2º Examen Entrenamiento

(1)
$$\begin{cases} u_t = (1+\ln(1+t))\, u_{xx} + u \\ u(x,0) = e^{-x^2} \end{cases}$$

Transformamos u en $\hat{u} = \mathcal{F}_x[u]$ (respecto a x):

$$\hat{u}_t = (1+\ln(1+t))\,\hat{u}\cdot(i\omega)^2 + \hat{u} \Rightarrow \qquad \hat{u}(\omega,0) = \mathcal{F}[e^{-x^2}] = \sqrt{\pi}\, e^{-\omega^2/4}$$

$$\Rightarrow \frac{d}{dt}(\ln(\hat{u})) = 1 - \omega^2(1+\ln(1+t)) \Rightarrow$$

$$\Rightarrow \ln(\hat{u}(\omega,t)) - \ln(\hat{u}(\omega,0)) = t - \omega^2 t - \omega^2\int_0^t \ln(1+s)\,ds \Rightarrow$$

$$\Rightarrow \hat{u}(\omega,t) = \sqrt{\pi}\, e^{-\omega^2/4 + t - \omega^2(t+1)\ln(t+1)}$$

Transformada inversa:

$$\mathcal{F}^{-1}[e^{-\omega^2/4b}] = \sqrt{\frac{b}{\pi}}\, e^{-bx^2}$$, luego escribimos:

$$\hat{u} = \underbrace{\sqrt{\pi}\, e^{t}}_{\text{constante respecto a } x}\, e^{-\omega^2\cdot\underbrace{\left[\frac{1}{4} + (t+1)\ln(t+1)\right]}_{1/4b}} \Rightarrow \quad 1/4b \Rightarrow b = \frac{1}{1+4(t+1)\ln(t+1)}$$

$$\Rightarrow u(x,t) = \sqrt{\pi}\, e^{t}\cdot\sqrt{\frac{b}{\pi}}\, e^{-bx^2} = e^{t}\sqrt{\frac{1}{1+4(1+t)\ln(1+t)}}\; e^{-\frac{1}{1+4(1+t)\ln(1+t)}x^2}$$

$$u(2, e^3-1) = \dots = \sqrt{\frac{1}{1+12e^3}}\; e^{e^3-1-\frac{4}{1+12e^3}} \qquad \boxed{\text{Es la (2)}}$$

↳ Usamos que $(1+t)\ln(1+t) = \ln\left((1+t)^{1+t}\right)$

(2)
$$\begin{cases} w_{tt} + 2w_t + 8w = g(t), \quad w(0)=0, \quad w'(0)=1 \\ g = \begin{cases} \pi - t & \text{en } (0,\pi) \\ \operatorname{sen} t & \text{en } (\pi,\infty) \end{cases} \end{cases}$$

Aplicamos $\mathcal{L}$ a la ecuación:

$$\hat{w} z^2 - \cancel{w(0)}^{0}\cdot z - w'\cancel{(0)}^{1} + 2(z\hat{w} - \cancel{w(0)}^{0}) + 8\hat{w} = \hat{g} \Rightarrow$$

$$\Rightarrow \hat{w} = (\hat{g}+1)\cdot\frac{1}{z^2+2z+8}$$

Calculemos $\hat{g}$:

$$g(t) = (\pi - t)\cdot\left[H(t) - H(t-\pi)\right] + \text{sen}\, t\cdot H(t-\pi) =$$

$$= (\pi - t)H(t) + (t-\pi)H(t-\pi) - \text{sen}(t-\pi)H(t-\pi) \Rightarrow$$

$$\Rightarrow \hat{g} = \frac{\pi}{z} - \frac{1}{z^2} + e^{-\pi z}\cdot\frac{1}{z^2} - e^{-\pi z}\cdot\frac{1}{1+z^2}$$

Luego $\hat{w}(4) = \frac{1}{32}\left(\frac{\pi}{4} - \frac{1}{16} + e^{-\pi 4}\left(\frac{1}{16} - \frac{1}{17}\right)\right) = \dots$

$$= \frac{1}{2^9}\left(15 + 4\pi + e^{-4\pi}\cdot\frac{1}{17}\right). \quad \boxed{\text{Opción (5)}}$$

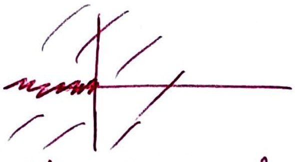
(3) $\sqrt{z} = \sqrt{|z|}\left(\cos\left(\frac{\text{Arg}(z)}{2}\right) + i\,\text{sen}\left(\frac{\text{Arg}(z)}{2}\right)\right) \to$ Rama principal de $\sqrt{z}$.

$$z(\cosh(\sqrt{z}) - 1)w'' + \frac{\text{sen}(2z)}{8}w' - \text{sen}(z)w = 0.$$

Para ver el tipo de singularidad del punto debemos, en primer lugar, decidir si los coeficientes son analíticos en un entorno de 0, y luego el tipo de polo que surge al dividir por el coeficiente de w''.

