11-2.2 Work of a Couple

The work done by a couple is obtained by calculating the work done by each force of the couple separately and adding them together. For example, consider a couple **C** acting on a rigid body as shown in Fig. 11-4*a*. During some small time dt the body translates and rotates. If the displacement of point A is $d\mathbf{r} = ds\,\mathbf{e}_t$, choose a second point B such that the line AB is perpendicular to $d\mathbf{r}$. Then the motion that takes A to A' will take B to B'. This motion may be considered in two parts: first a translation that takes the line AB to $A'\hat{B}$, followed by a rotation $d\theta$ about A' that takes $\hat{B}$ to B' (see Fig. 11-4*b*).

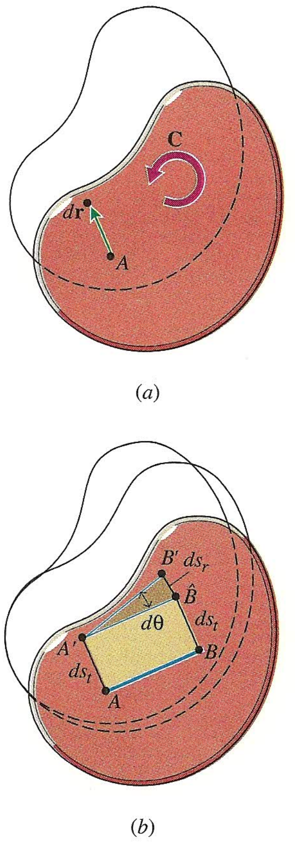
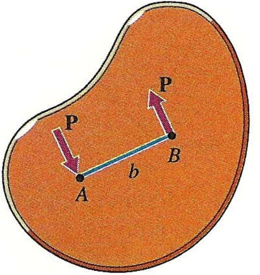


Figure 11-4 Action of a couple on a rigid body.



(c)

Now represent the couple by a pair of forces of magnitude $F = C/b$ in the direction perpendicular to the line AB (see Fig. 11-4c). During the translational part of the motion, one force will do positive work $F\,ds_t$ and the other will do negative work $-F\,ds_t$; therefore, the sum of the work done on the body by the pair of forces during the translational part of the motion is zero. During the rotational part of the motion, A' is a fixed point and the force applied at A' does no work. The work done by the force at B is $dU = F\,ds_r \cong Fb\,d\theta$, where $d\theta$ is in radians and $C = Fb$ is the magnitude of the moment of the couple. Therefore, when a body is simultaneously translated and rotated, the couple does work only as a result of the rotation.

The total work done by the couple during the differential motion is

$$dU = F\,ds_t - F\,ds_t + Fb\,d\theta = C\,d\theta \tag{11-5}$$

The work is positive if the angular displacement $d\theta$ is in the same direction as the sense of rotation of the couple and negative if the displacement is in the opposite direction. No work is done if the couple is translated or rotated about an axis parallel to the plane of the couple.

The work done on the body by the couple as the body rotates through a finite angle $\Delta\theta = \theta_2 - \theta_1$ is obtained by integrating Eq. 11-5:

$$U_{1\to 2} = \int_1^2 dU = \int_{\theta_1}^{\theta_2} C\,d\theta \tag{11-6}$$

If the couple is constant, then C can be taken outside the integral sign and Eq. 11-6 becomes

$$U_{1\to 2} = C\int_{\theta_1}^{\theta_2} d\theta = C(\theta_2 - \theta_1) = C\,\Delta\theta \tag{11-7}$$

If the body rotates in space, the component of the infinitesimal angular displacement $d\boldsymbol{\theta}$ in the direction of the couple $\mathbf{C}$ is required. For this case, the work done is determined by using the dot product relationship,

$$dU = \mathbf{C}\cdot d\boldsymbol{\theta} = M\,d\theta\cos\phi \tag{11-8}$$

where M is the magnitude of the moment of the couple, $d\theta$ is the magnitude of the infinitesimal angular displacement, and ϕ is the angle between $\mathbf{C}$ and $d\boldsymbol{\theta}$. For planar motion (in the xy-plane), $\mathbf{C} = C\,\mathbf{k}$, $d\boldsymbol{\theta} = d\theta\,\mathbf{k}$, $\mathbf{C}\cdot d\boldsymbol{\theta} = C\,d\theta$, and Eq. 11-8 gives the same result as Eq. 11-7.

Since work is a scalar quantity, the work done on a rigid body by a system of external forces and couples is the algebraic sum of work done by the individual forces and couples.

EXAMPLE PROBLEM 11-2

A constant couple $\mathbf{C} = 25\mathbf{i} + 35\mathbf{j} - 50\mathbf{k}$ N · m acts on a rigid body. The unit vector associated with the fixed axis of rotation of the body for an infinitesimal angular displacement $d\theta$ is $\mathbf{e}_\theta = 0.667\mathbf{i} + 0.333\mathbf{j} + 0.667\mathbf{k}$. Determine the work done on the body by the couple during an angular displacement of 2.5 rad.

SOLUTION

The magnitude (moment M) of the couple is

$$M = \sqrt{(25)^2 + (35)^2 + (-50)^2} = 65.95 \text{ N} \cdot \text{m}$$

The unit vector associated with the couple is

$$\mathbf{e}_C = \frac{+25}{65.95}\mathbf{i} + \frac{+35}{65.95}\mathbf{j} + \frac{-50}{65.95}\mathbf{k} = 0.379\mathbf{i} + 0.531\mathbf{j} - 0.758\mathbf{k}$$

The cosine of the angle between the axis of the couple and the axis of rotation of the body is

$$\begin{aligned} \cos\alpha &= \mathbf{e}_C \cdot \mathbf{e}_\theta \\ &= (0.379\mathbf{i} + 0.531\mathbf{j} - 0.758\mathbf{k}) \cdot (0.667\mathbf{i} + 0.333\mathbf{j} + 0.667\mathbf{k}) \\ &= -0.0760 \end{aligned}$$

Therefore,

$$\alpha = 94.36°$$

The work done by the couple during the finite rotation of 2.5 rad can be obtained by using Eq. 11-8. Thus,

The work done by the moment M of a couple during an angular displacement $d\theta$ in the plane of the couple is $dU = M\,d\theta$.

$$\begin{aligned} dU &= M\cos\alpha\,d\theta = 65.95(-0.0760)\,d\theta = -5.0122\,d\theta \\ U &= \int_0^{2.5} dU = \int_0^{2.5} (M\cos\alpha)\,d\theta \\ &= -5.0122\int_0^{2.5} d\theta = -12.53 \text{ N} \cdot \text{m} \end{aligned}$$ Ans.

Alternatively,

The work done by a couple $\mathbf{C}$ during an infinitesimal angular displacement $d\boldsymbol{\theta}$ is given by the dot product $dU = \mathbf{C} \cdot d\boldsymbol{\theta}$.

$$\begin{aligned} U &= \int_0^{2.5} \mathbf{C} \cdot d\boldsymbol{\theta} \\ &= \int_0^{2.5} (25\mathbf{i} + 35\mathbf{j} - 50\mathbf{k}) \cdot (0.667\mathbf{i} + 0.333\mathbf{j} + 0.667\mathbf{k})\,d\theta \\ &= \int_0^{2.5} -5.02\,d\theta = -12.55 \text{ N} \cdot \text{m} \end{aligned}$$ Ans.

The slight difference between this answer and the previous answer is due to the way the numbers were rounded off before being multiplied together.