



On Averaging Rotations

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Abstract. In this paper two common approaches to averaging rotations are compared to a more advanced approach based on a Riemannian metric. Very often the barycenter of the quaternions or matrices that represent the rotations are used as an estimate of the mean. These methods neglect that rotations belong to a non-linear manifold and re-normalization or orthogonalization must be applied to obtain proper rotations. These latter steps have been viewed as ad hoc corrections for the errors introduced by assuming a vector space. The article shows that the two approximative methods can be derived from natural approximations to the Riemannian metric, and that the subsequent corrections are inherent in the least squares estimation.

Keywords: averaging rotations, Riemannian metric, matrix, quaternion

1. Introduction

When we measure the pose of an object with different devices or different approaches we would like to integrate these measurements into a single mean pose. This could be the case in a robot vision system, where we measure the orientation of an object with multiple cameras, or it could occur in the registration of medical images, where several poses are computed from images of different modalities (Pennec et al., 1998). Figure 4 shows a typical ship block where robot manufacturing requires a high quality pose estimation with respect to a fixed frame. This can be obtained by the fusion of information from multiple cameras and/or sensors. In this article we only consider taking the mean of a set of three-dimensional rotations, and do not treat problems where there is a translational component.

The problem with calculating the mean of a set of rotations lies in the fact that rotations do not belong to a vector space, but lie on a non-linear manifold. If we choose to represent the rotation by a quaternion $\hat{\mathbf{q}} = (q, \mathbf{q})$ the proper rotations are constrained to the three-dimensional unit sphere $|\hat{\mathbf{q}}| = 1$ of a four-dimensional vector space. Another representation is obtained using a 3×3 matrix \mathbf{R} constrained to be orthogonal, $\mathbf{R}^T = \mathbf{R}^{-1}$, i.e. a three-dimensional manifold in a nine-dimensional space. Despite the non-linearity of rotations it has become customary to

calculate the average by the barycenter in the larger vector spaces. If we let underscore denote the mean, we have $\underline{\mathbf{R}} = \frac{1}{n} \sum_i \mathbf{R}_i$ or $\underline{\hat{\mathbf{q}}} = \frac{1}{n} \sum_i \hat{\mathbf{q}}_i$. There is, however, no guarantee that these means lie on the non-linear manifolds of allowable rotations. To ensure this, the rotation matrix must be orthogonalized, finding the closest orthogonal matrix to the calculated mean, or the mean quaternion must be re-normalized to lie on the unit sphere.

Recently, Pennec (1998) and Pennec et al. (1998) have addressed the problem of taking the mean of geometric features in a very general framework. Their approach is to consider the geodesic distance between points on the non-linear manifold instead of the vector distance. The geodesic distance is the length of the shortest curve lying on the manifold and connecting the two points. Their thorough work applies to all geometric features on non-linear manifolds and also considers Mahalanobis distances in place of the simple least squares with scalar weights. They use the particularly simple example of rotations to demonstrate their theory. Without questioning the importance of the theory described above, this paper will try to answer the following question: *how well do the barycentric means of rotations approximate the mean based on the Riemannian metric.*

The question if other linearizations are superior to the two barycentric means is not addressed here. In

fact, the paper will not address the problem of how to model the noise on the input rotations and no assumption will therefore be made that the least squares mean on the non-linear manifold is necessarily the maximum likelihood estimator. These considerations must be carried out by the researcher based on the actual origin of the sample of rotations. The argument stated in this paper is only that if the least squares mean is found to be appropriate, then the results obtained by averaging rotation matrices or quaternions prove to be good approximations. The comparison is based on the equally weighted average, but applies just as well to the weighted case.

2. Notation

In the above section, different notations were introduced. Scalars are written in lower case, vectors are lower case bold, matrices are bold capitals, quaternions are lower case bold with dots with the scalar part q first and the vectorial part \mathbf{q} last, and mean values are expressed with an underscore. The abbreviation *ort.* denotes the orthogonality constraint $\mathbf{R}^{-1} = \mathbf{R}^T$ on a matrix. The distance between two features is denoted $d(\cdot, \cdot)$, and $\theta(\cdot)$ is the angle of rotation induced by the quaternion or matrix argument. The cross product and dot product operators of vectors are ' \times ' and ' \cdot ', respectively. The determinant of a matrix is denoted $\det(\cdot)$ and $\text{diag}(\dots)$ is a diagonal matrix with the arguments on the diagonal.

3. The Invariant Metric

We have already argued that the vectorial distance $\|\dot{\mathbf{q}}_1 - \dot{\mathbf{q}}_2\|_2$ or $\|\mathbf{R}_1 - \mathbf{R}_2\|_2$ between two rotations ignores the curvature of the respective manifolds. We need a distance measure $d(\mathbf{R}_1, \mathbf{R}_2)$ that expresses the distance along a curve *on* the manifold. Rotations belong to a multiplicative group, i.e. rotations act on rotations by multiplication and form new rotations. It is therefore natural to require that the metric is invariant to rotations $d(\mathbf{R}_1, \mathbf{R}_2) = d(\mathbf{I}, \mathbf{R}_1^{-1}\mathbf{R}_2)$, such that the distance only depends on a single rotational argument. Invariance to orientation leads to the metric $d(\mathbf{R}_1, \mathbf{R}_2) = \theta(\mathbf{R}_1^{-1}\mathbf{R}_2)$, which is the metric used by Pennec (1998).

