Lower Bounds for the Polynomial Calculus via the "Pigeon Dance"

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Overview

We will present the result from A.A. Razborov, "Lower Bounds for the Polynomial Calculus", in: Computational Complexity 7.4 (Dec. 2, 1998).

1. Introduction
2. The Pigeonhole Principle
3. The Pigeon Dance
4. Conclusion

Background

- Lower bounds for proofs in various systems

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- Polynomial calculus is a strong proof system
- Provide a lower bound for it with the pigeonhole principle

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- Multiplication

$$\frac{f}{f \cdot x}$$

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- A proof of 1 exists if and only if $f_1, \ldots, f_n$ have no common zeroes
- We construct polynomials such that their zeroes correspond to satisfying assignments
- Proving 1 from them is a *refutation*

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- Closest to $xy + z$ we can prove is $xy + y + z$

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- Similarly V_I, Δ_I using subset of variables I

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Definition ($\neg\mathcal{PHP}^m_n$)

$$Q_i := 1 - \sum_{j \in [n]} x_{ij} \qquad \text{for each } i \in [m]$$

$$Q_{i_1,i_2,j} := x_{i_1 j} x_{i_2 j} \qquad \text{for each } i_1 \neq i_2 \in [m], j \in [n]$$

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Every polynomial calculus refutation of $\neg\mathcal{PHP}_n^m$ *must have degree at least* $n/2+1$.

- $\neg\mathcal{PHP}_n^m$ are constraints on pigeon assignments
- Proofs combine them into more complex ones
- Can only derive local constraints with small degrees
- Pigeons can fly away to escape local contradictions

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- Locally valid assignments are injections $I \hookrightarrow [n]$
- Corresponding variable assignments are M_I

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- Only works if whether $t \in \Delta_I$ is independent of $I \supseteq \mathrm{dom}(t)$

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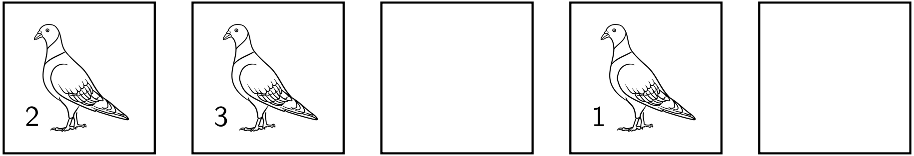
- Repeat until all pigeons have moved once
- If a pigeon cannot find an empty hole, the dance is aborted

Example



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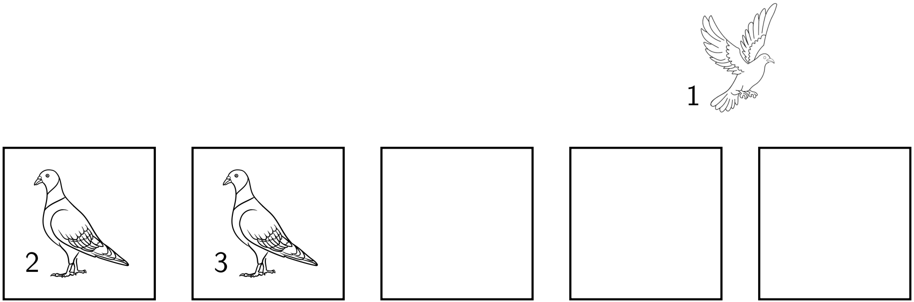

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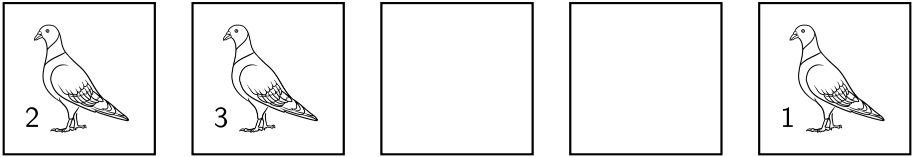

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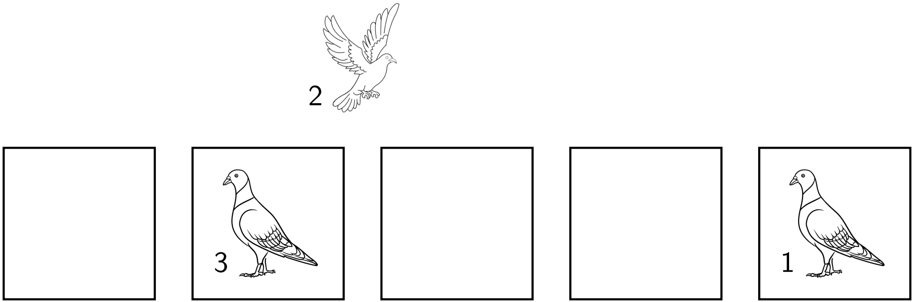

