8-6.3 Cables with Loads Uniformly Distributed Along Their Length

In the previous two sections, the weight of the cable was small in comparison to the loads being supported by the cable. For power transmission lines, telephone lines, and guy wires on radio and television towers, the weight of the cable is the only significant load being applied. In these applications, the weight is uniformly distributed along the length of the cable. When the supports are at the same elevation and the sag ratio is small (a taut cable), the curve assumed by the cable may be regarded with small error as being a parabola, since a uniformly distributed load along the cable does not differ significantly from the same load uniformly distributed along the horizontal. When the sag ratio is large ($h/a > 0.10$), the parabolic formulas of the previous section should not be used.

A uniform cable having a weight per unit length w over its length is illustrated in Fig. 8-27*a*. Equations that express relationships between the length L, span a, and sag h of the cable, the tension T in the cable, and the weight w of the cable can be developed by using equilibrium considerations and the free-body diagram of a portion of the cable shown in Fig. 8-27*b*. In this diagram, the lowest point on the cable (point O) has been taken as the origin of the xy-coordinate system.

The segment of cable shown in Fig. 8-27*b* is subjected to three forces; the tension $\mathbf{T}_O$ in the cable at point O, the tension $\mathbf{T}$ in the cable at an arbitrary point C, which is located a distance s from the origin, and the resultant $\mathbf{R}$ ($R = ws$) of the distributed load, whose line of action is located at an unknown distance x from point O. The weight acts at the centroid of the curve formed by the cable. The angle that $\mathbf{T}$ makes with the horizontal is denoted by θ. Since segment OC of the cable is in equilibrium under the action of these three forces, the three forces must be concurrent at point D as shown in Fig. 8-27*c*. The two equilibrium equations $\Sigma F_x = 0$ and $\Sigma F_y = 0$ for this concurrent force system yield

$$+\rightarrow \Sigma F_x = T\cos\theta - T_O = 0 \qquad T_x = T\cos\theta = T_O \qquad (a)$$

$$+\uparrow \Sigma F_y = T\sin\theta - R = 0 \qquad T_y = T\sin\theta = R = ws \qquad (b)$$

Equation *a* shows that the horizontal component T_x of the tension T at any point in the cable is constant and equal to the tension T_O at the

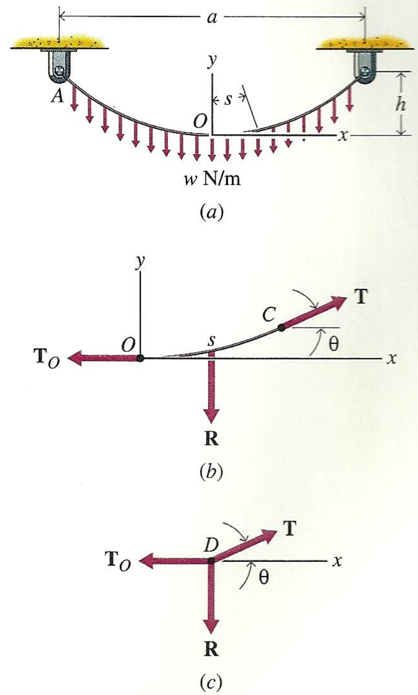

Figure 8-27 Flexible cable supporting a load uniformly distributed along its length.

lowest point on the cable. Solving Eqs. a and b for T and θ yields

$$T = [T_O^2 + w^2 s^2]^{1/2} \tag{8-19}$$

$$\theta = \tan^{-1} \frac{ws}{T_O} \tag{8-20}$$

Equation 8-19 shows that the tension is minimum at the lowest point in the cable (where $s = 0$) and maximum at the supports (where s is greatest). The distance s remains to be determined.

