Lower Bounds for the Polynomial Calculus via the “Pigeon Dance”

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Overview

We will present the result from A.A. Razborov, "Lower Bounds for the Polynomial Calculus", in: Computational Complexity 7.4 (Dec. 2, 1998).

1. Introduction
2. The Pigeonhole Principle
3. The Pigeon Dance
4. Conclusion

Background

- Lower bounds for proofs in various systems

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- In particular for the pigeonhole principle
- Polynomial calculus is a strong proof system
- Provide a lower bound for it with the pigeonhole principle

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- Multiplication

$$\frac{f}{f \cdot x}$$

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- We construct polynomials such that their zeroes correspond to satisfying assignments
- Proving 1 from them is a *refutation*

Example proof

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- There is no way to prove y from $x + 1$ and z
- Closest to $xy + z$ we can prove is $xy + y + z$

Algebraic view of proofs

- V are polynomials we can prove $\qquad xy + y + z$
- Δ are leading terms of ones we cannot prove $\qquad y$
- $S_n(\mathbb{K}) \cong \mathbb{K}\Delta \oplus V \qquad xy + z = -y + xy + y + z$
- R is the projection onto $\Delta \qquad R(xy + z) = y$
- Similarly:
  V_d, Δ_d, R_d for polynomials up to degree d
  V_I, Δ_I, R_I for polynomials using variables for subset of pigeons I

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Definition ($\neg\mathcal{PHP}^m_n$)

$$Q_i := 1 - \sum_{j \in [n]} x_{ij} \qquad \text{for each } i \in [m]$$

$$Q_{i_1,i_2,j} := x_{i_1 j} x_{i_2 j} \qquad \text{for each } i_1 \neq i_2 \in [m], j \in [n]$$

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- Derived polynomials are constraints on pigeon assignments
- Locally valid assignments are possible
- Polynomials need to have large degree to prove inconsitencies

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- Locally valid assignments are injections $I \hookrightarrow [n]$
- Corresponding varliable assignments are M_I
- V_I is the set of all polynomials with $a(f) = 0$ for all $a \in M_I$

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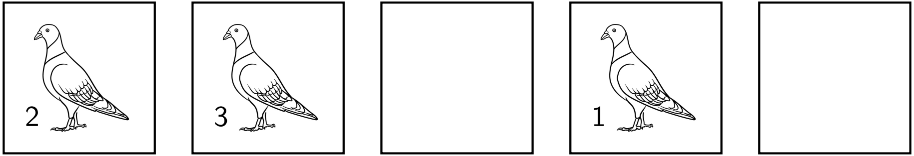
- It only works if $R_I(t)$ is independent of I
- Looks like this does not work at all

Example



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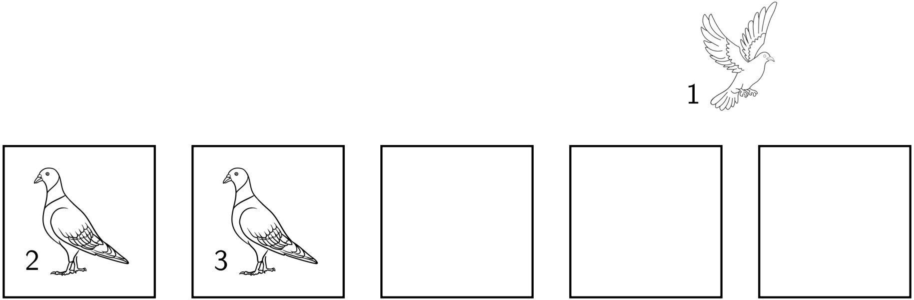

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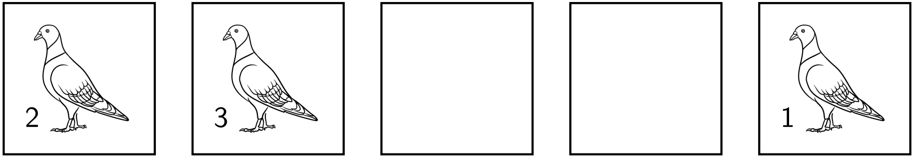

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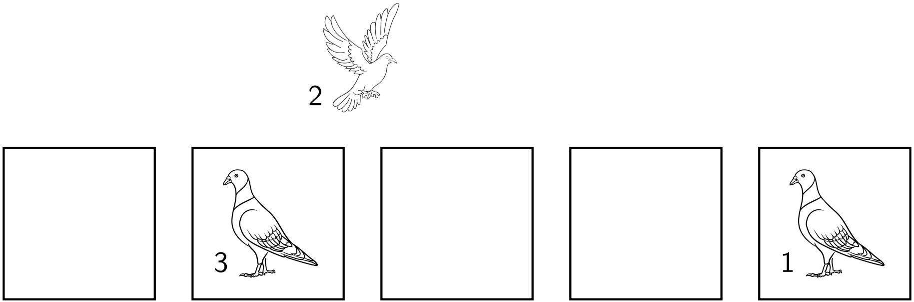

