# Quant Analysis of Financial Time Series

Maria Culjak

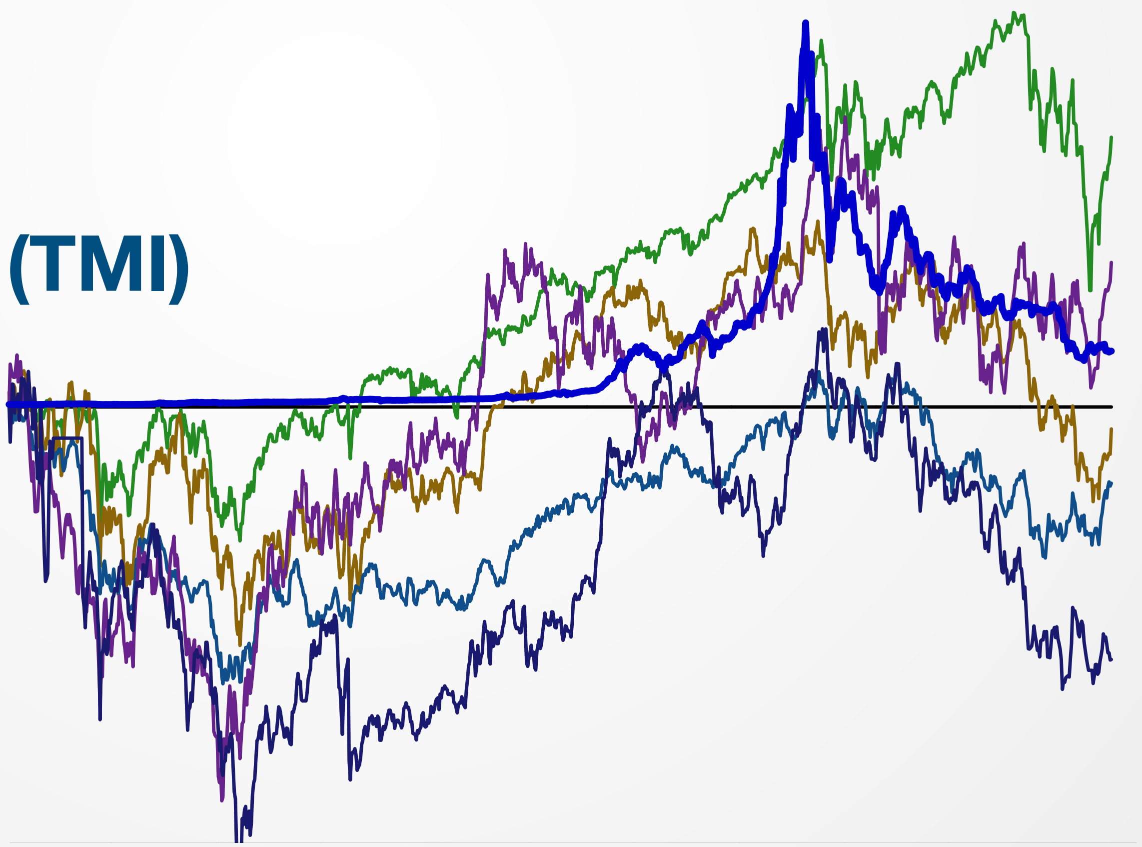
Wolfgang Karl Härdle

Ladislaus von Bortkiewicz Professor of Statistics
Humboldt-Universität zu Berlin
BRC Blockchain Research Center
lvb.wiwi.hu-berlin.de
Charles University, WISE XMU, NCTU 玉山学者

# Quant's daily question

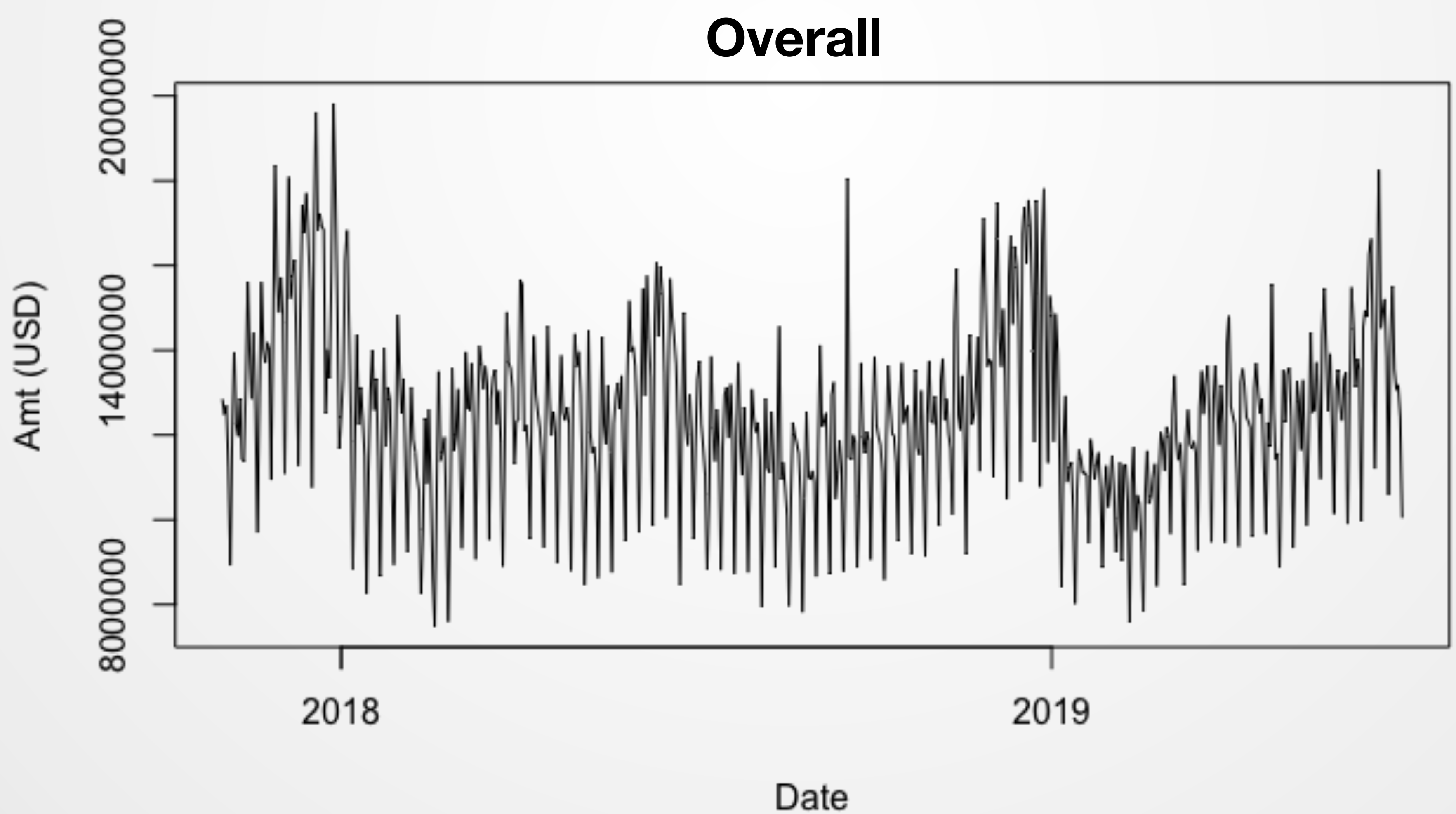
- How well does a certain model describe the data?
- Does the model help to predict the future development?
- How and when does one need recalibration of model/data?

**Total Market Index (TMI)**

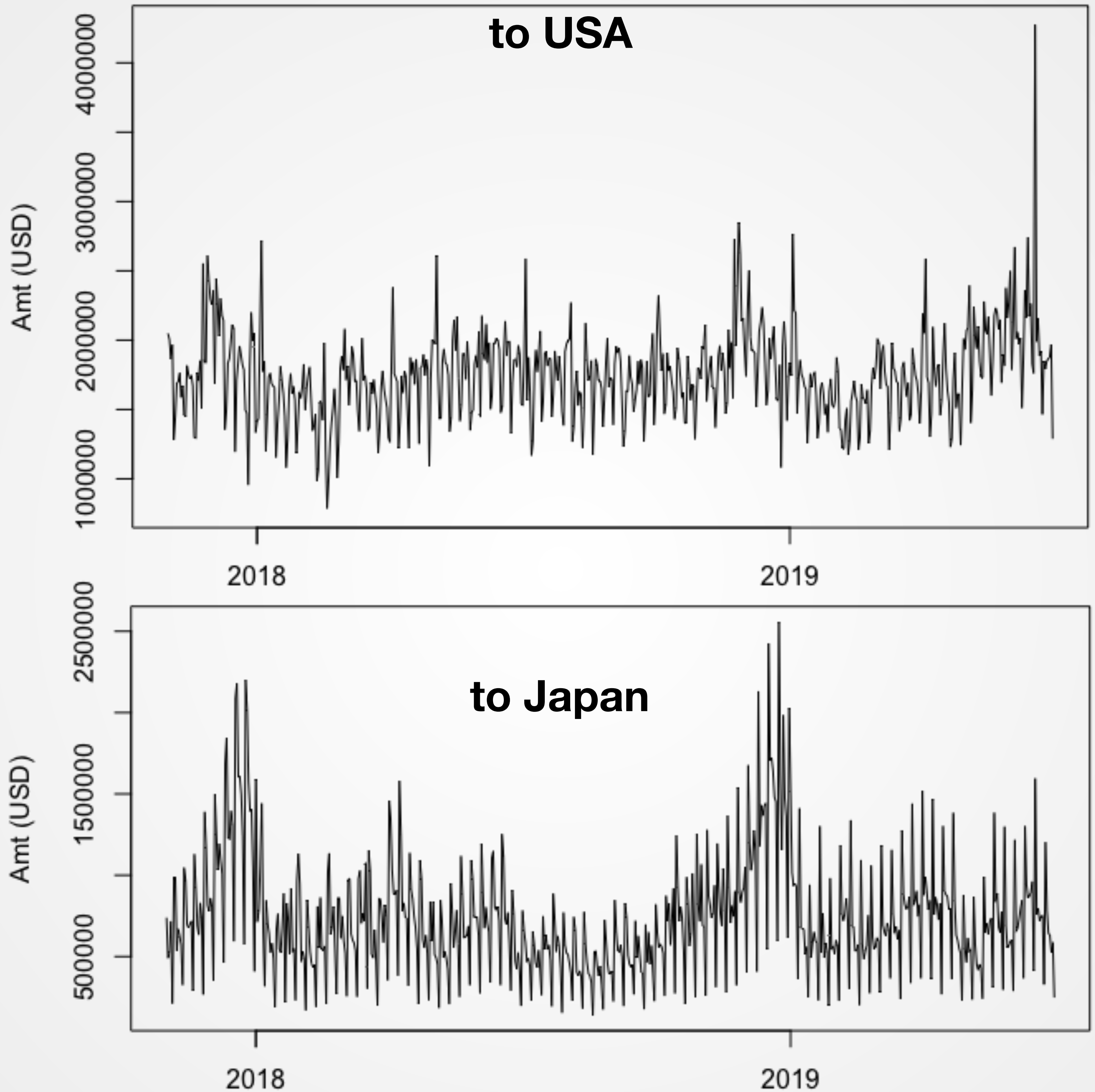


# VISA data

- Daily data for overseas spending by collective of SG visa card holders from Nov 1, 2017 to June 30, 2019.
- Can we predict overall overseas spending?
- Can we predict overseas spending by country?