The shape of the curve can be determined by using Eq. 8-20, which gives the slope. Thus,

$$\frac{dy}{dx} = \tan\theta = \frac{ws}{T_O} \tag{c}$$

Recall the relationship

$$(ds)^2 = (dx)^2 + (dy)^2$$

from which

$$\frac{dy}{dx} = \left[\left(\frac{ds}{dx}\right)^2 - 1\right]^{1/2} \tag{d}$$

Substituting Eq. d into Eq. c yields

$$\left[\left(\frac{ds}{dx}\right)^2 - 1\right]^{1/2} = \frac{ws}{T_O}$$

from which

$$\frac{ds}{dx} = \left[1 + \left(\frac{ws}{T_O}\right)^2\right]^{1/2}$$

Solving for dx yields

$$dx = \frac{(T_O/w)dx}{[(T_O/w)^2 + s^2]^{1/2}} \tag{e}$$

Integrating Eq. e yields

$$x = \left(\frac{T_O}{w}\right) \ln \left\{s + \left[\left(\frac{T_O}{w}\right)^2 + s^2\right]^{1/2}\right\} + C$$

The integration constant C can be determined by substituting the boundary condition $s = 0$ when $x = 0$. Thus,

$$C = -\left(\frac{T_O}{w}\right) \ln \left(\frac{T_O}{w}\right)$$

therefore,

$$x = \left(\frac{T_O}{w}\right) \ln \frac{s + [(T_O/w)^2 + s^2]^{1/2}}{(T_O/w)} \tag{8-21}$$

Equation 8-21 can also be written in exponential form as

$$\left(\frac{T_O}{w}\right) e^{wx/T_O} = s + \left[\left(\frac{T_O}{w}\right)^2 + s^2\right]^{1/2}$$

Solving for s yields

$$s = \left(\frac{T_O}{w}\right)\left[\frac{e^{wx/T_O} - e^{-wx/T_O}}{2}\right] = \left(\frac{T_O}{w}\right)\sinh\left(\frac{wx}{T_O}\right) \tag{8-22}$$

Once the distance s is known in terms of x, Eq. 8-19 yields

$$T = T_O[1 + \sinh^2(wx/T_O)]^{1/2} = T_O \cosh(wx/T_O) \tag{8-23}$$

Similarly, Eq. 8-22 can be substituted into Eq. c to obtain

$$\frac{dy}{dx} = \sinh\left(\frac{wx}{T_O}\right)$$

which can be integrated to give

$$y = \left(\frac{T_O}{w}\right)\cosh\left(\frac{wx}{T_O}\right) + C$$

Substituting the boundary condition $y = 0$ at $x = 0$ yields

$$C = -\left(\frac{T_O}{w}\right)$$

Thus,

$$y = \left(\frac{T_O}{w}\right)\left[\cosh\left(\frac{wx}{T_O}\right) - 1\right] \tag{8-24}$$

Equation 8-24 is the Cartesian equation of a catenary.

The tension T at an arbitrary point along the cable can be expressed in terms of T_O and the y-coordinate of the point by solving Eq. 8-24 for cosh (wx/T_O) and substituting the result into Eq. 8-23. The final result is

$$T = T_O + wy \tag{8-25}$$

Equation 8-24 can also be used to determine T_O if both coordinates of a point on the cable are known. For example, the coordinates at a support for a cable with span a and sag h and supports at the same elevation are $x = a/2$ and $y = h$. Thus,

$$h = \left(\frac{T_O}{w}\right)\left[\cosh\left(\frac{wa}{2T_O}\right) - 1\right] \tag{8-26}$$

Unfortunately, Eq. 8-26 cannot be solved directly for T_O; therefore, values for T_O must be obtained by using trial-and-error solutions, iterative procedures, numerical solutions, or graphs and tables.

EXAMPLE PROBLEM 8-14

A cable with supports at the same elevation has a span of 800 ft. The cable weighs 5 lb/ft and the sag at midspan is 200 ft. Determine

a. The tension in the cable at midspan.
b. The tension in the cable at a support.
c. The length of the cable.

