Optimal Strategy Proof

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1 Introduction to the Problem

We are given a problem:

- max $\sum_{i=1}^{n} c_i y_i$
- $\sum_{i=1}^{n} a_i y_i \leqslant b$
- $0 \leqslant y_i \leqslant 1$, $i = 1, ..., n$

We are also given a pseudocode that should be an the algorithm that finds an optimal solution to this problem:

sort indexes so that $i < j \Rightarrow \frac{c_i}{a_i} \geqslant \frac{c_j}{a_j}$
i ←1;
repeat
{
$y_i \leftarrow min\{1, \frac{b}{a_i}\};$
$b \leftarrow b - y_i a_i;$
$i \leftarrow i + 1;$
}
until b=0 or $i > n$

We have to prove that the solution is optimal using linear programming and complementary dual solution.

2 Solution

Assuming that $b > 0$ and $a_i > 0$ (seems reasonable).
The solution proposed (considering the sorting of (c_i, a_i) couples) is a vector $\hat{y}$ such that there is an index $1 \leqslant r \leqslant |\hat{y}|$ such that $\forall i < r : y_i = 1$, $y_r \in \{0, 1\}$ or $y_r = \frac{b - \sum_{i=0}^{r-1} a_i}{a_r}$, $\forall i > r : y_i = 0$.
To prove its optimality, the idea is to exploit the complementary slackness properties.

2.1 Definitions

Lets rewrite the problem a bit!
I will call $\boldsymbol{y}$ the column vector for our variable.

I will call $\boldsymbol{A}$ the matrix of size $1 + |\boldsymbol{y}| \cdot |\boldsymbol{y}|$, defined as follows:

$$\boldsymbol{A} = \begin{bmatrix} a_1 & a_2 & a_3 & ... & a_{|\boldsymbol{y}|} \\ 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & ... & 1 \end{bmatrix} \tag{1}$$

I will call $\boldsymbol{c}$ the row vector containing the c_i elements.
I will call $\boldsymbol{b}$ the column vector of length $1 + |\boldsymbol{y}|$ defined as follows:

$$\boldsymbol{b} = \begin{bmatrix} b \\ 1 \\ 1 \\ 1 \\ . \\ . \\ . \\ 1 \end{bmatrix} \tag{2}$$

The problem can now be rewritten as:

- max $\boldsymbol{cy}$
- $\boldsymbol{Ay} \leqslant \boldsymbol{b}$
- $\boldsymbol{y} \geqslant 0$

Given a column vector $\boldsymbol{x}$, the dual will be:

- min $\boldsymbol{xb}$
- $\boldsymbol{xA} = \boldsymbol{c}$
- $x \geqslant 0$

2.1.1 Proof

The proof leverages on

***Theorem*4.1**:
Let $\hat{p}$ and $\hat{d}$ be feasible solutions for the primal and the dual problem, respectively. Then the following properties are equivalent:
i) $\hat{p}$ and $\hat{d}$ are optimal solutions;
ii) $c\hat{p} = \hat{d}b$;
iii) $\hat{d}(b - A\hat{p}) = 0$;

and

***Theorem*4.2**:
Let $\hat{p}$ and $\hat{d}$ be feasible solutions for the primal and the dual problem, respectively: then $\hat{p}$ and $\hat{d}$ are optimal solutions if and only if:
$\hat{d}(b_i - A_i\hat{p} = 0)$, $i = 1, ..., m$
$(\hat{d}A^j - c_j)\hat{p}_j = 0$, $j = 1, ..., n$

As we said at the beginning:

$$\hat{p} = \begin{bmatrix} 1 & ... & 1 & k & 0 & ... & 0 \end{bmatrix} \tag{3}$$

Let's take $k \in [0, 1]$ that can vary its position between 1 and $|\hat{p}|$.
Let's call r the position of the element k.
Let's use the *Theorem 4.2* and write the system to check if our solution is feasible.

$$\begin{bmatrix} d_1 & d_2 & ... & d_{r+1} & ... & d_{1+|\hat{p}|} \end{bmatrix} \left(\begin{bmatrix} b \\ 1 \\ 1 \\ . \\ . \\ . \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} a_1 & a_2 & ... & a_r & ... & a_{|\hat{p}|} \\ 1 & 0 & ... & 0 & ... & 0 \\ 0 & 1 & ... & 0 & ... & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & ... & 0 & ... & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ . \\ k_r \\ . \\ . \\ 0_{|\hat{p}|} \end{bmatrix} \right) = 0$$

Becomes:

$$\begin{bmatrix} d_1 & d_2 & ... & d_{r+1} & ... & d_{1+|\hat{p}|} \end{bmatrix} \begin{bmatrix} b - (\sum_{i=0}^{r-1} a_i + a_r k) \\ 0 \\ 0 \\ ... \\ (1-k)_{r+1} \\ ... \\ 1 \\ 1 \end{bmatrix} = 0$$

From which we get the following system:

$$\begin{cases} d_1(\sum_{i=0}^{r-1} a_i + a_r k) = 0 \\ d_{r+1}(1-k) = 0 \\ \qquad d_{r+2} = 0 \\ \qquad \ldots \\ \qquad d_{1+|\hat{p}|} = 0 \end{cases}$$

We can have a look at the second set of equations:

$$\left(\begin{bmatrix} d_1 & d_2 & \ldots & d_{r+1} & \ldots & d_{1+|\hat{p}|} \end{bmatrix} \begin{bmatrix} a_1 & a_2 & \ldots & a_r & \ldots & a_{|\hat{p}|} \\ 1 & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & \ldots & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & \ldots & 0 & \ldots & 1 \end{bmatrix} - \begin{bmatrix} c_1 & c_2 & \ldots & c_r & \ldots & c_{|\hat{p}|} \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \\ . \\ k_r \\ . \\ . \\ 0_{|\hat{p}|} \end{bmatrix} = 0$$

From which we get another system:

$$\begin{cases} a_1 d_1 + d2 = c_1 \\ \ldots \\ a_i d_1 + d_{i+1} = c_i \\ \ldots \\ (a_r d_1 + d_{r+1} - c_r)k = 0 \end{cases}$$

Considering the two systems and the case in which $k \in (0,1)$ we can easily notice that $d_{r+1} = 0$.
So $d_1 = \frac{c_r}{a_r}$.

Follows that $d_1 > 0$ and from the first equation $k = \frac{b - \sum_{i=0}^{r-1} a_i}{a_r}$ which is perfectly in line with our algorithm.
We can than notice that for all other i: $d_i = c_i - \frac{a_i c_r}{a_r}$.
Considering the HP $i < j \Rightarrow \frac{c_i}{a_i} \geqslant \frac{c_j}{a_j}$, all d are positive and the solution is feasible therefore optimal.

Considering the case in which $k = 0$ we can spot $d_1(\sum_{i=0}^{r-1} a_i) = 0$ and $d_{r+1} = 0$. The other d are arbitrary or equal to a cost. The solution is feasible and optimal.

Considering the case in which $k = 1$ we can see that $d_1 = 0$, $d_{r_1} = c_r$ and the situation is exactly the same as the one above mentioned. Even in this case the solution provided by the algorithm is optimal.