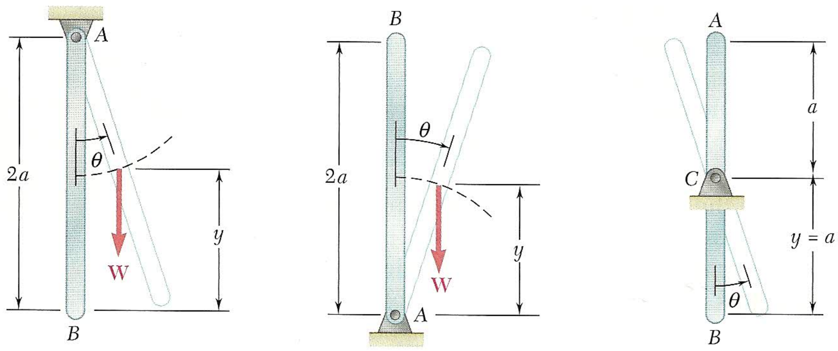
*10.9. STABILITY OF EQUILIBRIUM

Consider the three uniform rods of length $2a$ and weight **W** shown in Fig. 10.14. While each rod is in equilibrium, there is an important difference between the three cases considered. Suppose that each rod is slightly disturbed from its position of equilibrium and then released: rod *a* will move back toward its original position, rod *b* will keep moving away from its original position, and rod *c* will remain in its new position. In case *a*, the equilibrium of the rod is said to be *stable;* in case *b*, it is said to be *unstable;* and, in case *c*, it is said to be *neutral*.



(*a*) Stable equilibrium (*b*) Unstable equilibrium (*c*) Neutral equilibrium

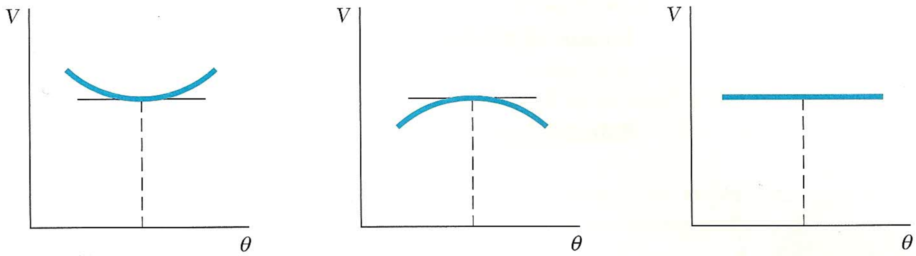
Fig. 10.14

Recalling from Sec. 10.7 that the potential energy V_g with respect to gravity is equal to Wy, where y is the elevation of the point of application of **W** measured from an arbitrary level, we observe that the potential energy of rod *a* is minimum in the position of equilibrium considered, that the potential energy of rod *b* is maximum, and that the potential energy of rod *c* is constant. Equilibrium is thus *stable, unstable,* or *neutral* according to whether the potential energy is *minimum, maximum,* or *constant* (Fig. 10.15).

That the result obtained is quite general can be seen as follows: We first observe that a force always tends to do positive work and thus to decrease the potential energy of the system on which it is applied. Therefore, when a system is disturbed from its position of equilibrium, the forces acting on the system will tend to bring it back to its original position if V is minimum (Fig. 10.15*a*) and to move it farther away if V is maximum (Fig. 10.15*b*). If V is constant (Fig. 10.15*c*), the forces will not tend to move the system either way.

Recalling from calculus that a function is minimum or maximum according to whether its second derivative is positive or negative, we can summarize the conditions for the equilibrium of a system with one degree of freedom (that is, a system the position of which is defined by a single independent variable θ) as follows:

$$\begin{aligned} \frac{dV}{d\theta} &= 0 \qquad \frac{d^2V}{d\theta^2} > 0\text{: stable equilibrium} \\ \frac{dV}{d\theta} &= 0 \qquad \frac{d^2V}{d\theta^2} < 0\text{: unstable equilibrium} \end{aligned} \tag{10.23}$$



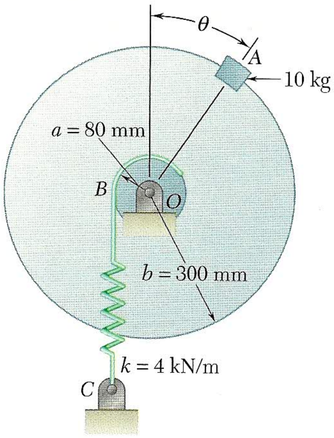
(*a*) Stable equilibrium (*b*) Unstable equilibrium (*c*) Neutral equilibrium

Fig. 10.15

If both the first and the second derivatives of V are zero, it is necessary to examine derivatives of a higher order to determine whether the equilibrium is stable, unstable, or neutral. The equilibrium will be neutral if all derivatives are zero, since the potential energy V is then a constant. The equilibrium will be stable if the first derivative found to be different from zero is of even order and positive. In all other cases the equilibrium will be unstable.

