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# Iterated Function Systems and Control Languages

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Valuations—morphisms from $(\Sigma^*, \cdot, e)$ to $((0, \infty), \cdot, 1)$—are a generalization of Bernoulli morphisms introduced by Eilenberg ["Automata, Languages, and Machines," Academic Press, New York, 1974]. Here, we show how to generalize the notion of entropy (of a language) in order to obtain new formulas to determine the Hausdorff dimension of fractal sets (also in Euclidean spaces), especially defined via regular ($\omega$-)languages. By doing this, we can sharpen and generalize earlier results in two ways: first, we treat the case where the underlying basic iterated function system contains noncontractive mappings and, second, we obtain results valid for nonregular languages as well. 

## 1. INTRODUCTION

There are a series of approaches which use formal languages in order to describe fractals or, more generally, pictures. Probably, the most prominent examples are L systems [30, 31] and finite automata [4–7]. Further links provide hypergraph-based ideas such as collage grammars [19, 44] and cellular automata [20, 43].

Another approach to fractals is via *Iterated Function Systems* (IFS) (cf. [2, 9, 11]). An IFS is composed of a metric space $\mathcal{X}$ and a set $\{\mathcal{F}(1), \ldots, \mathcal{F}(n)\}$ of mappings on $\mathcal{X}$. If all $\mathcal{F}(i)$ are contractive, the IFS $\mathcal{F}$ defines a fractal as the smallest nonempty and (topologically) closed solution of $K = \bigcup_{i=1}^{n} \mathcal{F}(i)(K)$. Similarly, $K$ can be defined in the following way (cf. [2]):

1. For every mapping $\xi : \mathbb{N} \to \{1, \ldots, n\}$, the sequence

$$((\mathcal{F}(\xi(1)) \circ \cdots \circ \mathcal{F}(\xi(j)))(a))_{j=1}^{\infty} \tag{1}$$

converges independently of the starting point $a \in \mathcal{X}$ to a limit point $a_\xi \in K$ depending only on $\xi$.

2. Thus, every mapping $\xi$ can be seen as an address of $\phi_{\mathcal{F}}(\xi) := a_\xi$.
3. The above-mentioned solution $K$ equals $\{\phi_{\mathcal{F}}(\xi) : \xi : \mathbb{N} \to \{1, \ldots, n\}\}$.

Mappings $\xi : \mathbb{N} \to \Sigma$, where $\Sigma$ is a finite set (alphabet), are known in the theory of formal languages as $\omega$-words (cf. [40]).

It also makes sense to consider IFS where not all mappings $\mathcal{F}(i)$ are contractive (cf. [1, 6, 26]). Should this be the case, Item 1 from above is not fulfilled for all $\xi$. Then one must single out those $\xi$ for which the sequence (1) converges independently of the start $a \in \mathcal{X}$. In practice, this condition is very intriguing and depends heavily on the mappings $\mathcal{F}(i)$. In the above-mentioned papers, finite graphs (automata) were used as control structures to guarantee the convergence of (1).

Evidently, finite automata are simple control structures and therefore leave out, in general, a great number of $\omega$-words $\xi$ for which convergence can be guaranteed (even if we take into account—as was



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done in [1, 6, 26]—just the contraction coefficients $\beta_{\mathcal{F}}(i)$ of the mappings $\mathcal{F}(i)$). Thus, it is natural to ask what happens if we allow for more general control structures. Here, we will not deal with a particular type of automata. Instead, we will consider languages (or $\omega$-languages) as devices controlling the process of convergence of (1).

One of the main problems encountered in fractal geometry is the determination of the Hausdorff dimension of fractals. We have already shown that, for certain types of fractals described by formal languages, this calculation may be simplified considerably by using results on unambiguous regular expressions and unambiguous context-free grammars. We will show how this task can be accomplished for more general types of languages.

To this end, we generalize here the notion of entropy.[1] Moreover, we investigate special metrizations (induced by valuations) of spaces of $\omega$-words; Hausdorff measure and dimension within these spaces are directly related to those entities within Euclidean spaces. Further material is contained in other works of the authors of this paper [12–15, 27, 36–39]. A preliminary version of this paper was presented at MFCS'98 [18].

Our paper is structured as follows. In Section 2, we introduce the notion of valuation $\beta$ and the derived concept of $\beta$-entropy, which is central to the whole of this paper. Section 3 shows how one can compute the $\beta$-entropy for regular languages and star languages, i.e., languages of the form $L^*$. In Section 4, we introduce the metric $\rho_\beta$ (derived from valuation $\beta$) on the space of all $\omega$-words $\Sigma^\omega$ and show how the Hausdorff dimension of certain $\omega$-languages may be computed with the aid of the $\beta$-entropy. Finally, in Section 5, we prove that so-called OSC-codes are a useful notion for the calculation of the Hausdorff dimension of certain language-based Euclidean fractals. Readers who are mainly interested in fractal geometry may prefer to read Section 5 before going into the sometimes technical details of the intermediate sections.

*Conventions.* $\Sigma_n = \{1, \ldots, n\} \subset \mathbb{N}$ denotes our standard alphabet. $\Sigma$ (without subscript) denotes some at most countable alphabet. A language $L$ is a subset of the word monoid $\Sigma^*$, generated by the alphabet $\Sigma$, where $e$ is the neutral element of the monoid and called the empty word. Mostly, the monoid operation called catenation is just denoted by juxtaposition, sometimes made explicit using $\cdot$ between the words. The monoid generated by the language $L \subseteq \Sigma^*$ is denoted by $L^*$, and the semigroup generated by $L$ is denoted by $L^+$.

We also consider $\omega$-languages $F$ over the alphabet $\Sigma$, i.e., sets of one-sided infinite words, $F \subseteq \Sigma^\omega$ for short. If $L \subseteq \Sigma^*$, then

$$L^\omega = \{v_0 \cdot v_1 \cdot v_2 \cdots : \forall i \in \mathbb{N}(v_i \in L \setminus \{e\})\}.$$

Further notions and denotations are introduced throughout the paper.

## 2. VALUATIONS

DEFINITION 2.1. We call a monoid morphism $\beta$ mapping from $(\Sigma_n^*, \cdot, e)$ to $((0, \infty), \cdot, 1)$ a *valuation*. Any valuation can be extended to languages $L \subseteq \Sigma_n^*$ by defining $\beta(L) = \sum_{w \in L} \beta(w)$.

Basic facts on valuations can be found in [14, 15]. Every valuation $\beta$ is uniquely defined by the values $\beta(a)$ for all $a \in \Sigma_n$.

As an example, consider the valuation $\beta_n$ defined by $\beta_n(a) = 1/n$ for every $a \in \Sigma_n$.

DEFINITION 2.2. Let $\beta\colon \Sigma_n^* \to (0, \infty)$ be a valuation, and $s \geq 0$ a real number. We call $\beta^s(L) := \sum_{w \in L} \beta(w)^s$ the *s-dimensional valuation* of the language $L \subseteq \Sigma_n^*$.

Here, we allow valuations $\beta$ having $\beta(w) \geq 1$ for some words, which we have not yet considered in detail.

For a fixed $L \subseteq \Sigma_n^*$, we consider the $s$-dimensional valuation as a function $\beta^{(\cdot)}(L) : [0, \infty) \to [0, \infty]$. We summarize some properties of $\beta^{(\cdot)}(L)$:

[1] This entity corresponds to the Besicovitch-Taylor index defined in connection with IFS [45].

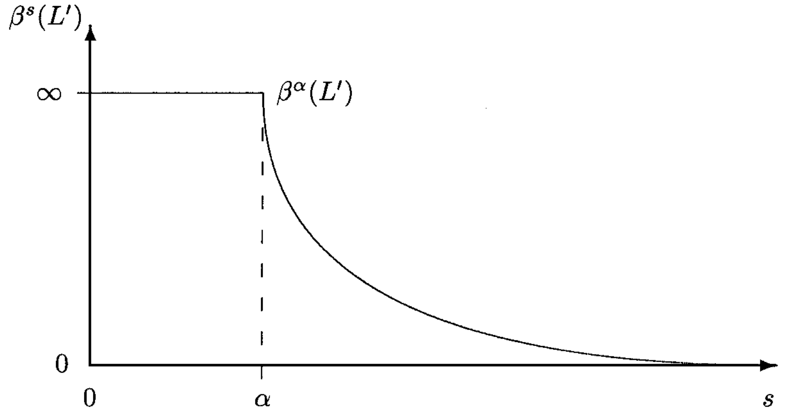


**FIG. 1.** Graph of $\beta^s(L')$ when $\beta^\alpha(L') = \infty$.

PROPERTIES 2.1. *Let $L \subseteq \Sigma_n^*$ and $\beta\colon \Sigma_n^* \to (0, \infty)$ be a valuation, and let $\beta^s(L) < \infty$ for some $s \in [0, \infty)$. Then the following properties are true*:

1. *There is an $\alpha \in [0, \infty)$ such that $\beta^s(L) = \infty$ for $s < \alpha$ and $\beta^s(L) < \infty$ for $s > \alpha$.*
2. *The function $\beta^{(\cdot)}(L)$ is continuous on $(\alpha, \infty)$.*
3. $\lim_{s\downarrow\alpha} \beta^s(L) = \beta^\alpha(L)$.[2]
4. *If, moreover, $\beta(w) < 1$ for all $w \in L$ and $L \neq \emptyset$, then the function $\beta^{(\cdot)}(L)$ is strictly decreasing on $[\alpha, \infty]$ and* $\lim_{s\to\infty} \beta^s(L) = 0$.

*Proof.* If $\beta^s(L) < \infty$ for some $s \in [0, \infty)$, then the set $\{w : \beta(w) \geq 1 \wedge w \in L\}$ is finite.

Since, for a finite language $W$, $\beta^s(W) = \sum_{w\in W} \beta^s(w)$ is a continuous function mapping from $(0, \infty)$ into $[0, \infty)$, we split $L$ into a disjoint union $L = L' \cup W$, where $W$ is finite and contains $\{w : \beta(w) \geq 1 \wedge w \in L\}$. Thus, in virtue of $\beta^s(L) = \beta^s(L') + \beta^s(W)$, it remains to verify the assertions for functions $\beta^s(L')$, where $\emptyset \neq L' \subseteq \{w : \beta(w) < 1 \wedge w \in L\}$.

Let $f : [0, \infty) \to [0, \infty]$ be defined by $f(s) := \beta^s(L') = \sum_{w\in L'}(\beta(w))^s$, and let $f_i(s) := \sum_{w\in U_i}(\beta(w))^s$, where $U_i := \{w : w \in L' \wedge |w| \leq i\}$. Then the functions $f_i$ are continuous mappings from $(0, \infty)$ to $(0, \infty)$.

Moreover, $f$ and $f_i$ have the following easily verified properties.

1. $s \leq s'$ implies $f(s) \geq f(s')$, that is, $f$ is nonincreasing.
2. $f(s) = \lim_{i\to\infty} f_i(s)$, that is, the family $(f_i)_{i\in\mathbb{N}}$ converges pointwise to the function $f$ (with the convention $f(s) = \infty$ if the family $(\beta^s(U_i))_{i\in\mathbb{N}}$ is unbounded).
3. $f(s) - f_i(s) \geq f(s') - f_i(s') \geq 0$ whenever $s \leq s'$. This follows from $f(s) - f_i(s) = \sum_{w\in L'\setminus U_i}(\beta(w))^s$ and $\beta(w) \leq 1$ for $w \in L'$.

Typical plots of the function $f(s) = \beta^s(L')$ are shown in Figs. 1 and 2.

Note that Assertion 1 follows directly from Item 1.