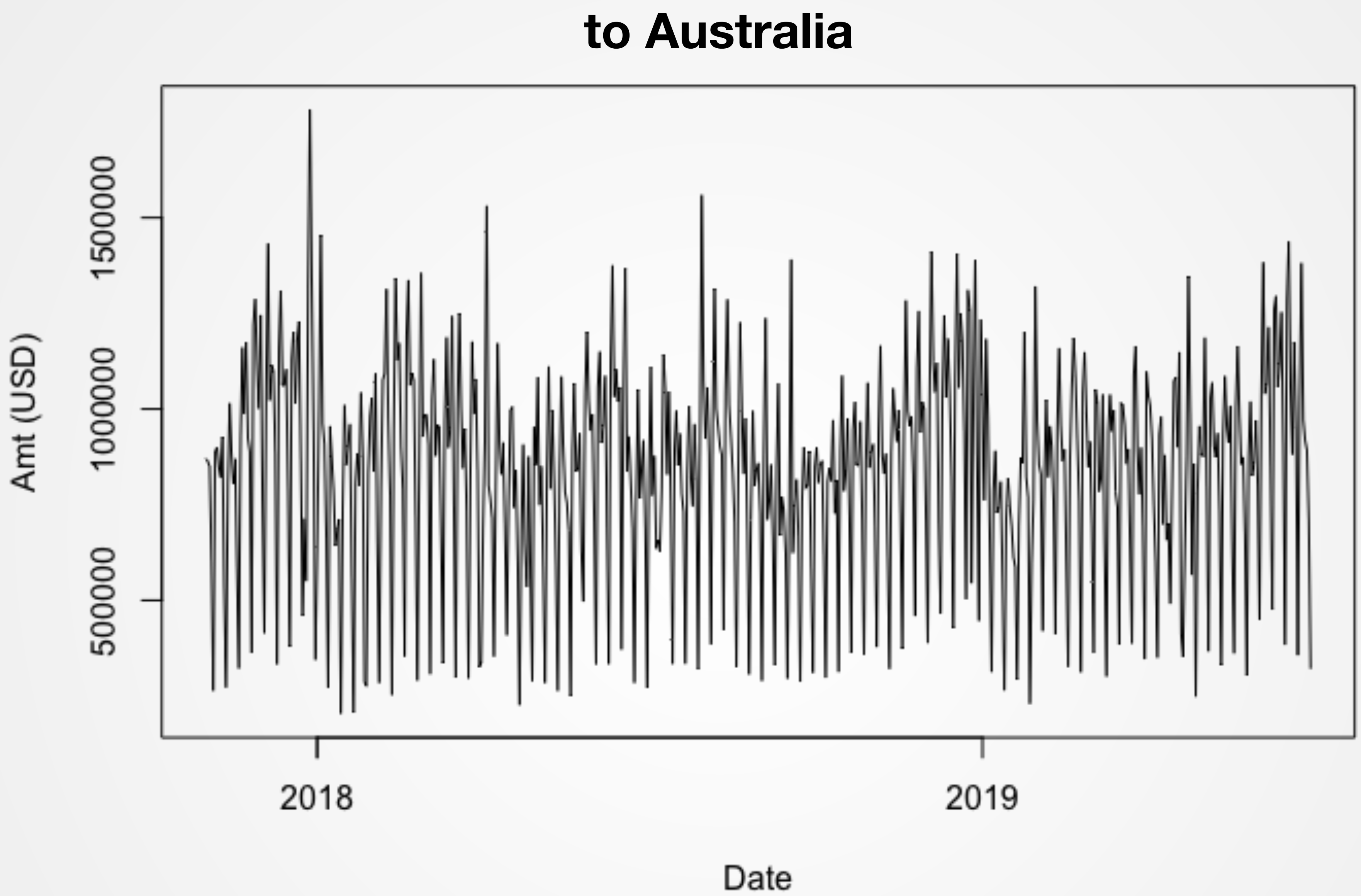



VISAplot


to USA
Amt (USD)
1000000
2000000
3000000
4000000
2018
2019
to Japan
Amt (USD)
500000
1500000
2500000
2018
2019

