

# ICDSS Lecture 4: Neural Networks

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We have free licenses for our members through our partnership!  
Requires filling in a form to assess financial need (5-10 minutes)  
Form will be available on Wednesday.

Not a mathematically rigorous lecture. We assume you know:

- Matrix Multiplication
- Chain Rule
- Linear and Logistic Regression

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix}$$

$$\frac{dL}{dW_1} = \frac{dL}{da_2} \frac{da_2}{dz_2} \frac{dz_2}{da_1} \frac{da_1}{dz_1} \frac{dz_1}{dW_1}$$

1

Linear  
Regression as a  
NN

2

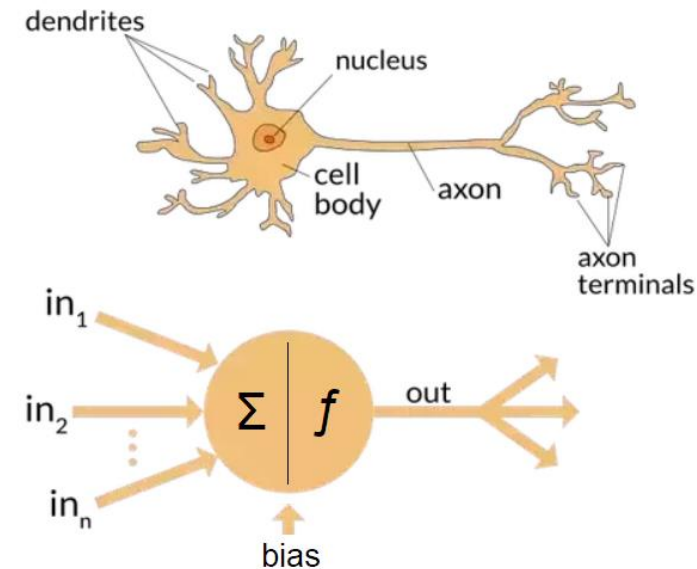
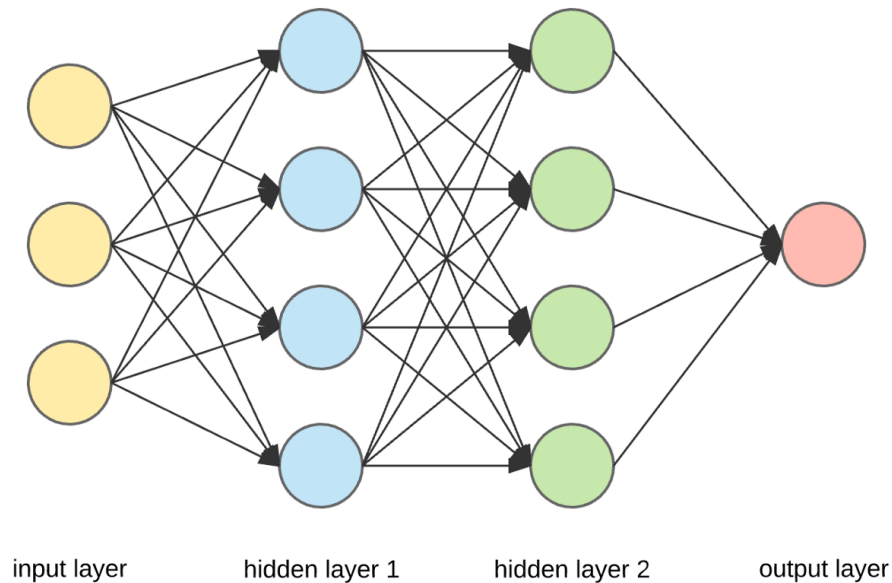
Logistic  
Regression as a  
NN

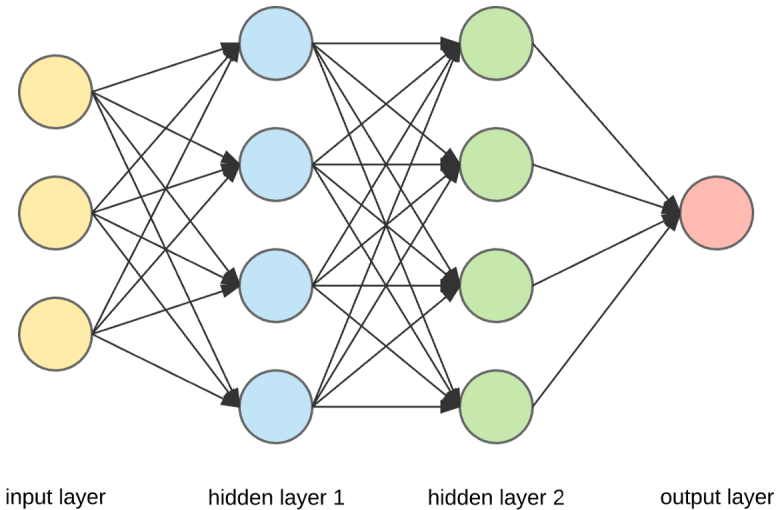
3

General NNs

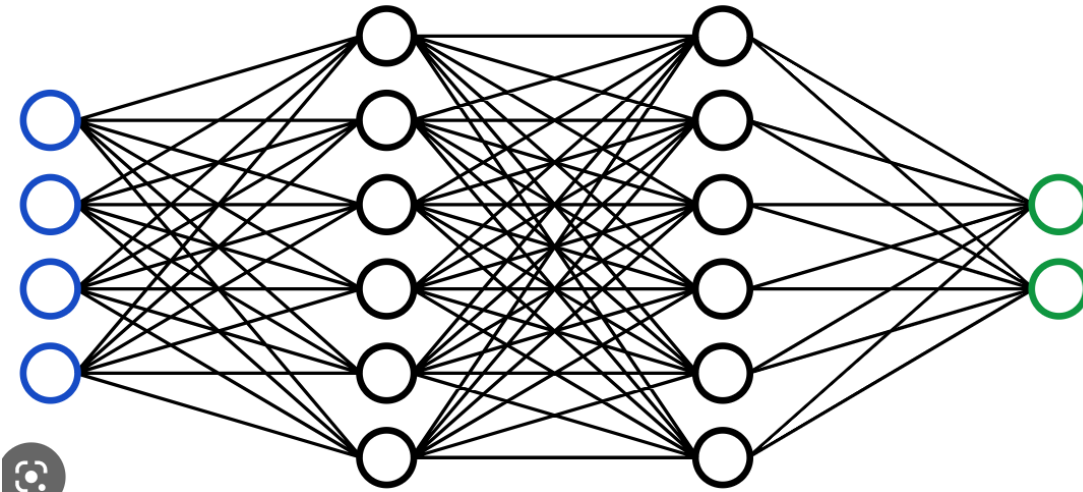
# Why are they called Neural Networks?

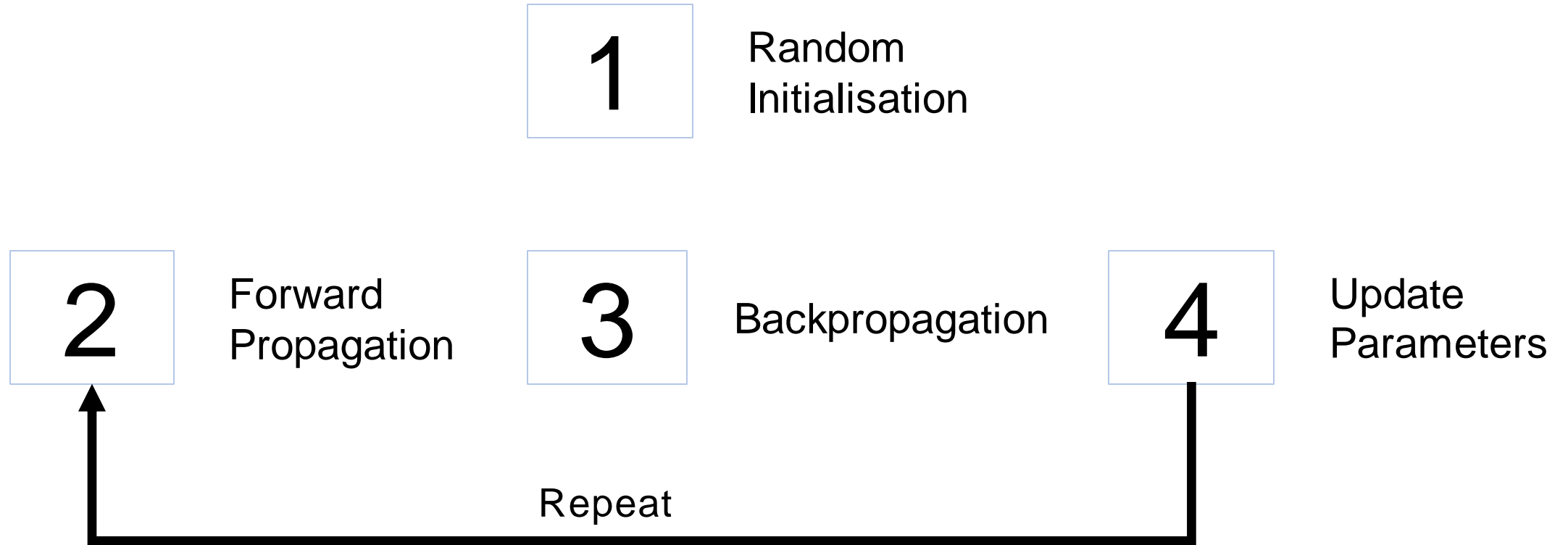
- Mimics the human brain (not exactly!)
- Output signals from other neurons as the input
- Each signal has a different “weighting”
- If signal is above some threshold value, it is “activated.”