In the following sections we will investigate how this metric is related to the barycentric mean approaches.

4. Quaternion-Based Mean

Rotations can be expressed using unit quaternions. A quaternion is a four-dimensional vector, that has a scalar part and a three dimensional vector part. We can also view it as a sort of complex number with a real part and three imaginary parts. A rotation through an angle θ around the unit vector \mathbf{n} is represented by the quaternion $\dot{\mathbf{q}} = (\cos \theta/2, \mathbf{n} \sin \theta/2)$. It is easily verified that $\dot{\mathbf{q}}$ and $-\dot{\mathbf{q}}$ give rise to the same rotation. Having a set of rotations, we may choose consistent representatives by requiring that the dot product with the first quaternion is positive. For all reasonable sets of rotations this approach will work. It will occasionally fail if the quaternions scatter all over the unit sphere, but in that case the mean does not make much sense anyway. The inverse of a rotation $\dot{\mathbf{q}}^{-1}$ is obtained by the conjugate quaternion $\dot{\mathbf{q}}^* = (q, -\mathbf{q})$.

The composition of rotations represented by quaternions is achieved by the quaternion product defined by

$$\dot{\mathbf{q}}_1 \dot{\mathbf{q}}_2 = (q_1 \cdot q_2 - \mathbf{q}_1 \cdot \mathbf{q}_2, q_1 \mathbf{q}_2 + q_2 \mathbf{q}_1 + \mathbf{q}_1 \times \mathbf{q}_2) \quad (1)$$

We see immediately that $\dot{\mathbf{q}}\dot{\mathbf{q}}^* = 1$, verifying that multiplication with the conjugate rotation yields the identity. According to the metric derived above the distance between two rotations is

$$d(\dot{\mathbf{q}}_1, \dot{\mathbf{q}}_2) = \theta(\dot{\mathbf{q}}_1^{-1} \dot{\mathbf{q}}_2) \quad (2)$$

$$= \theta(\dot{\mathbf{q}}_1^* \dot{\mathbf{q}}_2) \quad (3)$$

$$= \theta((\dot{\mathbf{q}}_1 \cdot \dot{\mathbf{q}}_2, q_1 \mathbf{q}_2 - q_2 \mathbf{q}_1 - \mathbf{q}_1 \times \mathbf{q}_2)) \quad (4)$$

$$= 2 \arccos(\dot{\mathbf{q}}_1 \cdot \dot{\mathbf{q}}_2), \quad (5)$$

where it was used in the last equality that the real part of a quaternion is the cosine of half the angle of rotation. It is not surprising that the distance along a geodesic takes this form. We know from classical geometry that the geodesics on a sphere are formed by the great circles and the distance between two points on the unit sphere is therefore the angle between the corresponding vectors. Neglecting the difference in scale, this is exactly what is derived in Eq. (5).

As expected the difference between two rotations is a non-linear function of the vectorial representations. To arrive at a closed form approximation to the mean, we need to linearize this expression. Consider the Taylor expansion of cosine

$$\cos \frac{\theta}{2} = 1 - \frac{1}{8}\theta^2 + o(\theta^4). \quad (6)$$

Rearranging this expression gives

$$\theta^2(\dot{\mathbf{q}}_1^* \dot{\mathbf{q}}_2) = 8 \left(1 - \cos \frac{\theta(\dot{\mathbf{q}}_1^* \dot{\mathbf{q}}_2)}{2} \right) + o(\theta^4) \quad (7)$$

$$\approx 8(1 - \dot{\mathbf{q}}_1 \cdot \dot{\mathbf{q}}_2) \quad (8)$$

Now, the mean rotation is the rotation $\dot{\mathbf{q}}$ that minimizes the quadratic cost function

$$\chi^2 = \sum_i \theta^2(\dot{\mathbf{q}}^* \dot{\mathbf{q}}_i) \quad (9)$$

We have not considered weights w_i in this expression, but they are straightforwardly included. Using the approximation above, the mean rotation is determined as

$$\underline{\dot{\mathbf{q}}} = \arg \min_{|\dot{\mathbf{q}}|=1} \sum_i \theta^2(\dot{\mathbf{q}}^* \dot{\mathbf{q}}_i) \quad (10)$$

$$\approx \arg \min_{|\dot{\mathbf{q}}|=1} \sum_i 8(1 - \dot{\mathbf{q}} \cdot \dot{\mathbf{q}}_i) \quad (11)$$

$$= \arg \max_{|\dot{\mathbf{q}}|=1} \sum_i \dot{\mathbf{q}} \cdot \dot{\mathbf{q}}_i \quad (12)$$

$$= \arg \max_{|\dot{\mathbf{q}}|=1} \dot{\mathbf{q}} \cdot \sum_i \dot{\mathbf{q}}_i \quad (13)$$

$$= \sum_i \dot{\mathbf{q}}_i / \left| \sum_i \dot{\mathbf{q}}_i \right| \quad (14)$$

i.e. the optimal quaternion is exactly the barycentric mean with renormalization. It is interesting to note, that in this formulation the re-normalization is part of the true optimization of an approximated cost function and not an ad hoc repair of an approximated optimization.