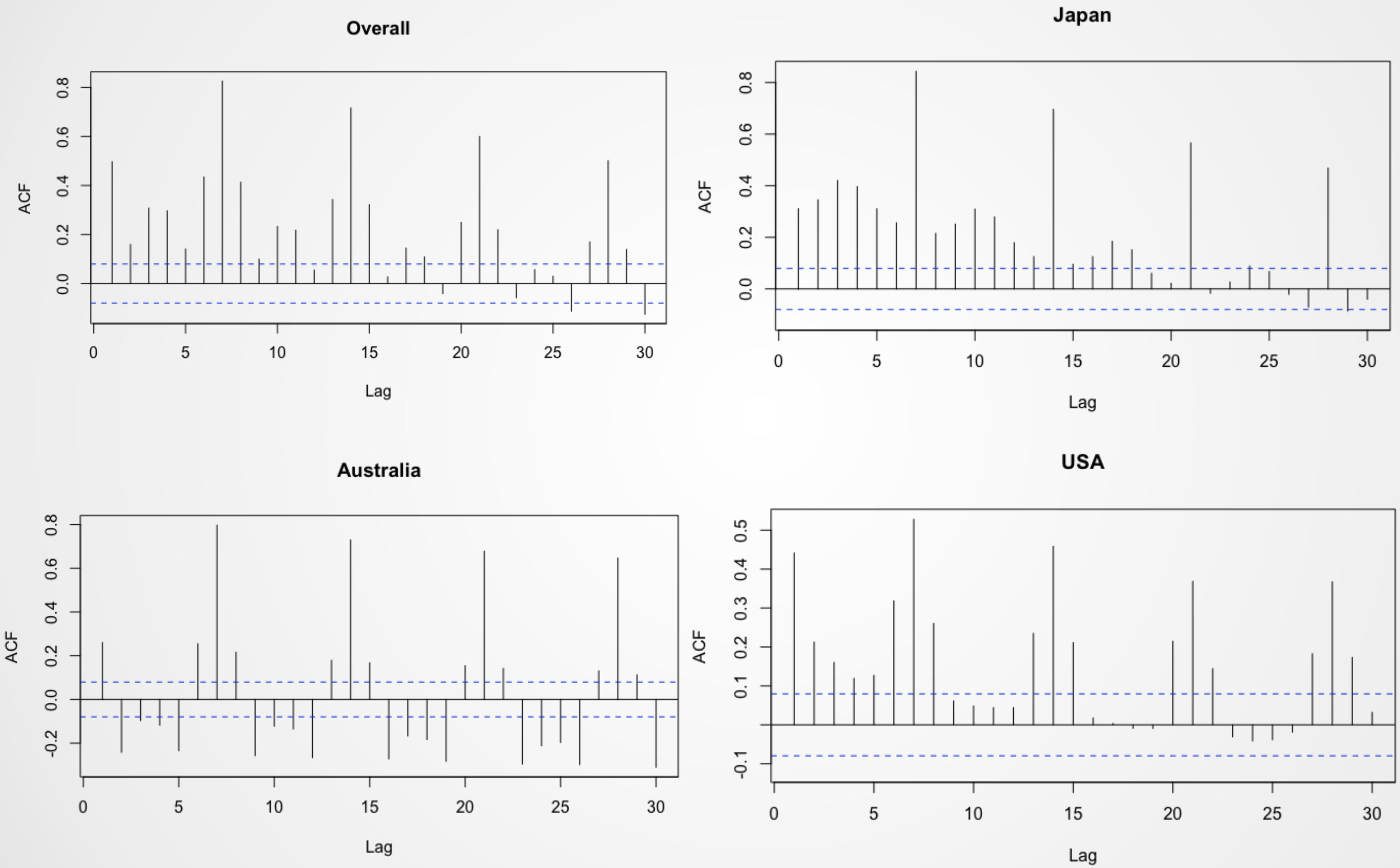
VISAplot

# VISA data



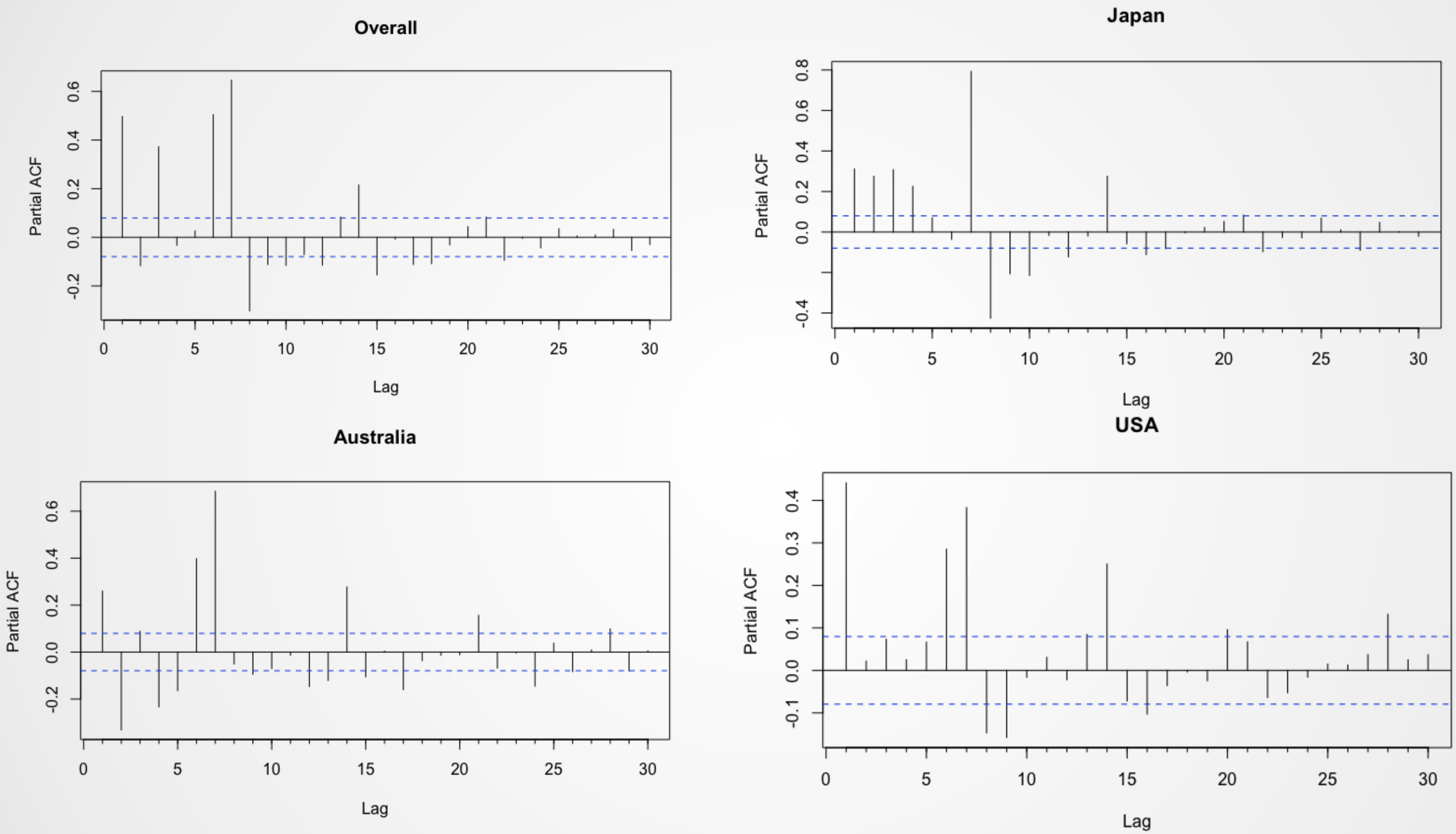

VISAplot

# VISA data - ACF




VISAplot

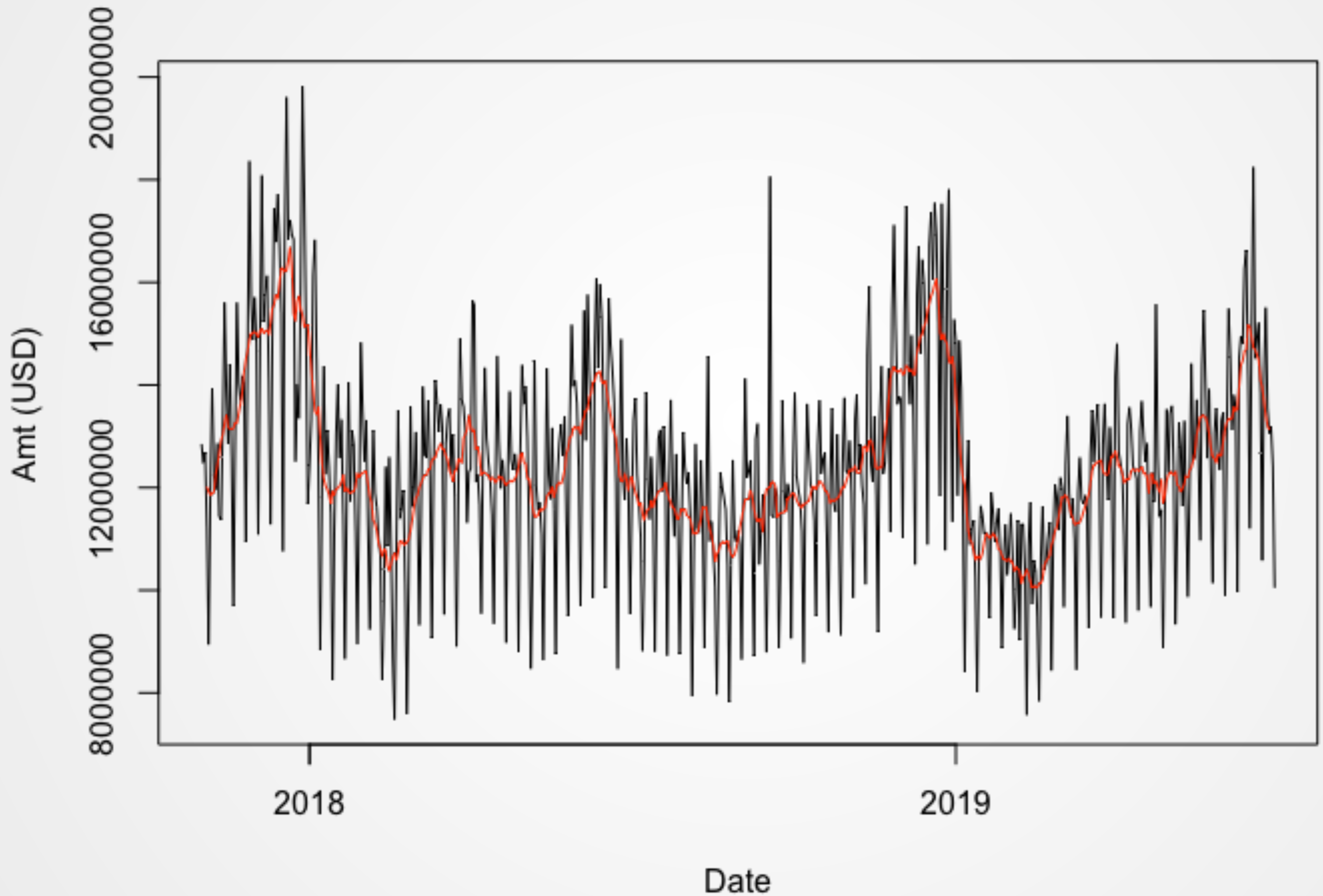
# VISA data - PACF




VISAplot

# VISA data - MA(7)

## Overall Transaction Amount

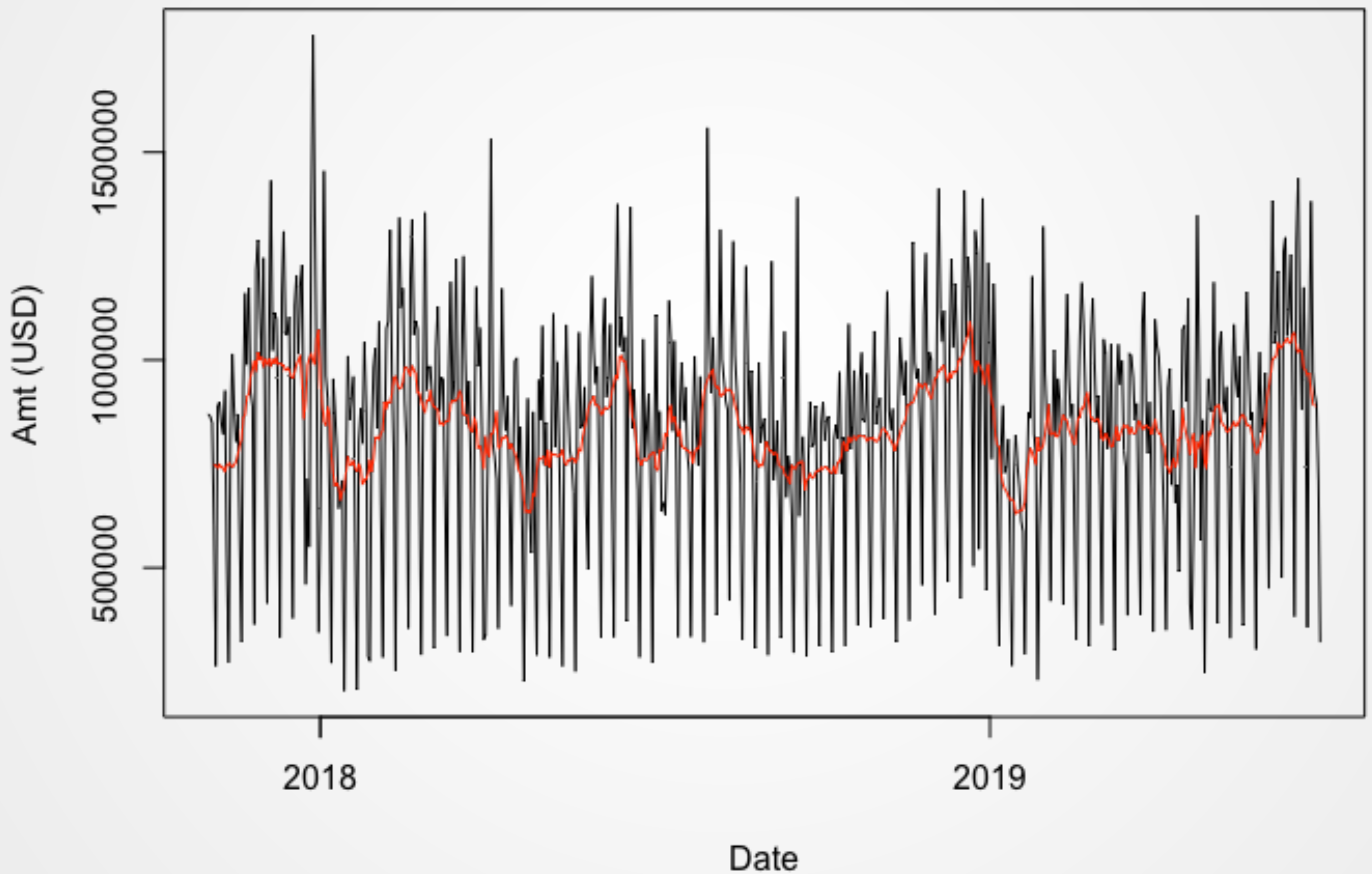



VISAplot

**Actual values and Rolling mean trend**

# VISA data - MA(7)

## Australia Transaction Amount

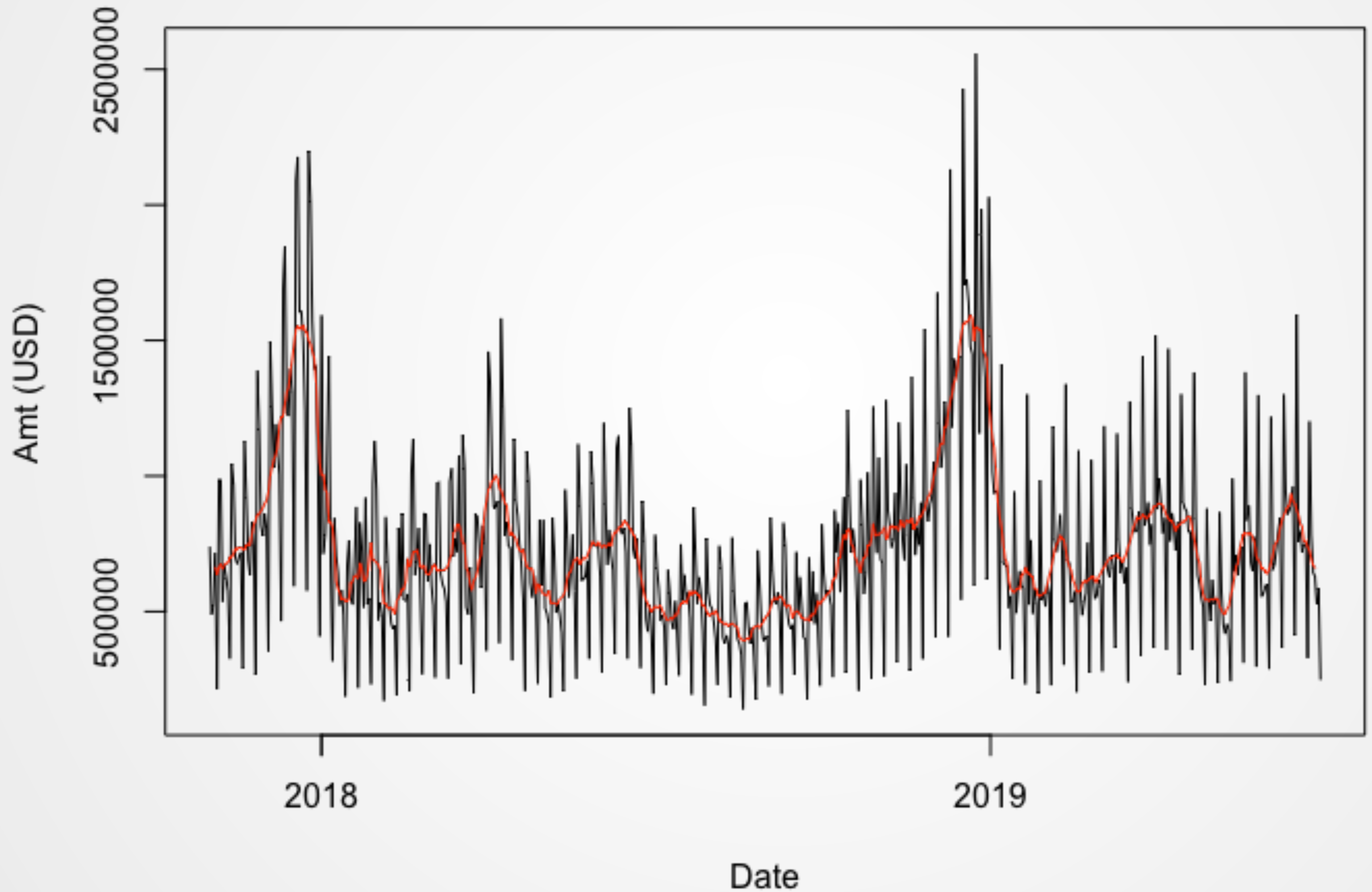



VISAplot

**Actual values and Rolling mean trend**

# VISA data - MA(7)

## Japan Transaction Amount

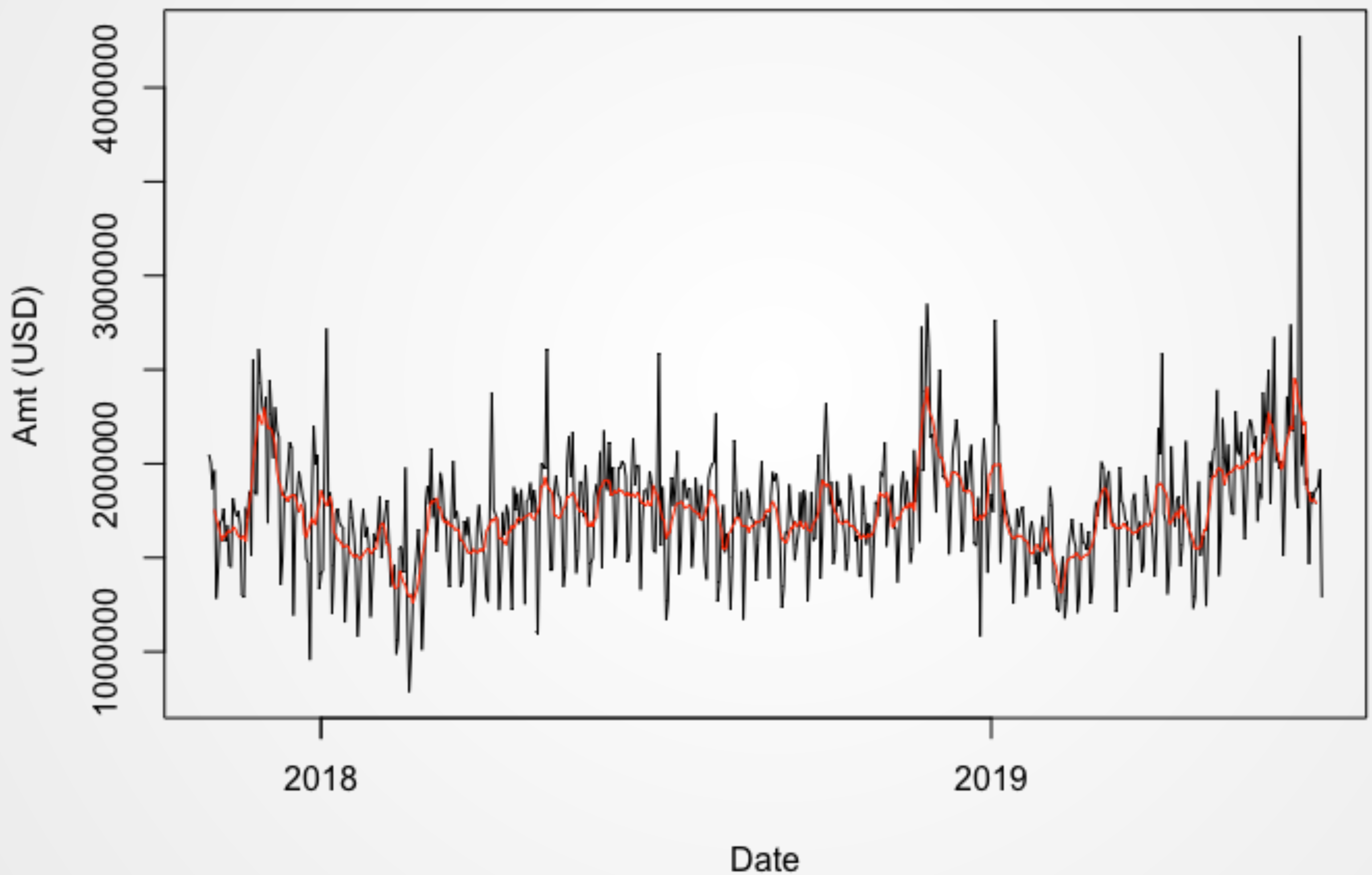



**Actual values and Rolling mean trend**

VISAplot