- **Input, Label Matrix:**  $X, Y$
- **Prediction Matrix:**  $\hat{y}$
- **Number Of Layers:** All layers excluding input
- **Input Layer:** Layer 0
- **Number of examples:**  $m$
- **Input, Label Size:**  $n_x, n_y$





# Linear Regression (Single Feature)

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# Linear Regression (Single Feature):

Consider a single feature linear regression problem:

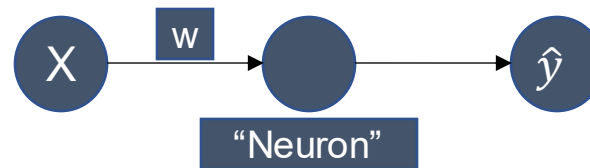
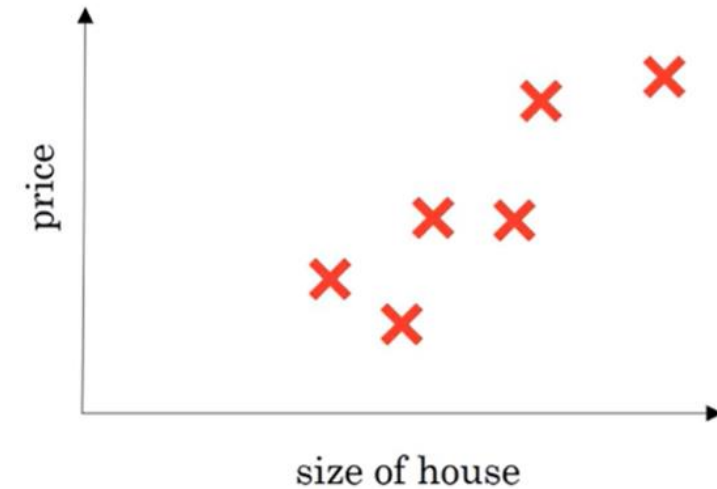
- Predicting House Price based on Total Size

Linear Regression attempts to fit data into the form:

$$y = wx + b$$

**w: Weighting, b: Bias**

This can be visualised as a “Neural Network”



# Linear Regression (Single Feature):

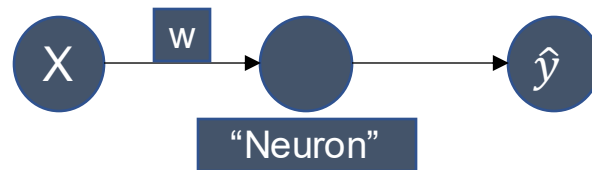
First, the "network" will randomly guess  $w$  and  $b$ .

Then, the model will calculate:

$$Z = wx + b$$

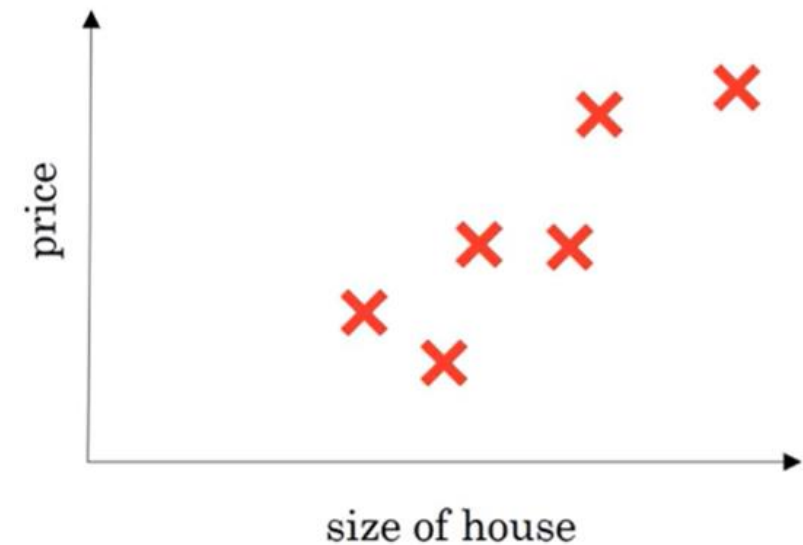
This value acts as the input of the Neuron

- **Forward Propagation!**



In Linear Regression, there is no special “**activation function**.”

$$\hat{y} = Z = wx + b$$



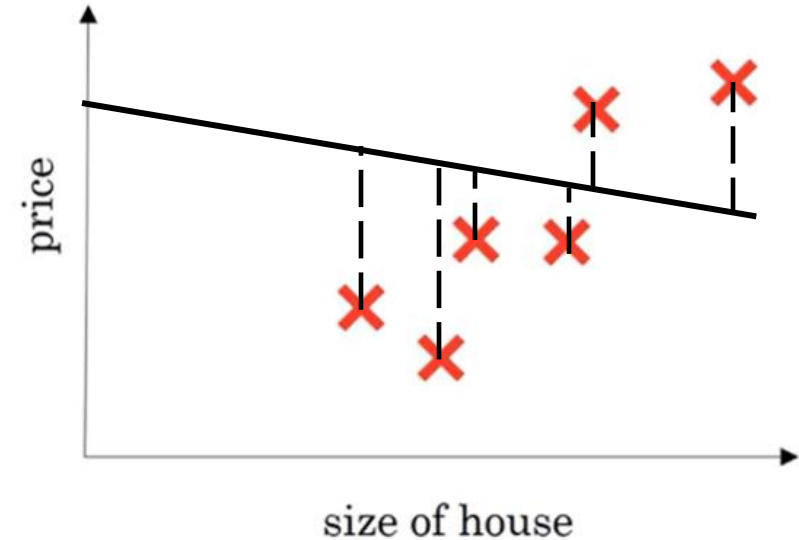
The “network” just guessed values, predictions can’t be accurate

- A loss function (performance measure) needs to be defined and minimised
- Use Mean Squared Error (MSE)

$$MSE = \frac{1}{m} \sum_{i=1}^m (y_i - \hat{y}_i)^2$$

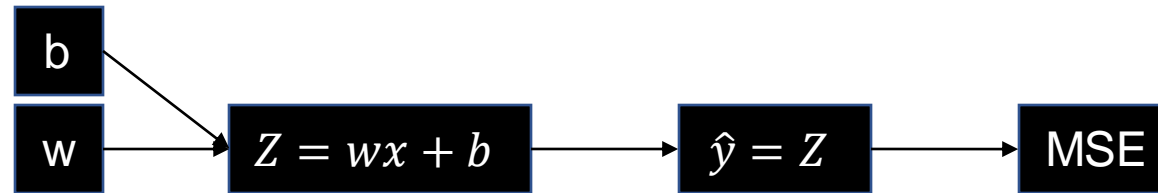
How do we change the parameters (such as  $w$ ) so that we minimize the MSE?

- Update the value of the parameter using its gradient (**gradient descent**)



How do we change the parameters (such as  $w$ ) of the network to minimize the MSE?

- Examine how MSE is calculated in terms of a **computation diagram**



Consider derivatives of MSE with respect to  $w$  and  $b$

- Use chain rule to get

$$\frac{d(MSE)}{dw} = \frac{d(MSE)}{d\hat{y}} \frac{d\hat{y}}{dZ} \frac{dZ}{dw}$$

$$\frac{d(MSE)}{db} = \frac{d(MSE)}{d\hat{y}} \frac{d\hat{y}}{dZ} \frac{dZ}{db}$$

See that we are calculating derivatives from the output layer to the input layer (backwards)

**Backpropagation!**

# Linear Regression (Single Feature):

Consider derivatives of MSE with respect to  $w$  and  $b$

- Use chain rule to get:

$$\frac{d(MSE)}{dw} = \frac{d(MSE)}{d\hat{y}} \frac{d\hat{y}}{dZ} \frac{dZ}{dw}$$

$$\frac{d(MSE)}{db} = \frac{d(MSE)}{d\hat{y}} \frac{d\hat{y}}{dZ} \frac{dZ}{db}$$

$$\frac{d(MSE)}{d\hat{y}} = \frac{d}{d\hat{y}} \left( \frac{1}{m} \sum_{i=1}^m (y_i - \hat{y}_i)^2 \right) \longrightarrow \frac{d(MSE)}{d\hat{y}} = \frac{1}{m} \sum_{i=1}^m \left( \frac{d}{d\hat{y}} (y_i - \hat{y}_i)^2 \right)$$

$$\frac{d(MSE)}{d\hat{y}} = -\frac{2}{m} \sum_{i=1}^m (y_i - \hat{y}_i)$$

Consider derivatives of MSE with respect to  $w$  and  $b$

$$Z = wx + b$$

$$\frac{d(MSE)}{d\hat{y}} = -\frac{2}{m} \sum_{i=1}^m (y_i - \hat{y}_i) \qquad \frac{dZ}{d\hat{y}} = 1; \frac{d\hat{y}}{dw} = x; \frac{d\hat{y}}{db} = 1$$

$$\frac{d(MSE)}{dw} = \frac{d(MSE)}{d\hat{y}} \frac{d\hat{y}}{dZ} \frac{dZ}{dw} = -\frac{2}{m} x \sum_{i=1}^m (y_i - \hat{y}_i)$$

$$\frac{d(MSE)}{db} = \frac{d(MSE)}{d\hat{y}} \frac{d\hat{y}}{dZ} \frac{dZ}{db} = -\frac{2}{m} \sum_{i=1}^m (y_i - \hat{y}_i)$$

The gradients can be used to **update parameters** (also known as “learning”)

$$\frac{d(MSE)}{dw} = \frac{d(MSE)}{d\hat{y}} \frac{d\hat{y}}{dZ} \frac{dZ}{dw}$$

$$\frac{d(MSE)}{db} = \frac{d(MSE)}{d\hat{y}} \frac{d\hat{y}}{dZ} \frac{dZ}{db}$$

Update as following:

$$w \rightarrow w - \alpha \frac{d(MSE)}{dw}$$

$$b \rightarrow b - \alpha \frac{d(MSE)}{db}$$

$\alpha$  Is the **learning rate**

- It is a **hyperparameter**: it needs to be specified when the network is created

