Parseval's Theorem and Fourier Inversion Theorem

Student Filipp Nikitin

Moscow institute of Physics and Technology
(state university)
Phystech school of applied mathematics and informatics

May 17, 2020

Plan

1. Introduction
2. Fourier Transforms
3. Fourier Inversion Theorem
4. Fourier Inversion Theorem(proof)
5. Parseval's Theorem
6. Parseval's Theorem(proof)
7. Discussion

Time and Frequency Domains

Two ways of process descibing:

- **Time domain**. Values of some quantity
- **Frequency Domain**. Complex number: amplitude and phase

Fourier transforms

The mapping between the domains

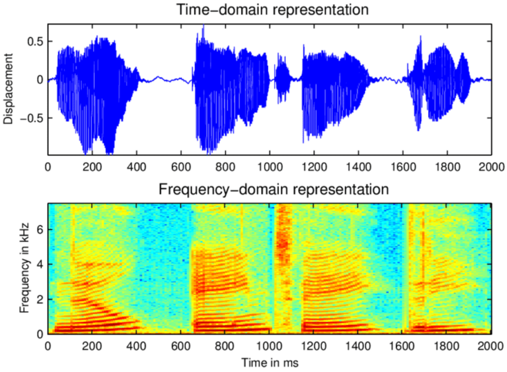

Figure: Two representations of speech.

Fourier Inversion Theorem

1. From time domain to frequency domain:

$$H(f) = \int_{-\infty}^{+\infty} h(t) \exp(-ift) dt$$

2. From frequency domain to time domain:

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(f) \exp(ift) df$$

Properties

1. FT is a linear operator
2. Time scaling: $h(at) \equiv \frac{1}{a} H(\frac{f}{a})$
3. Time shifting: $h(t - t_0) \equiv H(f) \exp(-ift_0)$

Fourier Inversion Theorem(proof)

Assume

$$H(f) = A \int_{-\infty}^{+\infty} h(t) \exp(-ift) dt$$

$$h(t) = B \int_{-\infty}^{+\infty} H(f) \exp(ift) df$$

When

$$h(t) = AB \int_{-\infty}^{+\infty} A \int_{-\infty}^{+\infty} h(\tau) \exp(-if\tau) \exp(ift) d\tau df =$$

$$= \int_{-\infty}^{+\infty} \left[AB \int_{-\infty}^{+\infty} \exp(-if\tau) \exp(ift) df \right] h(\tau) d\tau$$

QM flashback

$$AB \int_{-\infty}^{+\infty} \exp(-if\tau)\exp(ift)df = \delta(t-\tau)$$

where δ is a Dirac Delta function

When

$$AB \int_{a}^{b} \int_{-\infty}^{+\infty} \exp(if(t-\tau))dfd(t-\tau) = \frac{\mathrm{sign(b)} - \mathrm{sign(a)}}{2} =$$

$$= AB \int_{a}^{b} \int_{0}^{+\infty} \exp(ify) + \exp(-ify)dfdy =$$

$$4AB \int_{0}^{+\infty} \int_{a}^{b} cos(ky)dy \; d\tau = \mathrm{sign}(b) - \mathrm{sign}(a)$$

Fourier Inversion Theorem(proof)

$$4AB\int_{0}^{+\infty}\int_{a}^{b} cos(\tau y)dy\ d\tau = 4AB\int_{0}^{+\infty}\frac{\sin(\tau b)-\sin(\tau a)}{\tau}d\tau$$

Dirichlet integral

$$\int_{0}^{+\infty}\frac{\sin u}{u}du = \frac{1}{4AB};\ \ AB = \frac{1}{\pi}$$

Parseval's Theorem

Theorem

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}|H(f)|^2 df = \int_{-\infty}^{+\infty}|h(t)|^2 df$$

General Form(Plancherel's formula)

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}H(f)G^*(f) df = \int_{-\infty}^{+\infty}h(t)g^*(t) dt$$

Plancherel's formula(proof)

Proven statement

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} exp(ixt)dt$$

When

$$\int_{-\infty}^{+\infty} h(t)g^*(t)df =$$

$$\int_{-\infty}^{+\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} H(f)\exp(ift)df\right)\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} G^*(f')\exp(if't)df'\right)dt =$$

Plancherel's formula(proof)

$$\int_{-\infty}^{+\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} H(f)\exp(ift)df\right)\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} G^*(f')\exp(if't)df'\right)dt =$$

$$\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} H(f)G^*(f')\int_{-\infty}^{+\infty}(\exp(i(f-f')t)dt)\ df\ df' =$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} H(f)G^*(f')\delta(f-f')\ df\ df' = \frac{1}{2\pi}\int_{-\infty}^{+\infty} H(f)G^*(f)df$$

Note

Taking $h \equiv g$ we immediately obtain Parseval's Theorem

Summary

- Introduction
- Fourier Transforms
- Fourier Inversion Theorem
- Fourier Inversion Theorem(proof)
- Parseval's Theorem
- Parseval's Theorem(proof)
- Discussion

Thank you!

Основная литература I