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Lower Bounds for the Polynomial Calculus
via the "Pigeon Dance"

Imogen Hergeth

January 2023

Overview

We will present the result from A.A. Razborov, "Lower Bounds for the Polynomial Calculus", in: Computational Complexity 7.4 (Dec. 2, 1998).

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2. The Pigeonhole Principle

3. The Pigeon Dance

4. Conclusion

The polynomial calculus

Background

- Lower bounds for proofs in various systems

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The polynomial calculus

Background

- Lower bounds for proofs in various systems

- In particular for the pigeonhole principle

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- Polynomial calculus is a strong proof system

Background

- Lower bounds for proofs in various systems

- In particular for the pigeonhole principle

- Polynomial calculus is a strong proof system

- Provide a lower bound for it with the pigeonhole principle

The polynomial calculus

Definition

- Similar to sequent calculus, but lines are polynomials

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- We use multilinear polynomials $S(\mathbb{K})$
  $(xy + xz + v \equiv x^2 y + x^3 z^5 + v)$

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- Addition

$$\frac{f \qquad g}{af + bg}$$

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$$\frac{f \qquad g}{af + bg}$$

- Multiplication

$$\frac{f}{f \cdot x}$$

Refutations

- g is provable from $f_1, \ldots, f_n$ if and only if it is in the ideal generated by them

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Refutations

- g is provable from $f_1, \ldots, f_n$ if and only if it is in the ideal generated by them
- A proof of 1 exists if and only if $f_1, \ldots, f_n$ have no common zeroes
- We construct polynomials such that their zeroes correspond to satisfying assignments
- Proving 1 from them is a *refutation*

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The polynomial calculus

Example proof

- Try to prove $xy + z$ from $x + 1$ and z

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The polynomial calculus

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$$x + 1$$

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The polynomial calculus

Example proof

- Try to prove $xy + z$ from $x + 1$ and z

$$(\cdot y)\ \frac{x + 1}{xy + y}$$

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$$(\cdot y) \, \frac{\dfrac{x+1}{xy+y} \quad z}{xy+y+z} \, (+)$$

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Example proof

- Try to prove $xy + z$ from $x + 1$ and z

$$(\cdot y)\,\dfrac{x + 1}{\dfrac{xy + y}{xy + y + z}}\,z\,(+)$$

- We now want to subtract y

- There is no way to prove y from $x + 1$ and z

- Closest to $xy + z$ we can prove is $xy + y + z$

Algebraic view of proofs

- Think of polynomials not just as lines but as elements of $\mathbb{K}$-algebras

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- Think of polynomials not just as lines but as elements of $\mathbb{K}$-algebras
- V are polynomials we can prove
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- Think of polynomials not just as lines but as elements of $\mathbb{K}$-algebras
- V are polynomials we can prove $\qquad\qquad\qquad\qquad\qquad\qquad xy + y + z$
- Δ are leading terms of ones we cannot prove $\qquad\qquad\qquad\qquad\qquad y$
- $S(\mathbb{K}) \cong \mathbb{K}\Delta \oplus V$ $\qquad\qquad\qquad\qquad xy + z = -y + xy + y + z$

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- $S(\mathbb{K}) \cong \mathbb{K}\Delta \oplus V$ $xy + z = -y + xy + y + z$
- Similarly V_d, Δ_d for proofs of bounded degrees

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Algebraic view of proofs

- Think of polynomials not just as lines but as elements of $\mathbb{K}$-algebras

- V are polynomials we can prove $xy + y + z$

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- $S(\mathbb{K}) \cong \mathbb{K}\Delta \oplus V$ $xy + z = -y + xy + y + z$

- Similarly V_d, Δ_d for proofs of bounded degrees

- Investigate what V_d looks like

Overview

The pigeonhole principle

- If there are m pigeons, n pigeon holes, and $m > n$ then at least two pigeons have to share a hole

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- Formally: if $m > n$ there is no injection $[m] \hookrightarrow [n]$

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- Assignment of $x_{3,5}$ corresponds to pigeon 3 being in hole 5

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Definition ($\neg \mathcal{PHP}_n^m$)

$$Q_i := 1 - \sum_{j \in [n]} x_{ij} \qquad \text{for each } i \in [m]$$

$$Q_{i_1,i_2,j} := x_{i_1 j} x_{i_2 j} \qquad \text{for each } i_1 \neq i_2 \in [m], j \in [n]$$

Main result

Theorem

Every polynomial calculus refutation of $\neg\mathcal{PHP}_n^m$ must have degree at least $n/2 + 1$.