# VISA data - MA(7)

## USA Transaction Amount

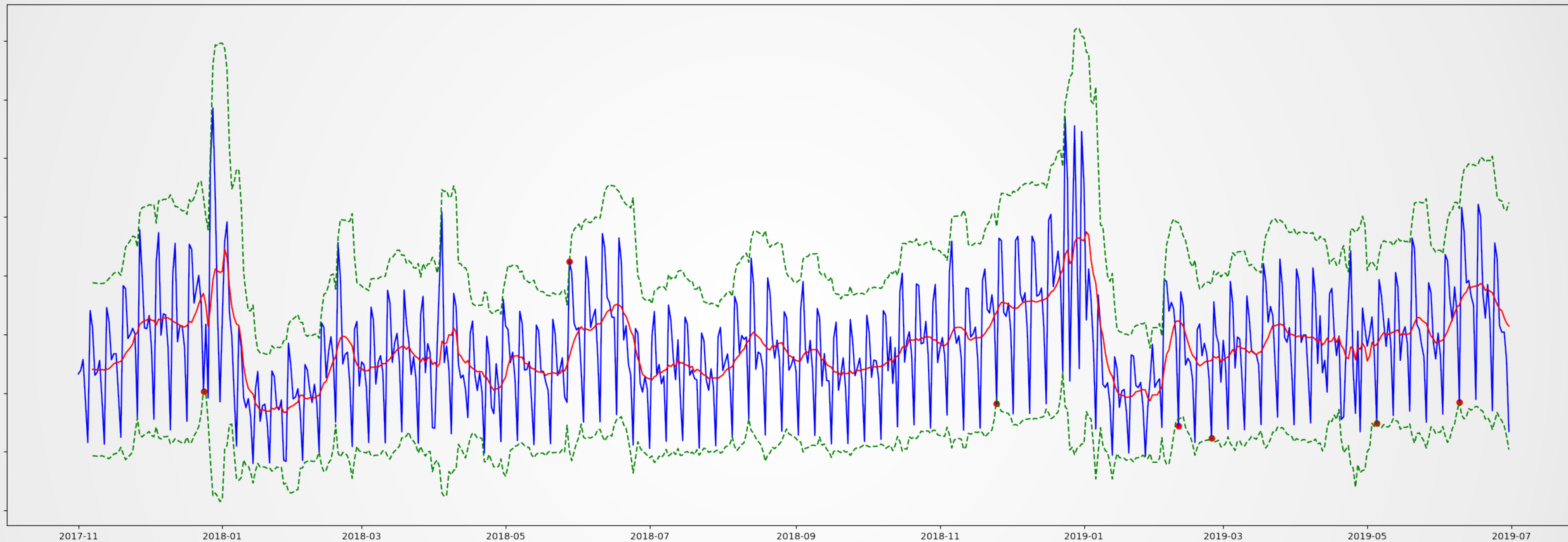



VISAplot

**Actual values and Rolling mean trend**

# VISA data - Outlier Check

## Australia Transaction Count

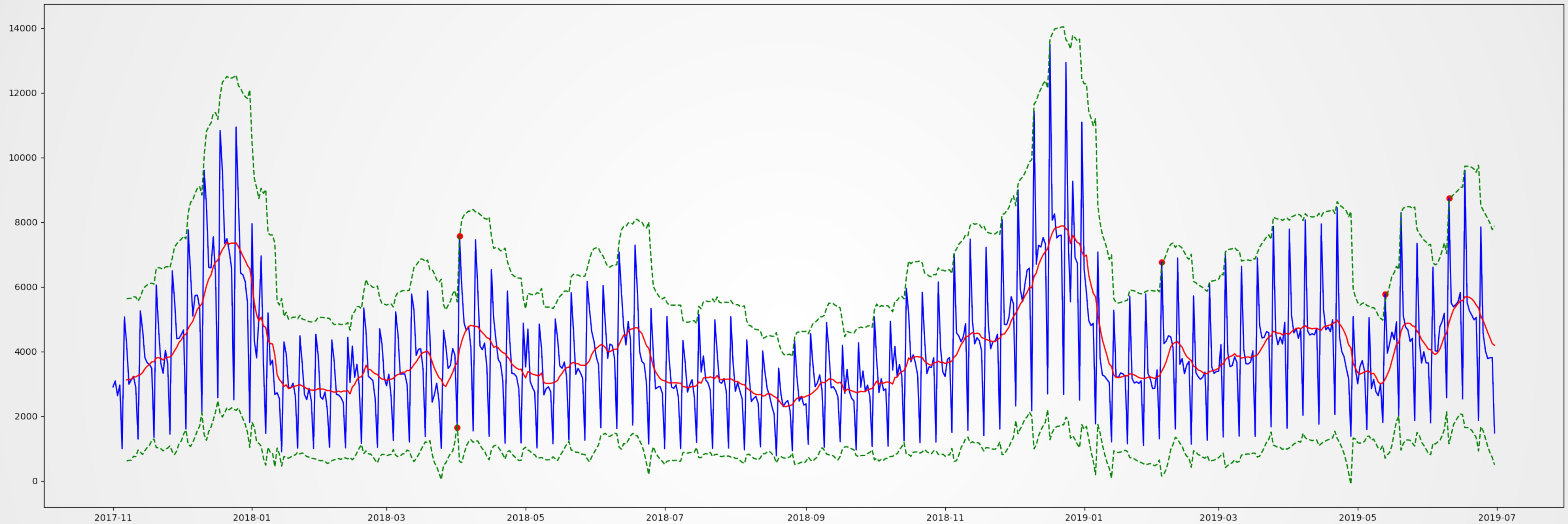



**Upper and lower bounds (95% Confidence Interval, CI)**
**Rolling mean trend (MA7)**

VISAoutliers

# VISA data - Outlier Check

## Japan Transaction Count

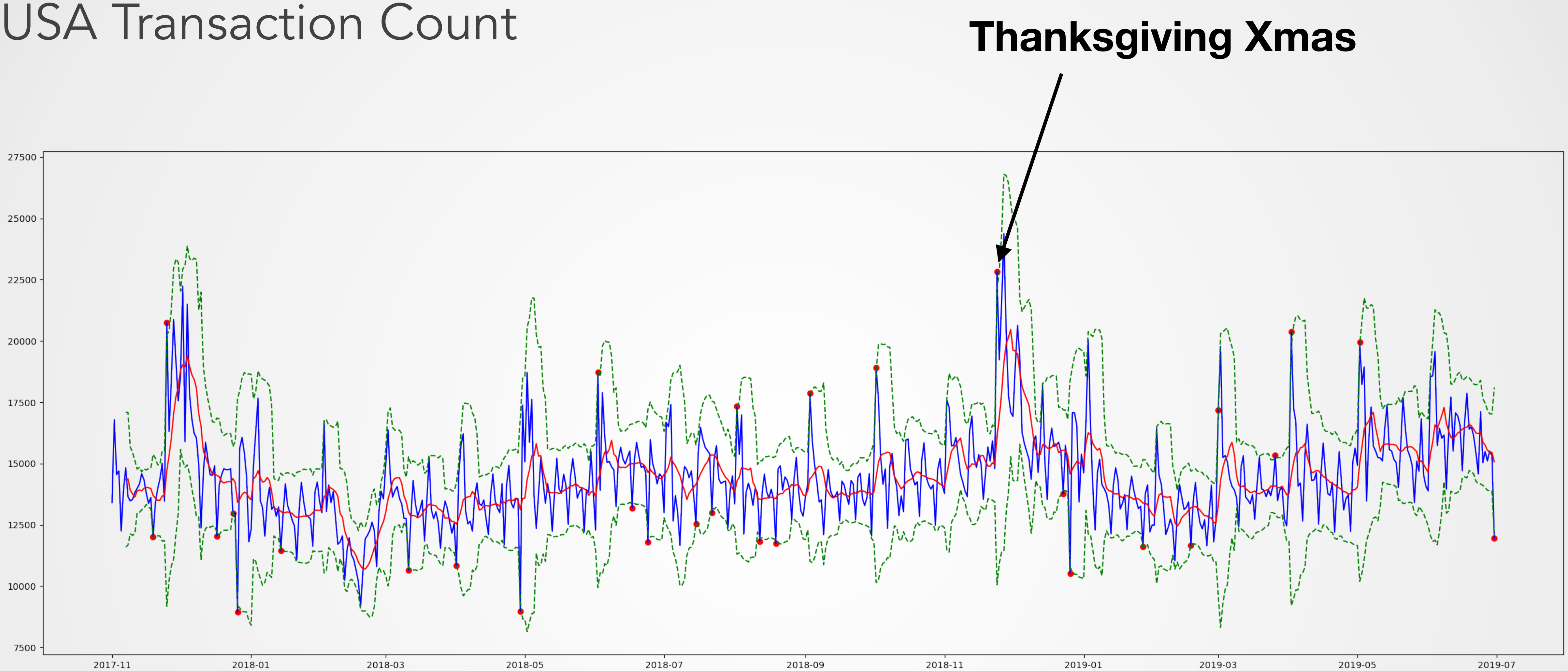



**Upper and lower bounds (95% Confidence Interval, CI)**
**Rolling mean trend (MA7)**

VISAoutliers

# VISA data - Outlier Check

## USA Transaction Count




**Upper and lower bounds (95% Confidence Interval, CI)**

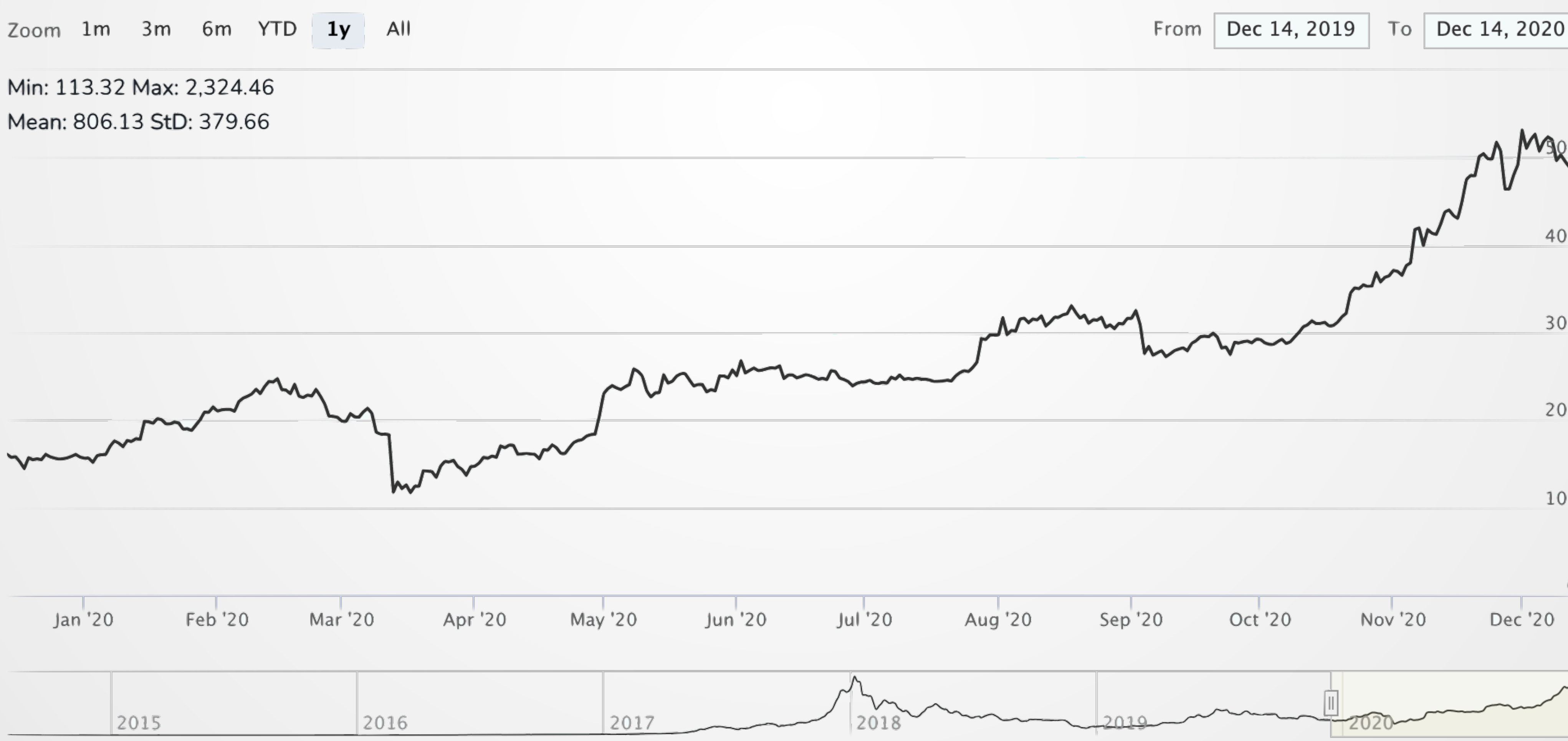
