2parcial-2019.pdf

Aeropro

Ampliación de Matemáticas

3º Grado en Ingeniería Aeroespacial

Escuela Técnica Superior de Ingeniería Aeronáutica y del Espacio
Universidad Politécnica de Madrid

KEEP
CALM
AND
ESTUDIA
UN POQUITO

Examen 20-12-2019 (2º parcial)

A Problema de Cauchy.

$$\frac{\partial u}{\partial t} = (1+\tanh(t))\frac{\partial^2 u}{\partial x^2} \quad \text{en } (x,t)\in \mathbb{R}\times]0,\infty[$$

$u(x,0) = \exp(-2x^2)$ con $x \in \mathbb{R}$

$u(x,t)$ uniformemente acotada en $\mathbb{R}\times(0,\infty)$

Nota:

$$u(x,t) = \int_{-\infty}^{\infty} u(x,t)\exp(-i\omega x)\,dx$$

La función $u(x,t)$ verifica que:

$$F\left(\frac{\partial u}{\partial t}(x,t)\right)(\omega) = F\left((1+\tanh(t))\frac{\partial^2 u}{\partial x^2}(x,t)\right)(\omega)$$

$$\frac{\partial \hat u}{\partial t} = (1+\mathrm{tah}(t))(-i\omega)^2\,\hat u(x,t))$$

Integro para obtener $\hat u(\omega)$:

$$\int\frac{\partial\hat u}{\hat u} = \int(\ +\tanh t)(-i\omega\ )^2\,dt$$

$$\ln\hat u = (i\omega)^2 t + \int \tanh t\,(i\omega)^2\,dt$$

$$\ln\hat u = (i\omega)^2 t + \int\frac{\operatorname{senh} t}{\cosh t}\, i^2\omega^2\,dt = (i\omega)^2 t + \ln(\cosh t)(i\omega)^2 + C$$

$$\hat u(x,t) = e^{-\omega^2 t - \omega^2\ln\cosh t}\cdot C$$

¿C?

$$u(x,0) = \exp(-2x^2) \longrightarrow \mathcal{F}(-2x^2) = \sqrt{\frac{\pi}{2}}\exp\left(\frac{-\omega^2}{4\cdot 2}\right) = \sqrt{\frac{\pi}{2}}\exp\left(\frac{-\omega^2}{8}\right)$$

$$\hat u(\omega,t) = \sqrt{\frac{\pi}{2}}\exp\left(-\omega^2\left(\frac{1}{8}+t+\ln\cosh t\right)\right)$$

Aplicando la transformada inversa ($\hat u(\omega,t) \longrightarrow u(x,t)$):

$$u(x,t) = \sqrt{\frac{\pi}{2}}\,\mathcal{F}^{-1}\left(\exp\left(-\omega^2(\underbrace{1/8+t+\ln\cosh t}_{b})\right)\right) \qquad F(\ ) = \sqrt{\frac{\pi}{b}}\exp\left(\frac{-x^2}{4b}\right)$$

$$u(x,t) = \sqrt{\frac{\cancel{\pi}}{2}}\cdot\frac{1}{2\cancel{\pi}}\sqrt{\frac{\cancel{\pi}}{1/8+t+\ln\cosh t}}\cdot\exp\left(\frac{-x^2}{4(1/8+t+\ln\cosh t)}\right)$$

$$u(x,t) = \sqrt{\frac{1}{1+8t+8\ln\cosh t}}\exp\left(\frac{-x^2}{1/2+4t+4\ln\cosh t}\right)$$

$$u(4,4) = \sqrt{\frac{1}{33+8\ln\cosh 4}}\exp\left(\frac{-16}{33/2+4\ln\cosh 4}\right) = \frac{1}{\sqrt{33+8\ln\cosh 4}}\exp\left(\frac{-32}{33+8\ln\cosh 4}\right)$$

B $$\frac{d^2w}{dt^2}(t) + 2\frac{dw}{dt}(t) + 8w(t) = g(t) \text{ en } (0,\infty)$$

$$w(0) = 0 \qquad g(t) = \cos t \text{ con } t \in [0, \pi/2) \text{ y } g(t) = 0 \text{ con } (\pi/2, \infty)$$

$$\frac{dw}{dt}(0) = 1$$

¿Transformada de Laplace de la función $w: [0,\infty) \longrightarrow \mathbb{R}$ es?