In order to prove Assertion 2, we consider a value $\theta \geq \alpha$ for which $f(\theta) < \infty$. Then, in virtue of Item 3, we have $|f(s') - f_i(s')| \leq |f(\theta) - f_i(\theta)|$ for all $s' \geq \theta$. Consequently, on $[\theta, \infty)$, the sequence $(f_i)_{i\in\mathbb{N}}$ of continuous functions converges uniformly to the function $f$. Hence, $f$ is continuous on every interval $[\theta, \infty) \subseteq (\alpha, \infty)$. This proves that $f$ is continuous on $(\alpha, \infty)$.

Next, we show Assertion 3. In the case where $f(\alpha) < \infty$, as we have seen in the preceding paragraph, the function $f$ is continuous on $[\alpha, \infty)$, and Assertion 3 follows.

Consider the case $f(\alpha) = \infty$. If $f$ is unbounded on $(\alpha, \infty)$, then $\lim_{s\downarrow\alpha} f(s) = \infty$, because $f$ is nonincreasing. Assume now that $f$ is bounded on $(\alpha, \infty)$, that is, there is a $c > 0$ such that $f(s) \leq c$ for every $s > \alpha$. Consequently, $f_i(s) \leq c$ for all $s > \alpha$ and all $i \in \mathbb{N}$. Since every function $f_i$ is continuous on $[\alpha, \infty)$, we have also $f_i(\alpha) \leq c$. Now Item 2 shows $f(\alpha) \leq c$, a contradiction.

[2] We permit the value $\infty$ for $\beta^\alpha(L)$.

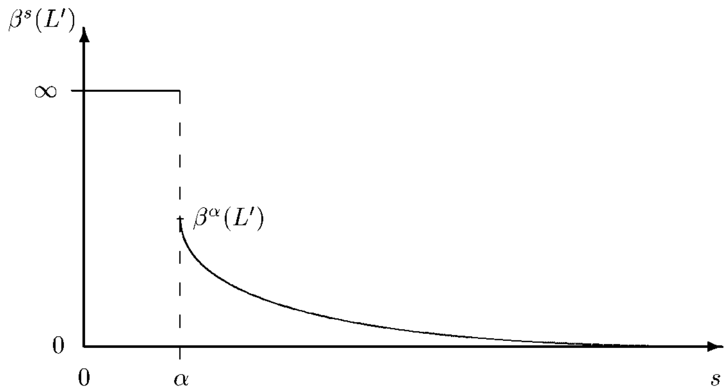


**FIG. 2.** Graph of $\beta^s(L')$ when $\beta^\alpha(L') < \infty$ .

Utilizing again Item 2 and the property that $\lim_{s\to\infty} f_i(s) = 0$, we can show that $f(s) = \beta^s(L')$ tends to zero as $s$ approaches infinity. ■

DEFINATION 2.3. The $\beta$-*entropy* of the language $L \subseteq \Sigma_n^*$, written $H_L^\beta$, is defined as the point $\alpha$ defined above, which is a "change-over-point" of the function $\beta^{(\cdot)}(L)$, i.e.,

$$H_L^\beta := \inf\{s : s \geq 0 \wedge \beta^s(L) < \infty\}.^3$$

Thus we have $H_L^\beta < \infty$ if and only if there is an $s \in (0, \infty)$ such that $\beta^s(L) < \infty$.

*Remark 2.1.* One can construct valuations $\beta$ and languages $L$ with $\beta(w) < 1$ for all $w \in L$, and still have $\beta^s(L) = \infty$ for all $s \in [0, \infty)$:

Let $n = 2$, $\beta(0) < 1$ and $\beta(0) \cdot \beta(1) = 1$. Choosing $L := 0 \cdot \{01, 10\}^*$ yields $\beta(w) = \beta(0) < 1$ for all $w \in L$ and $\beta^s(L) = \beta^s(0) \sum_{i=0}^{\infty} 2^i = \infty$

In what follows, however, we are not interested in such pathological cases. If $\beta(a) < 1$ for every $a \in \Sigma_n$, then there is a finite change-over-point $\alpha$ of the function $\beta^s(L)$ for any $L \subseteq \Sigma_n^*$.

It was shown in [27, 37, 38] that the entropy of languages introduced by Chomsky and Miller (cf. [22]) is a useful tool for the calculation of the Hausdorff dimension of certain subsets of the Cantor space $\Sigma_n^\omega$ or of the Euclidean space $\mathbb{R}^d$. Here, we will see the usefulness of the generalized notion of $\beta$-entropy, especially leading to similar calculation formulae for the Hausdorff dimension of subsets of $(\Sigma_n^\omega, \rho_\beta)$ and of $\mathbb{R}^d$, thereby generalizing results shown in [1, 26]. The main idea is to generalize properties of the entropy of languages (see [35], [38, Section 2]). Thus, analogously to the usual entropy of languages, we find the following easily verified identities:

$$H_{W\cup V}^\beta = H_{W\cdot V}^\beta = \max\left\{H_W^\beta, H_V^\beta\right\} \quad \text{if } W \cdot V \neq \emptyset, \text{ and} \tag{2}$$

$$H_L^\beta = 0 \quad \text{if } L \text{ is finite.} \tag{3}$$

## 3. THE $\beta$-ENTROPY OF LANGUAGES

In this section, we show for two classes of languages that their $\beta$-entropy can be computed. The first class is the class of regular languages. Here, we show that the finiteness of the $\beta$-entropy of a regular language $L$ is closely related to the values of $\beta$ on so-called $L$-cycles $w \in \Sigma_n^*$. Moreover, we show that the $\beta$-entropy of a regular language coincides with the $\beta$-entropy of its language of infixes. This allows for a calculation of the $\beta$-entropy of a regular language along the lines described in [1, 26].

In general, the $\beta$-entropy is in no way continuous, that is, $\lim_{i\to\infty} L_i = L$ does not necessarily imply that $H_{L_i}^\beta$ tends to $H_L^\beta$. In the second subsection we deal with a class of languages, the so-called star-languages or submonoids of $\Sigma_n^*$, for which an approximation of the $\beta$-entropy of the whole language

[3] Here, we follow the convention $\inf \emptyset = \infty$.

by the $\beta$-entropy of its finitely generated sublanguages is possible. We generalize a result obtained in [35] for the usual entropy of languages. It is interesting to note that this approximation is valid for arbitrary, i.e., not only for regular or star-languages.

Before we proceed to the consideration of the $\beta$-entropy of regular languages, we derive a condition for the finiteness of the entropy. As Remark 2.1 explains, the mere requirement that $\beta(w) < 1$ for almost all $w \in L$ does not guarantee that $H_L^\beta$ is finite. An exponential decrease of $\beta(w),\ w \in L,\ |w| \to \infty$, however, will be sufficient.

PROPERTY 3.1. *If there exist an $\ell \in \mathbb{N}$ and a positive constant $c < 1$ such that for all $w \in L$ with $|w| \geq \ell$ it is true that $\beta(w) \leq c^{|w|}$, then $H_L^\beta < \infty$.*

*Proof.* If $\beta(w) \leq c^{|w|}$ for some positive $c < 1$ and all $w \in L$ with $|w| \geq \ell$, then the inequality $n \cdot c^s < 1$ holds true for some $s \geq 0$. This implies

$$\beta^s(L) \leq \sum_{|w|<\ell} \beta^s(w) + \sum_{i \geq \ell} n^i \cdot c^{s \cdot i} < \infty. \qquad \blacksquare$$

## 3.1. *The $\beta$-Entropy of Regular Languages*

We can characterize the *regular* languages with finite $\beta$-entropy. It turns out to be the case that the sufficient condition of Property 3.1 will also be necessary for regular languages.

We briefly recall some important concepts regarding our study. The *state* (or *derivative*) of $M \subseteq \Sigma_n^* \cup \Sigma_n^\omega$ derived from $w \in \Sigma_n^*$ is defined as

$$M/w := \{p : p \in \Sigma_n^* \cup \Sigma_n^\omega \wedge w \cdot p \in M\}. \tag{4}$$

$M \subseteq \Sigma_n^* \cup \Sigma_n^\omega$ is called *finite-state* if it has a finite number of distinct states. It is well-known that $L \subseteq \Sigma_n^*$ is finite-state iff it is regular, whereas every regular $\omega$-language [4] is finite-state but the converse does not hold true (see e.g., [33]). A word $w \in \Sigma_n^*$ is called *prefix* of a string $p \in \Sigma_n^* \cup \Sigma_n^\omega$ provided $p = w \cdot p'$ for some $p' \in \Sigma_n^* \cup \Sigma_n^\omega$ (abbreviated by $w \sqsubseteq p$). For $M \subseteq \Sigma_n^* \cup \Sigma_n^\omega$, its set of finite prefixes is denoted by $\mathbf{A}(M)$ and its set of subwords (infixes) is

$$\mathbf{T}(M) := \{v : v \in \Sigma_n^* \wedge \exists p \exists w (w \in \Sigma_n^* \wedge w \cdot v \cdot p \in M)\}.$$

We call a word $v \in \Sigma_n^*$ an *L-cycle* provided $v \neq e$ and $L/w = L/w \cdot v \neq \emptyset$ for some $w \in \Sigma_n^*$. Evidently, every $L$-cycle $v$ belongs to $\mathbf{T}(L)$. Using the concept of $L$-cycles we obtain a necessary condition for the finiteness of the $\beta$-entropy of $L$.

PROPERTY 3.2. *Let $L \subseteq \Sigma_n^*$. If there is an $s \geq 0$ causing $\beta^s(L) < \infty$ then $\beta(v) < 1$ for every L-cycle $v$.*[5]

*Proof.* If there is an $L$-cycle $v$ such that $\beta(v) \geq 1$, then there exist words $w, u$ with $wvu \in L$ and $L/w = L/w \cdot v$. Then $L \supseteq wv^*u$ and, therefore, $\beta^s(L) \geq \beta^s(wv^*u) = \beta(wu)^s \cdot \sum_{i \in \mathbb{N}} (\beta(v)^s)^i = \infty$. $\blacksquare$

It turns out that, for regular languages, the conditions derived in Properties 3.1 and 3.2 are both necessary and sufficient.

In order to facilitate the proof, we consider the following concept. We call an $L$-cycle $v$ *reducible* provided there are $L$-cycles $v'$ and $u = u'u''$ such that $v' \neq e \neq u$ and $v = u' \cdot v' \cdot u''$; otherwise $v$ is called *irreducible*. We get the following sufficient reducibility condition for $L$-cycles.

PROPERTY 3.3. *Let $L \subseteq \Sigma_n^*$, and let $v$ be an L-cycle for which $L/w = L/wv \neq \emptyset$. If $v = u' \cdot v' \cdot u''$, $v' \neq e \neq u'u''$ and $L/wu' = L/wu' \cdot v'$ then $v$ is reducible.*

[4] Regular $\omega$-languages are defined as finite unions of sets of the form $W \cdot V^\omega$, where $W, V$ are regular languages.

[5] This condition is equivalent to the "contracting cycles property" of [1] and [26].

*Proof.* In fact, we have $L/wu' \neq \emptyset$ since $L/wu' \cdot v' \cdot u'' = L/wv \neq \emptyset$. Thus $v'$ is an $L$-cycle.

Moreover, $L/wu' = L/wu' \cdot v'$ implies $L/w \cdot v = L/wu' \cdot u''$ and, thus, $u = u' \cdot u''$ is also an $L$-cycle. ∎

We define the *average*[6] *per letter valuation* of a nonempty word $v$, $\overline{\beta}(v)$, as

$$\overline{\beta}(v) := \sqrt[|v|]{\beta(v)}\,,$$

and we define the *upper extent* of the average cycle letter valuation of a language $L \subseteq \Sigma_n^*$ as

$$\hat{c}_L := \sup\{\overline{\beta}(v) : v \text{ is an } L\text{-cycle}\}.$$

We have the following formulas.

$$\beta(v) \leq (\hat{c}_L)^{|v|} \quad \text{for all } L\text{-cycles}, \tag{5}$$

$$\hat{c}_L = \sup\{\overline{\beta}(v) : v \text{ is an irreducible } L\text{-cycle}\}. \tag{6}$$

We prove Eq. (6).

