11-2.3 Virtual Work

When a body being acted on by a force **F** moves through an infinitesimal linear displacement $d\mathbf{r}$ as described in Section 11-2.1, the body is not in equilibrium. In studying the equilibrium of bodies by the method of virtual work, it is necessary to introduce fictitious displacements, called virtual displacements. An infinitesimal virtual linear displacement will be represented by the first-order differential $\delta\mathbf{s}$ rather than $d\mathbf{s}$. The work done by a force **F** acting on a body during a virtual displacement $\delta\mathbf{s}$ is called virtual work δU and is represented mathematically as

$$\delta U = \mathbf{F} \cdot \delta\mathbf{s} \qquad \text{or} \qquad \delta U = F\, \delta s \cos \alpha \tag{11-9}$$

where F and δs are the magnitudes of the force $\mathbf{F}$ and virtual displacement $\delta\mathbf{s}$, respectively, and α is the angle between $\mathbf{F}$ and $\delta\mathbf{s}$.

A virtual displacement may also be a rotation of the body. The virtual work done by a couple $\mathbf{C}$ during an infinitesimal virtual angular displacement $\delta\boldsymbol{\theta}$ of the body is

$$\delta U = \mathbf{C} \cdot \delta\boldsymbol{\theta} \qquad \text{or} \qquad \delta U = M\, \delta\theta \cos \alpha \tag{11-10}$$

where M and $\delta\theta$ are the magnitudes of the couple $\mathbf{C}$ and virtual displacement $\delta\boldsymbol{\theta}$, respectively, and α is the angle between $\mathbf{C}$ and $\delta\boldsymbol{\theta}$. Since the infinitesimal virtual displacements δs and $\delta\theta$ in Eqs. 11-9 and 11-10 refer to fictitious movements, the equations cannot be integrated.

11-3 PRINCIPLE OF VIRTUAL WORK AND EQUILIBRIUM

The principle of virtual work can be stated as follows:

> If the virtual work done by all external forces (or couples) acting on a particle, a rigid body, or a system of connected rigid bodies with ideal (frictionless) connections and supports is zero for all virtual displacements of the system, the system is in equilibrium.

The principle of virtual work can be expressed mathematically as

$$\delta U = \sum_{i=1}^{m} \mathbf{F}_i \cdot \delta \mathbf{s}_i + \sum_{j=1}^{n} \mathbf{C}_j \cdot \delta \boldsymbol{\theta}_j = 0 \qquad (11\text{-}11)$$

11-3.1 Equilibrium of a Particle

Consider the particle shown in Fig. 11-6, which is acted on by several forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$. The work done on the particle by these forces during an arbitrary virtual displacement $\delta\mathbf{s}$ is

$$\begin{aligned}\delta U &= \mathbf{F}_1 \cdot \delta\mathbf{s} + \mathbf{F}_2 \cdot \delta\mathbf{s} + \cdots + \mathbf{F}_n \cdot \delta\mathbf{s} \\ &= (\mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n) \cdot \delta\mathbf{s} = \Sigma\mathbf{F} \cdot \delta\mathbf{s} = \mathbf{R} \cdot \delta\mathbf{s} \qquad (11\text{-}12)\end{aligned}$$

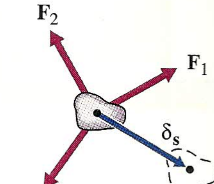


Figure 11-6 Virtual displacement of a particle acted on by several forces.

where $\mathbf{R}$ is the resultant of the forces acting on the particle.

Expressing the resultant $\mathbf{R}$ and the virtual displacement $\delta\mathbf{s}$ in Cartesian vector form and computing the vector scalar product yields

$$\begin{aligned}\delta U = \mathbf{R} \cdot \delta\mathbf{s} &= (\Sigma F_x\, \mathbf{i} + \Sigma F_y\, \mathbf{j} + \Sigma F_z\, \mathbf{k}) \cdot (\delta x\, \mathbf{i} + \delta y\, \mathbf{j} + \delta z\, \mathbf{k}) \\ &= \Sigma F_x\, \delta x + \Sigma F_y\, \delta y + \Sigma F_z\, \delta z \qquad (11\text{-}13)\end{aligned}$$

Applying the principle of virtual work by combining Eqs. 11-11 and 11-13 yields

$$\delta U = \mathbf{R} \cdot \delta\mathbf{s} = \Sigma F_x\, \delta x + \Sigma F_y\, \delta y + \Sigma F_z\, \delta z = 0 \qquad (11\text{-}14)$$

By considering virtual displacements (δx, δy, and δz) taken one at a time in each of the three mutually perpendicular coordinate directions,

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma F_z = 0$$

Thus, the virtual work equation $\delta U = 0$ is simply an alternative statement of the equilibrium equations for a particle. The principle of virtual work does not simplify the solution of problems involving equilibrium of a particle since the equations $\delta U = 0$ and $\Sigma F = 0$ are equivalent.

