Random Variables

Abel Isaias Gutierrez-Cruz

29/11/2021

Exercise 7.1

Consider the following cumulative distribution function of a random variable X :

$$F(x) = \begin{cases} 0 & if \;\; x < 2 \\ -\dfrac{1}{4}x^2 + 2x - 3 & if \;\; 2 \leq x \leq 4 \\ 1 & if \;\; x > 4 \end{cases}$$

a. What is the PDF of X ? $\frac{d}{dx}0 = 0$

$\frac{d}{dx} - \dfrac{1}{4}x^2 + 2x - 3 = -\frac{1}{2}x + 2$

$\frac{d}{dx}1 = 0$

$$\therefore F'(X) = \begin{cases} 0 & if \;\; x < 2 \\ -\dfrac{1}{2}x + 2 & if \;\; 2 \leq x \leq 4 \\ 0 & if \;\; x > 4 \end{cases}$$

b. Calculate $\mathbf{P(X < 3)}$ and $\mathbf{P(X = 4)}$.

- Given theorem $P(X = x_0) = 0$ $P(X = 4) = 0$
- $P(X < 3) = P(X \leq 3) - P(X = 3) = -\dfrac{1}{4} * 3^2 + 2 * 3 - 3 - 0 = 0.75$

c. Determine $E(X)$ and $Var(X)$ $E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{2} xf(x)dx + \int_{2}^{4} xf(x)dx + \int_{4}^{\infty} xf(x)dx$

$$= 0 + \int_2^4 (-\frac{1}{2}x^2 + 2x)dx + 0 = \left[-\frac{x^3}{6} + x^2\right]_2^4 = (-\frac{4^3}{6} + 4^2) - (-\frac{2^3}{6} + 2^2) = \frac{8}{3} = 2.6666667$$

$Var(X) = E(X^2) - [E(X)]^2$

$$E(X^2) = \int_2^4 (-\frac{1}{2}x^3 + 2x^2)dx = \left[-\frac{x^4}{8} + \frac{2}{3}x^3\right]_2^4 = (-\frac{4^4}{8} + \frac{2}{3}4^3) - (-\frac{2^4}{8} + \frac{2}{3}2^3) = \frac{22}{3} = 7.3333333$$

$Var(X) = \frac{22}{3} - (\frac{8}{3})^2 = \frac{2}{9}$

Exercise 7.2

Joey manipulates a die to increase his chances of winning a board game against his friends. In each round, a die is rolled and larger numbers are generally an advantage. Consider the random variable X denoting the outcome of the rolled die and the respective probabilities $P(X = 1 = 2 = 3 = 5) = 1/9$, $P(X = 4) = 2/9$, and $P(X = 6) = 3/9$

a. Calculate and interpret the expectation and variance of X . $E(X) = \sum_{i=1}^{k} x_i p_i = x_1 P(X = x_1) + x_2 P(X = x_2) + ... + x_k P(X = x_k)$

$E(X) = \sum_{i=1}^{6} x_i p_i = (1+2+3+5)\frac{1}{9} + 4 \cdot \frac{2}{9} + 6 \cdot \frac{3}{9} = \frac{37}{9} \approx 4.1111111$

$E(X^2) = \sum_{i=1}^{6} x_i p_i = (1^2+2^2+3^2+5^2)\frac{1}{9} + 4^2 \cdot \frac{2}{9} + 6^2 \cdot \frac{3}{9} = \frac{179}{9} \approx 19.8888889$

$Var(X) = E(X^2) - [E(X)]^2$

$Var(X) = \frac{179}{9} - (\frac{37}{9})^2 = \frac{241}{81} \approx 2.9753086$

When the players rolling the dies, they will have in average the following result: 4.11. Therefore, they will have higher number than with a normal die. On the other hand, the variance was not modified.

b. Imagine that the board game contains an action which makes the players use $1/X$ rather than X . What is the expectation of $Y = 1/X$? Is $E(Y) = E(1/X) = 1/E(X)$? Given the prior instructions: $P(X = \frac{1}{1} = \frac{1}{2} = \frac{1}{3} = \frac{1}{5}) = 1/9$, $P(X = \frac{1}{4}) = 2/9$, and $P(X = \frac{1}{6}) = 3/9$

$E(Y) = \sum_{i=1}^{6} x_i p_i = (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5})\frac{1}{9} + \frac{1}{4} \cdot \frac{2}{9} + \frac{1}{6} \cdot \frac{3}{9} = \frac{91}{270} \approx 0.337037$

