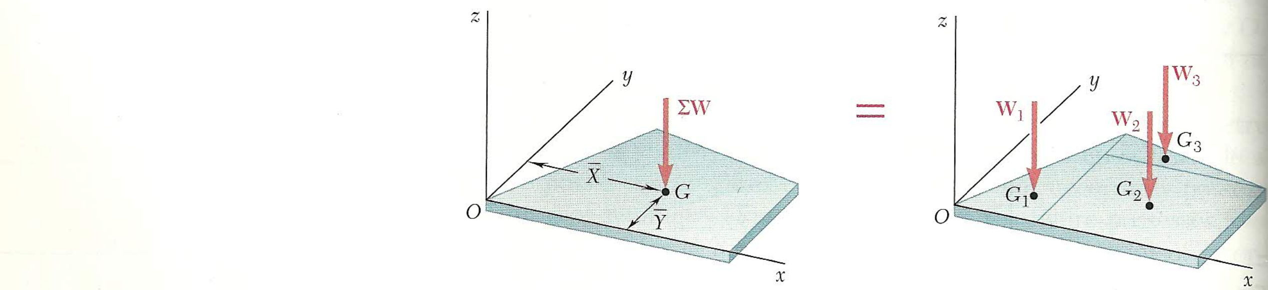
5.5. COMPOSITE PLATES AND WIRES

In many instances, a flat plate can be divided into rectangles, triangles, or the other common shapes shown in Fig. 5.8A. The abscissa $\overline{X}$ of its center of gravity G can be determined from the abscissas $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n$ of the centers of gravity of the various parts by expressing that the moment of the weight of the whole plate about the y axis is equal to the sum of the moments of the weights of the various parts about the same axis (Fig. 5.9). The ordinate $\overline{Y}$ of the center of gravity of the plate is found in a similar way by equating moments about the x axis. We write

$$\Sigma M_y: \quad \overline{X}(W_1 + W_2 + \cdots + W_n) = \overline{x}_1 W_1 + \overline{x}_2 W_2 + \cdots + \overline{x}_n W_n$$

$$\Sigma M_x: \quad \overline{Y}(W_1 + W_2 + \cdots + W_n) = \overline{y}_1 W_1 + \overline{y}_2 W_2 + \cdots + \overline{y}_n W_n$$



$$\Sigma M_y: \quad \overline{X}\,\Sigma W = \Sigma \overline{x}\, W$$

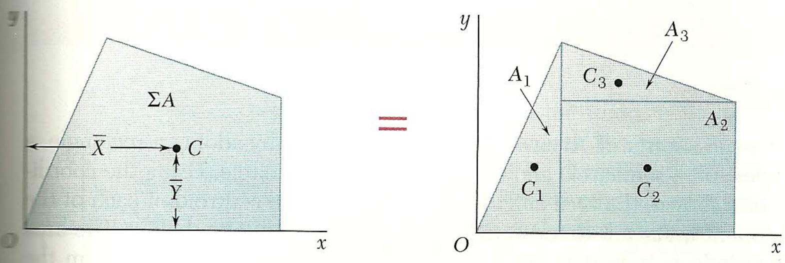
$$\Sigma M_x: \quad \overline{Y}\,\Sigma W = \Sigma \overline{y}\, W$$

Fig. 5.9 Center of gravity of a composite plate.

or, for short,

$$\overline{X}\Sigma W = \Sigma\overline{x}W \qquad \overline{Y}\Sigma W = \Sigma\overline{y}W \tag{5.7}$$

These equations can be solved for the coordinates $\overline{X}$ and $\overline{Y}$ of the center of gravity of the plate.



$$Q_y = \overline{X}\Sigma A = \Sigma\overline{x}A$$
$$Q_x = \overline{Y}\Sigma A = \Sigma\overline{y}A$$

Fig. 5.10 Centroid of a composite area.

