Lower Bounds for the Polynomial Calculus via the "Pigeon Dance"

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Overview

We will present the result from A.A. Razborov, "Lower Bounds for the Polynomial Calculus", in: Computational Complexity 7.4 (Dec. 2, 1998).

Background

- Lower bounds for proofs in various systems

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- In particular for the pigeonhole principle
- Polynomial calculus is a strong proof system
- Provide a lower bound for it with the pigeonhole principle

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- Multiplication

$$\frac{f}{f \cdot x}$$

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- We construct polynomials such that their zeroes correspond to satisfying assignments
- Proving 1 from them is a *refutation*

Example proof

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- We now want to subtract y
- There is no way to prove y from $x + 1$ and z
- Closest to $xy + z$ we can prove is $xy + y + z$

Algebraic view of proofs

- V are polynomials we can prove $xy + y + z$
- Δ are leading terms of ones we cannot prove y
- $S_n(\mathbb{K}) \cong \mathbb{K}\Delta \oplus V$ $xy + z = -y + xy + y + z$
- Similarly V_I, Δ_I using subset of variables I

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Definition ($\neg\mathcal{PHP}_n^m$)

$$Q_i := 1 - \sum_{j \in [n]} x_{ij} \qquad \text{for each } i \in [m]$$

$$Q_{i_1,i_2,j} := x_{i_1 j} x_{i_2 j} \qquad \text{for each } i_1 \neq i_2 \in [m], j \in [n]$$

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- $\neg\mathcal{PHP}_n^m$ are constraints on pigeon assignments
- Proofs combine them into more complex ones
- Can only derive local constraints with small degrees
- Pigeons can fly away to escape local contradictions

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- Corresponding varliable assignments are M_I

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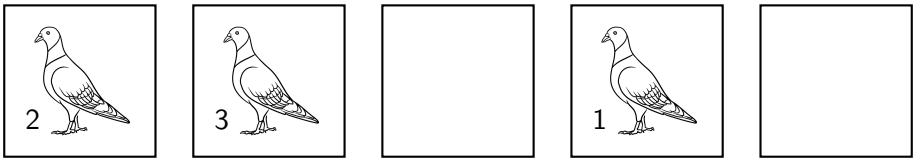
- But, this definition completely ignores the degrees of the proofs!
- It still works, but only if $t \in \Delta_I$ is independent of I
- Use pigeon dance to see this

Example



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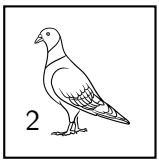

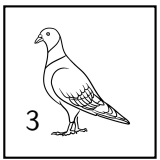

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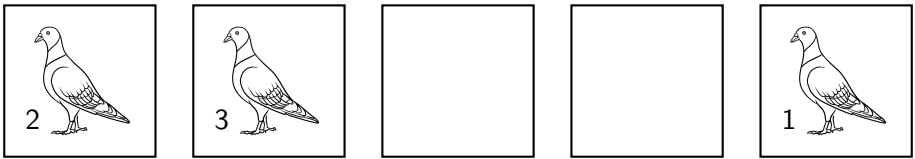

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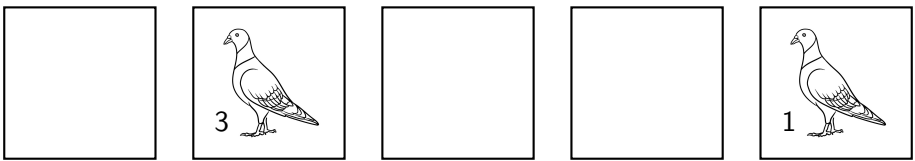

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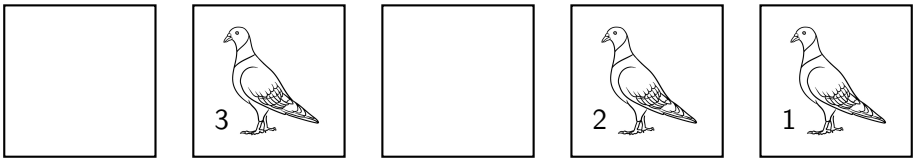

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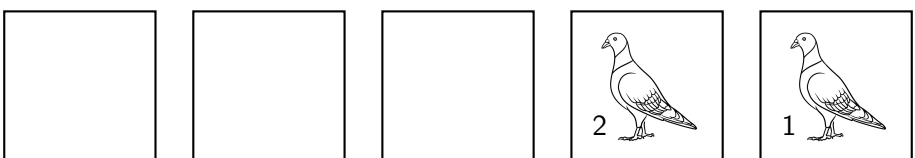

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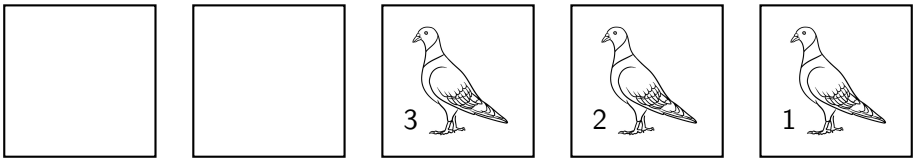

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- $t \in \Delta_I$ independent of I since pigeons not in the dance do not affect it

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- Kill the first pigeon and moves its hole to the left
- $\mathrm{Kill}(x_{i_1 j_1} \cdots x_{i_d, j_d}) = x_{i_2 j'_2} \cdots x_{i_d j'_d}$ with

$$j'_k := \begin{cases} j_k + 1, & \text{if } j_k < j_1 \\ j_k, & \text{if } j_k > j_1. \end{cases}$$

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Theorem

$x_{i_1 j_1} \cdots x_{i_d j_d} \in \Delta_I$ *if and only if there is a* $j' > j_1$ *such that* $\mathrm{Kill}(x_{i_1 j'} \cdots x_{i_d j_d}) \in \Delta_I$.

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Proof sketch

$\mathrm{Kill}(x_{i_1 j'} \cdots x_{i_d j_d})$ effectively moves the first pigeon to an empty hole to its right and then kills it. This is the same as each step in the dance, where the first pigeon flies to some free hole to its right and then occupies it. □

Closure of Δ_I

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Proof sketch

If $t \in \Delta_I$ then the pigeons can complete their dance. During this the first pigeon will start at j and fly to j'. Killing the pigeon frees up j' so any other pigeon that wanted to use j can use it instead. $\square$

The lower bound

Theorem

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At most

$$|\operatorname{dom}(t)| \leq |I \setminus \{i\}| \leq \frac{n-1}{2}$$

pigeons involved in the dance, each occupying two holes. Place pigeon at unused hole and kill it there. The remaining pigeons can complete the dance since the moved hole was not used. □

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- Extend assignment to I with $a(f) \neq 0$

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- If the degree is small, pigeons can complete the dance
- You need to have $n/2 + 1$ pigeons to force a contradiction