MYP Related Rates

Jake Bruner

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1 Introduction

1.1 Background

Mixology is the careful craft of creating hand-crafted (often alcoholic) drinks for people to enjoy. Mixologists gain their skill over years of experience and muscle memory; their job requires mastery in pouring and measuring out precise quantities of drink. Often those new to the prospect of drink making require measuring cups and spoons to ensure the height of drink and the amount of drink within the glass are appropriate. Mixologists often have a learnt ability to understand how much the height of drink in the glass will change given their pour rate. A novice, however, may require more careful mathematical planning to understand how the height and volume change with time given their pour rate. One notable challenge for aspiring mixologists is *martini glasses*!

Figure 1: Martini Glass

Due to their lack of cross-sectional symmetry (see Figure 1), the drink level within these glasses does not change proportionally to the change in volume (i.e., $dV \not\propto dh$). For instance, a person drinking a beverage at a constant two/three sips per minute will find that their drink level decreases very slowly at first, then speeds up as more of the drink is consumed up until the last sip at the bottom and its time to order another. Hence:

$$\frac{dV}{dt} \not\propto \frac{dh}{dt}$$

1.2 Intent

For these reasons, it is helpful and applicable to understand how the drink level in a martini glass may change with time in *relation* to the change in volume with time. In other words, I seek to model the relation between the rate of change in volume and height per unit time.

2 Methodology and Data

2.1 Method

Figure 2: Experimental Setup

To sufficiently solve for the change in height with respect to time, it requires that a few key measurements be taken. First the overall slant height and diameter will be determined in order to relate the drink height to the glass's radius from the center at any given drink height in the cone. Once this ratio of height to radius is determined, the function of the cone's volume is only in terms of height. From here, the experimental setup consists of a stopwatch, a measuring cup, and a conical glass with slant height markings per centimetre. Using these materials and a slo-motion recording, the predetermined 200mL of water will be poured by hand at a near-constant rate. Then using video software, the total time of the experiment and the flow rate will be determined by subtracting the time when the water first hits the glass from the time when the water supply is depleted. This will give the average flow rate and the total time of the experiment.

Following this setup, the video will be consulted and data will be recorded using the measurements on the slant. Using the slant height as the independent variable, the time will be recorded at each 1 centimetre interval. This time will again be manipulated to reflect the time since the start of the water flow. From this point, the Pythagorean ratio of slant height to height will be used to record the different heights. Plugging these values into the proper derived equations will then hopefully yield the change in height with time.

This data will be iteratively manipulated using a spreadsheet, but the process is also laid out in full detail in the following sections.

2.2 Video

The method requires that a video be taken. The video was done in 120fps and slowed down to 24fps to accurately measure the timestamps and the slant height. The video can be found at this link: `https://youtu.be/2ZIjisp_kNQ`

Figure 3: Screenshot of video

2.3 Data Collected

Cone's Dimensions

Using a measuring tape in metric units, measurements of the glass's exteriour were taken and recorded. Data was recorded to two decimal places: the most the measuring devices allowed. The top of the glass had a diameter of 11.75 cm, and therefore a radius of 5.875 cm. The slant height of the entire glass cone was about 10.16 cm. Since the radius and slant height form a right angle, the Pythagorean identity can be used to determine height:

$$
\begin{aligned}
a^2 + b^2 &= c^2 \\
h^2 + r^2 &= s^2 \\
h &= \sqrt{s^2 - r^2} \\
h &= \sqrt{(10.16cm)^2 - (5.875cm)^2} \\
h &= 8.29cm
\end{aligned}
$$

Hence to two significant figures, the height of the cone is about 8.29 cm.