11-3.2 Equilibrium of a Rigid Body

If a rigid body is in equilibrium, all particles forming the body must be in equilibrium. Therefore, according to the principle of virtual work, the total virtual work of all forces, both internal and external, acting on all of the particles must be zero. Since the internal forces between particles occur as equal, opposite, collinear pairs, the work done by each pair of forces sum to zero during any virtual displacement of the rigid body. Thus, only the external forces do work during any virtual displacement of the body. Any system of forces acting on a rigid body can be replaced by a resultant force $\mathbf{R}$ and a resultant couple $\mathbf{C}$. Therefore, the work done on a rigid body by the external forces during an arbitrary linear virtual displacement $\delta\mathbf{s}$ and an arbitrary angular virtual displacement $\delta\boldsymbol{\theta}$ is

$$\delta U = \mathbf{R} \cdot \delta\mathbf{s} + \mathbf{C} \cdot \delta\boldsymbol{\theta} \tag{11-15}$$

Expressing the resultant force **R**, the resultant couple **C**, and virtual displacements $\delta\mathbf{r}$ and $\delta\boldsymbol{\theta}$ in Cartesian vector form and computing the vector scalar products yields

$$\begin{aligned}
\delta U &= \mathbf{R} \cdot \delta\mathbf{s} + \mathbf{C} \cdot \delta\boldsymbol{\theta} \\
&= (\Sigma F_x\,\mathbf{i} + \Sigma F_y\,\mathbf{j} + \Sigma F_z\,\mathbf{k}) \cdot (\delta x\,\mathbf{i} + \delta y\,\mathbf{j} + \delta z\,\mathbf{k}) \\
&\quad + (\Sigma M_x\,\mathbf{i} + \Sigma M_y\,\mathbf{j} + \Sigma M_z\,\mathbf{k}) \cdot (\delta\theta_x\,\mathbf{i} + \delta\theta_y\,\mathbf{j} + \delta\theta_z\,\mathbf{k}) \\
&= \Sigma F_x\,\delta x + \Sigma F_y\,\delta y + \Sigma F_z\,\delta z + \Sigma M_x\,\delta\theta_x + \Sigma M_y\,\delta\theta_y + \Sigma M_z\,\delta\theta_z
\end{aligned} \tag{11-16}$$

Applying the principle of virtual work by combining Eqs. 11-11 and 11-16 yields

$$\delta U = \Sigma F_x\,\delta x + \Sigma F_y\,\delta y + \Sigma F_z\,\delta z + \Sigma M_x\,\delta\theta_x + \Sigma M_y\,\delta\theta_y + \Sigma M_z\,\delta\theta_z = 0 \tag{11-17}$$

By considering virtual linear displacements (δx, δy, and δz) and virtual angular displacements ($\delta\theta_x$, $\delta\theta_y$, and $\delta\theta_z$) taken one at a time,

$$\begin{aligned}
\Sigma F_x &= 0 \qquad & \Sigma F_y &= 0 \qquad & \Sigma F_z &= 0 \\
\Sigma M_x &= 0 \qquad & \Sigma M_y &= 0 \qquad & \Sigma M_z &= 0
\end{aligned}$$

Therefore, the virtual work equation $\delta U = 0$ is simply an alternative statement of the equilibrium equations for a rigid body. The principle of virtual work does not simplify the solution of problems involving equilibrium of a single rigid body since the equation $\delta U = 0$ is equivalent to the equilibrium equations $\Sigma F = 0$ and $\Sigma M = 0$.

11-3.3 Equilibrium of an Ideal System of Connected Rigid Bodies

The principle of virtual work can also be used to study systems of connected rigid bodies. Frequently it is possible to solve such problems by using the complete system rather than individual free-body diagrams of each member of the system.

