Crude Monte Carlo estimation

Let us first remind the crude Monte Carlo estimation procedure. Suppose that we want to approximate some integral I using simulations. I has the form

$$I = \int g(x)f(x)dx$$

where $g(x)$ is some function of a continuous r.v. X and $f(x)$ is its density. We can draw a sample of size m from the density $f(x)$ and then use the empirical average to get a Monte Carlo estimate of I

$$\hat{I} = \frac{1}{m}\sum_{i=1}^{m} g(x_i)$$

The standard error $se(\hat{I})$ is given by

$$se(\hat{I}) = \sqrt{\frac{1}{m^2}\sum_{i=1}^{m}\left(g(x_i) - \hat{I}\right)^2}$$

Antithetic variates method

Now let us introduce the antithetic variates method, for Monte Carlo variance reduction. For two iid r.v. X_1 and X_2, we have that

$$var\left(E[X]\right) = var\left(\frac{1}{2}\sum_{i=1}^{2} X_i\right) = \frac{1}{4}[var(X_1) + var(X_2)]$$

But when X_1 and X_2 are NOT independent, we have that

$$var\left(E[X]\right) = var\left(\frac{1}{2}\sum_{i=1}^{2} X_i\right) = \frac{1}{4}[var(X_1)+var(X_2)+2cov(X_1, X_2)]$$

so when $cov(X_1, X_2) < 0$, then the overall variance is lower compared to the case when X_1 and X_2 are independent.

Antithetic variates estimator

Suppose we first generate $U_1 \sim U[0,1]$, then $U_2 = 1 - U_1 \sim U[0,1]$ and the $\rho_{U_1,U_2} = -1$. We have two samples $U_{1(1)}, ...U_{1(m/2)}$ and $U_{2(1)}, ...U_{2(m/2)}$. Then, using the $m/2$ sample of U_1, we generate $m/2$ realizations of X_1, with PDF $f(x_1)$, and using the $m/2$ sample of U_2, we generate $m/2$ realizations of X_2, with PDF $f(x_2)$. This can be done for instance using the inverse CDF method: we would have $X_1 = F^{-1}(U_1)$ and $X_2 = F^{-1}(U_2)$. Then we compute $X = \big(g(X_1) + g(X_2)\big)/2$. Finally, our antithetic estimator is given gy

$$\hat{I}_A = \frac{2}{m}\sum_{i=1}^{m/2} x_i = \frac{2}{m}\sum_{i=1}^{m/2} \big(g(X_{1(i)}) + g(X_{2(i)})\big)/2$$

Antithetic Variates standard error estimator

The standard error $se(\hat{I}_A)$ is given by

$$se(\hat{I}_A) = \sqrt{\frac{4}{m^2} \sum_{i=1}^{m/2} \left(x_i - \hat{I}_A \right)^2}$$

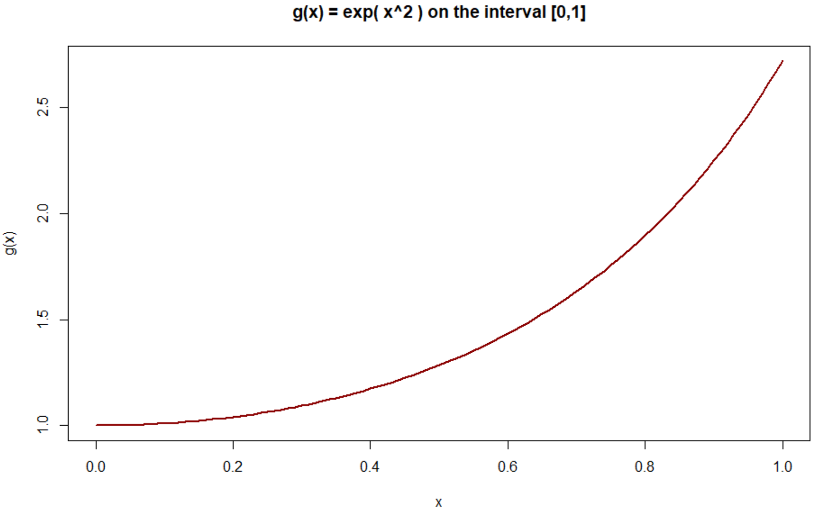
Working example (1/3)

