5.10. CENTER OF GRAVITY OF A THREE-DIMENSIONAL BODY. CENTROID OF A VOLUME

The *center of gravity* G of a three-dimensional body is obtained by dividing the body into small elements and by then expressing that the weight $\mathbf{W}$ of the body acting at G is equivalent to the system of distributed forces $\Delta\mathbf{W}$ representing the weights of the small elements. Choosing the y axis to be vertical with positive sense upward (Fig. 5.20) and denoting by $\bar{\mathbf{r}}$ the position vector of G, we write that

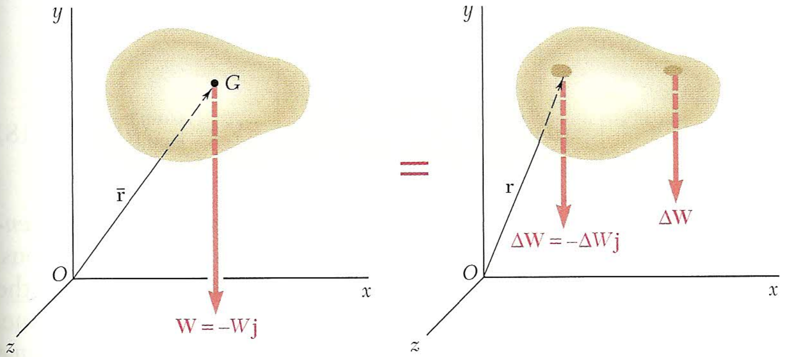

Fig. 5.20

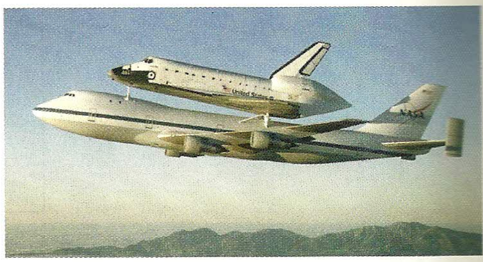

Photo 5.5 To predict the flight characteristics of the modified Boeing 747 when used to transport a space shuttle, the center of gravity of each craft had to be determined.

$\mathbf{W}$ is equal to the sum of the elemental weights $\Delta\mathbf{W}$ and that its moment about O is equal to the sum of the moments about O of the elemental weights:

$$\begin{aligned} &\Sigma\mathbf{F}: &\quad -W\mathbf{j} &= \Sigma(-\Delta W\mathbf{j}) \\ &\Sigma\mathbf{M}_O: &\quad \bar{\mathbf{r}} \times (-W\mathbf{j}) &= \Sigma[\mathbf{r} \times (-\Delta W\mathbf{j})] \end{aligned} \tag{5.13}$$

Rewriting the last equation in the form

$$\bar{\mathbf{r}}W \times (-\mathbf{j}) = (\Sigma\mathbf{r}\,\Delta W) \times (-\mathbf{j}) \tag{5.14}$$

we observe that the weight $\mathbf{W}$ of the body is equivalent to the system of the elemental weights $\Delta\mathbf{W}$ if the following conditions are satisfied:

$$W = \Sigma\,\Delta W \qquad \bar{\mathbf{r}}W = \Sigma\mathbf{r}\,\Delta W$$

Increasing the number of elements and simultaneously decreasing the size of each element, we obtain in the limit

$$W = \int dW \qquad \bar{\mathbf{r}}W = \int \mathbf{r}\,dW \tag{5.15}$$

We note that the relations obtained are independent of the orientation of the body. For example, if the body and the coordinate axes were rotated so that the z axis pointed upward, the unit vector $-\mathbf{j}$ would be replaced by $-\mathbf{k}$ in Eqs. (5.13) and (5.14), but the relations (5.15) would remain unchanged. Resolving the vectors $\bar{\mathbf{r}}$ and $\mathbf{r}$ into rectangular components, we note that the second of the relations (5.15) is equivalent to the three scalar equations

$$\bar{x}W = \int x\,dW \qquad \bar{y}W = \int y\,dW \qquad \bar{z}W = \int z\,dW \tag{5.16}$$

If the body is made of a homogeneous material of specific weight γ, the magnitude dW of the weight of an infinitesimal element can be expressed in terms of the volume dV of the element, and the magnitude W of the total weight can be expressed in terms of the total volume V. We write

$$dW = \gamma\, dV \qquad W = \gamma V$$

Substituting for dW and W in the second of the relations (5.15), we write

$$\bar{\mathbf{r}}V = \int \mathbf{r}\, dV \tag{5.17}$$

or, in scalar form,

$$\bar{x}V = \int x\, dV \qquad \bar{y}V = \int y\, dV \qquad \bar{z}V = \int z\, dV \tag{5.18}$$

The point whose coordinates are $\bar{x}$, $\bar{y}$, and $\bar{z}$ is also known as the *centroid C of the volume V* of the body. If the body is not homogeneous, Eqs. (5.18) cannot be used to determine the center of gravity of the body; however, Eqs. (5.18) still define the centroid of the volume.

The integral $\int x\, dV$ is known as the *first moment of the volume with respect to the yz plane.* Similarly, the integrals $\int y\, dV$ and $\int z\, dV$ define the first moments of the volume with respect to the zx plane and the xy plane, respectively. It is seen from Eqs. (5.18) that if the centroid of a volume is located in a coordinate plane, the first moment of the volume with respect to that plane is zero.

A volume is said to be symmetrical with respect to a given plane if for every point P of the volume there exists a point P' of the same volume, such that the line PP' is perpendicular to the given plane and is bisected by that plane. The plane is said to be a *plane of symmetry* for the given volume. When a volume V possesses a plane of symmetry, the first moment of V with respect to that plane is zero, and the centroid of the volume is located in the plane of symmetry. When a volume possesses two planes of symmetry, the centroid of the volume is located on the line of intersection of the two planes. Finally, when a volume possesses three planes of symmetry which intersect at a well-defined point (that is, not along a common line), the point of intersection of the three planes coincides with the centroid of the volume. This property enables us to determine immediately the locations of the centroids of spheres, ellipsoids, cubes, rectangular parallelepipeds, etc.

The centroids of unsymmetrical volumes or of volumes possessing only one or two planes of symmetry should be determined by integration (Sec. 5.12). The centroids of several common volumes are shown in Fig. 5.21. It should be observed that in general the centroid of a volume of revolution *does not coincide* with the centroid of its cross section. Thus, the centroid of a hemisphere is different from that of a semicircular area, and the centroid of a cone is different from that of a triangle.