If the plate is homogeneous and of uniform thickness, the center of gravity coincides with the centroid C of its area. The abscissa $\overline{X}$ of the centroid of the area can be determined by noting that the first moment Q_y of the composite area with respect to the y axis can be expressed both as the product of $\overline{X}$ and the total area and as the sum of the first moments of the elementary areas with respect to the y axis (Fig. 5.10). The ordinate $\overline{Y}$ of the centroid is found in a similar way by considering the first moment Q_x of the composite area. We have

$$Q_y = \overline{X}(A_1 + A_2 + \cdots + A_n) = \overline{x}_1A_1 + \overline{x}_2A_2 + \cdots + \overline{x}_nA_n$$
$$Q_x = \overline{Y}(A_1 + A_2 + \cdots + A_n) = \overline{y}_1A_1 + \overline{y}_2A_2 + \cdots + \overline{y}_nA_n$$

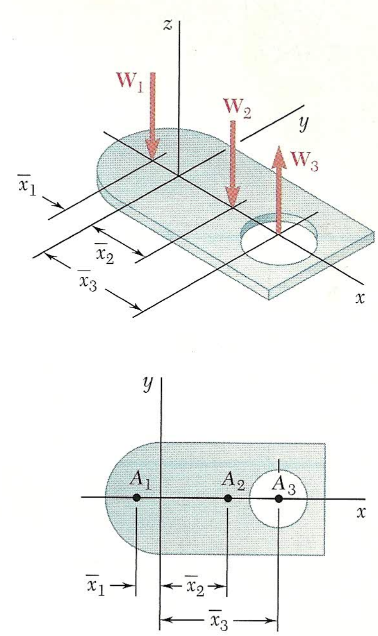
or, for short,

$$Q_y = \overline{X}\Sigma A = \Sigma\overline{x}A \qquad Q_x = \overline{Y}\Sigma A = \Sigma\overline{y}A \tag{5.8}$$

These equations yield the first moments of the composite area, or they can be used to obtain the coordinates $\overline{X}$ and $\overline{Y}$ of its centroid.

Care should be taken to assign the appropriate sign to the moment of each area. First moments of areas, like moments of forces, can be positive or negative. For example, an area whose centroid is located to the left of the y axis will have a negative first moment with respect to that axis. Also, the area of a hole should be assigned a negative sign (Fig. 5.11).

Similarly, it is possible in many cases to determine the center of gravity of a composite wire or the centroid of a composite line by dividing the wire or line into simpler elements (see Sample Prob. 5.2).



	$\overline{x}$	A	$\overline{x}A$
A_1 Semicircle	−	+	−
A_2 Full rectangle	+	+	+
A_3 Circular hole	+	−	−

Fig. 5.11

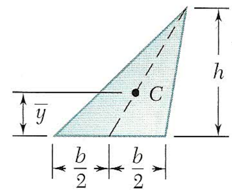
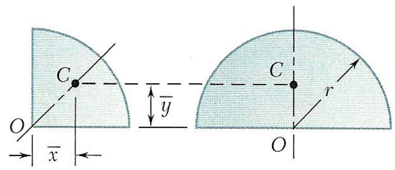
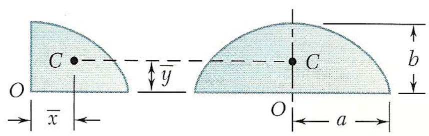
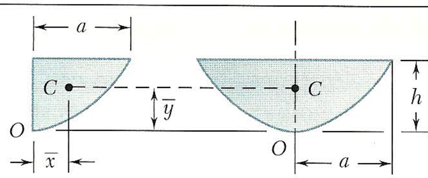
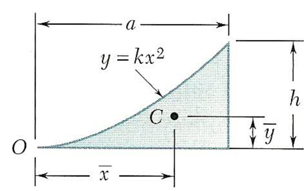
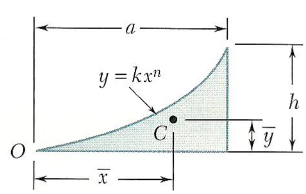
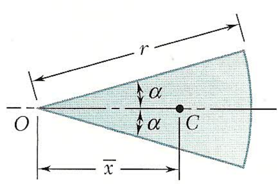
Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
General spandrel		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r \sin \alpha}{3\alpha}$	0	αr^2

Fig. 5.8A Centroids of common shapes of areas.