5. Matrix-Based Mean

Let us now look to another representation of rotations, namely the rotation matrix. This is a 3×3 orthogonal matrix with the rotated basis vectors as columns. Naturally, the composition of two rotations is obtained by usual matrix multiplication and the inverse rotation is consequently the matrix inverse. The eigenvalues of an orthogonal matrix are $(1, e^{i\theta}, e^{-i\theta})$, where the eigenvector corresponding to the unit eigenvalue represents the rotation axis. The trace of a square matrix is the sum of the eigenvalues, such that

$$\text{tr}(\mathbf{R}) = 1 + e^{i\theta} + e^{-i\theta} \quad (15)$$

$$= 1 + 2 \cos \theta \quad (16)$$

Using this relation to evaluate the distance between two rotations represented by matrices we get

$$d(\mathbf{R}_1, \mathbf{R}_2) = \theta(\mathbf{R}_1^{-1} \mathbf{R}_2) \quad (17)$$

$$= \arccos \left(\frac{\text{tr}(\mathbf{R}_1^{-1} \mathbf{R}_2) - 1}{2} \right) \quad (18)$$

This time we use the usual Taylor expansion of cosine

$$\cos \theta = 1 - \frac{1}{2} \theta^2 + o(\theta^4) \quad (19)$$

which gives the expression

$$\theta^2(\mathbf{R}_1^{-1} \mathbf{R}_2) = 2(1 - \cos \theta(\mathbf{R}_1^{-1} \mathbf{R}_2)) + o(\theta^4) \quad (20)$$

$$\approx 2 \left(1 - \frac{\text{tr}(\mathbf{R}_1^{-1} \mathbf{R}_2) - 1}{2} \right) \quad (21)$$

$$= 3 - \text{tr}(\mathbf{R}_1^{-1} \mathbf{R}_2) \quad (22)$$

As in the case of quaternions, the mean rotation is defined by

$$\underline{\mathbf{R}} = \arg \min_{\mathbf{R}_{\text{ort.}}} \sum_i \theta^2(\mathbf{R}^{-1} \mathbf{R}_i) \quad (23)$$

$$\approx \arg \min_{\mathbf{R}_{\text{ort.}}} \sum_i 3 - \text{tr}(\mathbf{R}^{-1} \mathbf{R}_i) \quad (24)$$

$$= \arg \max_{\mathbf{R}_{\text{ort.}}} \sum_i \text{tr}(\mathbf{R}^{-1} \mathbf{R}_i) \quad (25)$$

$$= \arg \max_{\mathbf{R}_{\text{ort.}}} \text{tr} \left(\mathbf{R}^{-1} \sum_i \mathbf{R}_i \right). \quad (26)$$

The latter optimization problem is actually the core of the 3D-3D pose problem, which has been solved among others by Horn (1987), Arun et al. (1987), and Umeyama (1991). The solution can be seen as the rotation that best rotates the identity matrix into the matrix consisting of the sum of rotations. In this light the solution is obviously minimizing the sum of squared distances of the individual elements of the matrix, and is what we understand of a best orthogonal approximation to the sum of rotations. Since the solution is scale invariant, we can also regard it as an orthogonalization of the barycentric mean. Just as for quaternions the orthogonalization is a natural part of the optimization, and is thus more than an ad hoc subsequent correction.

The solution to the above problem is most easily obtained by Singular Value Decomposition (SVD), $\sum_i \mathbf{R}_i = \mathbf{U} \mathbf{D} \mathbf{V}$, where the singular values are ordered in descending order. Introducing the matrix $\mathbf{S} = \text{diag}(1, 1, \det(\mathbf{U}) \det(\mathbf{V}))$ the mean rotation in the

