Normal model: introduction

Let us consider the following Normal model, for which μ and σ^2 are parameters.

$$\left\{N(\mu, \sigma^2); \mu \in \mathbb{R},\ \sigma^2 > 0\right\}$$

The empirical mean $\overline{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$ is a natural estimator for μ. As we have seen, it is the Maximum Likelihood Estimator (MLE) and the Method of Moments (MoM) estimator for μ.

Consistency of estimators (1/4)

Consistency: An estimator $\hat{\theta}_n$ is said to be consistent for θ if it converges in probability to the true value of the parameter. We have that $\hat{\theta}_n \xrightarrow{P} \theta$ as n goes to ∞.
By the Law of Large Numbers (LLN), we have that

$$\overline{x}_n \quad \xrightarrow{P} \quad E[X] \qquad \text{as } n \to \infty. \text{ Here, } E[X] = \mu$$

So $\overline{x}_n$ is a **consistent** estimator for μ.

Continuous Mapping Theorem: If we have $\hat{\theta}_n \xrightarrow{P} \theta$ as n goes to ∞, then for a continuous function $g(\hat{\theta}_n)$, the convergence in probability is peserved. We have that $g(\hat{\theta}_n) \xrightarrow{P} g(\theta)$ as n goes to ∞.

Consistency of estimators (2/4)

So by the Continuous Mapping Theorem (CMT), here for $g(x) = x^2$, we have that

$$(\overline{x}_n)^2 \quad \xrightarrow{\mathcal{P}} \quad \left(E[X]\right)^2 \qquad\qquad \text{as } n \to \infty. \text{ Here, } \left(E[X]\right)^2 = \mu^2$$

Slutsky Theorem: For two random variables X_n and Y_n, if $X_n \xrightarrow{P} X$ as n goes to ∞ and $Y_n \to c$, where c is a constant, then we have that $X_n + Y_n \xrightarrow{P} X + c$ as n goes to ∞.

Consistency of estimators (3/4)

We recall that the empirical variance is given by

$$\begin{aligned} S_n^2 &= \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x}_n)^2 \\ &= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - (\overline{x}_n)^2 \end{aligned}$$

And since, by the LLN, we have that

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 \quad \xrightarrow{\mathcal{P}} \quad E[X^2] \qquad \text{as } n \to \infty$$

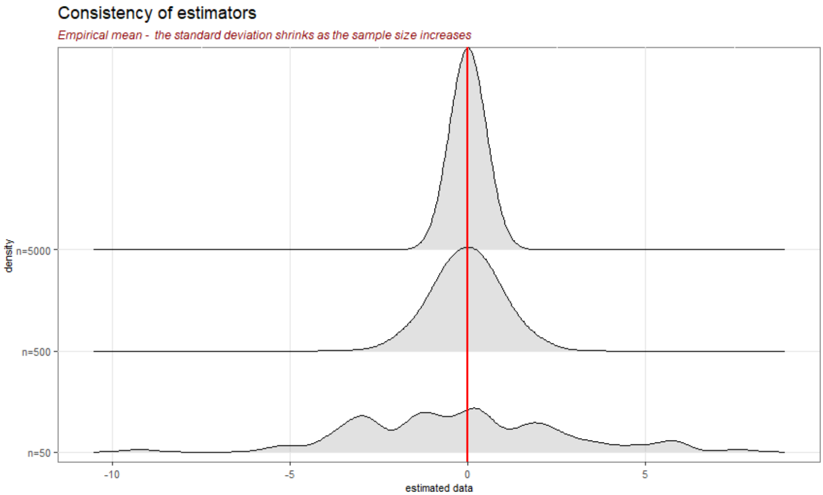
Consistency of estimators (4/4)

Therefore, by the Slutsky theorem, we have that

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 - (\overline{x}_n)^2 \quad \xrightarrow{\mathcal{P}} \quad E[X^2] - \big(E[X]\big)^2$$

So we conclude that S_n^2 is a **consistent** estimator for $var_\theta(X) = \sigma_\theta^2$.

