Mutation of Quivers with Potential and Dimer Models

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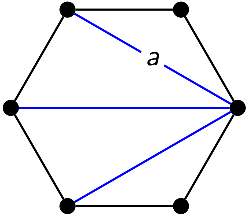
Outline

Triangulations of n-gon

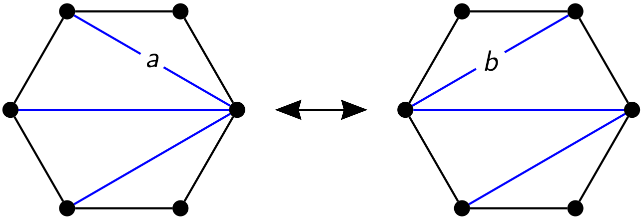
In this talk, a **triangulation** of an n-gon is a division of the n-gon into triangles by some choice of non-intersecting diagonals along with the sides of the n-gon.



These are special cases of **ideal triangulations** as described by Fomin, Shapiro, and Thurston in [3].

Flipping of a Triangulation

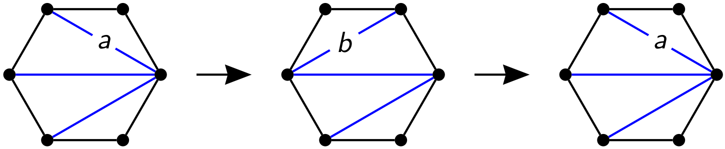
We can **flip** a diagonal of the triangulation by removing a diagonal and replacing it with the other diagonal of the resulting quadrilateral.



In the above, we flip the triangulation on the left by exchanging the diagonal a for the diagonal b.

Quick Properties of Flips

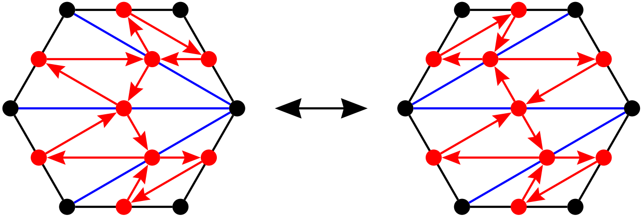
1. Flips are involutive (self-inverse):



2. Number of edges invariant under flips

Adding the Quiver

We associate a **quiver** (directed multigraph) to the triangulation by placing vertices and arrows so that for each triangle there is a clockwise 3-cycle between its edges:

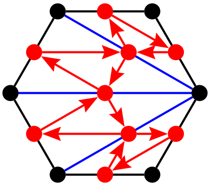


Notice that the two triagulations differ only in the top half of the hexagon (i.e. effects of flip on quiver are "local")

Adding the Quiver (cont.)

Definition

A **quiver** $Q = (Q_0, Q_1, s, t)$ where Q_0 is vertex set, Q_1 is the arrow set, $s : Q_1 \to Q_0$ gives the source vertex of an arrow, $t : Q_1 \to Q_0$ gives the target vertex of an arrow.



Definition

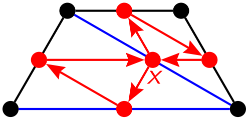
A **cluster quiver** is a quiver without 2-cycles and loops.

Generalizing Flips with Quiver Mutation

Quiver mutation at vertex x of cluster quiver Q gives new quiver $\mu_x(Q)$:

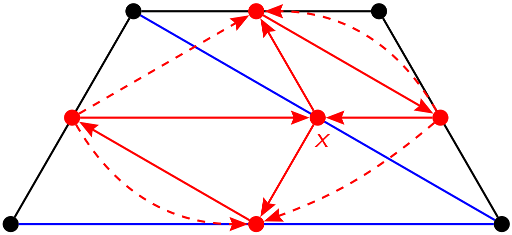
1. For each path $v \to x \to w$, add a new arrow $v \to w$
2. Reverse orientation of arrows incident to x
3. Remove any pairs of arrows forming 2-cycles. Repeat until no more 2-cycles are left

Demonstrate on the top half of the triangulation from last slide by mutating the center vertex of the quiver:



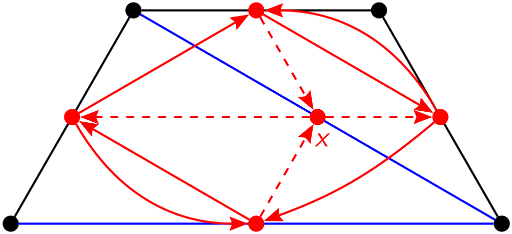
Example of Quiver Mutation

1. For each path $v \to x \to w$, add a new arrow $v \to w$.



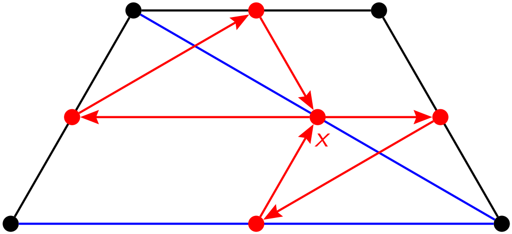
Example of Quiver Mutation (cont.)

2. Reverse orientation of arrows incident to x



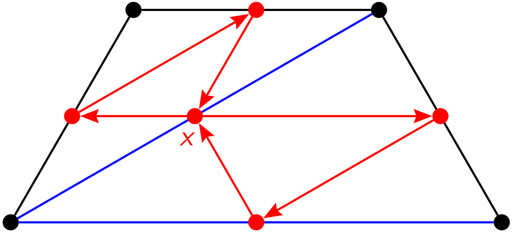
Example of Quiver Mutation (still cont.)

3. Remove any pairs of arrows forming 2-cycles. Repeat until no more 2-cycles left



Example of Quiver Mutation (still cont.)

Also update triangulation so it's consistent with quiver:

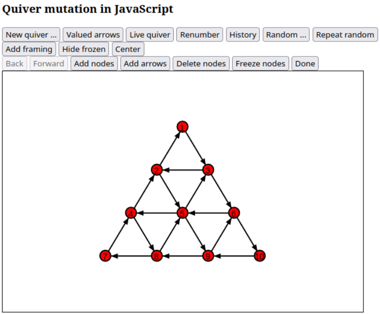


Flips of triangulation induce quiver mutation (at 4-valent vertices) in associated quiver

- See *Introduction to Cluster Algebras* by Fomin, Williams, and Zelevinsky [4] for alternative approach with **frozen** and **mutable** vertices

Mutate your own Quivers!

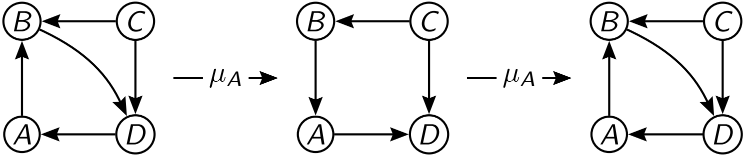
Dr. Bernhard Keller has a quiver mutation program [5] on his website that you can play with:



Google **"Keller Mutation App"**

Properties of Quiver Mutation

1. Quiver mutation is involutive: $\mu_A(\mu_A(Q)) = Q$



2. Number of vertices of quiver invariant under mutation

Dimer Models

Definition

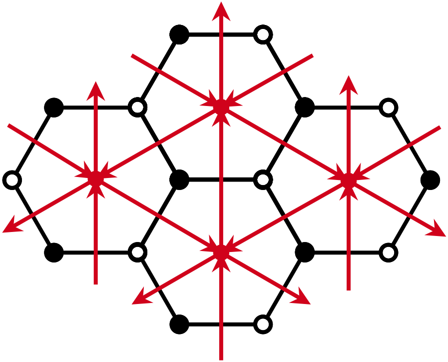
A **dimer model** is a finite bipartite graph embedded into a compact oriented surface whose nodes and edges form a polygonal cell decomposition of that surface.

- Actively studied in theoretical physics: connections to mirror symmetry
- Interesting source of quivers

Dimer Quiver

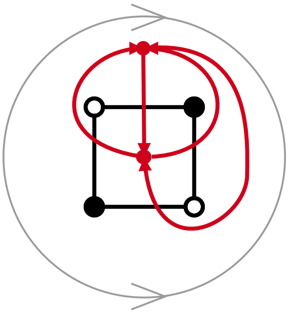
The **dimer quiver** of a dimer model is obtained by:

- Taking the dual of the dimer model and
- Orienting the faces of the dual clockwise about white vertices and counterclockwise about black vertices



Dimer Quiver (cont.)