*Proof.* It suffices to show that, for every reducible $L$-cycle, thereis a shorter $L$-cycle possessing at least the same average per letter valuation.

Let $v = u' \cdot v' \cdot u''$ with $L$-cycles $v'$ and $u := u'u''$ such that $v' \neq e \neq u$. Then, $|v'| + |u| = |v|$ and

$$\beta(v) = \beta(v') \cdot \beta(u) = (\overline{\beta}(v'))^{|v'|} \cdot (\overline{\beta}(u))^{|u|}.$$

Consequently, $\overline{\beta}(v) = \sqrt[|v|]{\beta(v)} \leq \max\{\overline{\beta}(v'), \overline{\beta}(u)\}$. ∎

PROPERTIES 3.4. *If $L \subseteq \Sigma_n^*$ is a regular language and $\beta : \Sigma_n^* \to (0, \infty)$ is a valuation, then the following conditions are equivalent*:

1. *There is an $s \geq 0$ causing $\beta^s(L) < \infty$.*
2. $\forall w, v(v \neq e \wedge L/w = L/w \cdot v \neq \emptyset \to \beta(v) < 1)$ .
3. $\hat{c}_L < 1$
4. *There are an $\ell \in \mathbb{N}$ and a positive constant $c < 1$ such that, for all $u \in L$ with $|u| \geq \ell$, it holds that $\beta(u) \leq c^{|u|}$.*

*Proof.* "1. → 2.:" follows from Property 3.2.

"2. → 3.:" If $L$ is regular, then the set $\{L/w : w \in \Sigma_n^*\}$ is finite. Let $k := \mathrm{card}\{L/w : w \in \Sigma_n^*\}$. Then, Property 3.3 implies that every $L$-cycle of length $> k$ is reducible. Thus, in virtue of Eq. (6), we have that $\hat{c}_L = \sup\{\overline{\beta}(v) : v \text{ is an } L\text{-cycle } \wedge |v| < k\}$. Since the set of $L$-cycles of length $< k$ is finite, there is an $L$-cycle $u$, $|u| < k$, with maximum average per letter valuation. Now, in view of Condition 2, $\hat{c}_L = \overline{\beta}(u) < 1$ follows.

"3. → 4.:" Again, let $k := \mathrm{card}\{L/w : w \in \Sigma_n^*\}$ and $\hat{c}_L < 1$. First, we show by induction on the word length that there are constants $c_0$ and $c_1$, $0 < c_1 < 1$, such that

$$\beta(w) \leq c_0 \cdot c_1^{|w|-k} \qquad \text{for every } w \in L. \tag{7}$$

Choose $c_1$, $0 < c_1 < 1$ in such a way that $c_1 \geq \hat{c}_L$ and define

$$c_0 := \max\left\{\beta(w) \cdot c_1^{k-|w|} : |w| \leq k\right\}.$$

Then Eq. (7) is valid for all words $w$, $|w| \leq k$.

Assume that the assertion is valid for all $w \in L$ with $|w| < m$, $k < m$, and let $v \in L$ with $|v| = m$. Since $m > k$ and $L/v \neq \emptyset$, there exist words $u', v', u''$ such that $v = u' \cdot v' \cdot u''$, $v' \neq e \neq u'u''$ and

[6] Since the valuation is a multiplicative function, here the natural mean value is the geometric mean.

$L/u' = L/u' \cdot v' \neq \emptyset$. Consequently, $v'$ is an $L$-cycle and $\beta(v') \leq c_1^{|v'|}$. Since $|u'u''| < m$, we have $\beta(u'u'') \leq c_0 \cdot c_1^{|u'u''|-k}$, whence $\beta(v) \leq c_0 \cdot c_1^{|v|-k}$.

Now, having first chosen $c$ such that $c_1 < c < 1$ and then a large enough $\ell$, the assertion follows immediately.

"4. $\rightarrow$ 1.:" follows from Property 3.1. ■

We obtain the following relations between the $\beta$-entropies of $L$, $\mathbf{A}(L)$ and $\mathbf{T}(L)$:

PROPERTY 3.5. *Let $L$ be regular. Then, $\beta^s(L) < \infty$ iff $\beta^s(\mathbf{T}(L)) < \infty$. Further, $H_L^\beta = H_{\mathbf{A}(L)}^\beta = H_{\mathbf{T}(L)}^\beta$.*

*Proof.* In view of $L \subseteq \mathbf{A}(L) \subseteq \mathbf{T}(L)$, the second assertion results from the first one.

Clearly, $\beta^s(\mathbf{T}(L)) < \infty$ implies $\beta^s(L) < \infty$.

Conversely, let $\beta^s(L) < \infty$. Since $L$ is regular, $k := \operatorname{card}\{L/w : w \in \Sigma_n^*\} \in \mathbb{N}$. Thus, the following condition holds true:

$$\forall v(v \in \mathbf{T}(L) \rightarrow \exists w, \hat{w}(\hat{w} \in L \wedge w \cdot v \sqsubseteq \hat{w} \wedge |w|, |\hat{w}| - |w \cdot v| \leq k)).$$

Hence, $\mathbf{T}(L) = \bigcup_{i,j=0}^{k} L_{i,j}$ where $L_{i,j} := \{v : \Sigma_n^i \cdot v \cdot \Sigma_n^j \cap L \neq \emptyset\}$, and the assertion results from the easily verified inequality

$$\beta^s(L_{i,j}) \cdot \min\{\beta(v)^s : |v| \leq 2k\} \leq \beta^s(L). \qquad ■$$

Next, we show how to compute the $\beta$-entropy for an infinite regular set $L$ provided $H_L^\beta < \infty$. Let $\{L_1 = L, L_2, \ldots, L_k\} = \{L/w : w \in \mathbf{A}(L)\}$ be its set of nonempty states. Define the matrix

$$\mathcal{A}_L^{\beta,s} = (a_{s;i,j})_{1 \leq i,j \leq k} \quad \text{by} \quad a_{s;i,j} := \sum_{L_i/a = L_j} (\beta(a))^s.$$

Then,

$$\beta^s\left(\mathbf{A}(L) \cap \Sigma_n^\ell\right) = (1, 0, \ldots, 0) \cdot \left(\mathcal{A}_L^{\beta,s}\right)^\ell \cdot \mathbb{1},$$

where $\mathbb{1}$ is the all ones column vector.

Let

$$\Phi_L(s) := \lim_{\ell \to \infty} \sqrt[\ell]{\left\| \left(\mathcal{A}_L^{\beta,s}\right)^\ell \right\|}$$

be the spectral radius of $\mathcal{A}_L^{\beta,s}$. Property 3.4.2 ensures that the assumptions of Theorem 2 in [26] are satisfied and, according to this theorem, $\Phi_L$ as a function of $s$ is strictly decreasing,[7] $\Phi_L(0) \geq 1$, and $\lim_{s \to \infty} \Phi_L(s) = 0$. Thus, the sum

$$\beta^s(\mathbf{A}(L)) = (1, 0, \ldots, 0) \cdot \sum_{\ell \in \mathbb{N}} \left(\mathcal{A}_L^{\beta,s}\right)^\ell \cdot \mathbb{1}$$

converges if $\Phi_L(s) < 1$ and, if $\Phi(s) \geq 1$, it diverges. So, $H_L^\beta = H_{\mathbf{A}(L)}^\beta = H_{\mathbf{T}(L)}^\beta = \alpha$ iff $\Phi_L(\alpha) = 1$. Hence, we have:

COROLLARY 3.1. *$\beta^\alpha(L) = \infty$ for $\alpha = H_L^\beta$ if $L$ is an infinite regular set.*

We end these considerations with the calculation of the $\beta$-entropy for regular languages in the simple case where $\beta$ is constant on $\Sigma_n$, not necessarily $\beta(a) = \frac{1}{n}$, and smaller than 1.

Let $\beta(a) = t < 1$ for all $a \in \Sigma_n$. Then $\mathcal{A}_L^{\beta,s} = t^s \cdot \hat{\mathcal{A}}_L$ where $\hat{\mathcal{A}}_L$ is the usual adjacency matrix of the language $L \subseteq \Sigma_n^*$, that is, has coefficients $\hat{a}_{ij} := \operatorname{card}\{a : L_i/a = L_j\}$. From this we obtain $\beta^s(\mathbf{A}(L) \cap \Sigma_n^\ell) = (1, 0, \ldots, 0) \cdot t^{s \cdot \ell} \cdot \hat{\mathcal{A}}_L^\ell \cdot \mathbb{1} = t^{s \cdot \ell} \cdot \operatorname{card} \mathbf{A}(L) \cap \Sigma_n^\ell$.

[7] More precisely, there is a $c$, $0 < c < 1$, such that for all $\varepsilon > 0$, the inequality $\Phi_L(s + \varepsilon) \leq c^\varepsilon \cdot \Phi(s)$ holds true.

Thus $\Phi_L(s) := \lim_{\ell\to\infty} \sqrt[\ell]{\|t^{s\cdot\ell} \cdot (\hat{\mathcal{A}}_L)^\ell\|} = t^s \cdot n^{H_{\mathbf{A}(L)}}$, where $H_{\mathbf{A}(L)} := \limsup_{\ell\to\infty} \ell^{-1} \cdot \log_n \mathrm{card}(\mathbf{A}(L) \cap \Sigma_n^\ell)$ is the usual entropy of the language $\mathbf{A}(L)$.

Consequently, the identity $\Phi_L(\alpha) = 1$ holds true iff $\alpha = H_{\mathbf{A}(L)} \cdot \log_n t^{-1}$.

### 3.2. *The $\beta$-Entropy of the Star Languages*

Next, we consider the relationship between the entropies of $L$ and $L^*$. Because $H_L^\beta \le H_{L^*}^\beta$ and $H_{L^*}^\beta = \infty$ whenever $\beta(w) \ge 1$ for some $w \in L\backslash\{e\}$, we are interested only in cases when $H_L^\beta < \infty$ and $\beta(w) < 1$ for $w \in L\backslash\{e\}$. This also implies that $\beta(w) < 1$ for $w \in L^*\backslash\{e\}$.

First, we give some general bounds on the $\beta$-entropy of $L^*$. Recall that a language $C \subseteq \Sigma_n^*$ is a *code* provided that $w_1 \cdots w_l = v_1 \cdots v_m$, where $w_1, \ldots, w_l, v_1, \ldots, v_m \in C$ implies $w_1 = v_1$.

PROPERTY 3.6. *Let $e \notin L$, $\alpha = H_L^\beta < \infty$ and $\beta(w) < 1$ for all $w \in L\backslash\{e\}$. Then*

1. $H_{L^*}^\beta \le \inf\{s : \beta^s(L) \le 1\}$ *and,*
2. *if $L$ is a code and $\beta^\alpha(L) \ge 1$, then $H_{L^*}^\beta$ is the unique solution of the equation $\beta^s(L) = 1$.*

*Proof.* 1. Since $\beta(w) < 1$ for all $w \in L$, the function $\beta^s(L)$ is strictly decreasing in $(\alpha, \infty)$. Consequently, if $\beta^s(L) < 1$, in view of the inequality $\beta^s(L^*) \le \sum_{i\in\mathbb{N}} (\beta^s(L))^i$, we have that $\beta^s(L^*) < \infty$.