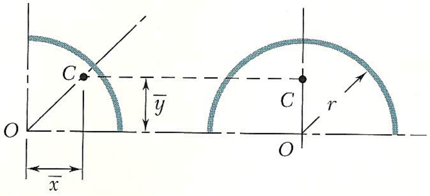
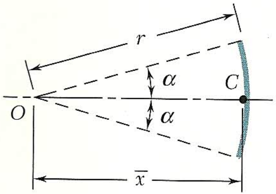
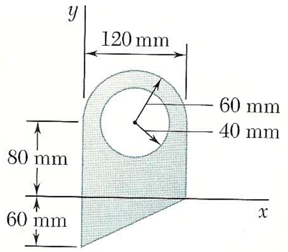
Shape		$\bar{x}$	$\bar{y}$	Length
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	πr
Arc of circle		$\frac{r \sin \alpha}{\alpha}$	0	$2\alpha r$

Fig. 5.8B Centroids of common shapes of lines.

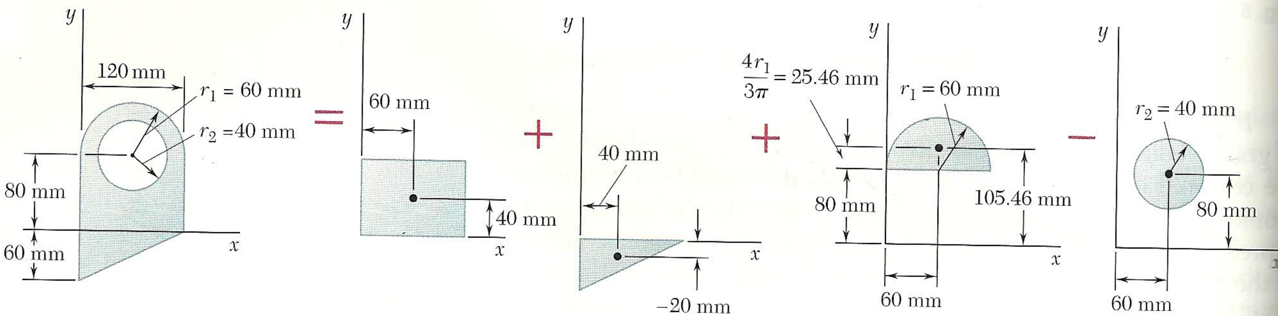


SAMPLE PROBLEM 5.1

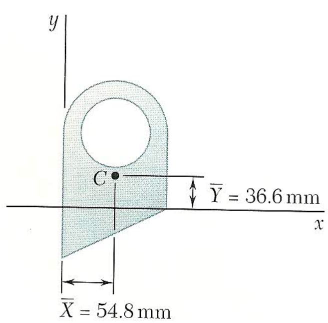
For the plane area shown, determine (*a*) the first moments with respect to the x and y axes, (*b*) the location of the centroid.

SOLUTION

Components of Area. The area is obtained by adding a rectangle, a triangle, and a semicircle and by then subtracting a circle. Using the coordinate axes shown, the area and the coordinates of the centroid of each of the component areas are determined and entered in the table below. The area of the circle is indicated as negative, since it is to be subtracted from the other areas. We note that the coordinate $\bar{y}$ of the centroid of the triangle is negative for the axes shown. The first moments of the component areas with respect to the coordinate axes are computed and entered in the table.



Component	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
Rectangle	$(120)(80) = 9.6 \times 10^3$	60	40	$+576 \times 10^3$	$+384 \times 10^3$
Triangle	$\frac{1}{2}(120)(60) = 3.6 \times 10^3$	40	−20	$+144 \times 10^3$	-72×10^3
Semicircle	$\frac{1}{2}\pi(60)^2 = 5.655 \times 10^3$	60	105.46	$+339.3 \times 10^3$	$+596.4 \times 10^3$
Circle	$-\pi(40)^2 = -5.027 \times 10^3$	60	80	-301.6×10^3	-402.2×10^3
	$\Sigma A = 13.828 \times 10^3$			$\Sigma\bar{x}A = +757.7 \times 10^3$	$\Sigma\bar{y}A = +506.2 \times 10^3$



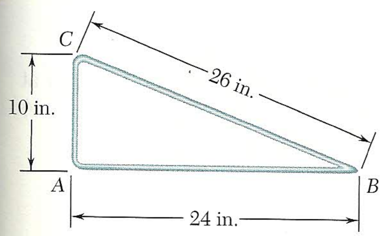
***a.* First Moments of the Area.** Using Eqs. (5.8), we write

$$Q_x = \Sigma\bar{y}A = 506.2 \times 10^3 \text{ mm}^3 \qquad Q_x = 506 \times 10^3 \text{ mm}^3 \blacktriangleleft$$
$$Q_y = \Sigma\bar{x}A = 757.7 \times 10^3 \text{ mm}^3 \qquad Q_y = 758 \times 10^3 \text{ mm}^3 \blacktriangleleft$$