**Rolling mean trend (MA7)**

VISAoutliers

# Some Definitions

**Definition 11.1 (Stochastic process)** *A stochastic process $X_t$, $t \in T$ is a family of random variables, defined on a probability space $(\Omega, F, P)$.*

For a fixed $t$, $X_t$ is a random variable with a cdf and for a fixed $\omega \in \Omega$, $X(\omega) = \{X_t(\omega), t \in Z\}$ is a realisation or path of the process.



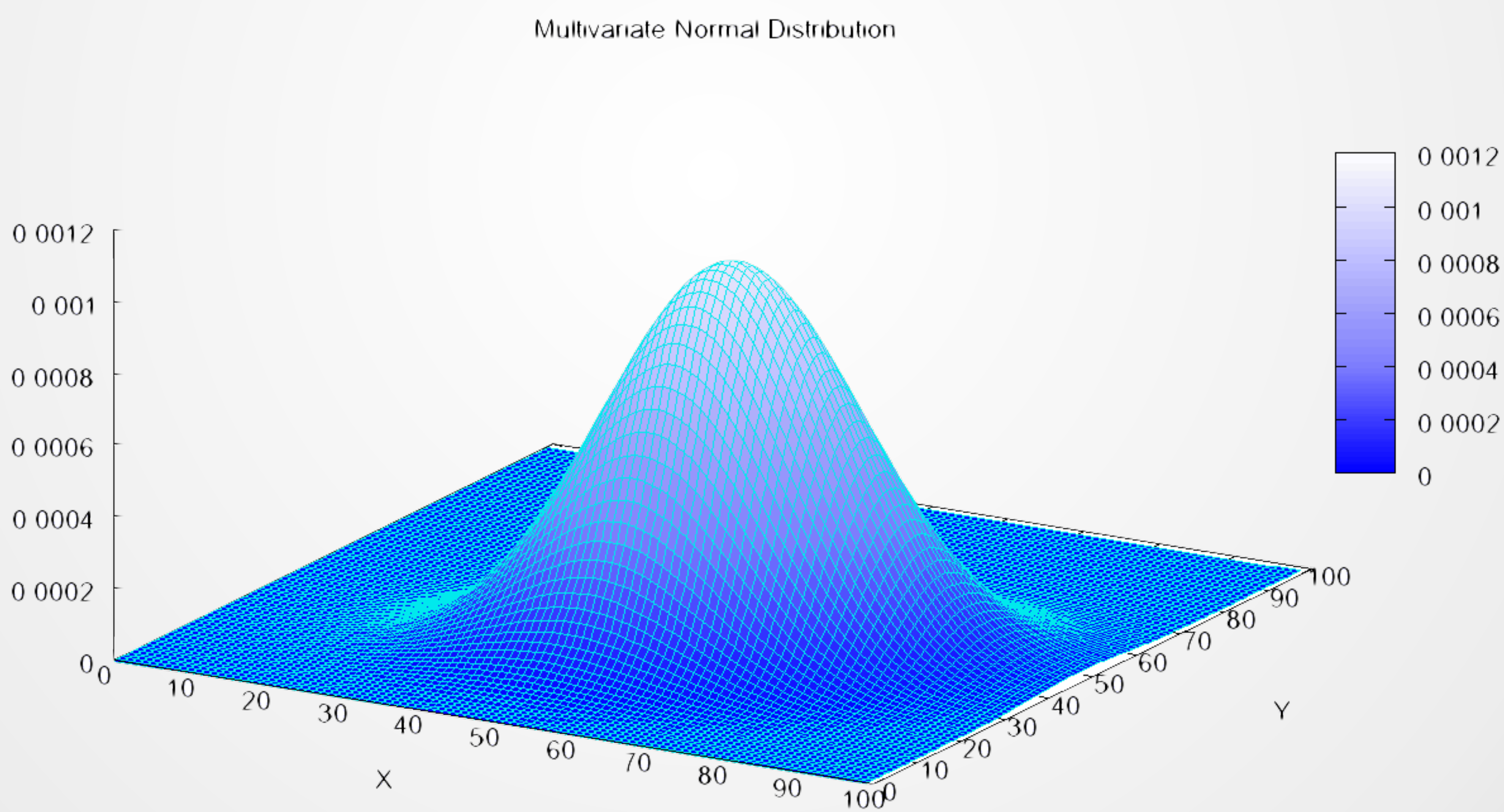
**Definition 11.2 (cdf of a stochastic process)** *The joint cdf of a stochastic process $X_t$ is defined as:*

$$F_{t1,\ldots,tn}(x_1, \ldots, x_n) = P(X_{t1} \leq x_1, \ldots, X_{tn} \leq x_n)$$