$$\cancel{g(t) = (\pi/2 - t)[H(t) - H(t - \pi/2)]}$$

$$g(t) = \cos t\,[H(t) - H(t - \pi/2)] = \cos t\, H(t) + H(t - \pi/2)\,\text{sen}(t - \pi/2)$$

$$\mathcal{L}\left(\frac{d^2w}{dt^2}\right)(z) + 2\mathcal{L}\left(\frac{dw}{dt}\right)(z) + 8\mathcal{L}[w](z) = \mathcal{L}[g(t)](z)$$

$$z\mathcal{L}\left(\frac{dw}{dt}\right)(z) - \frac{dw}{dt}(0) + 2\left(z\mathcal{L}(w)(z) - w(0)\right) + 8\mathcal{L}(w)(z) = \mathcal{L}(g(t))(z)$$

$$z\left(z\mathcal{L}(w)(z) - w(0)\right) - \frac{dw}{dt}(0) + 2\left(z\mathcal{L}(w)(z) - w(0)\right) + 8\mathcal{L}(w)(z) = \mathcal{L}(g(t))$$

$$\cancel{z^2\mathcal{L}(w)(z) - \frac{dw}{z} + 2z\mathcal{L}(w)(z) + 8\mathcal{L}(w)(z) = \mathcal{L}(g(t))(z)} + \ldots$$

$$(z^2 + 2z + 8)\mathcal{L}(w)(z) = \cancel{1} + \mathcal{L}(g(t))(z) + w(0) + z\,w(0) + \frac{dw}{dt}(0)$$

$$\mathcal{L}(w)(z) = \frac{1}{z^2+2z+8}\left(\underbrace{\mathcal{L}(g(t))(z)}_{\text{Homogenea}} + \underbrace{(1+z)w(0) + \frac{dw}{dt}(0)}_{\text{particular}}\right)$$

¿$\mathcal{L}(g(t))$?

$$\mathcal{L}(g(t)) = \mathcal{L}(\cos t) + e^{-\pi/2\, z}\mathcal{L}(\text{sen}(t - \pi/2))$$

$$= \frac{z}{z^2+1} + e^{-\pi/2\, z}\frac{1}{z^2+1}$$

$$\mathcal{L}(w)(z) = \frac{1}{z^2+2z+8}\left(\frac{1}{z^2+1}(z + e^{-\pi/2\, z}) + \cancel{(1+z)w(0)} + \frac{d(w)}{dt}(0)\right)$$

$$\mathcal{L}(w)(z) = \frac{1}{z^2+2z+8}\left(1 + \frac{z}{z^2+1} + \frac{e^{-\pi/2\, z}}{z^2+1}\right)$$

Solución:

$$\mathcal{L}(w(t))(2) = \frac{1}{16}\left(1 + \frac{2}{5} + \frac{e^{-\pi}}{4+1}\right) = \frac{1}{16}\left(\frac{7}{5} + \frac{e^{-\pi}}{5}\right) = \frac{1}{80}(7 + e^{-\pi})$$

$\boxed{C}$ $\frac{d^2w}{dz^2} - z^3 w = 0 \in \mathbb{C}$ $w(0)=0$ $\frac{dw}{dz}(0) = i$

$w(z) = \sum_{k=0}^{\infty} C_k z^k$ ¿La función $w(z)$ cumple que?

$$\sum_{k=2}^{\infty} C_k z^{k-2} k\cdot(k-1) - z^3 \sum_{k=0}^{\infty} C_k z^k = 0$$

$\rightarrow C_0 + C_1 z + C_2 z^2 + \dots$

$iz + \sum_{k=2}^{\infty} C_k z^k$

Aplicando las condiciones:

$$\sum_{k=2}^{\infty} C_k z^{k-2} k(k-1) - z^3\left(iz + \sum_{k=2}^{\infty} C_k z^k\right) = 0$$

Ya sabemos que $w(z) = iz + \sum_{k=2}^{\infty} C_k z^k$

El primero no nulo:

$C=6 \longrightarrow C_6 z^4 (6\cdot 5) - z^3(iz^0 + C_6 z^6) = 0 \longrightarrow 6\cdot 30 = i \rightarrow C_6 = \frac{i}{30}$