If we consider the part a), we will realize that both expression are different $\frac{1}{E(X)} = \frac{37}{9} \neq \frac{91}{270}$

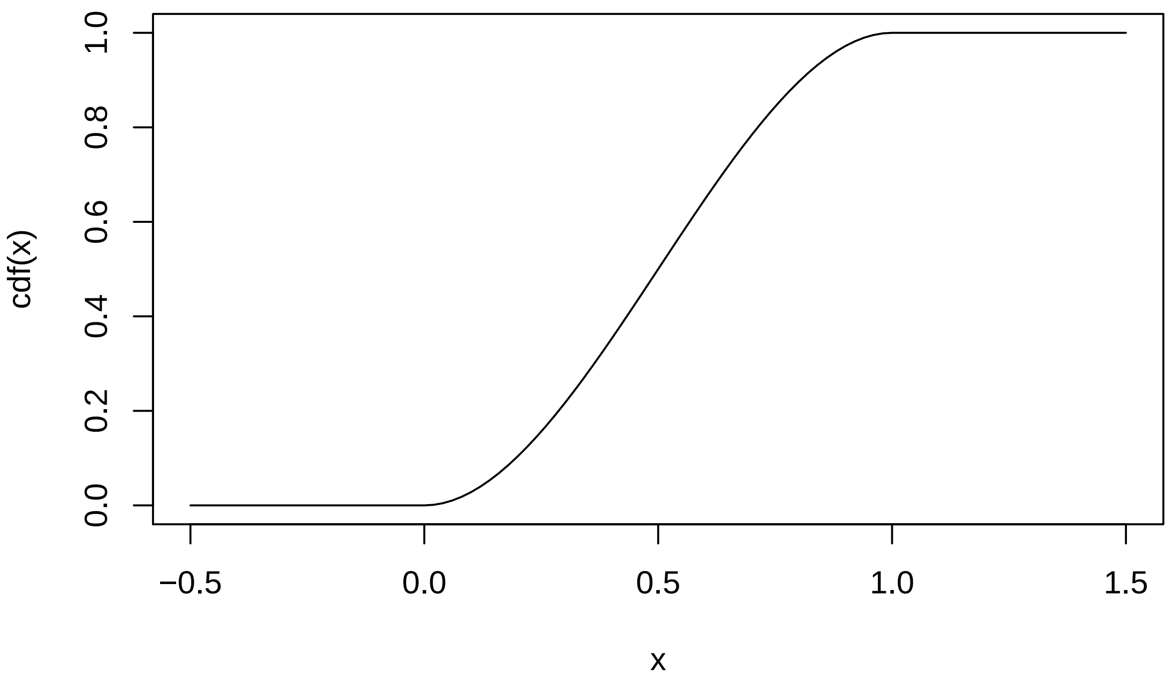
Exercise 7.3

An innovative winemaker experiments with new grapes and adds a new wine to his stock. The percentage sold by the end of the season depends on the weather and various other factors. It can be modeled using the random variable X with the CDF as

$$F(x) = \begin{cases} 0 & if \;\; x < 0 \\ 3x^2 - 2x^3 & if \;\; 0 \leq x \leq 1 \\ 1 & if x > 1 \end{cases}$$

a. Plot the cumulative distribution function with R. To consider the behavior in each interval of the function, we multiply each behavior of the data by a logical condition.

```
cdf <- function(x){(3*x^2 - 2*x^3) * (x >= 0 & x <= 1) + 1*(x>1) + 0*(x<0)}
curve(cdf, from=-0.5, to=1.5)
```



b. Determine $f(x)$ $\frac{d}{dx}F(x) = (3x^2 - 2x^3)d/dx = 6x - 6x^2$

$$\therefore f(x) = \begin{cases} 6(x - x^2) & if \;\; 0 \leq x \leq 1 \\ 0 & elsewhere \end{cases}$$

c. What is the probability of selling at least one-third of his wine, but not more than two thirds? $F(\frac{2}{3}) - F(\frac{1}{3}) = \left(3(\frac{2}{3})^2 + 2(\frac{2}{3})^3\right) - \left(3(\frac{1}{3})^2 + 2(\frac{1}{3})^3\right) = 0.4814815$

```
cdf(2/3) - cdf(1/3)
```

d. Define the CDF in R and calculate the probability of c) again.

```
## [1] 0.4814815
```

e. What is the variance of X? $E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{0} xf(x)dx + \int_{0}^{1} xf(x)dx + \int_{1}^{\infty} xf(x)dx =$