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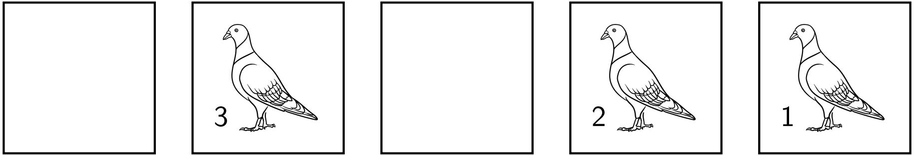

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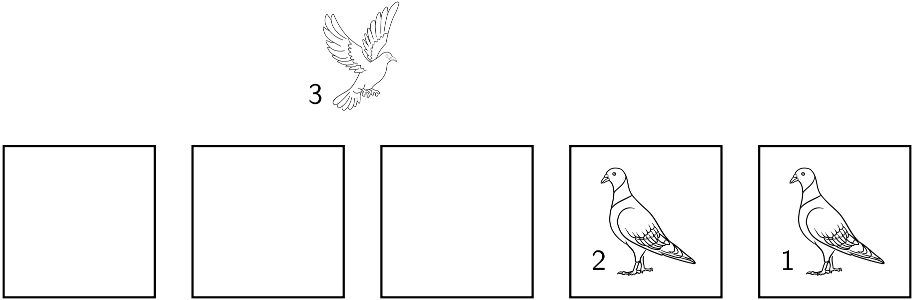

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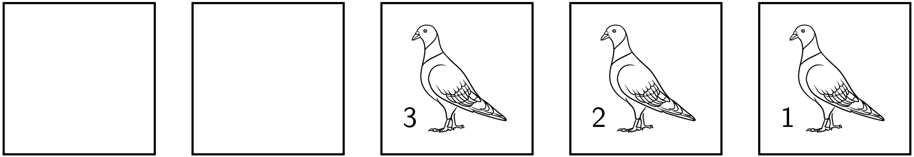

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- Encode pigeon positions as terms $\qquad (x_{1,4}\, x_{2,1}\, x_{3,2})$
- Δ_I is the set of terms that let pigeons complete the dance
- Whether $t \in \Delta_I$ is independent of I since pigeons not in the dance do not affect it

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- Idea: a way to block specific pigeon holes
- Kill the first pigeon and moves its hole to the left
- $\mathrm{Kill}(x_{i_1 j_1} \cdots x_{i_d, j_d}) = x_{i_2 j_2'} \cdots x_{i_d j_d'}$ with

$$j_k' := \begin{cases} j_k + 1, & \text{if } j_k < j_1 \\ j_k, & \text{if } j_k > j_1. \end{cases}$$

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Theorem

$x_{i_1 j_1} \cdots x_{i_d j_d} \in \Delta_I$ *if and only if there is a* $j' > j_1$ *such that* $\mathrm{Kill}(x_{i_1 j'} \cdots x_{i_d j_d}) \in \Delta_I$.

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Proof sketch

$\mathrm{Kill}(x_{i_1 j'} \cdots x_{i_d j_d})$ effectively moves the first pigeon to an empty hole to its right and then kills it. This is the same as each step in the dance, where the first pigeon flies to some free hole to its right and then occupies it. $\square$

Closure of Δ_I

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Proof sketch

If $t \in \Delta_I$ then the pigeons can complete their dance. During this the first pigeon will start at j and fly to j'. Killing the pigeon frees up j' so any other pigeon that wanted to use j can use it instead. $\square$

The lower bound

Theorem

If $|I| \leq (n+1)/2$ *and pigeons can complete the dance, then we can introduce a new pigeon such that* $\mathrm{Kill}(x_{ij}t) \in \Delta_I$.

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At most

$$|\operatorname{dom}(t)| \leq |I \setminus \{i\}| \leq \frac{n-1}{2}$$

pigeons involved in the dance, each occupying two holes. Place pigeon at unused hole and kill it there. The remaining pigeons can complete the dance since the moved hole was not used. □

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- Extend assignment to I with $a(f) \neq 0$

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- Characterize Δ_I through pigeon dance
- This works since Δ_I is linearly independent over M_I