Also, see that the negative value of the gradient is taken

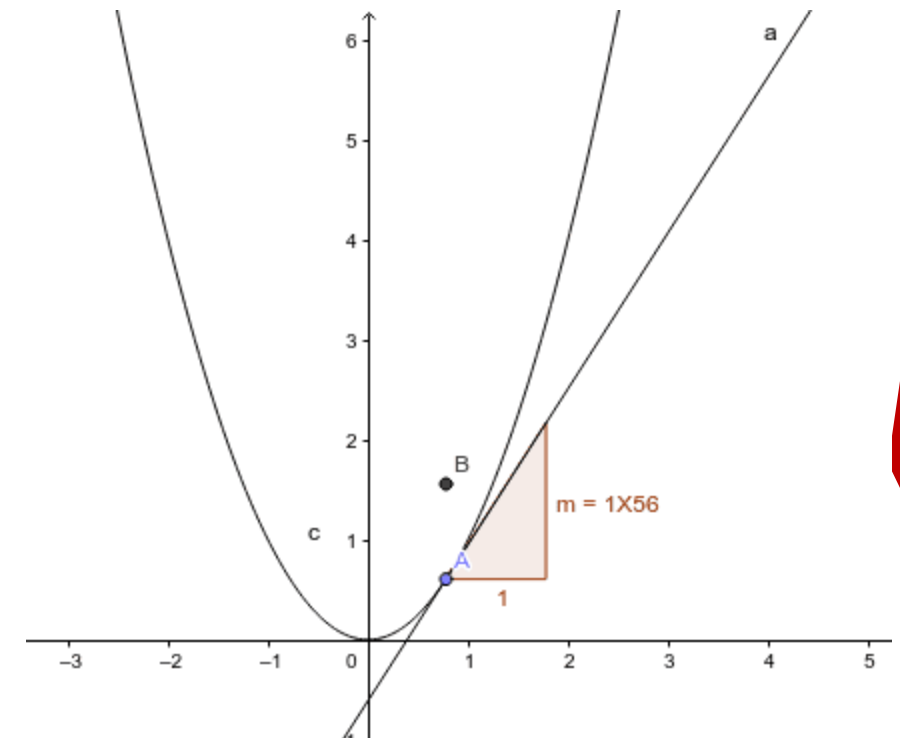
Why are the negative values of gradients used?

$$w \rightarrow w - \alpha \frac{d(MSE)}{dw}$$

$$b \rightarrow b - \alpha \frac{d(MSE)}{db}$$

- The gradient points towards the direction with the steepest increase
- To get direction of steepest decrease, the negative value should be taken

Forward, Backpropagation and Updating of parameters will repeat for a set number of “**epochs**”

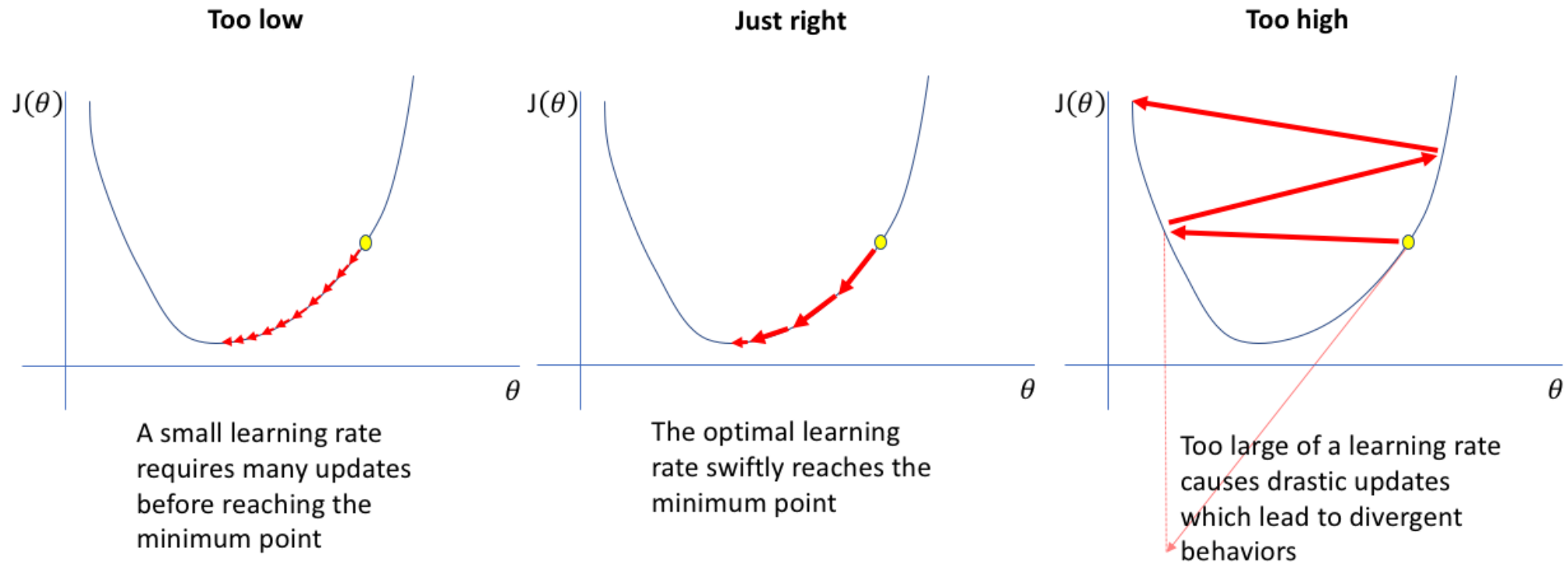




Finding the right value for the learning rate can be an iterative process

$$w \rightarrow w - \alpha \frac{d(MSE)}{dw}$$

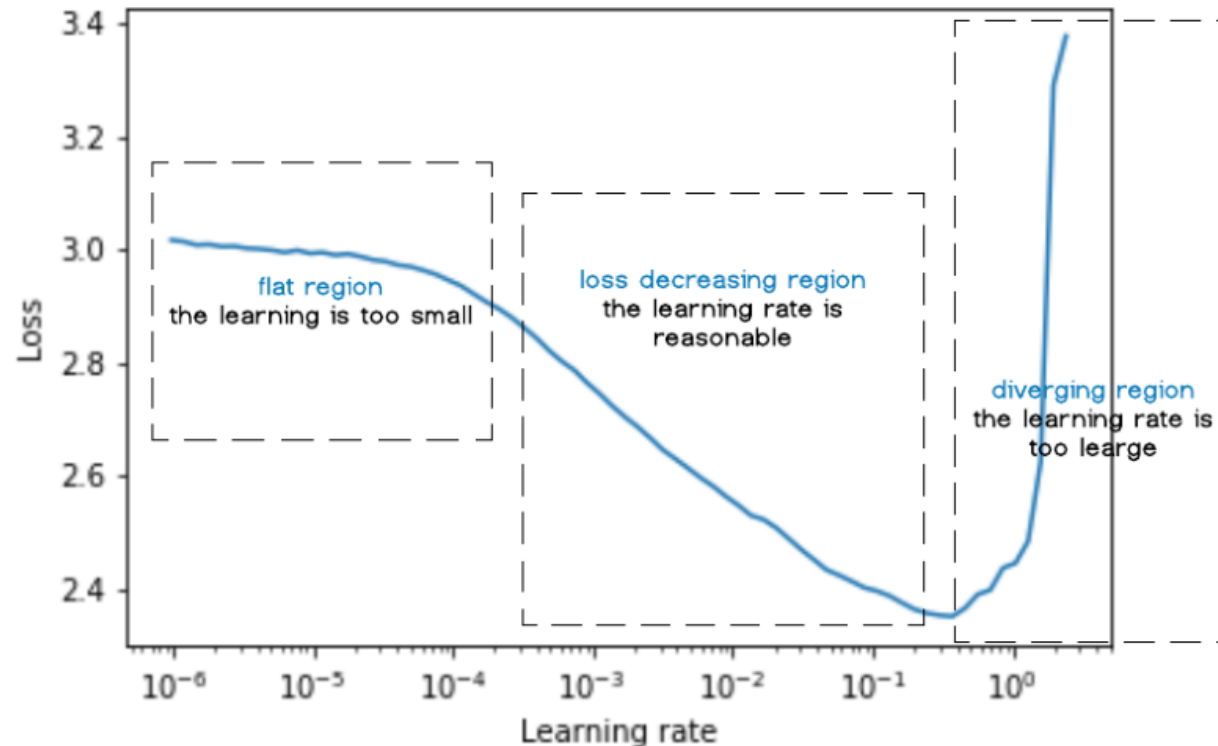
$$b \rightarrow b - \alpha \frac{d(MSE)}{db}$$



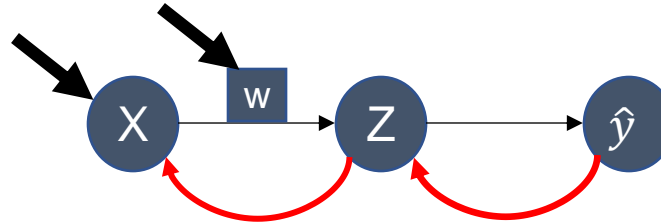
Starting from very small values, evaluate the loss of the model using a validation set

$$w \rightarrow w - \alpha \frac{d(MSE)}{dw}$$

$$b \rightarrow b - \alpha \frac{d(MSE)}{db}$$



In Summary:



1. **Random Initialisation:** Neural Network assigns random values to  $W$  and  $b$ .

2. **Forward Propagation:** The Network Calculates:

$$Z = wx + b ; \hat{y} = Z$$

3. **Backpropagation:** The network calculates derivatives to minimise the loss function

$$\frac{d(MSE)}{dw} = \frac{d(MSE)}{d\hat{y}} \frac{d\hat{y}}{dZ} \frac{dZ}{dw}$$

$$\frac{d(MSE)}{db} = \frac{d(MSE)}{d\hat{y}} \frac{d\hat{y}}{dZ} \frac{dZ}{db}$$

4. **Update Parameters:** Using the gradients, parameters ( $w, b$ ) are updated

$$w \rightarrow w - \alpha \frac{d(MSE)}{dw}$$

$$b \rightarrow b - \alpha \frac{d(MSE)}{db}$$

5. **Repeat 2-4 for a set number of “Epochs”**

# Linear Regression (Multiple Features)

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What if there are multiple features?