- $\frac{\text{sen}(2z)}{8}$ y $\text{sen}(z)$ son analíticos.
- ¿Es $\cosh(\sqrt{z}) - 1$ analítica?

Sabemos que $\sqrt{z}$ es analítica en:



En principio, $\cosh(\sqrt{z}) - 1$ no está definida en el semieje real negativo. Sin embargo, $\cosh(w) - 1 = \sum_{k=1}^{\infty} \frac{w^{2k}}{(2k)!}$, y esto es válido $\forall w$.

En particular, si $w = \sqrt{z}$ (donde $\sqrt{z}$ esté definida), se tiene:

$$\cosh(\sqrt{z}) - 1 = \sum_{k=1}^{\infty} \frac{z^k}{(2k)!}$$ $\to$ Esta serie converge en todo $\mathbb{C}$. Luego podemos extender la definición de $\cosh(\sqrt{z}) - 1$ a todo $\mathbb{C}$ de manera analítica.

Escribimos la ecuación en forma estándar:

$$w'' = -\underbrace{\frac{\operatorname{sen}(2z)}{8z(\cosh(\sqrt{z})-1)}}_{b(z)/z} w' + \underbrace{\frac{\operatorname{sen} z}{z(\cosh(\sqrt{z})-1)}}_{a(z)/z^2} w$$

$$b(z) = -\frac{\operatorname{sen}(2z)}{8(\cosh(\sqrt{z})-1)} \Rightarrow b(0) = \lim_{z\to 0} -\frac{\operatorname{sen}(2z)}{8(\cosh(\sqrt{z})-1)} = \lim_{z\to 0} \frac{-2z}{8\cdot(z/2)} = \frac{-1}{2}$$

$$a(z) = \frac{z\operatorname{sen} z}{\cosh(\sqrt{z})-1} \Rightarrow a(0) = \lim_{z\to 0} \frac{z\operatorname{sen} z}{\cosh(\sqrt{z})-1} = 0$$

El sistema queda determinado por $C = \begin{bmatrix} 0 & 1 \\ 0 & 1/2 \end{bmatrix}$.

Autovalores: $\lambda_1 = \frac{1}{2}$, $\lambda_2 = 0$.

Sol. general: $w(z) = C_1 z^{1/2} p_1(z) + C_2 p_2(z)$, $p_1(0) = p_2(0) = 1$.

La solución es la (14), porque $z^{1/2} p_1(z)$ es $o(z^{1/4})$, ya que

$$\frac{z^{1/2} p_1(z)}{z^{1/4}} \xrightarrow[z\to 0]{} 0$$

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(4) $(1+t^2)w'' + \frac{t^2}{1+t^2} w = 0$, $w(0)=0$, $w'(0)=1$.

Buscamos una solución $w = \sum c_k t^k$. Nota: la ecuación está definida en $t>0$. Pero si extendemos w a todo $\mathbb{R}$ usando la misma serie se podría ~~...~~ hablar del desarrollo de Taylor de w.

$$(1+t^2)w'' + \frac{t^2}{1+t^2} w = 0 \Rightarrow (1+t^2)^2 w'' = -t^2 w \Rightarrow$$

$$\Rightarrow (1+t^4+2t^2) w'' = -t^2 w \Rightarrow$$

$$\Rightarrow (1+t^4+2t^2) \sum_{n=2}^{\infty} n(n-1) t^{n-2} c_n = -t^2 \sum_{n=0}^{\infty} t^n \cdot c_n$$

$t^0 \to 2c_2 = 0$

$t^1 \to 6c_3 = 0$

$t^2 \to 4c_2 + 12c_4 = -c_0$

$t^3 \to 12c_3 + 20c_5 = -c_1$

$t^4 \to 2c_2 + 24c_4 + 30c_5 = -c_2$

$\vdots$

$t^n \to (n+2)(n+1)c_{n+2} + 2n(n-1)c_n + (n-2)(n-3)c_{n-2} = -c_{n-2}$

$$c_{k+2} = -\frac{c_{k-2}(1+(k-2)(k-3)) + 2k(k-1)c_k}{(k+2)(k+1)}.$$

$\boxed{\text{la recurrencia lleva a la (9)}}$ y $c_5 = -1/20$. Luego (9)

