8-6.2 Cables with Loads Uniformly Distributed Along the Horizontal

A cable carrying a load **W** distributed uniformly over a distance a is illustrated in Fig. 8-23a. Such a cable could be analyzed by using the procedures discussed in the previous section; however, the process would be time-consuming and tedious due to the large number of loads involved. Such a cable can also be analyzed by representing the large number of discrete loads as a uniformly distributed load $w(x) = W/a$ along the horizontal. Little error is introduced by this simplification if the weight of the cable is small in comparison to the weight being supported by the cable. The loads on the cables of a suspension bridge closely approximate this type of loading since the weights of the cables are usually small in comparison to the weight of the roadway. The weight of the roadway is uniformly distributed along the length of the roadway.

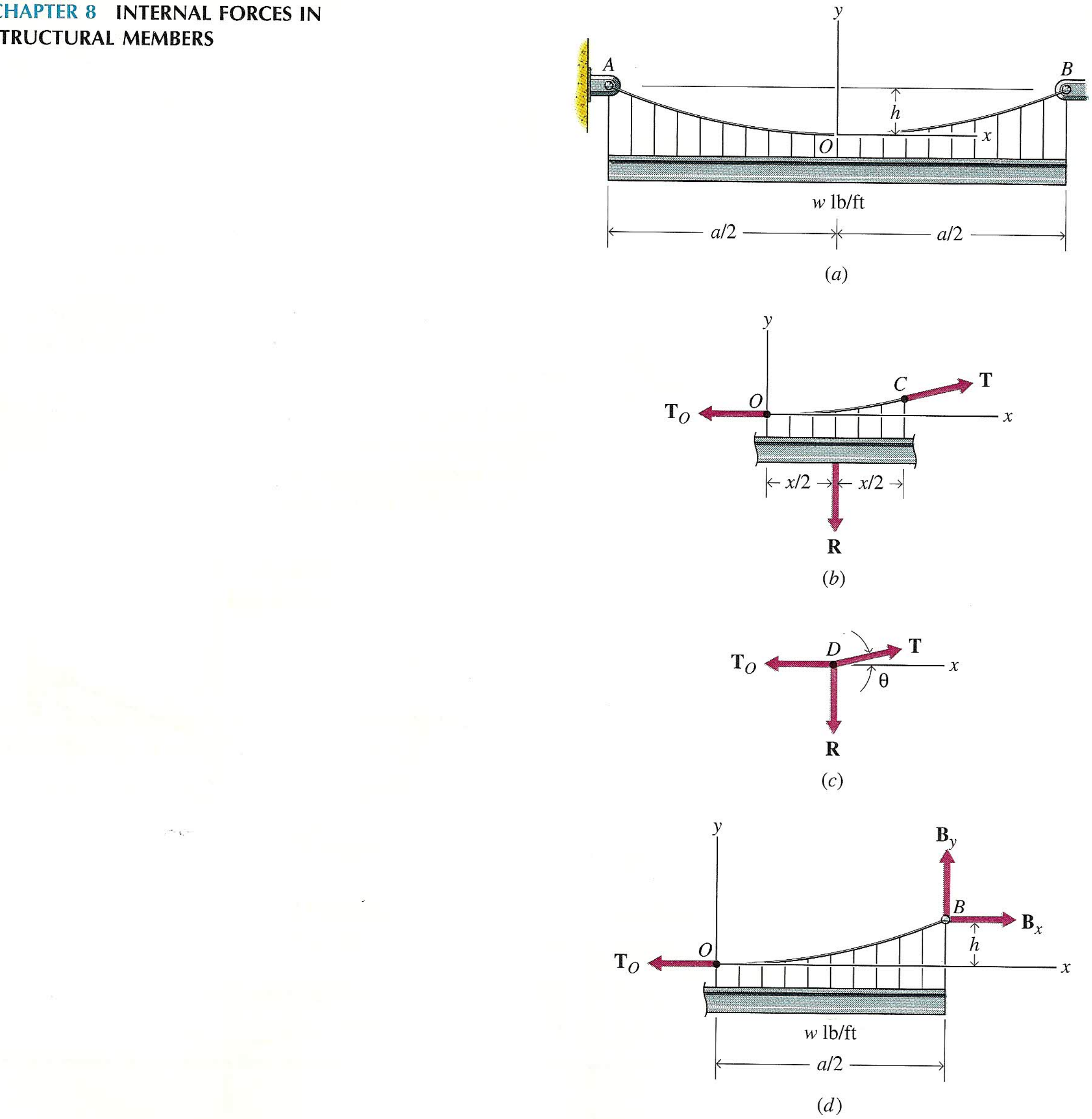

Figure 8-23 Flexible cable supporting a uniformly distributed load.