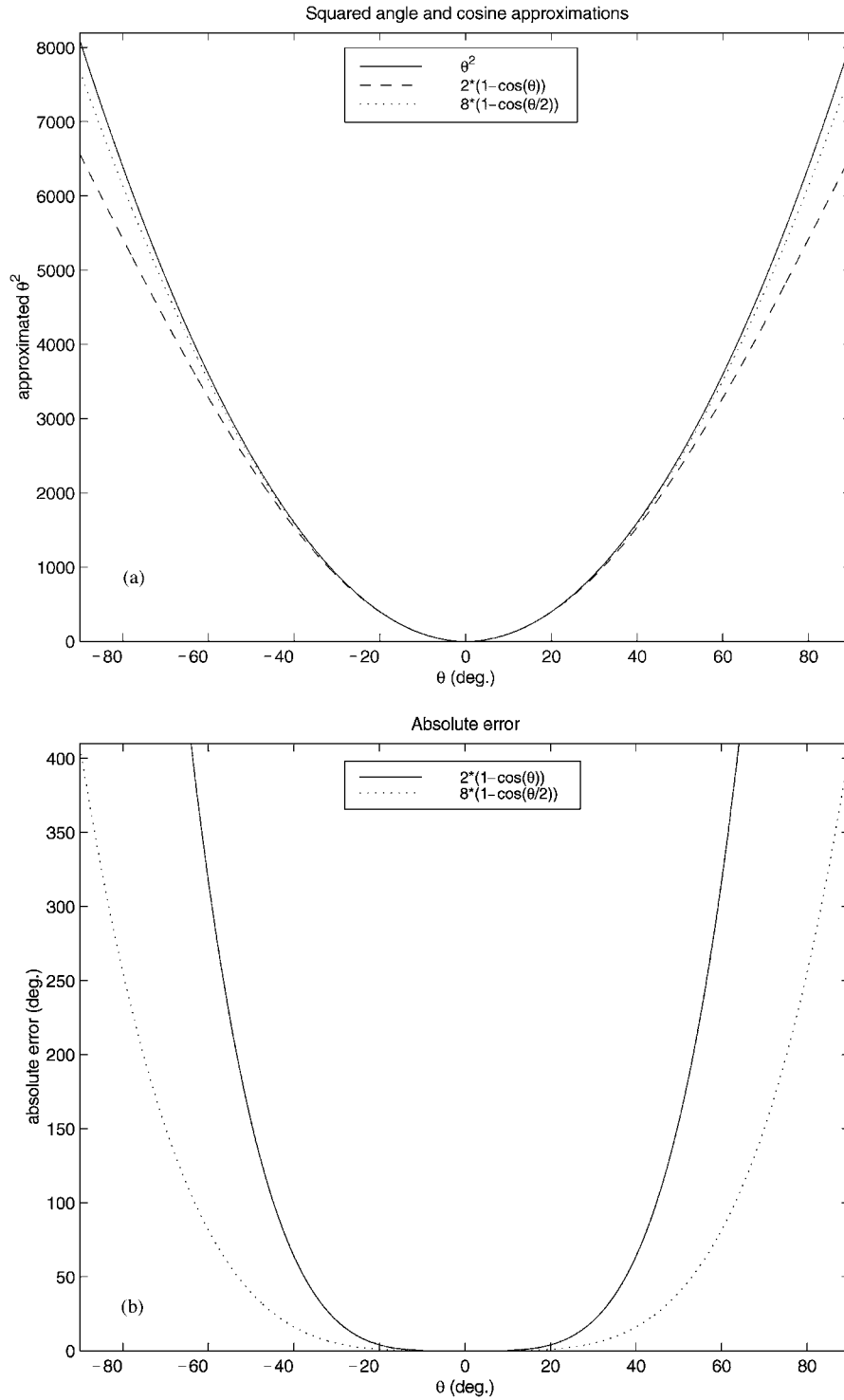


Figure 1. Evaluation of the cosine approximation. (a) The true squared angle and the two approximations. (b) The absolute error of the approximations. (c) The relative error of the approximations. For the quaternion and matrix approximations the angle corresponding to 1% relative error is approximately 40 deg. and 20 deg., respectively.

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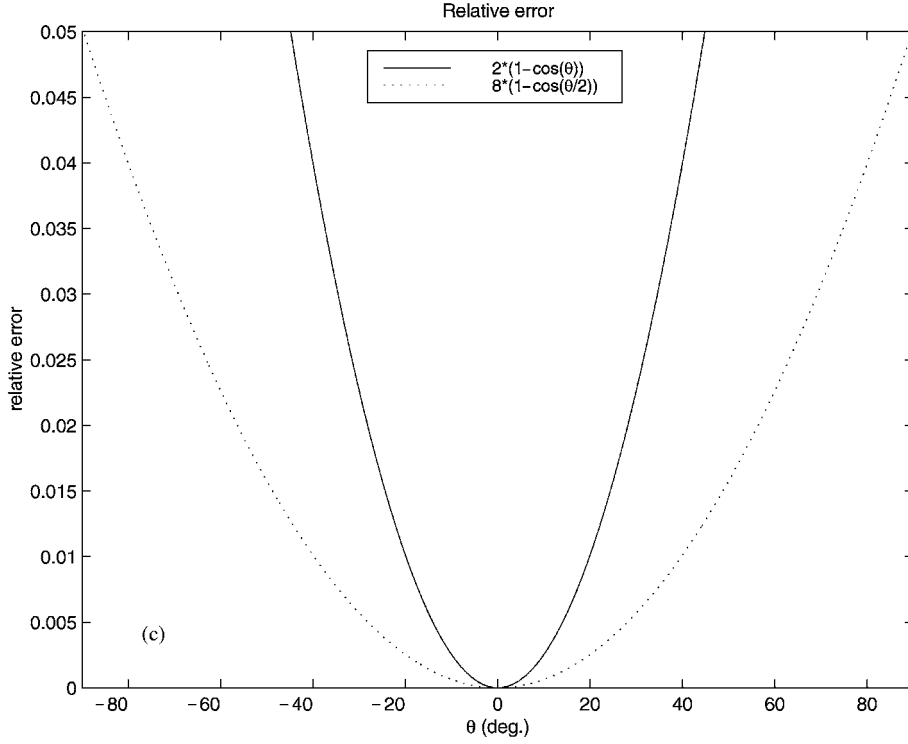


Figure 1. (Continued).

above problem is simply given by

$$\mathbf{R} = \mathbf{U}\mathbf{S}\mathbf{V} \quad (27)$$

Again, individual weights w_i on the rotation matrices were omitted, but they are easy to add.

