A Series Of SERIES

Just Some Series Notes.

Series is series is series.

1 NONSENCE

Cauthy 收敛定理

$$\forall\varepsilon\left(\exists N\left(\forall n,m>N\left(\left|\sum_{m}^{n} a_\circ\right|<\varepsilon\right)\right)\right)\iff \sum a_\circ \longrightarrow a.$$

为了简写记作 $\overset{\rightarrow}{\sum} a_\circ$, 在这里无穷级数的和定义为部分和的极限:

$$\sum a_\circ := \lim \sum^{n} a_\circ.$$

显然

$$\lim a_\circ \neq 0 \implies \overset{\nrightarrow}{\sum} a_\circ.$$

- 比如 $\overset{\nrightarrow}{\sum}(-)^\circ$.

另, 级数是否收敛只取决与无穷远处的形状, 与有限处无关.

2 审敛

为了方便记忆, 以下审敛法都写成极限形式, 这意味着判别力比原来要弱.

比较判别 若 $|a_n| = \mathcal{O}(|b_n|)$, 则:

$$\overset{\nrightarrow}{\sum} |a_\circ| \implies \overset{\nrightarrow}{\sum} |b_\circ|,$$

$$\overset{\rightarrow}{\sum} |b_\circ| \implies \overset{\rightarrow}{\sum} |a_\circ|.$$

p 级数法

$$\overset{\rightarrow}{\sum} \frac{1}{\circ^n}, \text{ iff } \mathfrak{R}(n) > 1, \text{ otherwise } \overset{\nrightarrow}{\sum} \frac{1}{\circ^n}.$$

D'Alembert 审敛 若

$$\limsup\left|\frac{a_{\circ+1}}{a_\circ}\right|<1\implies\stackrel{\longrightarrow}{\sum}a_\circ,$$

$$\liminf\left|\frac{a_{\circ+1}}{a_\circ}\right|>1\implies\stackrel{\not\longrightarrow}{\sum}a_\circ.$$

否则不能判定.

Cauthy 审敛 若

$$\limsup\left|\sqrt[\circ]{a_\circ}\right|<1\implies\stackrel{\longrightarrow}{\sum}a_\circ,$$

$$\liminf\left|\sqrt[\circ]{a_\circ}\right|>1\implies\stackrel{\not\longrightarrow}{\sum}a_\circ,$$

否则不能判定.

Raabe 审敛 若

$$\lim\circ\left(\left|\frac{a_\circ}{a_{\circ+1}}\right|-1\right)>1\implies\stackrel{\longrightarrow}{\sum}a_\circ,$$

$$\lim\circ\left(\left|\frac{a_\circ}{a_{\circ+1}}\right|-1\right)<1\implies\stackrel{\not\longrightarrow}{\sum}a_\circ,$$

否则不能判定.

积分判别法

$$\forall x\geqslant1(f(x)\searrow\wedge f(x)>0)\implies\sum f(\circ)\text{ 与 }\int_1^\infty f(x)\,\mathrm{d}x\text{ 共敛散}.$$

Leibniz 审敛

$$\forall x\geqslant1(f(x)\searrow\wedge f(x)>0\wedge\lim f(x)=0)\implies\stackrel{\longrightarrow}{\sum}(-)^\circ f(\circ).$$

Dirichlet 审敛 若 $a_\circ$ 单调, 则:

$$\exists M\left(\left|\sum^n b_\circ\right|<M(\lim a_\circ=0)\right)\implies\stackrel{\longrightarrow}{\sum}a_\circ b_\circ.$$

Abel 审敛 若 $a_\circ$ 单调, $\overrightarrow{\sum} b_\circ$, 则:

$$\exists M(a_\circ < M) \implies \overrightarrow{\sum} a_\circ b_\circ.$$

Ermakov 审敛

$$\exists F(\forall x > F(f(x) \searrow \wedge f(x) > 0)) \wedge \exists G(\forall x > G(g(x) \nearrow \wedge g(x) > \wedge g(x) \in \mathcal{C}^1))$$

$$\implies \exists X\left(\forall x > X\left(\begin{matrix} \dfrac{f(g)g'}{f} \leqslant q < 1 \implies \overrightarrow{\sum} f(\circ) \\ \dfrac{f(g)g'}{g} \geqslant 1 \implies \overset{\nrightarrow}{\sum} f(\circ) \end{matrix}\right)\right).$$

Cauthy 的凝聚判别

$$\exists N(n > N(a_n \searrow \wedge a_n \geqslant 0)) \implies \sum a_\circ \text{ 与 } \sum 2^\circ a_{2^\circ} \text{ 共敛散.}$$

3 绝对与条件收敛

$$\overrightarrow{\sum} |a_\circ| =: \sum a_\circ \text{绝对收敛},$$

$$\overset{\nrightarrow}{\sum} |a_\circ| \wedge \overrightarrow{\sum} a_\circ =: \sum a_\circ \text{条件收敛}.$$

绝对收敛的级数可以任意改变运算顺序, 但条件收敛不可以.

4 函数项级数

一致收敛

$$\forall \varepsilon > 0(\forall x \in \mathcal{I}(\exists N(\forall n, m > N(\left|f_n(x) - f_m(x)\right| < \varepsilon)))) =: \sum f_\circ(x) \text{在 } \mathcal{I} \text{ 上一致收敛.}$$

记作 $\sum f_\circ(x) \rightrightarrows f(x)$.

一致收敛的级数可以交换求和与积分, 求导, 极限. 显然连续函数项的级数若一致收敛, 则和函数必然连续. 一致收敛是对整个区间而言的, 比普通的逐点收敛强.

显然, 若 $f_\circ(x) \not\rightrightarrows 0$, 则 $\overset{\not\rightrightarrows}{\sum} f_\circ(x)$.

内闭一致收敛 实际上就是在开区间 $\mathcal{I}$ 内的任意一个闭区间上都一致收敛.

- 比如 $\sum \diamond \exp(-\diamond^2 x)$ 在 $(0,\infty)$ 上不一致收敛, 具体是靠近 $x=0$ 时收敛速度无限衰减, 但在 $(0,\infty)$ 上内闭一致收敛.

Weierstrass 判别法

$$\overset{\rightarrow}{\sum} \sup_{x\in\mathcal{I}} |f_\circ(x)| \implies \overset{\rightrightarrows}{\sum_{x\in\mathcal{I}}} f_\circ(x).$$

Dirichlet 判别 若 $f_n(x)$ 对 n 单调

$$\exists M\left(\forall n\left(\forall x\in\mathcal{I}\left(\left|\sum^{n} g_\circ(x)\right| < M\right)\right)\right) \wedge f_\circ(x) \rightrightarrows 0 \implies \overset{\rightrightarrows}{\sum} f_\circ(x) g_\circ(x).$$

其中我们称 $\exists M\left(\forall n\left(\forall x\in\mathcal{I}\left(\left|\sum^{n} g_\circ(x)\right| < M\right)\right)\right)$ 为 $g_\circ(x)$ 的部分和在 $\mathcal{I}$ 上一致有界.

Abel 判别 若 $f_n(x)$ 对 n 单调且在 $\mathcal{I}$ 上一致有界, 则:

$$\overset{\rightrightarrows}{\sum} b_\circ(x) \implies \overset{\rightrightarrows}{\sum} f_\circ(x) g_\circ(x).$$