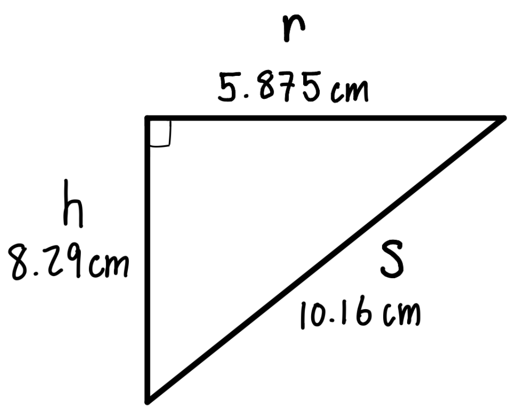

Figure 4: Triangular Cross-Section of the Glass

Flow Rate

Figure 5: First and Last Frame

From the video, the approximate first and last frames of water entering the cup were recorded. Although the flow rate may not have been perfectly accurate, the average flow rate will be a sufficient approximation. The first time the water enters the cone is when the stopwatch hits 7.53 s, and the last frame when water is flowing constantly is when the stopwatch hits 15.80s. To determine the total time , the initial stopwatch time was subtracted from the final time. The predetermined 200mL of water was measured out before the pour, so the average flow rate is the change in volume over change in time. I.e., 200mL water over 8.27 seconds is approx 24.2 mLs^{-1} a.k.a. 24.2 cm^3s^{-1} (since these units match).

2.4 Experimental Data Collected

independent	dependent	manipulated	
slant height (cm)	absolute time (s)	time since pour (s)	height (cm)
0.00	7.53	0.00	0.00
2.00	7.64	0.11	1.63
3.00	8.02	0.49	2.45
4.00	8.50	0.97	3.26
5.00	9.33	1.80	4.08
6.00	10.02	2.49	4.89
7.00	11.28	3.75	5.71
8.00	13.25	5.72	6.53
9.00	15.80	8.27	7.34

Figure 6: Data Table

Using the video recorded, the slant height and time can be measured directly. To make the measurements easier, the slant height was taken as the independent variable and the absolute time was determined dependently. In intervals of 1 cm (see Accuracy), the data was recorded in a spreadsheet. Iterating with equations, the absolute time was subtracted to give the time since the start of experiment, and the ratio between slant height and height was used to determine height.

3 Mathematical Analysis

3.1 Equation of a Cone's Volume

The volume of a cone is given by the equation:

$$V = \frac{1}{3}\pi r^2 h \tag{1}$$

From the measurements outlined in 2.1, the radius of the cone used was $\sim 0.709\times$ its height. So hence:

$$r = 0.709h$$

Substituting this into (1):

$$V = \frac{1}{3}\pi(0.709h)^2 h$$

Simplifying to:

$$V \approx 0.5259h^3 \tag{2}$$

Given the nature of this problem, it is helpful to take the derivative of (2), since both volume and height vary with time here. The derivative will be taken on both sides with respect to time:

$$\begin{aligned}\frac{d}{dt}(V) &= \frac{d}{dt}(0.5259h^3)\\ \frac{dV}{dt} &= 0.5259 * 2h^2\frac{dh}{dt}\\ \frac{dV}{dt} &\approx 1.052h^2\frac{dh}{dt}\end{aligned}$$

Since this problem seeks to model how height varies with time, it would be beneficial to isolate $\frac{dh}{dt}$:

$$\frac{dh}{dt} = \frac{1}{1.052h^2}\frac{dV}{dt} \tag{3}$$

Now that both (2) and (3) are determined, it is possible to input values from measured data to model $\frac{dh}{dt}$.