*The stochastic process is uniquely defined by its cdf.*

**Definition 11.3 (Conditional cdf)** *The conditional cdf of a stochastic process $X_t$ for arbitrary $t_1, \ldots, t_n$ with $t_1 < t_2 < \ldots < t_n$ is defined through:*

$$F_{t_n|t_{n-1},\ldots,t_1}(x_n|x_{n-1}, \ldots, x_1) = P(X_{t_n} \leq x_n | X_{t_{n-1}} = x_{n-1}, \ldots, X_{t_1} = x_1)$$



**Definition 11.4 (Mean function)** *The mean function $\mu_t$ of a stochastic process $X_t$ is defined as:*

$$\mu_t = \mathsf{E}[X_t] = \int_{\mathbb{R}} x dF_t(x)$$

**Definition 11.5 (Autocovariance function)** *The autocovariance function for $\tau \in T$ is defined as:*

$$\begin{aligned}\gamma(t,\tau) &= \mathsf{E}\left[\left(X_t - \mu_t\right)\left(X_{t-\tau} - \mu_{t-\tau}\right)\right] \\ &= \int_{\mathbb{R}^2} \left(x_1 - \mu_t\right)\left(x_2 - \mu_{t-\tau}\right) dF_{t,t-\tau}\left(x_1, x_2\right)\end{aligned}$$

**Definition 11.6 (Stationarity)** *A stochastic process is covariance stationary, if:*

1. $\mu_t = \mu$, *and*
2. $\gamma(t, \tau) = \gamma_\tau$

*A stochastic process* $X_t$ *is strictly stationary if for any* $t_1, \ldots, t_n$ *and for all* $n, s \in Z$ *it holds that*

$$F_{t_1,\ldots,t_n}(x_1, \ldots, x_n) = F_{t_1+s,\ldots,t_n+s}(x_1, \ldots, x_n)$$

**Definition 11.7 (ACF Autocorrelation function)** *The autocorrelation function* $\rho$ *of a covariance stationary stochastic process is defined as:*

$$\rho_\tau = \frac{\gamma_\tau}{\gamma_0}$$

## Example 1

Autocovariance function for $X_t = X_{t-1} + \varepsilon_t$,

$\varepsilon_t$ are iid, $\mathrm{Var}(\varepsilon_t) = \sigma^2$

$$\begin{aligned}
\gamma(t,\tau) = \mathrm{Cov}\left(X_t, X_{t-\tau}\right) &= \mathrm{Cov}\left(\sum_{i=1}^{t} \varepsilon_i, \sum_{j=1}^{t-\tau} \varepsilon_j\right) \\
&= \sum_{j=1}^{t-\tau} \sum_{i=1}^{t} \mathrm{Cov}\left(\varepsilon_i, \varepsilon_j\right) \\
&= \sum_{j=1}^{t-\tau} \sigma^2 = (t-\tau)\sigma^2
\end{aligned}$$

Autocorrelation function

$$\rho(t,\tau) = \frac{(t-\tau)\sigma^2}{\sqrt{t\sigma^2(t-\tau)\sigma^2}} = \frac{(t-\tau)}{\sqrt{t(t-\tau)}} = \sqrt{1-\frac{\tau}{t}}$$

**Definition 11.8 (WN White noise)** *A stochastic process $X_t$ is WN, if:*

1. $\mu_t = 0$ and
2. $\gamma(t,\tau) = \begin{cases} \sigma^2 & \text{if } \tau = 0 \\ 0 & \text{if } \tau \neq 0 \end{cases}$

SFEtimewn

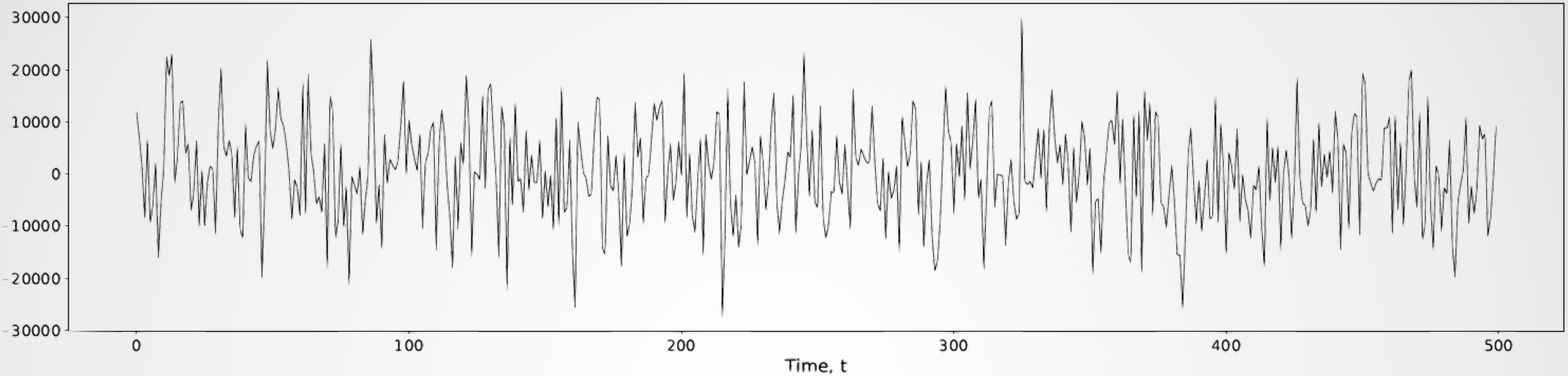
**Definition 11.9 (Random walk)** *The process $X_t$ is a random walk, if it can be written as:*

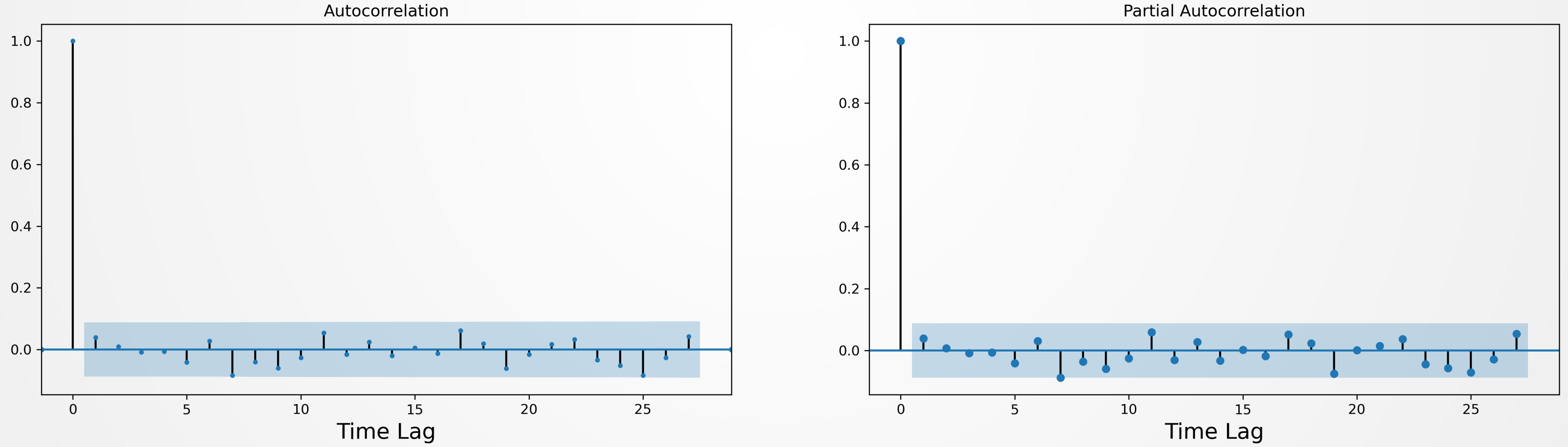
$$X_t = c + X_{t-1} + \varepsilon_t$$

*If $c \neq 0$ then the increments $Z_t = X_t - X_{t-1} = c + \varepsilon_t$ have an expectation different from zero. This random walk has a trend.*

SFEBinomp

# White noise

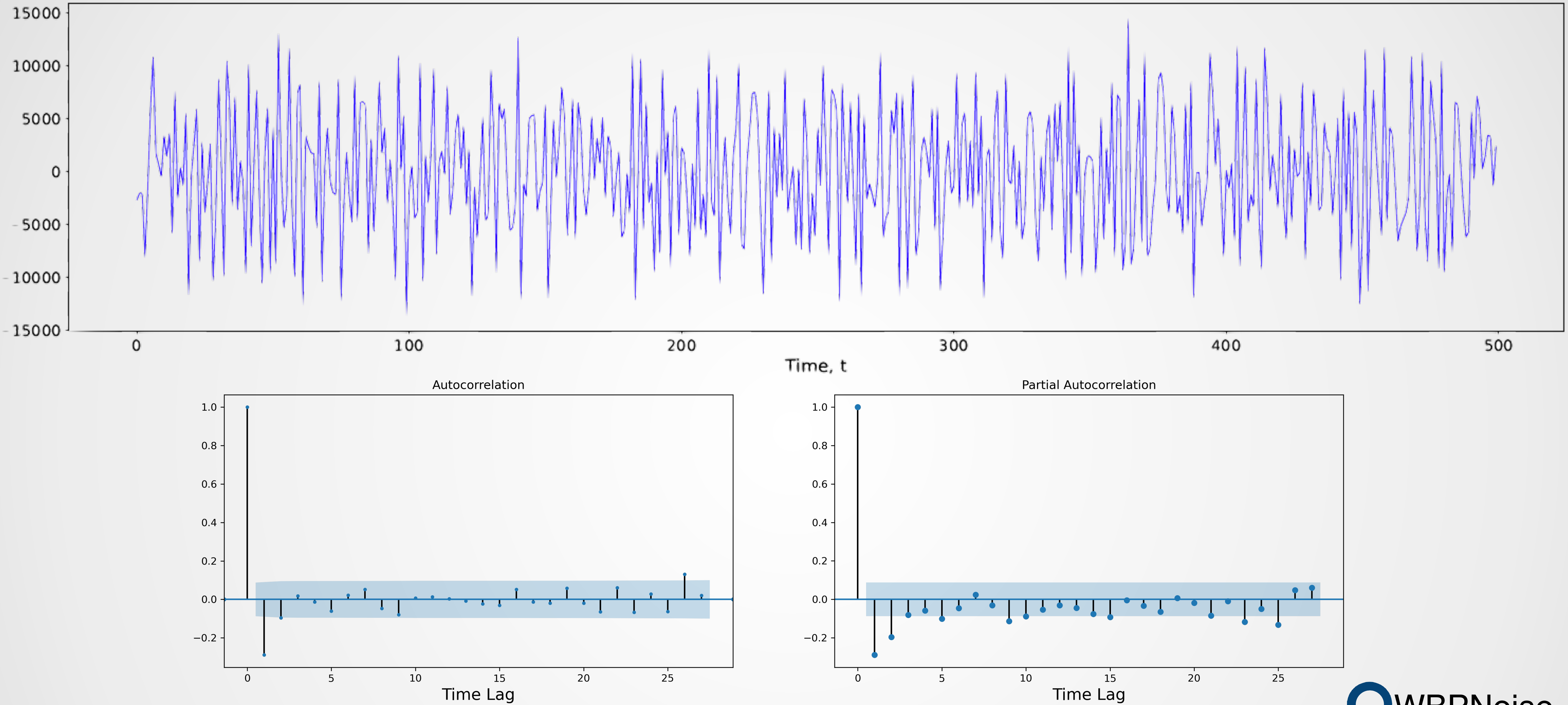



WBPNoise

- Stationary process, $AR(0)$ process, $\{X_t\}_{t=0}^{500} \sim \mathrm{N}\left(0,\sigma^2\right)$
- White noise dominates in whole spectrum