Both approximations to the Riemannian metric are based on a linearization of the non-linear manifold around the identity rotation. Consequently, the contact point of the linear sub-space is effectively chosen as the estimated mean. This is immediately verified by noticing that any rotation applied to our set of rotations before estimating the mean will, due to linearity, cause a similar rotation of the mean. Thus, the linear approximations have the same invariancy to rotations as the non-linear metric does.

6. Comparison

As derived, both averaging quaternions and rotation matrices can be seen as approximations to the true least squares problem. They are based on the same order of approximation, namely the second order Taylor expansion of cosine. Comparing the two representations we see, however, that they cannot give rise to the same

optimum. The rotation matrix that may be derived from a quaternion contains the elements of the quaternion quadratically. This difference is also reflected in the quality of the approximation. The quaternion solution is based on an approximation of $\cos \theta/2$, whereas the rotation matrix solution is based on an approximation of $\cos \theta$. For the sake of comparison we preserve the scale in the $o()$ notation

$$\text{quaternion : } \theta^2 \approx 8 - 8 \cos \frac{\theta}{2} + o\left(\frac{1}{48}\theta^4\right) \quad (28)$$

$$\text{matrix : } \theta^2 \approx 2 - 2 \cos \theta + o\left(\frac{1}{12}\theta^4\right) \quad (29)$$

For small values of θ , we see that the error caused by the latter approximation is four times larger than for the former approximation. It should be noted, though, that for any reasonable variation in the rotations either one of the approximations is very good. Figure 1 shows the errors related to the two different approximations. When using the quaternion-based method the relative error committed is less than 1% for angles smaller than 40 deg. For the matrix-based method the 1% limit is violated for angles greater than 20 deg.