Visualization of consistency



Bias of estimators (1/3)

We consider the empirical variance $S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x}_n)^2$. Is it an unbiased estimator for σ_θ^2 (denoted also equivalently $var_\theta(X)$). Let us first rearrange its expression, and then comptute its expectation.

$$S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x}_n)^2 = \frac{1}{n}\sum_{i=1}^{n}\Big(x_i - E_\theta[X] + E_\theta[X] - \overline{x}_n\Big)^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}\Big(x_i - E_\theta[X]\Big)^2 + \frac{1}{n}\sum_{i=1}^{n}\Big(E_\theta[X] - \overline{x}_n\Big)^2 + 2\frac{1}{n}\sum_{i=1}^{n}\Big(x_i - E_\theta[X]\Big)\Big(E_\theta[X] - \overline{x}_n\Big)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\Big(x_i - E_\theta[X]\Big)^2 + \underbrace{\Big(E_\theta[X] - \overline{x}_n\Big)^2 + 2\Big(E_\theta[X] - \overline{x}_n\Big)\underbrace{\frac{1}{n}\sum_{i=1}^{n}\Big(x_i - E_\theta[X]\Big)}_{\Big(\overline{x}_n - E_\theta[X]\Big)}}_{\Big(E_\theta[X] - \overline{x}_n\Big)^2 - 2\Big(E_\theta[X] - \overline{x}_n\Big)^2}$$

$$= \frac{1}{n}\sum\Big(x_i - E_\theta[X]\Big)^2 - \Big(E_\theta[X] - \overline{x}_n\Big)^2$$

Bias of estimators (2/3)

$$
\begin{aligned}
E_\theta[S_n^2] &= E_\theta\left[\frac{1}{n}\sum_{i=1}^n (x_i - \overline{x}_n)^2\right] \\
&= E_\theta\left[\frac{1}{n}\sum_{i=1}^n \Big(x_i - E_\theta[X]\Big)^2 - \Big(E_\theta[X] - \overline{x}_n\Big)^2\right] \\
&= E_\theta\left[\frac{1}{n}\sum_{i=1}^n \Big(x_i - E_\theta[X]\Big)^2\right] - E_\theta\left[\Big(E_\theta[X] - \overline{x}_n\Big)^2\right] \\
&= \frac{1}{n}\sum_{i=1}^n \underbrace{E_\theta\left[\Big(x_i - E_\theta[X]\Big)^2\right]}_{var_\theta(X)} - \underbrace{E_\theta\left[\Big(E_\theta[X] - \overline{x}_n\Big)^2\right]}_{var_\theta(\overline{x}_n)} \\
&= \sigma_\theta^2 - var_\theta(\overline{x}_n) \\
&= \sigma_\theta^2 - \frac{\sigma_\theta^2}{n} \\
&= \left(\frac{n-1}{n}\right)\sigma_\theta^2 \qquad\qquad \Big(\neq \sigma_\theta^2 \text{ for all } n\Big).
\end{aligned}
$$

Bias of estimators (3/3)

Therefore we conclude that S_n^2 is a **biased estimator** for $var_\theta(X) = \sigma_\theta^2$. But since $E_\theta[S_n^2] = \left(\frac{n-1}{n}\right) \sigma_\theta^2 \quad \xrightarrow[n\to\infty]{} \quad \sigma_\theta^2$

S_n^2 is an **asymptotically unbiased** estimator for $var_\theta(X) = \sigma_\theta^2$. Now we consider $S_n^{2*} = \frac{1}{n-1}\sum_{i=1}^n (x_i - \overline{x}_n)^2$. We have

$$E_\theta[S_n^{2*}] = E_\theta\left[\frac{1}{n-1}\sum_{i=1}^n (x_i - \overline{x}_n)^2\right] = \frac{n}{n-1}\ E_\theta\left[\frac{1}{n}\sum_{i=1}^n (x_i - \overline{x}_n)^2\right]$$

$$= \frac{n}{n-1}\ \underbrace{\underbrace{E_\theta\left[\sum_{i=1}^n (x_i - \overline{x}_n)^2\right]}_{var_\theta(X)}}_{S_n^2} = \frac{n}{n-1}\ E_\theta[S_n^2] = \frac{n}{n-1}\left(\frac{n-1}{n}\right)\sigma_\theta^2 \ = \ \sigma_\theta^2$$

So S_n^{2*} is an **unbiased** estimator for $var_\theta(X) = \sigma_\theta^2$.