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> **Theorem**
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> *Every polynomial calculus refutation of $\neg\mathcal{PHP}_n^m$ must have degree at least $n/2 + 1$.*

- Goal: if $d < n/2 + 1$, then $1 \notin V_d \Leftrightarrow V_d \neq S(\mathbb{K})$

Main result

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> *Every polynomial calculus refutation of $\neg \mathcal{PHP}_n^m$ must have degree at least $n/2 + 1$.*

- Goal: if $d < n/2 + 1$, then $1 \notin V_d \Leftrightarrow V_d \neq S(\mathbb{K})$
- Characterize V_d in a way that lets us see this

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Overview

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- Goal: if $d < n/2 + 1$, then $1 \notin V_d \Leftrightarrow V_d \neq S(\mathbb{K})$
- Characterize V_d in a way that lets us see this
- Prove this characterization is correct via the pigeon dance

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Overview

$\neg \mathcal{PHP}_n^m$ constraints

- Polynomials in $\neg \mathcal{PHP}_n^m$ are constraints on assignments

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- Locally valid assignments are injections $I \hookrightarrow [n]$

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- Possible for pigeons $I \subseteq [m]$ with $|I| \leq n$

- Locally valid assignments are injections $I \hookrightarrow [n]$

- Corresponding variable assignments are M_I

Overview

Locally valid assignments

- Define V_I as the set of all polynomials with $a(f) = 0$ for all $a \in M_I$

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Locally valid assignments

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- This definition completely ignores the degrees of the proofs!

- Only works if for all terms t, $t \in \Delta_I$ for either all or no $I \supseteq \mathrm{dom}(t)$

Ideas

Overview

- Goal:
  - Use pigeon dance to characterize Δ_I
  - Prove correctness

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- Goal:
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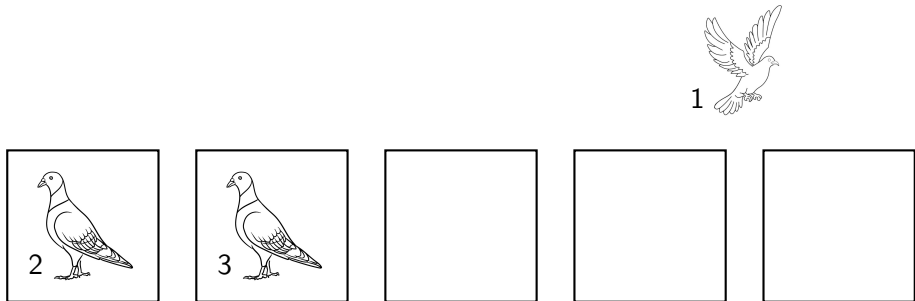- Start with pigeons sitting in their holes

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- Goal:
  - Use pigeon dance to characterize Δ_I
  - Prove correctness
- Start with pigeons sitting in their holes
- The first pigeon flies to an unoccupied hole to its right

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- Goal:
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- Start with pigeons sitting in their holes

- The first pigeon flies to an unoccupied hole to its right

- Repeat until all pigeons have moved once

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- Start with pigeons sitting in their holes

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- Repeat until all pigeons have moved once

