Lower Bounds for the Polynomial Calculus via the "Pigeon Dance"

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Overview

We will present the result from A.A. Razborov, "Lower Bounds for the Polynomial Calculus", in: Computational Complexity 7.4 (Dec. 2, 1998).

Background

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- In particular for the pigeonhole principle
- Polynomial calculus is a strong proof system
- Provide a lower bound for it with the pigeonhole principle

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- Multiplication

$$\frac{f}{f \cdot x}$$

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- A proof of 1 exists if and only if $f_1, \ldots, f_n$ have no common zeroes
- We construct polynomials such that their zeroes correspond to satisfying assignments
- Proving 1 from them is a *refutation*

Example proof

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- There is no way to prove y from $x + 1$ and z
- Closest to $xy + z$ we can prove is $xy + y + z$

Algebraic view of proofs

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- R is the projection onto Δ $\qquad R(xy + z) = y$
- Similarly:
  V_d, Δ_d, R_d for polynomials up to degree d
  V_I, Δ_I, R_I for polynomials using variables for subset of pigeons I

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Definition ($\neg\mathcal{PHP}_n^m$)

$$Q_i := 1 - \sum_{j \in [n]} x_{ij} \qquad \text{for each } i \in [m]$$

$$Q_{i_1,i_2,j} := x_{i_1 j} x_{i_2 j} \qquad \text{for each } i_1 \neq i_2 \in [m], j \in [n]$$

Main result

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- Characterize R_I syntactically

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For any $m > n$, every polynomial calculus refutation of $\neg\mathcal{PHP}_n^m$ must have degree at least $n/2 + 1$.

- Claim: derivable polynomials are locally consistent
- Works if R_I agree on their intersections
- Characterize R_I syntactically
- Show the different operators are identical

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- Pigeon assignments are variable assignments

$$a(x) = \begin{cases} 1, & \text{if } x \in \{x_{1,2}, x_{2,4}, x_{3,1}\} \\ 0, & \text{otherwise} \end{cases}$$

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- Polynomials are evaluated to 0 if they allow the assignment

$$a(1 - x_{1,1} - x_{1,2} - x_{1,3} - x_{1,4}) = 0$$
$$a(x_{1,1}x_{3,1}) = a(x_{1,2}x_{2,2}) = 0$$

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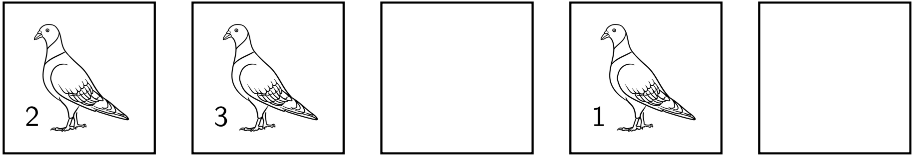
- Repeat until all pigeons have moved once
- If a pigeon cannot find an empty hole, the dance is aborted

Example



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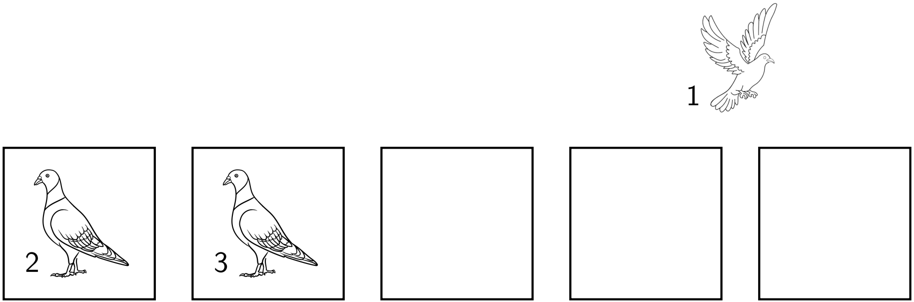

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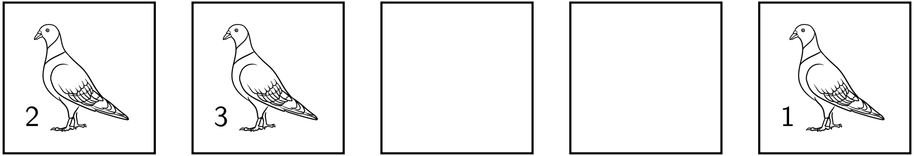

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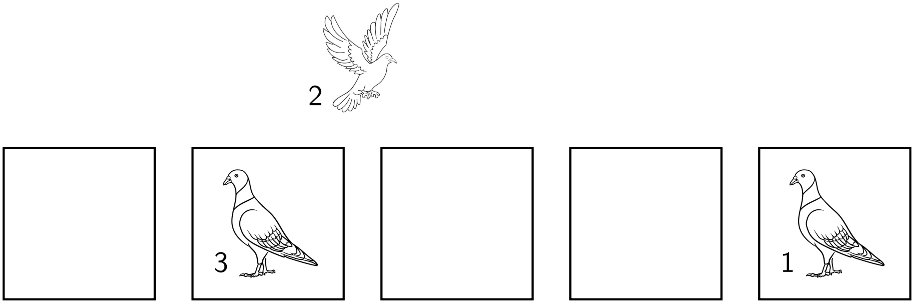