$$0 + \int_0^1 xf(x)dx + 0 = 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 6\left(\frac{1}{3} - \frac{1}{4}\right) = 0.5$$

$$E(X^2) = 0 + \int_0^1 x^2 f(x)dx + 0 = 6\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = 6\left(\frac{1}{4} - \frac{1}{5}\right) = 0.3$$

$$Var(X) = E(X^2) - [E(X)]^2 = 0.3 - 0.5^2 = 0.05$$

Exercise 7.4

A quality index summarizes different features of a product by means of a score. Different experts may assign different quality scores depending on their experience with the product. Let X be the quality index for a tablet. Suppose the respective probability density function is given as follows:

$$f(x) = \begin{cases} cx(2-x) & if\ \ 0 \le x \le 2 \\ 0 & elsewhere \end{cases}$$

a. Determine c such that $f(x)$ is a proper PDF. Conditions:

- $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_0^2 c \cdot x(2-x)dx = c\int_0^2 2x - x^2 = c \cdot \left[x^2 - \frac{x^3}{3}\right]_0^2 = c \cdot \frac{4}{3}$$

$c \cdot \frac{4}{3} = 1$

$c = \frac{3}{4}$

- $f(x) \ge 0\ \ for\ \ all\ \ x \in R$

$f(x) = \frac{3}{4}x(2-x) \ge 0\ \ \forall\ \ x \in [0,2]$

b. Determine the cumulative distribution function. $F(X) = \displaystyle\int_0^x f(t)dt = \int_0^x \frac{3}{4}x(2-x)dt = \frac{3}{4}\int_0^x 2t - t^2$

$$\frac{3}{4}(t^2 - \frac{t^3}{3}) = \frac{3}{4}x^2(1 - \frac{x}{3})$$

$$\therefore F(X) = \begin{cases} 0 & if\ \ x < 0 \\ \frac{3}{4}x^2(1-\frac{x}{3}) & if\ \ 0 \le x \le 2 \\ 1 & if\ \ x > 2 \end{cases}$$

c. Calculate the expectation and variance of X . $E(X) = \displaystyle\int_{-\infty}^{\infty} xf(x)dx = \frac{3}{4}\int_0^2 (2x^2 - x^3)dx = \frac{3}{4} \cdot \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 =$

$$\frac{3}{4} \cdot \left(\frac{16}{3} - 4\right) = \frac{3}{4} \cdot \frac{4}{3} = 1$$

$$E(X^2) = \int_{-\infty}^{\infty} xf(x)dx = \frac{3}{4} \cdot \left[\frac{2x^4}{4} - \frac{x^5}{5}\right]_0^2 = \frac{3}{4} \cdot \left(\frac{32}{4} - \frac{32}{5}\right) = \frac{6}{5}$$

$$Var(X) = \frac{6}{5} - 1^2 = \frac{1}{5}$$

d. Use Tschebyschev's inequality to determine the probability that X does not deviate more than 0.5 from its expectation. $P(|X-\mu| < 0.5) \ge 1 - \frac{\frac{1}{5}}{0.5^2} = 1 - 0.8 = 0.2$

Exercise 7.5

Consider the joint PDF for the type of customer service $X(0 = telephonic\ \ hotline,\ \ 1 = Email)$ and of satisfaction score $Y(1 = unsatisfied,\ \ 2 = satisfied,\ \ 3 = very\ \ satisfied)$:

X/Y	1	2	3
0	0	1/2	1/4
1	1/6	1/12	0

a. Determine and interpret the marginal distributions of both X and Y

X	$P(X = x_i)$
0	3/4
1	1/4

Y	$P(Y = y_i)$
1	1/6
2	7/12
3	1/4

b. Calculate the 75 % quantile for the marginal distribution of Y The 75 % have to follow:

$F(x_p) \geq p\ F(x) < p\ \ for\ \ x < x_p$

If we consider those properties, we will found that the third quartile is $X = 2$ $F(x = 2) = 1/6 + 7/12 = 9/12$ $F(x = 2) \geq 0.75$

c. Determine and interpret the conditional distribution of satisfaction level for $X = 1$. The conditional distribution is given as follows: $P(Y = y_i|X = 1) = p_{j|i} = \dfrac{p_{ij}}{p_{1+}} = \dfrac{p_{ij}}{1/4}$

$P(Y = 1|X = 1) = \frac{1/6}{1/4} = 2/3$

$P(Y = 2|X = 1) = \frac{1/12}{1/4} = 1/3$

$P(Y = 3|X = 1) = \frac{0}{1/4} = 0$

The customers using email like customer services are more uncomfortable. The most of them have a level of satisfaction poor (they are unsatisfied)

d. Are the two variables independent? To be independent they have to follow: $P(X = x_i, Y = y_j) = P(X = x_i)P(Y = y_i)$