3.2 Making Predictions from Data

From Section 2.3, $\frac{dV}{dt}$ will be given equal to the approximate average flow rate, 24.2 cubic metres per second. Similarly, the slant height measured in the video will be used to determine the approximate height of the cone at three given points in time. The equation for this in simplified form is:

$$\begin{aligned}s^2 &= h^2 + r^2\\ s &= \sqrt{h^2 + (0.709h)^2}\\ s &= \sqrt{h^2 + 0.502681h^2}\\ s &= \sqrt{1.502681h^2}\\ s &\approx 1.226h\\ h &\approx \frac{s}{1.226} \qquad (4)\end{aligned}$$

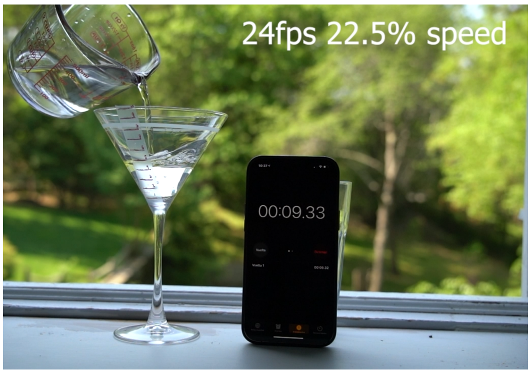

Figure 7: Approx slant height at approx 9.33s-7.53s = 1.80 seconds

Slant Height 5cm

Putting this 5cm slant height into (4):

$$h = \frac{5cm}{1.226}$$
$$h \approx 4.08cm$$

Putting this and the approximate flowrate $\frac{dV}{dt}$ into the first time derivative of the volume equation is:

$$\frac{dh}{dt} = \frac{24.1837cm^3 s^{-1}}{1.052(4.08cm)^2}$$
$$\frac{dh}{dt} = 1.38098\frac{cm^{\cancel{3}\ 1}}{cm^{\cancel{2}\ 0}s}$$
$$\frac{dh}{dt} = 1.38098\ cm\ s^{-1} \tag{5}$$

Hence, this math predicts that 6.47 seconds into the pour when the water has a slant height of 5cm, the height of the water is rising at a rate of ≈ 1.38 centimetres per second.

Slant Height 6cm

Putting this 6cm slant height into (4):

$$h = \frac{6cm}{1.226}$$
$$h \approx 4.89cm$$

Figure 8: Approx slant height at approx 10.02s - 7.53s = 2.49 seconds

Putting this and the approximate flowrate $\frac{dV}{dt}$ into the first time derivative of the volume equation is:

$$\frac{dh}{dt} = \frac{24.1837 cm^3 s^{-1}}{1.052(4.89cm)^2}$$
$$\frac{dh}{dt} = 0.961367\ cm\ s^{-1} \tag{6}$$

The results of doing this process with a greater height and slant height intuitively corresponds to a lower chance in height per time since the radius increases with increasing height. So, at 2.49 seconds from the start of the pour and at a slant height of 6cm, the water's height is rising at a rate of ≈ 0.96 centimetres per second.

Slant Height 7cm

Putting this 7cm slant height into (4):

$$h = \frac{7cm}{1.226}$$
$$h \approx 5.71cm$$

Putting this and the approximate flowrate $\frac{dV}{dt}$ into the first time derivative of the volume equation is:

$$\frac{dh}{dt} = \frac{24.1837 cm^3 s^{-1}}{1.052(5.71cm)^2}$$
$$\frac{dh}{dt} = 0.705074\ cm\ s^{-1} \tag{7}$$

Figure 9: Approx slant height at approx 11.28s - 7.53s = 3.75 seconds

At a slant height of 7cm, and 3.75 seconds into the pour, the change in height will again decrease compared to smaller slant heights coming in at ≈ 0.71 centimetres per second.

Summary of Manipulated Data

The process complete thrice above was then programmed into the spreadsheet so as to get data points for the rest of the experiment.

independent	dependent	manipulated		
slant height (cm)	**absolute time (s)**	**time since pour (s)**	**height (cm)**	**dh/dt (cm/s)**
0.00	7.53	0.00	0.00	#DIV/0!
2.00	7.64	0.11	1.63	8.64
3.00	8.02	0.49	2.45	3.84
4.00	8.50	0.97	3.26	2.16
5.00	9.33	1.80	4.08	1.38
6.00	10.02	2.49	4.89	0.96
7.00	11.28	3.75	5.71	0.71
8.00	13.25	5.72	6.53	0.54
9.00	15.80	8.27	7.34	0.43

Figure 10: Data Table 1 - All data collected and manipulated

Ignoring the divide by zero error, this process gave data points for each trial. At first glance this also fits a mental picture of decreasing change in height with increasing height.