When the system remains connected during the virtual displacement, only the work of forces external to the system need be considered, since the net work done by the internal forces at connections between members during any virtual displacement is zero because the forces exist as equal, opposite, collinear pairs. Such a condition exists when the connection is a smooth pin, a smooth roller, or an inextensible link or cable. When the reaction exerted by a support is to be determined, the restraint is replaced by a force and the body is given a virtual displacement with a component in the direction of the force. The virtual work done by the reaction and all other forces acting on the body is computed. If several forces are to be determined, the system of bodies can be given a series of separate virtual displacements in which only one of the unknown forces does virtual work during each displacement.

The problems in this chapter will be limited to a single degree of freedom, systems for which the virtual displacements of all points can be expressed in terms of a single variable (displacement).

CONCEPTUAL EXAMPLE 11-1: VIRTUAL WORK AND EQUILIBRIUM

A postal patron tries to close the door on a mail box after inserting a package. If the door is blocked as shown in Fig. CE11-1a, what force must the door hinge be able to resist if **P** *= 45 lb,* **a** *= 10 in., and* **b** *= 15 in.*

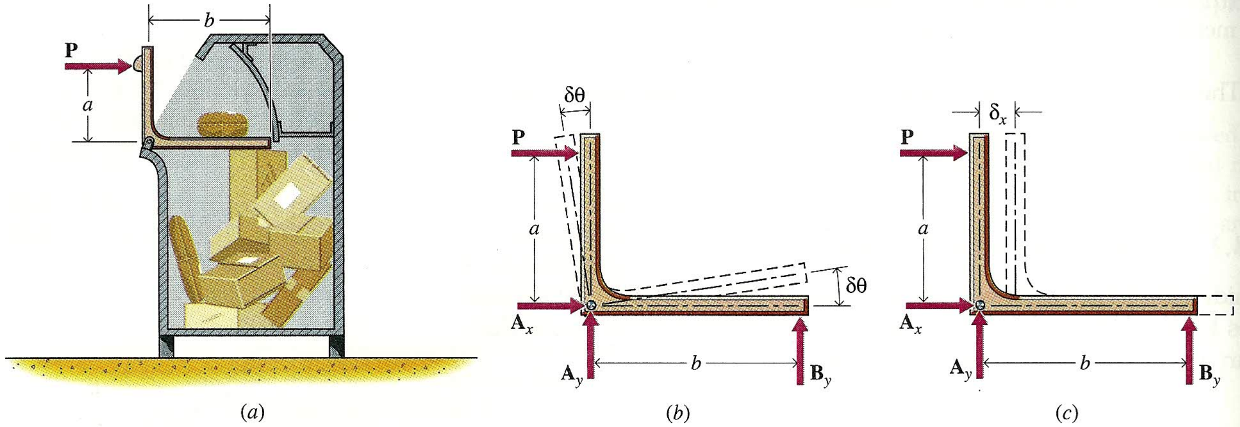
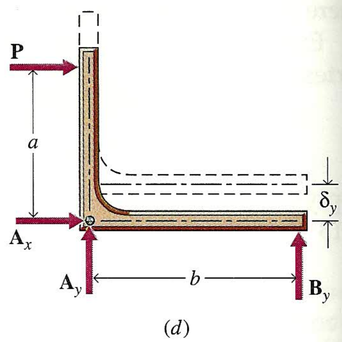


Fig. CE11-1



SOLUTION

If the door is in equilibrium, the virtual work done by all external forces acting on the door must be zero for all virtual displacements of the door. In Fig. CE11-1*b*, a small virtual rotation $\delta\theta$ of the door about the hinge at A is assumed. As a result of this rotation, $\delta x = \delta y = 0$ at hinge A, $\delta y = b\ \delta\theta$ at support B, and $\delta x = -a\ \delta\theta$ at door handle C. As a result, the virtual work principle $\delta U = 0$ gives

$$\delta U = A_x(0) + A_y(0) + P(-a\ \delta\theta) + B_y(b\ \delta\theta) = 0$$

or

$$B_y(b) - P(a) = 0 \qquad B_y = \frac{a}{b}P = \frac{10}{15}(45) = 30\ \text{lb}\uparrow$$

which is simply a statement of the equilibrium equation $\Sigma M_A = 0$.