(5) $$u_t = \left(1 + \frac{t}{\sqrt{2+2t+t^2}}\right) u_{xx}$$

$$u(x,0) = e^{-x^2}$$

$\hat{u}(w,t) = \mathcal{F}_x(u(x,t))$. Aplicamos $\mathcal{F}$ a la ecuación:

$$\hat{u}_t = \left(1 + \frac{t}{\sqrt{2+2t+t^2}}\right) \cdot (-w^2 \hat{u}) \Rightarrow$$

$$\Rightarrow \frac{\partial}{\partial t}(\ln(\hat{u})) = \left(1 + \frac{t}{\sqrt{2+2t+t^2}}\right) \cdot (-w^2) \Rightarrow$$

$$\Rightarrow \ln(\hat{u}(w,t)) - \ln(\hat{u}(w,0)) = -w^2 \int_0^t 1 + \frac{s}{\sqrt{2+2s+s^2}}\, ds =$$

$$= -w^2 t - w^2 \int_0^t \frac{s}{\sqrt{2+2s+s^2}}\, ds$$

$$\int \frac{x}{\sqrt{2+2x+x^2}}\, dx = \int \frac{x}{\sqrt{1+(x+1)^2}}\, dx = \frac{1}{2}\int \frac{2x}{\sqrt{1+(x+1)^2}} =$$

$$= \frac{1}{2}\left(\int \frac{2(x+1)}{\sqrt{1+(x+1)^2}}\, dx - \int \frac{2}{\sqrt{1+(x+1)^2}}\right) =$$

$$= \sqrt{1+(x+1)^2} - \operatorname{argsh}(x+1)$$

$$\int_0^t \frac{s}{\sqrt{s+2s+s^2}}\, ds = \sqrt{1+(t+1)^2} - \sqrt{2} - [\operatorname{argsh}(t+1) - \operatorname{argsh}(1)]$$

Luego $$\hat{u}(w,t) = \hat{u}(w,0) \cdot e^{-w^2\left(t + \sqrt{1+(t+1)^2} - \operatorname{argsh}(t+1) - \sqrt{2} + \operatorname{argsh}(1)\right)}$$

Con $\hat{u}(w,0) = \mathcal{F}[e^{-x^2}] = \sqrt{\pi}\, e^{-w^2/4}$ $\Rightarrow$

$$\Rightarrow \hat{u}(w,t) = \sqrt{\pi}\, e^{-w^2\left(\frac{1}{4} + t + \sqrt{1+(t+1)^2} - \operatorname{argsh}(t+1) - \sqrt{2} + \operatorname{argsh}(1)\right)}$$

Llamando $\frac{1}{4b} = \frac{1}{4} + t + \sqrt{1+(t+1)^2} - \operatorname{argsh}(t+1) - \sqrt{2} + \operatorname{argsh}(1)$, podemos hallar $\mathcal{F}^{-1}$

O mejor, $\mathcal{F}^{-1}\left[\sqrt{\pi}\, e^{-w^2 c}\right] = (?)$:

$$\mathcal{F}^{-1}\left[\sqrt{\frac{\pi}{b}}\, e^{-w^2/4b}\right] = e^{-bx^2} \Rightarrow c = 1/4b \Rightarrow b = \frac{1}{4c}$$

$$\Rightarrow \frac{1}{\sqrt{1/4c}}\mathcal{F}^{-1}\left[\sqrt{\pi}\, e^{-w^2 c}\right] = e^{-\frac{1}{4c}x^2} \Rightarrow$$

$$\Rightarrow \boxed{\mathcal{F}^{-1}\left[\sqrt{\pi}\, e^{-w^2 c}\right] = \frac{1}{2\sqrt{c}}\, e^{-x^2/4c}} \Rightarrow$$

$$\Rightarrow u(x,t) = \frac{e^{-x^2/4\cdot\left[\frac{1}{4}+t+\sqrt{1+(t+1)^2}-\operatorname{argsh}(t+1)-\sqrt{2}+\operatorname{argsh}(1)\right]}}{2\sqrt{\frac{1}{4}+t+\sqrt{1+(t+1)^2}-\operatorname{argsh}(t+1)-\sqrt{2}+\operatorname{argsh}(1)}}$$

Opción (2)