- Predicting House Price based on House Size ( $x_1$ ), Number of Toilets ( $x_2$ ), ..., ( $x_n$ )

Linear Regression attempts to fit data into the form:

$$y = w_1x_1 + w_2x_2 + w_3x_3 + \cdots + w_nx_n + \mathbf{b}$$

Let  $W$  be a vector containing all weightings and  $X$  be a vector containing all features:

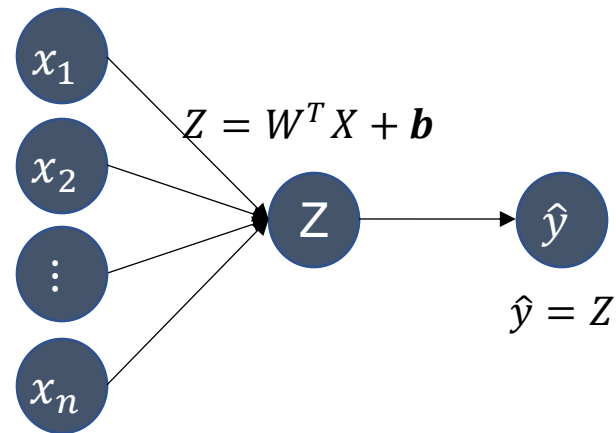
$$W = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} ; X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

In terms of Matrix Multiplication:

$$Y = W^T X + \mathbf{b}$$

# Linear Regression Example:

This can be illustrated as a neural network:



Forward Propagation:

$$w_1 x_1 + w_2 x_2 + w_3 x_3 + \dots + w_n x_n + b$$

Just like before, the network will first assign random values to  $w_1, w_2, \dots, w_n, b$

# Linear Regression Example:

$w_1, w_2, \dots, w_n, b$  need to be modified

$$\frac{d(MSE)}{dw_1}; \frac{d(MSE)}{dw_2}; \dots; \frac{d(MSE)}{dw_n}; \frac{d(MSE)}{db}$$

Follow similar workings to single variable linear regression:

$$\frac{d(MSE)}{dw_i} = \frac{d(MSE)}{d\hat{y}} \frac{d\hat{y}}{dZ} \frac{dZ}{dw_i}$$

$$\frac{d(MSE)}{dw_i} = \frac{d(MSE)}{d\hat{y}} \frac{d\hat{y}}{dZ} \frac{dZ}{dw_i} = -\frac{2}{m} x_i \sum_{k=1}^m (y_k - \hat{y}_k)$$

$$\frac{d(MSE)}{db} = \frac{d(MSE)}{d\hat{y}} \frac{d\hat{y}}{dZ} \frac{dZ}{db} = -\frac{2}{m} \sum_{k=1}^m (y_k - \hat{y}_k)$$

# Linear Regression Example:

With the gradients and the learning rate, update the parameters

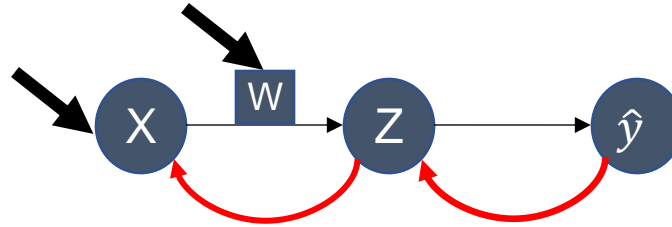
$$w_1 \rightarrow w_1 - \alpha \frac{d(MSE)}{dw_1} \quad w_2 \rightarrow w_2 - \alpha \frac{d(MSE)}{dw_2} \quad w_n \rightarrow w_n - \alpha \frac{d(MSE)}{dw_n} \quad b \rightarrow b - \alpha \frac{d(MSE)}{db}$$

In terms of vectors:

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \rightarrow \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} - \alpha \begin{pmatrix} \frac{d(MSE)}{dw_1} \\ \frac{d(MSE)}{dw_2} \\ \vdots \\ \frac{d(MSE)}{dw_n} \end{pmatrix} \quad \text{if } dW = \begin{pmatrix} \frac{d(MSE)}{dw_1} \\ \frac{d(MSE)}{dw_2} \\ \vdots \\ \frac{d(MSE)}{dw_n} \end{pmatrix} \equiv \nabla MSE$$
$$W \rightarrow W - \alpha dW$$



In Summary:



1. **Random Initialisation:** Neural Network assigns random values to  $W$  and  $b$ .

2. **Foreward Propagation:** The Network Calculates:

$$Z = W^T X + b ; \hat{y} = Z$$

3. **Backpropagation:** The network calculates derivatives to minimise the loss function

$$\frac{d(MSE)}{dw_i} = \frac{d(MSE)}{d\hat{y}} \frac{d\hat{y}}{dZ} \frac{dZ}{dw_i}$$

$$\frac{d(MSE)}{db} = \frac{d(MSE)}{d\hat{y}} \frac{d\hat{y}}{dZ} \frac{dZ}{db}$$

4. **Update Parameters:** Using the gradients, parameters ( $w, b$ ) are updated

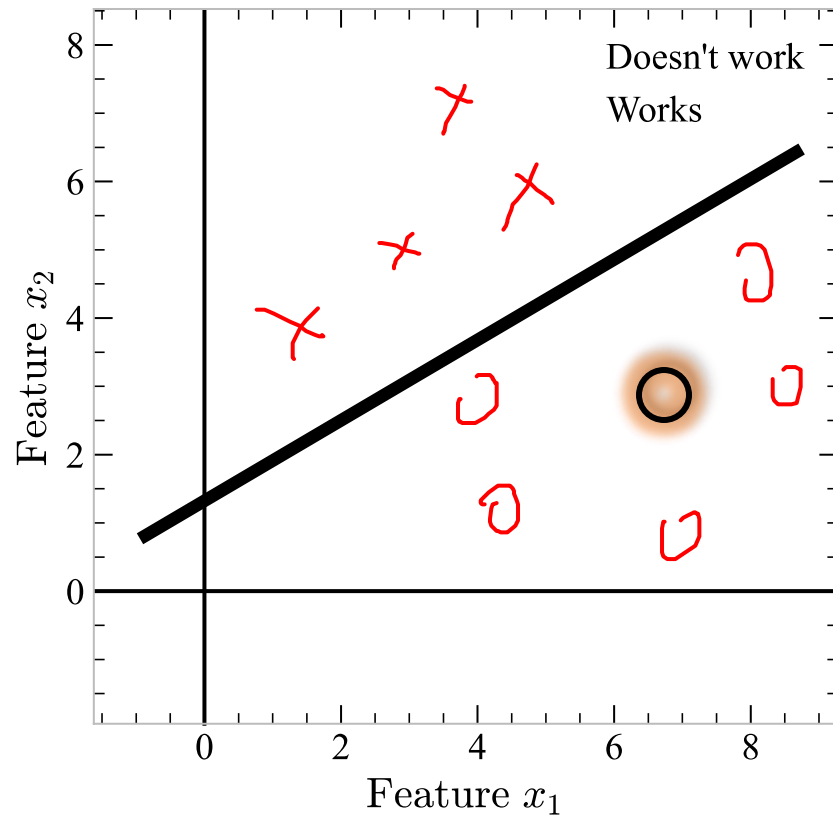
$$W \rightarrow W - \alpha dW$$

$$b \rightarrow b - \alpha \frac{d(MSE)}{db}$$

5. **Repeat 2-4 for a set number of “Epochs”**

# Logistic Regression

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We want to predict the probability that the engine will work.

 Will this engine work?

Do this using **Logistic Regression**

# Linear regression or Logistic Regression?

We want to predict a probability that is either 0 = 0% or 1 = 100%.

Linear regression gave us a value that **is not between 0 and 1**

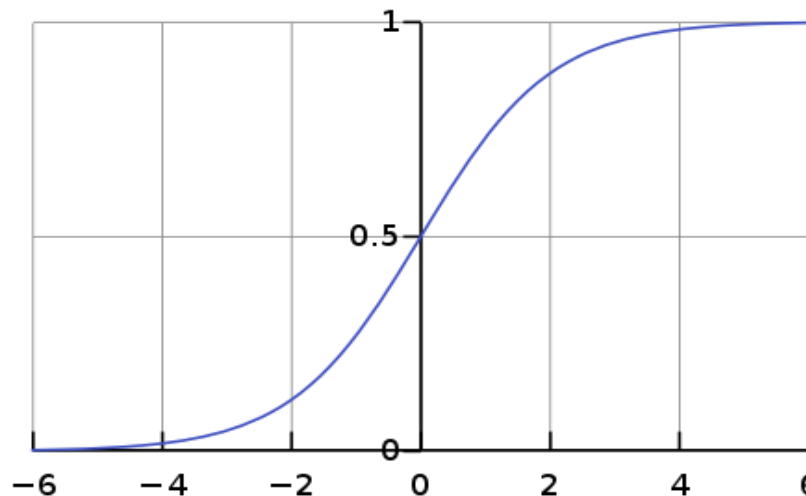
$$\hat{y} = w_1x_1 + w_2x_2 + w_3x_3 + \dots + w_nx_n + \mathbf{b} = \mathbf{W}^T\mathbf{X} + \mathbf{b}$$

Wrap the result of the linear regression with another function.