Similarly, for the virtual displacement δx of Fig. CE11-1*c*,

$$\delta U = A_x(\delta x) + A_y(0) + P(\delta x) + B_y(0) = 0$$

or

$$A_x + P = 0 \qquad A_x = -P = -45\ \text{lb} = 45\ \text{lb}\leftarrow$$

which is simply a statement of the equilibrium equation $\Sigma F_x = 0$.

Finally, for the virtual displacement δy of Fig. CE11-1*d*,

$$\delta U = A_x(0) + A_y(\delta y) + P(0) + B_y(\delta y) = 0$$

or

$$A_y + B_y = 0 \qquad A_y = -B_y = -30\ \text{lb} = 30\ \text{lb}\downarrow$$

which is simply a statement of the equilibrium equation $\Sigma F_y = 0$. Finally,

$$A = \sqrt{(A_x)^2 + (A_y)^2} = \sqrt{(45)^2 + (30)^2} = 54.1\ \text{lb}$$
$$\mathbf{A} = 54.1\ \text{lb} \measuredangle 33.7^\circ \qquad \text{Ans.}$$

EXAMPLE PROBLEM 11-3

A beam is loaded and supported as shown in Fig. 11-7*a*. Use the method of virtual work to determine the reaction at support *B*. Neglect the weight of the beam.

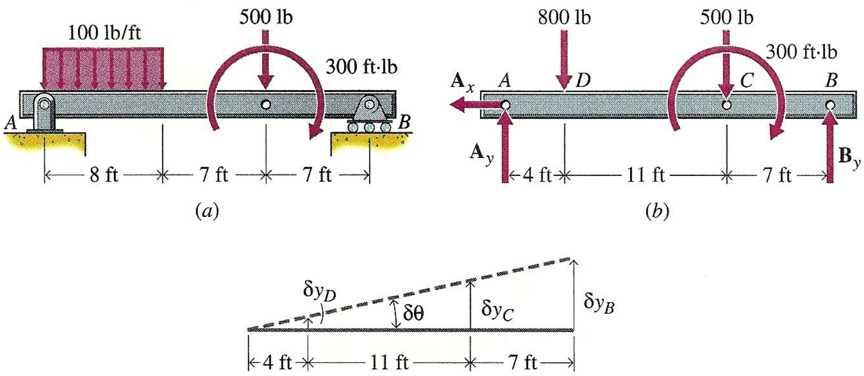


Fig. 11-7

SOLUTION

A free-body diagram for the beam is shown in Fig. 11-7*b*. The 100-lb/ft distributed load has been replaced by its resultant $R = w\ell = 100(8) = 800$ lb. The beam is given a counterclockwise virtual angular displacement $\delta\theta$ about support *A*, which produces the virtual linear displacements $\delta y_B = 22\ \delta\theta$, $\delta y_C = 15\ \delta\theta$, and $\delta y_D = 4\ \delta\theta$ at support *B* and load points *C* and *D*, respectively, as shown in the displacement diagram (Fig. 11-7*c*). The virtual work done as a result of these linear and angular virtual displacements is given by Eqs. 11-9 and 11-10 as

$$\delta U = F\ \delta s \cos\alpha \qquad \text{and} \qquad \delta U = M\ \delta\theta \cos\alpha$$

Thus,

$$\begin{aligned}
\delta U_B &= B_y\ \delta y_B \cos 0° = B_y\ (22\ \delta\theta) \cos 0° = 22B_y\ \delta\theta \\
\delta U_C &= F_C\ \delta y_C \cos 180° = 500\ (15\ \delta\theta) \cos 180° = -7500\ \delta\theta \\
\delta U_D &= F_D\ \delta y_D \cos 180° = 800\ (4\ \delta\theta) \cos 180° = -3200\ \delta\theta \\
\delta U_M &= M\ \delta\theta \cos 180° = 300\ \delta\theta \cos 180° = -300\ \delta\theta
\end{aligned}$$

The total work on the beam is zero when the beam is in equilibrium. Thus,

$$\delta U_{\text{total}} = \delta U_B + \delta U_C + \delta U_D + \delta U_M = (22B_y - 7500 - 3200 - 300)\ \delta\theta = 0$$

Since $\delta\theta \neq 0$

$$(22B_y - 7500 - 3200 - 300) = 0 \qquad \mathbf{B}_y = 500\ \text{lb}\uparrow \qquad \text{Ans.}$$

Solutions to problems of this type are much simpler if the equations of equilibrium are used. As an example, for this problem,

$$+\circlearrowleft \Sigma M_A = 0 \qquad B_y(22) - 500(15) - 300 - 800(4) = 0 \qquad \mathbf{B}_y = 500\ \text{lb}\uparrow \quad \text{Ans.}$$

A force does positive virtual work if its linear virtual displacement has the same direction as the force.