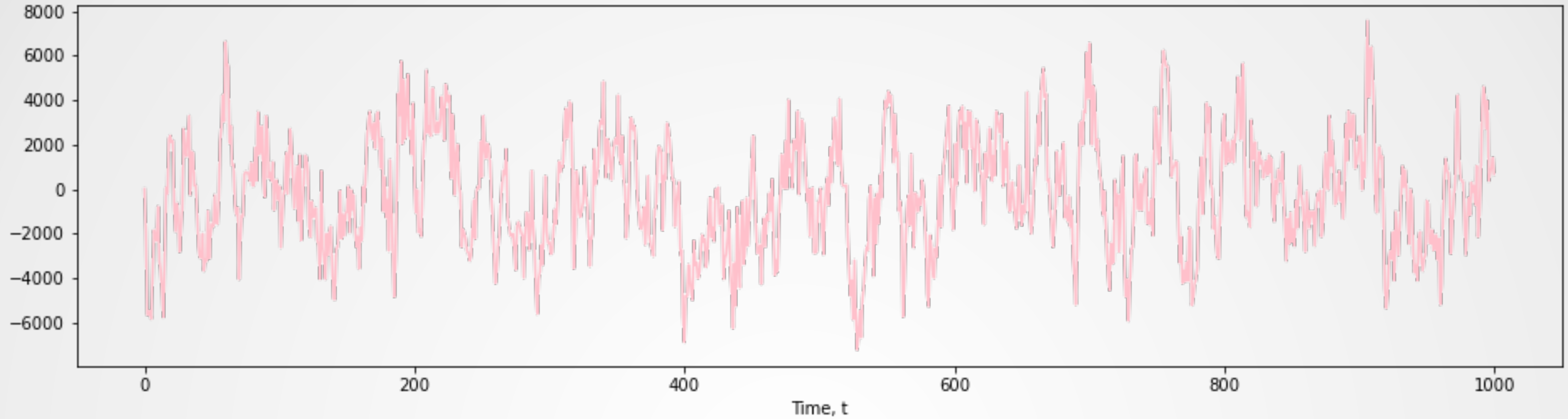
# Blue noise

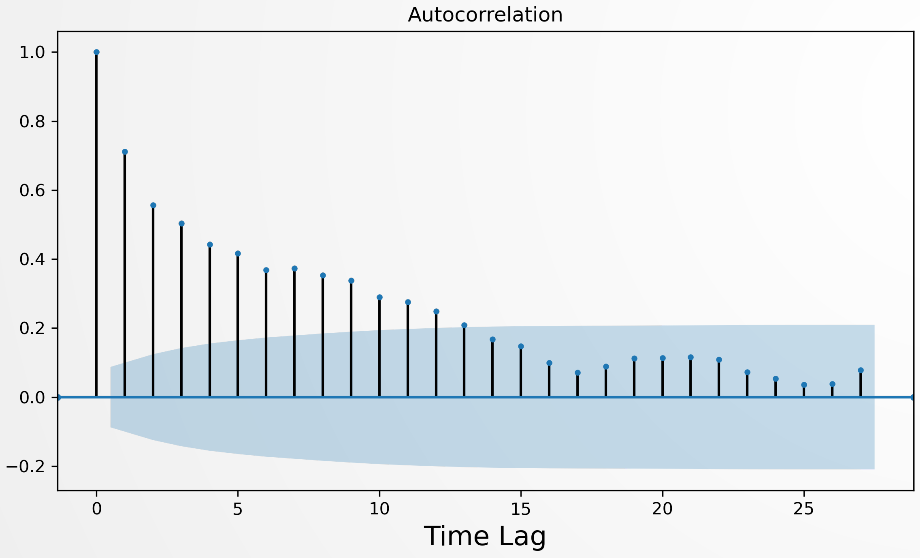


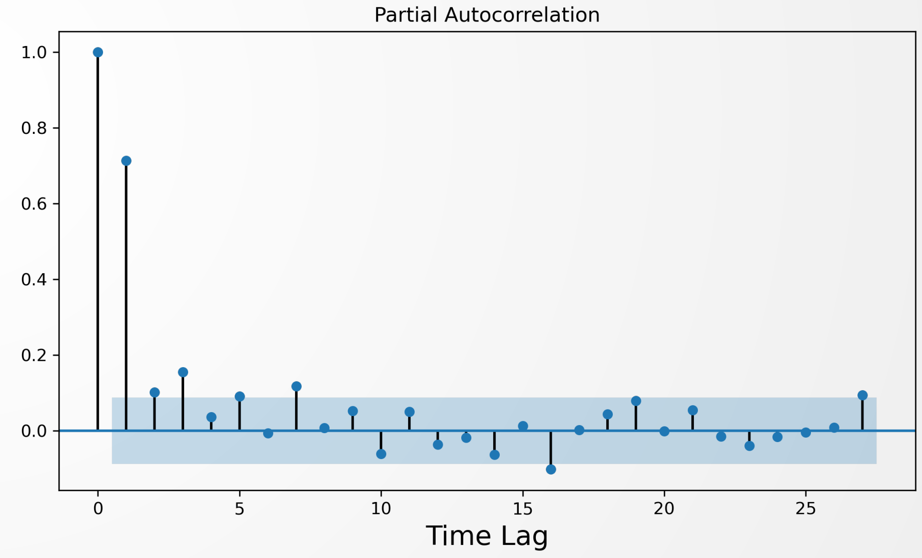

WBPNoise

- Stationary process, $AR(1)$ process, non-normal distributed
- Blue noise has more energy in higher frequency domain

# Pink noise




PNoise

- Stationary process, $AR(p)$ process, $p > 0$
- Pink noise is low frequency signal with no high frequency energy

Moments of a random walk, if $c = 0$ and $X_0 = 0$

$$X_t = \varepsilon_t + \varepsilon_{t-1} + \ldots + \varepsilon_1$$

Mean function

$$\mu_t = \mathsf{E}[X_t] = \sum_{i=1}^{t} \mathsf{E}[\varepsilon_i] = 0$$

Variance function

$$\mathrm{Var}(X_t) = \mathrm{Var}\left(\sum_{i=1}^{t} \varepsilon_i\right) = \sum_{i=1}^{t} \mathrm{Var}(\varepsilon_i) = t\sigma^2$$

**Example:** Mean function and ACF of an AR(1) process

The stochastic process $X_t$ is an AR(1) process if

$$X_t = c + \alpha X_{t-1} + \varepsilon_t$$

with a constant parameter $\alpha$, $|\alpha| < 1$

$$X_t = c(1 + \alpha + \alpha^2 + \ldots + \alpha^{k-1}) + \alpha^k X_{t-k}$$

$$+\varepsilon_t + \alpha\varepsilon_{t-1} + \ldots + \alpha^{k-1}\varepsilon_{t-k+1}$$

$$= c\left(\sum_{i=0}^{k-1} \alpha^i\right) + \alpha^k X_{t-k} + \sum_{i=0}^{k-1} \alpha^i \varepsilon_{t-i}$$

$$= c\frac{1 - \alpha^k}{1 - \alpha} + \alpha^k X_{t-k} + \sum_{i=0}^{k-1} \alpha^i \varepsilon_{t-i}$$

Since $|\alpha| < 1$ the sequence $\alpha^k \to 0$ for $k \to \infty$. There is a limit:

$$X_t = \frac{c}{1-\alpha} + \sum_{i=0}^{\infty} \alpha^i \varepsilon_{t-i}$$

Mean function:

$$\mu_t = \frac{c}{1-\alpha}$$

Autocovariance function:

$$\gamma_\tau = \sum_{i,j=0}^{\infty} \alpha^i \alpha^j \,\mathrm{Cov}(\varepsilon_{t-i}, \varepsilon_{t-(j+\tau)}) = \sigma^2 \sum_{j=0}^{\infty} \alpha^{\tau+j} \alpha^j$$

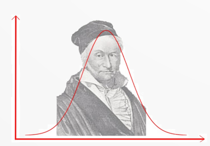
$$= \sigma^2 \alpha^\tau \sum_{j=0}^{\infty} (\alpha^2)^j = \frac{\sigma^2}{1-\alpha^2} \alpha^\tau$$

The ACF is therefore simply $\rho_\tau = \alpha^\tau$

For positive $\alpha$ this function is positive, for negative $\alpha$ it alternates around zero.

**Definition 11.10 (Markov process)** *A stochastic process* $X_t$ *is Markov process, if for the cdf* $F_{t|t-1,\ldots,t-k}$ *and all* $t \in \mathbb{Z}$ *with* $k \geq 1$ *holds:*

$$F_{t|t-1,\ldots,t-k}(x_t \,|\, x_{t-1}, \ldots, x_{t-k}) = F_{t|t-1}(x_t \,|\, x_{t-1})$$

*Andrei A. Markov on BBI:* 

**Definition 11.11 (Martingale)** *The stochastic process $X_t$ is a martingale if the following holds:*

$$\mathrm{E}[X_t \,|\, X_{t-1} = x_{t-1}, \ldots, X_{t-k} = x_{t-k}] = x_{t-1}$$

*for every $k > 0$*

**Definition 11.12 (Fair game)** *The process $X_t$ is a fair game if the following holds:*

$$\mathrm{E}[X_t \,|\, X_{t-1} = x_{t-1}, \ldots, X_{t-k} = x_{t-k}] = 0$$

*for every $k > 0$*

*If $X_t$ is martingale, then $Z_t = X_t - X_{t-1}$ is a fair game*

**Definition 11.13 (Lag operator)** *The operator $L$ moves the process $X_t$ back by one unit of time, i.e., $LX_t = X_{t-1}$. In addition we define the difference operator $\Delta$ as $\Delta = 1 - L$, i.e., $\Delta X_t = X_t - X_{t-1}$, and $\Delta^k = (1 - L)^k$.*

**Definition 11.14 (Simple returns)** *For the stochastic process of prices $P_t$ we define the rate of return:*

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}}$$

*Average simple return over $k$ periods (geometric mean of $R_t$):*

$$R_t(k) = \prod_{j=0}^{k-1} (1 + R_{t-j})^{1/k} - 1$$

**Definition 11.15 (Log returns)** *Log return $r_t$ is defined as:*

$$r_t = \log \frac{P_t}{P_{t-1}} = \log(1 + R_t)$$

*Average log return over $k$ periods (arithmetic mean of $r_t$):*

$$r_t(k) = \log\{R_t(k) + 1\} = \frac{1}{k} \log \prod_{j=0}^{k-1} \left(1 + R_{t-j}\right)$$

$$= \frac{1}{k} \sum_{j=0}^{k-1} \log\left(1 + R_{t-j}\right) = \frac{1}{k} \sum_{j=0}^{k-1} r_{t-j}$$

For small price changes $(R_t < 10\%)$ the different definitions of the returns are not important, particularly if the frequency of the observed financial time series is high (for example intra-day observations), since:

$$\begin{aligned}\log(1+x) &= \log(1) + \frac{\partial \log x}{\partial x}(1)x + \frac{\partial^2 \log x}{\partial x^2}(1)\frac{x^2}{2!} \\ &= x - \frac{x^2}{2!} + \frac{x^3}{3!} + \dots .\end{aligned}$$

For $x \approx 0$, we have $\log(1+x) \approx x$.

