa_A e^k t_1^k \cdot \frac{\partial (X_b - X_a)}{\partial X_{b,y}} = \kappa_A e^k t_{1,y}^k$$

$$F_{A,z_{a,z}}^k = \kappa_A e^k t_1^k \cdot \frac{\partial (X_b - X_a)}{\partial X_{a,z}} = -\kappa_A e^k t_{1,z}^k$$

$$F_{A,z_{b,z}}^k = \kappa_A e^k t_1^k \cdot \frac{\partial (X_b - X_a)}{\partial X_{b,z}} = \kappa_A e^k t_{1,z}^k$$

2.2. Torsional force

The partial derivative of the angle $\theta_{1,j}^k$ is expressed as

$$\frac{\partial \theta_{1,j}^k}{\partial u_i} = \frac{\partial \left(t_2^k \cdot n_j\right)}{\partial u_i} = -\frac{\partial t_2^k}{\partial u_i} \cdot n_j - t_2^k \cdot \frac{\partial n_j}{\partial u_i} \tag{17}$$

From eq.17 we can express the partial derivative of the difference in the θ_1 angles as

$$\frac{\partial \left(\theta_{1,b}^k - \theta_{1,a}^k\right)}{\partial u_i} = \frac{\partial \left(t_2^k \cdot n_a\right)}{\partial u_i} - \frac{\partial \left(t_2^k \cdot n_b\right)}{\partial u_i} = \frac{\partial t_2^k}{\partial u_i} \cdot n_a + t_2^k \frac{\partial n_a}{\partial u_i} - \frac{\partial t_2^k}{\partial u_i} n_b - t_2^k \frac{\partial n_b}{\partial u_i} = \frac{\partial t_2^k}{\partial u_i} \cdot (n_a - n_b) + t_2^k \left(\frac{\partial n_a}{\partial u_i} - \frac{\partial n_b}{\partial u_i}\right) \tag{18}$$

and then by substituting in eq. 9

$$F_T^k, i = \kappa_T \left[t_2^k \cdot (n_a - n_b)\right] \cdot \left[\frac{\partial t_2^k}{\partial u_i} (n_a - n_b) + t_2^k \left(\frac{\partial n_a}{\partial u_i} - \frac{\partial n_b}{\partial u_i}\right)\right] \tag{19}$$

The computation of the torsional force involves the derivatives $\frac{\partial t_2^k}{\partial u_i}$, $\frac{\partial n_a}{\partial u_i}$ and $\frac{\partial n_b}{\partial u_i}$ which are non-zero only for the degrees of freedom of the endpoints of element k. In total we compute 10 torsional force components for each element k. If $u_i \in (X_{a,x}, X_{a,y}, X_{a,z}, X_{b,x}, X_{b,y}, X_{b,z})$ we get:

$$F_{T,b_i}^k = F_{T,a_i}^k = \kappa_T \left(t_2^k \cdot (n_a - n_b)\right) \cdot \left(\frac{\partial t_2^k}{\partial u_i} \cdot (n_a - n_b)\right) \tag{20}$$

Similarly for the translational dof of node b.
The remaining 4 non-zero components correspond to the dof n_x and n_y of nodes a and b and for which the partial derivatives using the formulas of eq.12 and 13 are required. For $a_i \in (n_{a,x}, n_{a,y})$ and $b_i \in (n_{b,x}, n_{b,y})$ we compute the torsional force as:

$$F_{T,a_i}^k = \kappa_T \left[t_2^k \cdot (n_a - n_b)\right] \cdot \left[\frac{\partial t_2^k}{\partial a_i} (n_a - n_b) + t_2^k \frac{\partial n_a}{\partial a_i}\right] \tag{21}$$

$$F_{T,b_i}^k = \kappa_T \left[t_2^k \cdot (n_a - n_b)\right] \cdot \left[\frac{\partial t_2^k}{\partial n_{b,x}} (n_a - n_b) - t_2^k \frac{\partial n_b}{\partial b_i}\right] \tag{22}$$

(23)

2.3. First bending force

We express the partial derivative of angle $\theta_{2,j}^k$ as

$$\frac{\partial \theta_{2,j}^k}{\partial u_i} = \frac{\partial \left(t_1^k \cdot n_j\right)}{\partial u_i} = \frac{\partial t_1^k}{\partial u_i} \cdot n_j + t_1^k \cdot \frac{\partial n_j}{\partial u_i} \tag{24}$$