4 Analysis

4.1 Graphical Representations of Data

Graph 1.1 plots the height of the water with respect to time. Each point features a blue line representing the derivative (as slope) at the point. These

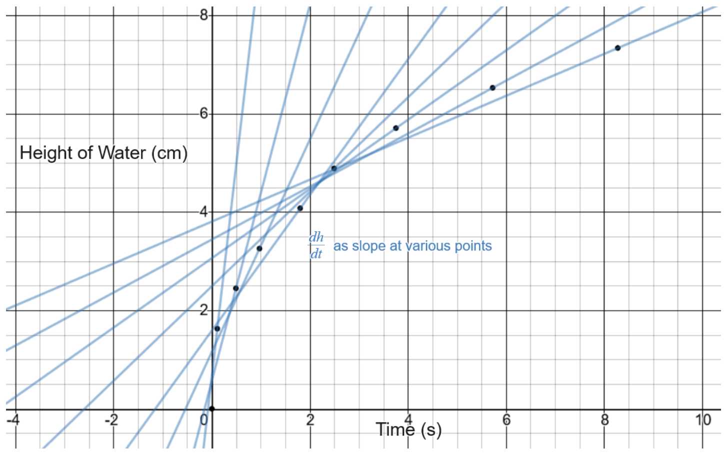

Figure 11: Graph 1.1 - Height of Water vs. Time with $\frac{dh}{dt}$ as slope

lines are expected to become less accurate as the distance from the point increases, but they still represent a modest indication of where the height of water is going over time.

This interpretation is valuable because it depicts the extent to which the calculated $\frac{dh}{dt}$ was accurate. The blue lines successfully demonstrate decreasing slope over increasing time, corroborating the hypothesized relationship.

Graph 1.2 provides a less intuitive perspective, but perhaps a more tangible insight into the relationship between height and change in height. In a form of phase space, this graph demonstrates how this system will evolve with time from any initial condition. *(one crucial note is that this represents phase space* ***only*** *when given the exact same constant change in volume over time, and ratio of radius to height.)* This graph shows the relationship between height and change in height of this system. It supports the conclusion that change in height is inversely proportional to the height of water in the glass. Furthermore the regression line hints that the relationship may be that of an inverse square root. The R-squared value also demonstrates that this regression fits the data with a high degree of accuracy.

4.2 Significance of Results

As outlined above, the data collected fits the picture of the non-linear relationship between the rate of change in volume and the rate of change in water height. Graph 1.2 demonstrates convincingly that the change in height decreases with increasing height. This relationship fits the intuition that a martini glass undergoing constant change in volume will have a non-constant change in height. The parts of the glass with a greater radius on their horizontal cross section will hold more volume, so it follows the vol-