# German and British Stock Returns

Statistical analysis: linear chronological (in)dependence of the returns and distribution characteristics from 2014 to 2021

Results:

- ACF of first order is close to zero for all stock returns
- ALLIANZ- highest autocorrelation (0.0754)
- UNILEVER UK - lowest autocorrelation(−0.1420)

- More autocorrelations are negative (9 pos., 11 neg.)
- ACF of squared and absolute returns are positive and significantly larger than zero

# Consequences

- Autocorrelations of the squared returns are positive: small absolute returns are followed sequentially by small absolute returns and vice versa
- There are quiet periods with low volatility and turbulent periods with high volatility
- These periods are relatively long: the ACF of squared returns is still positive for high orders. These effects have been detected by Mandelbrot and Fama in the 60's

| | $\hat{\rho}_1(r_t)$ | $\hat{\rho}_1(r_t^2)$ | $\hat{\rho}_1(\lvert r_t \rvert)$ | $BJ$ | $\hat{S}$ | $\widehat{Kurt}$ |
|---|---|---|---|---|---|---|
| adidas | 0.0643 | 0.3096 | 0.1648 | 4240.3765 | -0.3744 | 11.1536 |
| allianz | 0.0754 | 0.0549 | 0.1791 | 18987.6886 | -0.6368 | 20.2794 |
| basf | 0.0346 | 0.0455 | 0.0869 | 2417.0837 | -0.4198 | 9.1245 |
| bayer | 0.0118 | 0.0380 | 0.0906 | 3062.6832 | -0.6944 | 9.8186 |
| beiersdorf | -0.0138 | 0.0436 | 0.1268 | 2100.6159 | -0.3515 | 8.7199 |
| bmw | 0.0356 | 0.0641 | 0.1501 | 3395.6868 | -0.2960 | 10.3032 |
| continental | 0.0400 | 0.0335 | 0.1090 | 5204.0672 | -0.6690 | 11.9715 |
| daimler | 0.0298 | 0.0199 | 0.1306 | 27362.0840 | -0.0518 | 23.7988 |
| dax | -0.0589 | 0.1251 | 0.1878 | 8234.8063 | -1.0598 | 14.2117 |
| deutsche bank | -0.0012 | 0.1174 | 0.1479 | 1758.2422 | -0.1434 | 8.2646 |
| deutsche post | -0.0335 | 0.1914 | 0.1806 | 3187.1432 | -0.4533 | 10.0404 |
| deutsche telekom | -0.0763 | 0.1148 | 0.1901 | 12048.3574 | -1.3564 | 16.5325 |

Table 42: Estimated first order AC of the returns, the squared returns ($\hat{\rho}_1(r_t^2)$, absolute returns ($\hat{\rho}_1(\lvert r_t \rvert)$, estimated skewness ($\hat{S}$), estimated kurtosis($\widehat{Kurt}$), Bera-Jarque test statistic ($BJ$) for daily returns of the DAX index and German stocks from 2014 to 2021

SFEReturns

| | $\hat{\rho}_1(r_t)$ | $\hat{\rho}_1(r_t^2)$ | $\hat{\rho}_1(\lvert r_t \rvert)$ | $BJ$ | $\hat{S}$ | $\widehat{Kurt}$ |
|---|---|---|---|---|---|---|
| national grid | -0.0455 | 0.4228 | 0.3535 | 10524.8646 | -0.3704 | 15.8784 |
| prudential | -0.0810 | 0.3607 | 0.4431 | 22440.4359 | -1.0110 | 21.7270 |
| reckitt benckiser | -0.0734 | 0.2832 | 0.2122 | 5493.7646 | -0.0456 | 12.3193 |
| royal dutch shell | 0.0409 | 0.1987 | 0.3196 | 23052.2873 | -0.9285 | 22.0004 |
| rio tinto | -0.0299 | 0.2744 | 0.2059 | 862.8632 | 0.0355 | 6.6928 |
| standard chartered | 0.0263 | 0.1188 | 0.1943 | 1380.0734 | 0.1326 | 7.6636 |
| unilever UK | -0.1420 | 0.3327 | 0.2731 | 12526.6440 | -0.0012 | 17.0730 |
| vodafone | -0.0390 | 0.2760 | 0.2220 | 9149.2657 | -0.5760 | 14.9719 |

SFEReturns

## Skewness, $S$ and Kurtosis, $Kurt$

Under the null hypothesis of normality the estimators of $\widehat{S}$ and $\widehat{\text{Kurt}}$ are asymptotically normal and independent.

$$\sqrt{n}\widehat{S} \xrightarrow{\mathcal{L}} \mathrm{N}(0,6)$$

$$\sqrt{n}(\widehat{\text{Kurt}} - 3) \xrightarrow{\mathcal{L}} \mathrm{N}(0,24)$$

Bera and Jarque test statistics, $BJ$:

$$BJ = n\left\{\frac{\widehat{S}^2}{6} + \frac{(\widehat{\text{Kurt}} - 3)^2}{24}\right\}$$

$$BJ \sim \mathcal{X}^2(2)$$

## Results

- Estimated skewness is in most cases close to zero
- Estimated kurtosis is considerably larger than three
- RIO TINTO - smallest kurtosis (6.69); DAIMLER - highest kurtosis (23.79)

## Consequences

- Normality is strongly rejected
- Overkurtosis - due to outliers (extreme returns)
- Leptokurtic distributions - stylized fact
- Platykurtic distributions - rare

# Unit Root Tests

We have discussed the AR(1) process

$$X_t = c + \alpha X_{t-1} + \varepsilon_t$$

If $|\alpha| < 1$, then $X_t$ stationary

If $\alpha = 1$, then $X_t$ non stationary random walk

Hence we are interested in testing $\alpha = 1$

# Dickey-Fuller Test

Dickey and Fuller developed a unit root test of $H_0 : \alpha = 1$ vs $H_1 : \alpha \neq 1$. The basic idea is the regression

$$\Delta X_t = (\alpha - 1)X_{t-1} + \varepsilon_t$$

Under $H_0$ the coefficient of $X_{t-1}$ equals zero. Under $H_1$ this coefficient is negative.

## Dickey-Fuller (DF) test statistics

$$\hat{t}_n = \frac{\hat{\alpha} - 1}{\sqrt{\hat{\sigma}^2 \left( \sum_{t=2}^{n} X_{t-1}^2 \right)^{-1}}},$$

where $(\hat{\alpha}; \hat{\sigma}^2)$ are the LS-estimators of $(\alpha, \sigma^2)$.

# Asymptotic Distribution

$$\hat{t}_n \xrightarrow{\mathcal{L}} \frac{W^2(1) - 1}{2\left\{\int_0^1 W^2(u)du\right\}^{1/2}}$$

*W*-Standard Wiener process

The critical values for $\alpha = 1\,\%, 5\,\%, 10\,\%$ are -2.58, -1.58, and -1.62 respectively. Problem of DF Test: If $\varepsilon_t$ are autocorrelated, the level (e.g. $\alpha = 5\,\%$) is strongly affected.

# Augmented DF (ADF) Test

$$\Delta X_t = c + (\alpha - 1)X_{t-1} + \sum_{i=1}^{p} \alpha_i \Delta X_{t-i} + \varepsilon_t$$

Reject $H_0$ if $\hat{t}_n$ (constructed with this $\hat{\alpha}$) is smaller than the critical value.

Problem: Choice of *p*.
If we augment *p* we hold the level but we lose power!

**Example**

$\varepsilon_t \sim$ MA(1) $\quad \varepsilon_t = \beta\xi_{t-1} + \xi_t \quad \xi_t$ i.i.d $(0,\sigma^2)$

$$\mathrm{Var}(\varepsilon_t) = \sigma^2(1+\beta^2)$$

$$\gamma_1(\varepsilon_t) = \mathrm{Cov}(\varepsilon_t, \varepsilon_{t-1}) = \beta\sigma^2$$

$$\gamma_\tau(\varepsilon_t) = 0 \text{ for } \tau \geq 2$$

Hence the ACF of $\varepsilon_t$ is:

$$\rho_\tau(\varepsilon_t) = \begin{cases} \frac{\beta}{1+\beta^2} & \text{if } \tau = 1 \\ 0 & \text{if } \tau \geq 2. \end{cases}$$

For the process

$$X_t = \alpha X_{t-1} + \varepsilon_t = \alpha X_{t-1} + \beta \xi_{t-1} + \xi_t$$

we make simulations for the ADF test.

| | | | | $\beta$ | | |
|---|---|---|---|---|---|---|
| $\alpha$ | $p$ | -0.990 | -0.900 | 0.000 | 0.900 | 0.990 |
| 0,9 | 3 | 0.998 | 0.995 | 0.130 | 0.258 | 0.288 |
| | 11 | 0.322 | 0.222 | 0.056 | 0.081 | 0.103 |
| 1 | 3 | 0.992 | 0.933 | 0.043 | 0.100 | 0.112 |
| | 11 | 0.219 | 0.111 | 0.040 | 0.050 | 0.072 |

SFESimADF

Table 44: ADF-test: simulated rejection probability for the process with a nominal level of 5% (Friedmann, 1992).

Conditions:

- level is better held for larger *p*
- power is worse for larger *p*

Conflict remains between validity and power of test.

An additional test includes a linear time trend:

$$\Delta X_t = c + \mu t + (\alpha - 1)X_{t-1} + \sum_{i=1}^{p} \alpha_i \Delta X_{t-i} + \varepsilon_t$$

The ADF test (with time trend) has power against a trend stationary process. On the other hand we lose power w.r.t. the simple ADF test if the time process is a stationary AR(1) process.

# Quant Analysis of Financial Time Series

Maria Culjak

Wolfgang Karl Härdle

Ladislaus von Bortkiewicz Professor of Statistics
Humboldt-Universität zu Berlin
BRC Blockchain Research Center
lvb.wiwi.hu-berlin.de
Charles University, WISE XMU, NCTU 玉山学者