Element of Probability Theory Exercises

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Exercise 6.1

Suppose $\Omega = \{0, 1, ..., 15\}$**,** $A = \{0, 8\}$**,** $B = \{1, 2, 3, 5, 8, 10, 12\}$**,** $C = \{0, 4, 9, 15\}$**. Determine** $A \cap B, B \cap C, A \cup C, C\ \ A, \Omega\ \ (B \cup A \cup C)$**.**

- $A \cap B = \{8\}$
- $B \cap C = \emptyset$
- $A \cup C = \{0, 4, 8, 9, 15\}$
- $C \setminus A = \{4, 9, 15\}$
- $\Omega \setminus (B \cup A \cup C) = \{6, 7, 11, 13, 14\}$

b. Now consider the three pairwise disjoint events E, F, G **with** $\Omega = E \cup F \cup G$ **and** $P(E) = 0.2$ **and** $P(F) = 0.5$**. Calculate** $P(\bar{F}), P(G), P(E \cap G), P(E \setminus E)$**, and** $P(E \cup F)$**.**

- $P(\bar{F}) = 0.5$
- $P(G) = 0.3$
- $P(E \cap G) = \emptyset$ because they are disjoint
- $P(E \setminus E) = 0$
- $P(E \cup F) = 0.7$

Exercise 6.2

A driving licence examination consists of two parts which are based on a theoretical and a practical examination. Suppose 25 % of people fail the practical examination, 15 % of people fail the theoretical examination, and 10 % of people fail both the examinations. If a person is randomly chosen, then what is the probability that this person

- $P(FP) = 0.25$
- $P(FT) = 0.15$
- $P(FP \cap FT) = 0.10$

a. fails at least one of the examinations? $P(FP \cup FT) = P(FP) + P(Ft) - P(FP \cap FT) = 0.25 + 0.15 - 0.10 = 0.3$

b. only fails the practical examination, but not the theoretical examination? $P(FP) \setminus P(FT) = P(FP) - P(FP \cap FT) = 0.25 - 0.10 = 0.15$

c.successfully passes both the tests? $P(\overline{FP \cup FT}) = 1 - P(FP \cup FT) = 1 - 0.3 = 0.7$

d. fails any of the two examinations? $P(FO) = P(FP \cup FT) - P(FP \cap FT) = 0.3 - 0.1 = 0.2$

Exercise 6.3

A new board game uses a twelve-sided die. Suppose the die is rolled once, what is the probability of getting

$\Omega = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ $A_{even} = \{2, 4, 6, 8, 10, 12\}$ $B = \{10, 11, 12\}$ $C = \{10, 12\}$ $A \cup B = \{2, 4, 6, 8, 10, 11, 12\}$

a. an even number? $P(A_{even}) = \frac{6}{12} = 0.5$

b. a number greater than 9? $P(B) = \frac{3}{12}$

c. an even number greater than 9? $P(C) = \frac{2}{12}$

d. an even number or a number greater than 9? $P(A \cup B) = \frac{7}{12}$

Exercise 6.4

The Smiths are a family of six. They are celebrating Christmas and there are 12 gifts, two for each family member. The name tags for each family member have been attached to the gifts. Unfortunately the name tags on the gifts are damaged by water. Suppose each family member draws two gifts at random. What is the probability that someone

a. gets his/her two gifts, rather than getting the gifts for another family member? $|\Omega| = \binom{12}{2}$

```
Omega <- choose(12, 2)
```

$\frac{1}{66} = 66$

b. gets none of his/her gifts, but rather gets the gifts for other family members? $P(B) = \frac{\binom{10}{2}}{\binom{12}{2}} =$

0.6818182

To get the possible events we have to remove the "correct" states of the gift's distribution

Exercise 6.5

A chef from a popular TV cookery show sometimes puts too much salt in his pumpkin soup and the probability of this happening is 0.2. If he is in love (which he is with probability 0.3), then the probability of using too much salt is 0.6. $P(MS) = 0.2$ $P(\overline{MS}) = 0.8$ $P(IL) = 0.3$ $P(\overline{IL}) = 1 - P(IL) = 0.7$ $P(MS|IL) = 0.6$ $P(MS \cap IL) = P(MS|IL) \cdot P(IL) = 0.6 * 0.3 = 0.18$ $P(\overline{MS} \cap IL) = P(IL) - P(MS \cap IL) = 0.3 - 0.18 = 0.12$ $P(MS \cap \overline{IL}) = P(MS) - P(MS \cap IL) = 0.2 - 0.18 = 0.02$ $P(\overline{MS} \cap \overline{IL}) = P(\overline{MS}) - P(\overline{MS} \cap IL) = 0.8 - 0.12 = 0.68$

```
data_exercise65 <- data.frame(much_salt = c(0.18, 0.02),
                              not_much_salt = c(0.12, 0.68),
                              row.names = c("In love", "not In Love"))
data_exercise65
```

a. Create a contingency table for the probabilities of the two variables "in love" and "too much salt".

```
##              much_salt not_much_salt
## In love           0.18          0.12
## not In Love       0.02          0.68
```

b. Determine whether the two variables are stochastically independent or not. Given $P(A \cap B) = P(A)P(B)$, two events are independent. But, $P(MS \cap IL) = 0.18 \neq P(MS) \cdot P(IL) = 0.2 \cdot 0.3 = 0.06$ so, they are not independent

