9.2. SECOND MOMENT, OR MOMENT OF INERTIA, OF AN AREA

In the first part of this chapter, we consider distributed forces $\Delta\mathbf{F}$ whose magnitudes ΔF are proportional to the elements of area ΔA on which the forces act and at the same time vary linearly with the distance from ΔA to a given axis.

Consider, for example, a beam of uniform cross section which is subjected to two equal and opposite couples applied at each end of the beam. Such a beam is said to be in *pure bending*, and it is shown in mechanics of materials that the internal forces in any section of the beam are distributed forces whose magnitudes $\Delta F = ky\,\Delta A$ vary linearly with the distance y between the element of area ΔA and an axis passing through the centroid of the section. This axis, represented by the x axis in Fig. 9.1, is known as the *neutral axis* of the section. The forces on one side of the neutral axis are forces of compression, while those on the other side are forces of tension; on the neutral axis itself the forces are zero.

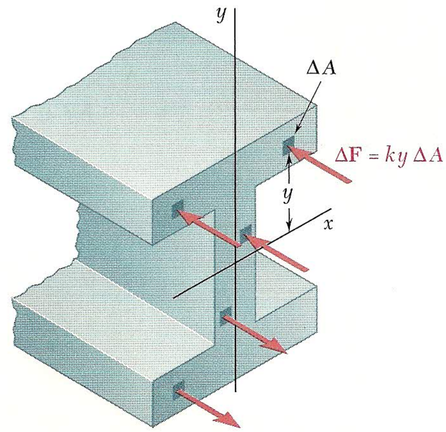


Fig. 9.1

The magnitude of the resultant $\mathbf{R}$ of the elemental forces $\Delta\mathbf{F}$ which act over the entire section is

$$R = \int ky\,dA = k\int y\,dA$$

The last integral obtained is recognized as the *first moment* Q_x of the section about the x axis; it is equal to $\bar{y}A$ and is thus equal to zero, since the centroid of the section is located on the x axis. The system of the forces $\Delta\mathbf{F}$ thus reduces to a couple. The magnitude M of this couple (bending moment) must be equal to the sum of the moments $\Delta M_x = y\,\Delta F = ky^2\,\Delta A$ of the elemental forces. Integrating over the entire section, we obtain

$$M = \int ky^2\,dA = k\int y^2\,dA$$

The last integral is known as the *second moment*, or *moment of inertia*,† of the beam section with respect to the x axis and is denoted by I_x. It is obtained by multiplying each element of area dA by the *square of its distance* from the x axis and integrating over the beam section. Since each product $y^2\,dA$ is positive, regardless of the sign of y, or zero (if y is zero), the integral I_x will always be positive.

Another example of a second moment, or moment of inertia, of an area is provided by the following problem from hydrostatics: A vertical circular gate used to close the outlet of a large reservoir is submerged under water as shown in Fig. 9.2. What is the resultant of the forces exerted by the water on the gate, and what is the moment of the resultant about the line of intersection of the plane of the gate and the water surface (x axis)?

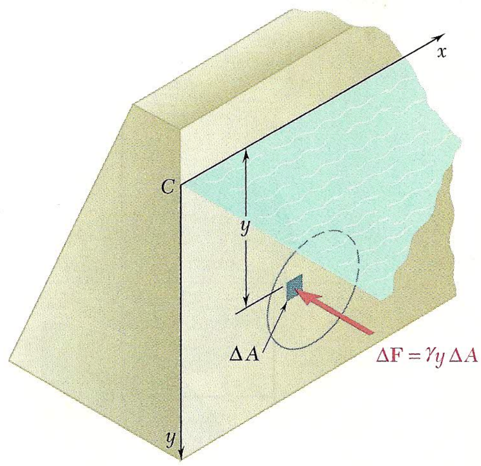


Fig. 9.2

†The term *second moment* is more proper than the term *moment of inertia,* since, logically, the latter should be used only to denote integrals of mass (see Sec. 9.11). In engineering practice, however, moment of inertia is used in connection with areas as well as masses.

If the gate were rectangular, the resultant of the forces of pressure could be determined from the pressure curve, as was done in Sec. 5.9. Since the gate is circular, however, a more general method must be used. Denoting by y the depth of an element of area ΔA and by γ the specific weight of water, the pressure at the element is $p = \gamma y$, and the magnitude of the elemental force exerted on ΔA is $\Delta F = p\,\Delta A = \gamma y\,\Delta A$. The magnitude of the resultant of the elemental forces is thus

$$R = \int \gamma y\, dA = \gamma \int y\, dA$$

and can be obtained by computing the first moment $Q_x = \int y\, dA$ of the area of the gate with respect to the x axis. The moment M_x of the resultant must be equal to the sum of the moments $\Delta M_x = y\,\Delta F = \gamma y^2\,\Delta A$ of the elemental forces. Integrating over the area of the gate, we have

$$M_x = \int \gamma y^2\, dA = \gamma \int y^2\, dA$$

Here again, the integral obtained represents the second moment, or moment of inertia, I_x of the area with respect to the x axis.