We express the derivatives of angles θ_2 of an element's endpoints as

$$\begin{aligned} \frac{\partial \theta_{2,b}^k}{\partial u_i} &= \frac{\partial t_1^k}{\partial u_i} \cdot n_b + t_1^k \cdot \frac{\partial n_b}{\partial u_i} \\ \frac{\partial \theta_{2,a}^k}{\partial u_i} &= \frac{\partial t_1^k}{\partial u_i} \cdot n_a + t_1^k \cdot \frac{\partial n_a}{\partial u_i} \end{aligned} \tag{25}$$

Substituting eq. 25 in eq. 9 we express the first bending force with respect to the node normals n_a and n_b and the vector t_1^k as

$$F^k_{B_2,i} = \kappa_{B_2}\left(2\left(n_a \cdot t_1^k\right)\left(\frac{\partial n_a}{\partial u_i}\cdot t_1^k + n_a\frac{\partial t_1^k}{\partial u_i}\right)+\right. \tag{26}$$

$$+\left(n_a \cdot t_1^k\right)\cdot\left(\frac{\partial n_b}{\partial u_i}\cdot t_1^k + n_b\frac{\partial t_1^k}{\partial u_i}\right) \tag{27}$$

$$+\left(n_b \cdot t_1^k\right)\left(\frac{\partial n_a}{\partial u_i}\cdot t_1^k + n_a\cdot\frac{\partial t_1^k}{\partial u_i}\right) \tag{28}$$

$$\left.+2\left(n_b \cdot t_1^k\right)\left(\frac{\partial n_b}{\partial u_i}\cdot t_1^k + n_b\cdot\frac{\partial t_1^k}{\partial u_i}\right)\right) \tag{29}$$

The only non-zero derivatives for the bending force are the ones with respect to the dof of the two endpoints of element k. As such if $u_i \in (X_{a,x}, X_{a,y}, X_{a,z}, X_{b,x}, X_{b,y}, X_{b,z})$ we compute the bending force as:

$$\begin{aligned} F^k_{B_2,i} = \kappa_{B_2}\Bigg(&2\left(n_a \cdot t_1^k\right)\left(n_a\frac{\partial t_1^k}{\partial u_i}\right)+\\ &+\left(n_a \cdot t_1^k\right)\cdot\left(n_b\frac{\partial t_1^k}{\partial u_i}\right)\\ &+\left(n_b \cdot t_1^k\right)\left(n_a\cdot\frac{\partial t_1^k}{\partial u_i}\right)\\ &+2\left(n_b \cdot t_1^k\right)\left(n_b\cdot\frac{\partial t_1^k}{\partial u_i}\right)\Bigg) \end{aligned} \tag{30}$$

For $a_i \in (n_{a,x}, n_{a,y})$ and $b_i \in (n_{b,x}, n_{b,y})$ we compute the bending force as:

$$\begin{aligned} F^k_{B_2,a_i} &= \kappa_{B_2}\left(2\left(n_a \cdot t_1^k\right)\left(\frac{\partial n_a}{\partial u_i}\cdot t_1^k\right) \qquad +\left(n_b \cdot t_1^k\right)\left(\frac{\partial n_a}{\partial u_i}\cdot t_1^k\right)\right)\\ F^k_{B_2,b_i} &= \kappa_{B_2}\Bigg(\left(n_a \cdot t_1^k\right)\cdot\left(\frac{\partial n_b}{\partial u_i}\cdot t_1^k\right)\\ &\qquad +2\left(n_b \cdot t_1^k\right)\left(\frac{\partial n_b}{\partial u_i}\cdot t_1^k\right)\Bigg) \end{aligned} \tag{31}$$

2.4. Second bending force

For the second bending force its important to note that the number of non-zero components per beam element is higher compared to the other force components we saw. This is because in addition to the 12 non-zero derivatives wrt the dof of the n^{th} element's end-nodes a and b we need to compute additionally 6 derivatives wrt to the dof of the non-common node of elements k and n.