Many dimer quivers are not cluster quivers. Example on sphere:



Multiple 2-cycles above. Mutating non-cluster quivers not allowed!

Quivers with Potential

Potential of a dimer quiver Q is the formal quantity [1]

$$W = \sum(\text{cycles of } Q \text{ about } \bullet) - \sum(\text{cycles of } Q \text{ about } \circ).$$

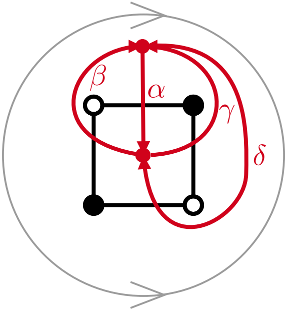
Cycles are determined up to cyclical equivalence: *abc* is equivalent to *bca*.

- Rigorously defined using path algebras: W is an element of the quotient space $\mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$ where $[\mathbb{C}Q, \mathbb{C}Q]$ is the subspace of $\mathbb{C}Q$ spanned by commutators.
  - A way for representation theory to enter the picture! We will ignore this perspective entirely, though :(

The pair (Q, W) is called a **quiver with potential** [2].

Example

Example of computing potential of following dimer on sphere:



Here $W = \alpha\gamma + \beta\delta - \alpha\beta - \gamma\delta$

Mutation of Quivers with Potential

Recall steps of quiver mutation:

1. For each path $v \to x \to w$, add a new arrow $v \to w$
2. Reverse orientation of arrows incident to x
3. Remove any pairs of arrows forming 2-cycles. Repeat until no more 2-cycles are left

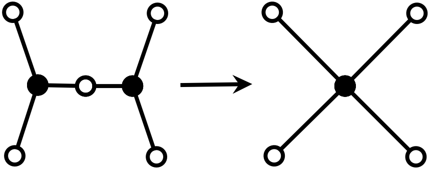
To **mutate** (Q, W), modify step 1 to add newly created cycles to W and step 3 to only delete 2-cycles that are in W. The resultant quiver is denoted $\mu_x(Q, W)$.

This modified variant is known as **QP mutation** as described by Derksen, Weyman, and Zelevinsky [2]. The potential allows us to control which 2-cycles are allowed to be deleted and which are left alone.

Join Moves

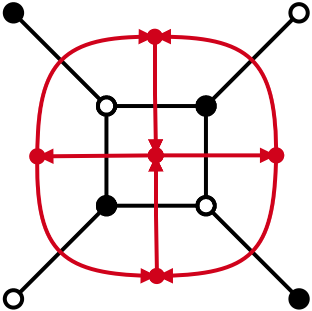
Here, we introduce the dimer equivalent of deleting 2-cycles from the quiver.

A **join move** removes a degree-2 (bivalent) node of a dimer model and joins the two nodes connected to it.



Dimer Models and QP Mutation

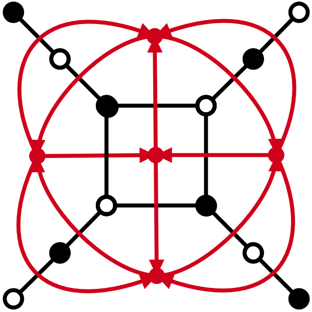
We motivate (a special case of) dimer model version of QP mutation using QP mutation.



We will do QP mutation on the center vertex of the quiver.

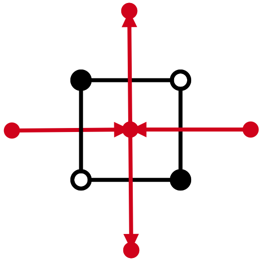
Dimer Models and QP Mutation (cont.)

Steps 1 and 2 of QP mutation:



Dimer Models and QP Mutation (cont.)