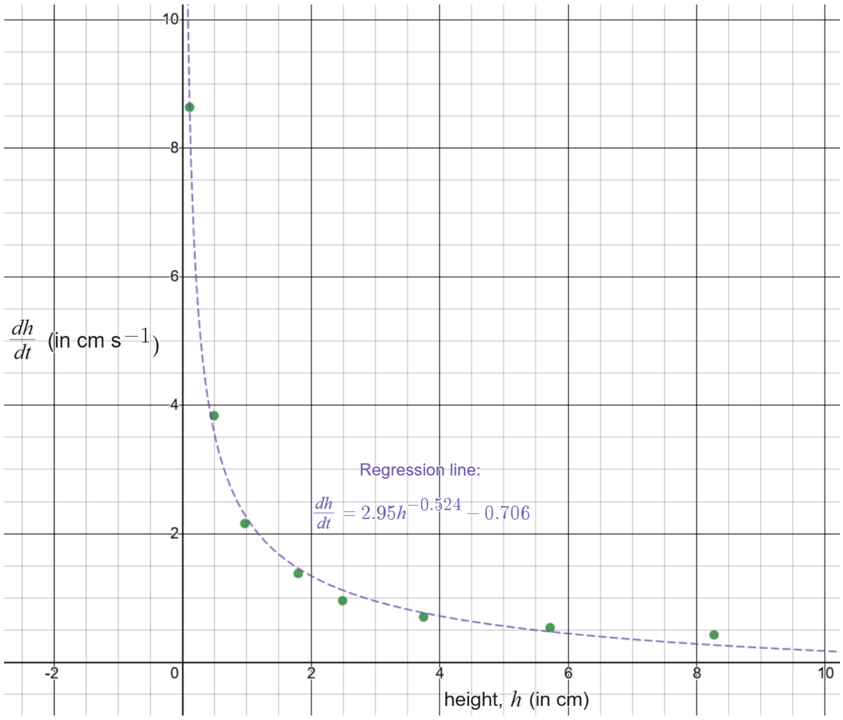

Figure 12: Graph 1.2 - Phase Space of the system ($R^2 = 0.997$)

ume will take longer to be added/removed than a comparatively smaller cross section.

5 Conclusion

5.1 Accuracy of Results

From the results achieved, it is clear that the mathematical model was effective at quantifying how the system evolved with respect to time. The data demonstrates the hypothesised link between change in height and change in volume. Within the data, there were no significant outliers and little obvious error. For these reasons, this method was–to a great extent–effective in illustrating how the system evolved with time.

One area where the method could be improved was the rigor of the rounding during the calculations. Most of the measurement devices used within the experiment were accurate to two decimal places, yet once mathematical operations began to be preformed, the rounding was disorganized and sometimes arbitrary. For instance, $\frac{1}{3}\pi(0.709)^2$–an equation involving an already rounded ratio accurate to three significant figures–became 0.5259.

Another flaw in the method is the measurements involving the slant height of the water level. These measurements were taken with a strip of clear tape marked at 1 centimetre intervals. Since this was only accurate to one significant figure, the method attempted to circumvent this possible error by dependently measuring a variable measured to more precision: time. However, determining when the slant height was exactly an integer value using a video was an error-prone method. In the slow-motion video, multiple frames could have qualified as a certain slant height, so some of the time measurements are likely slightly inaccurate.

Another significant limitation of this process within the real-world is that martini glasses are often not exactly conical. Their bottom usually terminates in a roundish section. This means that within the method, about half a centimetre of space at the bottom consisted of glass and could not be filled. For this reason, the video could not be used to accurately timestamp at 1 cm slant height. This slight error was also demonstrated through the regression of the phase space. During the regression, the equation proved much more accurate when the function included a constant term. This term ended up being -0.706 cms^{-1}, meaning that some constant error was creating a different change in height versus measured height. One could conjecture this error came from a difference in the height to the actual volume of water the cup could hold, creating a slightly altered change in height over time in the mathematical versus the real.

5.2 Context Within The Real World

Within the real world, the results achieved from this experiment are modestly applicable. Many of the assumptions made in the experiment, despite being mathematically accurate and acceptable, don't directly translate to a real-world version of this experiment. For instance, it is difficult to pour liquid at a flow rate since it often decreases as the liquid source depletes. In addition, it is only helpful to preform this model when the total amount of liquid one seeks to pour is known. For instance, this method would not sufficiently answer how long one should pour in order to achieve a certain water level. Now despite these limitations, this model does serve to provide meaningful intuition for how the height in a conical glass evolves with time. As outlined in the Background, an aspiring mixologist could benefit from the familiarity the model provides.