A couple does positive virtual work if its angular virtual displacement has the same direction as the sense of rotation of the couple.

EXAMPLE PROBLEM 11-4

The slender bar shown in Fig. 11-8*a* is 7.2 m long and has a mass of 100 kg. The bar rests against smooth surfaces at supports *A* and *B*. Use the method of virtual work to determine the magnitude of the force **F** required to maintain the bar in the equilibrium position shown in the figure.

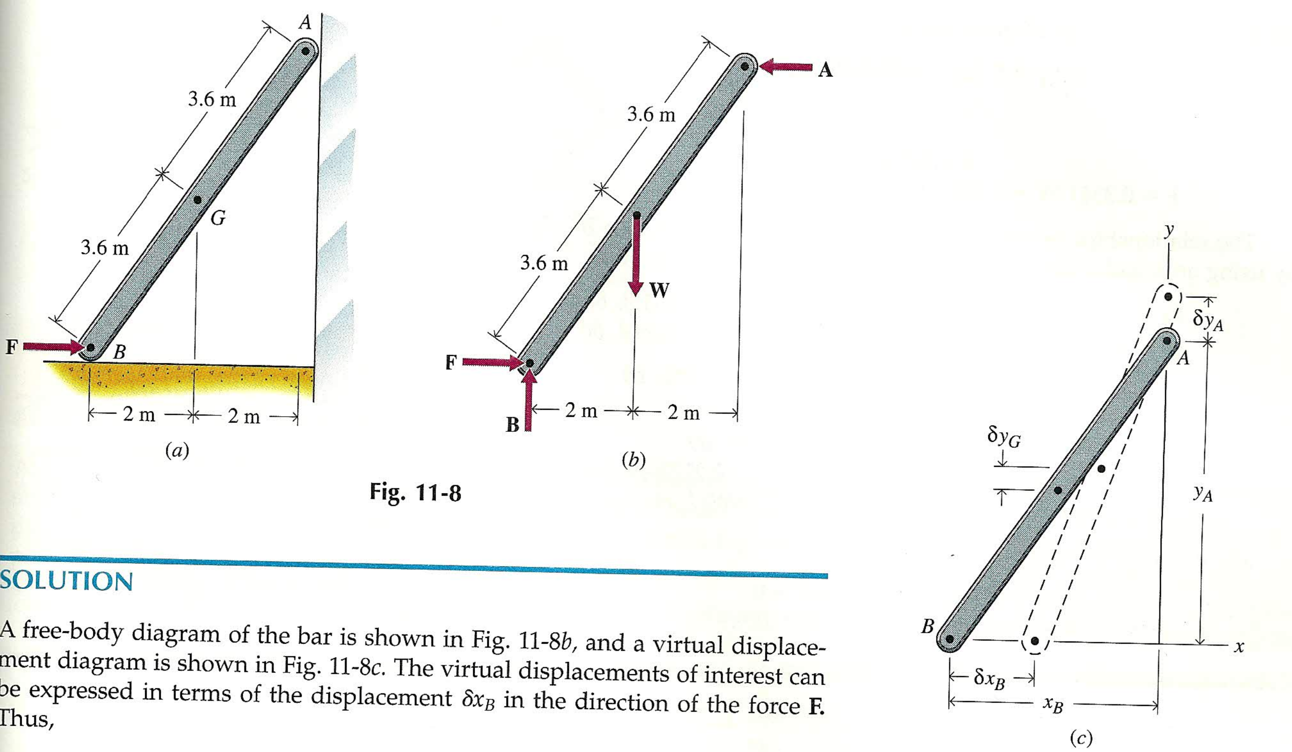


Fig. 11-8

SOLUTION

A free-body diagram of the bar is shown in Fig. 11-8*b*, and a virtual displacement diagram is shown in Fig. 11-8*c*. The virtual displacements of interest can be expressed in terms of the displacement δx_B in the direction of the force **F**. Thus,

$$(x_B - \delta x_B)^2 + (y_A + \delta y_A)^2 = x_B^2 + y_A^2 = L^2 \quad \text{(a)}$$

from which

$$x_B^2 - 2x_B\delta x_B + \delta x_B^2 + y_A^2 + 2y_A\delta y_A + \delta y_A^2 = x_B^2 + y_A^2 \quad \text{(b)}$$