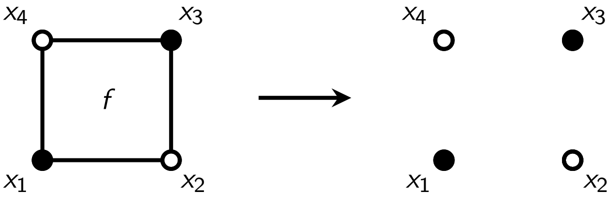
Step 3 of QP mutation corresponds to join moves:



General n-face Urban Renewal

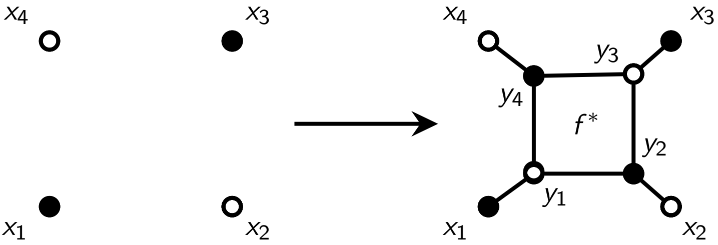
Let Γ be a dimer model with no bivalent nodes. **General n-face urban renewal** of Γ at the n-face f with nodes $x_1, \ldots, x_n$ consists of the following steps:

1. Remove the edges of f from Γ



General n-face Urban Renewal (cont.)

2. Add an n-cycle with distinct nodes $y_1, \ldots, y_n$ such that each y_i has opposite color of x_i for each i and add edges $x_i y_i$ for each i.



3. Apply a join move at each x_i that is bivalent

Properties of General n-face Urban Renewal

Can be thought of as a generalization (loosely speaking) of traditional urban renewal ($n = 4$ case) [4, Section 2.5]

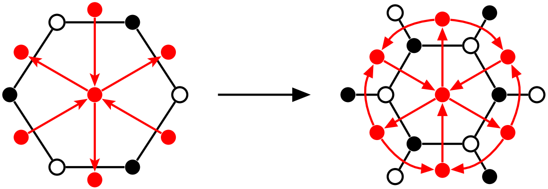
Theorem

General n-face urban renewal ρ_f at face f has the following properties:

1. *It is an involution: $\rho_{f^*}(\rho_f(\Gamma)) \cong \Gamma$*
2. *Number of faces invariant (i.e. does not annihilate vertices of dimer quiver)*
3. *Effect on dimer quiver coincides with QP mutation when $n = 4$*

Future Research Possibilities?

1. General n-face urban renewal corresponds to QP mutation for the special case $n = 4$. What properties arise when $n \neq 4$?



2. Formulating general n-face urban renewal in terms of a ribbon graph framework. Avoids "picture-proofs" and some topology pathologies [1]

3. Connections to Postnikov diagrams and plabic graphs (and possibly positroids)?

Acknowledgements

1. Dr. Alex Dugas for introducing presenter to quivers and dimer models

2. University of the Pacific Summer Undergraduate Research Fellowships program for funding

3. Pi Mu Epsilon for funding expenses

References I

[1] Raf Bocklandt. *A Dimer ABC*. arXiv.org. Oct. 14, 2015. DOI: 10.1112/blms/bdv101. URL: https://arxiv.org/abs/1510.04242v1 (visited on 05/31/2023).

[2] Harm Derksen, Jerzy Weyman, and Andrei Zelevinsky. *Quivers with Potentials and Their Representations I: Mutations*. arXiv.org. Apr. 4, 2007. URL: https://arxiv.org/abs/0704.0649v4 (visited on 05/31/2023).

[3] Sergey Fomin, Michael Shapiro, and Dylan Thurston. *Cluster Algebras and Triangulated Surfaces. Part I: Cluster Complexes*. arXiv.org. Aug. 15, 2006. URL: https://arxiv.org/abs/math/0608367v3 (visited on 05/31/2023).

References II

[4] Sergey Fomin, Lauren Williams, and Andrei Zelevinsky. *Introduction to Cluster Algebras. Chapters 1-3*. Aug. 29, 2021. DOI: 10.48550/arXiv.1608.05735. arXiv: 1608.05735 [math]. URL: http://arxiv.org/abs/1608.05735 (visited on 11/05/2023). preprint.

[5] Bernhard Keller. *Mutation App*. URL: https://webusers.imj-prg.fr/~bernhard.keller/quivermutation/.

Thanks for Listening!

The slides can be found on my GitHub:

`https://github.com/ItzSomebody/jmm2024-slides`