Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
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Introduction to University Mathematics **MATH40001/MATH40009**

Problem Sheet 3: Functions

KMB, 5/10/19

To try these questions in Lean, go to https://tinyurl.com/Lean-M40001-Example-Sheet-3. Remember that this is completely optional, and there will be no Lean in any M40001/M40009 tests or exams. But note that one of the main problems that students have with this material – thinking they have done the question when they have not – can't happen when using Lean. If your solution runs, you've definitely got it right, and if it doesn't, you haven't.

1. * Say X, Y and Z are sets, and $f : X \to Y$ and $g : Y \to Z$ are functions. In lectures we proved that if f and g are injective, then $g \circ f$ is also injective, and we will prove on Monday that if f and g are surjective, then $g \circ f$ is surjective. But what about the other way?

   (a) If $g \circ f$ is injective, then is f injective? Give a proof or a counterexample.
   (b) If $g \circ f$ is injective, then is g injective? Give a proof or a counterexample.
   (c) If $g \circ f$ is surjective, then is f surjective? Give a proof or a counterexample.
   (d) If $g \circ f$ is surjective, then is g surjective? Give a proof or a counterexample.

2. For each of the following functions, decide whether or not they are injective, surjective, bijective. Proofs required!

   (a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1/x$ if $x \neq 0$ and $f(0) = 0$.
   (b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 1$.
   (c) $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 2x + 1$.
   (d) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3 - x$ if the Riemann hypothesis is true, and $f(x) = 2 + x$ if not. [NB the Riemann Hypothesis is a hard unsolved problem in mathematics; nobody currently knows if it is true or false.]
   (e) $f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = n^3 - 2n^2 + 2n - 1$.

3. For each of the following "functions", explain why I just lost a mark.

   (a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1/x$.
   (b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sqrt{x}$.
   (c) $f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = (n + 1)^2/2$.
   (d) $f : \mathbb{R} \to \mathbb{R}$, $f(x)$ is a solution to $y^3 - y = x$.
   (e) $f : \mathbb{R}\backslash\{1\} \to \mathbb{R}$, $f(x) = 1 + x + x^2 + x^3 + \cdots$.

4. Prove the claim I will make in lecture on Monday, saying that if $f : X \to Y$ is a function, and $g : Y \to X$ is a two-sided inverse of f, then f is a two-sided inverse for g. Deduce that if X and Y are sets, and there exists a bijection from X to Y, then there exists a bijection from Y to X.

5. Let Z be a set. If $f : X \to Z$ and $g : Y \to Z$ are injective functions, let's say that f *is friends with* g if there is a bijection $h : X \to Y$ such that $f = g \circ h$. Prove that f is friends with g if and only if the image of f equals the image of g. NB: by the *image* of $f : X \to Z$ I mean the subset of Z consisting of things "hit" by f, in other words the set $\{z \in Z : \exists x \in X, f(x) = z\}$. Some people call this the "range" of f, although other people use "range" to mean the same thing as "codomain" :-/