Equations that express relationships between the length L, span a and sag h of a cable with supports at the same height, the tension T in the cable, and the distributed load $w(x)$ applied to the cable can be developed by using equilibrium considerations and the free-body diagram of a portion of the cable shown in Fig. 8-23*b*. In this diagram, the lowest point on the cable (point O) has been taken as the origin of the xy-coordinate system.

The segment of cable shown in Fig. 8-23*b* is subjected to three forces; the tension $\mathbf{T}_O$ in the cable at point O, the tension $\mathbf{T}$ in the cable at an arbitrary point C, located a distance x from the origin, and the resultant $\mathbf{R}$ ($R = wx$) of the distributed load whose line of action is located a distance $x/2$ from point O. Since segment OC of the cable is in equilibrium under the action of these three forces, the three forces must be concurrent at point D, as shown in Fig. 8-23*c*. The two equi-

[equilib]rium equations $\Sigma F_x = 0$ and $\Sigma F_y = 0$ for this concurrent force system [y]ield

$$+\rightarrow \Sigma F_x = T\cos\theta - T_O = 0 \qquad T_x = T\cos\theta = T_O \tag{a}$$
$$+\uparrow \Sigma F_y = T\sin\theta - R = 0 \qquad T_y = T\sin\theta = R = wx \tag{b}$$

[E]quation a shows that the horizontal component T_x of the tension T at [an]y point in the cable is constant and equal to the tension T_O at the [lo]west point on the cable. Solving Eqs. a and b for T and θ yields

$$T = \sqrt{T_O^2 + w^2x^2} \tag{8-5}$$

$$\theta = \tan^{-1}\frac{wx}{T_O} \tag{8-6}$$

[E]quation 8-5 shows that the tension is minimum at the lowest point in [th]e cable (where $x = 0$) and maximum at the supports (where $x = a/2$). [T]hus,

$$T_{\text{max}} = \sqrt{T_O^2 + \frac{w^2a^2}{4}} \tag{8-7}$$

The shape of the curve can be determined by using Eq. 8-6, which [g]ives the slope. Thus,

$$\frac{dy}{dx} = \tan\theta = \frac{wx}{T_O} \tag{c}$$

[I]ntegrating Eq. c yields

$$y = \frac{wx^2}{2T_O} + C$$

[T]he constant C is determined by using the boundary conditions re[s]ulting from the choice of axes. For the xy-coordinate system being [u]sed, y is equal to zero when x is equal to zero; therefore, C is equal [t]o zero, and the equation of the loaded cable is

$$y = \frac{wx^2}{2T_O} = kx^2 \tag{8-8}$$

[E]quation 8-8 indicates that the shape of the loaded cable is a parabola [w]ith its vertex at the lowest point of the cable. By applying the bound[a]ry condition $y = h$ (the sag) where $x = a/2$, an expression for the ten[s]ion T_O is obtained in terms of the applied load w, the span a, and the [s]ag h. Thus

$$T_O = \frac{wa^2}{8h} \tag{8-9}$$

[I]f Eq. 8-9 is substituted into Eq. 8-7, the maximum tension in the cable T_{max} is obtained in terms of the applied load w, the span a, and the sag [h]. Thus,

$$\begin{aligned} T_{max} &= \sqrt{\frac{w^2a^4}{64h^2} + \frac{w^2a^2}{4}} \\ &= \frac{wa}{8h}\sqrt{a^2 + (4h)^2} \end{aligned} \tag{8-10}$$

Equations 8-9 and 8-10 can also be determined by using the free-body diagram of the right half of the cable shown in Fig. 8-23*d*. From the equilibrium equation $\Sigma M_B = 0$,

$$+\circlearrowleft M_B = \frac{wa}{2}\left(\frac{a}{4}\right) - T_O(h) = 0 \qquad T_O = \frac{wa^2}{8h} \tag{8-9}$$