Efficiency of estimators (1/5)

The **mean square error** (MSE), is defined as

$$R(T(\boldsymbol{x}),\theta) = E_\theta\left[\Big(T(\boldsymbol{x}) - g(\theta)\Big)^2\right] = var_\theta(T(\boldsymbol{x})) + b_T^2(\theta)$$

$$\begin{aligned}
MSE = R\Big(T(\boldsymbol{x}),\theta\Big) &= E_\theta\left[\Big(T(\boldsymbol{x}) - g(\theta)\Big)^2\right] \\
&= E_\theta\left[\Big(T(\boldsymbol{x}) - E_\theta[T(\boldsymbol{x})] + E_\theta[T(\boldsymbol{x})] - g(\theta)\Big)^2\right] \\
&= \underbrace{E_\theta\left[\Big(T(\boldsymbol{x}) - E_\theta[T(\boldsymbol{x})]\Big)^2\right]}_{var_\theta(T(\boldsymbol{x}))} + \underbrace{E_\theta\left[\Big(E_\theta[T(\boldsymbol{x}) - g(\theta)\Big)^2\right]}_{constant} \\
&\quad + 2\,\underbrace{E_\theta\left[\Big(T(\boldsymbol{x}) - E_\theta[T(\boldsymbol{x})]\Big)\right]}_{=0} + \underbrace{E_\theta\left[\Big(E_\theta[T(\boldsymbol{x}) - g(\theta)\Big)\right]}_{constant} \\
&= var_\theta(T(\boldsymbol{x})) + \left(E_\theta\Big[T(\boldsymbol{x}) - g(\theta)\Big]\right)^2 + 0 \\
&= var_\theta(T(\boldsymbol{x})) + b_T^2(\theta)
\end{aligned}$$

Efficiency of estimators (2/5)

The MSE for the empirical mean is therefore

$$\begin{aligned} MSE = R(\overline{x}), \theta) &= var_\theta(\overline{x}) + b^2_{\overline{x}}(\theta) \\ &= \frac{var_\theta(\overline{x})}{n} + 0 \\ &= \frac{\sigma^2_\theta}{n} \end{aligned}$$

We notice that it decreases to 0 as n gets large (a MSE should always decrease to 0).

$$\frac{\sigma^2_\theta}{n} \quad \xrightarrow[n \to \infty]{} \quad 0$$

Efficiency of estimators (3/5)

The **Fisher information** for one observation is denoted $I(\theta)$. The Fisher information for the sample $x_1, x_2, ..., x_n$ is denoted $I_n(\theta)$. It is defined as follow

definition I : $$var_\theta(S(x,\theta)) = E[S^2(x,\theta)]$$

definition II: $$-E\left[\frac{\partial^2 ln\mathcal{L}(\theta|\mathbf{x})}{\partial\theta^2}\right] = -E[l'']$$

For the model (i), we first recall the expression for the second order derivative

$$\frac{\partial^2 ln\mathcal{L}(\mu \mid \mathbf{x})}{\partial\mu^2} = \frac{1}{\sigma^2}\,(-n) = -\frac{n}{\sigma^2}$$

Efficiency of estimators (4/5)

As a consequence, we have that (using the first definition I)

$$I(\mu) = var_\mu(S(x,\mu)) = E[S^2(x,\mu)] = E\left[\frac{(x,\mu)^2}{(\sigma^2)^2}\right] = \sigma^4 \underbrace{E\left[(x,\mu)^2\right]}_{var(x)=\sigma^2} = \frac{\sigma^2}{\sigma^4} = \frac{1}{\sigma^2}$$

Finally, $I_n(\mu) = n\ I(\mu) = n/\sigma^2$.

Or equivalently, using the definition II

$$I_n(\mu) = var_\mu(S(x,\mu)) = -\underbrace{E\left[\frac{\partial^2 ln\mathcal{L}(\mu \mid \mathbf{x})}{\partial\mu^2}\right]}_{=n\ \frac{1}{\sigma^2}} = -\left(-\frac{n}{\sigma^2}\right) = \frac{n}{\sigma^2}$$

Efficiency of estimators (5/5)

An estimator is said to be **efficient** if it attains the **Cramér-Rao bound** (CRB), defined as follow

$$CRB = \frac{\left(g'(\theta)\right)^2}{I_n(\theta)}$$

where $I_n(\theta)$ is the **Fisher information** of the sample. Moreover, since we are dealing here with the identity link function for the parameter, i.e. $g(\mu) = \mu$, we have $g'(\mu) = (\mu)' = 1$.

We obtain the following expression for the Cramér-Rao bound

$$CRB = \frac{1}{\frac{n}{\sigma^2}} = \frac{g'(\mu)}{I_n(\mu)} = \frac{\sigma^2}{n} \qquad \text{or, for 1 observation } CRB = \sigma^2.$$

We conclude that our estimator $\overline{x}$ is an efficient estimator for μ.

References

Course notes

The R Project for Statistical Computing:
https://www.r-project.org/