Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
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Introduction to University Mathematics **MATH40001/MATH40009**

Problem Sheet 2: Sets

KMB, 3/10/19

To try these questions in Lean, go to https://tinyurl.com/Lean-M40001-Example-Sheet-2. Remember that this is completely optional, and there will be no Lean in any M40001/M40009 tests or exams. But I think that trying to do problems like Q5 in Lean will teach you how to think logically about the difference between 5a and 5b.

1. How could you give a formal proof that if X and Y are sets, then $X \cap Y = Y \cap X$? I know it's obvious – but if someone asked you to prove it, and wanted you to say something, what would you say? The technical term for this result is "commutativity of $\cap$".

2. Let U be the set $\{1, 2, 3, 4, 5\}$. Which of the following statements are true and which are false? (just write T or F).

   (a) $1 \in U$.
   (b) $\{1\} \in U$.
   (c) $\{1\} \subseteq U$.
   (d) $\{1, 2\} \subseteq U$.
   (e) $\{1, 2, 1\} \subseteq U$.
   (f) $\{1, 1\} \in U$.
   (g) $U \in U$.
   (h) $U \supseteq U$.

3. Define $A = \{x \in \mathbb{R} : x^2 < 3\}$, $B = \{x \in \mathbb{Z} : x^2 < 3\}$ and $C = \{x \in \mathbb{R} : x^3 < 3\}$. For each statement below, either prove it or disprove it!

   (a) $\frac{1}{2} \in A \cap B$.
   (b) $\frac{1}{2} \in A \cup B$.
   (c) $A \subseteq C$.
   (d) $B \subseteq C$.
   (e) $C \subseteq A \cup B$.
   (f) $(A \cap B) \cup C = (A \cup B) \cap C$

4. Let $P(x)$ and $Q(x)$ be propositions which depends on a variable x in a set X, and let $R(x, y)$ be a proposition which depends on two variables $x \in X$ and $y \in Y$. What are the logical negations of the following statements? Try to move the $\neg$ as far into the formulae as you can.

   (a) $\forall x \in X, P(x) \wedge \neg Q(x)$
   (b) $\exists x \in X, (\neg P(x)) \wedge Q(x)$
   (c) $\forall x \in X, \exists y \in Y, R(x, y)$.

   These questions are surprisingly important. If $f : \mathbb{R} \to \mathbb{R}$ is a function, then, as you will learn later on this term, we say that f is *continuous* at $x \in \mathbb{R}$ if $\forall \epsilon \in \mathbb{R}_{>0}, \exists \delta \in \mathbb{R}_{>0}, \forall y \in \mathbb{R}, |y - x| < \delta \implies |f(y) - f(x)| < \epsilon$. This is the correct formal definition of continuity, which avoids having to say any waffle about being able to draw the graph without taking your pencil off the paper. Hence to prove that a function is *not* continuous, one has to figure out what the logical negation of the above proposition is! Can you do it?

5. * Are the following statements true or false? Proofs or counterexamples are required!

   (a) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y = 2$.
   (b) $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x + y = 2$.

6. Let $\varnothing$ be the empty set. Are the following propositions true or false?

   (a) $\exists x \in \varnothing, 2+2=5$

   (b) $\forall x \in \varnothing, 2+2=5$

   Hint: think about logical negations.