From the equilibrium equation $\Sigma F_x = 0$,

$$+\rightarrow \Sigma F_x = B_x - T_O = 0 \qquad B_x = T_O = \frac{wa^2}{8h}$$

From the equilibrium equation $\Sigma F_y = 0$,

$$+\uparrow \Sigma F_y = B_y - \frac{wa}{2} = 0 \qquad B_y = \frac{wa}{2}$$

Thus,

$$T_{\text{max}} = B = \sqrt{B_x^2 + B_y^2} = \sqrt{\left(\frac{wa^2}{8h}\right)^2 + \left(\frac{wa}{2}\right)^2}$$

$$= \frac{wa}{8h}\sqrt{a^2 + (4h)^2} \tag{8-10}$$

For any curve, the length dL of an arc of the curve can be obtained from the equation

$$dL = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{d}$$

Thus, for the cable, from Eqs. 8-8 and 8-9,

$$y = kx^2 = \left(\frac{w}{2T_O}\right)x^2 = \left(\frac{4h}{a^2}\right)x^2$$

$$\frac{dy}{dx} = 2kx = \left(\frac{8h}{a^2}\right)x$$

$$L = \int_{\text{span}} dL = 2\int_0^{a/2} \sqrt{1 + 4k^2x^2}\, dx$$

$$= \left[x\sqrt{1 + 4k^2\, x^2} + \frac{1}{2k}\ln\left(2kx + \sqrt{1 + 4k^2x^2}\right)\right]_0^{a/2}$$

$$= \frac{a}{2}\sqrt{1 + k^2a^2} + \frac{1}{2k}\ln\left(ka + \sqrt{1 + k^2a^2}\right)$$

Finally, substituting $k = 4h/a^2$ yields

$$L = \frac{1}{2}\sqrt{a^2 + 16h^2} + \frac{a^2}{8h}\ln\frac{1}{a}\left(4h + \sqrt{a^2 + 16h^2}\right) \tag{8-11}$$

When the supports have different elevations, as shown in Fig. 8-24, the location of the lowest point on the cable (the origin of the *xy*-coordinate system) is not known and must be determined. The equation of the loaded cable (Eq. 8-8) remains valid since the free-body diagram (Fig. 8-23*b*) used in its development would be identical for a cable with supports at different elevations as shown in Fig. 8-24. If the sags h_A and h_B of the cable with respect to supports A and B are known, Eq. 8-8 can be used to locate the origin of the coordinate system. Thus, from Eq. 8-8,

$$y = kx^2$$
$$k = \frac{y}{x^2} = \frac{h_B}{x_B^2} = \frac{h_A}{x_A^2} \qquad (e)$$

Since $x_A = x_B - a$, Eq. e can be written

$$k = \frac{y}{x^2} = \frac{h_B}{x_B^2} = \frac{h_A}{(x_B - a)^2}$$

which leads to the quadratic equation

$$(h_B - h_A)x_B^2 - 2ah_Bx_B + h_Ba^2 = 0$$

Solving for x_B yields

$$x_B = \frac{h_B - \sqrt{h_Bh_A}}{h_B - h_A}a \qquad (8\text{-}12)$$

Also, since $x_A = x_B - a$,

$$x_A = \frac{h_A - \sqrt{h_Bh_A}}{h_B - h_A}a \qquad (8\text{-}13)$$

The horizontal distance from support A to the origin O of the coordinate system is shown as distance d in Fig. 8-24. At support A, $x_A = -d$. Thus,

$$d = \frac{\sqrt{h_Bh_A} - h_A}{h_B - h_A}a \qquad (8\text{-}14)$$

The tension T_O in the cable can be found by using Eq. 8-8. Thus,

$$T_O = \frac{wx^2}{2y} = \frac{wx_A^2}{2h_A} = \frac{wx_B^2}{2h_B} \qquad (8\text{-}15)$$

Equation 8-5 shows that the tension in the cable varies from a minimum value T_O at the lowest point in the cable to a maximum value at the highest (most distant) support. By substituting the magnitudes of x_A and x_B in Eq. 8-5 and using Eq. 8-15, we find that the tensions in the cable at the supports are

$$T_A = wd_A\left[1 + \frac{d_A^2}{4h_A^2}\right]^{1/2} \qquad (8\text{-}16a)$$
$$T_B = wd_B\left[1 + \frac{d_B^2}{4h_B^2}\right]^{1/2} \qquad (8\text{-}16b)$$

where $d_A = |x_A|$ and $d_B = |x_B|$.

