Lower Bounds for the Polynomial Calculus via the "Pigeon Dance"

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January 2023

Overview

We will present the result from A.A. Razborov, “Lower Bounds for the Polynomial Calculus”, in: Computational Complexity 7.4 (Dec. 2, 1998).

1. Introduction
2. The Pigeonhole Principle
3. The Pigeon Dance
4. Conclusion

Background

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- In particular for the pigeonhole principle
- Polynomial calculus is a strong proof system
- Provide a lower bound for it with the pigeonhole principle

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- Multiplication

$$\frac{f}{f \cdot x}$$

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- We construct polynomials such that their zeroes correspond to satisfying assignments
- Proving 1 from them is a *refutation*

Example proof

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$$x + 1$$

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- Closest to $xy + z$ we can prove is $xy + y + z$

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- Similarly V_d, Δ_d for proofs of bounded degrees
- Investigate what V_d looks like

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Definition ($\neg\mathcal{PHP}_n^m$)

$$Q_i := 1 - \sum_{j \in [n]} x_{ij} \qquad \text{for each } i \in [m]$$

$$Q_{i_1,i_2,j} := x_{i_1 j} x_{i_2 j} \qquad \text{for each } i_1 \neq i_2 \in [m], j \in [n]$$

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- Goal: if $d < n/2 + 1$, then $1 \notin V_d \Leftrightarrow V_d \neq S(\mathbb{K})$
- Characterize V_d in a way that lets us see this
- Prove this characterization is correct via the pigeon dance

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- Corresponding variable assignments are M_I

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- This definition completely ignores the degrees of the proofs!
- Only works if for all terms t, $t \in \Delta_I$ for either all or no $I \supseteq \mathrm{dom}(t)$

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- Start with pigeons sitting in their holes
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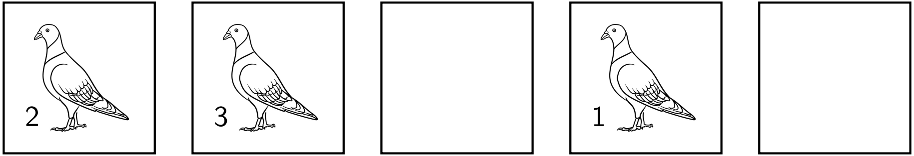
- Repeat until all pigeons have moved once
- If a pigeon cannot find an empty hole, the dance is aborted

Example



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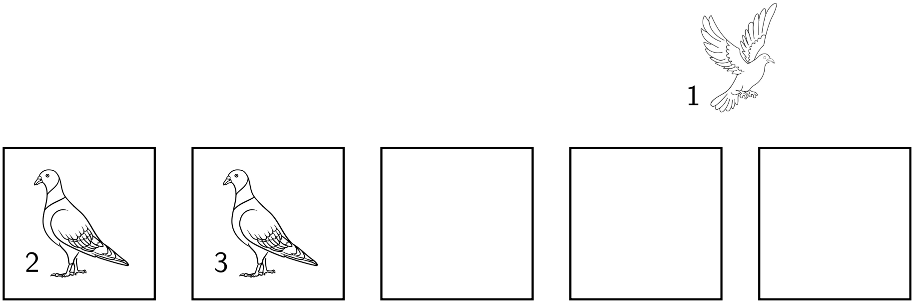

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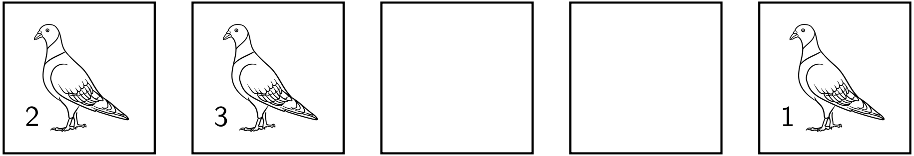

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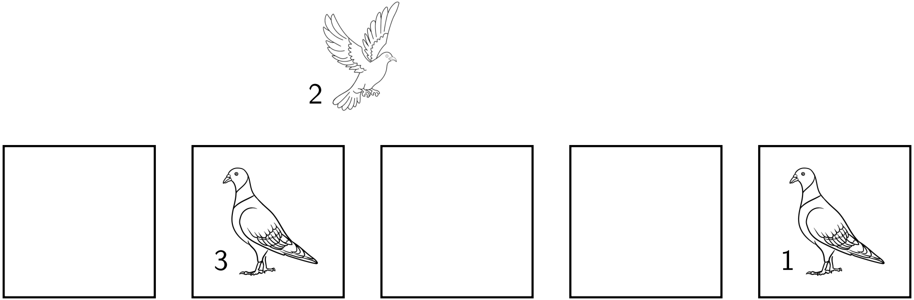