(6)
$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{(x-t)^2+y^2}\,dt = \frac{y}{\pi}\int_0^1\frac{t(1-t)}{(x-t)^2+y^2}\,dt \Rightarrow$$

$$\Rightarrow u(1,\alpha) = \frac{\alpha}{\pi}\int_0^1\frac{t(1-t)}{(1-t)^2+\alpha^2}\,dt.$$

Integral indefinida $\int\frac{t(1-t)}{(1-t)^2+\alpha^2}\,dt$

1) Dividimos

$$\begin{array}{l|l} -t^2+t+0 & t^2-2t+1+\alpha^2 \\ +t^2-2t+1+\alpha^2 & -1 \\ \hline -t+1+\alpha^2 & \end{array}$$

$$\int\frac{t(1-t)}{(1-t)^2+\alpha^2}\,dt = -t + \int\frac{-t+1+\alpha^2}{(1-t)^2+\alpha^2} =$$

$$= -t + \int\frac{-t+1}{(1-t)^2+\alpha^2} + \int\frac{\alpha^2}{(1-t)^2+\alpha^2} = -t + \frac{\ln\left(\alpha^2+(1-t)^2\right)}{2} +$$

$$+ \boxed{\int\frac{\alpha^2}{(1-t)^2+\alpha^2}} \longrightarrow \int\frac{\alpha^2}{\alpha^2\left(\left(\frac{1-t}{\alpha}\right)^2+1\right)} = \int\frac{1}{\left(\frac{1-t}{\alpha}\right)^2+1} = \operatorname{atan}\left(\frac{1-t}{\alpha}\right)\cdot(-\alpha)$$

2) Es decir:

$$\int \frac{t(1-t)}{(1-t)^2+\alpha^2}\,dt = -t - \frac{\ln(\alpha^2+(1-t)^2)}{2} - \alpha\,\text{atan}\left(\frac{1-t}{\alpha}\right) \Rightarrow$$

$$\Rightarrow U(1,\alpha) = \frac{1}{\pi}\left(-\alpha + \frac{\alpha}{2}\left(-\ln(\alpha^2)+\ln(\alpha^2+1)\right) - \alpha^2\left(\cancel{\text{atan}(0)}^{\,0} - \text{atan}\left(\frac{1}{\alpha}\right)\right)\right) =$$

$$= \frac{1}{\pi}\left(-\alpha + \frac{\alpha}{2}\left(\ln\left(\frac{\alpha^2+1}{\alpha^2}\right)\right) + \alpha^2\,\text{atan}(1/\alpha)\right) =$$

$$= \frac{1}{\pi}\left(-\alpha + \frac{\alpha}{2}\ln\left(1+\frac{1}{\alpha^2}\right) + \alpha^2\,\text{atan}(1/\alpha)\right). \quad \boxed{\text{Solución! (18)}}$$

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$$\begin{cases} U_t = U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} \\ U(1,\theta,t) = 0 \\ U(r,\theta,0) = J_1(\alpha r)\cos\theta + J_2(\beta r)\,\text{sen}(2\theta) \end{cases}$$

Con $\alpha \neq \beta$, α cero de J_1 y β cero de J_2.

$$U(r,\theta,t) = \cos\theta \sum w_k(t) J_1(\lambda_{k_1} r) + \text{sen}(2\theta) \sum h_k(t) J_2(\lambda_{k_2} r)$$

- Con esta definición, $U(1,\theta,t) = 0$ siempre.
- $U(r,\theta,0) = \cos\theta \sum w_k(0) J_1(\lambda_{k_1} r) + \text{sen}(2\theta)\sum h_k(0) J_2(\lambda_{k_2} r) =$
$$= J_1(\alpha r)\cos\theta + J_2(\beta r)\,\text{sen}(2\theta).$$

Por un lado, $\cos\theta$ y $\text{sen}(2\theta)$ son funciones linealmente independientes en $[-\pi,\pi]$ (de hecho, ortogonales). Luego esta igualdad implica que:

- $\sum w_k(0) J_1(\lambda_{k_1} r) = J_1(\alpha r)$
- $\sum h_k(0) J_2(\lambda_{k_2} r) = J_2(\beta r)$

Ahora, aplicando unicidad de serie de Bessel:

$$\begin{cases} w_k(0) = 0 \text{ si } \lambda_{k_1} \neq \alpha; \quad w_k(0) = 1 \text{ si } \lambda_{k_1} = \alpha \\ \beta_k(0) = 0 \text{ si } \lambda_{k_2} \neq \beta; \quad \beta_k(0) = 1 \text{ si } \lambda_{k_2} = \beta. \end{cases}$$