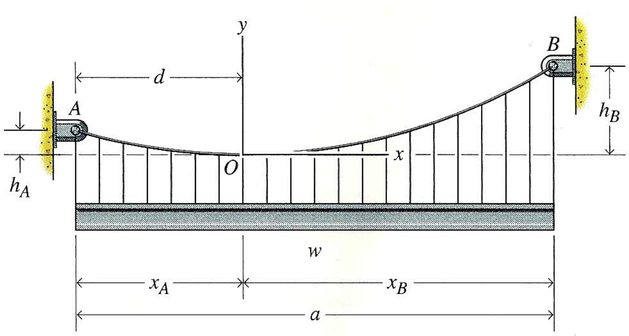

Figure 8-24 Flexible cable with supports at different elevations.

The length of the curve between the lowest point O and a supp
can be determined by using Eq. d. Thus, for support B,

$$
\begin{aligned}
L_{OB} &= \int_0^{d_B} \sqrt{1 + 4k^2x^2}\, dx \\
&= \left[\frac{x}{2}\sqrt{1 + 4k^2x^2} + \frac{1}{4k} \ln \left(2kx + \sqrt{1 + 4k^2x^2} \right) \right]_0^{d_B} \\
&= \frac{d_B}{2} \sqrt{1 + 4k^2 d_B^2} + \frac{1}{4k} \ln \left(2kd_B + \sqrt{1 + 4k^2 d_B^2} \right)
\end{aligned}
$$

Finally, substituting $k = h_B/d_B^2$ yields

$$
L_{OB} = \frac{1}{2}\sqrt{d_B^2 + 4h_B^2} + \frac{d_B^2}{4h_B} \ln \frac{1}{d_B}\left(2h_B + \sqrt{d_B^2 + 4h_B^2}\right)
$$

Similarly,

$$
L_{OA} = \frac{1}{2}\sqrt{d_A^2 + 4h_A^2} + \frac{d_A^2}{4h_A} \ln \frac{1}{d_A}\left(2h_A + \sqrt{d_A^2 + 4h_A^2}\right)
$$

The total length L of the cable is the sum of the arc lengths L_{OA} and
L_{OB} measured from the origin to each support.

EXAMPLE PROBLEM 8-12

A cable supports a uniformly distributed load of 300 lb/ft along the horizontal as shown in Fig. 8-25*a*. Determine

a. The minimum tension in the cable.
b. The tension in the cable at the supports and the angle it makes with the horizontal.
c. The length L of the cable.

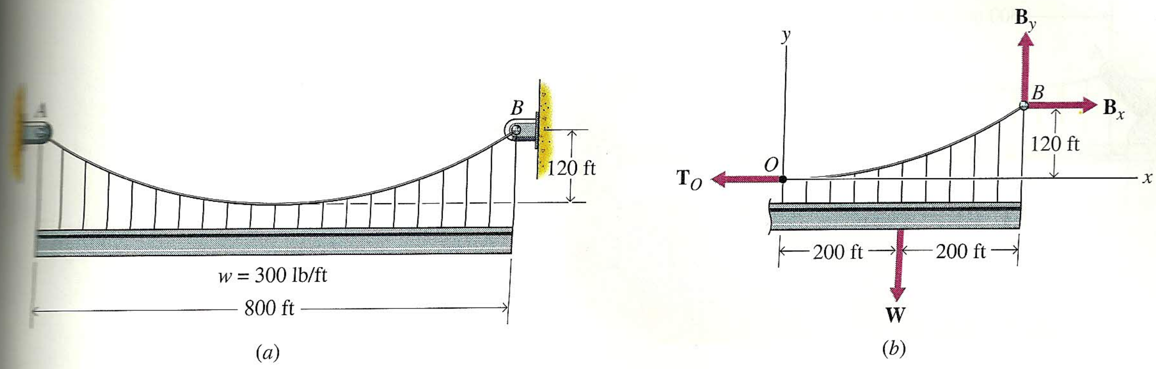

Fig. 8-25

SOLUTION

Parts a and b of the problem will be solved by using the free-body diagram shown in Fig. 8-25b. Thus,

$$W = w(400) = 300(400) = 120{,}000 \text{ lb} = 120 \text{ kip}$$

a. From the equilibrium equation $\Sigma M_B = 0$,

$$+\circlearrowleft M_B = W(200) - T_O(120) = 120(200) - T_O(120) = 0$$
$$T_O = 200 \text{ kip} \qquad \text{Ans.}$$

> Since the supports are at the same elevation, the low point in the cable occurs at the middle of the span.