$P(X = 0, Y = 1) = 3/4 \times 1/6 \neq 0$

e. Calculate and interpret the covariance of X and Y . $Cov(X, Y) = E(XY) - E(X)E(Y)$

$E(XY) = \sum_i \sum_j x_i y_i p_{ij}$

$$E(X) = \sum_{i=1}^{k} x_i p_i = x_1 P(X = x_1) + x_2 P(X = x_2) + ... + x_k P(X = x_k)$$

$E(X) = 0 \cdot 3/4 + 1 \cdot 3/12 = 1/4$

$E(Y) = 1 \cdot 1/6 + 2 \cdot 7/12 + 3 \cdot 1/4 = \frac{25}{12}$

$E(XY) = 0 \cdot 1 \cdot 0 + 0 \cdot 2 \cdot 1/2 + 0 \cdot 3 \cdot 1/4 + 1 \cdot 1 \cdot 1/6 + 1 \cdot 2 \cdot 1/12 + 1 \cdot 3 \cdot 0 = \frac{2}{6}$

$Cov(X, Y) = \frac{2}{6} - \frac{25}{12} \cdot \frac{1}{4} =$ -0.1875

The value of covariance between X and Y is negative, so we could say that high values of X have low values of Y.

Exercise 7.6

Consider a continuous random variable X with expectation 15 and variance 4. Determine the smallest interval $[15c, 15 + c]$ which contains at least 90 % of the values of X .

$P(|X - 15| < c) = 0.9 \geq 1 - \frac{4}{c^2}$

$c = \sqrt{\frac{4}{0.1}}$

$c = \pm$ 6.3245553

[15$-$ 6.3245553 , 15+ 6.3245553]

Exercise 7.7

Let X and Y be two random variables for which only 6 possible events— $A1, A2, A3, A4, A5, A6$—are defined:

	i	1	2	3	4	5	6
$P(A_i)$	0.3	0.1	0.1	0.2	0.2	0.1	
X_i	-1	2	2	-1	-1	2	
Y_i	0	2	0	1	2	1	

a. What is the joint PDF of X and Y?

	X /Y	0	1	2
-1	0.3	0.2	0.2	
2	0.1	0.1	0.1	

b. Calculate the marginal distributions of X and Y $P(X_{-1}) = 0.7$

$P(X_2) = 0.3$

$P(Y_0) = 0.4$

$P(Y_1) = 0.3$

$P(Y_2) = 0.3$

c. Are both variables independent? To be independent they have to follow: $P(X = x_i, Y = y_j) = P(X = x_i)P(Y = y_i)$

$P(X = x_{-1}, Y = y_0) = 0.7 \times 0.4 \neq 0.3$

d. Determine the joint PDF for $U = X + Y$ -1 | 0 | 1 | 2 | 3 | 4

P(U) | 0.3 | 0.2 | 0.2 | 0.1 | 0.1 | 0.1

e. Calculate $E(U)$ and $Var(U)$ and compare it with $E(X)+E(Y)$ and $Var(X)+Var(Y)$, respectively.
$E(U) = -1 \cdot 0.3 + 0 \cdot 0.2 + 1 \cdot 0.2 + 2 \cdot 0.1 + 3 \cdot 0.1 + 4 \cdot 0.1 = 0.8$

$E(U^2) = (-1)^2 \cdot 0.3 + 0^2 \cdot 0.2 + 1^2 \cdot 0.2 + 2^2 \cdot 0.1 + 3^2 \cdot 0.1 + 4^2 \cdot 0.1 = 3.4$

$Var(U) = E(X^2) - [E(X)]^2 = 3.4 - 0.8^2 = 2.76$

$E(X) = -1 \cdot 0.7 + 2 \cdot 0.3 =$ -0.1

$E(X^2) = -1^2 \cdot 0.7 + 2^2 \cdot 0.3 = 1.9$

$Var(X) = E(X^2) - [E(X)]^2 = 1.89$

$E(Y) = 0 \cdot 0.4 + 1 \cdot 0.3 + 2 \cdot 0.3\ 0.9$

$E(Y^2) = 0^2 \cdot 0.4 + 1^2 \cdot 0.3 + 2^2 \cdot 0.3\ 1.5$

$Var(Y) = E(Y^2) - [E(Y)]^2 = 0.69$