Convolution, Correlation and Fourier Transforms

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Plan

1. Introduction
2. Fourier Transforms
3. Convolution transform. Convolution theorem
4. Proof of the convolution theorem
5. Correlation and auto-correlation
6. CNN: Advanced insights
7. Bonus

Time and Frequency Domains

Two ways of process descibing:

- **Time domain**. Values of some quantity
- **Frequency Domain**. Complex number: amplitude and phase

Fourier transforms

The mapping between the domains

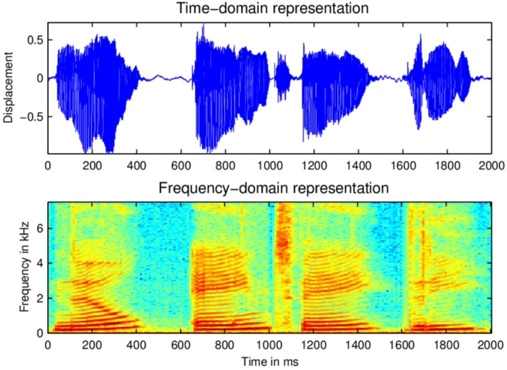


Figure: Two representations of speech.

Fourier Transforms

1. From time domain to frequency domain:

$$H(f) = \int_{-\infty}^{+\infty} h(t) \exp(-2\pi ift) dt$$

2. From frequency domain to time domain:

$$h(t) = \int_{-\infty}^{+\infty} H(f) \exp(2\pi ift) df$$

Properties

1. FT is a linear operator
2. Time scaling: $h(at) \equiv \frac{1}{a} H(\frac{f}{a})$
3. Time shifting: $h(t - t_0) \equiv H(f) \exp(-2\pi ift_0)$

Convolution

Functions $h(t)$ and $g(t)$ is given. $H(f)$ and $G(f)$ are corresponding Fourier transforms.

Convolution

$$f(t) = g * h \equiv \int_{-\infty}^{+\infty} f(v)g(t-v)dv$$

For functions f, g supported on only $[0, \infty)$, the integration limits can be truncated:

$$f(t) = g * h \equiv \int_{0}^{t} f(t-v)g(v)dv$$

Properties:

- $g * h = h * g$
- $g * h \equiv G(f)H(f)$

Convolution Theorem

Theorem

The Fourier transform of the convolution is the product of the two Fourier transforms.

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$$

Where $\mathcal{F}$ is a Fourier transform.

Modifications:

1. Laplace transform
2. Two-sided Laplace transform
3. Z-transform
4. Mellin transform

Proof for Laplace transform

Laplace transform:

$$\mathcal{L}(f) = \int_0^\infty \exp(-st) f(t) dt$$

Convolution theorem:

$$\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$$

Proof:

$$\begin{aligned}\mathcal{L}(f * g) &= \int_0^\infty \exp(-st) \left(\int_0^t f(t-v) g(v) dv \right) dt = \\ &= \int_0^\infty \exp(-st) \left(\int_0^\infty f(t-v) g(v) u(t-v) dv \right) dt\end{aligned}$$

Proof

$$
\begin{aligned}
\mathcal{L}(f * g) &= \int_0^\infty \exp(-st) \left(\int_0^\infty f(t-v) g(v) u(t-v) dv \right) dt = \\
&= \int_0^\infty \left(\int_0^\infty \exp(-st) f(t-v) g(v) u(t-v) dv \right) dt = \\
&= \int_0^\infty \left(\int_0^\infty \exp(-st) f(t-v) g(v) u(t-v) dt \right) dv = \\
&= \int_0^\infty g(v) \left(\int_0^\infty \exp(-st) f(t-v) u(t-v) dt \right) dv
\end{aligned}
$$

Proof

Note:

$$\mathcal{L}(f(t-v)u(t-v)) = \exp(-vs)\mathcal{L}(f(t)) = \exp(-vs)F(s)$$

Then:

$$\mathcal{L}(f*g) = \int_0^\infty g(v)\exp(-sv)F(s)dv = F(s)G(s) = \mathcal{L}(f(t))\mathcal{L}(g(t))$$

Correlation

Definition

The correlation of g and h. $H(f)$ and $G(f)$ are corresponding Fourier transforms.

$$Corr(g,h) \equiv \int_{-\infty}^{+\infty} g(v+t)h(t)dv$$

Correlation theorem:

$$Corr(g,h) \equiv G(f)H^*(f)$$

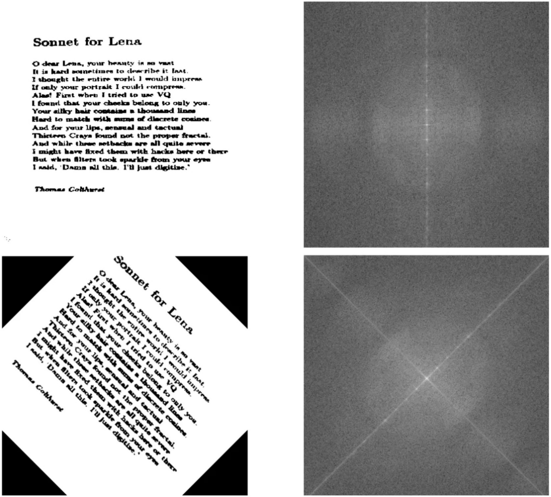
If g and h are real: $H(-f) = H^*(f)$.

Wiener-Khinchin Theorem:

The correlation of a function with itself is called its autocorrelation.

$$Corr(g,g) \equiv |G(f)|^2$$

Example of Image transform



CNN: Advanced concepts

Continuous form:

$$h(x) = f * g = \int_{-\infty}^{\infty} f(x-v)g(v)dv = \mathcal{F}^{-1}(\mathcal{F}(f)\mathcal{F}(g))$$

Discrete form:

$$y[m,n] = h[m,n] * \; x[m,n] = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} h[i,j]\; x[m-i, n-j] =$$
$$= \mathcal{F}^{-1}(\mathcal{F}(x)\mathcal{F}(k))$$

Note

A Fourier transform contains a lot of information about the orientation of an object in an image.

Open question:

Can we imagine that convolutional nets operate on images in the Fourier domain?

CNN: Cross-correlation

Cross-correlation for real functions g, h:

$$Corr(g, h) \equiv G(f)H(-f)$$

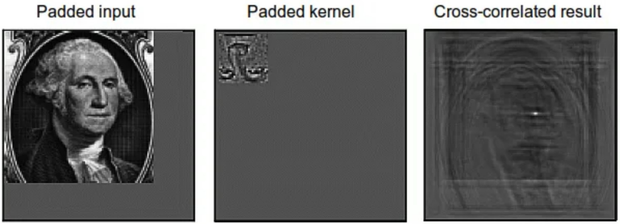


Figure: Cross-correlation via convolution: The input and kernel are padded with zeros and the kernel is rotated by 180 degrees.

Bonus: Complex networks

Reasons:

1. Improve phase prediction in existed pipelines
2. Work directly in Fourier domain
3. Use previously developed theory

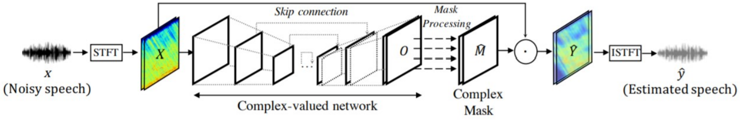


Figure: Deep complex U-net

Summary

- Fourier Transform
- Convolution in a general view
- Convolution theorem with proof
- Convolution and cross-correlation
- CNN: Advanced insights
- Complex Networks

Thank you!

Основная литература I