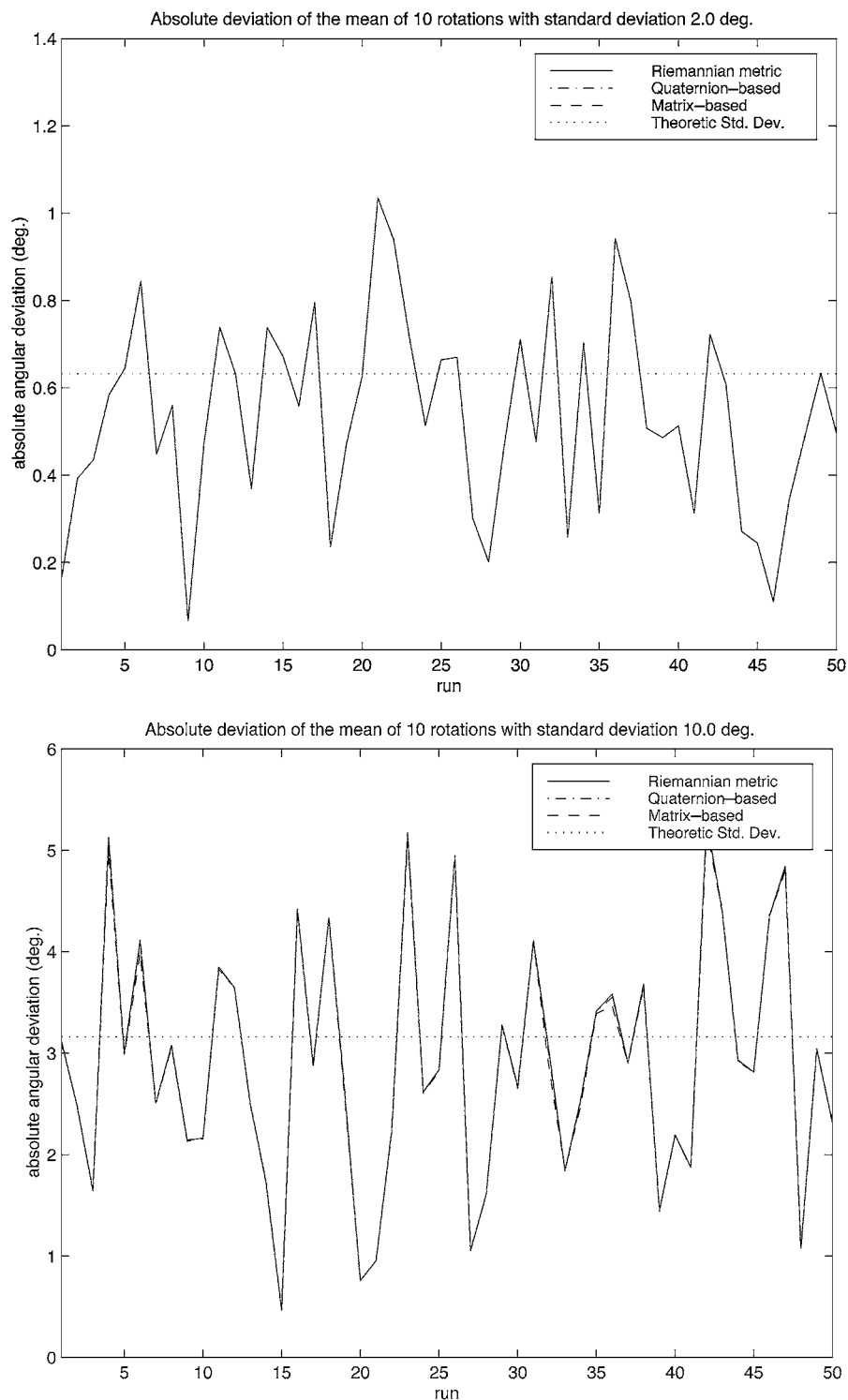


Figure 2. The absolute angular deviation from the identity of the average of 10 rotations for three different values of the standard deviation of the angle of rotation: 2 deg., 10 deg., and 40 deg. In each case the experiment has been repeated 50 times.

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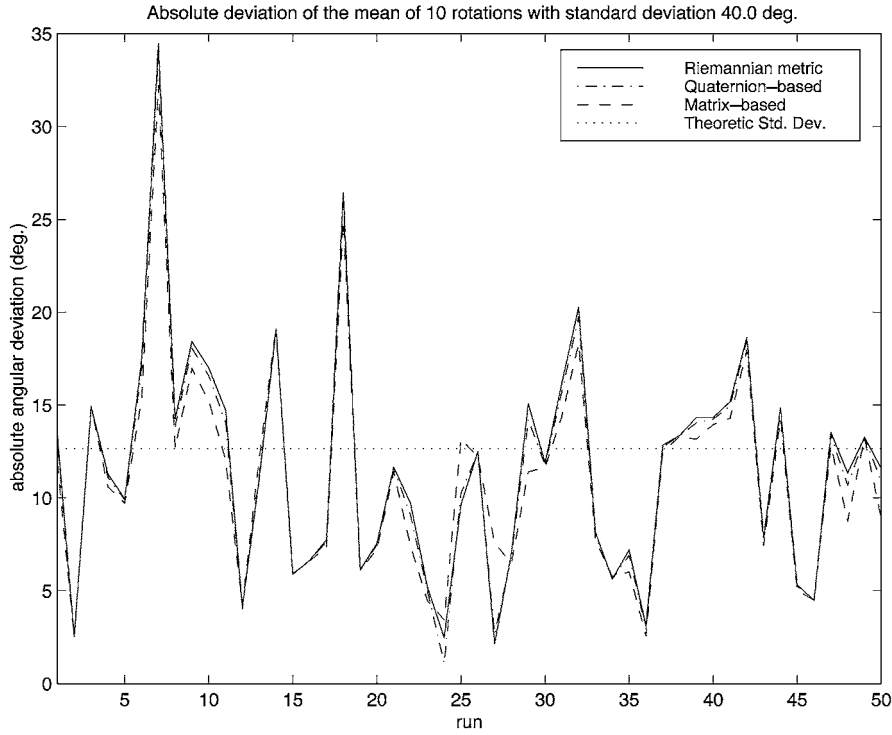


Figure 2. (Continued).

7. Test

A series of tests have been carried out to verify the similarity of the three approaches to taking the mean of rotations: Riemannian metric-based, quaternion-based, and matrix-based. Synthetic samples of rotations have been constructed to compare the different approaches. The rotation space is sampled by sampling the direction and the angle independently. The angle is sampled from a Gaussian distribution. The direction is sampled uniformly over all directions, which seems fair owing to symmetry. Note, that this distribution produces samples lying on the non-linear manifold, such that all samples are true rotations. However, there is no reason to believe that the distribution has the properties of the Gaussian in linear space, where the least squares mean is known to be the maximum likelihood estimator, and we can therefore not expect that the true least squares estimate is necessarily superior to the two linear approximations. Nonetheless, the distribution is a fair choice in the present setup, where the aim is to show that the three methods have a very similar behaviour.

In all experiments the rotations are sampled with the identity as mean. The comments at the end of

Section 5 justifies that the tests are invariant to this choice.

In the first series of experiments we take the mean of only 10 rotations to investigate the robustness for a small number of samples. The experiment is repeated 50 times for each of three different choices of the standard deviation of the angle of rotation: 2 deg., 10 deg., and 40 deg. The quality of the estimation of the mean is measured by the angle of rotation required to bring the mean into the identity. The results are seen in Fig. 2. For the small value of the standard deviation there is no difference to be seen in the estimation. The plots lie on top of each other. As the standard deviation is increased the difference becomes visible. In the experiments the quaternion-based and matrix-based approximations actually seem to be superior to the method based on the Riemannian metric. This could either be an effect of the more robust weighting function caused by the approximation, see Fig. 1, or a mere result of the chosen sampling distribution, but the difference is too insignificant to make final conclusions. Note, that the difference between the methods in each run is significantly smaller than the variation between runs. Note also, that the quaternion approximation is closer to the non-linear