***b.* Location of Centroid.** Substituting the values given in the table into the equations defining the centroid of a composite area, we obtain

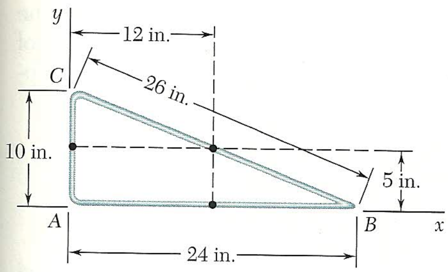
$$\bar{X}\Sigma A = \Sigma\bar{x}A: \qquad \bar{X}(13.828 \times 10^3 \text{ mm}^2) = 757.7 \times 10^3 \text{ mm}^3$$
$$\bar{X} = 54.8 \text{ mm} \blacktriangleleft$$

$$\bar{Y}\Sigma A = \Sigma\bar{y}A: \qquad \bar{Y}(13.828 \times 10^3 \text{ mm}^2) = 506.2 \times 10^3 \text{ mm}^3$$
$$\bar{Y} = 36.6 \text{ mm} \blacktriangleleft$$



SAMPLE PROBLEM 5.2

The figure shown is made from a piece of thin, homogeneous wire. Determine the location of its center of gravity.



SOLUTION

Since the figure is formed of homogeneous wire, its center of gravity coincides with the centroid of the corresponding line. Therefore, that centroid will be determined. Choosing the coordinate axes shown, with origin at A, we determine the coordinates of the centroid of each line segment and compute the first moments with respect to the coordinate axes.

Segment	L, in.	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}L$, in^2	$\bar{y}L$, in^2
AB	24	12	0	288	0
BC	26	12	5	312	130
CA	10	0	5	0	50
	$\Sigma L = 60$			$\Sigma \bar{x}L = 600$	$\Sigma \bar{y}L = 180$

Substituting the values obtained from the table into the equations defining the centroid of a composite line, we obtain

$\bar{X}\Sigma L = \Sigma \bar{x}L$: $\quad \bar{X}(60 \text{ in.}) = 600 \text{ in}^2 \qquad \bar{X} = 10 \text{ in.}$ ◀

$\bar{Y}\Sigma L = \Sigma \bar{y}L$: $\quad \bar{Y}(60 \text{ in.}) = 180 \text{ in}^2 \qquad \bar{Y} = 3 \text{ in.}$ ◀

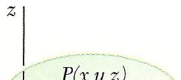
5.11. COMPOSITE BODIES

If a body can be divided into several of the common shapes shown in Fig. 5.21, its center of gravity G can be determined by expressing that the moment about O of its total weight is equal to the sum of the moments about O of the weights of the various component parts. Proceeding as in Sec. 5.10, we obtain the following equations defining the coordinates $\bar{X}$, $\bar{Y}$, and $\bar{Z}$ of the center of gravity G:

$$\bar{X}\Sigma W = \Sigma \bar{x} W \qquad \bar{Y}\Sigma W = \Sigma \bar{y} W \qquad \bar{Z}\Sigma W = \Sigma \bar{z} W \tag{5.19}$$

If the body is made of a homogeneous material, its center of gravity coincides with the centroid of its volume, and we obtain

$$\bar{X}\Sigma V = \Sigma \bar{x} V \qquad \bar{Y}\Sigma V = \Sigma \bar{y} V \qquad \bar{Z}\Sigma V = \Sigma \bar{z} V \tag{5.20}$$



