Kruskal-Wallis test: introduction

The Kruskal-Wallis test is a nonparametric test in the sense that we do NOT make the assumption that the data were generated from a parametric distribution.

This test uses the idea of the so-called 'one-way ANOVA', where inter-class variation is compared to the total variation. In this non-parametric alternative, we consider the variations of the ranks. It is a generalization of the Mann-Whitney-Wilcoxon test.

Test statistic

Assume that we have k samples $s_1, ..., s_k$ and let us consider a Kruskal-Wallis test, which has null hypothesis H_0: 'All $k \geq 2$ samples come from the same generating distribution' versus H_1: 'At least two samples have a different generating distribution', or equivalently H_0: 'All samples $k \geq 2$ come from the same population' versus H_1: 'At least two samples come from a different population'.

Let n_j be the size of the sample s_j, for $j = 1, ..., k$ Let $N = \sum_{j=1}^{k} n_j$. Let S be the pooled sample, with all observations from $s_1, ..., s_k$ and $R = rank(S)$. Finally, let $\bar{R}_j$ be the average rank of the observations from the sample s_j in the pooled sample S. Then, the Kruskal-Wallis test statistic, KW, is given by

$$KW = \frac{12}{N(N-1)} \sum_{j=1}^{k} n_j \left(\bar{R}_j - \frac{N+1}{2} \right)^2 = \frac{12}{N(N-1)} \sum_{j=1}^{k} n_j \bar{R}_j^2 - 3\left(N+1\right)$$

Distribution of the test statistic and decision rule

The distribution of the test statistic KW under H_0 depends on the n_j's and is thus difficult to find in tables. However, the distribution of KW can be simulated from (see example).

The critical region R_α has the form $R_\alpha = \left\{ q_{1-\alpha}, ..., c \right\}$,

where $q_{1-\alpha}$ is the $(1-\alpha)$-quantile of the distribution of KW under H_0 and c is the maximum value that can take the test statistic.

When the n_j's are large, then, under H_0, the test statistic KW has the following distribution asymptotically

$$KW \sim \chi^2_{(k-1)}$$

H_0 is then rejected with an approximate α level of significance if $KW > \chi^2_{1-\alpha,(k-1)}$.

Working example (1/3)

Example: With the data of the Seabelts dataset of R, we compute casualty rates for the first semesters of years 1978 to 1981. The data are shown in the table below. Compute manually and in R the Kruskal-Wallis test statistic. Then, simulate from the exact distribution of the test statistic under H_0 and determine if H_0 is rejected for those data at $\alpha = 5\%$ level of significance.

i	s_1	s_2	s_3	s_4
1	15.791	16.193	11.870	9.681
2	12.595	11.937	9.400	9.764
3	10.405	11.968	9.322	9.154
4	9.836	9.376	8.200	8.330
5	8.729	9.227	8.020	8.388
6	9.608	8.539	8.671	7.888

Working example (2/3)

The ranks R_j in the pooled sample and the average ranks $\bar{R}_j$ are given in the following tables.

i	R_1	R_2	R_3	R_4
1	23	24	19	15
2	22	20	13	16
3	18	21	11	9
4	17	12	3	4
5	8	10	2	5
6	14	6	7	1

$\bar{R}_1$	$\bar{R}_2$	$\bar{R}_3$	$\bar{R}_4$
17	15.5	9.17	8.33

Working example (3/3)

In addition, we have $n_1 = n_2 = n_3 = n_4 = 6$ and N $= \sum_{j=1}^{k} n_j = 24$. The value of our test statistic KW is thus

$$\begin{aligned}\frac{12}{N(N-1)} \sum_{j=1}^{k} n_j \left(\bar{R}_j - \frac{N+1}{2}\right)^2 &= \left(\frac{12}{24(24-1)}\right) \Big[6(17-12.5)^2 + \\ &\quad 6(15.5-12.5)^2 + 6(9.17-12.5)^2 + 6(8.33-12.5)^2 \Big] \\ &= 6.927\end{aligned}$$

Since the 0.95-quantile of a $\chi_{(3)} = 7.815$, we can not reject H_0 in this case.

References

Bagdonavičius V., Kruopis J., Nikulin M. S., Non-parametric Tests for Complete Data (2011), Wiley, ISBN 978-1-84821-269-5 (hardback)

The R Project for Statistical Computing: https://www.r-project.org/

course notes