- In a neural network, this is called the **Activation Function**

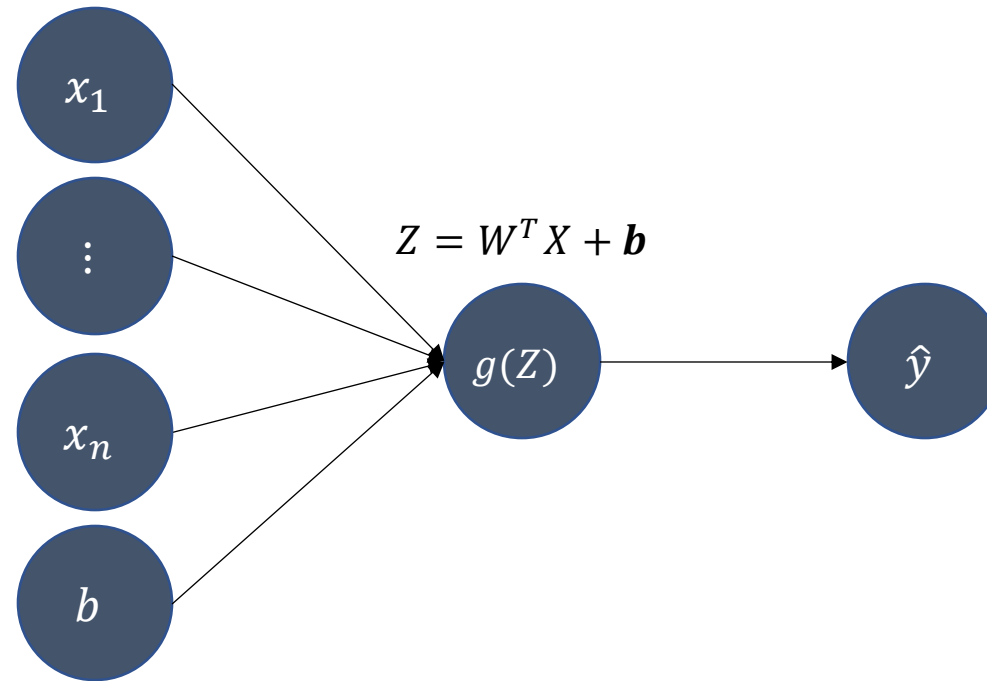
$$g(x) = \frac{1}{1 + e^{-x}}$$

Logistic Function

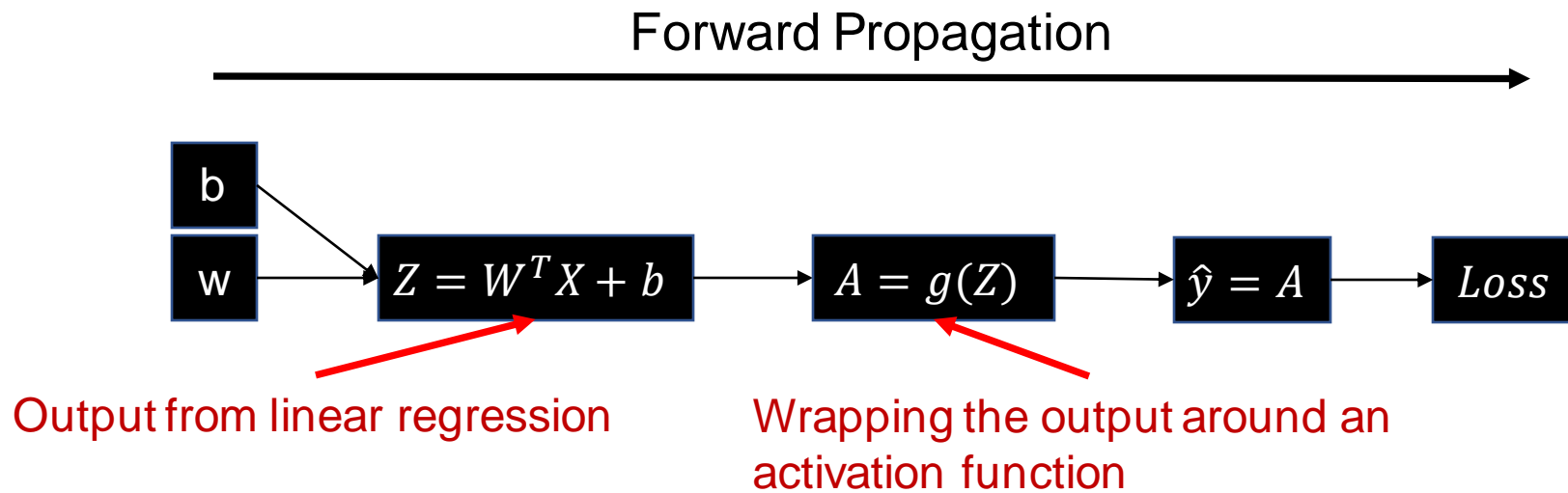


We can represent Logistic Regression as a Simple Neural Network

$$g(x) = \frac{1}{1 + e^{-x}}$$



Logistic regression model:



Sigmoid function:

$$g(x) = \frac{1}{1 + e^{-x}}$$

The probability that the engine works is represented by  $g(Z)$

We still need to define a Loss function

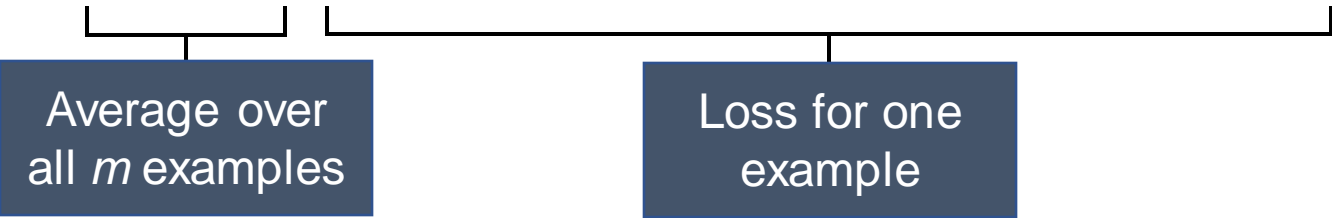
A measure of how "badly" the model performed.

- Logistic Regression uses a "Cross Entropy" Loss Function

$$J = -\frac{1}{m} \sum_{i=1}^m y_i \log \hat{y}_i + (1 - y_i) \log(1 - \hat{y}_i)$$

$y_i$  Label for the i-th example

$\hat{y}_i$  Prediction for the i-th example

$$J = -\frac{1}{m} \sum_{i=1}^m y_i \log \hat{y}_i + (1 - y_i) \log(1 - \hat{y}_i)$$


Average over all  $m$  examples

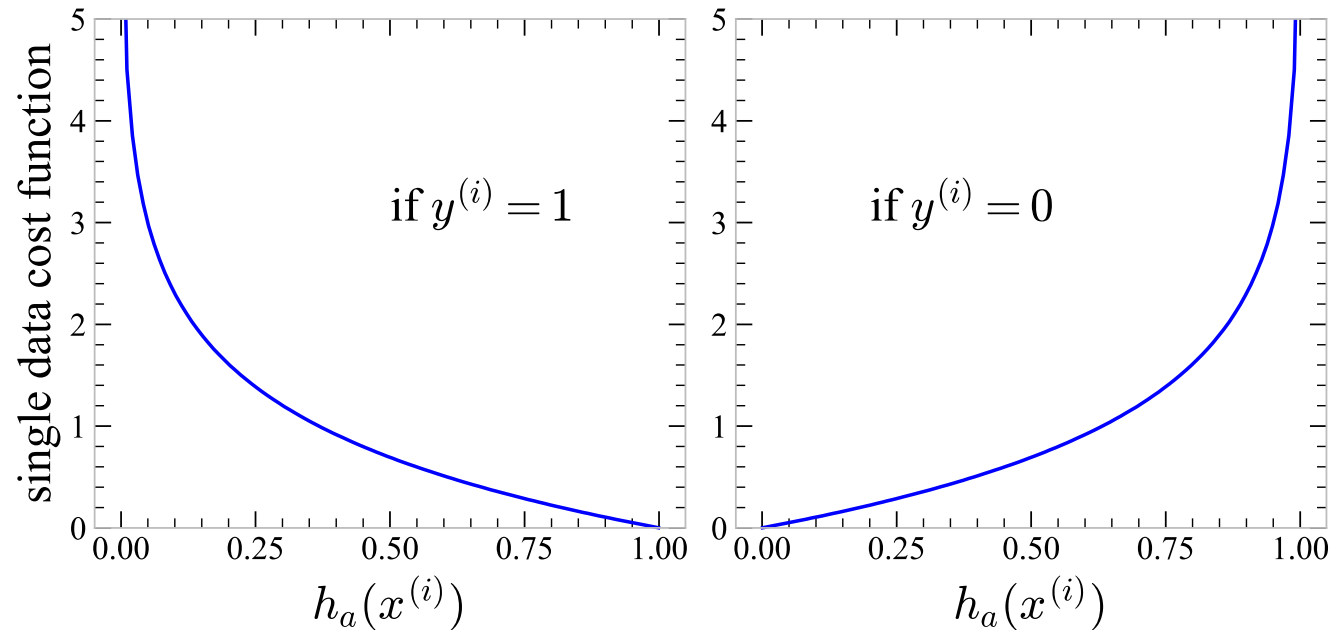
Loss for one example

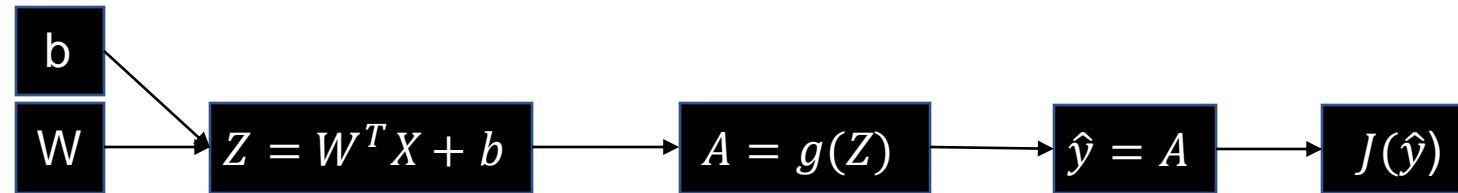
Note that:  $0 \leq \hat{y}_i \leq 1$

- Meaning both  $\log \hat{y}_i$  and  $\log(1 - \hat{y}_i)$  are negative
- A larger value of  $J$  should mean worse performance: minus sign added



$$J = -\frac{1}{m} \sum_{i=1}^m y_i \log \hat{y}_i + (1 - y_i) \log(1 - \hat{y}_i)$$