Example We want to approximate the following integral using the crude Monte Carlo estimator and the antithetic estimator

$$I = \int_0^1 g(x)dx \qquad \text{with } g(x) = e^{x^2}$$

We will generate $m = 10,000$ realizations of $U \sim U[0,1]$ to approximate this quantity. $m = 5,000$ realizations of $U \sim U[0,1]$, then we take $1 - U$ for the antithetic method. We obtain the following results using R.

Working example (2/3)



Working example (3/3)

	$\hat{I}$	$se(\hat{I})$
crude MC estimate	1.463267	0.004767316
Antithetic estimate	1.463597	0.002362512

Clearly, the antithetic estimator has a lower variance, thus converging faster.

Antithetic variates in R

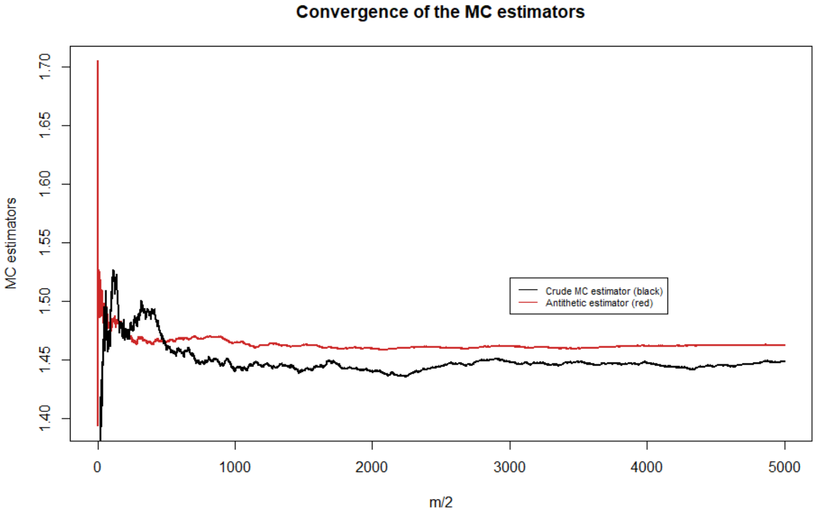
```r
m <- 10000 # number of simulations
set.seed(1986) # for replication

# MC crude estimation
u <- runif(m)
gx_c <- exp(u^2)
I_hat_crude <- mean(gx_c)
I_hat_crude_2 <- (1/m) * sum(gx_c)
se_I_hat_crude <- sqrt(var(gx_c) / m)
se_I_hat_crude_2 <- sqrt(((1/m)^2)*sum((gx_c - I_hat_crude_2)^2 ))

# Antithetic estimation
u <- runif(m/2)
gx1 <- exp(u^2)
gx2 <- exp((1-u)^2)
gx <-(gx1+gx2)/2  # length(gx) [1] 5000
I_hat_anti <- mean(gx)
I_hat_anti_2 <- 2*(1/m) * sum(gx)
se_I_hat_anti <- sqrt(var(gx)/(m/2) )
se_I_hat_anti_2 <- sqrt((4* (1/m)^2)*sum((gx - I_hat_anti_2)^2 ))

# results
results <- matrix(c(I_hat_crude,I_hat_anti,se_I_hat_crude, se_I_hat_anti),nrow=2)
colnames(results) <- c("estimate", "sd")
rownames(results) <- c("crude_MC", "Antithetic")
results
#              estimate          sd
# crude MC    1.463267 0.004767316
# Antithetic 1.463597 0.002362512
```

Visualizing convergence



References

Rizzo, M.L. (2019). Statistical Computing with R, Second Edition (2nd ed.). Chapman and Hall/CRC. https://doi.org/10.1201/9780429192760

The R Project for Statistical Computing: https://www.r-project.org/

Python: https://www.python.org/

course notes