SOLUTION

a. The tension in the cable at midspan can be determined by using Eq. 8-26. Thus,

$$h = \left(\frac{T_O}{w}\right)\left[\cosh\left(\frac{wa}{2T_O}\right) - 1\right]$$

$$200 = \frac{T_O}{5}\left[\cosh\frac{5(800)}{2T_O} - 1\right]$$

A trial-and-error solution for the above equation is shown in the following table. Equation 8-9 for a parabolic cable was used to obtain a first estimate. Thus $T_O = wa^2/8h = 5(800)^2/8(200) = 2000$ lb.

The equation for T_O cannot be solved directly; therefore, values for T_O must be obtained by using trial-and-error methods, iterative procedures, or numerical solutions.

T_O	$\frac{T_O}{5}$	$\cosh\frac{2000}{T_O}$	$\frac{T_O}{5}\left[\cosh\frac{2000}{T_O} - 1\right]$	h	% Error
2000	400	1.54308	217.23	200	+8.62
2100	420	1.48885	205.32	200	+2.66
2200	440	1.44248	194.69	200	−2.66
2150	430	1.46478	199.86	200	−0.07

Thus,

$$T_O = 2150 \text{ lb} \qquad \text{Ans.}$$

The tension T_O in the cable at midspan can also be determined by using the Newton-Raphson Method. Equation 8-26 is first written in the form

$$f(T_O) = T_O\left[\cosh\left(\frac{2000}{T_O}\right) - 1\right] - 1000 = 0$$

Using the Newton-Raphson Method

$$(T_O)_{n+1} = (T_O)_n + \delta_n$$

$$\delta_n = -\frac{f(T_O)_n}{f'(T_O)_n}$$

$$f'(T_O) = \left[\cosh\left(\frac{2000}{T_O}\right) - 1\right] + T_O\left(-\frac{2000}{T_O^2}\right)\sinh\left(\frac{2000}{T_O}\right)$$

Using the initial value $(T_O)_0 = 2000$

n	$(T_O)_n$	$f(T_O)_n$	$f'(T_O)_n$	δ_n
0	2000	86.1614	−0.6321	136.3053
1	2136.31	6.6034	−0.5390	12.2505
2	2148.56	0.0440	−0.5317	0.0845
3	2148.64	0.0000		

Thus,

$$T_O = 2149 \text{ lb} \qquad \text{Ans.}$$

Once the tension at midspan is known, Eq. 8-25 can be used to determine the tension at a support. Thus,

$$T_B = T_{max} = T_O + wh = 2150 + 5(200) = 3150 \text{ lb} \qquad \text{Ans.}$$

The distance s along the cable from the low point to a support can be determined by using Eq. 8-22. Thus,

$$s = \left(\frac{T_O}{w}\right) \sinh\left(\frac{wx}{T_O}\right)$$

$$s_B = \frac{2150}{5} \sinh \frac{5(400)}{2150} = 460.2 \text{ ft}$$

Therefore

$$L = 2s_B = 2(460.2) = 920.4 \cong 920 \text{ ft} \qquad \text{Ans.}$$

The equivalent results obtained by using the parabolic equations are

$$T_B = \frac{1}{2}wa\left[1 + \left(\frac{a}{4h}\right)^2\right]^{1/2}$$

$$= \frac{1}{2}(5)(800)\left[1 + \left(\frac{800}{4(200)}\right)^2\right]^{1/2} = 2830 \text{ lb}$$

The length of the cable is determined by using Eq. 8-11. Thus,

$$L = \frac{1}{2}\sqrt{a^2 + 16h^2} + \frac{a^2}{8h} \ln \frac{1}{a}\left(4h + \sqrt{a^2 + 16h^2}\right)$$

$$= \frac{1}{2}\sqrt{(800)^2 + 16(200)^2}$$

$$+ \frac{(800)^2}{8(200)} \ln \frac{1}{800}\left[4(200) + \sqrt{(800)^2 + 16(200)^2}\right]$$

$$= 918.2 \cong 918 \text{ ft} \qquad \text{Ans.}$$

The results obtained by using the equations for parabolic and catenary cables are summarized in the following table.

Quantity	Catenary	Parabolic
T_O	2150	2000
T_{max}	3150	2830
L	920	918