2. The additional claim that $\beta^s(L) = 1$ implies $H_{L^*}^\beta \ge s$ follows from the fact that, for codes $L$, the identity $\beta^s(L^*) = \sum_{i\in\mathbb{N}} (\beta^s(L))^i$ holds true. ■

Summarizing 1 and 2 for codes $C \subseteq \Sigma_n^*$ yields the formula

$$H_{C^*}^\beta = \inf\{s : \beta^s(C) \le 1\}\,. \tag{8}$$

We obtain a condition sufficient for the inequality $H_{L^*}^\beta > H_{L^\cdot}^\beta$.

LEMMA 3.1. *If $L$ is a finite union of $k$ codes, $\alpha = H_L^\beta < \infty$ and $\beta^\alpha(L) > k$, then $H_{L^*}^\beta > H_L^\beta$.*

*Proof.* Let $L = C_1 \cup \cdots \cup C_k$, where all $C_i$ are codes. It follows that there is an $i \in \mathbb{N}$ such that $\beta^\alpha(C_i) > 1$. Because $\beta^s(C_i)$ is continuous on $(\alpha, \infty)$, we have $\inf\{s : \beta^s(C_i) \le 1\} > \alpha$ and, therefore, $\alpha < H_{C_i^*}^\beta \le H_{L^*}^\beta$. ■

In connection with Corollary 3.1 we obtain:

COROLLARY 3.2. *If $L \subseteq \Sigma_n^*$ is regular and a finite union of codes and $H_L^\beta < \infty$, then $H_{L^*}^\beta > H_L^\beta$.*

Next, we consider the approximation of the $\beta$-entropy of $L^*$, $H_{L^*}^\beta$, via $H_{U^*}^\beta$, where $U$ is a finite subset of $L$. We aim to derive a result analogous to the theorem of [35]. There, we used the real numbers $\lambda_m$ defined as the smallest (positive) roots of the equation $1 = \lambda_m + (\lambda_m)^m$.[8]

In what follows, we assume that there is a positive constant $c < 1$ such that every word $w \in L$ satisfies $\beta(w) \le c^{|w|}$. In other words, $L \subseteq V_{\beta,c}$, where $V_{\beta,c} := \{w : w \in \Sigma_n^* \wedge \beta(w) \le c^{|w|}\}$. Observe that $V_{\beta,c}^* \subseteq V_{\beta,c}$ and, therefore, $L \subseteq V_{\beta,c}$ implies $L^* \subseteq V_{\beta,c}$.

THEOREM 3.7. *Let $L$ be a nonempty subset of $V_{\beta,c}$. Then, for $m \le \min\{|w| : w \in L\backslash\{e\}\}$ and $\varepsilon_m := \log_c \lambda_m$, we have*

$$\beta^s(L^*) \le \sum_{i\in\mathbb{N}} (\beta^s(L))^i \le \beta^{s-\varepsilon_m}(L^*)$$

*whenever $s \ge \varepsilon_m$.*

[8] It is well known that $\lambda_m^{-\ell}$ is an upper bound on the number of compositions (ordered partitions) $p_m(\ell)$ of the number $\ell$ into parts not smaller than $m$, and it holds that $0 < \lambda_m < \lambda_{m+1} < 1$ and $\lim_{m\to\infty} \lambda_m = 1$. These compositions come up here because the number of representations of a word $w \in L^*$ as a product of nonempty words $v \in L$ cannot be smaller than $p_m(|w|)$ when $m = \min\{|v| : v \in L \wedge v \ne e\}$(cf. [35]).

*Proof.* As in [35], one obtains

$$\beta^s(L^*) \le \sum_{i\in\mathbb{N}} (\beta^s(L))^i \le \sum_{w\in L^*} \lambda_m^{-|w|} \cdot (\beta(w))^s.$$

Now $\beta(w) \le c^{|w|}$ implies $(\beta(w))^{\varepsilon_m} \le (c^{|w|})^{\varepsilon_m} = \lambda_m^{|w|}$ and, consequently,

$$\sum_{w\in L^*} \lambda_m^{-|w|} \cdot (\beta(w))^s \le \beta^{s-\varepsilon_m}(L^*). \qquad \blacksquare$$

COROLLARY 3.3. *Let* $L \subseteq V_{\beta,c}$ *for some* $c < 1$, $e \notin L$ *and* $\min\{|w| : w \in L\} \ge m > 0$. *Then,*

$$0 \le \theta - H^{\beta}_{L^*} \le \varepsilon_m \textit{ whenever } \beta^{\theta}(L) = 1.$$

*Proof.* If $\beta^\theta(L) = 1$, then $H^{\beta}_{L^*} \le \theta$. On the other hand, Theorem 3.7 shows $\beta^{\theta-\varepsilon_m}(L^*) = \infty$, that is, $H^{\beta}_{L^*} \ge \theta - \varepsilon_m$. $\blacksquare$

We obtain the announced analogue to the theorem of [35].

THEOREM 3.8. *Let* $L \subseteq V_{\beta,c}$ *for some* $c < 1$. *Then, for every* $\varepsilon > 0$, *there is a finite subset* $U \subseteq L$ *such that*

$$H^{\beta}_{L^*} - H^{\beta}_{U^*} < \varepsilon.$$

*Proof.* Let $H^{\beta}_{L^*} = \alpha$. It is sufficient to show that for every $\varepsilon > 0$ there is a finite subset $U \subseteq L$ such that $\beta^{\alpha-2\cdot\varepsilon}(U^*) = \infty$.

If $H^{\beta}_{L^*} = \alpha$, then $\beta^{\alpha-\varepsilon}(L^*) = \infty$ for all $\varepsilon > 0$. Now choose $m \in \mathbb{N}$ such that $\varepsilon > \varepsilon_m := \log_c \lambda_m$. Since $\beta^{\alpha-\varepsilon}(L^*) = \infty$, there is a finite subset $V \subseteq \{w : w \in L^* \wedge |w| \ge m\}$ satisfying $\beta^{\alpha-\varepsilon}(V) > 1$. Hence, by Theorem 3.7,

$$\infty = \sum_{i\in\mathbb{N}} (\beta^{\alpha-\varepsilon}(V))^i \le \beta^{\alpha-\varepsilon-\varepsilon_m}(V^*) \le \beta^{\alpha-2\cdot\varepsilon}(V^*).$$

Finally, we may choose $U$ to be any finite subset of $L$ satisfying $V \subseteq U^*$. $\blacksquare$

As a concluding remark to this section we derive an upper bound to the $\beta$-entropy of the languages $V_{\beta,c}$, where $c < 1$.

$$H^{\beta}_{V_{\beta,c}} \le -\log_c n \quad \text{for } V_{\beta,c} \subseteq \Sigma_n^* \text{ and } c < 1. \tag{9}$$

*Proof.* We have

$$\beta^s(V_{\beta,c}) \le \sum_{i\in\mathbb{N}} n^i \cdot c^{s\cdot i} = \sum_{i\in\mathbb{N}} (n \cdot c^s)^i < \infty$$

only when $n \cdot c^s < 1$. $\blacksquare$

## 4. $\omega$-LANGUAGES AND HAUSDORFF DIMENSION

Now, we apply our results on valuations of languages to the calculation of the Hausdorff dimension in the spaces $(\Sigma_n^\omega, \rho_\beta)$, where the metric $\rho_\beta$ is derived from the valuation $\beta : \Sigma_n \to (0, \infty)$ in the following manner.

DEFINITION 4.1. Let $\beta : \Sigma_n \to (0, \infty)$ be a valuation. Define

$$\rho_\beta(\xi, \eta) = \begin{cases} 0, & \text{if } \xi = \eta, \text{ and} \\ \min\{\beta(w) : w \in \mathbf{A}(\xi) \cap \mathbf{A}(\eta)\}, & \text{otherwise.} \end{cases} \tag{10}$$

The case when $\beta(a) = \beta_n(a) := \frac{1}{n}$ for $a \in \Sigma_n$ was investigated in [38]; here, we generalize those results. Particular results for arbitrary valuations (with finite automata as a control structure) were obtained in [1, 26].

### 4.1. *Metric Properties of the Space* $(\Sigma_n^\omega, \rho_\beta)$

Here, we list some properties of the metric $\rho_\beta$. We will assume some familiarity with basic notions of metric spaces on the side of the reader.

It turns out that there is a crucial distinction between the behaviour of the metrics derived from various valuations $\beta$, mainly depending on the fact whether $\beta(a) < 1$ for all $a \in \Sigma_n$[9] or not.

1. $\rho_\beta$ satisfies the *ultra-metric inequality*

$$\rho_\beta(\xi, \eta) \leq \max\{\rho_\beta(\xi, \zeta), \rho_\beta(\eta, \zeta)\}, \tag{11}$$

because $\mathbf{A}(\xi) \cap \mathbf{A}(\eta)$ contains at least one of the sets $\mathbf{A}(\xi) \cap \mathbf{A}(\zeta)$ or $\mathbf{A}(\eta) \cap \mathbf{A}(\zeta)$.

2. Sets of the form $w \cdot \Sigma_n^\omega$ are always open. Consequently, $\omega$-languages $E \subseteq \Sigma_n^\omega$ satisfying $E = \{\xi : \mathbf{A}(\xi) \subseteq \mathbf{A}(E)\} = \Sigma_n^\omega \setminus (\Sigma_n^* \setminus \mathbf{A}(E)) \cdot \Sigma_n^\omega$ are closed in every space $(\Sigma_n^\omega, \rho_\beta)$; therefore, we call them *strongly closed*.

3. If $\beta$ is noncontractive, $(\Sigma_n^\omega, \rho_\beta)$ need not be *compact*, and there may be *isolated points*, i.e., points $\xi \in \Sigma_n^\omega$ with $\rho_\beta(\xi, \eta) > \epsilon$ for some $\epsilon > 0$ and all $\eta \neq \xi$.

On the other hand, if $\beta$ is contractive, then $(\Sigma_n^\omega, \rho_\beta)$ is always compact.

4. *Balls* with center $\xi$ and radius $\epsilon > 0$ in $(\Sigma_n^\omega, \rho_\beta)$ are defined as

$$\overline{\mathbb{B}}_\epsilon(\xi) = \{\eta : \rho_\beta(\xi, \eta) \leq \epsilon\}.$$

They are characterized by words in $\Sigma_n^*$ as

$$\overline{\mathbb{B}}_\epsilon(\xi) = \begin{cases} \{\xi\}, & \text{if } \forall\eta(\xi \neq \eta \to \rho_\beta(\xi, \eta) > \epsilon), \text{ and} \\ w_\beta(\xi, \epsilon) \cdot \Sigma_n^\omega, & \text{otherwise (when } w_\beta(\xi, \epsilon) \text{ exists!),} \end{cases} \tag{12}$$

where $w_\beta(\xi, \epsilon)$ is the shortest prefix (provided that it exists) $w \sqsubset \xi$ with $\beta(w) \leq \epsilon$.

Since $\rho_\beta$ satisfies the ultrametric inequality (11), balls are simultaneously open and closed and, moreover, any $\eta \in \overline{\mathbb{B}}_\epsilon(\xi)$ can be chosen to be the center, showing that its radius is an upper bound of its *diameter*.

In contrast to the contractive case, neither are all balls subsets of the form $w \cdot \Sigma_n^\omega$ nor are all subsets of the form $w \cdot \Sigma_n^\omega$ balls; e.g., take $12 \cdot \Sigma_2^\omega$ in $(\Sigma_2^\omega, \rho_\beta)$ where $\beta(1) < 1$ and $\beta(2) \geq 1$.

