Sign test: introduction

The Sign test is a nonparametric test in the sense that we do NOT make the assumption that the data were generated from a parametric distribution.

The Sign test allows us to test hypotheses on a location parameter, typically the median, but it could be another quantile of the distribution. We also do NOT assume that this distribution is continuous.
0
The biliateral test only exists when we do the test on the median. Unilateral tests could be done on other quantiles.

Test statistic (1/2)

Let us consider a Sign test, which has null hypothesis $H_0 : med(X) = \theta_0$ with $\theta_0 \in \mathbb{R}$, i.e. H_0: 'The median of the data is θ_0'. We observe a sample $x_1, ..., x_n$. The Sign test statistic is

$$T(x_1, ..., x_n) = \sum_{i=1}^{n} \mathbb{1}_{x_i > \theta_0} = \sum_{i=1}^{n} \mathbb{1}_{x_i - \theta_0 > 0}$$

where

$$\mathbb{1}_{x_i > \theta_0} = \begin{cases} 1, & \text{if } x_i > \theta_0. \\ 0, & \text{otherwise.} \end{cases}$$

Test statistic (2/2)

When none of the x_i are equal to θ_0, then the test statistic can be rewritten as $T(x_1, ..., x_n) = (S + n)/2$ where S is equal to

$$S = \sum_{i=1}^{n} sign(x_i - \theta_0)$$

with $sign(x_i - \theta_0) = 1$ if $x_i > \theta_0$ and -1 if $x_i < \theta_0$. Those two versions of the Sign test are equivalent.

Distribution of the test statistic

Under H_0, the distribution of $\mathbb{1}_{x_i>\theta_0}$ is Bernoulli with parameter $p = 1 - F_X(\theta_0)$, where F_X is the CDF of X. It follows that the test statistic is Binomial, with $p = 1/2$ if we do test on the median. We have indeed

$$T(x_1, ..., x_n) \sim B(n, p = 1/2)$$

So that we have

$E[\mathbb{1}_{x_i>\theta_0}] = p = 1/2$ and variance $var(\mathbb{1}_{x_i>\theta_0}) = p(1-p) = 1/4$

and $E[T(x_1, ..., x_n)] = E[\sum_{i=1}^n \mathbb{1}_{x_i>\theta_0}] = np = n/2$ and variance $var(T(x_1, ..., x_n)) = np(1-p) = n/4$.

Asymptotic distribution of the test statistic

In addition, from the CLT, it follows that

$$\frac{\sum_{i=1}^{n} x_i - E[X]}{\sqrt{n}\ sd(X)} \sim N(0,1)$$

$$\sum_{i=1}^{n} x_i \sim N(nE[X], nvar(X))$$

$$\frac{1}{n}\sum_{i=1}^{n} x_i \sim N(E[X], \frac{var(X)}{n})$$

$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{x_i > \theta_0} \sim N(1/2, 1/4n)$$

$$T(x_1, ..., x_n) = \sum_{i=1}^{n} \mathbb{1}_{x_i > \theta_0} \sim N(n/2, n/4)$$

Asymptotic critical region

Let us in addition consider an alternative hypothesis $H_1 : med(X) \neq \theta_0$, that allows us to define a critical region R_α. If $T(x_1, ..., x_n) \in R_\alpha$, H_0 is rejected at level α. Else, there is not enough evidence against H_0 at level α. The critical region is given by

$$R_\alpha = \left\{0, ..., k\right\} \bigcup \left\{n - k, ..., n\right\}$$

where k is the largest integer such that $P_{H_0}(T(x_1, ..., x_n) \in R_\alpha) \leq \alpha$. Using the CLT, the large sample critical region is then becomes

$$R_\alpha = \left\{0, ..., n/2 - q_{1-\frac{\alpha}{2}}\sqrt{n/4}\right\} \bigcup \left\{n/2 + q_{1-\frac{\alpha}{2}}\sqrt{n/4}, ..., n\right\}$$

where $q_{1-\frac{\alpha}{2}} \approx 1.96$ if $\alpha = 0.05$.

Working example 1 (1/3)

Example 1: We test the hypothesis $H_0 : med(X) = 280$. We want to perform the test at level $\alpha = 5\%$ against the alternative hypothesis $H_1 : med(X) \neq 280$. Suppose that we observe the following data.

n	x_i	$\mid x_i - 280 \mid$	$sign(x_i - 280)$
1	275	15	-
2	292	12	$+$
3	281	1	$+$
4	284	4	$+$
5	285	5	$+$
6	283	3	$+$
7	290	10	$+$
8	294	14	$+$
9	300	20	$+$
10	284	4	$+$

Working example 1 (2/3)

The Sign test statistic is therefore equal to $T(x_1, ...x_{10}) = (S + n)/2 = (9 + 10)/2 \approx 9$ or $\sum_{i=1}^{n} \mathbb{1}_{x_i - \theta_0 > 0} = 9$. If we choose $k = 1$, the critical region at level $\alpha = 5\%$ is then

$$R_{0.05} = \left\{0, ..., 1\right\} \bigcup \left\{9, ..., 10\right\}$$

Since $T(x_1, ...x_{10}) \in R_{0.05}$, we conclude that H_0 is rejected and that the median of the data is NOT equal to 280. We note that the true level of the test (probability of type I error) is equal to 2.15% and not 5%.

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> 2*pbinom(q = 1, size = 10, prob = 1/2)
[1] 0.02148438
```

Working example 1 (3/3)

If we choose $k = 2$, the critical region becomes

$$R_{0.05} = \Big\{0, ..., 2\Big\} \bigcup \Big\{8, ..., 10\Big\},$$

and the level of the test is

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> 2*pbinom(q = 2, size = 10, prob = 1/2)
[1] 0.109375
```

So we conclude that $k = 1$ is optimal in this case.

Working example 2

Example 2: We consider a sample of $n = 64$ observations. Out of theses 64 values, 20 are larger than some median θ_0. What is the critical region R_α and the p-value of a bilateral test at $\alpha = 5\%$?

The critical region at level $\alpha = 5\%$ is then

$$\begin{aligned} R_{0.05} = &\left\{0, ..., 64/2 - q_{1-\frac{0.05}{2}}\sqrt{64/4}\right\} \bigcup \\ &\left\{64/2 + q_{1-\frac{0.05}{2}}\sqrt{64/4}, ..., 64\right\} \\ = &\left\{0, ..., 24\right\} \bigcup \left\{40, ..., 64\right\}, \end{aligned}$$

which is centered on $n/2 = 64/2 = 32$. Since $T(x_1, ..., x_n) \in R_\alpha$, we reject H_0 at 0.05 level of significance. The p-value is $2 * P(T \leq 20) \Leftrightarrow 2 * P\left(\frac{T-32}{4} \leq \frac{20-32}{4}\right) \Leftrightarrow 2 * P(z \leq -3) \Leftrightarrow 2 * \Phi(-3) = 2 * 0.01 = 0.02$

References

Bagdonavičius V., Kruopis J., Nikulin M. S., Non-parametric Tests for Complete Data (2011), Wiley, ISBN 978-1-84821-269-5 (hardback)

The R Project for Statistical Computing: https://www.r-project.org/

course notes