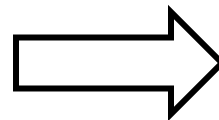
We need to compute:

$$\frac{d(J)}{dw_1}; \frac{d(J)}{dw_2}; \dots; \frac{d(J)}{dw_n}; \frac{d(J)}{db}$$

$$\frac{d(J)}{dw_i} = \frac{d(J)}{d\hat{y}} \frac{d\hat{y}}{dA} \frac{dA}{dZ} \frac{dZ}{dw_i}$$

$$\frac{d(J)}{db} = \frac{d(J)}{d\hat{y}} \frac{d\hat{y}}{dA} \frac{dA}{dZ} \frac{dZ}{db}$$

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \rightarrow \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} - \alpha \begin{pmatrix} \frac{d(J)}{dw_1} \\ \frac{d(J)}{dw_2} \\ \vdots \\ \frac{d(J)}{dw_n} \end{pmatrix}$$



$$\begin{aligned} W &\rightarrow W - \alpha dW \\ b &\rightarrow b - \alpha \frac{d(MSE)}{db} \end{aligned}$$

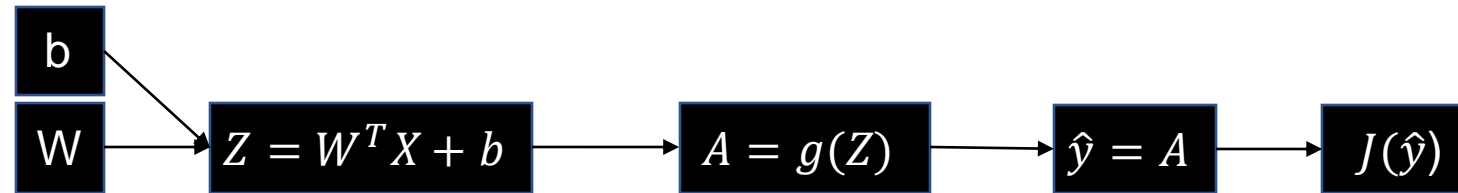
$$J = -\frac{1}{m} \sum_{i=1}^m y_i \log \hat{y}_i + (1 - y_i) \log(1 - \hat{y}_i)$$

Determine  $\frac{dJ}{d\hat{y}}$ : Let  $L(y_i, \hat{y}_i) = y_i \log \hat{y}_i + (1 - y_i) \log(1 - \hat{y}_i)$

$$\frac{dJ}{d\hat{y}} = \frac{d}{d\hat{y}} \left( -\frac{1}{m} \sum_{i=1}^m L(y_i, \hat{y}_i) \right) \longrightarrow \frac{dJ}{d\hat{y}} = -\frac{1}{m} \sum_{i=1}^m \frac{d}{d\hat{y}} (L(y_i, \hat{y}_i))$$

$$\frac{d}{d\hat{y}} (L(y_i, \hat{y}_i)) = \frac{y_i}{\hat{y}_i} - \frac{(1 - y_i)}{1 - \hat{y}_i}$$

$$\frac{dJ}{d\hat{y}} = -\frac{1}{m} \sum_{i=1}^m \left\{ \frac{y_i}{\hat{y}_i} - \frac{(1 - y_i)}{1 - \hat{y}_i} \right\}$$



Now compute the derivatives

$$\frac{d(J)}{dw_i} = \frac{d(J)}{d\hat{y}} \frac{d\hat{y}}{dA} \frac{dA}{dZ} \frac{dZ}{dw_i}$$

$$\frac{d(J)}{db} = \frac{d(J)}{d\hat{y}} \frac{d\hat{y}}{dA} \frac{dA}{dZ} \frac{dZ}{db}$$

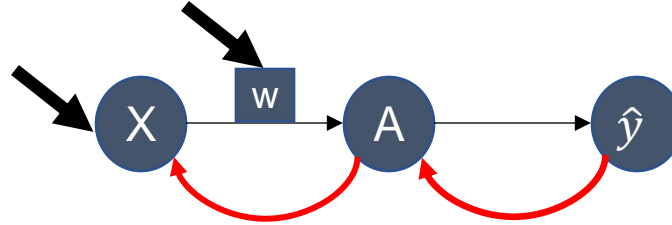
$$\frac{dJ}{d\hat{y}} = -\frac{1}{m} \sum_{i=1}^m \left\{ \frac{y_i}{\hat{y}_i} - \frac{(1 - y_i)}{1 - \hat{y}_i} \right\}$$

$$\frac{d\hat{y}}{dA} = 1; \frac{dA}{dZ} = A(1 - A); \frac{dZ}{dw_i} = x_i; \frac{dZ}{db} = 1$$

$$\frac{d(J)}{dw_i} = -\frac{1}{m} A(1 - A) x_i \sum_{k=1}^m \left\{ \frac{y_k}{\hat{y}_k} - \frac{(1 - y_k)}{1 - \hat{y}_k} \right\}$$

$$\frac{d(J)}{db} = -\frac{1}{m} A(1 - A) \sum_{k=1}^m \left\{ \frac{y_k}{\hat{y}_k} - \frac{(1 - y_k)}{1 - \hat{y}_k} \right\}$$

In Summary:



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$$Z = W^T X + b ; A = g(Z); \hat{y} = A$$

3. **Backpropagation:** The network calculates derivatives to minimise the loss function

$$\frac{d(J)}{dw_i} = \frac{d(J)}{d\hat{y}} \frac{d\hat{y}}{dA} \frac{dA}{dZ} \frac{dZ}{dw_i}$$

$$\frac{d(MSE)}{db} = \frac{d(MSE)}{d\hat{y}} \frac{d\hat{y}}{dA} \frac{dA}{dZ} \frac{dZ}{db}$$

4. **Update Parameters:** Using the gradients, parameters ( $w, b$ ) are updated

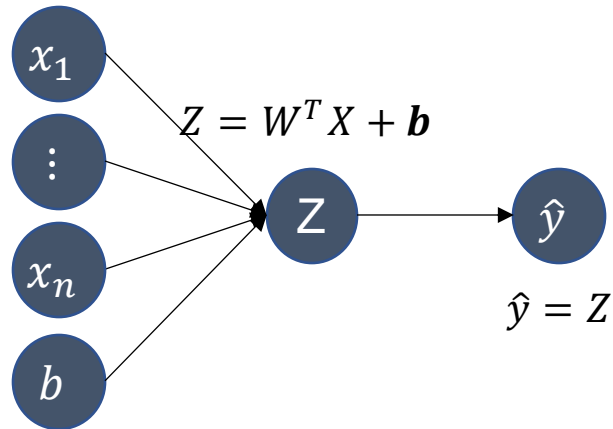
$$w \rightarrow w - \alpha dW$$

$$b \rightarrow b - \alpha \frac{d(MSE)}{db}$$

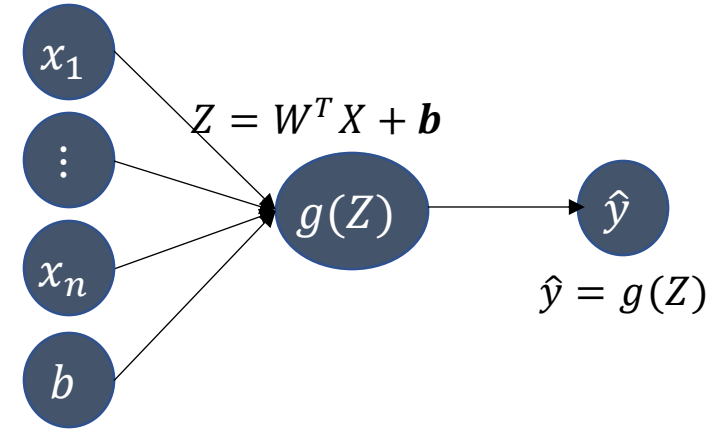
5. **Repeat 2-4 for a set number of “Epochs”**

# What about General NNs?

Linear Regression



Logistic Regression (Classification)



See that a neural network can function as a regressor or a classifier

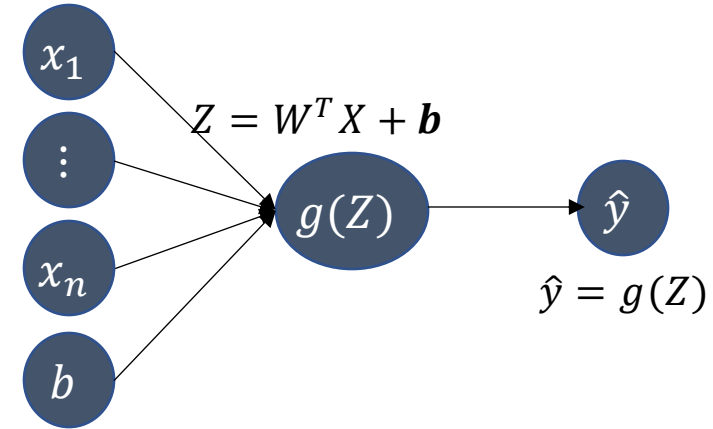
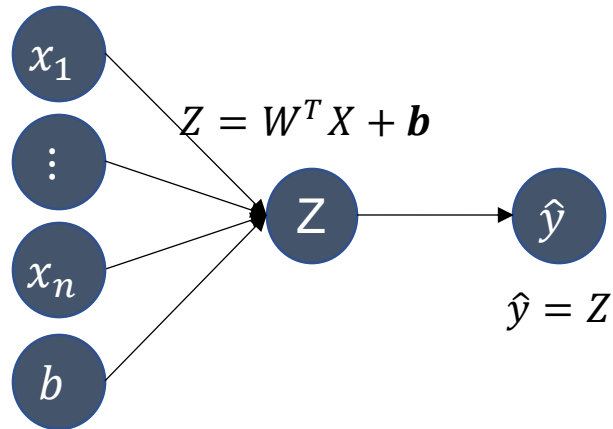
It only depends on which activation function we are using!