5. Let $\mathbb{I}_\beta := \{\xi : \inf\{\beta(w) : w \sqsubset \xi\} > 0\}$ be the set of all isolated points. $\mathbb{I}_\beta$ is open. $\mathbb{I}_\beta = \emptyset$ iff $\beta$ is contractive. For noncontractive valuations $\beta$, we have $\mathbb{I}_\beta = \Sigma_n^* \cdot a^\omega$ iff $\beta(a) = 1$ and $\beta(b) < 1$ for $b \in \Sigma_n \setminus \{a\}$ and, otherwise, $\mathbb{I}_\beta$ is uncountable.

6. We call $\mathbb{F}_\beta := \Sigma_n^\omega \setminus \mathbb{I}_\beta$ the $\beta$-*fundamental set* of $(\Sigma_n^\omega, \rho_\beta)$. As the set of isolated points $\mathbb{I}_\beta$ is open, its complement $\mathbb{F}_\beta$ is closed. If $\mathbb{F}_\beta \neq \emptyset$, it is not strongly closed unless $\beta$ is contractive, since $\mathbb{F}_\beta = \mathbb{F}_\beta / w$ for all $w \in \Sigma_n^*$ and only two strongly closed one-state $\omega$-languages exist in $\Sigma_n^\omega$: $\Sigma_n^\omega$ itself and $\emptyset$.

[9] Valuations having this property will be called *contractive*. Since $\Sigma_n$ is finite, for contractive valuations, the space $(\Sigma_n^\omega, \rho_\beta)$ is compact and balls are exactly the sets of the form $w \cdot \Sigma_n^\omega$.

4.2. *Hausdorff Dimension in* $(\Sigma_n^\omega, \rho_\beta)$

In order to introduce the Hausdorff dimension of subsets of $(\Sigma_n^\omega, \rho_\beta)$, we define the *α-dimensional outer Hausdorff measure* induced by $\rho_\beta$ as

$$\nu_\beta^\alpha(F) := \lim_{\epsilon\to 0}\inf\left\{\sum_{i\in\mathbb{N}}(\mathrm{diam}F_i)^\alpha : F \subseteq \bigcup_{i\in\mathbb{N}} F_i \wedge \mathrm{diam}F_i < \epsilon\right\}.^{10} \tag{13}$$

Then the *Hausdorff dimension* (HD) of $F \subseteq \Sigma_n^\omega$ in $(\Sigma_n^\omega, \rho_\beta)$ is defined as $\dim^{(\beta)} F := \inf\{\alpha : \nu_\beta^\alpha(F) = 0\} = \sup\{\alpha : \alpha = 0 \vee \nu_\beta^\alpha(F) = \infty\}$ . Here we mention that the HD is countably stable, that is,

$$\dim^{(\beta)} \bigcup_{i\in\mathbb{N}} F_i = \sup_{i\in\mathbb{N}} \dim^{(\beta)} F_i. \tag{14}$$

If $F \subseteq \mathbb{F}_\beta$, we could show the following characterization of $\nu_\beta^\alpha$:

LEMMA 4.1. *Let* $F \subseteq \mathbb{F}_\beta$. *Then, we have*

$$\nu_\beta^\alpha(F) = \lim_{\epsilon\to 0}\inf\left\{\beta^\alpha(L) : F \subseteq L\cdot\Sigma_n^\omega \wedge \forall w(w\in L \to \beta(w)\le\epsilon)\right\}. \tag{15}$$

*Proof.* On the one hand, $\beta(w) \ge \mathrm{diam}\, w\cdot\Sigma_n^\omega$, so the inequality "≤" follows.

On the other hand, let $F \subseteq \bigcup_{i\in M} F_i$ and $\sum_{i\in M}(\mathrm{diam}F_i)^\alpha \le \nu_\beta^\alpha(F)+\epsilon$ for some $M\subseteq\mathbb{N}$ and $\epsilon > 0$. Without loss of generality we may assume $F\cap F_i \ne \emptyset$. We consider two cases.

If $\epsilon_i := \mathrm{diam}F_i > 0$ and $\xi_i \in F$, then $\{\xi_i\} \ne F_i$ and $F_i \subseteq \mathbb{B}_{\epsilon_i}(\xi_i) = \{\eta : \rho_\beta(\xi_i, \eta) \le \epsilon_i\}$. According to Eq. (12), $\mathbb{B}_{\epsilon_i}(\xi_i) = w_i\cdot\Sigma_n^\omega$ for some $w_i \in \Sigma_n^*$.

If $\mathrm{diam}F_i = 0$, that is, if $F_i = \{\xi_i\} \subseteq \mathbb{F}_\beta$, then we may find a $w_i \sqsubseteq \xi_i$ such that $\beta(w_i)^\alpha \le \epsilon\cdot 2^{-(i+1)}$.

Consequently, $F \subseteq \bigcup_{i\in M} w_i\cdot\Sigma_n^\omega$ and

$$\beta^\alpha(\{w_i : i\in M\}) \le \sum_{i\in M}\max\left\{(\mathrm{diam}F_i)^\alpha, \epsilon\cdot 2^{-(i+1)}\right\} \le \nu_\beta^\alpha(F) + 2\cdot\epsilon$$

and the assertion results, since $\epsilon$ can be made arbitrarily small. ■

Next, we derive some relations between the $\beta$-entropy of languages and the HD of $\omega$-languages in the case $\beta^\alpha(V) < \infty$. First, we get analogues to [38, Lemmas 3.8 and 3.10]. To this end, we introduce the *δ-limit*

$$V^\delta := \left\{\xi : \xi\in\Sigma_n^\omega \wedge \mathbf{A}(\xi)\cap V \text{ is infinite}\right\}$$

of a language $V \subseteq \Sigma_n^*$.

LEMMA 4.2. *If* $\beta^\alpha(V) < \infty$, *then* $\nu_\beta^\alpha(V^\delta) = 0$.

*Proof.* In order to be able to apply Lemma 4.1, we must show that $V^\delta \subseteq \mathbb{F}_\beta$ whenever $\beta^\alpha(V) < \infty$. To this end, observe that for $\xi \in \mathbb{I}_\beta$, we have $\beta(w) \ge c_\xi > 0$ for some $c_\xi > 0$ and all $w \sqsubset \xi$. Hence, $\beta^\alpha(V) < \infty$ implies that $\mathbf{A}(\xi)\cap V$ is finite.

Now, proceeding as in [38], we use the partition of $V$ into $V^{(i)} := \{v : v\in V$ and $v$ has exactly $i$ prefixes in $V\}$. Then, $V^\delta \subseteq V^{(i)}\cdot\Sigma_n^\omega$ and, in view of Lemma 4.1, $\nu_\beta^\alpha(V^\delta) \le \beta^\alpha(V^{(i)})$. Since $\beta^\alpha(V) < \infty$, $\beta^\alpha(V^{(i)})$ tends to 0 as $i$ approaches infinity, and the assertion results. ■

LEMMA 4.3. *Let* $F \subseteq \mathbb{F}_\beta$. *Then,* $\nu_\beta^\alpha(F) = 0$ *iff there is a language* $L \subseteq \Sigma_n^*$ *such that* $F \subseteq L^\delta$ *and* $\beta^\alpha(L) < \infty$.

[10] If $F$ contains uncountably many isolated points, we have always that $\nu_\beta^\alpha(F) = \infty$.

*Proof.* Let $\nu_\beta^\alpha(F) = 0$. For $\alpha = 0$, we have $\nu_\beta^\alpha(F) = \operatorname{card} F$. So $F = \emptyset$, and we may choose $L = \emptyset$.

Let $\alpha > 0$. According to Eq. (15), for every $i \in \mathbb{N}$, we can find a language $L_i$ such that $F \subseteq L_i \cdot \Sigma_n^\omega$ and $\beta^\alpha(L_i) < n^{-i}$. This, in particular, implies that $\beta^\alpha(w) < n^{-i}$ whenever $w \in L_i$.

Suppose now that $\beta_{\min} := \min\{\beta(a) : a \in \Sigma_n\} \geq \frac{1}{k}$ for some $k \in \mathbb{N}$ , $k > 1$. Thus,

$$\frac{1}{k^{\alpha \cdot |w|}} \leq \beta^\alpha(w) < n^{-i}$$

and, consequently,

$$|w| \geq \frac{i}{\alpha \cdot \log_n k} \qquad \text{for } w \in L_i.$$

Now it is easy to see that $L := \bigcup_{i \in \mathbb{N}} L_i$ satisfies $F \subseteq L^\delta$ and $\beta^\alpha(L) < \infty$.
The converse is proved in Lemma 4.2. ■

As consequences of the HD definition, we get the following relationships between the $\beta$-entropy of languages and the HD of its $\delta$-limits.

$$\dim^{(\beta)} V^\delta \leq H_V^\beta, \quad \text{and} \tag{16}$$

$$\dim^{(\beta)} F \;= \inf\left\{H_w^\beta : F \subseteq W^\delta\right\}, \quad \text{if } F \subseteq \mathbb{F}_\beta . \tag{17}$$

4.3. *The Hausdorff Dimension of ω-Languages*

Utilizing the results of [1, 26] and Property 3.5, we can relate the Hausdorff dimension and the Hausdorff measure of strongly closed finite-state $F \subseteq \Sigma_n^\omega$ to the $\beta$-entropy of $\mathbf{A}(F)$.

LEMMA 4.4.

1. *If $c < 1$ then $V_{\beta,c}^\delta \subseteq \mathbb{F}_\beta$; this inclusion is proper if $\mathbb{F}_\beta \neq \emptyset$ and $\beta$ is not contractive.* ($V_{\beta,c}$ *was defined in Subsection* 3.2).

2. *For every finite-state and strongly closed ω-language $E \subseteq \mathbb{F}_\beta$, there are $c \in (0, 1)$ and $\ell \in \mathbb{N}$ such that $E \subseteq \{w : |w| = \ell \wedge \beta(w) \leq c^\ell\}^\omega$.*

*Proof.* 1. The first part is immediate. From the additional assumption, it follows that $\beta(a) < 1$ and $\beta(b) \geq 1$ for some letters $a, b \in \Sigma_n$. Depending on these values and $c < 1$, it is easy to construct a $\xi \in \{a, b\}^\omega$ such that $\inf\{\beta(w) : w \sqsubset \xi\} = 0$ but $\beta(w) > c^{|w|}$ for all but finitely many $w \sqsubset \xi$.

2. If $E$ is finite-state, its prefix language $\mathbf{A}(E)$ and its infix language $\mathbf{T}(E) = \mathbf{T}(\mathbf{A}(E))$ are also finite-state, i.e., regular languages. Let $\emptyset \neq \mathbf{A}(E)/w = \mathbf{A}(E)/w \cdot v$ for some $w, v$ with $v \neq e$. Then $w \cdot v^* \subseteq \mathbf{A}(E)$ and, since $E$ is strongly closed $w \cdot v^\omega \in E \subseteq \mathbb{F}_\beta$ whence $\beta(v) < 1$. According to Properties 3.4 and 3.5, it follows that $\beta(v) \leq c^\ell$ for some $c < 1$ and all $v \in \mathbf{T}(E) \cap \Sigma_n^\ell$, where $\ell$ is sufficiently large. Now, the assertion follows from the obvious inclusion $E \subseteq (\mathbf{T}(E) \cap \Sigma_n^m)^\omega$ which holds true for an arbitrary $m \in \mathbb{N}\backslash\{0\}$. ■

In [1, 26], it is shown that $\alpha = \dim^{(\beta)}(F)$ is the solution of $\Phi_{\mathbf{A}(F)}(s) = 1$, and that $\nu_\beta^\alpha(F) > 0$. Together with our ideas on how to compute $\beta^\alpha(\mathbf{A}(L))$, this yields:

THEOREM 4.1. *If $\emptyset \neq F \subseteq \mathbb{F}_\beta$ is finite-state and strongly closed, then $H_{\mathbf{A}(F)}^\beta = \dim^{(\beta)}(F)$; furthermore, if $\alpha = \dim^{(\beta)}(F)$, then $\nu_\beta^\alpha(F) > 0$.* ■

Since $U^\omega$ is finite-state and strongly closed only when $U$ is finite, in view of the identity $\mathbf{A}(U^\omega) = \mathbf{A}(U^*)$ and Property 3.5, the HD of any $U^\omega \subseteq \mathbb{F}_\beta$ is obtained as $\dim^{(\beta)} U^\omega = H_{U^*}^\beta$. By an approximation as in Theorem 3.8, we get a general formula for the HD of $L^\omega$:

LEMMA 4.5. *If $c \in (0, 1)$ and $L \subseteq V_{\beta,c}$, then* $\dim^{(\beta)} L^\omega = \dim^{(\beta)}(L^*)^\delta = H_{L^*}^\beta$.