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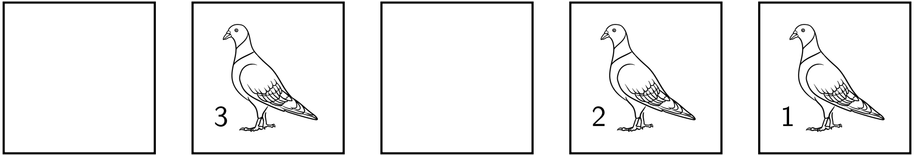

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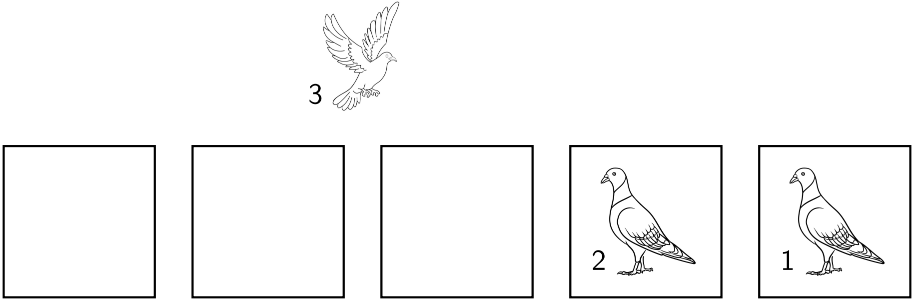

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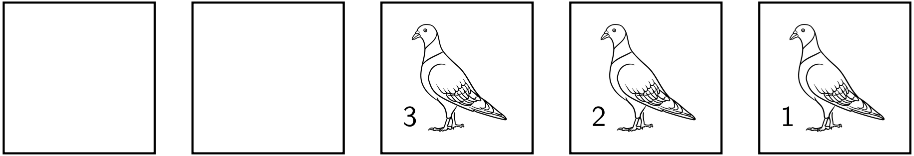

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- Δ_I is the set of terms that let pigeons complete the dance
- $t \in \Delta_I$ independent of I since pigeons not in the dance do not affect it

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- $\mathrm{Kill}(x_{i_1 j_1} \cdots x_{i_d, j_d}) = x_{i_2 j'_2} \cdots x_{i_d j'_d}$ with

$$j'_k := \begin{cases} j_k + 1, & \text{if } j_k < j_1 \\ j_k, & \text{if } j_k > j_1. \end{cases}$$

The dance in terms of Kill

Theorem

$x_{i_1 j_1} \cdots x_{i_d j_d} \in \Delta_I$ *if and only if there is a* $j' > j_1$ *such that* $\mathrm{Kill}(x_{i_1 j'} \cdots x_{i_d j_d}) \in \Delta_I$.

Proof sketch

This operator effectively moves the first pigeon to an empty hole to its right and then kills it. This is the same as each step in the dance, where the first pigeon flies to some free hole to its right and then occupies it. $\square$

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Δ_I *is closed under* Kill.

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Proof sketch

If $t \in \Delta_I$ then the pigeons can complete their dance. During this the first pigeon will start at j and fly to j'. Killing the pigeon frees up j' so any other pigeon that wanted to use j can use it instead. $\square$

The lower bound

Theorem

If $|I| \leq (n+1)/2, t \in \Delta_I$ *and the minimal element* i *of* I *is not in* $\mathrm{dom}(t)$*, then there exists a* $j \in [n]$ *such that* $\mathrm{Kill}(x_{ij}t) \in \Delta_I$.

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At most

$$|\mathrm{dom}(t)| \leq |I \setminus \{i\}| \leq \frac{n-1}{2}$$

pigeons involved in the dance, each occupying two holes. Thus the total number of holes is $n-1$ and one hole j remains free. For the purposes of the dance, $\mathrm{Kill}(x_{ij}t)$ is the same as t since the only difference is j being moved to the left. $\square$

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- Extend assignment to I with $a(f) \neq 0$

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- If the degree is small, pigeons can complete the dance
- You need to have $n/2+1$ pigeons to block enough holes
- Only polynomials corresponding to locally consistent assignments are derivable