Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
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Introduction to University Mathematics **MATH40001/MATH40009**

Problem Sheet 4: Binary relations

KMB, 7/10/19

To try these questions in Lean, go to https://tinyurl.com/Lean-M40001-Example-Sheet-4. Remember that this is completely optional, and there will be no Lean in any M40001/M40009 tests or exams.

1. * For each of the sets X and binary relations R below, figure out whether R is (a) reflexive, (b) symmetric, (c) antisymmetric, (d) transitive.

   (a) Let X be the set $\{1, 2\}$ and define R like this: $R(1, 1)$ is true, $R(1, 2)$ is true, $R(2, 1)$ is true and $R(2, 2)$ is false.

   (b) Let $X = \mathbb{R}$ and define $R(a, b)$ to be the proposition $a = -b$.

   (c) Let $X = \mathbb{R}$ and define $R(a, b)$ to be false for all real numbers a and b.

   (d) Let X be the empty set and define R to be the empty binary relation (we don't have to say what its value is on any pair (a, b) because no such pairs exist).

2. Let X be the set of subsets of the integers. So, for example, examples of elements of X would be the even numbers, or the prime numbers.

   (a) Prove that the binary relation $\subseteq$ on X is a partial order.

   (b) Is $\subseteq$ a total order? Give a proof or a counterexample.

3. (a) Is the binary relation $<$ on $\mathbb{R}$ symmetric?

   (b) Is the binary relation $<$ on the empty subset $\varnothing$ of $\mathbb{R}$ symmetric?

4. Let X be a set, and say R is a binary relation on X which is symmetric and transitive. We will show that R is reflexive. So let $x \in X$ be arbitrary, and choose $y \in X$ such that $R(x, y)$ is true. Then $R(y, x)$ is true by symmetry, and now because $R(x, y) \wedge R(y, x)$ is true, by transitivity we deduce that $R(x, x)$ is true. Because x was arbitrary, this means that R is reflexive, QED.

   This contradicts Q1c. Where's the mistake?

5. Let X be a fixed set. We say two surjections $f : X \to Y$ and $g : X \to Z$ are *pals* if there exists a bijection $h : Y \to Z$ such that $g = h \circ f$. Prove that f and g are pals if and only if the equivalence relations R_f and R_g on X induced by f and g are equal (by which I mean $R_f(a, b) \iff R_g(a, b)$). Recall that the equivalence relation R_f induced by f is the relation on X defined by $R_f(a, b)$ is true if and only if $f(a) = f(b)$.

   NB if X is the plastic shapes, Y is {red,green,yellow,blue} and Z is the set consisting of the four piles that the blind person made, the natural maps $X \to Y$ and $X \to Z$ are pals.