*Proof.* The inequality $\dim^{(\beta)} L^\omega \leq \dim^{(\beta)}(L^*)^\delta \leq H_{L^*}^\beta$ follows from the inclusion $L^\omega \subseteq (L^*)^\delta$ and Eq. (16).

In order to show the reverse inequality, observe that, in view of Theorem 3.8,we have

$$\begin{aligned} H^{\beta}_{L^*} &= \sup\left\{H^{\beta}_{U^*} : U \subseteq L \text{ and } U \text{ finite}\right\} \\ &= \sup\left\{\dim^{(\beta)} U^{\omega} : U \subseteq L \text{ and } U \text{ finite}\right\} \\ &\leq \dim^{(\beta)} L^{\omega}. \end{aligned}$$

■

Next, we obtain a general bound on $\nu^{\alpha}_{\beta}(L^{\omega})$ for $\alpha = H^{\beta}_{L^*}$.

LEMMA 4.6. *If $c \in (0, 1)$ and $L \subseteq V_{\beta,c}$, then $\nu^{\alpha}_{\beta}((L^*)^{\delta}) \leq 1$ for $\alpha = H^{\beta}_{L^*}$.*

*Proof.* Define $L^{(i)} := \{w : w \in L^* \wedge |w| \geq i \wedge \forall v(v \sqsubset w \wedge |v| \geq i \rightarrow v \notin L^*)\}$. Then $L^{(i)}$ is a prefix-code contained in $L^*$ satisfying $(L^*)^{\delta} \subseteq L^{(i)} \cdot \Sigma^{\omega}_n$.

Now on the one hand, following Property 3.6,

$$H^{\beta}_{(L^{(i)})^*} = \inf\left\{s : \beta^s\left(L^{(i)}\right) \leq 1\right\} \leq H^{\beta}_{(L^*)}.$$

On the other hand, let $\alpha := H^{\beta}_{(L^*)}$ and $\gamma_i := H^{\beta}_{(L^{(i)})^*}$. Then $\beta^{\gamma_i}(L^{(i)}) \leq 1$ and, therefore, $\alpha \geq \gamma_i$ and $(L^*)^{\delta} \subseteq L^{(i)} \cdot \Sigma^{\omega}_n$ imply

$$\nu^{\alpha}_{\beta}((L^*)^{\delta}) \leq \liminf_{i\to\infty} \beta^{\alpha}\left(L^{(i)}\right) \leq \liminf_{i\to\infty} \beta^{\gamma_i}\left(L^{(i)}\right) \leq 1,$$

because the function $\beta^s(L^{(i)})$ is continuous on $[\gamma_i, \infty)$. ■

Define the *strong closure* of an $\omega$-language $E$ as $\mathrm{cl}(E) := (\mathbf{A}(E))^{\delta}$; it is the smallest strongly closed $\omega$-language containing $E$. Thus, independently of $\beta$, it contains the smallest $\rho_{\beta}$-closed set $F \subseteq \Sigma^{\omega}_n$ with $E \subseteq F$. According to Lemma 4.5 and Property 3.5, we can see:

COROLLARY 4.1. *If $c \in (0, 1)$ and $L \subseteq V_{\beta,c}$ is a regular language, then $\dim^{(\beta)} L^{\omega} = \dim^{(\beta)} \mathrm{cl}(L^{\omega})$.*

COROLLARY 4.2. *If $c \in (0, 1)$ and $L \subseteq V_{\beta,c}$ is regular and a finite union of codes and $\alpha = H^{\beta}_{L^*}$, then $0 < \nu^{\alpha}_{\beta}(L^{\omega}) = \nu^{\alpha}_{\beta}(\mathrm{cl}(L^{\omega})) \leq 1$.*

*Remark 4.2.* More involved calculations such as those in [27, Theorem 6] show that

$$0 < \nu^{\alpha}_{\beta}((L^*)^{\delta}) = \nu^{\alpha}_{\beta}(\mathrm{cl}(L^{\omega})) \leq 1$$

for arbitrary regular $L \subseteq V_{\beta,c}$, but in the case of nonregular languages $W$, one might even have $\dim^{(\beta)} W^{\omega} < \dim^{(\beta)} \mathrm{cl}(W^{\omega})$ (cf. [38, Examples 6.3 and 6.5]).

## 5. APPLICATIONS TO FRACTAL GEOMETRY

One of the most popular ways to describe fractals is via iterated function systems (IFS), see [2].

### 5.1. *Iterated Function Systems*

We restrict ourselves in the following to compact *Euclidean spaces* $\mathcal{X} \subseteq \mathbb{R}^m$ equipped with the Euclidean distance $\rho_E$. Denoting the set of contractive similitudes $f : \mathcal{X} \to \mathcal{X}$ by $\mathcal{S}(\mathcal{X})$, we can describe an IFS $\mathcal{F}$ as a map $\mathcal{F} : \Sigma_n \to \mathcal{S}(\mathcal{X})$. We sketch out some well-known properties of IFS in the following: An IFS $\mathcal{F}$ gives a contractive valuation $\beta_{\mathcal{F}} : \Sigma^*_n \to (0, \infty)$, where $\beta_{\mathcal{F}}(i)$ (for $i \in \Sigma_n$) denotes the similarity factor of $\mathcal{F}(i)$. So, $w \in \Sigma^+_n$ can be seen as a similitude $\phi_{\mathcal{F}}(w) \in \mathcal{S}(\mathcal{X})$, where $\phi_{\mathcal{F}}$ is a semigroup morphism $(\Sigma^+_n, \cdot) \to (\mathcal{S}(\mathcal{X}), \circ)$. Recall the notion of *address* derived from Eq. (1) in the Introduction. Furthermore, the map $\phi_{\mathcal{F}} : (\Sigma^{\omega}_n, \rho_{\beta_{\mathcal{F}}}) \to (\mathcal{X}, \rho_E)$ is Lipschitz. Call $A_{\mathcal{F}} = \phi_{\mathcal{F}}(\Sigma^{\omega}_n)$ the *limit set* of $\mathcal{F}$.

Given an IFS $\mathcal{F} : \Sigma_n \to \mathcal{S}(\mathcal{X})$, we interpret a finite ($m$-element) language $L = \{w_1, \ldots, w_m\} \subset \Sigma_n^+$ as an IFS $\mathcal{F}_L : \Sigma_m \to \mathcal{S}(\mathcal{X})$, $i \mapsto \phi_{\mathcal{F}}(w_i)$. We have $A_{\mathcal{F}_L} = \phi_{\mathcal{F}}(L^\omega)$. Similarly, an infinite $L$ leads to an infinite IFS (IIFS) [13, 25, 46] whose theory is more involved but analogous to IFS theory. We can still define a set described by an IIFS $\mathcal{F}_L$ (based on the IFS $\mathcal{F}$ and the language $L$), called the *limit set* $\phi_{\mathcal{F}}(L^\omega)$.[11] In $\mathcal{X} = (\Sigma_n^\omega, \rho_{\beta_n})$, we interpret any $L \subseteq \Sigma_n^+$ as an (I)IFS by $w : \Sigma_n^\omega \to \Sigma_n^\omega, \xi \mapsto w \cdot \xi$, with limit set $L^\omega$. For $L \subset \Sigma_n^+$, call $\mathrm{vd}_\beta(L) = \inf\{s : \beta^s(L) \leq 1\}$ the *valuation dimension* (VD). Property 3.6 shows how close the relationship between $\mathrm{vd}_\beta(L)$ and $H_{L^*}^\beta$ is. VD corresponds to the similarity dimension in IFS theory.

We denote the $s$-dimensional outer Hausdorff measure on $(\mathcal{X}, \rho_E)$ by $\mathcal{H}^s$, and the corresponding HD by $\dim_H$. For IFS, *Moran's open set condition* (*OSC*) is well known as an assumption alleviating the determination of the HD of $A_{\mathcal{F}}$ [2, 9, 11]: Provided that there is an open bounded nonempty test set $M \subseteq \mathcal{X}$ such that $\mathcal{F}(i)(M) \subseteq M$ for any $i \in \Sigma_n$ and that, furthermore, for any $i, j \in \Sigma_n$, $i \neq j$, $\mathcal{F}(i)(M) \cap \mathcal{F}(j)(M) = \emptyset$, then, for $\alpha = \mathrm{vd}_{\beta_{\mathcal{F}}}(\Sigma_n)$, $0 < \mathcal{H}^\alpha(A_{\mathcal{F}}) < \infty$ and $\alpha = \dim_H(A_{\mathcal{F}})$. Generally, it is not trivial to find a test set for some $\mathcal{F}$. But, if we were to know that $\mathcal{F}$ fulfils an OSC, (when) could we say anything about $\mathcal{F}_L$?

Here, we need two further notions [39].

DEFINITION 5.1. Call $V \subseteq \Sigma_n^*$ *OSC-code* iff there is a $\emptyset \neq W \subseteq \Sigma_n^*$ (OSC-*witness*) verifying:

$$\forall v\big(v \in V \to v \cdot W \cdot \Sigma_n^\omega \subseteq W \cdot \Sigma_n^\omega\big), \quad \text{and} \tag{18}$$

$$\forall v, v'\big(v, v' \in V \wedge v \neq v' \to v \cdot W \cdot \Sigma_n^\omega \cap v' \cdot W \cdot \Sigma_n^\omega = \emptyset\big). \tag{19}$$

Any OSC-code is a code, and any prefix-code is an OSC-code. Moreover, any regular code is an OSC-code [39]. The correspondence with the Euclidean case is as follows: Interpreting $V$ as an (I)IFS in $\Sigma_n^\omega$, $V$ satisfies the OSC with open test set $W \cdot \Sigma_n^\omega$ iff $V$ is an OSC-code with OSC-witness $W$.

