Kronecker Product

The Kronecker product is a special operation on matrices (arrays). It is defined as follows.

If A is an $m \times n$ matrix and $\mathbf{B}$ is a $p \times q$ matrix, then the Kronecker product $A \otimes B$ $pm \times qn$ block matrix:

$$A \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix},$$

where a_{ij} is the (i, j)-th entry of A and $\mathbf{B}$ is the matrix B. More explicitly, let's use:

- $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, a 2×3 matrix, and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, a 2×2 matrix. Then, the Kronecker product $A \otimes B$ is computed as follows:

$$A \otimes \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \otimes \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & 2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & 3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ 4\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & 5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & 6\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 4 & 3 & 6 \\ 3 & 4 & 6 & 8 & 9 & 12 \\ 4 & 8 & 5 & 10 & 6 & 12 \\ 12 & 16 & 15 & 20 & 18 & 24 \end{bmatrix}$$

In R, this is easy to implement

```
A <- matrix(c(1, 2, 3, 4, 5, 6), nrow = 2, byrow = TRUE)
B <- matrix(c(1, 2, 3, 4), nrow = 2, byrow = TRUE)
A %x% B
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6]
## [1,]    1    2    2    4    3    6
## [2,]    3    4    6    8    9   12
## [3,]    4    8    5   10    6   12
## [4,]   12   16   15   20   18   24
```

Kronecker Product Use-Case From My Consulting Practice

Suppose I want to simulate a quasi-poisson variable based on given mean parameters λ and ϕ. For a quasi-poisson, the variance is equal to $\phi \cdot \lambda$. Suppose I have a matrix of means λ, that are determined by a combination of 2-variables. I also have a set of parameters ϕ. Now, I want all the possible combinations of $\lambda \cdot phi$, i.e all the possible variances. I can create this using the Kronecker product. The matrix is given by:

```
lambda <- matrix(c(1, 2, 3, 4, 5, 6), nrow = 2, byrow = TRUE)
phi <- c(3, 4)
phi %x% lambda
```

```
##      [,1] [,2] [,3]
## [1,]    3    6    9
## [2,]   12   15   18
## [3,]    4    8   12
## [4,]   16   20   24
```

Now, I can iterate over these variance parameters for my analysis.