- If a pigeon cannot find an empty hole, the dance is aborted
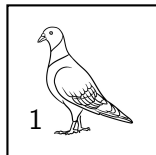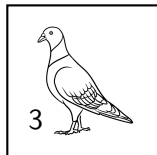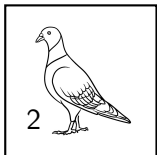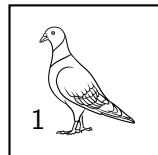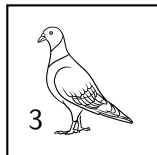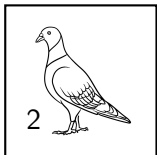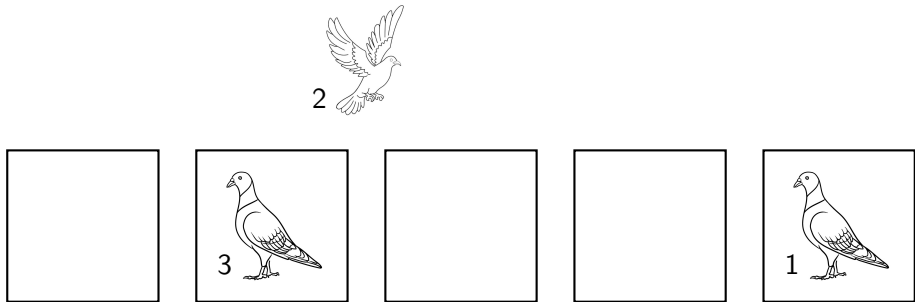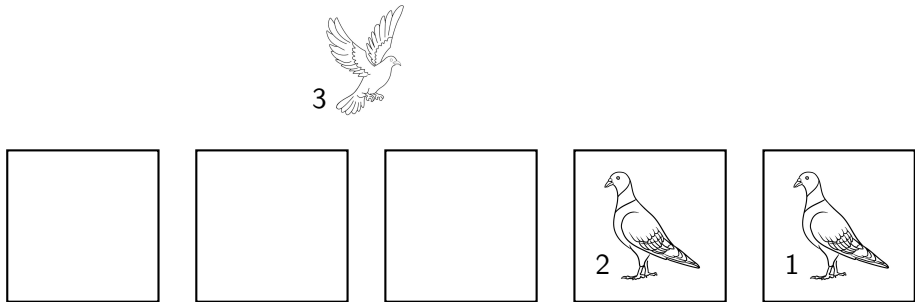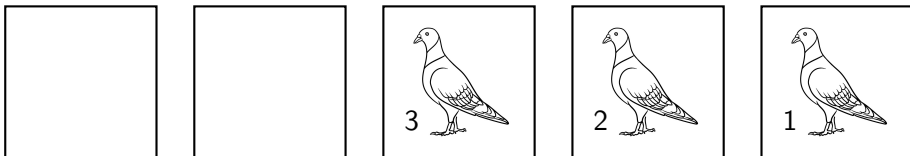
Ideas

Example

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Ideas

Formalization

- Consider partial injections $I \hookrightarrow [m]$ $\qquad (1 \mapsto 4, \; 2 \mapsto 1, \; 3 \mapsto 2)$

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- Encode pigeon positions as terms $\quad (x_{1,4} \, x_{2,1} \, x_{3,2})$

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- Consider partial injections $I \hookrightarrow [m]$ $\quad (1 \mapsto 4, \ 2 \mapsto 1, \ 3 \mapsto 2)$
- Encode pigeon positions as terms $\quad (x_{1,4} \, x_{2,1} \, x_{3,2})$
- Δ_I is the set of terms that let pigeons complete the dance

Ideas

Formalization

- Consider partial injections $I \hookrightarrow [m]$ $(1 \mapsto 4, \ 2 \mapsto 1, \ 3 \mapsto 2)$

- Encode pigeon positions as terms $(x_{1,4} \, x_{2,1} \, x_{3,2})$

- Δ_I is the set of terms that let pigeons complete the dance

- Membership in Δ_I is independent of I since pigeons not in the dance do not affect it

The Kill operator

- We need to further understand the dance to prove it correct

Properties

The Kill operator

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- Idea: a way to block specific pigeon holes

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The Kill operator

- We need to further understand the dance to prove it correct

- Idea: a way to block specific pigeon holes

- Kill the first pigeon and moves its hole to the left

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Properties

The Kill operator

- We need to further understand the dance to prove it correct

- Idea: a way to block specific pigeon holes

- Kill the first pigeon and moves its hole to the left

- $\mathrm{Kill}(x_{i_1 j_1} \cdots x_{i_d, j_d}) = x_{i_2 j_2'} \cdots x_{i_d j_d'}$ with

$$j_k' \coloneqq \begin{cases} j_k + 1, & \text{if } j_k < j_1 \\ j_k, & \text{if } j_k > j_1. \end{cases}$$

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Properties

Simulating the pigeon dance with Kill

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Properties

Simulating the pigeon dance with Kill

Theorem

$x_{i_1 j_1} \cdots x_{i_d j_d} \in \Delta_I$ *if and only if there is a* $j' > j_1$ *such that* $\mathrm{Kill}(x_{i_1 j'} \cdots x_{i_d j_d}) \in \Delta_I$.