Since the virtual displacements may be considered to be very small, the terms δx_B^2 and δy_A^2 can be neglected and Eq. *b* simplifies to

$$\delta y_A = \frac{x_B}{y_A}\,\delta x_B$$

The *y*-component of the virtual displacement of the center of gravity is

$$\delta y_G = \frac{1}{2}\,\delta y_A = \frac{1}{2}\frac{x_B}{y_A}\,\delta x_B$$

For the given geometry

$$y_A = \sqrt{L^2 - x_B^2} = \sqrt{(7.2)^2 - (4)^2} = 5.987 \text{ m}$$

Therefore,

$$\delta y_G = \frac{1}{2}\frac{x_B}{y_A}\,\delta x_B = \frac{1}{2}\frac{4}{5.987}\,\delta x_B = 0.3341\ \delta x_B$$

Forces **A** and **B** undergo no displacements in the directions of the forces as a result of displacement δx_B; therefore, they do no work. The work of the force

The length *L* of the bar remains constant; therefore, the Pythagorean theorem can be used to relate the virtual displacements at ends *A* and *B* of the bar.

F and the work of the weight **W** on the bar as a result of displacement δx_B are as follows:

Since F and δx_B are in the same direction,

$$\delta U_F = F\ \delta x_B$$

Since W and δy are in opposite directions,

$$\delta U_W = W\,(-\delta y_G) = -0.3341W\ \delta x_B$$

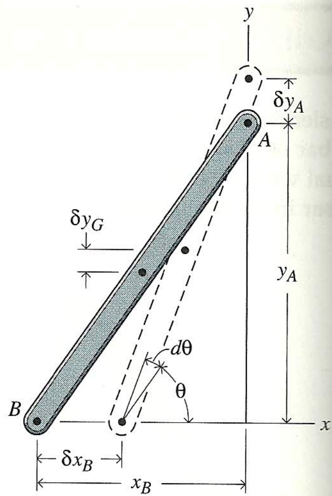
For the bar to be in equilibrium, the total work on the bar must be zero. Thus,

$$\delta U_{\text{total}} = \delta U_F + \delta U_W = F\ \delta x_B - 0.3341W\ \delta x_B = (F - 0.3341W)\ \delta x_B = 0$$

Since $\delta x_B \neq 0$,

$$F - 0.3341W = 0$$

$$F = 0.3341\ W = 0.3341(mg) = 0.3341(100)(9.81) = 328\ \text{N} \qquad \text{Ans.}$$



(*d*)

The relationships between the virtual displacements can also be obtained by using an angular displacement $\delta\theta$ as shown in Fig. 11-8*d*. Thus,

$$x_B = -L\cos\theta \quad \text{and} \quad \delta x_B = L\sin\theta\ \delta\theta = 0.8315L\ \delta\theta$$

$$y_A = L\sin\theta \quad \text{and} \quad \delta y_A = L\cos\theta\ \delta\theta = 0.5556L\ \delta\theta$$

$$\delta y_G = \frac{1}{2}\ \delta y_A = 0.2778L\ \delta\theta$$

$$\delta U_F = F\ \delta x_B = F(0.8315L\ \delta\theta) = 0.8315FL\ \delta\theta$$

$$\delta U_W = W(-\delta y_G) = W(-0.2778L\ \delta\theta) = -0.2778WL\ \delta\theta$$

$$\delta U_{\text{total}} = \delta U_F + \delta U_W = (0.8315F - 0.2778W)\ L\ \delta\theta = 0$$

Since L and $\delta\theta \neq 0$

$$0.8315F - 0.2778W = 0$$

$$F = 0.3341W = 0.3341(mg) = 0.3341(100)(9.81) = 328\ \text{N} \qquad \text{Ans.}$$

A force does positive virtual work if its linear virtual displacement has the same direction as the force.