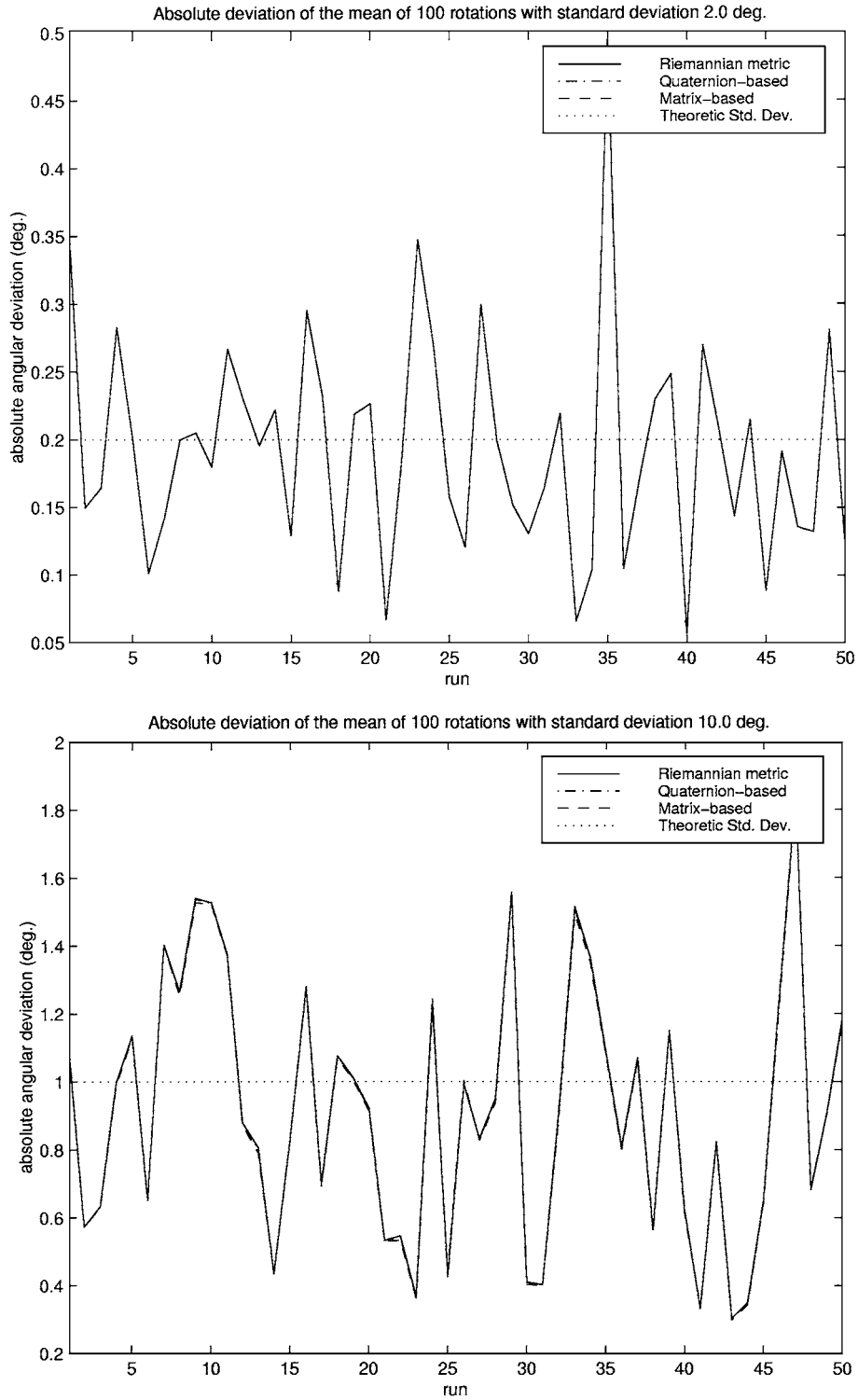


Figure 3. The absolute angular deviation from the identity of the average of 100 rotations for three different values of the standard deviation of the angle of rotation: 2 deg., 10 deg., and 40 deg. In each case the experiment has been repeated 50 times.

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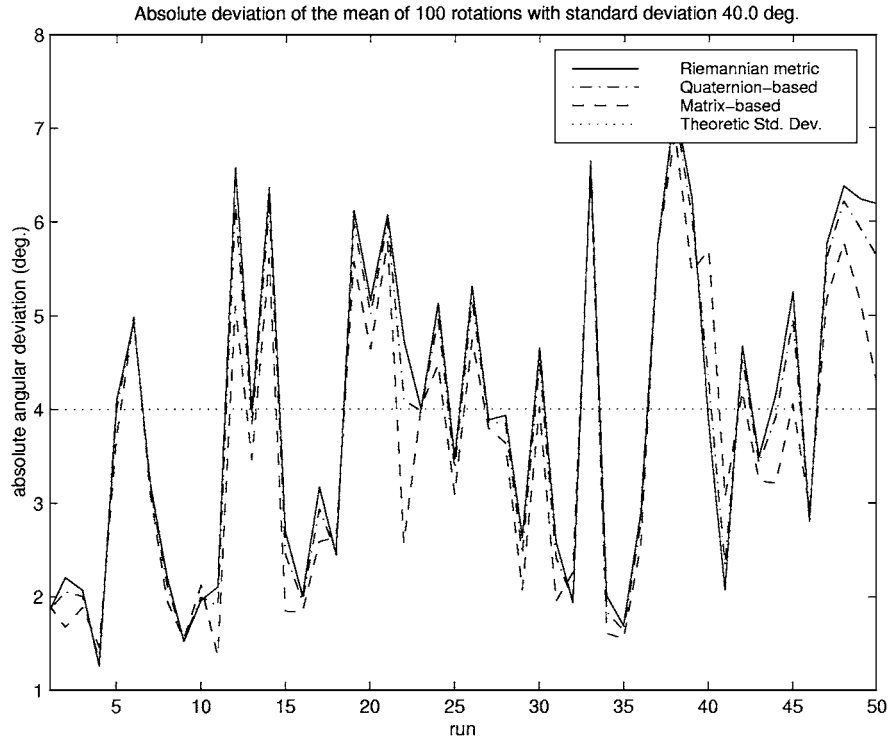