9.3. DETERMINATION OF THE MOMENT OF INERTIA OF AN AREA BY INTEGRATION

We defined in the preceding section the second moment, or moment of inertia, of an area A with respect to the x axis. Defining in a similar way the moment of inertia I_y of the area A with respect to the y axis, we write (Fig. 9.3a)

$$I_x = \int y^2\, dA \qquad I_y = \int x^2\, dA \tag{9.1}$$

These integrals, known as the *rectangular moments of inertia* of the area A, can be more easily evaluated if we choose dA to be a thin strip parallel to one of the coordinate axes. To compute I_x, the strip is chosen parallel to the x axis, so that all of the points of the strip are at the same distance y from the x axis (Fig. 9.3b); the moment of inertia dI_x of the strip is then obtained by multiplying the area dA of the strip by y^2. To compute I_y, the strip is chosen parallel to the y axis so that all of the points of the strip are at the same distance x from the y axis (Fig. 9.3c); the moment of inertia dI_y of the strip is $x^2\, dA$.

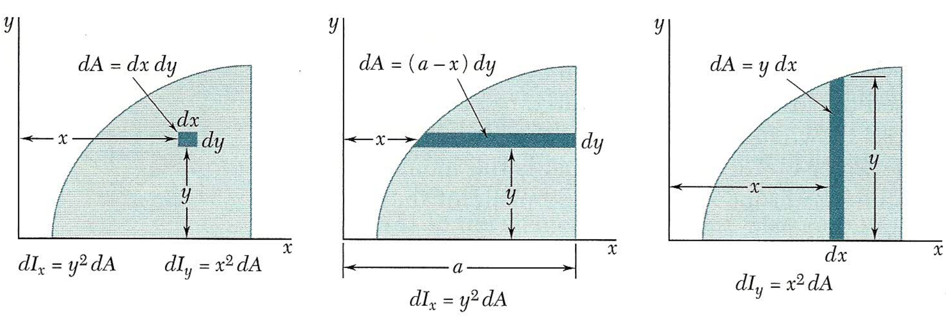


Fig. 9.3 (a) (b) (c)

Moment of Inertia of a Rectangular Area. As an example, let us determine the moment of inertia of a rectangle with respect to its base (Fig. 9.4). Dividing the rectangle into strips parallel to the x axis, we obtain

$$dA = b\,dy \qquad dI_x = y^2 b\,dy$$

$$I_x = \int_0^h by^2\,dy = \tfrac{1}{3}bh^3 \tag{9.2}$$

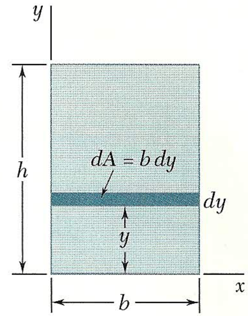


Fig. 9.4

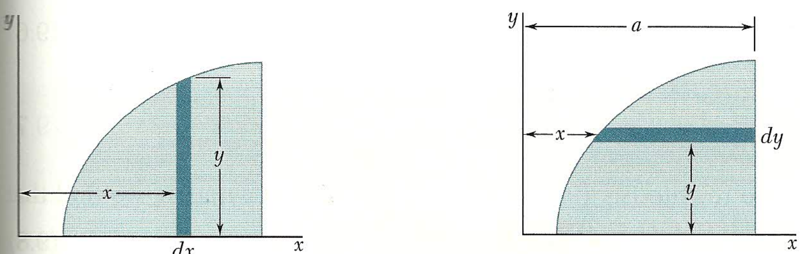
Computing I_x and I_y Using the Same Elemental Strip. The formula just derived can be used to determine the moment of inertia dI_x with respect to the x axis of a rectangular strip which is parallel to the y axis, such as the strip shown in Fig. 9.3c. Setting $b = dx$ and $h = y$ in formula (9.2), we write

$$dI_x = \tfrac{1}{3}y^3\,dx$$

On the other hand, we have

$$dI_y = x^2\,dA = x^2 y\,dx$$

The same element can thus be used to compute the moments of inertia I_x and I_y of a given area (Fig. 9.5a). The analogous results for the area of Fig. 9.3b are shown in Fig. 9.5b.