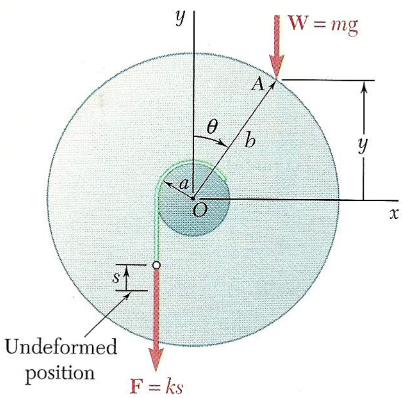
If the system considered possesses *several degrees of freedom,* the potential energy V depends upon several variables, and it is thus necessary to apply the theory of functions of several variables to determine whether V is minimum. It can be verified that a system with 2 degrees of freedom will be stable, and the corresponding potential energy $V(\theta_1, \theta_2)$ will be minimum, if the following relations are satisfied simultaneously:

$$\begin{gathered} \frac{\partial V}{\partial \theta_1} = \frac{\partial V}{\partial \theta_2} = 0 \\ \left(\frac{\partial^2 V}{\partial \theta_1\, \partial \theta_2}\right)^2 - \frac{\partial^2 V}{\partial \theta_1^2}\frac{\partial^2 V}{\partial \theta_2^2} < 0 \\ \frac{\partial^2 V}{\partial \theta_1^2} > 0 \quad \text{or} \quad \frac{\partial^2 V}{\partial \theta_2^2} > 0 \end{gathered} \tag{10.24}$$



SAMPLE PROBLEM 10.4

A 10-kg block is attached to the rim of a 300-mm-radius disk as shown. Knowing that spring BC is unstretched when $\theta = 0$, determine the position or positions of equilibrium, and state in each case whether the equilibrium is stable, unstable, or neutral.



SOLUTION

Potential Energy. Denoting by s the deflection of the spring from its undeformed position and placing the origin of coordinates at O, we obtain

$$V_e = \tfrac{1}{2}ks^2 \qquad V_g = Wy = mgy$$

Measuring θ in radians, we have

$$s = a\theta \qquad y = b\cos\theta$$

Substituting for s and y in the expressions for V_e and V_g, we write

$$V_e = \tfrac{1}{2}ka^2\theta^2 \qquad V_g = mgb\cos\theta$$
$$V = V_e + V_g = \tfrac{1}{2}ka^2\theta^2 + mgb\cos\theta$$

Positions of Equilibrium. Setting $dV/d\theta = 0$, we write

$$\frac{dV}{d\theta} = ka^2\theta - mgb\sin\theta = 0$$
$$\sin\theta = \frac{ka^2}{mgb}\theta$$

Substituting $a = 0.08$ m, $b = 0.3$ m, $k = 4$ kN/m, and $m = 10$ kg, we obtain

$$\sin\theta = \frac{(4\text{ kN/m})(0.08\text{ m})^2}{(10\text{ kg})(9.81\text{ m/s}^2)(0.3\text{ m})}\theta$$
$$\sin\theta = 0.8699\,\theta$$

where θ is expressed in radians. Solving numerically for θ, we find

$$\theta = 0 \quad\text{and}\quad \theta = 0.902\text{ rad}$$
$$\theta = 0 \quad\text{and}\quad \theta = 51.7° \quad ◀$$

Stability of Equilibrium. The second derivative of the potential energy V with respect to θ is

$$\frac{d^2V}{d\theta^2} = ka^2 - mgb\cos\theta$$
$$= (4\text{ kN/m})(0.08\text{ m})^2 - (10\text{ kg})(9.81\text{ m/s}^2)(0.3\text{ m})\cos\theta$$
$$= 25.6 - 29.43\cos\theta$$

For $\theta = 0$: $\quad \dfrac{d^2V}{d\theta^2} = 25.6 - 29.43\cos 0 = -3.83 < 0$

The equilibrium is unstable for $\theta = 0$ ◀

For $\theta = 51.7°$: $\quad \dfrac{d^2V}{d\theta^2} = 25.6 - 29.43\cos 51.7° = +7.36 > 0$

The equilibrium is stable for $\theta = 51.7°$ ◀