b. From the equilibrium equation $\Sigma F_x = 0$,

$$+\rightarrow \Sigma F_x = B_x - T_O = B_x - 200 = 0 \quad B_x = 200 \text{ kip}$$

From the equilibrium equation $\Sigma F_y = 0$,

$$+\uparrow \Sigma F_y = B_y - W = B_y - 120 = 0 \quad B_y = 120 \text{ kip}$$
$$T_A = T_B = T_{max} = \sqrt{B_x^2 + B_y^2}$$
$$= \sqrt{(200)^2 + (120)^2} = 233 \text{ kip} \qquad \text{Ans.}$$
$$\theta_x = \tan^{-1}\frac{B_y}{B_x} = \tan^{-1}\frac{120}{200} = 30.96 \cong 31.0° \qquad \text{Ans.}$$

c. The length of the cable is determined by using Eq. 8-11. Thus,

$$L = \frac{1}{2}\sqrt{a^2 + 16h^2} + \frac{a^2}{8h}\ln\frac{1}{a}\left(4h + \sqrt{a^2 + 16h^2}\right)$$
$$= \frac{1}{2}\sqrt{(800)^2 + 16(120)^2}$$
$$+ \frac{(800)^2}{8(120)}\ln\frac{1}{800}\left[4(120) + \sqrt{(800)^2 + 16(120)^2}\right]$$
$$= 845.7 \cong 846 \text{ ft} \qquad \text{Ans.}$$

EXAMPLE PROBLEM 8-13

The cable shown in Fig. 8-26*a* supports a pipeline as it passes over a river. The support at the middle of the river is 25 m above the supports on the two sides. The lowest points on the cable are 45 m below the middle support. Determine

a. The minimum tension in the cable.
b. The maximum tension in the cable.
c. The length L of the cable.

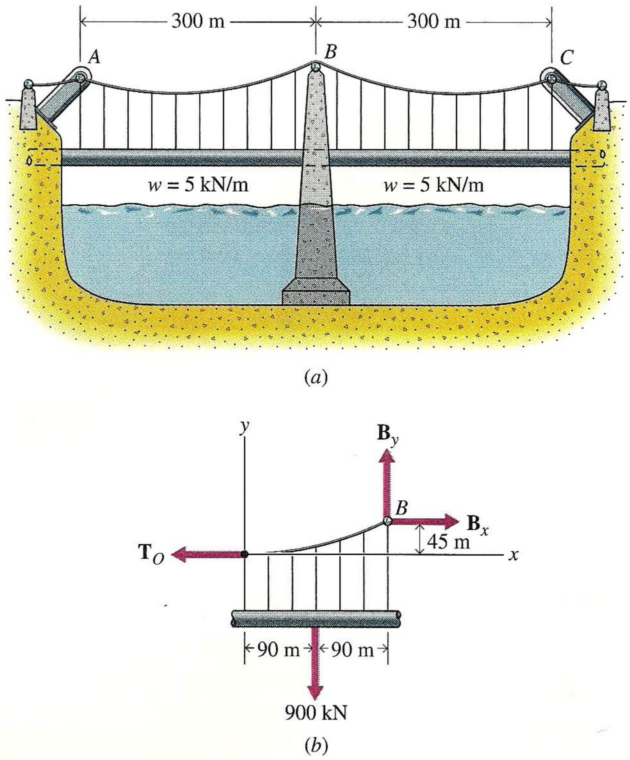

Fig. 8-26

SOLUTION

The lowest point in the cable (origin of the *xy*-coordinate system) can be located by using Eq. 8-14. Thus,

$$d = \frac{\sqrt{h_B h_A} - h_A}{h_B - h_A} a = \frac{\sqrt{45(20)} - 20}{45 - 20}(300) = 120.0 \text{ m}$$

If the ends of the cable are not at the same level, it is not possible to use symmetry to locate the lowest point on the cable.

Once the distance *d* is known, the free-body diagram shown in Fig. 8-26*b* can be used to solve parts *a* and *b* of the problem. The distance from the origin of coordinates to support *B* is

$$x_B = a - d = 300 - 120 = 180 \text{ m}$$
$$W = w(x_B) = 5(180) = 900 \text{ kN}$$

a. From the equilibrium equation $\Sigma M_B = 0$,

$$+\circlearrowleft M_B = W\left(\frac{x_B}{2}\right) - T_O(h_B) = 900(90) - T_O(45) = 0$$

$$T_O = T_{\min} = 1800 \text{ kN} \qquad \text{Ans.}$$