# General NNs

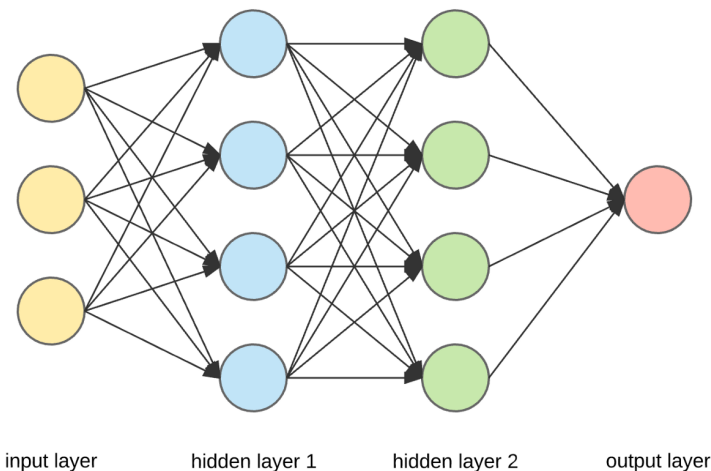
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# What about General NNs?

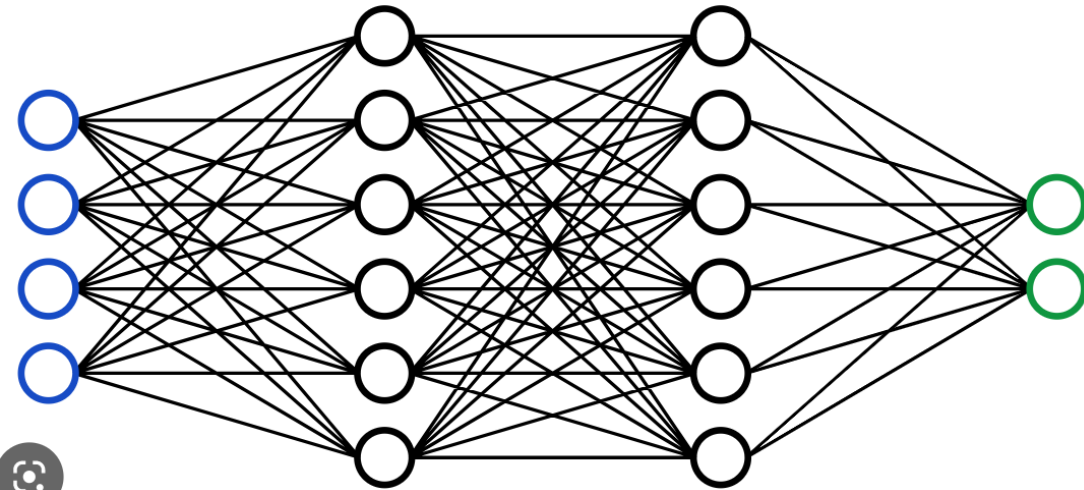
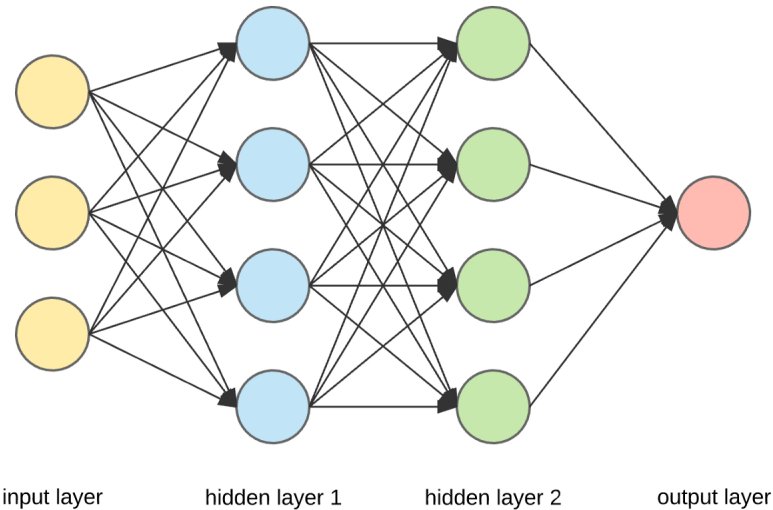
Until now, we only considered problems where there are only 1 neuron and 1 hidden layer.



What about general neural networks?

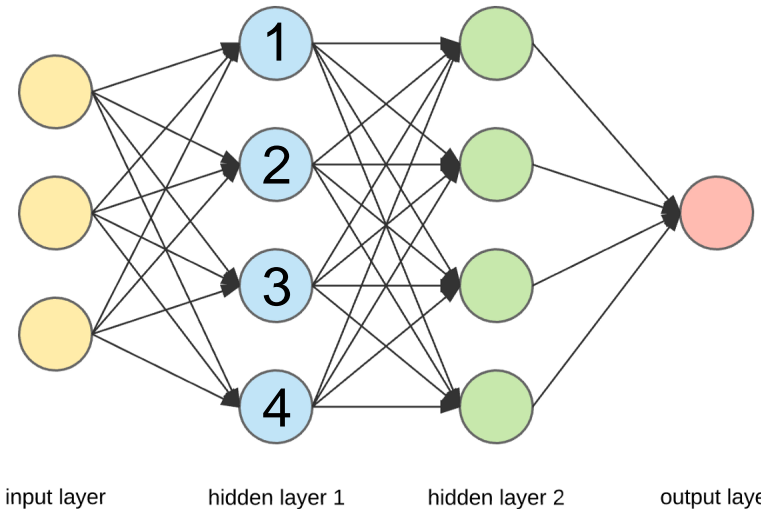






- **Input, Label Matrix:**  $X, Y$
- **Prediction Matrix:**  $\hat{y}$
- **Number Of Layers:** All layers excluding input
- **Input Layer:** Layer 0
- **Number of examples:**  $m$
- **Input, Label Size:**  $n_x, n_y$
- **Values in layer L** is denoted by:  $W^{[L]}$
- **Number of neurons in (hidden) layer L:**  $n_h^{[L]}$

Consider Input Layer to hidden layer 1 and number each hidden neuron 1 to 4 with some activation function  $g^{[1]}(x)$



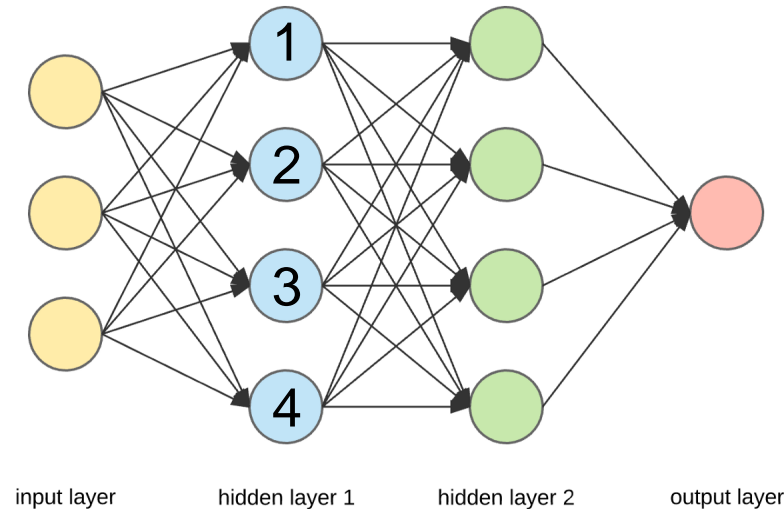
$$A_1^{[1]} = g^{[1]}(Z_1^{[1]}); \text{ where } Z_1^{[1]} = w_{1,1}x_1 + w_{2,1}x_2 + w_{3,1}x_3 + b_1 = W_1^{[1]T}X + b_1$$

$$A_2^{[1]} = g^{[1]}(Z_2^{[1]}); \text{ where } Z_2^{[1]} = w_{1,2}x_1 + w_{2,2}x_2 + w_{3,2}x_3 + b_2 = W_2^{[1]T}X + b_2$$

$$A_3^{[1]} = g^{[1]}(Z_3^{[1]}); \text{ where } Z_3^{[1]} = w_{1,3}x_1 + w_{2,3}x_2 + w_{3,3}x_3 + b_3 = W_3^{[1]T}X + b_3$$

$$A_4^{[1]} = g^{[1]}(Z_4^{[1]}); \text{ where } Z_4^{[1]} = w_{1,4}x_1 + w_{2,4}x_2 + w_{3,4}x_3 + b_4 = W_4^{[1]T}X + b_4$$

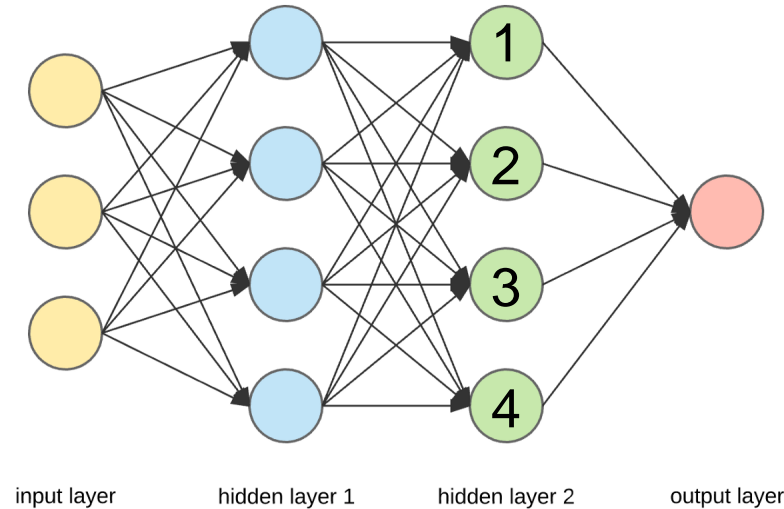
Consider Input Layer to hidden layer 1 and number each hidden neuron 1 to 4 with some activation function  $g(x)$



$$A^{[1]} = \begin{bmatrix} A_1^{[1]} \\ A_2^{[1]} \\ A_3^{[1]} \\ A_4^{[1]} \end{bmatrix} = \begin{bmatrix} g^{[1]}(W_1^{[1]T} X + b_1^{[1]}) \\ g^{[1]}(W_2^{[1]T} X + b_2^{[1]}) \\ g^{[1]}(W_3^{[1]T} X + b_3^{[1]}) \\ g^{[1]}(W_4^{[1]T} X + b_4^{[1]}) \end{bmatrix}$$