THEOREM 5.1. *Let $\mathcal{F} = (\varphi_1, \ldots, \varphi_n)$, where $\varphi_i : \mathbb{R}^d \to \mathbb{R}^d$, be an IFS satisfying OSC, and let $C \subseteq \Sigma_n^*$ be an OSC-code. Then (I)IFS $\mathcal{F}_C$ satisfies OSC, too.*

*Proof.* Let $\mathcal{F} = (\varphi_1, \ldots, \varphi_n)$ satisfy the OSC with test set $M \subseteq \mathbb{R}^d$, and let $W \subseteq \Sigma_n^*$ be an OSC-witness for $C$. Let $\mathcal{F}_C = (\varphi_v)_{v \in C}$, where $\varphi_v := \varphi_{v_1} \circ \cdots \circ \varphi_{v_\ell}$ for $v = v_1 \cdots v_\ell$. Define $M' := \bigcup_{w \in W} \varphi_w(M)$. The set $M'$ is nonempty and open, because all $\varphi_i$ are similitudes and $M$ is nonempty and open and, moreover, $\varphi_v(M') = \varphi_v(\bigcup_{w \in W} \varphi_w(M)) = \bigcup_{u \in v \cdot W} \varphi_u(M) \subseteq M'$ for $v \in C$. Now consider, for $v, v' \in C$, $v \neq v'$,

$$\begin{aligned}\varphi_v(M') \cap \varphi_{v'}(M') &= \left(\bigcup_{w \in W} \varphi_{v \cdot w}(M)\right) \cap \left(\bigcup_{w \in W} \varphi_{v' \cdot w}(M)\right) \\ &= \bigcup_{w, w' \in W} (\varphi_{v \cdot w}(M) \cap \varphi_{v' \cdot w'}(M)).\end{aligned}$$

Since $v \cdot w \cdot \Sigma_n^\omega \cap v' \cdot w' \cdot \Sigma_n^\omega = \emptyset$, neither $v \cdot w \sqsubseteq v' \cdot w'$ nor $v' \cdot w' \sqsubseteq v \cdot w$. Hence, we have a first position where $v \cdot w$ and $v' \cdot w'$ do not coincide, that is, $u \cdot i \sqsubseteq v \cdot w$ and $u \cdot j \sqsubseteq v' \cdot w'$ where $i, j \in \Sigma_n$, $i \neq j$, and we have $\varphi_{u \cdot i}(M) \supseteq \varphi_{v \cdot w}(M)$ and $\varphi_{u \cdot j}(M) \supseteq \varphi_{v' \cdot w'}(M)$.

Now, from the inclusions $\varphi_i(M), \varphi_j(M) \subseteq M$ (one part of the OSC) and the fact that $\varphi_i(M) \cap \varphi_j(M) = \emptyset$, we can readily conclude our assertion. ∎

Together with [13, Theorem 3.11], we obtain $\dim_H(\phi_{\mathcal{F}}(L^\omega)) = \mathrm{vd}_{\beta_{\mathcal{F}}}(L)$ if the IFS $\mathcal{F} : \Sigma_n \to \mathcal{S}(\mathcal{X})$ satisfies the OSC and $L$ is an OSC-code. The previous sections together with Theorem 3 of [1], however, allow the strengthening of the aforementioned result, and make it possible to generalize it to not necessarily contractive valuations.

In [1, 26], IFS have been generalized to systems $\mathcal{F}$ with arbitrary similitudes. To guarantee the convergence of (1), one must restrict the set of admissible $\omega$-words $\xi$. In [1, Theorem 3], it is shown that $\phi_{\mathcal{F}} : (E, \rho_{\beta_{\mathcal{F}}}) \to (\mathcal{X}, \rho_E)$ is Lipschitz whenever $E$ is a strongly closed finite-state subset of $\mathbb{F}_{\beta_{\mathcal{F}}}$.

[11] When restricting one's attention to compact sets, consider its closure instead.

In connection with this, the following generalization of the OSC for pairs $(\mathcal{F}, E)$ satisfying the above-mentioned property is introduced.

Let $\mathcal{M}$ be a finite set of open subsets of $(\mathcal{X}, \rho_E)$. To every $w \in \Sigma_n^*$, we assign a set $M_w \in \mathcal{M}$. We say that the assignment is *compatible* with $E$ iff

$$M_w = \emptyset \to w \notin \mathbf{A}(E), \tag{20}$$

$$\bigcup_{i=1}^{n} \varphi_i(M_{w\cdot i}) \subseteq M_w, \quad \text{and} \tag{21}$$

$$\varphi_i(M_{w\cdot i}) \cap \varphi_j(M_{w\cdot j}) = \emptyset, \quad \text{for } i \neq j. \tag{22}$$

We say that a pair $(\mathcal{F}, E)$ satisfies the *Generalized Open Set Condition (GOSC)* iff $E$ is a finite-state strongly closed subset of $\mathbb{F}_{\beta_\mathcal{F}}$ and there is an $\mathcal{M}$ as well as an assignment $w \mapsto M_w \in \mathcal{M}$ compatible with $E$. By the first condition, for every finite-state strongly closed subset $F \subseteq E$ the pair $(\mathcal{F}, F)$ satisfies GOSC if $(\mathcal{F}, E)$ satisfies GOSC. Theorem 3 of [1] gives:

THEOREM 5.2. *Let $E$ be a finite-state strongly closed subset of $\mathbb{F}_{\beta_\mathcal{F}}$ so that $(\mathcal{F}, E)$ satisfies GOSC. Then, we have*: $\dim_H \phi_\mathcal{F}(E) = H^{\beta_\mathcal{F}}_{\mathbf{A}(E)} = \dim^{(\beta_\mathcal{F})} E =: \alpha$ *with* $\mathcal{H}^\alpha(\phi_\mathcal{F}(E)) > 0$.

We proceed with the strengthening of $\dim_H(\phi_\mathcal{F}(L^\omega)) = \mathrm{vd}_{\beta_\mathcal{F}}(L)$.

THEOREM 5.3. *Let $(\mathcal{X}, \rho_E)$ be a Euclidean space, $\mathcal{F} : \Sigma_n \to \mathcal{S}(\mathcal{X})$, let $E$ be a finite-state and strongly closed subset of $\mathbb{F}_\beta$, and let $L \subseteq \Sigma_n^*$ such that $L^\omega \subseteq E$. Assume the pair $(\mathcal{F}, E)$ satisfies the GOSC. Then,* $\dim_H(\phi_\mathcal{F}(L^\omega)) = \dim^{(\beta_\mathcal{F})} L^\omega$ *and, provided $L$ is a code, we have* $\dim_H(\phi_\mathcal{F}(L^\omega)) = \mathrm{vd}_{\beta_\mathcal{F}}(L)$.

*Proof.* Since $\phi_\mathcal{F} : E \to X$ is Lipschitz, clearly $\dim_H(\phi_\mathcal{F}(L^\omega)) \leq \dim^{(\beta)} L^\omega = H^{\beta_\mathcal{F}}_{L^*} \leq \mathrm{vd}_{\beta_\mathcal{F}}(L)$.

For each of the finite languages $L_m = L \cap \{w \in \Sigma_n^* : |w| \leq m\}$, the $\omega$-language $L_m^\omega \subseteq E$ is finite-state and strongly closed, hence $(\mathcal{F}, L_m^\omega)$ satisfies the GOSC and, according to Theorem 5.2, we have $\dim_H \phi_\mathcal{F}(L_m^\omega) = H^{\beta_\mathcal{F}}_{\mathbf{A}(L_m^*)} = H^{\beta_\mathcal{F}}_{L_m^*}$. Since $(L_m)_{m\in\mathbb{N}}$ is an increasing chain of sets with $\bigcup_{m\in\mathbb{N}} L_m = L$, by Theorem 3.8, $\lim_{m\to\infty} H^{\beta_\mathcal{F}}_{L_m^*} = H^{\beta_\mathcal{F}}_{L^*}$, which in turn equals $\dim^{(\beta)}(L^\omega)$ by Lemma 4.5. Hence, $\dim_H(\phi_\mathcal{F}(L^\omega)) \geq \sup_{m\in\mathbb{N}} \dim_H(\phi_\mathcal{F}(L_m^\omega)) = \dim^{(\beta)}(L^\omega)$.

The additional assertion —if $L$ is a code—follows from Eq. (8). ∎

*Remark 5.3.* An analogue for IIFS satisfying the OSC (using the notion of topological pressure function) is given in [25, Theorem 3.15]. Compare also [17, Theorem 10].

In the case of a regular language $L$, we can strengthen our result.

THEOREM 5.4. *Let $(\mathcal{X}, \rho_E)$ be a Euclidean space, $\mathcal{F} : \Sigma_n \to \mathcal{S}(\mathcal{X})$, and let $L \subseteq \Sigma_n^*$ be a regular language such that $\beta_\mathcal{F}(w) < 1$ for all $w \in L \setminus \{e\}$. Then, $cl(L^\omega) \subseteq \mathbb{F}_{\beta_\mathcal{F}}$ and $dim_H(\phi_\mathcal{F}(cl(L^\omega)) = dim_H(\phi_\mathcal{F}(L^\omega)) = dim^{(\beta_\mathcal{F})} L^\omega$. If, moreover, $L$ is a finite union of codes, then $\mathcal{H}^s(\phi_\mathcal{F}(L^\omega)) = \mathcal{H}^s(\phi_\mathcal{F}(cl(L^\omega)))$ for $s \in [0, \infty)$.*

*Proof.* If $\beta_\mathcal{F}(w) < 1$ for all $w \in L\setminus\{e\}$, then, according to Properties 3.4 and 3.5, we have $\beta_\mathcal{F}(w) \leq c_L^{|w|}$ for some $c_L < 1$ and for all $w \in \mathbf{T}(L^*)$ with $|w| \geq \ell$ for a suitably chosen $\ell \in \mathbb{N}$. In particular, $\mathbf{A}(\xi) \subseteq \mathbf{A}(L^\omega) \subseteq \mathbf{T}(L^*)$ implies $\xi \in \mathbb{F}_{\beta_\mathcal{F}}$, which proves our first assertion. Since $L$ is regular, $\dim^{(\beta_\mathcal{F})} L^\omega = \dim^{(\beta_\mathcal{F})} \mathrm{cl}(L^\omega)$ according to Corollary 4.1, and the second assertion follows from Theorems 5.2 and 5.3.

Now, let $L$ be a finite union of codes and $\alpha = \dim^{(\beta)} L^\omega$. Applying the fact that $\dim^{(\beta)}(\mathbf{A}(L))^\delta < \alpha = \dim^{(\beta)} L^\omega$ utilized in the proof of Corollary 4.2, we have $\dim_H(\phi_\mathcal{F}((\mathbf{A}(L))^\delta)) < \alpha$ and, therefore, $\mathcal{H}^s(\phi_\mathcal{F}((\mathbf{A}(L))^\delta)) = 0$ for $s \geq \alpha$. In the case where $s < \alpha$, we obviously find $\mathcal{H}^s(\phi_\mathcal{F}(L^\omega)) = \mathcal{H}^s(\phi_\mathcal{F}(\mathrm{cl}(L^\omega))) = \infty$. ∎

*Note 5.4.* In [13, Remark 3.12], the question was raised whether requiring an OSC for each IFS-part $\mathcal{F}_n = (\mathcal{F}(1), \ldots, \mathcal{F}(n))$ of a given IIFS $\mathcal{F}$ is weaker than requiring an OSC for $\mathcal{F}$ itself. We can show the following here: If all $\mathcal{F}_n$ fulfil an OSC, then $\mathcal{F}$ itself does not necessarily satisfy an OSC.

*Proof.* Consider the basic IFS $\mathcal{F} : \Sigma_2 \to \mathcal{S}(([0, 1], \rho_E))$ defined by $\mathcal{F}(1)(x) = x/2$ and $\mathcal{F}(2)(x) = x/2 + 1/2$. It is clear that $A_\mathcal{F} = [0, 1]$. Consider the suffix-code $L = \{w12^{|w|} : w \in \Sigma_2^*\}$ (which is not an OSC-code) from [39, Example 1]. Assume the IIFS $\mathcal{F}_L$ satisfies an OSC with test set $M$. Then,

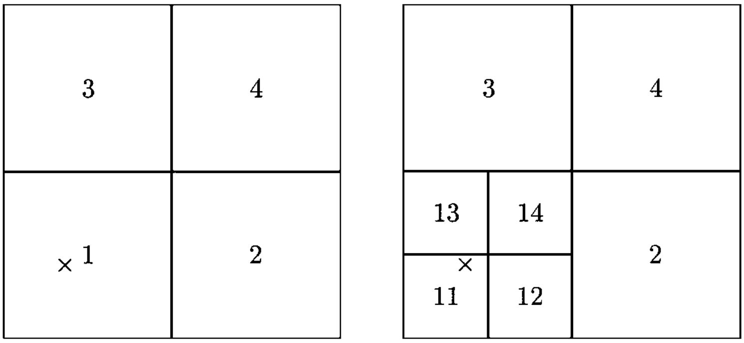


**FIG. 3.** Quadtree.

$\phi_{\mathcal{F}}^{-1}(M) \neq \emptyset$ is open in the topology of $(\Sigma_2^\omega, \rho_2)$, and defines $\emptyset \neq W \subseteq \Sigma_2^*$ by $W\Sigma_2^\omega = \phi_{\mathcal{F}}^{-1}(M)$. We show that $W$ is an OSC-witness for $L$, contradicting [39].