Figure 3. (Continued).

least squares in all runs, as was predicted by the theory in the previous sections.

In another series of experiments the sample size has been increased to 100 to see how the approximations to the true Riemannian metric behave for large sample sizes. The result is displayed in Fig. 3. The three methods seem equally good in this experiment as well.

For all tests the theoretical standard deviation of the mean was calculated as $\sigma_x = \sigma_x / \sqrt{n}$, where n is the number of samples. This value is plotted as a horizontal line in Figs. 2 and 3. As expected about 70% of the mean values lie within one standard deviation.

In one of the Riemannian metric experiments with a standard deviation of 40 deg in the sample of rotations, the *MATLAB*-routine **fmins** failed to converge in 800 iterations. The starting point of the optimization was in all cases the output from the linear quaternion method. When the identity rotation was used as starting point the test showed 34 cases of divergence. Of course, there is no convergence issue in the closed form solutions.

The relative pose of a ship part has been determined from a single image to give an idea about the quantities discussed above. To have a measure of ground truth, the example is purely synthetic. The CAD model of the part is shown in Fig. 4 where the features that are

used for pose determination have been circled. The part measures $80 \times 90 \times 50$ cm and is viewed from a distance of 2.5 m. The synthetic camera has a focal length of $f = 1040$ pixels (8 mm) and an image size of

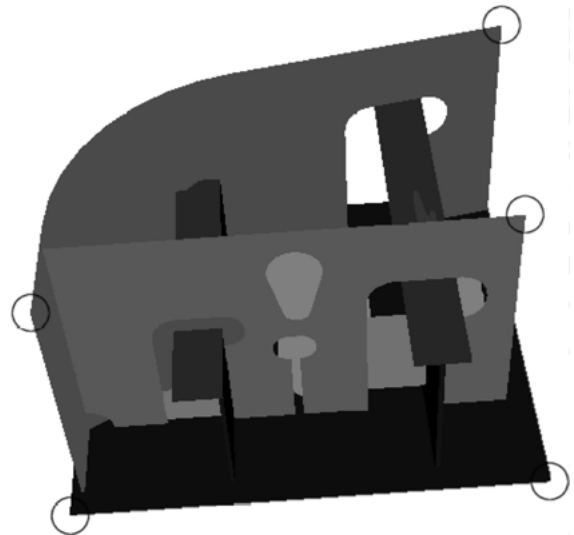


Figure 4. CAD model of the ship part that is used for pose determination, see the text.

768×576 . The location of the features have been perturbed with a uniform distribution in the interval $[-5; 5]$ pixels, which is a rather large noise level. The drawn circles have a radius of 15 pixels. The pose was determined 20 times for different noise realizations. On average the absolute angular deviation between the found pose and the nominal pose was 1.3 deg. and the maximum deviation was 2.2 deg. The example shows that variations in estimated orientation are rather modest when reasonable features are chosen. Even when the noise level is high. It is therefore fair to conclude that the three methods for averaging rotations discussed in this paper are equally good for practical purposes.

8. Conclusion

In this paper the problem of averaging rotations has been investigated. The customary approaches based on the barycenter of either the quaternion representations or the matrix representations have been compared to the 'exact' least squares method based on the Riemannian metric. Theoretical and empirical analysis have shown that both linear approaches give very good approximations to the non-linear average. It is therefore fair to use either of the approximations, if the estimation problem in question justifies the choice of the least squares estimator.

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References

- Arun, K., Huang, T., and Blostein, S. 1987. Least-squares fitting of two 3-D point sets. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 9(5):698–700.
- Horn, B. 1987. Closed-form solution of absolute orientation using unit quaternions. *J. Opt. Soc. Am. A*, 4(4):629–642.
- Pennec, X. 1998. Computing the mean of geometric features—Application to the mean rotation. Institut National de Recherche en Informatique et en Automatique, Rapport de Recherche 3371.
- Pennec, X., Guttman, C., and Thirion, J.-P. 1998. Feature-based registration of medical images: Estimation and validation of the pose accuracy. In *Medical Image Computing and Computer-Assisted Intervention (MICCAI'98)*, Cambridge, MA, pp. 1107–1114.
- Umeyama, S. 1991. Least-squares estimation of transformation parameters between two point patterns. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 13(4):376–380.