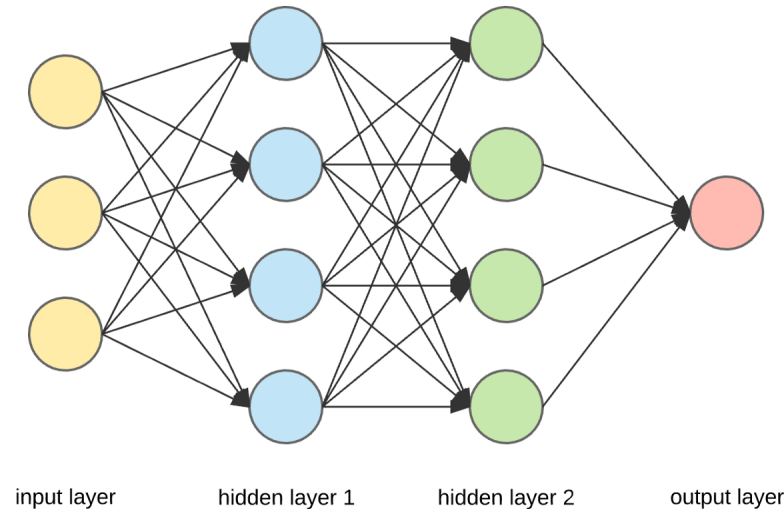
Now  $A^{[1]}$  acts as the input for hidden layer 2

Now consider Hidden Layer 1 to Hidden Layer 2. Follow similar logic to previous slides, using  $g^{[2]}(x)$  as the activation function



$$A^{[2]} = \begin{bmatrix} A_1^{[2]} \\ A_2^{[2]} \\ A_3^{[2]} \\ A_4^{[2]} \end{bmatrix} = \begin{bmatrix} g^{[2]}(W_1^{[2]T} A_1^{[1]} + b_1^{[2]}) \\ g^{[2]}(W_2^{[2]T} A_2^{[1]} + b_2^{[2]}) \\ g^{[2]}(W_3^{[2]T} A_3^{[1]} + b_3^{[2]}) \\ g^{[2]}(W_4^{[2]T} A_4^{[1]} + b_4^{[2]}) \end{bmatrix}$$

Finally for the output layer with activation function  $g^{[3]}(x)$  :



$$\hat{y} = g^{[3]}(W^{[3]T}A^{[2]} + b^{[3]})$$

A general neural network will have to assign random values for:  $W^{[1]}; W^{[2]}; W^{[3]}; b^{[1]}; b^{[2]}; b^{[3]}$

And then calculate derivatives for all parameters and then update using:

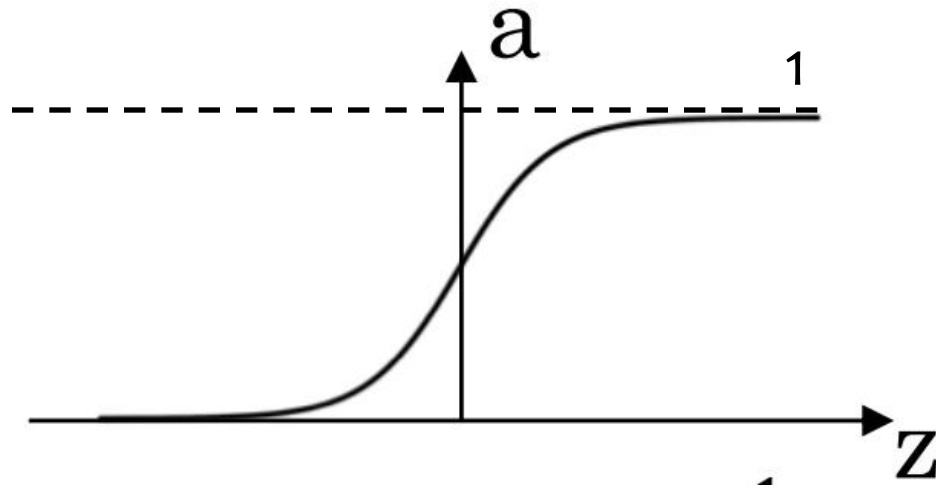
$$W \rightarrow W - \alpha dW$$

$$b \rightarrow b - \alpha \frac{d(Loss)}{db}$$

# Activation Functions

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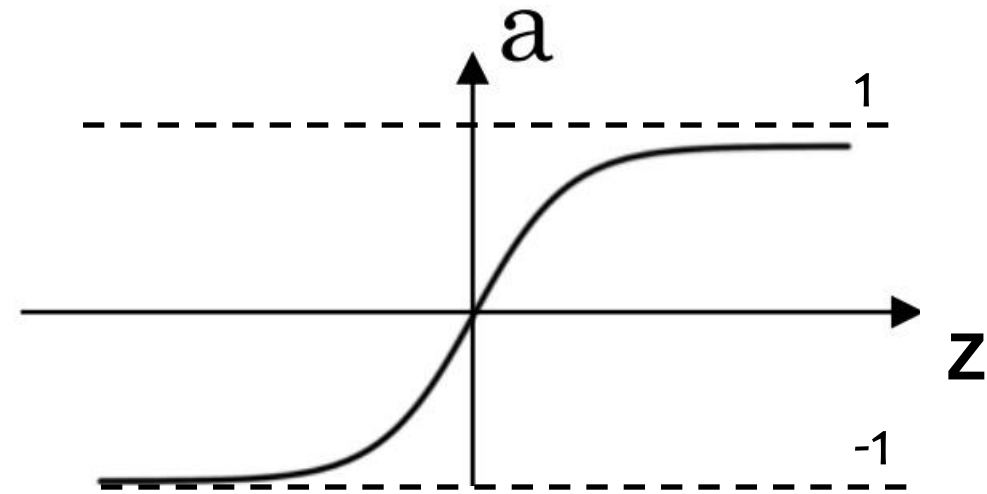
## Sigmoid Function



$$g(Z) = \frac{1}{1 + e^{-Z}}$$

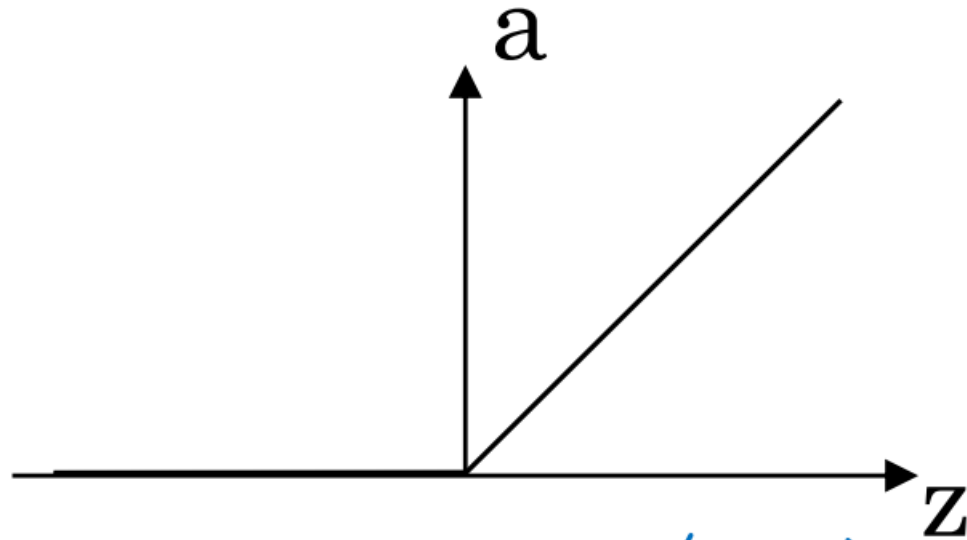
Generally, **only** used for the output layer in a binary classification problem

## tanh(Z)



$$g(Z) = \tanh(Z) = \frac{e^Z - e^{-Z}}{e^Z + e^{-Z}}$$

## ReLU

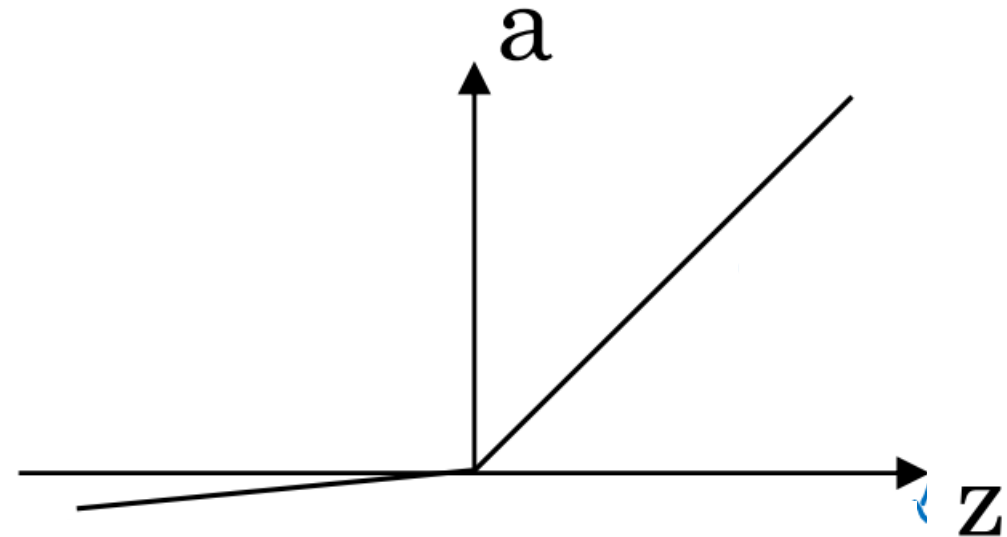


$$g(Z) = \begin{cases} Z & \text{if } Z \geq 0 \\ 0 & \text{if } Z < 0 \end{cases}$$

### When in doubt, use ReLU

Easy to implement, closest to how a Neuron actually works

## Leaky ReLU



$$g(Z) = \begin{cases} Z & \text{if } Z \geq 0 \\ \beta Z & \text{if } Z < 0 \end{cases}$$

Tends to perform better than just ReLU

But 1 more hyperparameter to tune



# Questions?