Trigonometric relations and standard mathematical rules of differentiation can be used to relate virtual displacements at different points on the bar.

EXAMPLE PROBLEM 11-5

A system of pin-connected bars supports a system of loads shown in Fig. 11-9*a*. Use the method of virtual work to determine the horizontal component of the reaction at support *G*. Neglect the weights of the bars.

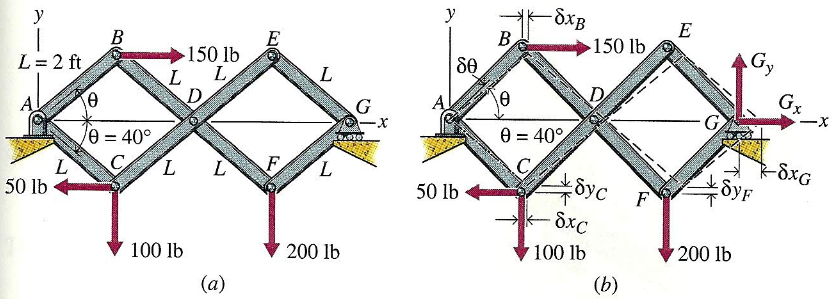


Fig. 11-9

SOLUTION

A virtual displacement diagram of the system with the loads superimposed is shown in Fig. 11-9*b*. The virtual displacements δx and δy at each of the load points and at the support *G* can be expressed in terms of a virtual angular displacement $\delta\theta$ about support *A*. Thus,

Trigonometric relations and standard mathematical rules of differentiation can be used to relate virtual displacements at different points on the system of bars.

$$x_B = L\cos\theta \qquad \delta x_B = -L\sin\theta\,\delta\theta = -2(\sin 40°)\,\delta\theta = -1.2856\,\delta\theta$$
$$x_C = L\cos\theta \qquad \delta x_C = -L\sin\theta\,\delta\theta = -2(\sin 40°)\,\delta\theta = -1.2856\,\delta\theta$$
$$x_G = 4L\cos\theta \qquad \delta x_G = -4L\sin\theta\,\delta\theta = -4(2)(\sin 40°)\,\delta\theta = -5.1423\,\delta\theta$$

$$y_C = -L\sin\theta \qquad \delta y_C = -L\cos\theta\,\delta\theta = -2(\cos 40°)\,\delta\theta = -1.5321\,\delta\theta$$
$$y_F = -L\sin\theta \qquad \delta y_F = -L\cos\theta\,\delta\theta = -2(\cos 40°)\,\delta\theta = -1.5321\,\delta\theta$$

The virtual work done by each of the forces can then be computed by carefully noting the directions of the forces and the directions of the associated virtual displacements. Thus,

A force does positive virtual work if its linear virtual displacement has the same direction as the force.

$$\delta U_{Bx} = B_x(\delta x_B) = 150(-1.2856\,\delta\theta) = -192.84\,\delta\theta$$
$$\delta U_{Cx} = C_x(\delta x_C) = 50(+1.2856\,\delta\theta) = +64.28\,\delta\theta$$
$$\delta U_{Cy} = C_y(\delta y_C) = 100(+1.5321\,\delta\theta) = +153.21\,\delta\theta$$
$$\delta U_{Fy} = F_y(\delta y_F) = 200(+1.5321\,\delta\theta) = +306.42\,\delta\theta$$
$$\delta U_{Gx} = G_x(\delta x_G) = G_x(-5.1423\,\delta\theta) = -5.1423G_x\,\delta\theta$$

Force G_y does not undergo a virtual displacement in the direction of the force as a result of the virtual angular displacement $\delta\theta$; therefore, it does no work. For the system to be in equilibrium, the total work on the system must be zero. Thus

$$\begin{aligned}\delta U_{\text{total}} &= \delta U_{Bx} + \delta U_{Cx} + \delta U_{Cy} + \delta U_{Fy} + \delta U_{Gx}\\ &= (-192.84 + 64.28 + 153.21 + 306.42 - 5.1423G_x)\,\delta\theta\\ &= (331.07 - 5.1423G_x)\,\delta\theta = 0\end{aligned}$$

Since $\delta\theta \neq 0$,

$$331.07 - 5.1423G_x = 0$$
$$G_x = \frac{331.07}{5.1423} = 64.4 \text{ lb} \qquad \text{Ans.}$$