The partial derivatives of f_1,$\hat{f}_1$,f_2,f_3 and $\theta^n_{3,j}$ from [SOA20] are:

$$\frac{\partial f_1}{\partial u_i} = \frac{\partial t_1^k}{\partial u_i} - \left[\left(\frac{\partial t_1^k}{\partial u_i}\cdot n_j + t_1^k\cdot\frac{\partial n_j}{\partial u_i}\right)n_j + \left(t_1^k\cdot n_j\right)\frac{\partial n_j}{\partial u_i}\right] \tag{32}$$

$$\frac{\partial \hat{f}_1}{\partial u_i} = -\frac{f_1}{\|f_1\|^2}\frac{\partial\|f_1\|}{\partial u_i} + \frac{1}{\|f_1\|}\frac{\partial f_1}{\partial u_i} \tag{33}$$

$$\frac{\partial f_2}{\partial u_i} = \frac{\partial f_1}{\partial u_i} \cos\left(n_{j,r} + \alpha\right) + f_1 \left(-\sin\left(n_{j,r} + \alpha\right) \frac{\partial n_{j,r}}{\partial u_i}\right) \tag{34}$$

$$+ \left(\frac{\partial n_j}{\partial u_i} \times f_1 + n_j \times \frac{\partial f_1}{\partial u_i}\right) \sin\left(n_{j,r} + \alpha\right) \tag{35}$$

$$+ \left(n_j \times f_1\right) \cos\left(n_{j,r} + \alpha\right) \frac{\partial n_{j,r}}{\partial u_i} \tag{36}$$

$$\tag{37}$$

$$\frac{\partial f_3}{\partial u_i} = \frac{\partial t_1^n}{\partial u_i} \times f_2 + t_1^n \times \frac{\partial f_2}{\partial u_i} \tag{38}$$

$$\frac{\partial \theta_{3,j}^n}{\partial u_i} = \frac{\partial f_3}{\partial u_i} \cdot n_j + f_3 \cdot \frac{\partial n_j}{\partial u_i} \tag{39}$$

For $u_i \in (X_{a,x}, X_{a,y}, X_{a,z}, X_{b,x}, X_{b,y}, X_{b,z})$ we get:

$$\frac{\partial f_1}{\partial u_i} = \frac{\partial t_1^k}{\partial u_i} - \left(\frac{\partial t_1^k}{\partial u_i} \cdot n_j\right) n_j \tag{40}$$

$$\frac{\partial f_2}{\partial u_i} = \frac{\partial f_1}{\partial u_i} \cos\left(n_{j,r} + \alpha\right) \tag{41}$$

Using eq.40 and 41 we compute the derivative of $\theta_{3,j}^n$ from eq. 50.

For $u_i \in (n_{a,x}, n_{a,y}, n_{b,x}, n_{b,y})$ we get:

$$\frac{\partial f_1}{\partial u_i} = -\left(t_1^k \cdot \frac{\partial n_j}{\partial u_i}\right) n_j + \left(t_1^k \cdot n_j\right) \frac{\partial n_j}{\partial u_i} \tag{42}$$

$$\frac{\partial f_2}{\partial u_i} = \frac{\partial f_1}{\partial u_i} \cos\left(n_{j,r} + \alpha\right) + \left(\frac{\partial n_j}{\partial u_i} \times f_1 + n_j \times \frac{\partial f_1}{\partial u_i}\right) \sin\left(n_{j,r} + \alpha\right) \tag{43}$$

Using eq.42 and 43 we compute the derivative of $\theta_{3,j}^n$ from eq. 50.

If $u_i \in (n_{a,r}, n_{b,r})$

$$\frac{\partial f_1}{\partial u_i} = -\left(t_1^k \cdot n_j\right) \frac{\partial n_j}{\partial u_i} \tag{44}$$

$$\frac{\partial f_2}{\partial u_i} = \frac{\partial f_1}{\partial u_i} \cos\left(n_{j,r} + \alpha\right) + f_1 \left(-\sin\left(n_{j,r} + \alpha\right) \frac{\partial n_{j,r}}{\partial u_i}\right) \tag{45}$$

$$+ \left(\frac{\partial n_j}{\partial u_i} \times f_1 + n_j \times \frac{\partial f_1}{\partial u_i}\right) \sin\left(n_{j,r} + \alpha\right) \tag{46}$$

$$+ \left(n_j \times f_1\right) \cos\left(n_{j,r} + \alpha\right) \frac{\partial n_{j,r}}{\partial u_i} \tag{47}$$

$$\tag{48}$$

$$\frac{\partial f_3}{\partial u_i} = t_1^n \times \frac{\partial f_2}{\partial u_i} \tag{49}$$

Using eq.44, 45 and 49 we compute the derivative of $\theta_{3,j}^n$ as:

$$\frac{\partial \theta_{3,j}^n}{\partial u_i} = \frac{\partial f_3}{\partial u_i} \cdot n_j \tag{50}$$

We compute the second bending force by making use of eq. 11 and the expression presented in this section.