Assume to the contrary that $W$ is not an OSC-witness. Then either condition (18) or (19) is violated. Now, assume there are $w, w' \in W$ and $v, v' \in L$ such that $\xi \in v \cdot w \cdot \Sigma_2^\omega \cap v' \cdot w' \cdot \Sigma_2^\omega \neq \emptyset$. Then, $\phi_{\mathcal{F}}(\xi) \in \phi_{\mathcal{F}}(v)(M) \cap \phi_{\mathcal{F}}(v')(M)$, contradicting the assumption that $M$ is a test set. Finally, if $v \cdot W \cdot \Sigma_2^\omega \not\subseteq W \cdot \Sigma_2^\omega$ for some $v \in L$, then there is a $\xi \in v \cdot W \cdot \Sigma_2^\omega$ such that $\xi \notin W \cdot \Sigma_2^\omega$. $\phi_{\mathcal{F}}(\xi) \in \phi_{\mathcal{F}}(v)(M)$ implies $\phi_F(\xi) \in M$, since $M$ is a test set. Hence, $\xi \in W \cdot \Sigma_2^\omega = \phi_{\mathcal{F}}^{-1}(M)$, which is a contradiction. ∎

## 5.2. *Calculating Dimensions of Some Fractals*

We show how to apply our results in the following examples. As a basic IFS, we take the quadtree IFS $\mathcal{F} : \Sigma_4 \to \mathcal{S}([0,1]^2)$. The effect of its four mappings is indicated in Fig. 3; e.g., $\mathcal{F}(2)$ maps the unit square onto its lower right quarter. The words in the center of the subsquares are the prefixes of the addresses of its points. For example, any address of the point indicated by the $\times$-symbol starts with 11. Of course, $A_{\mathcal{F}} = [0,1]^2$ is of little interest. We remark that $V = (0,1)^2$ may serve as a test set for OSC. First, taking $L = \{1, 2, 3\}$, we get the well-known Sierpiński triangle. Since $L$ is a code, we may solve $\beta_{\mathcal{F}}^s(L) = 3(0.5)^s = 1$, making $\alpha = \log_2 3 \approx 1.5850$ the Hausdorff dimension.

The next example is more involved. Consider the fractals presented in Fig. 4. They are obtained in the following way: $F_0 = L_0^\omega$, where

$$L_0 := \{1, 4\} \cup 3 \cdot 2^* \cdot (\{1, 3\} \cup 4 \cdot 3^* \cdot 2) \cup 2 \cdot 3^* \cdot 2.^{12}$$

According to Theorem 5.4 and Corollary 4.1 we calculate its Hausdorff dimension as follows: $\dim_H \phi_{\mathcal{F}}(L_0^\omega) = \dim_H \phi_{\mathcal{F}}(\mathrm{cl}(L_0^\omega)) = \dim^{(\beta_{\mathcal{F}})} \mathrm{cl}(L_0^\omega)$. This dimension is estimated in Example 1 of [27] as $\log_2(2 + \sqrt{2}) \approx 1.772$.

If we omit the transition labelled **3** (boldface) in the automaton of Fig. 5, we obtain $F' := L'^\omega$ where $L' := \{1, 4\} \cup 3 \cdot 2^* \cdot (\{1\} \cup 4 \cdot 3^* \cdot 2) \cup 2 \cdot 3^* \cdot 2$ . The fractal generated by $F' := L'^\omega$ has dimension $\dim_H \phi_{\mathcal{F}}(F') = 1.654 < \dim_H \phi_{\mathcal{F}}(F_0)$.

In Theorem 3.8 we proved that the entropy of $L^*$ can be approximated by the entropy of $U^*$ where $U \subseteq L$ is finite. Below we present the two approximations $F_1$ and $F_2$ of the fractal $F_0$ based on this fact.

If we consider $L_1 := \{1, 4, 22, 31, 33, 232, 321, 323, 342, 3242, 3432, 32432\} \subseteq L_0$, that is, the finite language obtained by passing each loop in the automaton of Fig. 5 at most once, we obtain an approximation $F_1 := L_1^\omega$ of $F_0$.

Another approximation of $F_0$ is obtained as $F_2 := L_2^\omega$, where

$$L_2 := L_1 \cup \{2332, 3221, 3223, 23332, 32221, 32223, 32242\}$$

is the set of words of length $\leq 5$ in $L_0$. Both fractals are presented in Fig. 6.

Their Hausdorff dimensions are $\dim_H \phi_{\mathcal{F}}(F_1) = 1.734 < \dim_H \phi_{\mathcal{F}}(F_2) = 1.766 < \dim_H \phi_{\mathcal{F}}(F_0)$.

Further examples of calculating the Hausdorff dimension of fractals using the approach presented here can be found, e.g., in [14] where variations of the Sierpiński triangle are presented and in [27] where a relationship between dimension and density is investigated.

[12] A finite automaton accepting $L_0$ is given in Fig. 5.

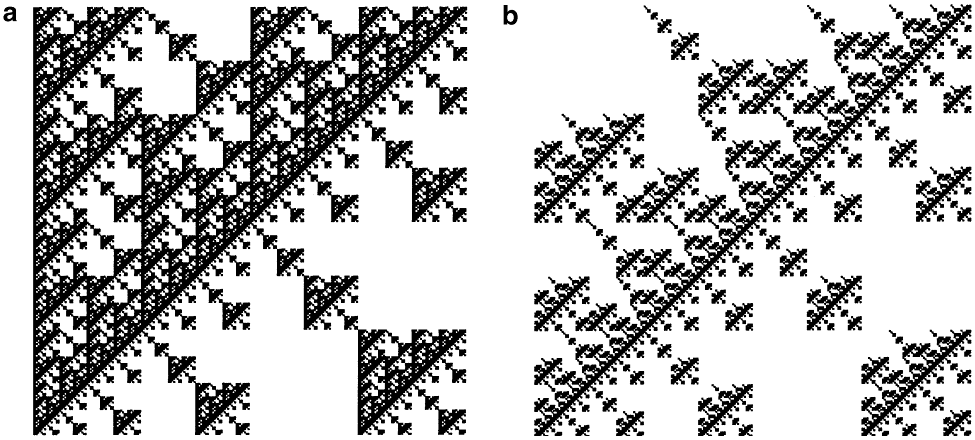


**FIG. 4.** The fractal (a) $F_0$ and its relative (b) $F'$.

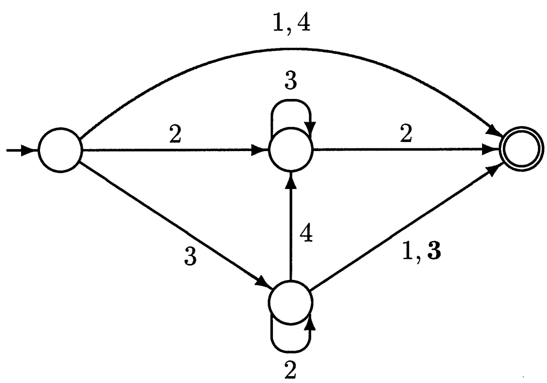


**FIG. 5.** An automaton accepting $L_0$.

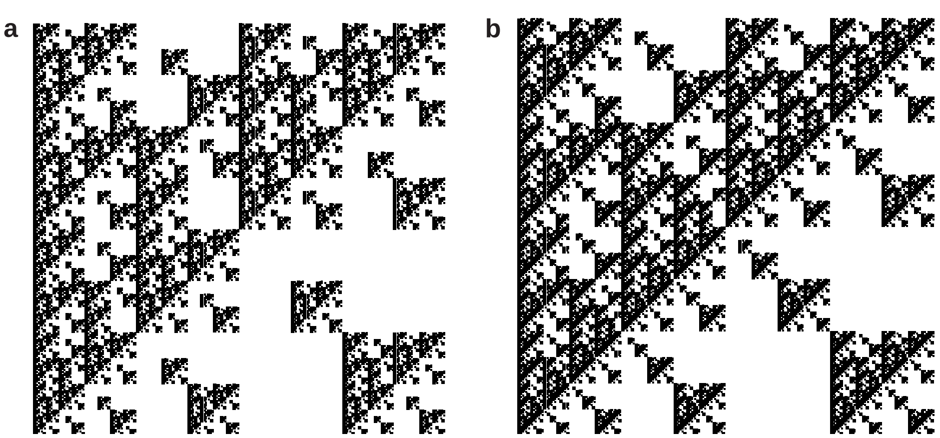


**FIG. 6.** "Finite" approximations of the fractal $F_0$: (a) $F_1$, (b) $F_2$.

## 6. CONCLUSIONS

We remark that there is another related approach connecting languages and fractals (based on an IFS $\mathcal{F} : \Sigma_n \to \mathcal{S}(X)$): Take an $\omega$-language $F \subseteq \Sigma_n^\omega$ and consider the (fractal) set $\phi_{\mathcal{F}}(F)$. For example, using regular (or finite-state strongly closed) $\omega$-languages, we obtain in such a way a class of fractals known under different names: Generalized recurrent systems [8], graph directed constructions [26], recurrent IFS [3], MRFS [5], and hierarchical IFS [29], see also [28]. By using the well-known McNaughton theorem, a regular $\omega$-language $F$ can be represented in the form $F = \bigcup_{i=1}^m W_i \cdot V_i^\omega$, where the $V_i$'s are regular prefix codes. Hence, $\dim_H(\phi_{\mathcal{F}}(F)) = \max_{i=1}^m \dim_H(\phi_{\mathcal{F}}(V_i^\omega))$, where the latter dimensions can be computed easily presuming the $V_i$'s are given by unambiguous regular expressions. By such means, we obtain another way of determining the Hausdorff dimension of those fractals as well (cf. [12, 42]).

Our method is not restricted to the calculation of the Hausdorff dimension of regular $\omega$-languages and their fractal counterparts in Euclidean space: It is well known that other $\omega$-languages, e.g., context-free $\omega$-languages, are also of the form $F = \bigcup_{i=1}^m W_i \cdot V_i^\omega$ (cf. [34]). (Here the languages $W_i$, $V_i$ are not necessarily regular.) It is clear from the formulas derived in this and in the preceding sections that, for fractals related to $\omega$-languages of this shape, the Hausdorff dimension can be calculated as soon as we are able to calculate the $\beta$-entropy of the corresponding languages, thus leading to a problem related to formal language theory.

We mention that basic IFS, where not all mappings are contractive, come across quite naturally. For example, one could extend the quadtree IFS employed in the previous section by introducing a fifth letter, indicating a $90^\circ$ clockwise rotation of a square around its midpoint. So, one would like to calculate the valuation of a regular set in particular. Given some finite description of the regular set, one can do this either by the techniques explained after Proposition 3.5 or by exploiting unambiguities in representations as done in [14, 15, 17].

Finally, note that one of the assumptions in Corollary 3.2, Corollary 4.2, Theorem 5.1, and Theorem 5.4 was the (OSC-)code property of the considered language. Similar conditions are present in [14, 17]. In order to have the possibility of an automatic dimension calculation, the computability of such conditions should be guaranteed. Results regarding this direction can be found in [16, 21].

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