$C=11 \longrightarrow C_{11} z^9 (11\cdot 10) - z^3(iz + C_{11} z^{11}) = 0$

$C = k \longrightarrow C_k z^{k-2}(k(k-1)) - z^3(iz + C_k z^k) = 0$

$C_k\left[z^{k-2}(k-1)k - z^{3+k}\right] \neq iz^4$

chapuza

$C_k z^{k-2} k(k-1) = z^3 C_k z^k \longrightarrow C_k (k-1) k = z^5 C_k$

$\boxed{C_k = \frac{C_{k-5}}{k(k-1)}}$ $\xrightarrow[\text{a } 0]{\text{tiende}} 0$

$w(z) = C_5 z^5 = z^5 C_k \longrightarrow C_{k-5}$

$C_{11} = \frac{C_6}{11\cdot 10} = \frac{i}{30\cdot 11\cdot 10}$

$w(z) = iw + \frac{i}{30} z^6 + \frac{i}{30\cdot 11\cdot 10} z^{11}$

$\lim_{z\to 0} \left(\quad \right) \longrightarrow \frac{i}{3300}$

D

$$z^2\frac{d^2w}{dz^2}+\frac{\operatorname{senh}(2z)}{4}\frac{dw}{dz}-\operatorname{sen}(z)\,w=0 \qquad -\frac{\operatorname{senh}(2z)}{4z}=-\frac{\cosh z\operatorname{senh} z}{2z}$$

$$\frac{d^2w}{dz^2}=-\frac{\operatorname{senh}(2z)}{4z^2}\frac{dw}{dz}+\frac{\operatorname{sen}(z)\,w}{z^2}\longrightarrow\frac{d^2w}{dz}=-\underbrace{\frac{\operatorname{senh}(2z)}{4z}}_{b(z)}\frac{1}{z}\frac{dw}{dz}+\frac{\overbrace{\operatorname{sen}(z)}^{a(z)}w}{z^2}$$

punto singular regular

$$G=\begin{pmatrix}0&1\\a(0)&b+1\end{pmatrix}=\begin{pmatrix}0&1\\0&-1/2+1\end{pmatrix}\qquad\lambda=0\quad\lambda=1/2$$

~~$w(z)=C\sqrt{z}\,p_1+C_2p_2=C_1\sqrt{z}+C_2$~~ $\qquad w(z)=C_1\sqrt{z}\,p_1+C_2p_2(z)$

$$\forall C\neq 0\longrightarrow\lim_{z\to 0}C\sqrt{z}\,p_1=0$$

$$\xrightarrow{\text{sol}}\lim\frac{C_1\sqrt{z}\,p_1+C_2p_2}{\ln z}=0$$

En este caso la función es no nula pero el lim. es nulo

$$\lim_{z\to 0}\frac{C_1\sqrt{z}\,p_1(z)+C_2p_2(z)}{\sqrt{z}}=0$$

solo si $C_1=C_2=0$.

Función y límite nulo.

E

$$\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}=0\quad\text{con}\quad u(x,0)=1-x^2\quad\text{para}\ -1\le x\le 1$$

$$u(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\frac{y}{(x-t)+y^2}dt\qquad ¿u(3,1)?$$

$$u(x,y)=\frac{1}{\pi}\int_{-1}^{1}\frac{1-t^2}{(3-t)+1}dt=\frac{1}{\pi}\int_{-1}^{1}\frac{1-t^2-6t+6t+10-10}{1+9+t^2-6t}dt$$

$$\frac{1}{\pi}\int_{-1}^{1}\left(-1+\frac{-6t+11}{1+(t-3)^2}\right)dt=\frac{1}{\pi}\int_{-1}^{1}-1+\frac{-6(t-3)+7}{1+(t-3)^2}dt$$

Necesito la derivada

$$\frac{1}{\pi}\left(-t-3\ln\left((t-3)^2+1\right)+7\operatorname{arctg}(t-3)\right)\Big|_{-1}^{1}=$$

$$\frac{1}{\pi}\left(-2-3\ln(5)+3\ln(17)+7\operatorname{arctg}(-2)-7\operatorname{arctg}(-4)\right)$$

$$\boxed{\frac{1}{\pi}\left(-2+3\ln\frac{17}{5}-(\operatorname{arctg}(4)-\operatorname{arctg}2)\right)}$$