Veamos ahora las EDOs que ~~cumplen~~ cumplen w_k y β_k.

$$\cos\theta \sum w_k'(t)\, J_1(\lambda_{k_1} r) + \text{sen}(2\theta) \sum \beta_k'(t)\, J_2(\lambda_{k_2} r) =$$

$$= \cos\theta \sum w_k(t)\, \lambda_{k_1}^2 J_1''(\lambda_{k_1} r) + \text{sen}(2\theta) \sum \beta_k(t)\, \lambda_{k_2}^2 J_2''(\lambda_{k_2} r) +$$

$$+ \frac{1}{r}\left[\cos\theta \sum w_k(t)\, \lambda_{k_1} J_1'(\lambda_{k_1} r) + \text{sen}(2\theta) \sum \beta_k(t)\, \lambda_{k_2} J_2'(\lambda_{k_2} r)\right] +$$

$$+ \frac{1}{r^2}\left[-\cos\theta \sum w_k(t)\, J_1(\lambda_{k_1} r) - 4\,\text{sen}(2\theta) \sum \beta_k(t)\, J_2(\lambda_{k_2} r)\right].$$

Como $\cos\theta$ y $\text{sen}(2\theta)$ son independientes, esta igualdad implica:

$$(*) \begin{cases} \sum w_k'(t)\, J_1(\lambda_{k_1} r) = \sum w_k(t)\, \lambda_{k_1}^2 J_1''(\lambda_{k_1} r) + \frac{1}{r} \sum w_k(t)\, \lambda_{k_1} J_1'(\lambda_{k_1} r) - \\ \quad - \frac{1}{r^2} \sum w_k(t)\, J_1(\lambda_{k_1} r) \\ \sum \beta_k'(t)\, J_2(\lambda_{k_2} r) = \sum \beta_k(t)\, \lambda_{k_2}^2 J_2''(\lambda_{k_2} r) + \frac{1}{r} \sum \beta_k(t)\, \lambda_{k_2} J_2'(\lambda_{k_2} r) - \\ \quad - \frac{4}{r^2} \sum \beta_k(t)\, J_2(\lambda_{k_2} r) \end{cases}$$

J_1 cumple: $x^2 J_1'' + x J_1' + (x^2 - 1) J_1 = 0$ $\Rightarrow$ Evaluamos en $\lambda_{k_i} r$

J_2 cumple: $x^2 J_2'' + x J_2' + (x^2 - 4) J_2 = 0$

$$\Rightarrow \begin{cases} \lambda_{k_1}^2 r^2 J_1''(\lambda_{k_1} r) + \lambda_{k_1} r J_1'(\lambda_{k_1} r) + (\lambda_{k_1}^2 r^2 - 1) J_1(\lambda_{k_1} r) = 0 \\ \lambda_{k_2}^2 r^2 J_2''(\lambda_{k_2} r) + \lambda_{k_2} r J_2'(\lambda_{k_2} r) + (\lambda_{k_2}^2 r^2 - 4) J_2(\lambda_{k_2} r) = 0 \end{cases}$$

Luego el sistema (*) queda:

$$\begin{cases} \sum w_k'(t)\, J_1(\lambda_{k_1} r) = -\sum \lambda_{k_1}^2 w_k(t)\, J_1(\lambda_{k_1} r) \\ \sum \beta_k'(t)\, J_2(\lambda_{k_2} r) = -\sum \lambda_{k_2}^2 \beta_k(t)\, J_2(\lambda_{k_2} r) \end{cases}$$

Por unicidad de Bessel:

$$\begin{cases} w_k' = -\lambda_{k_1}^2 w_k(t) \\ \beta_k' = -\lambda_{k_2}^2 \beta_k(t) \end{cases} \Bigg/ \quad \begin{array}{l} w_k(0) = 0 \text{ salvo en } k_1 = k_\alpha, \text{ que vale } 1 \\ \beta_k(0) = 0 \text{ salvo en } k_2 = k_\beta, \text{ que vale } 1 \end{array}$$

La solución es $w_k(t) = \begin{cases} e^{-\alpha^2 t}, & k = k_\alpha \\ 0, & k \neq k_\alpha \end{cases}$, $\beta_k(t) = \begin{cases} e^{-\beta^2 t}, & k = k_\beta \\ 0, & k \neq k_\beta \end{cases}$

Luego $u(r,\theta,t) = e^{-\alpha^2 t} \cos\theta\, J_1(\alpha r) + e^{-\beta^2 t}\, \text{sen}(2\theta)\, J_2(\beta r)$. $\boxed{\text{Es la (17)}}$