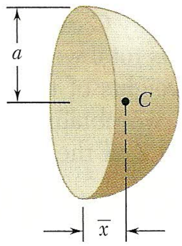
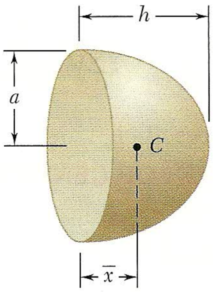
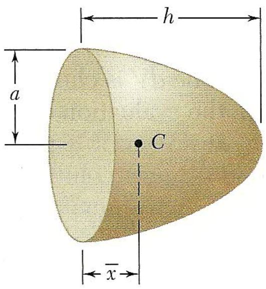
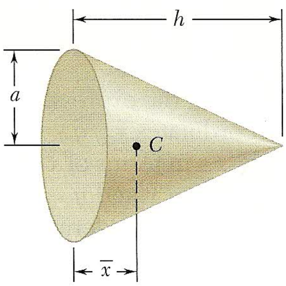
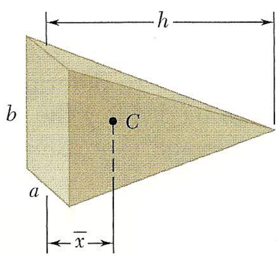
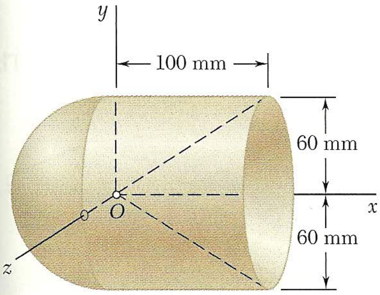
Shape		$\bar{x}$	Volume
Hemisphere		$\frac{3a}{8}$	$\frac{2}{3}\pi a^3$
Semiellipsoid of revolution		$\frac{3h}{8}$	$\frac{2}{3}\pi a^2 h$
Paraboloid of revolution		$\frac{h}{3}$	$\frac{1}{2}\pi a^2 h$
Cone		$\frac{h}{4}$	$\frac{1}{3}\pi a^2 h$
Pyramid		$\frac{h}{4}$	$\frac{1}{3}abh$

Fig. 5.21 Centroids of common shapes and volumes.

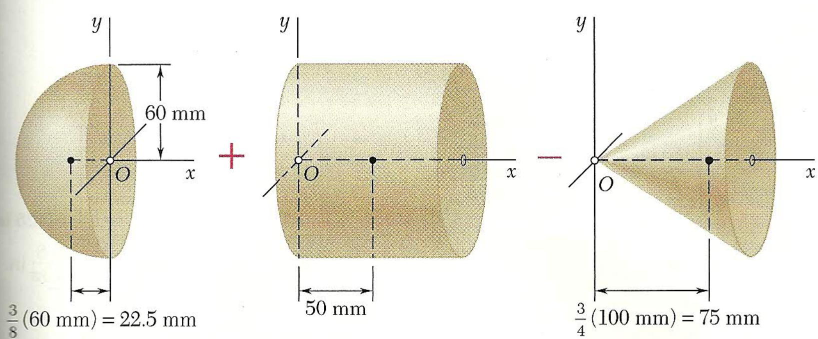


SAMPLE PROBLEM 5.11

Determine the location of the center of gravity of the homogeneous body of revolution shown, which was obtained by joining a hemisphere and a cylinder and carving out a cone.

SOLUTION

Because of symmetry, the center of gravity lies on the x axis. As shown in the figure below, the body can be obtained by adding a hemisphere to a cylinder and then subtracting a cone. The volume and the abscissa of the centroid of each of these components are obtained from Fig. 5.21 and are entered in the table below. The total volume of the body and the first moment of its volume with respect to the yz plane are then determined.



Component	Volume, mm^3	$\bar{x}$, mm	$\bar{x}V$, mm^4
Hemisphere	$\frac{1}{2}\frac{4\pi}{3}(60)^3 = \quad 0.4524 \times 10^6$	-22.5	-10.18×10^6
Cylinder	$\pi(60)^2(100) = \quad 1.1310 \times 10^6$	$+50$	$+56.55 \times 10^6$
Cone	$-\frac{\pi}{3}(60)^2(100) = -0.3770 \times 10^6$	$+75$	-28.28×10^6
	$\Sigma V = \quad 1.206 \times 10^6$		$\Sigma\bar{x}V = +18.09 \times 10^6$

Thus,

$\bar{X}\Sigma V = \Sigma\bar{x}V$: $\quad \bar{X}(1.206 \times 10^6 \text{ mm}^3) = 18.09 \times 10^6 \text{ mm}^4$

$$\bar{X} = 15 \text{ mm} \quad ◀$$