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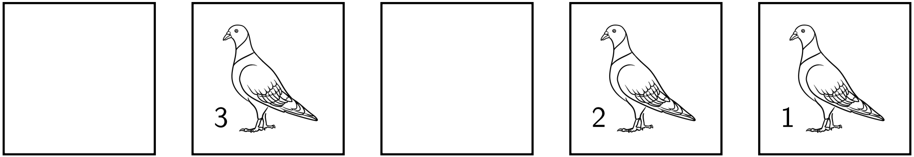

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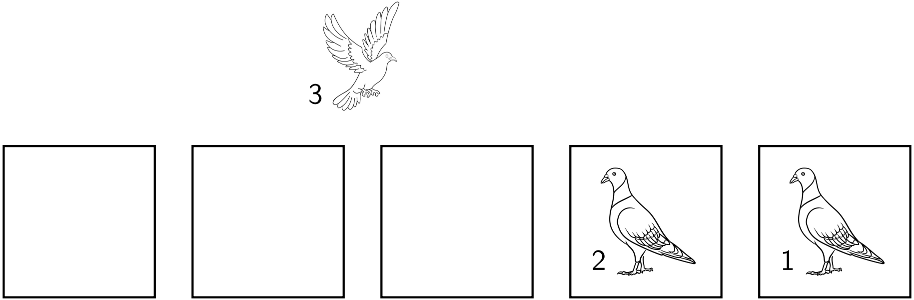

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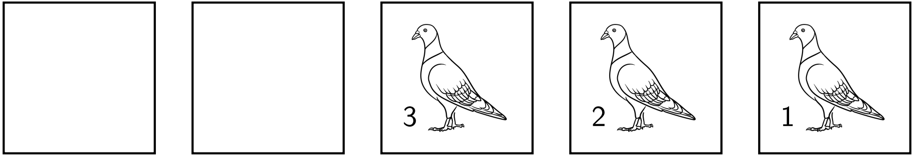

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- Encode pigeon positions as terms $\qquad (x_{1,4}\, x_{2,1}\, x_{3,2})$
- Δ_I is the set of terms that let pigeons complete the dance
- Membership in Δ_I is independent of I since pigeons not in the dance do not affect it

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- Idea: a way to block specific pigeon holes
- Kill the first pigeon and moves its hole to the left
- $\mathrm{Kill}(x_{i_1 j_1} \cdots x_{i_d, j_d}) = x_{i_2 j'_2} \cdots x_{i_d j'_d}$ with

$$j'_k := \begin{cases} j_k + 1, & \text{if } j_k < j_1 \\ j_k, & \text{if } j_k > j_1. \end{cases}$$

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Theorem

$x_{i_1 j_1} \cdots x_{i_d j_d} \in \Delta_I$ *if and only if there is a* $j' > j_1$ *such that* $\mathrm{Kill}(x_{i_1 j'} \cdots x_{i_d j_d}) \in \Delta_I$.

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Proof sketch

$\mathrm{Kill}(x_{i_1 j'} \cdots x_{i_d j_d})$ effectively moves the first pigeon to an empty hole to its right and then kills it. This is the same as each step in the dance, where the first pigeon flies to some free hole to its right and then occupies it. $\square$

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Proof sketch

If $t \in \Delta_I$ then the pigeons can complete their dance. During this the first pigeon will start at j and fly to j'. Killing the pigeon frees up j' so any other pigeon that wanted to use j can use it instead. $\square$

The lower bound

Theorem

If $|I| \leq (n+1)/2$ *and* $t \in \Delta_I$*, then there is a hole* j *such that* $\mathrm{Kill}(x_{ij}t) \in \Delta_I$*.*

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There are at most

$$|\operatorname{dom}(t)| \leq |I \setminus \{i\}| \leq \frac{n-1}{2}$$

pigeons involved in the dance, each occupying two holes. Place pigeon at unused hole and kill it there. The remaining pigeons can complete the dance since the moved hole was not used. $\square$

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- Extend assignment to I with $a(f) \neq 0$

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- Correct since Δ_I is linearly independent over M_I