Exercise 6.6

Dr. Obermeier asks his neighbour to take care of his basil plant while he is away on leave. He assumes that his neighbour does not take care of the basil with a probability of $\frac{1}{3}$. The basil dies with probability $\frac{1}{2}$ when someone takes care of it and with probability $\frac{3}{4}$ if no one takes care of it. $P(NTCare) = \frac{1}{3}$

$P(\overline{NTCare}) = \frac{2}{3}$

$P(Death|\overline{NTCare}) = \frac{1}{2}$

$P(Death|NTCare) = \frac{3}{4}$

a. Calculate the probability of the basil plant surviving after its owner's leave. $P(B) = \sum_{i=1}^{2} P(B|A_i)P(A_i)$

$P(Death) = P(Death|\overline{NTCare}) \cdot P(\overline{NTCare}) + P(Death|NTCare) \cdot P(NTCare)$

$P(Death) = \left(\frac{1}{2}\right) \cdot \left(\frac{2}{3}\right) + \left(\frac{3}{4}\right) \cdot \left(\frac{1}{3}\right)$

```
pdeath <- ((1/2) * (2/3)) + ((3/4*(1/3)))
```

$P(Death) = 0.5833333$

$P(\overline{Death}) = 1 - 0.5833333 = 0.4166667$

b. It turns out that the basil eventually dies. What is the probability that Dr. Obermeier's neighbour did not take care of the plant?

$$P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum_i P(B|A_i)P(A_i)}$$

$P(NTCare) = \dfrac{P(Death|NTCare) \cdot P(NTCare)}{P(Death|\overline{NTCare}) \cdot P(\overline{NTCare}) + P(Death|NTCare) \cdot P(NTCare)}$

$P(NTCare) = \dfrac{\left(\frac{3}{4}\right) \cdot \left(\frac{1}{3}\right)}{\left(\frac{1}{2}\right) \cdot \left(\frac{2}{3}\right) + \left(\frac{3}{4}\right) \cdot \left(\frac{1}{3}\right)}$

$P(NTCare) = 0.4285714$

Exercise 6.7

A bank considers changing its credit card policy. Currently 5 % of credit card owners are not able to pay their bills in any month, i.e. they never pay their bills. Among those who are generally able to pay their bills, there is still a 20 % probability that the bill is paid too late in a particular month.

$P(NeverPay) = 0.05$

$P(\overline{NeverPay}) = 0.95$

$P(Late|\overline{NeverPay}) = 0.20$

$P(Late|NeverPay) = 1$

a. What is the probability that someone is not paying his bill in a particular month? $P(B) = \sum_{i=1}^{m} P(B|A_i)P(A_i)$

$P(Late) = P(Late|NeverPay) \cdot P(NeverPay) + P(Late|\overline{NeverPay}) \cdot P(\overline{NeverPay})$

$P(Late) = 0.24$

b. A credit card owner did not pay his bill in a particular month. What is the probability that he never pays back the money? $P(NeverPay|Late) =?$

$$P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum_i P(B|A_i)P(A_i)}$$

$P(NeverPay|Late) = \dfrac{P(Late|NeverPay) \cdot P(NeverPay)}{P(Late|NeverPay) \cdot P(NeverPay) + P(Late|\overline{NeverPay}) \cdot P(\overline{NeverPay})}$ $P(NeverPay|Late) = \dfrac{1 * 0.05}{1 * 0.05 + 0.20 * 0.95} = 0.2083333$

c. Should the bank consider blocking the credit card if a customer does not pay his bill on time? If we consider the probability of someone never pay, if he/she does not pay on time (20%), it is lower compared with the rest probability. So, if that bank block to all its customer will lose a lot of them, even though they could pay in the future.

Exercise 6.8

There are epidemics which affect animals such as cows, pigs, and others. Suppose 200 cows are tested to see whether they are infected with a virus or not. Let event A describe whether a cow has been transported by a truck recently or not and let B denote the event that a cow has been tested positive with a virus. The data are summarized in the following table:

.	B	$\bar{B}$	Total
A	40	60	100
$\bar{A}$	20	80	100
Total	60	140	200

a. What is the probability that a cow is infected and has been transported by a truck recently?
$P(A \cap B) = \dfrac{40}{200} = 0.2$

b. What is the probability of having an infected cow given that it has been transported by the truck? $P(B|A) = \dfrac{P(A \cap B)}{P(A)}$

$$P(Infected|Truck) = \frac{40/200}{100/200} = 0.4$$

c. Determine and interpret P(B). $P(Truck) = P(Truck|Infected) \cdot P(Infected) + P(Truck|\overline{Infected}) \cdot P(\overline{Infected})$

$$P(Truck) = \frac{0.2}{0.5} \cdot 0.5 + \frac{0.1}{0.5} \cdot 0.5 = 0.3$$

That means the there are 0.3 % of probabilities of a cow being infected.

Exercise 6.9

A football practice target is a portable wall with two holes (which are the target) in it for training shots. Suppose there are two players A and B. The probabilities of hitting the target by A and B are 0.4 and 0.5, respectively. $P(A) = 0.4$ $P(B) = 0.5$

a. What is the probability that at least one of the players succeeds with his shot? If we consider independence: $P(A \cap B) = P(A)P(B) = 0.2$ $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.4 + 0.5 - 0.2 = 0.7$

b. What is the probability that exactly one of the players hits the target? $P(A \setminus B \cup B \setminus A) = 0.4 - 0.2 + 0.5 - 0.2 = 0.5$

c. What is the probability that only B scores? $P(B \setminus A) = 0.5 - 0.2 = 0.3$