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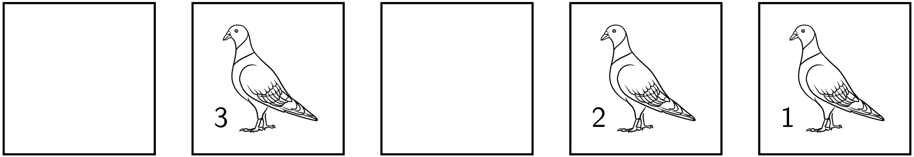

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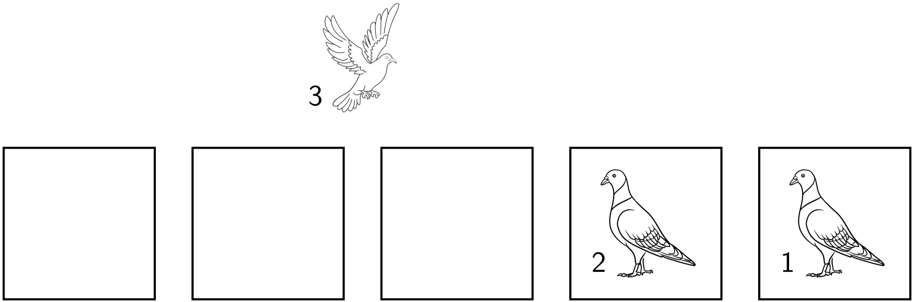

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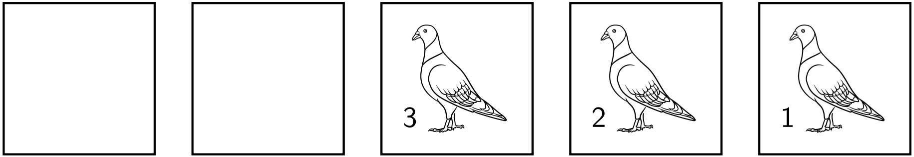

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- Consider terms $x_{i_1 j_1} \cdots x_{i_d j_d}$ with all i and j pairwise different
- Variables occuring in term correspond to pigeon positions
- Define Δ_I to be the set of terms that let pigeons complete the dance
- $t \in \Delta_I$ independent of I since pigeons not in the dance do not affect it

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- The first two summands are $\prec t$ and can be ignored
- Any terms with $j' \in \{j_2, \ldots, j_d\}$ are 0 and can be ignored

Defining R_I (cont.)

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- This is the pigeon dance!
- Since $t \notin \Delta_I$ the dance cannot be completed
- Process terminates with $\mathrm{LT}(f) \prec t$ and $t = f \mod V_I$

The Kill operator

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- The Kill operator kills the first pigeon and moves its hole to the left
- $\mathrm{Kill}(x_{i_1 j_1} \cdots x_{i_d, j_d}) = x_{i_2 j_2'} \cdots x_{i_d j_d'}$ with

$$j_k' := \begin{cases} j_k + 1, & \text{if } j_k < j_1 \\ j_k, & \text{if } j_k > j_1. \end{cases}$$

The dance in terms of Kill

Theorem

$x_{i_1 j_1} \cdots x_{i_d j_d} \in \Delta_I$ *if and only if there is a* $j' > j_1$ *such that* $\mathrm{Kill}(x_{i_1 j'} \cdots x_{i_d j_d}) \in \Delta_I$.

Proof sketch

This operator effectively moves the first pigeon to an empty hole to its right and then kills it. This is the same as each step in the dance, where the first pigeon flies to some free hole to its right and then occupies it. □

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Proof sketch

If $t \in \Delta_I$ then the pigeons can complete their dance. During this the first pigeon will start at j and fly to j'. Killing the pigeon frees up j' so any other pigeon that wanted to use j can use it instead. $\square$

The lower bound

Theorem

If $|I| \leq (n+1)/2, t \in \Delta_I$ *and the minimal element* i *of* I *is not in* $\mathrm{dom}(t)$*, then there exists a* $j \in [n]$ *such that* $\mathrm{Kill}(x_{ij}t) \in \Delta_I$.

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Proof sketch

At most

$$|\mathrm{dom}(t)| \leq |I \setminus \{i\}| \leq \frac{n-1}{2}$$

pigeons involved in the dance, each occupying two holes. Thus the total number of holes is $n-1$ and one hole j remains free. For the purposes of the dance, $\mathrm{Kill}(x_{ij}t)$ is the same as t since the only difference is j being moved to the left. $\square$

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- Inductive assumption gives us $a' \in M_{I\setminus\{i\}}$ with $a'(f') \neq 0$
- Pick a j such that $\mathrm{Kill}(x_{ij}t) \in \Delta_I$
- Extend assignment to I with $a(f) \neq 0$

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- Pigeons not in the dance do not affect its success, so $R_I(t) = R_{\mathrm{dom}(t)}(t)$
- V_d is precisely polynomials identically zero on M_I
- $R_d \neq 0$
- There is no refutation of $\neg \mathcal{PHP}_n^m$ with $d \leq n/2 + 1$