Properties

Simulating the pigeon dance with Kill

Theorem

$x_{i_1 j_1} \cdots x_{i_d j_d} \in \Delta_I$ *if and only if there is a* $j' > j_1$ *such that* $\mathrm{Kill}(x_{i_1 j'} \cdots x_{i_d j_d}) \in \Delta_I$.

Proof sketch

$\mathrm{Kill}(x_{i_1 j'} \cdots x_{i_d j_d})$ effectively moves the first pigeon to an empty hole to its right and then kills it. This is the same as each step in the dance, where the first pigeon flies to some free hole to its right and then occupies it. □

Properties

Closure of Δ_I

Theorem

Δ_I *is closed under* Kill.

Properties

Closure of Δ_I

Theorem

Δ_I *is closed under* Kill.

Proof sketch

If $t \in \Delta_I$ then the pigeons can complete their dance. During this the first pigeon will start at j and fly to j'. Killing the pigeon frees up j' so any other pigeon that wanted to use j can use it instead. $\qquad\square$

Properties

The lower bound

Theorem

If $|I| \leq (n+1)/2$ and $t \in \Delta_I$, then there is a hole j such that $\mathrm{Kill}(x_{ij}t) \in \Delta_I$.

The lower bound

Theorem

If $|I| \leq (n+1)/2$ and $t \in \Delta_I$, then there is a hole j such that $\mathrm{Kill}(x_{ij}t) \in \Delta_I$.

Proof sketch

There are at most

$$| \operatorname{dom}(t)| \leq |I \setminus \{i\}| \leq \frac{n-1}{2}$$

pigeons involved in the dance, each occupying two holes. Place pigeon at unused hole
and kill it there. The remaining pigeons can complete the dance since the moved hole
was not used. □

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Putting things together

- Goal: show pigeon dance correctly characterizes $S(\mathbb{K})/V_I$

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- Goal: show pigeon dance correctly characterizes $S(\mathbb{K})/V_I$
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- Remove variables x_{ij} for minimal $i \in I$ from f
- Inductive assumption gives us $a' \in M_{I\setminus\{i\}}$ with $a'(f') \neq 0$ closure of Δ_I

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- Remove variables x_{ij} for minimal $i \in I$ from f
- Inductive assumption gives us $a' \in M_{I\setminus\{i\}}$ with $a'(f') \neq 0$ closure of Δ_I
- Pick a j such that $\mathrm{Kill}(x_{ij}t) \in \Delta_I$ lower bound

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- Remove variables x_{ij} for minimal $i \in I$ from f
- Inductive assumption gives us $a' \in M_{I \setminus \{i\}}$ with $a'(f') \neq 0$ closure of Δ_I
- Pick a j such that $\mathrm{Kill}(x_{ij}t) \in \Delta_I$ lower bound
- Extend assignment to I with $a(f) \neq 0$

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Summary

- Proven a lower bound $d \geq n/2 + 1$ for polynomial calculus refutations of $\neg\mathcal{PHP}_n^m$

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- Characterize derivable polynomials V_d through locally consistent assignments M_I

Summary

- Proven a lower bound $d \geq n/2 + 1$ for polynomial calculus refutations of $\neg\mathcal{PHP}_n^m$
- Characterize derivable polynomials V_d through locally consistent assignments M_I
- This only works if membership in Δ_I is independent of I

Summary

- Proven a lower bound $d \geq n/2 + 1$ for polynomial calculus refutations of $\neg\mathcal{PHP}_n^m$
- Characterize derivable polynomials V_d through locally consistent assignments M_I
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- Characterize Δ_I through pigeon dance

Introduction
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The Pigeonhole Principle
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The Pigeon Dance
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Conclusion
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- Correct since Δ_I is linearly independent over M_I