$dI_x = \frac{1}{3}y^3\,dx$

$dI_y = x^2 y\,dx$

(a)

$dI_x = y^2(a - x)dy$

$dI_y = (\frac{1}{3}a^3 - \frac{1}{3}x^3)dy$

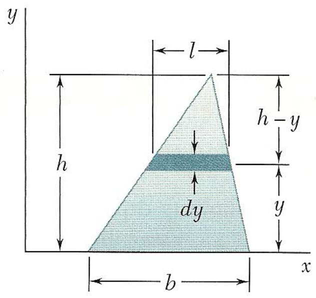
(b)

Fig. 9.5

SAMPLE PROBLEM 9.1

Determine the moment of inertia of a triangle with respect to its base.

SOLUTION



A triangle of base b and height h is drawn; the x axis is chosen to coincide with the base. A differential strip parallel to the x axis is chosen to be dA. Since all portions of the strip are at the same distance from the x axis, we write

$$dI_x = y^2\,dA \qquad dA = l\,dy$$

Using similar triangles, we have

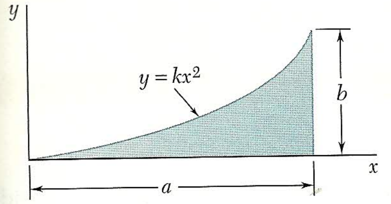
$$\frac{l}{b} = \frac{h-y}{h} \qquad l = b\frac{h-y}{h} \qquad dA = b\frac{h-y}{h}\,dy$$

Integrating dI_x from $y = 0$ to $y = h$, we obtain

$$I_x = \int y^2\,dA = \int_0^h y^2 b\frac{h-y}{h}\,dy = \frac{b}{h}\int_0^h (hy^2 - y^3)\,dy$$

$$= \frac{b}{h}\left[h\frac{y^3}{3} - \frac{y^4}{4}\right]_0^h \qquad I_x = \frac{bh^3}{12} \blacktriangleleft$$

SAMPLE PROBLEM 9.3

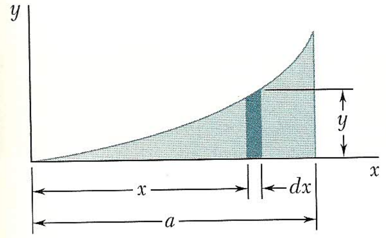


(a) Determine the moment of inertia of the shaded area shown with respect to each of the coordinate axes. (Properties of this area were considered in Sample Prob. 5.4.) (b) Using the results of part a, determine the radius of gyration of the shaded area with respect to each of the coordinate axes.

SOLUTION

Referring to Sample Prob. 5.4, we obtain the following expressions for the equation of the curve and the total area:

$$y = \frac{b}{a^2}x^2 \qquad A = \tfrac{1}{3}ab$$



Moment of Inertia I_x. A vertical differential element of area is chosen to be dA. Since all portions of this element are *not* at the same distance from the x axis, we must treat the element as a thin rectangle. The moment of inertia of the element with respect to the x axis is then

$$dI_x = \tfrac{1}{3}y^3\,dx = \frac{1}{3}\left(\frac{b}{a^2}x^2\right)^3 dx = \frac{1}{3}\frac{b^3}{a^6}x^6\,dx$$

$$I_x = \int dI_x = \int_0^a \frac{1}{3}\frac{b^3}{a^6}x^6\,dx = \left[\frac{1}{3}\frac{b^3}{a^6}\frac{x^7}{7}\right]_0^a$$

$$I_x = \frac{ab^3}{21} \quad ◀$$

Moment of Inertia I_y. The same vertical differential element of area is used. Since all portions of the element are at the same distance from the y axis, we write

$$dI_y = x^2\,dA = x^2(y\,dx) = x^2\left(\frac{b}{a^2}x^2\right)dx = \frac{b}{a^2}x^4\,dx$$

$$I_y = \int dI_y = \int_0^a \frac{b}{a^2}x^4\,dx = \left[\frac{b}{a^2}\frac{x^5}{5}\right]_0^a$$

$$I_y = \frac{a^3b}{5} \quad ◀$$

Radii of Gyration k_x and k_y. We have, by definition,

$$k_x^2 = \frac{I_x}{A} = \frac{ab^3/21}{ab/3} = \frac{b^2}{7} \qquad k_x = \sqrt{\tfrac{1}{7}}\,b \quad ◀$$

and

$$k_y^2 = \frac{I_y}{A} = \frac{a^3b/5}{ab/3} = \tfrac{3}{5}a^2 \qquad k_y = \sqrt{\tfrac{3}{5}}\,a \quad ◀$$